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Dupin's indicatrix: a tool for quantifying periclinal folds on maps

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Abstract – The elliptical and hyperbolic outcrop patterns characteristic of periclinal folds can be used to classify structures according to different curvature attributes. Elliptical patterns indicate domal-basinal structures with synclastic curvature, that is, principal curvatures of the same sign. Hyperbolic patterns are diagnostic of anticlastic curvature (saddle-like structures). Such outcrop geometries are geological examples of Dupin's indicatrix, the geometrical figure obtained by sectioning a curved surface on a plane parallel and almost coincident with the tangent plane. The aspect ratio of Dupin's indicatrix is theoretically related to the ratio of the principal curvature values for the part of the structure being considered. This new method allows quantitative assessment of structures on maps and on remote sensing images. Illustrations are given from Wyoming, USA, and Yorkshire, England.

Keywords: folds, style, geometry, maps, cross-sections.

1. Introduction

Fleuty (1964) provided a terminological framework for the description of folds that has served structural geologists well for four decades. Fleuty's classic paper represented a significant advance because of the way existing basic descriptive terms such as 'hinge', 'limb' and 'interlimb angle' were given exact definitions based on geometry. It enabled structural geologists to apply familiar terms to fold structures in a more objective fashion.

Most of the definitions in that paper, including the fundamental terms 'fold hinge line', 'fold limb', 'axial surface' and 'inflection line' are founded on a simple geometrical model of fold geometry: the cylindrical model. A cylindrical fold is one whose surface everywhere contains a line of fixed orientation; its surface can be generated or swept out by a line which moves while remaining parallel to itself. Such folds show a curvature only in one sense. The assumption of cylindrical geometry lies at the heart of the geometrical analysis of folds. The methods based on this assumption have even been applied to structures of more complex geometry such as refolded folds, by subdividing these non-cylindrical structures into smaller regions with approximately cylindrical properties.

There is a general awareness amongst structural geologists that all natural folds deviate from a cylindrical form to some degree. For predicting fold geometry in the subsurface from scant information obtained at the surface, the cylindrical model may be a satisfactory approximation. However, where more complete data are available, as in regions investigated by 3D seismic surveys, the cylindrical model is seriously inadequate for characterizing the fold structures. This is particularly so in the analysis of folds in hydrocarbon provinces, where structural closure implies the presence of doubly curved non-cylindrical fold forms. Basic morphological terms such as 'inflection line', 'hinge line' and 'limb' cannot be employed in a strict sense when describing such folds.

In summary, there is a general lack of tools for the geometrical analysis of non-cylindrical folds. It is well known that such structures give rise to distinctive outcrop patterns on geological maps or satellite images. For example, many textbooks describe the concentric patterns that are diagnostic of domes and basins and the oval disposition of formations that characterizes doubly-plunging brachyanticlines and brachysynclines (e.g. Billings, 1954; Roberts, 1982; Lisle, 1995). This paper explains how more information can be extracted from such features and outlines the theoretical basis for a semi-quantitative interpretation of such patterns in terms of the geometry of the surface fold involved.

2. Folded surfaces and their curvature

Once the assumption that geological folds have cylindrical forms is relaxed, a more general approach is required to describe the geometry of their surfaces. The theoretical framework for a general treatment of surfaces is to be found in the literature on differential geometry.

At any point P on a folded surface the curvature can be defined with reference to a cross-section oriented normal to the surface, that is, a plane passing through the surface normal N at P (Fig. 1). The value of normal curvature is given by $k_n = 1/r$, where r is the radius of curvature of the curve obtained in the section plane.

