SIAM J. MATH. ANAL.
Vol. 49, No. 5, pp. 4024–4063

INSTABILITIES OF THE RELATIVISTIC VLASOV–MAXWELL SYSTEM ON UNBOUNDED DOMAINS*

JONATHAN BEN-ARTZI† AND THOMAS HOLDING‡

Abstract. The relativistic Vlasov–Maxwell system describes the evolution of a collisionless plasma. The problem of linear instability of this system is considered in two physical settings: the so-called *one and one-half dimensional* case, and the three dimensional case with cylindrical symmetry. Sufficient conditions for instability are obtained in terms of the spectral properties of certain Schrödinger operators that act on the spatial variable alone (and not in full phase space). An important aspect of these conditions is that they do not require any boundedness assumptions on the domains, nor do they require monotonicity of the equilibrium.

Key words. kinetic theory, Vlasov–Maxwell, linear instability

AMS subject classification. 35Q83

DOI. 10.1137/15M1025396

1. Introduction.

We obtain new linear instability results for plasmas governed by the relativistic Vlasov–Maxwell (RVM) system of equations. The main unknowns are two functions $f^{\pm} = f^{\pm}(t, \boldsymbol{x}, \boldsymbol{v}) \geq 0$ measuring the density of positively and negatively charged particles that at time $t \in [0, \infty)$ are located at the point $\boldsymbol{x} \in \mathbb{R}^d$ and have momentum $\boldsymbol{v} \in \mathbb{R}^d$. The densities $f^{\pm}$ evolve according to the Vlasov equations

$$\frac{\partial f^{\pm}}{\partial t} + \hat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{x}} f^{\pm} + \mathbf{F}^{\pm} \cdot \nabla_{\boldsymbol{v}} f^{\pm} = 0, \tag{1.1}$$

where $\hat{\boldsymbol{v}} = \boldsymbol{v}/\sqrt{1 + |\boldsymbol{v}|^2}$ is the relativistic velocity (the speed of light is taken to be 1 for simplicity) and where $\mathbf{F}^{\pm} = \mathbf{F}^{\pm}(t, \boldsymbol{x}, \boldsymbol{v})$ is the Lorentz force, given by

$$\mathbf{F}^{\pm} = \pm \left(\mathbf{E} + \mathbf{E}^{ext} + \hat{\boldsymbol{v}} \times (\mathbf{B} + \mathbf{B}^{ext}) \right)$$

with $\mathbf{E} = \mathbf{E}(t, \boldsymbol{x})$ and $\mathbf{B} = \mathbf{B}(t, \boldsymbol{x})$ being the electric and magnetic fields, respectively, and $\mathbf{E}^{ext}(t, \boldsymbol{x})$, $\mathbf{B}^{ext}(t, \boldsymbol{x})$ being external fields. The *self-consistent* fields obey Maxwell's equations,

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t},$$

where

$$\rho = \rho(t, \boldsymbol{x}) = \int (f^{+} - f^{-})\, d\boldsymbol{v} \tag{1.2}$$

*Received by the editors June 11, 2015; accepted for publication (in revised form) July 31, 2017; published electronically October 12, 2017.
http://www.siam.org/journals/sima/49-5/M102539.html
Funding: The first author's work was supported by the Department of Mathematics at Imperial College London, where he was a Junior Research Fellow while the majority of this paper was written, and by EPSRC grant EP/N020154/1. The second author's work was supported by the EPSRC funded (EP/H023348/1) Cambridge Centre for Analysis, where he was a doctoral student during the preparation of this article.
†School of Mathematics, Cardiff University, Cardiff, Wales CF24 4AG, UK (Ben-ArtziJ@cardiff.ac.uk).
‡Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK (T.Holding@warwick.ac.uk).

is the charge density and

$$\mathbf{j} = \mathbf{j}(t, \boldsymbol{x}) = \int \hat{\boldsymbol{v}}(f^+ - f^-)\, d\boldsymbol{v} \tag{1.3}$$

is the current density. In addition to the speed of light, we have taken all other constants that typically appear in these equations (such as the particle masses) to be 1 so as to keep the notation simple.

Novelty of the results. Let us mention the main novel aspects of our instability results:

Unbounded domains. Our problems are set in unbounded domains (as opposed to domains with boundaries or periodic domains). One consequence is that the spectrum of the Laplacian (which shall appear prominently) has an essential part, and there is no spectral gap.

Nonmonotone equilibrium. We do not assume that the equilibrium in question is (strongly) monotone (see (1.7) below). Many estimates in previous works rely heavily on monotonicity assumptions.

Existence of equilibria. In section 7 we prove the existence of nontrivial equilibria in the *unbounded, compactly supported* 1.5d case. Previously, this was done in the periodic setting by means of a perturbation argument about the trivial solution which is a center (in the dynamical systems sense). The proof here relies on a fixed point argument.

1.1. Main results. For the convenience of the reader, we provide the full statements of our results here, although some necessary definitions are too cumbersome. We shall refer to the later sections for these definitions.

The physical setting. As is explained in detail below, we consider two problems: the 1.5 *dimensional* case and the 3 *dimensional* case with *cylindrical symmetry.* We shall refer to these two settings as the 1.5d *case* and the 3d *case*, respectively, for brevity. In a nutshell, we consider these settings because they provide enough structure so that basic existence and uniqueness results hold and because they possess well-known conserved quantities which may be written explicitly.

The equilibrium. The conserved quantities mentioned above—the microscopic energy $e^\pm$ and momentum $p^\pm$—are the subject of further discussion below (see (1.17) for the 1.5d case and (1.28) for the 3d case); however, we stress the fact that they are functions that satisfy the *time-independent* Vlasov equations. Hence any functions of the form

$$f^{0,\pm}(\boldsymbol{x}, \boldsymbol{v}) = \mu^\pm(e^\pm, p^\pm) \tag{1.4}$$

are equilibria of the corresponding Vlasov equations. The converse statement—that any equilibrium may be written in this form—is called Jeans' theorem [9] (see Remark 1.1 below). In section 7 we prove that there exist such equilibria. When there is no room for confusion we simply write $\mu^\pm(e, p)$ or $\mu^\pm$ instead of $\mu^\pm(e^\pm, p^\pm)$.

Remark 1.1 (Jeans' "theorem"). Jeans' theorem is commonly referred to as such in the literature, though it is not (strictly speaking) a theorem. For instance, for the so-called Vlasov–Einstein system it has been shown to be false [19], while for the gravitational Vlasov–Poisson system it has indeed been proven rigorously [1]. As far as the authors know, there are no other proofs (or disproofs), though it is often easy to give a formal justification of this "theorem" by counting degrees of freedom. Indeed,

if one can argue that (due to symmetries) an equilibrium $f^0(\boldsymbol{x},\boldsymbol{v})$ can have at most $d \in \{1,\dots,6\}$ degrees of freedom and find d conserved quantities (that is, d quantities that are constant along the Vlasov flow), then formally it could be argued that f^0 may be rewritten as a function of these quantities.

We shall always assume that

$$0 \le f^{0,\pm}(\boldsymbol{x},\boldsymbol{v}) \in C^1 \quad \text{have compact support } \Omega \text{ in the } \boldsymbol{x}\text{-variable.}$$

Again, the existence of such equilibria is the subject of section 7. Note that in the $3d$ case, Ω must be cylindrically symmetric. In addition, we must assume that

$$\begin{aligned}&\text{there exist weight functions } w^{\pm} = c(1+|e^{\pm}|)^{-\alpha},\\ &\text{where } \alpha > \text{dimension of momentum space and } c>0,\end{aligned} \tag{1.5}$$

such that the integrability condition

$$\left(\left|\frac{\partial \mu^{\pm}}{\partial e}\right| + \left|\frac{\partial \mu^{\pm}}{\partial p}\right|\right)(e^{\pm},p^{\pm}) < w^{\pm}(e^{\pm}) \tag{1.6}$$

holds. This implies that $\int \left(|\mu_e^{\pm}| + |\mu_p^{\pm}|\right) d\boldsymbol{x}d\boldsymbol{v} < \infty$ in both the $1.5d$ and $3d$ cases, where we have abbreviated the writing of the partial derivatives of $\mu^{\pm}$. This abbreviated notation shall be used throughout the paper. It is often assumed that

$$\mu_e^{\pm} < 0 \quad \text{whenever } \mu^{\pm} > 0. \tag{1.7}$$

We call this a *strong monotonicity condition. We do* not *make any such assumption.* Monotonicity assumptions are natural in the study of both Vlasov systems [11, 12, 13, 15, 17] and the $2d$ Euler equations [5, 21], as they typically lead to stability. A famous exception to this rule is Penrose's result [18], often referred to as the "Penrose criterion." In many of the aforementioned works monotonicity assumptions play an important role throughout. It is therefore not always clear whether such conditions can be relaxed, or altogether dropped, as this would require extensive reformulation of the existing proofs.

The main results. To facilitate the understanding of our main results we state them now, trying not to obscure the big picture with technical details. Hence we attempt to extract only those aspects of the statements that are crucial for understanding, while referring to later sections for some additional definitions. First we define our precise notion of instability as follows.

Definition 1.1 (spectral instability). *We say that a given equilibrium $\mu^{\pm}$ is* spectrally unstable *if the system linearized around it has a purely growing mode solution of the form*

$$\left(e^{\lambda t} f^{\pm}(\boldsymbol{x},\boldsymbol{v}), e^{\lambda t}\mathbf{E}(\boldsymbol{x}), e^{\lambda t}\mathbf{B}(\boldsymbol{x})\right), \quad \lambda > 0. \tag{1.8}$$

We also need the following definition.

Definition 1.2. *Given a (bounded or unbounded) self-adjoint operator $\mathcal{A}$, we denote by* $\operatorname{neg}(\mathcal{A})$ *the number of negative eigenvalues (counting multiplicity) that it has whenever there is a finite number of such eigenvalues.*

In our first result we obtain a sufficient condition for spectral instability of equilibria in the $1.5d$ case. The condition is expressed in terms of spectral properties of certain operators that act on functions of the spatial variable alone.

THEOREM 1.1 (spectral instability: 1.5d case). *Let $f^{0,\pm}(x,\boldsymbol{v}) = \mu^{\pm}(e,p)$ be an equilibrium of the 1.5d system (1.15) satisfying (1.6). There exist self-adjoint Schrödinger operators $\mathcal{A}_1^0$ and $\mathcal{A}_2^0$ and a bounded operator $\mathcal{B}^0$ (all defined in (1.23)) acting only on functions of the spatial variable (and not the momentum variable) such that the equilibrium is spectrally unstable if 0 is not in the spectrum of $\mathcal{A}_1^0$ and*

$$\operatorname{neg}\left(\mathcal{A}_2^0 + \left(\mathcal{B}^0\right)^* \left(\mathcal{A}_1^0\right)^{-1} \mathcal{B}^0\right) > \operatorname{neg}\left(\mathcal{A}_1^0\right). \tag{1.9}$$

The second result provides a similar statement in the 3d case with cylindrical symmetry, as discussed in further detail in subsection 1.4 below.

THEOREM 1.2 (spectral instability: 3d case). *Let $f^{0,\pm}(\boldsymbol{x},\boldsymbol{v}) = \mu^{\pm}(e,p)$ be a cylindrically symmetric equilibrium of the RVM system satisfying (1.6). There exist self-adjoint operators $\widetilde{\mathcal{A}}_1^0$, $\widetilde{\mathcal{A}}_2^0$, and $\widetilde{\mathcal{A}}_3^0$ and a bounded operator $\widetilde{\mathcal{B}}_1^0$ (all defined in (1.33)) acting in the spatial variable alone (and not the momentum variable) such that the equilibrium is spectrally unstable if 0 is not an L^6-eigenvalue of $\widetilde{\mathcal{A}}_3^0$ (see Definition 1.3 below), 0 is not an eigenvalue of $\widetilde{\mathcal{A}}_1^0$ (0 will always lie in the essential spectrum of $\widetilde{\mathcal{A}}_1^0$, but this is not the same as 0 being an eigenvalue), and*

$$\operatorname{neg}\left(\widetilde{\mathcal{A}}_2^0 + \left(\widetilde{\mathcal{B}}_1^0\right)^* \left(\widetilde{\mathcal{A}}_1^0\right)^{-1} \widetilde{\mathcal{B}}_1^0\right) > \operatorname{neg}\left(\widetilde{\mathcal{A}}_1^0\right) + \operatorname{neg}\left(\widetilde{\mathcal{A}}_3^0\right). \tag{1.10}$$

Let us make precise the notion of an L^6-eigenvalue.

DEFINITION 1.3 (L^6-eigenvalue). *We say that $\lambda \in \mathbb{R}$ is an L^6-eigenvalue of a self-adjoint Schrödinger operator $\boldsymbol{\mathcal{A}} : H^2(\mathbb{R}^n;\mathbb{R}^m) \subset L^2(\mathbb{R}^n;\mathbb{R}^m) \to L^2(\mathbb{R}^n;\mathbb{R}^m)$ given by $\boldsymbol{\mathcal{A}} = -\boldsymbol{\Delta} + \boldsymbol{\mathcal{K}}$ if there exists a function $0 \neq \boldsymbol{u}^\lambda \in H^2_{loc}(\mathbb{R}^n;\mathbb{R}^m) \cap L^6(\mathbb{R}^n;\mathbb{R}^m)$, with $\boldsymbol{\nabla}\boldsymbol{u}^\lambda \in L^2(\mathbb{R}^n;\mathbb{R}^m)^n$, such that $\boldsymbol{\mathcal{A}}\boldsymbol{u}^\lambda = \lambda \boldsymbol{u}^\lambda$ in the sense of distributions. The function $\boldsymbol{u}^\lambda$ is called an L^6-eigenfunction.*

Remark 1.2. We note that L^6 is a natural space to consider in three dimensions due to the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, where $\Omega \subset \mathbb{R}^3$ is a bounded and smooth domain. In fact, any function which decays at infinity and whose first derivatives are square integrable also belongs to $L^6(\mathbb{R}^3)$. Therefore this is a natural condition for the potential formulation of Maxwell's equations where there is no physical reason for the potentials to be square integrable but where the condition that the fields are square integrable corresponds to the physical condition that the electromagnetic fields have finite energy.