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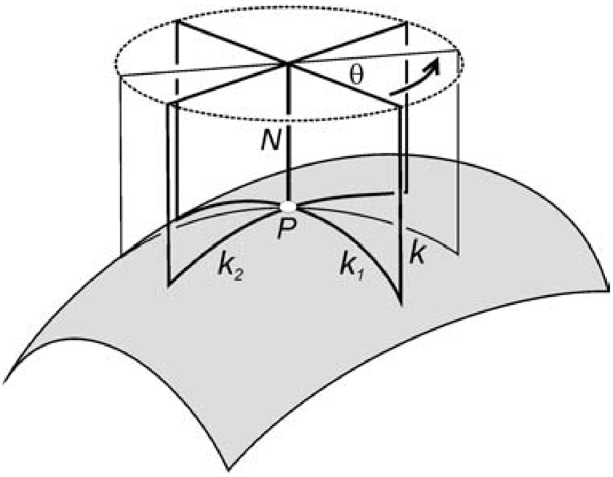

Figure 1. Curvature of a folded surface at some point, P. Cross-sections perpendicular to the surface, that is, sections parallel to surface normal N, display different values of curvature depending on the direction of the section. The normal curvature adopts extreme values in two perpendicular directions: the principal curvature directions.

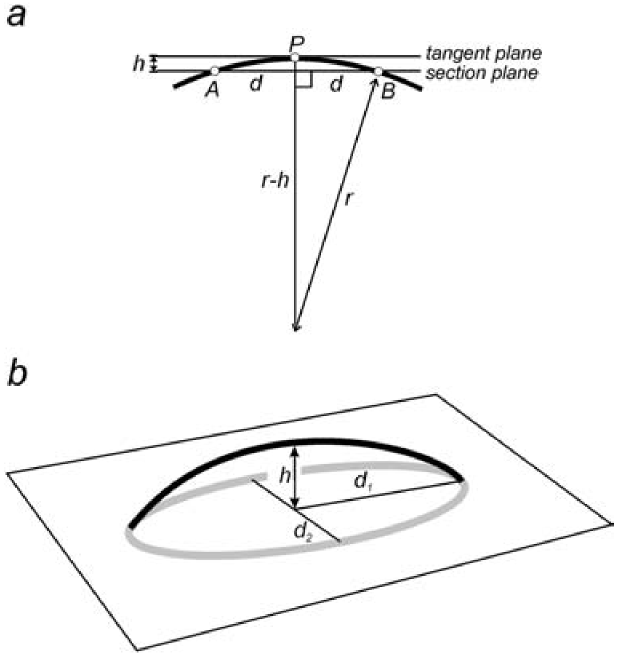

Figure 2. (a) Normal section of a surface. The surface crops out at points A and B, distance $2d$ apart. The distance depends on the radius of curvature, r, and the distance between the tangent plane and the section plane, h. (b) In the section plane, the distances d_1 and d_2 are in a ratio that relates to the ratio of the values of the principal curvatures.

Unless the local surface geometry around point P is a flat plane or forms a part of a spherical surface, the curvature obtained in the normal section will vary depending on the direction of the section plane. For all points, regardless of the surface to which they belong, the change of curvature with change in the direction of the section is orderly and systematic; there is a direction in which the curvature is greatest and another perpendicular direction along which the normal curvature has a minimum value. The extreme values are called the principal curvatures at P and have values k_1 and k_2, respectively. These values can take positive or negative values depending on whether the corresponding centre of curvature lies below or above the surface, that is, whether the curvature is convex upwards or concave upwards. For normal sections through P in other directions the curvature has an intermediate value given by Euler's equation (see Weatherburn, 1947, p. 73):

$$k_n = k_1 \cos^2 \theta + k_2 \sin^2 \theta, \tag{1}$$

where θ is the angle between the normal section and the direction of the maximum curvature (Fig. 1). Although the curvature properties of all points on all surfaces obey equation (1), three classes of points can be distinguished on the basis of the signs of the principal curvatures. Points where the principal curvatures have the same sign, like a small fragment of an eggshell, are called synclastic points. Where the curvature is convex in one principal direction but concave in the other, like a saddle, the point is classified as anticlastic ('oppositely curved'). A third class of points which has one principal curvature equal to zero, like a part of a cylindrical or conical fold, could be termed monoclastic because the surface around such a point is curved in one direction only.