The proofs of these two theorems appear in subsections 4.1 and 4.2, respectively. Let us describe the main ideas of the proofs. For brevity, we omit the $\pm$ signs distinguishing between positively and negatively charged particles in this paragraph. Since we are interested in linear instability, we linearize the Vlasov equation around f^0. The only nonlinear term is the forcing term $\mathbf{F} \cdot \nabla_v f$, so that the linearization of (1.1) becomes

$$\frac{\partial f}{\partial t} + \hat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{x}} f + \mathbf{F}^0 \cdot \nabla_{\boldsymbol{v}} f = -\mathbf{F} \cdot \nabla_{\boldsymbol{v}} f^0, \tag{1.11}$$

where $\mathbf{F}^0$ is the equilibrium self-consistent Lorentz force and $\mathbf{F}$ is the linearized Lorentz force. We make the following growing-mode ansatz:

$$\begin{aligned} &\textit{Ansatz:}\ \text{the perturbations } (f,\mathbf{E},\mathbf{B}) \text{ have} \\ &\qquad\qquad \text{time dependence } e^{\lambda t}, \text{ where } \lambda > 0. \end{aligned} \tag{1.12}$$

Equation (1.11) can therefore be written as

$$(\lambda + \mathcal{D})\, f = -\mathbf{F} \cdot \nabla_{\boldsymbol{v}} f^0, \tag{1.13}$$

where

$$\mathcal{D} = \hat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{x}} + \mathbf{F}^0 \cdot \nabla_{\boldsymbol{v}} \tag{1.14}$$

is the linearized Vlasov transport operator. We then invert expression (1.13) by applying $\lambda\,(\lambda + \mathcal{D})^{-1}$, which is an ergodic averaging operator along the trajectories of $\mathcal{D}$ (depending upon λ as a parameter); see [2, eq. (2.10)]. Hence we obtain an expression of f in terms of a certain average of the right-hand side $-\mathbf{F}\cdot\nabla_{\boldsymbol{v}} f^0$ depending upon the parameter λ (see (3.2) and (3.9)). This expression for f is substituted into Maxwell's equations through the charge and current densities, resulting in a system of (elliptic) equations for the spatial variable alone (recall that the momentum variable is integrated in the expressions for ρ and $\mathbf{j}$). The number of linearly independent equations is less than one would expect, due to the imposed symmetries. However, in both cases the equations can be written so that they form a self-adjoint system denoted $\boldsymbol{\mathcal{M}}^\lambda$ (see (3.5) for the $1.5d$ case and (3.14) for the $3d$ case) that has the general form

$$\boldsymbol{\mathcal{M}}^\lambda = \begin{bmatrix} -\Delta + 1 & 0 \\ 0 & \Delta - 1 \end{bmatrix} + \boldsymbol{\mathcal{K}}^\lambda$$

acting on the electric and magnetic potentials, where $\boldsymbol{\mathcal{K}}^\lambda$ is a uniformly bounded and symmetric family.

The problem then reduces to showing that the equation $\boldsymbol{\mathcal{M}}^\lambda \boldsymbol{u} = 0$ has a nontrivial solution for some value of $\lambda > 0$. The difficulty here is twofold: first, the spectrum of $\boldsymbol{\mathcal{M}}^\lambda$ is unbounded (not even semibounded) and includes essential spectrum extending to both $+\infty$ and $-\infty$. Second, for each λ, the operator $\boldsymbol{\mathcal{M}}^\lambda$ has a different spectrum: one must analyze a family of spectra that depends upon the parameter λ. In [4] we address the following related problem.

Problem 1.1. *Consider the family of self-adjoint unbounded operators*

$$\boldsymbol{\mathcal{M}}^\lambda = \boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{K}}^\lambda = \begin{bmatrix} -\Delta + 1 & 0 \\ 0 & \Delta - 1 \end{bmatrix} + \begin{bmatrix} \mathcal{K}^\lambda_{++} & \mathcal{K}^\lambda_{+-} \\ \mathcal{K}^\lambda_{-+} & \mathcal{K}^\lambda_{--} \end{bmatrix}, \quad \lambda \in [0,1],$$

acting in (an appropriate subspace of) $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$, where $\{\boldsymbol{\mathcal{K}}^\lambda\}_{\lambda\in[0,1]}$ is a uniformly bounded, symmetric, and strongly continuous family. Is it possible to construct explicit finite dimensional symmetric approximations of $\boldsymbol{\mathcal{M}}^\lambda$ whose spectrum in $(-1,1)$ converges to that of $\boldsymbol{\mathcal{M}}^\lambda$ for all λ simultaneously?

A solution to this problem allows us to construct finite dimensional approximations to $\boldsymbol{\mathcal{M}}^\lambda$. We discuss this problem in subsection 2.2. The conditions (1.9) and (1.10) appearing in the main theorems above translate into analogous conditions on the approximations, and those, in turn, guarantee the existence of a nontrivial *approximate* solution. Since the approximate problems converge (in an appropriate sense) to the original problem, this is enough to complete the proof. A crucial ingredient is the self-adjointness of all operators: this guarantees that the spectrum is restricted to the real line. The strategy is to "track" eigenvalues as a function of the parameter λ and conclude that they cross through 0 for some value $\lambda > 0$. To do so, we require knowledge of the spectrum of the operator $\boldsymbol{\mathcal{M}}^\lambda$ for small positive λ, which is obtained

from the assumptions (1.9) and (1.10), and for large λ which arises naturally from the form of the problem.

Yet even with a solution to this problem at hand, some difficulties remain. In the cylindrically symmetric case there is a geometric difficulty. Namely, cylindrical symmetries must be respected, a fact that requires a somewhat more cumbersome functional setup. In particular, the singular nature of the coordinate chart along the axis of symmetry requires special attention. To circumvent this issue we shall do all computation in Cartesian coordinates and use carefully chosen subspaces to decompose the magnetic potential. The second difficulty is the lack of a spectral gap, which is due to the unbounded nature of the problem in physical space. As a consequence, the dependence of the spectrum of $\boldsymbol{\mathcal{M}}^\lambda$ on λ is delicate, especially as $\lambda \to 0$, and needs careful consideration.

1.2. Previous results.

Existence theory. The main difficulty in attaining existence results for Vlasov systems is in controlling particle acceleration due to the nonlinear forcing term. Hence existence and uniqueness has only been proved under various symmetry assumptions. In [7] global existence in the $1.5d$ case was established, and in [6] the cylindrically symmetric case was considered. Local existence and uniqueness is due to [22].

Stability theory. One of the important early results on (linear) stability of plasmas is that of Penrose [18]. Two notable later results are [8, 14]. We refer the reader to [2] for additional references. The current result continues a program initiated by Lin and Strauss [12, 13] and continued by the first author [2, 3]. In [12, 13] the equilibria were always assumed to be strongly monotone, in the sense of (1.7). This added sign condition (which is widely used within the physics community, and is believed to be crucial for stability results) allowed them to obtain in [12] a linear *stability* criterion which was complemented by a linear *instability* criterion in [13]. Combined, these two results produced a necessary and sufficient criterion for stability in the following sense: there exists a Schrödinger operator $\mathcal{L}^0$ acting only in the spatial variable such that $\mathcal{L}^0 \geq 0$ implies spectral stability and $\mathcal{L}^0 \ngeq 0$ implies spectral instability. In [2, 3] the monotonicity assumption was removed, which mainly impacted the ability to obtain stability results. The instability results are similar to those of Lin and Strauss, though the author considers only the $1.5d$ case with periodicity. This is due to his methods which crucially require a Poincaré inequality. We remark that our results recover all previous results when one restricts the analysis to the monotone case.

1.3. The $1.5d$ case. First we treat the so-called $1.5d$ case, which is the lowest dimensional setting that permits nontrivial electromagnetic fields. In this setting, the plasma is assumed to have certain symmetries in phase-space that render the distribution function as a function of only one spatial variable x and two momentum variables $\boldsymbol{v} = (v_1, v_2)$, with v_1 being aligned with x. The only nontrivial components of the fields are the first two components of the electric field and the third component of the magnetic field, $\mathbf{E} = (E_1, E_2, 0)$ and $\mathbf{B} = (0, 0, B)$, and similarly for the equilibrium fields. The RVM system becomes the following system of scalar equations:

$$\partial_t f^\pm + \hat{v}_1 \partial_x f^\pm \pm (E_1 + \hat{v}_2 B)\partial_{v_1} f^\pm \pm (E_2 - \hat{v}_1 B)\partial_{v_2} f^\pm = 0, \tag{1.15a}$$

$$\partial_t E_1 = -j_1, \tag{1.15b}$$

$$\partial_t E_2 + \partial_x B = -j_2, \tag{1.15c}$$

$$\partial_x E_1 = \rho, \tag{1.15d}$$

$$\partial_t B = -\partial_x E_2, \tag{1.15e}$$

where ρ and j_1, j_2 are defined by (1.2) and (1.3).

1.3.1. Equilibrium. In section 7 we prove that there exist equilibria $f^{0,\pm}(x, \boldsymbol{v})$ which can be written as functions of the energy and momentum,

$$f^{0,\pm}(x, \boldsymbol{v}) = \mu^{\pm}(e^{\pm}, p^{\pm}), \tag{1.16}$$

as in (1.4). The energy and momentum are defined as

$$e^{\pm} = \langle \boldsymbol{v} \rangle \pm \phi^0(x) \pm \phi^{ext}(x), \qquad p^{\pm} = v_2 \pm \psi^0(x) \pm \psi^{ext}(x), \tag{1.17}$$

where $\langle \boldsymbol{v} \rangle = \sqrt{1 + |\boldsymbol{v}|^2}$ and where ϕ^0 and ψ^0 are the equilibrium electric and magnetic potentials (both scalar, in this case), respectively,

$$\partial_x \phi^0 = -E_1^0, \qquad \partial_x \psi^0 = B^0, \tag{1.18}$$

and similarly ϕ^{ext} and ψ^{ext} are external electric and magnetic potentials that give rise to external fields E_1^{ext} and B^{ext}. It is a straightforward calculation to verify that $e^{\pm}$ and $p^{\pm}$ are conserved quantities of the Vlasov flow, i.e., that $\mathcal{D}_{\pm} e^{\pm} = \mathcal{D}_{\pm} p^{\pm} = 0$, where the operators $\mathcal{D}_{\pm}$ are defined below, in (1.20).

LEMMA 1.1. *For compactly supported equilibria $E_2^0 \equiv 0$.*

Proof. First we note that, due to (1.15e), E_2^0 is a constant function. Let $f^{0,\pm}$ be a compactly supported (in x) equilibrium of positively and negatively charged particles. Let $\zeta(x)$ be a smooth test function, and define Z to be an antiderivative of ζ. Testing the Vlasov equation with Z gives

$$\iint \hat{v}_1 f^{0,\pm} \, dv \, \zeta \, dx = 0.$$

As ζ is arbitrary, we deduce that $\int f^{0,\pm} \hat{v}_1 \, dv = 0$ for all x (and both of $\pm$). Now, by testing the Vlasov equation with v_2, we obtain that

$$\iint (E_2^0 - \hat{v}_1 B^0) f^{0,\pm} \, dx \, dv = 0.$$

This implies that

$$\int \left[E_2^0 \int f^{0,\pm} \, dv - B^0 \int f^{0,\pm} \hat{v}_1 \, dv \right] dx = 0.$$

As $\int f^{0,\pm} \hat{v}_1 \, dv = 0$, and as $f^{0,\pm}$ is nonnegative (and nontrivial), we deduce that E_2^0, being a constant, is zero. ∎

1.3.2. Linearization. Let us discuss the linearization of (1.15) about a steady-state solution $(f^{0,\pm}, \mathbf{E}^0, \mathbf{B}^0)$. Using ansatz (1.12) and Jeans' theorem (1.4), together with (1.17) and (1.18), the linearized system becomes

$$(\lambda + \mathcal{D}_{\pm}) f^{\pm} = \mp \mu_e^{\pm} \hat{v}_1 E_1 \pm \mu_p^{\pm} \hat{v}_1 B \mp (\mu_e^{\pm} \hat{v}_2 + \mu_p^{\pm}) E_2, \tag{1.19a}$$

$$\lambda E_1 = -j_1, \tag{1.19b}$$

$$\lambda E_2 + \partial_x B = -j_2, \tag{1.19c}$$

$$\partial_x E_1 = \rho, \tag{1.19d}$$

$$\lambda B = -\partial_x E_2, \tag{1.19e}$$

where

$$\mathcal{D}_\pm = \hat{v}_1\partial_x \pm (E_1^0 + E_1^{ext} + \hat{v}_2(B^0 + B^{ext}))\partial_{v_1} \mp \hat{v}_1(B^0 + B^{ext})\partial_{v_2} \tag{1.20}$$

are the linearized transport operators as in (1.14), and

$$\rho = \int (f^+ - f^-)\, d\boldsymbol{v}, \qquad j_i = \int \hat{v}_i(f^+ - f^-)\, d\boldsymbol{v}$$

are the charge and current densities, respectively.

We now construct electric and magnetic potentials ϕ and ψ, respectively, as in (1.18). Equation (1.19b) implies that E_1 has the same spatial support as j_1, which is a moment of $f^\pm$, which, in turn, has the same x support as $\mu^\pm$ (this can be seen from (1.19a), for instance). We deduce that E_1 is compactly supported in Ω and choose an electric potential $\phi \in H^2(\Omega)$ such that $E_1 = -\partial_x\phi$ in Ω and $E_1 = 0$ outside Ω. Since E_1 vanishes at the boundary of Ω, we must impose Neumann boundary conditions on ϕ, and as E_1 depends only on the derivative of ϕ, we may impose that ϕ has mean zero. The magnetic potential ψ is chosen to satisfy $B = \partial_x\psi$ and $E_2 = -\lambda\psi$ (this is due to (1.19e)). Then the remaining Maxwell's equations (1.19b)–(1.19d) become

$$\lambda\partial_x\phi = -\lambda E_1 = j_1 \qquad \text{in } \Omega, \tag{1.21a}$$

$$(-\partial_x^2 + \lambda^2)\psi = -\partial_x B - \lambda E_2 = j_2 \qquad \text{in } \mathbb{R}, \tag{1.21b}$$

$$-\partial_x^2\phi = \partial_x E_1 = \rho \qquad \text{in } \Omega, \tag{1.21c}$$

where (1.21c) is complemented by the Neumann boundary condition

$$-\partial_x\phi = E_1 = 0 \quad \text{on } \partial\Omega.$$

The linearized Vlasov equations can now be written as

$$\begin{aligned}(\lambda + \mathcal{D}_\pm)f^\pm &= \pm\mu_e^\pm\hat{v}_1\partial_x\phi \pm \mu_p^\pm\hat{v}_1\partial_x\psi \pm \lambda(\mu_e^\pm\hat{v}_2 + \mu_p^\pm)\psi \\ &= \pm\mu_e^\pm\mathcal{D}_\pm\phi \pm \mu_p^\pm\mathcal{D}_\pm\psi \pm \lambda(\mu_e^\pm\hat{v}_2 + \mu_p^\pm)\psi,\end{aligned} \tag{1.22}$$

where we have used the fact that $\mathcal{D}_\pm u = \hat{v}_1\partial_x u$ if u is a function of x only.

Now let us specify the functional spaces that we shall use. For the scalar potential ϕ we define the space

$$L_0^2(\Omega) := \left\{ f \in L^2(\Omega) \ : \ \int_\Omega f = 0 \right\},$$

while for the magnetic potential ψ we simply use $L^2(\mathbb{R})$, the standard space of square integrable functions. We denote by $H^k(\mathbb{R})$ (resp., $H^k(\Omega)$) the usual Sobolev space of functions whose first k derivatives are in $L^2(\mathbb{R})$ (resp., $L^2(\Omega)$). Moreover, we naturally define

$$H_0^k(\Omega) := \left\{ f \in H^k(\Omega) \ : \ \int_\Omega f = 0 \right\}$$

and the corresponding version which imposes Neumann boundary conditions

$$H_{0,n}^k(\Omega) := \left\{ f \in H_0^k(\Omega) \ : \ \partial_x f = 0 \text{ on } \partial\Omega \right\}.$$

Finally, to allow us to consider functions that do not decay at infinity we use the conditions (1.5) and (1.6) to define weighted spaces $\mathfrak{L}_\pm$ as follows: we take the closure of the smooth and compactly supported functions of $(x, \boldsymbol{v})$ (with the x support contained in Ω) under the weighted-L^2 norm given by

$$\|u\|^2_{\mathfrak{L}_\pm} = \int_{\Omega\times\mathbb{R}^2} w^\pm |u|^2 \, d\boldsymbol{v}dx,$$

and we denote the inner product by $\langle\cdot,\cdot\rangle_{\mathfrak{L}_\pm}$. In particular we can view any function $u(x) \in L^2(\Omega)$ or $L^2_0(\Omega)$ as being in $\mathfrak{L}_\pm$ by considering u as a function of $(x, \boldsymbol{v})$ which does not depend on $\boldsymbol{v}$. We can extend this to functions in $L^2(\mathbb{R})$ by multiplying them by the characteristic function $\mathbb{1}_\Omega$ of the set Ω. Hence the function $\mathbb{1}_\Omega$ itself may be regarded as an element in $\mathfrak{L}_\pm$.

1.3.3. The operators. Finally, we define the operators used in the statement of Theorem 1.1. First define the following projection operators.

Definition 1.4 (projection operators). *We define $\mathcal{Q}^0_\pm$ to be the orthogonal projection operators in $\mathcal{L}_\pm$ onto* $\ker(\mathcal{D}_\pm)$.

Remark 1.3. Although this definition makes reference to the spaces $\mathfrak{L}_\pm$, the operators $\mathcal{Q}^0_\pm$ do not depend on the exact choice of weight functions $w^\pm$. This may be seen by writing $(\mathcal{Q}^0_\pm h)(x, \boldsymbol{v})$ as the pointwise limit of ergodic averages along trajectories (see Remark 3.1 and Lemma 6.1).

This allows us to define the following operators acting on functions of the spatial variable x, not the full phase-space variables:

$$\mathcal{A}^0_1 h = -\partial_x^2 h + \int \sum_\pm \mu_e^\pm (\mathcal{Q}^0_\pm - 1) h \, d\boldsymbol{v}, \tag{1.23a}$$

$$\mathcal{A}^0_2 h = -\partial_x^2 h - \left(\sum_\pm \int \mu_p^\pm \hat{v}_2 \, d\boldsymbol{v}\right) h - \int \sum_\pm \hat{v}_2 \mu_e^\pm \mathcal{Q}^0_\pm \, [\hat{v}_2 h] \, d\boldsymbol{v}, \tag{1.23b}$$

$$\mathcal{B}^0 h = \left(\int \sum_\pm \mu_p^\pm \, d\boldsymbol{v}\right) h + \int \sum_\pm \mu_e^\pm \mathcal{Q}^0_\pm \, [\hat{v}_2 h] \, d\boldsymbol{v}, \tag{1.23c}$$

$$\left(\mathcal{B}^0\right)^* h = \left(\int \sum_\pm \mu_p^\pm \, d\boldsymbol{v}\right) h + \int \sum_\pm \mu_e^\pm \hat{v}_2 \mathcal{Q}^0_\pm h \, d\boldsymbol{v}. \tag{1.23d}$$

Their precise properties are discussed in subsection 6.1. For future reference, we mention the important identity

$$\int_{\mathbb{R}} \left(\mu_p^\pm + \hat{v}_2 \mu_e^\pm\right) dv_2 = 0, \tag{1.24}$$

which is due to the fact that $\frac{\partial \mu^\pm}{\partial v_2} = \mu_e^\pm \hat{v}_2 + \mu_p^\pm$.

1.4. The cylindrically symmetric case. Since notation can be confusing when multiple coordinate systems are in use, we start this section by making clear what our conventions are.

Vector transformations and notational conventions. We let $\boldsymbol{x} = (x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ denote the representation of the point $\boldsymbol{x} \in \mathbb{R}^3$ in terms of the standard Cartesian coordinates. We define the usual cylindrical coordinates as

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan(y/x), \qquad z = z,$$

and the local cylindrical coordinates as

$$\mathbf{e}_r = r^{-1}(x, y, 0), \qquad \mathbf{e}_\theta = r^{-1}(-y, x, 0), \qquad \mathbf{e}_z = (0, 0, 1).$$

By *cylindrically symmetric* we mean that in what follows no quantity depends upon θ (which does not imply that the θ component is zero!). When writing $f(\boldsymbol{x})$, we mean the value of the function f at the point $\boldsymbol{x}$ in Cartesian coordinates. We shall often abuse notation and write $f(r, \theta, z)$ to mean the value of f at the point (r, θ, z) in cylindrical coordinates. A point $\boldsymbol{v} \in \mathbb{R}^3$ in momentum space shall either be expressed in Cartesian coordinates as

$$\boldsymbol{v}_{xyz} = (v_x, v_y, v_z) = (\boldsymbol{v} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\boldsymbol{v} \cdot \mathbf{e}_2)\mathbf{e}_2 + (\boldsymbol{v} \cdot \mathbf{e}_3)\mathbf{e}_3$$

or in cylindrical coordinates *(depending upon the point $\boldsymbol{x} \in \mathbb{R}^3$ in physical space)* as

$$\boldsymbol{v}_{rz\theta} = (v_r, v_\theta, v_z) = (\boldsymbol{v} \cdot \mathbf{e}_r)\mathbf{e}_r + (\boldsymbol{v} \cdot \mathbf{e}_\theta)\mathbf{e}_\theta + (\boldsymbol{v} \cdot \mathbf{e}_z)\mathbf{e}_z.$$

However, we shall not be too pedantic about this notation and shall use $\boldsymbol{v}$ (rather than $\boldsymbol{v}_{xyz}$ or $\boldsymbol{v}_{rz\theta}$) when there's no reason for confusion.

A vector-valued function $\mathbf{F}$ shall be understood to be represented in Cartesian coordinates. That is, unless otherwise said, $\mathbf{F} = (F_x, F_y, F_z) = F_x\mathbf{e}_1 + F_y\mathbf{e}_2 + F_z\mathbf{e}_3$. Its expression in cylindrical coordinates shall typically be written as $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_z\mathbf{e}_z$.

Differential operators. Partial derivatives in Cartesian coordinates are written as ∂_x, ∂_y, and ∂_z, while in cylindrical coordinates they are ∂_r, ∂_θ, and ∂_z. They transform in the standard manner. It is important to note that since we work in phase space, we shall require derivatives with respect to $\boldsymbol{v}$ as well. One important factor appearing in the Vlasov equation is $\hat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{x}}$, which transforms as

$$\begin{aligned}(\hat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{x}})h &= \hat{v}_x\partial_x h + \hat{v}_y\partial_y h + \hat{v}_z\partial_z h \\ &= \hat{v}_r\partial_r h + r^{-1}\hat{v}_\theta\partial_\theta h + \hat{v}_z\partial_z h \\ &= \hat{v}_r\partial_r h + r^{-1}\hat{v}_\theta(v_\theta\partial_{v_r}h - v_r\partial_{v_\theta}h) + \hat{v}_z\partial_z h.\end{aligned}$$

However, the next term in the Vlasov equation transforms "neatly":

$$\begin{aligned}(\mathbf{F} \cdot \nabla_{\boldsymbol{v}})h &= F_x\partial_{v_x}h + F_y\partial_{v_y}h + F_z\partial_{v_z}h \\ &= (F_x\cos\theta + F_y\sin\theta)\partial_{v_r}h + (-F_x\sin\theta + F_y\cos\theta)\partial_{v_\theta}h + F_z\partial_{v_z}h \\ &= F_r\partial_{v_r}h + F_\theta\partial_{v_\theta}h + F_z\partial_{v_z}h.\end{aligned}$$

1.4.1. The Lorenz gauge. As opposed to system (1.19), here we do not get a system of scalar equations. It is well known that there is some freedom in defining the electromagnetic potentials φ (we use φ in the cylindrically symmetric case rather than ϕ to avoid confusion) and $\mathbf{A}$, satisfying

$$\partial_t\mathbf{A} + \nabla\varphi = -\mathbf{E}, \qquad \nabla \times \mathbf{A} = \mathbf{B}.$$

Remark 1.4. Whenever the differential operator ∇ appears without any subscript, it is understood to be $\nabla_{\boldsymbol{x}}$, that is, the operator $(\partial_x, \partial_y, \partial_z)$ acting on functions of the spatial variable in Cartesian coordinates. The same holds for the Laplacian Δ.

We choose to impose the *Lorenz gauge* $\nabla \cdot \mathbf{A} + \frac{\partial \varphi}{\partial t} = 0$, hence transforming Maxwell's equations into the hyperbolic system

$$\frac{\partial^2}{\partial t^2}\varphi - \Delta\varphi = \rho, \tag{1.25a}$$

$$\frac{\partial^2}{\partial t^2}\mathbf{A} - \boldsymbol{\Delta}\mathbf{A} = \mathbf{j}. \tag{1.25b}$$

We remark that this is not unique to the cylindrically symmetric case, and the expressions above are written in Cartesian coordinates.

1.4.2. Equilibrium and the linearized system. We define the steady-state potentials $\varphi^0 : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{A}^0 : \mathbb{R}^3 \to \mathbb{R}^3$ through

$$\nabla\varphi^0 = -\mathbf{E}^0, \qquad \nabla \times \mathbf{A}^0 = \mathbf{B}^0, \tag{1.26}$$

which become

$$\mathbf{E}^0 = -\partial_r\varphi^0\mathbf{e}_r - \partial_z\varphi^0\mathbf{e}_z, \qquad \mathbf{B}^0 = -\partial_z A^0_\theta\mathbf{e}_r + \frac{1}{r}\partial_r(rA^0_\theta)\mathbf{e}_z. \tag{1.27}$$

The energy and momentum may be defined (analogously to (1.17)) as

$$\begin{aligned} e^\pm_{cyl} &= \langle \boldsymbol{v}\rangle \pm \varphi^0(r,z) \pm \varphi^{ext}(r,z), \\ p^\pm_{cyl} &= r(v_\theta \pm A^0_\theta(r,z) \pm A^{ext}_\theta(r,z)), \end{aligned} \tag{1.28}$$

where we recall that $\langle \boldsymbol{v}\rangle = \sqrt{1+|\boldsymbol{v}_{xyz}|^2}$. It is straightforward to verify that they are indeed conserved along the Vlasov flow (which is given by the integral curves of the differential operators $\widetilde{\mathcal{D}}_\pm$, defined in (1.30) below). To maintain simple notation we won't insist on writing the *cyl* subscript where it is clear which energy and momentum are meant. The external fields are also assumed to be cylindrically symmetric, and their potentials satisfy equations analogous to (1.27). We recall (1.4), namely that any equilibrium is assumed to be of the form

$$f^{0,\pm}(\boldsymbol{x},\boldsymbol{v}) = \mu^\pm(e^\pm, p^\pm).$$

Considering the Lorenz gauge, and applying the ansatz (1.12) and Jeans' theorem (1.4), the linearization of the RVM system about a steady-state solution $(f^{0,\pm}, \mathbf{E}^0, \mathbf{B}^0)$ is

$$(\lambda + \widetilde{\mathcal{D}}_\pm)f^\pm = \pm(\lambda + \widetilde{\mathcal{D}}_\pm)(\mu^\pm_e\varphi + r\mu^\pm_p(\mathbf{A}\cdot\mathbf{e}_\theta)) \pm \lambda\mu^\pm_e(-\varphi + \mathbf{A}\cdot\hat{\boldsymbol{v}}), \tag{1.29a}$$

$$\lambda^2\varphi - \Delta\varphi = \int (f^+ - f^-)\,d\boldsymbol{v}, \tag{1.29b}$$

$$\lambda^2\mathbf{A} - \boldsymbol{\Delta}\mathbf{A} = \int (f^+ - f^-)\hat{\boldsymbol{v}}\,d\boldsymbol{v}, \tag{1.29c}$$

where

$$\begin{aligned} \widetilde{\mathcal{D}}_\pm &= \hat{\boldsymbol{v}}_{xyz}\cdot\nabla_{\boldsymbol{x}} \pm (\mathbf{E}^0 + \mathbf{E}^{ext} + \hat{\boldsymbol{v}}_{xyz}\times(\mathbf{B}^0 + \mathbf{B}^{ext}))\cdot\nabla_{\boldsymbol{v}} \\ &= \hat{v}_r\partial_r + \hat{v}_z\partial_z + (\pm E^0_r \pm E^{ext}_r \pm \hat{v}_\theta(B^0_z + B^{ext}_z) + r^{-1}\hat{v}_\theta v_\theta)\partial_{v_r} \\ &\quad + (\pm\hat{v}_z(B^0_r + B^{ext}_r) \mp \hat{v}_r(B^0_z + B^{ext}_z) + r^{-1}\hat{v}_\theta v_r)\partial_{v_\theta} \\ &\quad \pm (E^0_z + E^{ext}_z + \hat{v}_\theta(B^0_r + B^{ext}_r))\partial_{v_z} \end{aligned} \tag{1.30}$$

are the linearized transport operators. The Lorenz gauge condition under the growing mode ansatz is

$$\nabla\cdot\mathbf{A} + \lambda\varphi = 0. \tag{1.31}$$

1.4.3. Functional spaces. Even more so than in the $1.5d$ case, choosing convenient functional spaces is crucial, due to the singular nature of the correspondence between Cartesian and cylindrical coordinates. We define

> $L^2_{cyl}(\mathbb{R}^3)$ = the smallest closed subspace of $L^2(\mathbb{R}^3)$ comprised of functions which have cylindrical symmetry.

A short computation using cylindrical coordinates shows that the decomposition $L^2(\mathbb{R}^3) = L^2_{cyl}(\mathbb{R}^3) \oplus (L^2_{cyl}(\mathbb{R}^3))^\perp$ reduces the Laplacian. This means that the Laplacian commutes with the orthogonal projection of $L^2(\mathbb{R}^3)$ onto $L^2_{cyl}(\mathbb{R}^3)$. Hence the Laplacian is decomposed as

$$\Delta = \Delta_{cyl} + \Delta_{cyl^\perp}.$$

As we have no use for $(L^2_{cyl}(\mathbb{R}^3))^\perp$, we shall abuse notation slightly and denote Δ_{cyl} as simply Δ. We now consider vector-valued functions

$$\mathbf{A} \in L^2_{cyl}(\mathbb{R}^3;\mathbb{R}^3) := (L^2_{cyl}(\mathbb{R}^3))^3.$$

We decompose such functions as

$$\begin{aligned} \mathbf{A} &= (\mathbf{A}\cdot\mathbf{e}_\theta)\mathbf{e}_\theta + ((\mathbf{A}\cdot\mathbf{e}_r)\mathbf{e}_r + (\mathbf{A}\cdot\mathbf{e}_z)\mathbf{e}_z) \\ &= \mathbf{A}_\theta + \mathbf{A}_{rz} \in L^2_\theta(\mathbb{R}^3;\mathbb{R}^3) \oplus L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3). \end{aligned} \tag{1.32}$$

By computing with cylindrical coordinates, we once again discover that this decomposition reduces the vector Laplacian $\mathbf{\Delta}$ on $L^2_{cyl}(\mathbb{R}^3;\mathbb{R}^3)$. Note that this reduction does not occur for $\mathbf{\Delta}$ on $L^2(\mathbb{R}^3;\mathbb{R}^3)$ (i.e., without the cylindrical symmetry).