If we accept that natural folded surfaces are generally non-cylindrical, then points of all three kinds might be expected to occur upon them. The question that now arises is how such points may be recognized from the two-dimensional outcrop patterns observed on outcrop surfaces or geological maps.

3. Planar sections through non-cylindrical folds

It is well known that a cross-section of a fold reveals geometrical features of the fold. For a cylindrical fold, the most logical and informative cross-section is one oriented perpendicular to the fold axis. For surfaces that are more complex, the equivalent natural cross-section does not exist because the curvature attributes vary from point to point in such a way that cannot be displayed in a single planar slice. Nevertheless, a cross-section almost coincident with one of the tangent planes of the surface (Fig. 2) is most revealing about the local curvature characteristics. It is demonstrated below that such unorthodox cross-sections correspond to certain outcrop patterns observed on geological maps.

Figure 2a illustrates how a planar cut through a surface on a section plane AB just below the plane of tangency at point P will intersect the surface twice, at points A and B. The folded surface with radius of curvature r will crop out at these two points with

Figure 3. Air photograph of a periclinal fold in Wyoming (U.S. Geological Survey photo). The elliptical outcrop has an aspect ratio of 4.2 indicating the principal curvatures are in the ratio of $4.2^2 = 18.0$.

distance $2d$ apart. Referring to Figure 2a, by Pythagoras we have

$$r^2 = (r - h)^2 + d^2. \tag{2}$$

Provided the section plane *AB* is close to the tangent plane, that is, when h is very small and therefore h^2 is of negligible magnitude, the curvature is approximately given by

$$k_n = \frac{1}{r} = \frac{2h}{d^2}. \tag{3}$$

If this relationship is applied in the two principal directions at a synclastic point (Fig. 2b), the distances of separation of the two outcrop positions $2d_1$ and $2d_2$ will be in the ratio

$$R_d = \frac{2d_2}{2d_1}. \tag{4}$$

From (3) it can be deduced that the ratio of the aspect ratio of the oval outcrop, R_d, relates to the ratio of the principal curvatures in absolute values:

$$R_d^2 = \left(\frac{d_2}{d_1}\right)^2 = \frac{\max|k|}{\min|k|}. \tag{5}$$

This is an important result because it permits an interpretation of the geometry of the oval outcrop shape in terms of the principal curvatures at the synclastic point on the folded surface. This is illustrated in Figure 3, where the elliptical outcrop has an aspect ratio of 4.2, indicating a curvature ratio of 18.0:1.

4. Dupin's indicatrix

The result above can be generalized for all three types of points. By combining (1) with (3) we obtain the equation in polar co-ordinates of the curve corresponding to the 'glancing slice' of a fold, that is, the polar distance d for each direction θ:

$$d = \sqrt{\frac{2h}{k_1 \cos^2\theta + k_2 \sin^2\theta}}. \tag{6}$$

The curve (d, θ), illustrated in Figure 4, is a conic called Dupin's indicatrix after Charles Dupin, the French mathematician who investigated its properties in the early nineteenth century. It is the figure obtained by taking a very thin slice off a folded surface, parallel to the tangent plane. The radius of the indicatrix in any direction is proportional to the square root of the radius of normal curvature in this direction (Hilbert & Cohn-Vossen, 1952). If the point is synclastic (k_1 and k_2 have the same sign) the curve is an ellipse (Fig. 4a). If the principal curvatures differ in sign, as they do in the case of an anticlastic point, the real values of d define a pair of hyperbolas (Fig. 4b). Finally if the local curvature is monoclastic (k_1 or k_2 are zero), the figure consists of a pair of straight parallel lines (Fig. 4c).

Where Dupin's indicatrix of elliptical, hyperbolic or straight-line types can be recognized on a geological map or cross-section, we have the opportunity to investigate principal curvatures and their directions for that small part of the folded structure. The practicalities of the method are explained in the next section.

5. Method

The first stage in the analysis of local curvature from maps or sections is to correctly identify Dupin's indicatrix, that is, elliptical, hyperbolic or straight-line outcrop patterns. In the case of observations from maps, care must be taken not to confuse the patterns with similar shapes arising from the effects of topography, in particular where there are isolated hills (elliptical shapes) and mountain cols (hyperbolic forms). In general, the accuracy of the method will be detrimentally affected by any non-planarity of the cross-section surface. Inspection of topographic contours in the vicinity of the shape should help check whether the pattern is indeed a valid Dupin's indicatrix, that is, whether the pattern is an expression of folding.

Where geological boundaries exhibit a nested arrangement of ellipses or hyperbolas, the shape of the innermost boundary should be taken as the Dupin indicatrix (DI) (Fig. 4). These are shapes produced by surfaces whose tangent plane is closest to the section plane, a condition that produces an outcrop pattern closest in form to a DI. To obtain a true DI, the section plane must be close to the tangent plane (h is small; see Fig. 2a). In this respect, it is not simply the distance h but this value as a proportion of the radius of curvature that influences the degree to which the outcrop shape departs from the ideal DI. Where h/r is large the outcrop pattern may deviate significantly from the ideal shape of a DI. This is illustrated in the fold interference