We further define the corresponding Sobolev spaces $H^k_{cyl}(\mathbb{R}^3)$, $H^k_\theta(\mathbb{R}^3;\mathbb{R}^3)$, $H^k_{rz}(\mathbb{R}^3;\mathbb{R}^3)$ of functions whose first k weak derivatives lie in $L^2_{cyl}(\mathbb{R}^3)$, $L^2_\theta(\mathbb{R}^3;\mathbb{R}^3)$, and $L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)$, respectively. Note that, because of the reductions above, Δ is self-adjoint on $L^2_{cyl}(\mathbb{R}^3)$ with domain $H^2_{cyl}(\mathbb{R}^3)$, and $\mathbf{\Delta}$ is self-adjoint on each of $L^2_\theta(\mathbb{R}^3;\mathbb{R}^3)$ and $L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)$ with domains $H^2_\theta(\mathbb{R}^3;\mathbb{R}^3)$ and $H^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)$, respectively.

As in the $1.5d$ case, we shall require certain weighted spaces $\mathfrak{N}_\pm$ that allow us to include functions that do not decay. We define $\mathfrak{N}_\pm$ as the closure of the smooth compactly supported functions $u : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ which are cylindrically symmetric in the $\boldsymbol{x}$ variable, and have $\boldsymbol{x}$-support contained in Ω, under the norms

$$\|u\|_{\mathfrak{N}_\pm} = \int_{\mathbb{R}^3\times\Omega} w^\pm |u|^2 \, d\boldsymbol{v} d\boldsymbol{x},$$

where the weight functions $w^\pm$ are those introduced in (1.5).

1.4.4. The operators. We now define the operators used in the statement of Theorem 1.2. As in the $1.5d$ case, we shall require the following definition of projection operators.

Definition 1.5 (projection operators). *We define $\widetilde{\mathcal{Q}}^0_\pm$ to be the orthogonal projection operators in $\mathfrak{N}_\pm$ onto* $\ker(\widetilde{\mathcal{D}}_\pm)$.

As in the $1.5d$ case, the operators $\widetilde{\mathcal{Q}}^0_\pm$ do not depend upon the exact choice of weights $w^\pm$. Now we are ready to define the operators of the cylindrically symmetric case. For brevity, given $\hat{\boldsymbol{v}} = (\hat{v}_r, \hat{v}_\theta, \hat{v}_z)$, we define $\hat{\boldsymbol{v}}_\theta = \hat{v}_\theta \mathbf{e}_\theta$ and $\hat{\boldsymbol{v}}_{rz} = \hat{v}_r\mathbf{e}_r + \hat{v}_z\mathbf{e}_z$. All operators act on functions of the spatial variables only: the operator $\widetilde{\mathcal{A}}^0_1$ acts on functions in $L^2_{cyl}(\mathbb{R}^3)$, $\widetilde{\mathcal{A}}^0_2$ on functions in $L^2_\theta(\mathbb{R}^3;\mathbb{R}^3)$, $\widetilde{\mathcal{A}}^0_3$ on functions in $L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)$,

and $\widetilde{\mathcal{B}}_1^0$ on functions in $L^2_\theta(\mathbb{R}^3;\mathbb{R}^3)$ with range $L^2_{cyl}(\mathbb{R}^3)$. We have

$$\widetilde{\mathcal{A}}_1^0 h = -\Delta h + \int \sum_{\pm} \mu_e^{\pm}(\widetilde{\mathcal{Q}}_{\pm}^0 - 1) h \, d\boldsymbol{v}, \tag{1.33a}$$

$$\widetilde{\mathcal{A}}_2^0 \boldsymbol{h} = -\boldsymbol{\Delta h} - \left(r \int \sum_{\pm} \mu_p^{\pm} \hat{v}_\theta \, d\boldsymbol{v} \right) \boldsymbol{h} - \int \sum_{\pm} \hat{\boldsymbol{v}}_\theta \mu_e^{\pm} \widetilde{\mathcal{Q}}_{\pm}^0 [\boldsymbol{h} \cdot \hat{\boldsymbol{v}}_\theta] \, d\boldsymbol{v}, \tag{1.33b}$$

$$\widetilde{\mathcal{A}}_3^0 \boldsymbol{h} = -\boldsymbol{\Delta h} - \int \sum_{\pm} \hat{\boldsymbol{v}}_{rz} \mu_e^{\pm} \widetilde{\mathcal{Q}}_{\pm}^0 [\boldsymbol{h} \cdot \hat{\boldsymbol{v}}_{rz}] \, d\boldsymbol{v}, \tag{1.33c}$$

$$\widetilde{\mathcal{B}}_1^0 \boldsymbol{h} = \int \sum_{\pm} \mu_e^{\pm} (\widetilde{\mathcal{Q}}_{\pm}^0 - 1) [\boldsymbol{h} \cdot \hat{\boldsymbol{v}}_\theta] \, d\boldsymbol{v}, \tag{1.33d}$$

$$(\widetilde{\mathcal{B}}_1^0)^* h = \int \sum_{\pm} \mu_e^{\pm} \hat{\boldsymbol{v}}_\theta (\widetilde{\mathcal{Q}}_{\pm}^0 - 1) h \, d\boldsymbol{v}. \tag{1.33e}$$

The precise properties of these operators are discussed in subsection 6.2. We also mention an identity analogous to (1.24),

$$\int_{\mathbb{R}^3} (r\mu_p^{\pm} + \hat{v}_\theta \mu_e^{\pm}) \, d\boldsymbol{v} = 0, \tag{1.34}$$

which is due to the integrand being a perfect derivative: $\frac{\partial \mu^{\pm}}{\partial v_\theta} = r\mu_p^{\pm} + \hat{v}_\theta \mu_e^{\pm}$.

1.5. Organization of the paper. In section 2 we provide some necessary background, including the crucial result on approximating spectra found in [4]. Then we treat the two problems—the $1.5d$ and $3d$ cases—in parallel: in section 3 we formulate the two problems as an equivalent family of self-adjoint problems, which we then successively solve in section 4. The proofs of the main theorems are concluded in section 5. In section 6 we provide the rigorous treatment of the various operators appearing throughout the paper, and in section 7 we show that there exist nontrivial equilibria.

2. Background, definitions, and notation. In this section we remind the reader of the various notions of convergence in Hilbert spaces in order to avoid confusion. For a Hilbert space $\mathfrak{H}$ we denote its norm and inner product by $\|\cdot\|_{\mathfrak{H}}$ and $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$, respectively. When there is no ambiguity we drop the subscript. We denote the set of bounded linear operators from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{G}$ as $\mathfrak{B}(\mathfrak{H}, \mathfrak{G})$, and when $\mathfrak{H} = \mathfrak{G}$ we simply write $\mathfrak{B}(\mathfrak{H})$. The operator norm is denoted $\|\cdot\|_{\mathfrak{H}\to\mathfrak{G}}$, where, again, when there is no ambiguity we may drop the subscript.

Definition 2.1 (convergence in $\mathfrak{B}(\mathfrak{H}, \mathfrak{G})$). *Let $\mathcal{T}, \mathcal{T}_n \in \mathfrak{B}(\mathfrak{H}, \mathfrak{G})$, where $n \in \mathbb{N}$.*

(a) *We say that the sequence $\mathcal{T}_n$ converges to $\mathcal{T}$ in norm (or uniformly) as $n \to \infty$ whenever $\|\mathcal{T}_n - \mathcal{T}\|_{\mathfrak{H}\to\mathfrak{G}} \to 0$ as $n \to \infty$. In this case we write $\mathcal{T}_n \to \mathcal{T}$.*

(b) *We say that the sequence $\mathcal{T}_n$ converges to $\mathcal{T}$ strongly as $n \to \infty$ whenever we have the pointwise convergence $\mathcal{T}_n u \to \mathcal{T} u$ in $\mathfrak{G}$ for all $u \in \mathfrak{H}$. In this case we write $\mathcal{T}_n \xrightarrow{s} \mathcal{T}$.*

Now let us recall some important notions related to unbounded self-adjoint operators.

Definition 2.2 (convergence of unbounded operators). *Let $\mathcal{A}$ and $\mathcal{A}_n$ be self-adjoint, where $n \in \mathbb{N}$.*

(a) *We say that the sequence* $\mathcal{A}_n$ *converges to* $\mathcal{A}$ *in the norm resolvent sense as* $n \to \infty$ *whenever* $(\mathcal{A}_n - z)^{-1} \to (\mathcal{A} - z)^{-1}$ *for any* $z \in \mathbb{C} \setminus \mathbb{R}$. *In this case we write* $\mathcal{A}_n \xrightarrow{n.r.} \mathcal{A}$.

(b) *We say that the sequence* $\mathcal{A}_n$ *converges to* $\mathcal{A}$ *in the strong resolvent sense as* $n \to \infty$ *whenever* $(\mathcal{A}_n - z)^{-1} \xrightarrow{s} (\mathcal{A} - z)^{-1}$ *for any* $z \in \mathbb{C} \setminus \mathbb{R}$. *In this case we write* $\mathcal{A}_n \xrightarrow{s.r.} \mathcal{A}$.

Remark 2.1. Notice that for any self-adjoint operator $\mathcal{A}$, the resolvent $(\mathcal{A} - z)^{-1}$ is a bounded operator for any $z \in \mathbb{C} \setminus \mathbb{R}$.

2.1. Basic facts. The subsequent results will be used throughout the paper without explicit reference.

LEMMA 2.1. *Let* $\mathfrak{H}, \mathfrak{G}$ *be Banach spaces,* $\mathcal{T}, \mathcal{T}_n \in \mathfrak{B}(\mathfrak{H}, \mathfrak{G})$ *and* $u, u_n \in \mathfrak{H}$ *where* $n \in \mathbb{N}$, *and assume that* $\mathcal{T}_n \xrightarrow{s} \mathcal{T}$ *and* $u_n \to u$. *Then* $\mathcal{T}_n u_n \to \mathcal{T} u$ *as* $n \to \infty$.

Proof. We compute

$$\begin{aligned}\|\mathcal{T}_n u_n - \mathcal{T} u\| &\leq \|\mathcal{T}_n (u_n - u)\| + \|(\mathcal{T}_n - \mathcal{T}) u\| \\ &\leq \left(\sup_{n \in \mathbb{N}} \|\mathcal{T}_n\|\right) \|u_n - u\| + \|(\mathcal{T}_n - \mathcal{T}) u\|.\end{aligned}$$

This supremum is finite by the uniform boundedness principle, so the first term converges to zero since $u_n \to u$. The second term converges to zero since $\mathcal{T}_n \xrightarrow{s} \mathcal{T}$. ∎

COROLLARY 2.1. *If* $\mathcal{T}_n \xrightarrow{s} \mathcal{T}$ *and* $\mathcal{S}_n \xrightarrow{s} \mathcal{S}$ *as* $n \to \infty$, *then* $\mathcal{T}_n \mathcal{S}_n \xrightarrow{s} \mathcal{T}\mathcal{S}$ *as* $n \to \infty$.

The following result complements Weyl's theorem (see [10, Chapter IV, Theorem 5.35]) on the stability of the essential spectrum under a relatively compact perturbation. In our setting we know more about the perturbation than its being merely relatively compact.

LEMMA 2.2. *Let* $\mathcal{A} = -\Delta + \mathcal{K} : H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *be a self-adjoint Schrödinger operator with* $\mathcal{K} \in \mathfrak{B}(L^2(\mathbb{R}^n))$ *and* $\mathcal{K} = \mathcal{K}\mathcal{P}$, *where* $\mathcal{P} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *is the multiplication operator by the characteristic function* $\mathbb{1}_\Omega$ *of some bounded domain* $\Omega \subset \mathbb{R}^n$. *Then* $\mathcal{A}$ *has a finite number of negative eigenvalues (counting multiplicity).*

Proof. By Weyl's theorem (see [10, Chapter IV, Theorem 5.35]), there are at most countably many negative eigenvalues, and they may only accumulate at 0. Denote these as the increasing sequence $\lambda_1 \leq \lambda_2 \leq \cdots$, where equality comes from multiplicity. As $\mathcal{A}$ is self-adjoint, the corresponding normalized eigenfunctions $e_1, e_2, \ldots$ form an orthonormal set. Let $\mathcal{E}$ be their linear span; i.e.,

$$\mathcal{E} = \operatorname{span}\{e_i : i = 1, 2, \ldots\}.$$

Note that $\mathcal{E}$ is a linear subspace of $L^2(\mathbb{R}^n)$ but is not necessarily closed. Also, from elliptic regularity, using that $\mathcal{K} \in \mathfrak{B}(L^2(\mathbb{R}^n))$, we have that $\mathcal{E} \subset H^2(\mathbb{R}^n)$.

We claim that there exists an injective linear map from $\mathcal{E}$ into a finite dimensional space, and hence $\mathcal{E}$ is finite dimensional, proving the lemma. Indeed, we define the map $\mathcal{T} : \mathcal{E} \to H^2(\Omega)$ by $u \mapsto \mathbb{1}_\Omega u$, with image $\mathcal{T}(\mathcal{E})$. $\mathcal{T}$ is manifestly linear, so it remains to check the other claimed properties.

Step 1. $\mathcal{T}$ *is injective into its image.* By linearity it suffices to show that $\mathcal{T} u = 0$

implies that $u = 0$ for any $u \in \mathcal{E}$. Since $u \in \mathcal{E}$ it must hold that

$$\int_{\mathbb{R}^n} u\mathcal{A}u\,dx \le 0.$$

However, $\mathcal{T}u = 0$ implies that $\mathcal{K}u = 0$; hence we have

$$\begin{aligned}\int_{\mathbb{R}^n} u\mathcal{A}u\,dx &= \int_{\mathbb{R}^n} u(-\Delta u + \mathcal{K}u)\,dx \\ &= \int_{\mathbb{R}^n} u(-\Delta)u\,dx = \int_{\mathbb{R}^n} |\nabla u|^2\,dx \ge 0.\end{aligned}$$

Therefore $\nabla u = 0$, which together with $u \in L^2(\mathbb{R}^n)$ implies that $u = 0$.

Step 2. *The image of* $\boldsymbol{\mathcal{T}}$ *is finite dimensional.* Let $v = \sum_{i=1}^m a_i \mathcal{T} e_i = \mathcal{T}u$ for scalar a_i and m finite be an arbitrary element of $\mathcal{T}(\mathcal{E})$. Then we have

$$\begin{aligned}\|\nabla v\|^2_{L^2(\Omega)} - \|\mathcal{K}\|\,\|v\|^2_{L^2(\Omega)} &\le \left\|\nabla \sum_{i=1}^m a_i e_i\right\|^2_{L^2(\mathbb{R}^n)} - \|\mathcal{K}\| \left\|\mathcal{P}\sum_{i=1}^m a_i e_i\right\|^2_{L^2(\mathbb{R}^n)} \\ &\le \left\langle \mathcal{A}\sum_{i=1}^m a_i e_i, \sum_{i=1}^m a_i e_i\right\rangle_{L^2(\mathbb{R}^n)} \\ &= \sum_{i=1}^m \lambda_i |a_i|^2\,\|e_i\|^2_{L^2(\mathbb{R}^n)} \le 0,\end{aligned}$$

where we have used orthogonality of the eigenfunctions e_i and that $\lambda_i < 0$. Hence any $v \in \mathcal{T}(\mathcal{E})$ obeys the bound

$$\|\nabla v\|^2_{L^2(\Omega)} \le C\,\|v\|^2_{L^2(\Omega)}$$

with C that does not depend upon v or m. By the Poincaré inequality on the bounded domain Ω (enlarging Ω as needed to ensure that its boundary is smooth), $\mathcal{T}(\mathcal{E})$ is finite dimensional. As discussed above, these two steps complete the proof. ∎

2.2. Approximating strongly continuous families of unbounded operators. Here we summarize the main results of [4] on properties of approximations of strongly continuous families of unbounded operators. We shall require these results in what follows. We refer the reader to [4] for full details, including proofs. Let $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ be a (separable) Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let

$$\mathcal{A}^\lambda = \begin{bmatrix}\mathcal{A}^\lambda_+ & 0 \\ 0 & -\mathcal{A}^\lambda_-\end{bmatrix} \quad \text{and} \quad \mathcal{K}^\lambda = \begin{bmatrix}\mathcal{K}^\lambda_{++} & \mathcal{K}^\lambda_{+-} \\ \mathcal{K}^\lambda_{-+} & \mathcal{K}^\lambda_{--}\end{bmatrix}, \qquad \lambda \in [0,1],$$

be two families of operators on $\mathfrak{H}$ depending upon the parameter $\lambda \in [0,1]$ (the range $[0,1]$ of values of the parameter is, of course, arbitrary), where the family $\mathcal{A}^\lambda$ is also assumed to be defined for λ in an open neighborhood D_0 of $[0,1]$ in the complex plane. The families $\mathcal{A}^\lambda$ and $\mathcal{K}^\lambda$ satisfy the following:

(1) *Sectoriality:* The sesquilinear forms $\mathfrak{a}^\lambda_\pm$ corresponding to $\mathcal{A}^\lambda_\pm$ are sectorial and closed for $\lambda \in D_0$, symmetric for real λ, have dense domains $\mathfrak{D}(\mathfrak{a}^\lambda_\pm)$ independent of $\lambda \in D_0$, and $D_0 \ni \lambda \mapsto \mathfrak{a}^\lambda_\pm[u,v]$ are holomorphic for any $u, v \in \mathfrak{D}(\mathfrak{a}^\lambda_\pm)$. (In the terminology of [10], $\mathfrak{a}^\lambda_\pm$ are *holomorphic families of type* (a), and $\mathcal{A}^\lambda$ are *holomorphic families of type* (B).)

(ii) *Gap:* $\mathcal{A}^\lambda_\pm > 1$ for every $\lambda \in [0,1]$. We let $\alpha > 1$ be a lower bound to all $\mathcal{A}^\lambda_\pm$.

(iii) *Bounded perturbation:* $\{\mathcal{K}^\lambda\}_{\lambda\in[0,1]} \subset \mathfrak{B}(\mathfrak{H})$ is a symmetric strongly continuous family.

(iv) *Compactness:* There exist symmetric operators $\mathcal{P}_\pm \in \mathfrak{B}(\mathfrak{H}_\pm)$ which are relatively compact with respect to the forms $\mathfrak{a}^\lambda_\pm$, satisfying $\mathcal{K}^\lambda = \mathcal{K}^\lambda\mathcal{P}$ for all $\lambda \in [0,1]$, where

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_+ & 0 \\ 0 & \mathcal{P}_- \end{bmatrix}.$$

Finally, if the family $\mathcal{A}^\lambda$ does not have a compact resolvent, we assume the following.

(v) *Compactification of the resolvent:* There exist holomorphic forms $\{\mathfrak{w}^\lambda_\pm\}_{\lambda\in D_0}$ of type (a) and associated operators $\{\mathcal{W}^\lambda_\pm\}_{\lambda\in D_0}$ of type (B) such that, for $\lambda \in [0,1]$, $\mathcal{W}^\lambda_\pm$ are self-adjoint and nonnegative, and if $\mathfrak{w}^\lambda$ is the form associated with

$$\mathcal{W}^\lambda = \begin{bmatrix} \mathcal{W}^\lambda_+ & 0 \\ 0 & -\mathcal{W}^\lambda_- \end{bmatrix}, \quad \lambda \in D_0,$$

then $\mathfrak{D}(\mathfrak{w}^\lambda)\cap\mathfrak{D}(\mathfrak{a}_\pm)$ are dense for all $\lambda \in D_0$ and the inclusion $(\mathfrak{D}(\mathfrak{w}^\lambda)\cap\mathfrak{D}(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}^\lambda_\varepsilon}) \to (\mathfrak{H}, \|\cdot\|)$ is compact for some $\lambda \in D_0$ and all $\varepsilon > 0$, where $\mathfrak{a}^\lambda_\varepsilon$ is the form associated with

$$\mathcal{A}^\lambda_\varepsilon := \mathcal{A}^\lambda + \varepsilon\mathcal{W}^\lambda, \quad \lambda \in D_0,\ \varepsilon \geq 0. \tag{2.1}$$

Define the family of (unbounded) operators $\{\mathcal{M}^\lambda_\varepsilon\}_{\lambda\in[0,1],\varepsilon\geq 0}$, acting in $\mathfrak{H}$ as

$$\mathcal{M}^\lambda_\varepsilon = \mathcal{A}^\lambda_\varepsilon + \mathcal{K}^\lambda, \quad \lambda \in [0,1].$$

For $\varepsilon > 0$, let

- $\{e^\lambda_{\varepsilon,k}\}_{k\in\mathbb{N}} \subset \mathfrak{H}$ be a complete orthonormal set of eigenfunctions of $\mathcal{A}^\lambda_\varepsilon$,
- $\mathcal{G}^\lambda_{\varepsilon,n} : \mathfrak{H} \to \mathfrak{H}$ be the orthogonal projection operators onto $\operatorname{span}(e^\lambda_{\varepsilon,1}, \dots, e^\lambda_{\varepsilon,n})$,
- $\widetilde{\mathcal{M}}^\lambda_{\varepsilon,n}$ be the n-dimensional operator defined as the restriction of $\mathcal{M}^\lambda_\varepsilon$ to $\mathcal{G}^\lambda_{\varepsilon,n}(\mathfrak{H})$.

Now, for $\lambda \in [0,1]$, $\varepsilon \geq 0$, and $n \in \mathbb{N}$ we define the measures (where we *always* take multiplicities into account!)

$$\nu_{\lambda,\varepsilon} = \sum_{x\in\mathrm{sp}_{\mathrm{pp}}(\mathcal{M}^\lambda_\varepsilon)\setminus\mathrm{sp}_{\mathrm{ess}}(\mathcal{M}^\lambda_\varepsilon)} \delta_x,$$

and for any $\varepsilon > 0$ and $n \in \mathbb{N}$ the measures

$$\widetilde{\nu}_{\lambda,\varepsilon,n} = \sum_{x\in\mathrm{sp}(\widetilde{\mathcal{M}}^\lambda_{\varepsilon,n})} \delta_x.$$

Consider a cutoff function ϕ_η satisfying

$$\phi_\eta(x) = \begin{cases} 1, & x \in [-1,1], \\ 0, & x \in \mathbb{R}\setminus(-1-\eta, 1+\eta), \end{cases} \qquad \phi_\eta \in C(\mathbb{R},[0,1]),\ \eta \in (0,\alpha).$$

Finally, define the measures

$$\mu^\eta_{\lambda,\varepsilon} = \phi_\eta\nu_{\lambda,\varepsilon}$$

and

$$\widetilde{\mu}^{\eta}_{\lambda,\varepsilon,n} = \phi_\eta \widetilde{\nu}_{\lambda,\varepsilon,n}.$$

Recall that the space of finite positive Borel measures equipped with the topology of weak convergence is metrizable, for example with the bounded Lipschitz distance

$$d_{BL}(\mu,\nu) := \sup_{\|\psi\|_{\mathrm{Lip}}\le 1, |\psi|\le 1} \int \psi \,\mathrm{d}(\mu-\nu).$$

Our main result in [4][1] is the following.

THEOREM 2.1. *The mappings* $(\lambda,\varepsilon)\mapsto \mu^{\eta}_{\lambda,\varepsilon}$ *and* $\lambda \mapsto \widetilde{\mu}^{\eta}_{\lambda,\varepsilon,n}$ *are weakly continuous, and as* $n\to\infty$, $d_{BL}(\widetilde{\mu}^{\eta}_{\lambda,\varepsilon,n}, \mu^{\eta}_{\lambda,\varepsilon}) \to 0$ *uniformly in* $\lambda\in[0,1]$.

3. An equivalent problem.

3.1. The 1.5*d* case. We will now reduce the linearized Vlasov–Maxwell system (1.19) to a self-adjoint problem in $L^2(\mathbb{R})\times L^2_0(\Omega)$ depending continuously (in the norm resolvent sense) on the parameter $\lambda>0$.

3.1.1. Inverting the linearized Vlasov equation. Rearranging the terms in (1.22), we obtain

$$(\lambda+\mathcal{D}_\pm)f^\pm = \pm(\lambda+\mathcal{D}_\pm)(\mu^\pm_e\phi+\mu^\pm_p\psi) \pm \lambda\mu^\pm_e(-\phi+\hat{v}_2\psi), \tag{3.1}$$

where we use the fact that $\mu^\pm$ are constant along trajectories of the vector-fields $\mathcal{D}_\pm$. In order to obtain an expression for $f^\pm$ in terms of the potentials ϕ,ψ we invert the operators $(\lambda+\mathcal{D}_\pm)$, and to do this we must study the operators $\mathcal{D}_\pm$.

LEMMA 3.1. *The operators* $\mathcal{D}_\pm$ *on* $\mathfrak{L}_\pm$ *satisfy the following:*

(a) $\mathcal{D}_\pm$ *are skew-adjoint, and the resolvents* $(\lambda+\mathcal{D}_\pm)^{-1}$ *are bounded linear operators for* $\operatorname{Re}\lambda\neq 0$ *with norm bounded by* $1/|\operatorname{Re}\lambda|$.

(b) $\mathcal{D}_\pm$ *flip parity with respect to the variable* v_1*; i.e., if* $h(x,v_1,v_2)\in\mathfrak{D}(\mathcal{D}_\pm)$ *is an even function of* v_1*, then* $\mathcal{D}_\pm h$ *is an odd function of* v_1 *and vice versa.*

(c) *For real* $\lambda\neq 0$ *the resolvents of* $\mathcal{D}_\pm$ *split as follows:*

$$(\lambda+\mathcal{D}_\pm)^{-1} = \lambda(\lambda^2-\mathcal{D}_\pm^2)^{-1} - \mathcal{D}_\pm(\lambda^2-\mathcal{D}_\pm^2)^{-1},$$

where the first part is symmetric and preserves parity with respect to v_1*, and the second part is skew-symmetric and inverts parity with respect to* v_1.

Proof. Skew-adjointness follows from integration by parts, noting that $w^\pm$ are in the kernels of $\mathcal{D}_\pm$. (To be fully precise, only skew-*symmetry* follows. However, skew-adjointness is a simple extension; see, e.g., [20] and in particular Exercise 28 therein.) The existence of bounded resolvents follows. The statement regarding parity follows directly from the formulas for $\mathcal{D}_\pm$ term by term. Finally, for the last part we use functional calculus formalism to compute

$$\frac{1}{\lambda+\mathcal{D}_\pm} = \frac{\lambda-\mathcal{D}_\pm}{\lambda^2-\mathcal{D}_\pm^2} = \frac{\lambda}{\lambda^2-\mathcal{D}_\pm^2} - \frac{\mathcal{D}_\pm}{\lambda^2-\mathcal{D}_\pm^2}.$$

As $\mathcal{D}_\pm$ are skew-adjoint, $\mathcal{D}_\pm^2$ are self-adjoint, and hence the first term is self-adjoint and the second skew-adjoint. For the parity properties we note that as $\mathcal{D}_\pm$ flip parity, $\mathcal{D}_\pm^2$ preserve parity and hence so do $\lambda^2-\mathcal{D}_\pm^2$ and their inverses. ∎

[1] This formulation of the theorem appears in an erratum to the original result.

Applying $(\lambda + \mathcal{D}_\pm)^{-1}$ to (3.1) yields

$$f^\pm = \pm\mu_e^\pm \phi \pm \mu_p^\pm \psi \pm \lambda(\lambda + \mathcal{D}_\pm)^{-1}[\mu_e^\pm(-\phi + \hat{v}_2\psi)]. \tag{3.2}$$

Furthermore, using Lemma 3.1, we split $f^\pm$ into even and odd functions of v_1:

$$\begin{aligned} f_{ev}^\pm &= \pm\mu_e^\pm \phi \pm \mu_p^\pm \psi \pm \mu_e^\pm \lambda^2(\lambda^2 - \mathcal{D}_\pm^2)^{-1}[-\phi + \hat{v}_2\psi], \\ f_{od}^\pm &= \mp\mu_e^\pm \lambda \mathcal{D}_\pm(\lambda^2 - \mathcal{D}_\pm^2)^{-1}[-\phi + \hat{v}_2\psi], \end{aligned}$$

using the fact that ϕ, ψ, and μ are all even functions of v_1. For brevity, we define operators $\mathcal{Q}_\pm^\lambda : \mathfrak{L}_\pm \to \mathfrak{L}_\pm$ as

$$\mathcal{Q}_\pm^\lambda = \lambda^2(\lambda^2 - \mathcal{D}_\pm^2)^{-1}, \quad \lambda > 0.$$

When $\lambda \to 0$, their strong limits exist and are defined in Definition 1.4 (this convergence is proved in Lemma 6.1).

Remark 3.1. Operators $\mathcal{Q}_\pm^\lambda$ also appeared in the prior works [2, 3, 12, 13]. In each of these, $\mathcal{Q}_\pm^\lambda$ were defined as integrated averages over the characteristics of the operators $\mathcal{D}_\pm$. In fact, as the Laplace transform of a semigroup is the resolvent of its generator, we see that the operators $\mathcal{Q}_\pm^\lambda$ in these prior works have the rule

$$\mathcal{Q}_\pm^\lambda h = \int_{-\infty}^0 \lambda e^{\lambda s} e^{s\mathcal{D}_\pm} h \, ds = \lambda \int_0^\infty e^{-\lambda s} e^{-s\mathcal{D}_\pm} h \, ds = \lambda(\lambda + \mathcal{D}_\pm)^{-1} h.$$

Here we have defined the operators $\mathcal{Q}_\pm^\lambda$ directly from the resolvents of $\mathcal{D}_\pm$, as this makes some of their properties clearer, although both approaches have advantages. In particular we are able to split $\lambda(\mathcal{D}_\pm + \lambda)^{-1}$ into symmetric and skew-symmetric parts in Lemma 3.1, which simplifies some computations.

3.1.2. Reformulating Maxwell's equations. Now we substitute the expressions (3.2) into Maxwell's equations (1.21). This will result in an equivalent system of equations for ϕ and ψ. Due to the integration $d\boldsymbol{v}$ we notice that $f_{od}^\pm$ and $f_{od}^\pm \hat{v}_2$ both integrate to zero, so that ρ and j_2 depend only on $f_{ev}^\pm$.