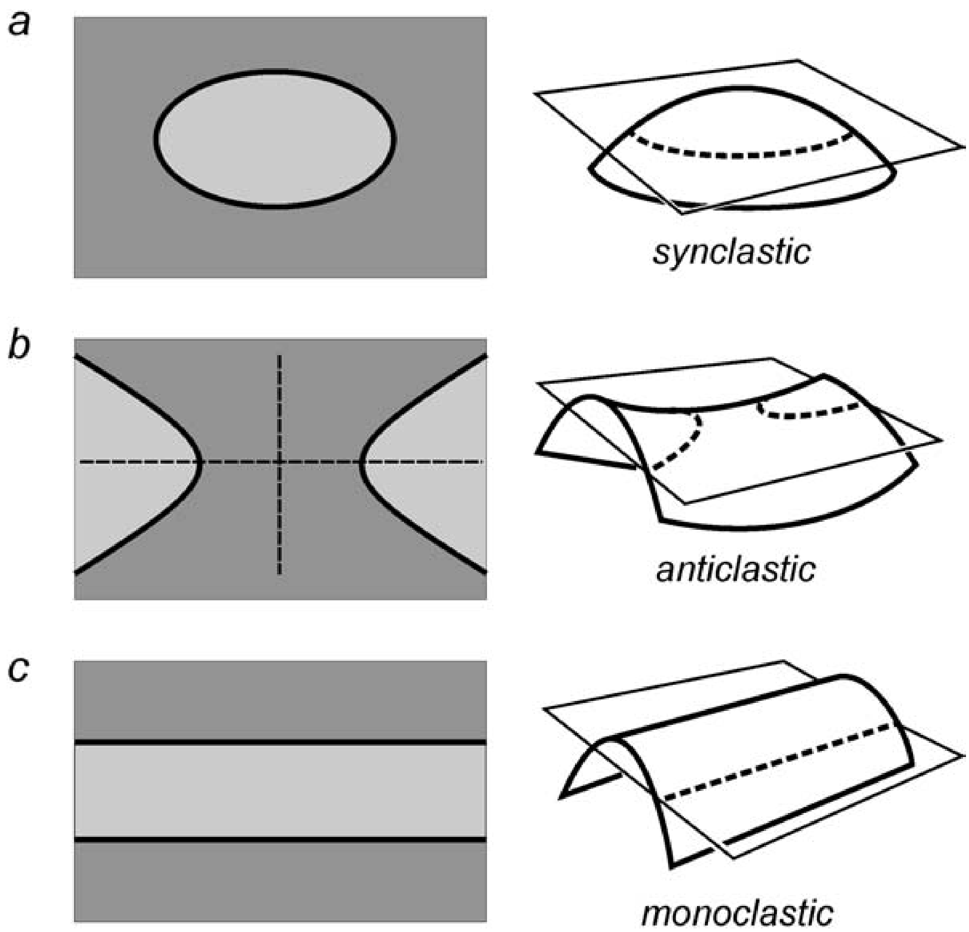

Figure 4. Dupin's indicatrix. (a) Elliptical type indicating synclastic curvature. (b) Hyperbolic type indicating anticlastic curvature. (c) Pair of straight lines indicating a fold with a zero principal curvature value ('monoclastic').

pattern in Figure 5 by the crescentic shapes of the traces of outermost surfaces.

5.a. Elliptical patterns

Once an elliptical DI is identified, we can conclude that the local curvature of fold is synclastic in nature. Furthermore, the directions of the minor and major axes of the ellipse indicate the principal curvature directions, k_1 and k_2. The ratio of the lengths of minor and major axes R_d can be used to calculate the ratio of the principal curvatures using equation (4). The elliptical pattern on its own does not allow the sign of the curvatures to be determined, though the short axis direction indicates the direction of greatest curvature (ignoring the sign). Therefore, without additional information, the DI will not indicate whether the structure is convex (antiformal) or concave (synformal).

5.b. Hyperbolic patterns

Dupin indicatrices of hyperbolic type indicate that the folded layering in the vicinity has anticlastic curvature, that is, they are associated with points on saddle-

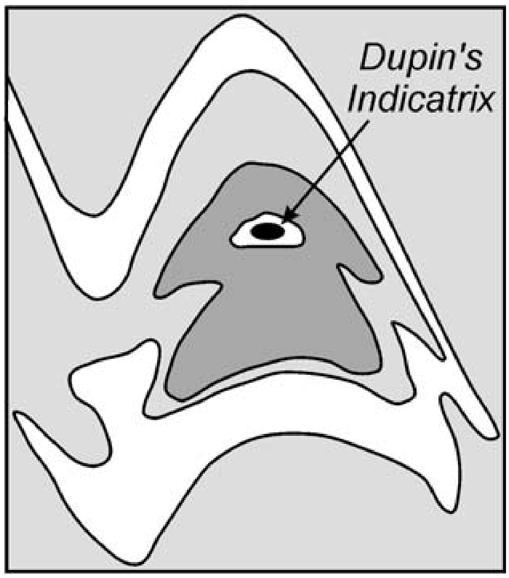

Figure 5. Interference pattern due to refolding (based on Roberts, 1989, fig. 241). Innermost closed traces of banding define an elliptical Dupin's indicatrix, whereas outer traces show more complex forms.

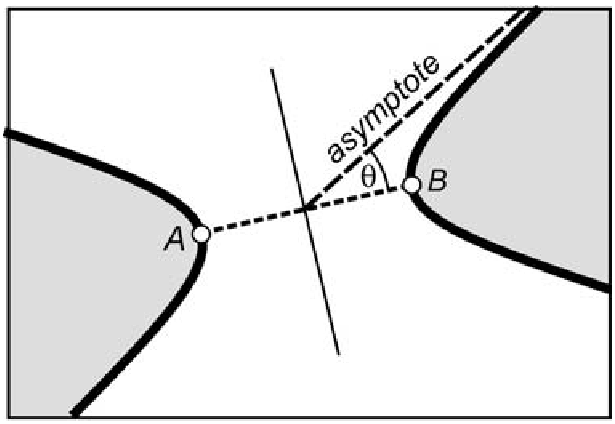

Figure 6. The analysis of hyperbolic patterns. See text for explanation.

shaped structures, or their upside-down equivalents, called 'shoe horns' by Lisle (1995, p. 35). The principal curvature directions are determined from the symmetry of the hyperbolic pattern, but from the DI geometry alone it is not possible to distinguish k_1 from k_2. The ratio of the absolute maximum and minimum curvatures is found as follows (Fig. 6):

(1) Locate the points of maximum curvature on each arm of the hyperbola (points A and B in Fig. 6). One of the principal curvature directions is parallel to the line AB, whilst the other principal direction is perpendicular to AB. We refer to these principal curvatures as k_p and k_q respectively.