Remark 3.2. It is important to note that due to the continuity equation it is possible to express either (1.21a) or (1.21b) using the remaining two equations in (1.21). See Lemma 5.4.

Gauss' equation (1.21c). Gauss' equation becomes

$$\begin{aligned} -\partial_x^2 \phi &= \int (f_{ev}^+ - f_{ev}^-) \, d\boldsymbol{v} \\ &= \int \sum_\pm \left(\mu_e^\pm \phi + \mu_p^\pm \psi + \mathcal{Q}_\pm^\lambda[\mu_e^\pm(-\phi + \hat{v}_2\psi)]\right) \, d\boldsymbol{v} \\ &= \int \sum_\pm (\mu_p^\pm + \mu_e^\pm \hat{v}_2)\psi \, d\boldsymbol{v} + \int \sum_\pm \mu_e^\pm (\mathcal{Q}_\pm^\lambda - 1)[-\phi + \hat{v}_2\psi] \, d\boldsymbol{v}, \end{aligned} \tag{3.3}$$

where we have pulled $\mu_e^\pm$ outside the application of $\mathcal{Q}_\pm^\lambda$ as they belong to $\ker(\mathcal{D}_\pm)$.

Ampère's equation (1.21b). Similarly, Ampère's equation becomes

$$
\begin{aligned}
(-\partial_x^2+\lambda^2)\psi &= \int \hat{v}_2(f_{ev}^+ - f_{ev}^-)\,d\boldsymbol{v} \\
&= \int \sum_{\pm} \hat{v}_2\left(\mu_e^{\pm}\phi + \mu_p^{\pm}\psi + \mathcal{Q}_{\pm}^{\lambda}[\mu_e^{\pm}(-\phi+\hat{v}_2\psi)]\right)\,d\boldsymbol{v} \\
&= \int \sum_{\pm} \hat{v}_2(\mu_p^{\pm}+\mu_e^{\pm}\hat{v}_2)\psi\,d\boldsymbol{v} \\
&\quad + \int \sum_{\pm} \hat{v}_2\mu_e^{\pm}(\mathcal{Q}_{\pm}^{\lambda}-1)[-\phi+\hat{v}_2\psi]\,d\boldsymbol{v}.
\end{aligned}
\tag{3.4}
$$

An equivalent formulation. We write the two new expressions (3.3) and (3.4) abstractly in the compact form

$$
\boldsymbol{\mathcal{M}}^{\lambda}\begin{bmatrix}\psi\\ \phi\end{bmatrix} = \begin{bmatrix}-\partial_x^2\psi+\lambda^2\psi - j_2\\ \partial_x^2\phi+\rho\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},
\tag{3.5}
$$

where, for $\lambda>0$, $\boldsymbol{\mathcal{M}}^{\lambda}$ is a self-adjoint matrix of operators mapping $L^2(\mathbb{R})\times L_0^2(\Omega)\to L^2(\mathbb{R})\times L_0^2(\Omega)$ (see Lemma 6.4). We claim that this operator may be written either as

$$
\boldsymbol{\mathcal{M}}^{\lambda} = \begin{bmatrix}-\partial_x^2+\lambda^2 & 0\\ 0 & \partial_x^2\end{bmatrix} - \boldsymbol{\mathcal{J}}^{\lambda}
\tag{3.6}
$$

or, equivalently, as

$$
\boldsymbol{\mathcal{M}}^{\lambda} = \begin{bmatrix}\mathcal{A}_2^{\lambda} & (\mathcal{B}^{\lambda})^*\\ \mathcal{B}^{\lambda} & -\mathcal{A}_1^{\lambda}\end{bmatrix},
\tag{3.7}
$$

where the various operators appearing above are given by

$$
\begin{aligned}
\boldsymbol{\mathcal{J}}^{\lambda}\begin{bmatrix}h\\ g\end{bmatrix} &= -\left(\sum_{\pm}\int \mu^{\pm}\frac{1+v_1^2}{\langle \boldsymbol{v}\rangle^3}\,d\boldsymbol{v}\right)\begin{bmatrix}h\\0\end{bmatrix} \\
&\quad + \sum_{\pm}\int \begin{bmatrix}\hat{v}_2\\ -1\end{bmatrix}\mu_e^{\pm}(\mathcal{Q}_{\pm}^{\lambda}-1)\left(\begin{bmatrix}\hat{v}_2\\ -1\end{bmatrix}\cdot\begin{bmatrix}h\\ g\end{bmatrix}\right)\,d\boldsymbol{v},
\end{aligned}
\tag{3.8a}
$$

$$
\mathcal{A}_1^{\lambda}h = -\partial_x^2 h + \int \sum_{\pm}\mu_e^{\pm}(\mathcal{Q}_{\pm}^{\lambda}-1)h\,d\boldsymbol{v},
\tag{3.8b}
$$

$$
\mathcal{A}_2^{\lambda}h = -\partial_x^2 h + \lambda^2 h - \left(\sum_{\pm}\int \mu_p^{\pm}\hat{v}_2\,d\boldsymbol{v}\right)h - \int\sum_{\pm}\hat{v}_2\mu_e^{\pm}\mathcal{Q}_{\pm}^{\lambda}\,[\hat{v}_2 h]\,d\boldsymbol{v},
\tag{3.8c}
$$

$$
\mathcal{B}^{\lambda}h = \left(\int\sum_{\pm}\mu_p^{\pm}\,d\boldsymbol{v}\right)h + \int\sum_{\pm}\mu_e^{\pm}\mathcal{Q}_{\pm}^{\lambda}\,[\hat{v}_2 h]\,d\boldsymbol{v},
\tag{3.8d}
$$

$$
(\mathcal{B}^{\lambda})^* h = \left(\int\sum_{\pm}\mu_p^{\pm}\,d\boldsymbol{v}\right)h + \int\sum_{\pm}\mu_e^{\pm}\hat{v}_2\mathcal{Q}_{\pm}^{\lambda}h\,d\boldsymbol{v}.
\tag{3.8e}
$$

Remark 3.3. Though $\lambda>0$ in the foregoing discussion, all operators can be defined for $\lambda=0$, as we have already done for some (see (1.23)).

The expression (3.7) is no more than a rewriting of (3.3) and (3.4). However, the expression (3.6) requires some attention. In particular, to obtain it one has to use (1.24) as well as the integration by parts

$$\int \frac{\partial \mu^{\pm}}{\partial v_2} \hat{v}_2 \, d\boldsymbol{v} = -\int \mu^{\pm} \frac{\partial \hat{v}_2}{\partial v_2} \, d\boldsymbol{v} = -\int \mu^{\pm} \frac{1+v_1^2}{\langle \boldsymbol{v} \rangle^3} \, d\boldsymbol{v}.$$

The properties of the operators appearing in (3.8) are discussed in detail in Lemmas 6.2 and 6.3. Let us briefly summarize:

- $\mathcal{A}_1^{\lambda} : H^2_{n,0}(\Omega) \subset L^2_0(\Omega) \to L^2_0(\Omega)$ is self-adjoint and has a purely discrete spectrum with finitely many negative eigenvalues.
- $\mathcal{A}_2^{\lambda} : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint and has essential spectrum in $[\lambda^2, \infty)$ and finitely many negative eigenvalues.
- $\mathcal{B}^{\lambda} : L^2(\mathbb{R}) \to L^2_0(\Omega)$ is a bounded operator, with bound independent of λ.
- $\boldsymbol{\mathcal{J}}^{\lambda} : L^2(\mathbb{R}) \times L^2_0(\Omega) \to L^2(\mathbb{R}) \times L^2_0(\Omega)$ is a bounded symmetric operator, with bound independent of λ.

3.2. The cylindrically symmetric case. Our approach here is fully analogous to that presented in subsection 3.1; hence we shall keep it brief, omitting repetitions as much as possible. For convenience we denote analogous operators by the same letter, but we shall add a *tilde* to any such operator in this section. Hence, e.g., the operators analogous to $\mathcal{D}_{\pm}$ shall be denoted $\widetilde{\mathcal{D}}_{\pm}$.

3.2.1. Inverting the linearized Vlasov equation. Recall the linearized Vlasov equation (1.29a):

$$(\lambda + \widetilde{\mathcal{D}}_{\pm}) f^{\pm} = \pm(\lambda + \widetilde{\mathcal{D}}_{\pm})(\mu_e^{\pm} \varphi + r \mu_p^{\pm} (\mathbf{A} \cdot \boldsymbol{e}_{\theta})) \pm \lambda \mu_e^{\pm} (-\varphi + \mathbf{A} \cdot \hat{\boldsymbol{v}}).$$

Inverting, we get the expression

$$f^{\pm} = \pm \mu_e^{\pm} \varphi \pm r \mu_p^{\pm} (\mathbf{A} \cdot \boldsymbol{e}_{\theta}) \pm \mu_e^{\pm} \lambda (\lambda + \widetilde{\mathcal{D}}_{\pm})^{-1} (-\varphi + \mathbf{A} \cdot \hat{\boldsymbol{v}}), \tag{3.9}$$

and, recalling that we care only about the quantity $f^+ - f^-$, we write it for future reference as

$$f^+ - f^- = \sum_{\pm} \mu_e^{\pm} \varphi + \sum_{\pm} r \mu_p^{\pm} (\mathbf{A} \cdot \boldsymbol{e}_{\theta}) + \sum_{\pm} \mu_e^{\pm} \lambda (\lambda + \widetilde{\mathcal{D}}_{\pm})^{-1} (-\varphi + \mathbf{A} \cdot \hat{\boldsymbol{v}}). \tag{3.10}$$

LEMMA 3.2. *The operators $\widetilde{\mathcal{D}}_{\pm}$ on $\mathfrak{N}_{\pm}$ satisfy the following:*

(a) *$\widetilde{\mathcal{D}}_{\pm}$ are skew-adjoint, and the resolvents $(\lambda + \widetilde{\mathcal{D}}_{\pm})^{-1}$ are bounded linear operators for $\operatorname{Re} \lambda \neq 0$ with norm bounded by $1/|\operatorname{Re} \lambda|$.*

(b) *$\widetilde{\mathcal{D}}_{\pm}$ flip parity with respect to the pair of variables (v_r, v_z); i.e., if $h \in \mathfrak{D}(\widetilde{\mathcal{D}}_{\pm})$ is an even function of the pair (v_r, v_z), then $\widetilde{\mathcal{D}}_{\pm} h$ is an odd function of (v_r, v_z) and vice versa (see Remark 3.4 below).*

(c) *For real $\lambda \neq 0$ the resolvents of $\widetilde{\mathcal{D}}_{\pm}$ split as follows:*

$$(\lambda + \widetilde{\mathcal{D}}_{\pm})^{-1} = \lambda (\lambda^2 - \widetilde{\mathcal{D}}_{\pm}^2)^{-1} - \widetilde{\mathcal{D}}_{\pm} (\lambda^2 - \widetilde{\mathcal{D}}_{\pm}^2)^{-1}, \tag{3.11}$$

where the first part is symmetric and preserves parity with respect to (v_r, v_z) and the second part is skew-symmetric and inverts parity with respect to (v_r, v_z).

We leave the proof, which is analogous to the proof of Lemma 3.1, to the reader.

Remark 3.4. For a function h expressed in cylindrical coordinates as $h(\boldsymbol{x}, v_r, v_z, v_\theta)$, we say that h is an even function of the pair (v_r, v_z) if $h(\boldsymbol{x}, v_r, v_z, v_\theta) = h(\boldsymbol{x}, -v_r, -v_z, v_\theta)$, where we flip the sign of both variables simultaneously. Note that this is a weaker property than both being an even function of v_r and an even function of v_z. Odd functions of (v_r, v_z) are defined similarly.

As in the $1.5d$ case, we define averaging operators. However, in this case both the symmetric and skew-symmetric parts are required. The operators $\widetilde{\mathcal{Q}}^\lambda_{\pm,sym}$ and $\widetilde{\mathcal{Q}}^\lambda_{\pm,skew}$ map $\mathfrak{N}_\pm$ to $\mathfrak{N}_\pm$ and are defined by the rules

$$\begin{aligned}\widetilde{\mathcal{Q}}^\lambda_{\pm,sym} &= \lambda^2(\lambda^2 - \widetilde{\mathcal{D}}^2_\pm)^{-1}, \qquad && \lambda > 0,\\ \widetilde{\mathcal{Q}}^\lambda_{\pm,skew} &= -\lambda\widetilde{\mathcal{D}}_\pm(\lambda^2 - \widetilde{\mathcal{D}}^2_\pm)^{-1}, && \lambda > 0.\end{aligned}$$

Note that by (3.11) we have $\lambda(\lambda + \widetilde{\mathcal{D}}_\pm)^{-1} = \widetilde{\mathcal{Q}}^\lambda_{\pm,sym} + \widetilde{\mathcal{Q}}^\lambda_{\pm,skew}$.

3.2.2. Reformulating Maxwell's equations. We now rewrite Maxwell's equations (1.29b)–(1.29c) as an equivalent self-adjoint problem using the expression (3.10). We start with (1.29b):

(3.12)
$$\begin{aligned}0 &= \lambda^2\varphi - \Delta\varphi - \int (f^+ - f^-)\, d\boldsymbol{v}\\ &= \lambda^2\varphi - \Delta\varphi - \int \sum_\pm \Big(\mu_e^\pm\varphi + r\mu_p^\pm(\mathbf{A}\cdot\boldsymbol{e}_\theta) + \mu_e^\pm\lambda(\lambda + \widetilde{\mathcal{D}}_\pm)^{-1}(-\varphi + \mathbf{A}\cdot\hat{\boldsymbol{v}})\Big)\, d\boldsymbol{v},\end{aligned}$$

where $\varphi \in \mathfrak{H}_\varphi$. Next, the system of equations (1.29c) becomes

(3.13)
$$\begin{aligned}0 &= \lambda^2\mathbf{A} - \boldsymbol{\Delta}\mathbf{A} - \int (f^+ - f^-)\hat{\boldsymbol{v}}\, d\boldsymbol{v}\\ &= \lambda^2\mathbf{A} - \boldsymbol{\Delta}\mathbf{A} - \int \sum_\pm \Big(\mu_e^\pm\varphi + r\mu_p^\pm(\mathbf{A}\cdot\boldsymbol{e}_\theta) + \mu_e^\pm\lambda(\lambda + \widetilde{\mathcal{D}}_\pm)^{-1}(-\varphi + \mathbf{A}\cdot\hat{\boldsymbol{v}})\Big)\, \hat{\boldsymbol{v}}\, d\boldsymbol{v},\end{aligned}$$

where $\mathbf{A} = (\mathbf{A}_\theta, \mathbf{A}_{rz}) \in L^2_\theta(\mathbb{R}^3;\mathbb{R}^3) \times L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)$. As in (3.5), we shall write these equations as a single system of the form

$$\widetilde{\boldsymbol{\mathcal{M}}}^\lambda \begin{bmatrix}\mathbf{A}_\theta\\ \varphi\\ \mathbf{A}_{rz}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, \tag{3.14}$$

which is a self-adjoint operator in $L^2_\theta(\mathbb{R}^3;\mathbb{R}^3)\times L^2_{cyl}(\mathbb{R}^3)\times L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)$; see Lemma 6.8. In analogy with (3.7), we define

$$\widetilde{\boldsymbol{\mathcal{M}}}^\lambda = \begin{bmatrix}\widetilde{\mathcal{A}}^\lambda_2 & (\widetilde{\mathcal{B}}^\lambda_1)^* & (\widetilde{\mathcal{B}}^\lambda_2)^*\\ \widetilde{\mathcal{B}}^\lambda_1 & -\widetilde{\mathcal{A}}^\lambda_1 & -(\widetilde{\mathcal{B}}^\lambda_3)^*\\ \widetilde{\mathcal{B}}^\lambda_2 & -\widetilde{\mathcal{B}}^\lambda_3 & -\widetilde{\mathcal{A}}^\lambda_3\end{bmatrix}. \tag{3.15}$$

With $\hat{\boldsymbol{v}} = (\hat{v}_r, \hat{v}_\theta, \hat{v}_z)$, we recall the notation $\hat{\boldsymbol{v}}_\theta = \hat{v}_\theta \boldsymbol{e}_\theta$ and $\hat{\boldsymbol{v}}_{rz} = \hat{v}_r \boldsymbol{e}_r + \hat{v}_z \boldsymbol{e}_z$ introduced before. Then the components of $\widetilde{\boldsymbol{\mathcal{M}}}^\lambda$ are now given by

$$\widetilde{\mathcal{A}}_1^\lambda h = -\Delta h + \lambda^2 h + \int \sum_{\pm} \mu_e^{\pm} (\widetilde{\mathcal{Q}}_{\pm,sym}^\lambda - 1) h \, d\boldsymbol{v}, \tag{3.16a}$$

$$\widetilde{\mathcal{A}}_2^\lambda \boldsymbol{h} = -\boldsymbol{\Delta} \boldsymbol{h} + \lambda^2 \boldsymbol{h} - \left(r \int \sum_{\pm} \mu_p^{\pm} \hat{v}_\theta \, d\boldsymbol{v} \right) \boldsymbol{h} - \int \sum_{\pm} \hat{\boldsymbol{v}}_\theta \mu_e^{\pm} \widetilde{\mathcal{Q}}_{\pm,sym}^\lambda [\boldsymbol{h} \cdot \hat{\boldsymbol{v}}_\theta] \, d\boldsymbol{v}, \tag{3.16b}$$

$$\widetilde{\mathcal{A}}_3^\lambda \boldsymbol{h} = -\boldsymbol{\Delta} \boldsymbol{h} + \lambda^2 \boldsymbol{h} - \int \sum_{\pm} \hat{\boldsymbol{v}}_{rz} \mu_e^{\pm} \widetilde{\mathcal{Q}}_{\pm,sym}^\lambda [\boldsymbol{h} \cdot \hat{\boldsymbol{v}}_{rz}] \, d\boldsymbol{v}, \tag{3.16c}$$

$$\widetilde{\mathcal{B}}_1^\lambda \boldsymbol{h} = \int \sum_{\pm} \mu_e^{\pm} (\widetilde{\mathcal{Q}}_{\pm,sym}^\lambda - 1) [\boldsymbol{h} \cdot \hat{\boldsymbol{v}}_\theta] \, d\boldsymbol{v}, \tag{3.16d}$$

$$(\widetilde{\mathcal{B}}_1^\lambda)^* h = \int \sum_{\pm} \mu_e^{\pm} \hat{\boldsymbol{v}}_\theta (\widetilde{\mathcal{Q}}_{\pm,sym}^\lambda - 1) h \, d\boldsymbol{v}, \tag{3.16e}$$

$$\widetilde{\mathcal{B}}_2^\lambda \boldsymbol{h} = \int \sum_{\pm} \mu_e^{\pm} \hat{\boldsymbol{v}}_{rz} \widetilde{\mathcal{Q}}_{\pm,skew}^\lambda [\boldsymbol{h} \cdot \hat{\boldsymbol{v}}_\theta] \, d\boldsymbol{v}, \tag{3.16f}$$

$$(\widetilde{\mathcal{B}}_2^\lambda)^* \boldsymbol{h} = -\int \sum_{\pm} \mu_e^{\pm} \hat{\boldsymbol{v}}_\theta \widetilde{\mathcal{Q}}_{\pm,skew}^\lambda \, [\hat{\boldsymbol{v}}_{rz} \cdot \boldsymbol{h}] \, d\boldsymbol{v}, \tag{3.16g}$$

$$\widetilde{\mathcal{B}}_3^\lambda h = \int \sum_{\pm} \mu_e^{\pm} \hat{\boldsymbol{v}}_{rz} \widetilde{\mathcal{Q}}_{\pm,skew}^\lambda \, h \, d\boldsymbol{v}, \tag{3.16h}$$

$$(\widetilde{\mathcal{B}}_3^\lambda)^* \boldsymbol{h} = -\int \sum_{\pm} \mu_e^{\pm} \hat{\boldsymbol{v}}_{rz} \widetilde{\mathcal{Q}}_{\pm,skew}^\lambda \, [\boldsymbol{h} \cdot \hat{\boldsymbol{v}}_{rz}] \, d\boldsymbol{v}. \tag{3.16i}$$

These are derived from (3.12) and (3.13), where some terms vanish due to parity in (v_r, v_z) (see Lemma 3.2(c)). In particular, in every occurrence of $\lambda(\lambda + \widetilde{\mathcal{D}}_\pm)^{-1} = \widetilde{\mathcal{Q}}_{\pm,sym}^\lambda + \widetilde{\mathcal{Q}}_{\pm,skew}^\lambda$, exactly one of these operators vanishes after integration $d\boldsymbol{v}$. In addition, we have made use of (1.34). We further define an operator $\widetilde{\boldsymbol{\mathcal{J}}}^\lambda$ as

$$\widetilde{\boldsymbol{\mathcal{J}}}^\lambda = \begin{bmatrix} \lambda^2 - \boldsymbol{\Delta} & 0 & 0 \\ 0 & -\lambda^2 + \Delta & 0 \\ 0 & 0 & -\lambda^2 + \boldsymbol{\Delta} \end{bmatrix} - \widetilde{\boldsymbol{\mathcal{M}}}^\lambda.$$

Let us briefly discuss these operators in further detail (their precise properties are treated in subsection 6.2):

- The operators

$$\begin{aligned} &\widetilde{\mathcal{A}}_1^\lambda : H_{cyl}^2(\mathbb{R}^3) \subset L_{cyl}^2(\mathbb{R}^3) \to L_{cyl}^2(\mathbb{R}^3), \\ &\widetilde{\mathcal{A}}_2^\lambda : H_\theta^2(\mathbb{R}^3; \mathbb{R}^3) \subset L_\theta^2(\mathbb{R}^3; \mathbb{R}^3) \to L_\theta^2(\mathbb{R}^3; \mathbb{R}^3), \\ &\widetilde{\mathcal{A}}_3^\lambda : H_{rz}^2(\mathbb{R}^3; \mathbb{R}^3) \subset L_{rz}^2(\mathbb{R}^3; \mathbb{R}^3) \to L_{rz}^2(\mathbb{R}^3; \mathbb{R}^3) \end{aligned}$$

  are self-adjoint, and have essential spectrum in $[\lambda^2, \infty)$ and a finite number of eigenvalues in $(-\infty, \lambda^2)$.

- The operators
$$\widetilde{\mathcal{B}}_1^\lambda : L^2_\theta(\mathbb{R}^3;\mathbb{R}^3) \to L^2_{cyl}(\mathbb{R}^3),$$
$$\widetilde{\mathcal{B}}_2^\lambda : L^2_\theta(\mathbb{R}^3;\mathbb{R}^3) \to L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3),$$
$$\widetilde{\mathcal{B}}_3^\lambda : L^2_{cyl}(\mathbb{R}^3) \to L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)$$
are bounded, with bound independent of λ.
- $\widetilde{\boldsymbol{\mathcal{J}}}^\lambda : L^2_\theta(\mathbb{R}^3;\mathbb{R}^3)\times L^2_{cyl}(\mathbb{R}^3)\times L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3) \to L^2_\theta(\mathbb{R}^3;\mathbb{R}^3)\times L^2_{cyl}(\mathbb{R}^3)\times L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)$ is a bounded symmetric operator with bound independent of λ.

4. Solving the equivalent problem. The problem is now reduced to finding some $\lambda \in (0,\infty)$ for which the operators $\boldsymbol{\mathcal{M}}^\lambda$ (in the 1.5d case) and $\widetilde{\boldsymbol{\mathcal{M}}}^\lambda$ (in the cylindrically symmetric case) have nontrivial kernels (not the same λ in both cases, of course). Our method is to compare their spectrum for $\lambda = 0$ and λ very large and use spectral continuity arguments to deduce that as λ varies, an eigenvalue must cross through 0. (Both operators are self-adjoint (see Lemmas 6.4 and 6.8 below); hence the spectrum lies on the real axis.)

4.1. The 1.5d case.

4.1.1. Continuity of the spectrum at $\lambda = 0$. Recall the condition (1.9) which we require for instability:

$$\operatorname{neg}(\mathcal{A}_2^0 + (\mathcal{B}^0)^*(\mathcal{A}_1^0)^{-1}\mathcal{B}^0) > \operatorname{neg}(\mathcal{A}_1^0). \tag{4.1}$$

We wish to move this condition to values of λ greater than 0, as follows.

LEMMA 4.1. *Assume that* (4.1) *holds and that zero is in the resolvent set of* $\mathcal{A}_1^0$. *Then there exists* $\lambda_* > 0$ *such that for all* $\lambda \in [0,\lambda_*]$
$$\operatorname{neg}(\mathcal{A}_2^\lambda + (\mathcal{B}^\lambda)^*(\mathcal{A}_1^\lambda)^{-1}\mathcal{B}^\lambda) > \operatorname{neg}(\mathcal{A}_1^\lambda).$$

Proof. The proof follows immediately from the following three steps:

Step 1. $\mathcal{A}_1^\lambda$ *is invertible for small* $\lambda \geq 0$. We know from Lemma 6.3 (below) that $\mathcal{A}_1^\lambda$ is continuous in the norm resolvent sense and has discrete spectrum. The norm resolvent continuity implies that its spectrum varies continuously in λ, so as 0 is not in its spectrum at $\lambda = 0$, there exists λ_* such that 0 is not in the spectrum for $0 \leq \lambda \leq \lambda_*$. Hence for all such λ, $\mathcal{A}_1^\lambda$ is invertible and the operator $\mathcal{A}_2^\lambda + (\mathcal{B}^\lambda)^*(\mathcal{A}_1^\lambda)^{-1}\mathcal{B}^\lambda$ is well defined.

Step 2. $\operatorname{neg}(\mathcal{A}_1^\lambda) = \operatorname{neg}(\mathcal{A}_1^0)$ *for all* $\lambda \in [0,\lambda_*]$. The spectrum of $\mathcal{A}_1^\lambda$ is purely discrete, and 0 is in its resolvent set. This means that none of its eigenvalues can cross 0 for small values of λ.

Step 3. $\operatorname{neg}(\mathcal{A}_2^\lambda + (\mathcal{B}^\lambda)^*(\mathcal{A}_1^\lambda)^{-1}\mathcal{B}^\lambda) \geq \operatorname{neg}(\mathcal{A}_2^0 + (\mathcal{B}^0)^*(\mathcal{A}_1^0)^{-1}\mathcal{B}^0)$ *for all* $\lambda \in [0,\lambda_*]$. Observe that

- $[0,\infty) \ni \lambda \mapsto \mathcal{A}_2^\lambda + (\mathcal{B}^\lambda)^*(\mathcal{A}_1^\lambda)^{-1}\mathcal{B}^\lambda$ is norm resolvent continuous,
- $\mathcal{A}_2^\lambda + (\mathcal{B}^\lambda)^*(\mathcal{A}_1^\lambda)^{-1}\mathcal{B}^\lambda$ has essential spectrum in $[\lambda^2,\infty)$,
- $\mathcal{A}_2^\lambda + (\mathcal{B}^\lambda)^*(\mathcal{A}_1^\lambda)^{-1}\mathcal{B}^\lambda$ has finitely many negative eigenvalues.

These statements follow from arguments similar to those appearing in the proof of Lemma 6.3(a)–(c), the last by the boundedness of the perturbation and the location of the essential spectrum (see Lemma 2.2). Since 0 is not in the resolvent set at $\lambda = 0$ we pick $\sigma < 0$ larger than all the (finitely many) negative eigenvalues of $\mathcal{A}_2^0 + (\mathcal{B}^0)^*(\mathcal{A}_1^0)^{-1}\mathcal{B}^0$. The continuous dependence of the spectrum (as a set) on the parameter λ implies that for small values of λ no eigenvalues cross σ and the number of negative eigenvalues can grow only as λ increases. ∎

4.1.2. Truncation. We follow the plan hinted at in subsection 2.2: first we discretize the spectrum, and then we truncate. The only continuous part in the spectrum of $\boldsymbol{\mathcal{M}}^\lambda$ is due to $\mathcal{A}_2^\lambda$; hence we let $W(x)$ be a smooth positive potential function satisfying $W(x) \to \infty$ as $x \to \pm\infty$, which we shall add to $\mathcal{A}_2^\lambda$. It is well known that the Schrödinger operator $-\partial_x^2 + W$ on $L^2(\mathbb{R})$ is self-adjoint (on an appropriate domain therein) with compact resolvent (and therefore discrete spectrum). Moreover, $C_0^\infty(\mathbb{R})$ is a core for both $\partial_x^2 + W$ and ∂_x^2. Thus our approximating operator family is $\{\boldsymbol{\mathcal{M}}_\varepsilon^\lambda\}_{\lambda\in[\lambda_*,\infty),\varepsilon\in[0,\infty)}$, where

$$\boldsymbol{\mathcal{M}}_\varepsilon^\lambda = \begin{bmatrix} \mathcal{A}_{2,\varepsilon}^\lambda & (\mathcal{B}^\lambda)^* \\ \mathcal{B}^\lambda & -\mathcal{A}_1^\lambda \end{bmatrix} = \underbrace{\begin{bmatrix} -\partial_x^2 + \varepsilon W & 0 \\ 0 & \partial_x^2 \end{bmatrix} + \begin{bmatrix} \lambda^2 & 0 \\ 0 & 0 \end{bmatrix}}_{\boldsymbol{\mathcal{A}}_\varepsilon^\lambda} - \boldsymbol{\mathcal{J}}^\lambda$$

defined on $L^2(\mathbb{R}) \times L_0^2(\Omega)$ and where λ_* is as given in Lemma 4.1. For $\varepsilon > 0$ this operator has discrete spectrum. As indicated in the statement of Theorem 2.1 and the preceding definitions, we define truncated versions using the eigenspaces of the operator $\boldsymbol{\mathcal{A}}_\varepsilon^\lambda$. As this operator is diagonal, we can choose the eigenvectors to lie in exactly one of $L^2(\mathbb{R})$ or $L_0^2(\Omega)$. We denote the nth truncation, a projection onto an eigenspace of dimension $2n$ consisting of n eigenvectors in each of $L^2(\mathbb{R})$ and $L_0^2(\Omega)$, as $\boldsymbol{\mathcal{M}}_{\varepsilon,n}^\lambda$, which is self-adjoint and defined for $\varepsilon > 0$, $\lambda \geq 0$, $n \in \mathbb{N}$. Moreover, the mapping $\lambda \mapsto \operatorname{sp}(\boldsymbol{\mathcal{M}}_{\varepsilon,n}^\lambda)$ is continuous (that is, the set of eigenvalues varies continuously). In particular, if there are $\lambda_* < \lambda^*$ for which $\operatorname{neg}(\boldsymbol{\mathcal{M}}_{\varepsilon,n}^{\lambda_*}) \neq \operatorname{neg}(\boldsymbol{\mathcal{M}}_{\varepsilon,n}^{\lambda^*})$, then there must exist $\lambda_{\varepsilon,n} \in (\lambda_*, \lambda^*)$ for which $0 \in \operatorname{sp}(\boldsymbol{\mathcal{M}}_{\varepsilon,n}^\lambda)$. We have therefore just proved the following result.