(2) From the mid-point of line AB, sketch the lines that are asymptotic to the hyperbola. These represent the directions of lines of zero normal curvature ($k_n = 0$) within the folded surface.

(3) Measure the angle, θ, between the asymptote and the line AB (see Fig. 6).

The angle of inclination of the asymptotic lines, θ, relates to the ratio of the principal curvatures. By setting k_n to zero in equation (1), and rearranging, we obtain

$$\left|\frac{k_p}{k_q}\right| = \tan^2\theta. \tag{7}$$

The maximum curvature (ignoring sign) is therefore in the q direction when θ is less than 45° and in the p direction when θ exceeds 45°.

5.c. Straight line patterns

Where the outcrop of a geological boundary consists of two closely spaced parallel lines, the structural interpretation is straightforward. This figure implies a monoclastic geometry for the folded surface, that is, the case where one of the principal curvatures equals zero and ratio of the principal curvatures is infinite. The direction of the bisector of the straight lines indicates the zero principal curvature direction.

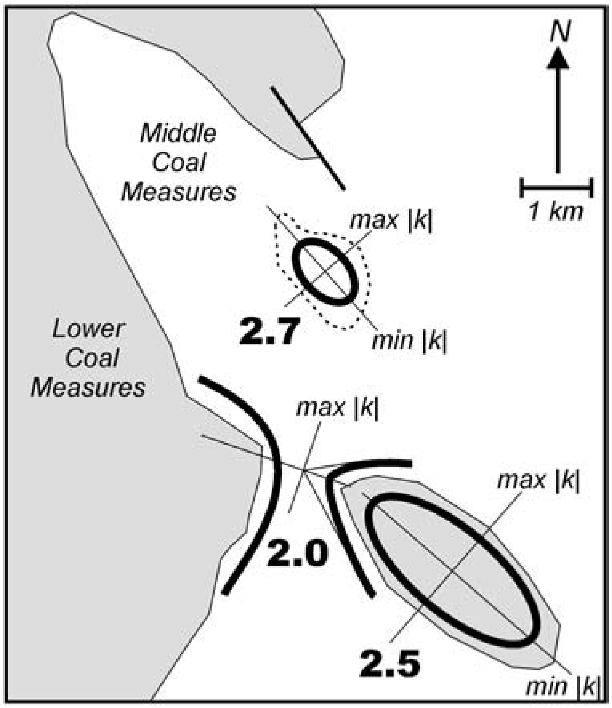

Figure 7. Curvature analysis from Dupin's patterns in the Chesterfield area, England. Geological boundaries taken from Sheet 112, 1 inch scale map, Geological Survey of Great Britain. The ratio of the principal curvatures are shown, together with the directions of the principal curvature axes.

6. Examples

Dupin patterns are common. This is because the sheet dip of folded horizons is frequently close to horizontal so that the topographic surface often intersects such horizons tangentially. Examples abound on geological maps of the Variscan fold belt of the Ardennes, the Cantabrian Mountains and South Wales and on Landsat images of the Appalachian fold belt in Pennsylvania. Figure 7 is a geological map of a region 8 km southeast of Chesterfield, England, where Dupin patterns are defined by the outcrop pattern of contacts within the Upper Carboniferous Coal Measures. Elliptical and hyperbolic outcrop shapes both indicate that direction of greatest absolute curvature varies from 019° and 049°N. The ratio of the principal curvatures varies between 2.0 and 2.7 and relates to the relative intensities of Variscan folding in two directions analysed by other means (Lisle, 1999).

Dupin patterns are also abundant on structure contour maps because each structure contour portrays the shape of a horizontal slice of a geological surface. Figure 8 illustrates examples of elliptical and hyperbolic types on the structure contour map (top of El Abra Limestone) of the Atn field, Veracruz, Mexico. These can be interpreted in terms of principal curvatures and their directions.

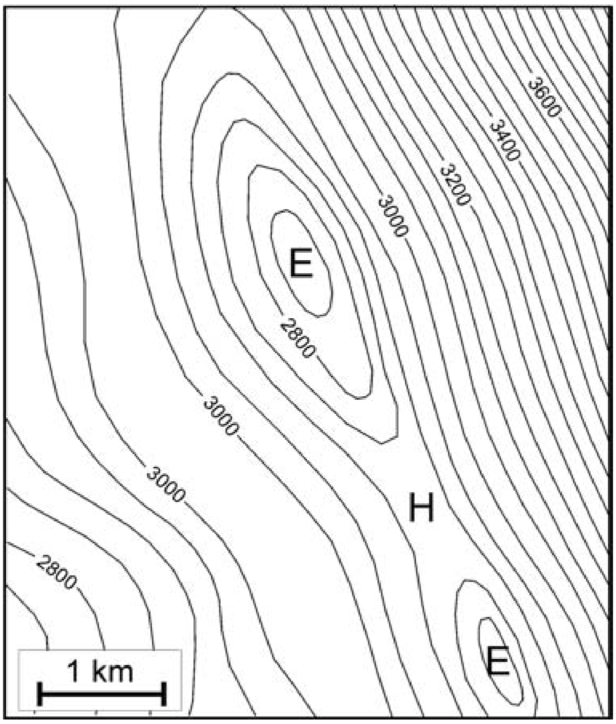

Figure 8. Examples of elliptical (E) and hyperbolic (H) Dupin indicatrices on structure contour maps. Atún oilfield, Veracruz, Mexico (from Viniegra & Castillo-Tejero, 1970).

7. Conclusions

The qualitative significance of the various patterns referred to in this paper is already well known. However, the new method outlined here delivers quantitative information of three-dimensional geometry from such two-dimensional structural images from outcrop maps, structure contours, seismic time slices and satellite images. It is stressed that information relates to local structural characteristics and its accuracy is adversely affected by irregularities of the cross-section surface (that is, the flatness of the land surface) and the distance separating the section surface and the tangent plane to the folded surface.

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References

BILLINGS, M. P. 1954. *Structural Geology.* Englewood Cliffs, N.J.: Prentice-Hall.

FLEUTY, M. J. 1964. The description of folds. *Proceedings, Geologists Association* **75**, 461–91.

HILBERT, D. & COHN-VOSSEN, S. 1952. *Geometry and the imagination*. New York: Chelsea Publishing Co.

LISLE, R. J. 1995. *Geological Structures and Maps.* 2nd edition. Oxford: Butterworth-Heinemann.

LISLE, R. J. 1999. A new Variscan deformation map of England and Wales. *Tectonophysics* **309**, 27–39.

ROBERTS, J. L. 1982. *Introduction to geological maps and structures.* Oxford: Pergamon Press.

ROBERTS, J. L. 1989. *A field guide to geological structures.* London: Macmillan.

VINIEGRA, F. O. & CASTILLO-TEJERO, C. 1970. Golden Lane Fields, Veracruz, Mexico. In *Geology of Giant Petroleum Fields* (ed. M.T. Halbouty), pp. 309–25. American Association of Petroleum Geologists, Memoir no. 14.

WEATHERBURN, C. E. 1947. *Differential geometry of three dimensions.* Volume 1. Cambridge: Cambridge University Press.