LEMMA 4.2. *Fix $\varepsilon > 0$, $n \in \mathbb{N}$. Suppose that there exist $0 < \lambda_* < \lambda^* < \infty$ such that $\operatorname{neg}(\boldsymbol{\mathcal{M}}_{\varepsilon,n}^{\lambda_*}) \neq \operatorname{neg}(\boldsymbol{\mathcal{M}}_{\varepsilon,n}^{\lambda^*})$. Then there is a $\lambda_{\varepsilon,n} \in (\lambda_*, \lambda^*)$ for which $\ker(\boldsymbol{\mathcal{M}}_{\varepsilon,n}^\lambda)$ is nontrivial.*

The next step is thus to establish estimates on $\operatorname{neg}(\boldsymbol{\mathcal{M}}_{\varepsilon,n}^\lambda)$.

4.1.3. The spectrum for large $\boldsymbol{\lambda}$. We begin by looking at $\operatorname{neg}(\boldsymbol{\mathcal{M}}_{\varepsilon,n}^\lambda)$ when λ is large. This turns out to be relatively simple due to the block form of the untruncated operator.

LEMMA 4.3. *There is $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$, $\varepsilon > 0$, and $n \in \mathbb{N}$ the truncated operator $\boldsymbol{\mathcal{M}}_{\varepsilon,n}^\lambda$ has spectrum composed of exactly n positive and n negative eigenvalues. In particular, $\operatorname{neg}(\boldsymbol{\mathcal{M}}_{\varepsilon,n}^\lambda) = n$.*

Proof. Take $\boldsymbol{u} = (u_1, 0) \in L^2(\mathbb{R}) \times L_0^2(\Omega)$ with $u_1 \in \mathfrak{D}(\mathcal{A}_{2,\varepsilon}^\lambda)$, $\|\boldsymbol{u}\|_{L^2(\mathbb{R})\times L^2(\Omega)} = 1$, and $\boldsymbol{u}$ in the $2n$-dimensional subspace associated with the truncation. Then,

$$\begin{aligned} \left\langle \boldsymbol{\mathcal{M}}_{\varepsilon,n}^\lambda \boldsymbol{u}, \boldsymbol{u} \right\rangle_{L^2(\mathbb{R})\times L_0^2(\Omega)} &= \left\langle \mathcal{A}_{1,\varepsilon,n}^\lambda u_1, u_1 \right\rangle_{L^2(\mathbb{R})} = \left\langle \mathcal{A}_{1,\varepsilon}^\lambda u_1, u_1 \right\rangle_{L^2(\mathbb{R})} \\ &= \left\langle \mathcal{A}_1^\lambda u_1, u_1 \right\rangle_{L^2(\mathbb{R})} + \varepsilon \left\| \sqrt{W} u_1 \right\|_{L^2(\mathbb{R})}^2 . \end{aligned}$$

As the second term is nonnegative, we may apply Lemma 6.3(d) to see that, for all large enough λ (independently of n and ε), $\boldsymbol{\mathcal{M}}_{\varepsilon,n}^\lambda$ is positive definite on a subspace of dimension n, and so it has n positive eigenvalues. Performing the same computation on $\boldsymbol{u} = (0, u_2)$ in the subspace associated with the truncation and with $u_2 \in \mathfrak{D}(\mathcal{A}_{1,\varepsilon}^\lambda)$, we obtain that, for large enough λ, $\boldsymbol{\mathcal{M}}_{\varepsilon,n}^\lambda$ is negative definite on a subspace of dimension n. As $\boldsymbol{\mathcal{M}}_{\varepsilon,n}^\lambda$ has exactly $2n$ eigenvalues, the proof is complete. ∎

4.1.4. The spectrum for small $\boldsymbol{\lambda}$. We now consider $\mathrm{sp}(\boldsymbol{\mathcal{M}}_{\varepsilon,n}^{\lambda_*})$. We recall the result on spectra of real block matrix operators in [3].

LEMMA 4.4. *Let M be the real symmetric block matrix*

$$M = \begin{bmatrix} A_2 & B^T \\ B & -A_1 \end{bmatrix}$$

with A_1 invertible. Then M has the same number of negative eigenvalues as the matrix

$$N = \begin{bmatrix} A_2 + B^T A_1^{-1} B & 0 \\ 0 & -A_1 \end{bmatrix}.$$

LEMMA 4.5. *Assume that* (4.1) *holds and that zero is in the resolvent set of $\mathcal{A}_1^0$. Then there exist $\lambda_*, \varepsilon_* > 0$ such that for all $\varepsilon \in (0, \varepsilon_*)$ there is $N > 0$ such that for all $n > N$ the operator $\boldsymbol{\mathcal{M}}_{\varepsilon,n}^{\lambda_*}$ satisfies*

$$\mathrm{neg}(\boldsymbol{\mathcal{M}}_{\varepsilon,n}^{\lambda_*}) \geq \mathrm{neg}(\mathcal{A}_2^0 + (\mathcal{B}^0)^*(\mathcal{A}_1^0)^{-1}\mathcal{B}^0) + n - \mathrm{neg}(\mathcal{A}_1^0).$$

Proof. The value of λ_* is that given in Lemma 4.1 and satisfies that for all $\lambda \in [0, \lambda_*]$ the kernel of $\mathcal{A}_1^\lambda$ is trivial. Since eigenvalues (counting multiplicity) are stable under strong resolvent perturbations (see [10, Chapter VIII, section 3.5, Theorem 3.15]), there exists $\varepsilon_* > 0$ such that $\mathrm{neg}(\mathcal{A}_{2,\varepsilon}^0 + (\mathcal{B}^0)^*(\mathcal{A}_1^0)^{-1}\mathcal{B}^0) \geq \mathrm{neg}(\mathcal{A}_2^0 + (\mathcal{B}^0)^*(\mathcal{A}_1^0)^{-1}\mathcal{B}^0)$ for all $\varepsilon \in [0, \varepsilon_*]$. The result then follows from Lemma 4.4, since $\mathrm{neg}(\boldsymbol{\mathcal{M}}_{\varepsilon,n}^{\lambda_*}) = \mathrm{neg}(\mathcal{A}_{2,\varepsilon}^{\lambda_*} + (\mathcal{B}^{\lambda_*})^*(\mathcal{A}_1^{\lambda_*})^{-1}\mathcal{B}^{\lambda_*}) + n - \mathrm{neg}(\mathcal{A}_1^{\lambda_*})$. ∎

4.2. The cylindrically symmetric case. For brevity we write

$$\widetilde{\boldsymbol{\mathcal{M}}}^\lambda = \begin{bmatrix} \widetilde{\mathcal{A}}_2^\lambda & (\widetilde{\boldsymbol{\mathcal{B}}}_4^\lambda)^* \\ \widetilde{\boldsymbol{\mathcal{B}}}_4^\lambda & -\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda \end{bmatrix},$$

where

$$\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda = \begin{bmatrix} \widetilde{\mathcal{A}}_1^\lambda & (\widetilde{\mathcal{B}}_3^\lambda)^* \\ \widetilde{\mathcal{B}}_3^\lambda & \widetilde{\mathcal{A}}_3^\lambda \end{bmatrix} \quad \text{and} \quad \widetilde{\boldsymbol{\mathcal{B}}}_4^\lambda = \begin{bmatrix} \widetilde{\mathcal{B}}_1^\lambda \\ \widetilde{\mathcal{B}}_2^\lambda \end{bmatrix}.$$

4.2.1. Continuity of the spectrum at $\boldsymbol{\lambda} = 0$.

LEMMA 4.6. *Assume that* (1.10) *holds, that $\widetilde{\mathcal{A}}_3^0$ does not have* 0 *as an L^6-eigenvalue (see Definition* 1.3*), and that $\widetilde{\mathcal{A}}_1^0$ does not have* 0 *as an eigenvalue. Then there exists $\lambda_* > 0$ such that for $\lambda \in [0, \lambda_*]$,*

$$\mathrm{neg}(\widetilde{\mathcal{A}}_2^\lambda + (\widetilde{\boldsymbol{\mathcal{B}}}_4^\lambda)^*(\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda)^{-1}\widetilde{\boldsymbol{\mathcal{B}}}_4^\lambda) > \mathrm{neg}(\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda).$$

Proof. We first note that, as the mean perturbed charge is zero (because $\int \rho \, d\boldsymbol{x}$ is an invariant of the linearized system), it follows from direct computation on the Green's function of the Laplacian that any L^6-eigenfunction of $\widetilde{\mathcal{A}}_1^0$ will also be square integrable and so be a proper eigenfunction. Indeed, for any L^6 eigenfunction u of $\widetilde{\mathcal{A}}_1^0$, we may define ρ by $\rho = -\Delta u$, which satisfies $\int \rho \, d\boldsymbol{x} = 0$ and has compact support. Then, for $\boldsymbol{x}$ outside the support of ρ, we have

$$\begin{aligned} u(\boldsymbol{x}) &= \frac{1}{4\pi} \int \frac{\rho(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|} \, d\boldsymbol{y} \\ &= \frac{1}{4\pi|\boldsymbol{x}|} \int \rho(\boldsymbol{y}) \, d\boldsymbol{y} + \frac{1}{4\pi|\boldsymbol{x}|^2} \int \rho(\boldsymbol{y}) \frac{|\boldsymbol{x}|(|\boldsymbol{x}| - |\boldsymbol{x} - \boldsymbol{y}|)}{|\boldsymbol{x} - \boldsymbol{y}|} \, d\boldsymbol{y}. \end{aligned}$$

The first term vanishes, and using the compact support of ρ, it is easily seen that the second integral is bounded independently of $\boldsymbol{x}$. Hence $u(\boldsymbol{x})$ decays like $C|\boldsymbol{x}|^{-2}$ for large $|\boldsymbol{x}|$, and square integrability follows.

Note also that $\widetilde{\mathcal{B}}_3^0 = 0$. Thus $\widetilde{\boldsymbol{\mathcal{A}}}_4^0$ having an L^6-eigenfunction of 0 contradicts our assumptions, a fact that we will use later.

We model the proof on that of Lemma 4.1, splitting it into four steps.

Step 1. *$\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda$ is invertible for small $\lambda \geq 0$ when restricted to functions supported in Ω.* Let $\boldsymbol{\mathcal{P}} \in \mathfrak{B}\big(L^2_{cyl}(\mathbb{R}^3) \times L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)\big)$ be multiplication by the indicator function of Ω. We claim that for all small enough $\lambda > 0$, $\boldsymbol{\mathcal{P}}(\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda)^{-1}\boldsymbol{\mathcal{P}}$ is a well-defined bounded operator that is strongly continuous in $\lambda > 0$ and has a strong limit as $\lambda \to 0$. To prove this, we argue that if this were not the case, then 0 would be an L^6-eigenvalue of $\widetilde{\boldsymbol{\mathcal{A}}}_4^0$, a contradiction.

As $L^2_{cyl}(\mathbb{R}^3) \times L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)$ is a closed subspace of $L^2(\mathbb{R}^3;\mathbb{R}^4)$, we may work in the larger space to ease notation. To this end, let $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the $L^2(\mathbb{R}^3;\mathbb{R}^4)$ norm and inner product. We can express $\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda$ in the form

$$\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda \boldsymbol{u} = -\boldsymbol{\Delta}\boldsymbol{u} + \lambda^2\boldsymbol{u} + \boldsymbol{\mathcal{K}}^\lambda\boldsymbol{u},$$

where $\boldsymbol{\mathcal{K}}^\lambda$ is uniformly bounded, strongly continuous in $\lambda \geq 0$, and $\boldsymbol{\mathcal{K}}^\lambda = \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{K}}^\lambda\boldsymbol{\mathcal{P}}$.

Step 1.1. *$\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda$ is bounded from below when restricted to functions supported in Ω.* First we claim that there exist constants $\lambda' > 0$ and $C > 0$ such that we have the uniform lower bound

$$\left\|\mathbb{1}_\Omega \widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda \boldsymbol{u}^\lambda\right\| \geq C\left\|\mathbb{1}_\Omega \boldsymbol{u}^\lambda\right\| \qquad \forall \lambda \in (0,\lambda'], \tag{4.2}$$

where the constant C does not depend on λ or on $\boldsymbol{u}^\lambda$ and where $\boldsymbol{u}^\lambda$ satisfies $\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda \boldsymbol{u}^\lambda = 0$ outside Ω. Indeed, if not, there would be sequences $\lambda_n \to 0$ and $\{\boldsymbol{u}_n\}_{n=1}^\infty$ with $\|\mathbb{1}_\Omega \boldsymbol{u}_n\|_{L^2} = 1$ that satisfy

$$\widetilde{\boldsymbol{\mathcal{A}}}_4^{\lambda_n}\boldsymbol{u}_n = -\boldsymbol{\Delta}\boldsymbol{u}_n + \lambda_n^2\boldsymbol{u}_n + \boldsymbol{\mathcal{K}}^{\lambda_n}\boldsymbol{u}_n = \boldsymbol{f}_n \to 0 \tag{4.3}$$

as $n \to \infty$, with $\boldsymbol{f}_n$ supported in Ω. Hence,

$$\|\nabla\boldsymbol{u}_n\|^2 + \lambda_n^2\|\boldsymbol{u}_n\|^2 + \left\langle \boldsymbol{\mathcal{K}}^{\lambda_n}\boldsymbol{u}_n, \boldsymbol{u}_n\right\rangle = \langle \boldsymbol{f}_n, \boldsymbol{u}_n\rangle \to 0 \tag{4.4}$$

so that $\|\nabla\boldsymbol{u}_n\|^2$ is uniformly bounded for large enough n. Therefore, there exists a subsequence (we abuse notation and keep the same sequence) such that $\nabla\boldsymbol{u}_n \rightharpoonup \boldsymbol{v}$ weakly in $L^2(\mathbb{R}^3;\mathbb{R}^4)$ for some $\boldsymbol{v} \in L^2(\mathbb{R}^3;\mathbb{R}^4)$. By the standard Sobolev inequality $\|\varphi\|_{L^6(\mathbb{R}^3)} \leq C\|\nabla\varphi\|_{L^2(\mathbb{R}^3)}$ we have a uniform bound on $\|\boldsymbol{u}_n\|_{L^6(\mathbb{R}^3;\mathbb{R}^4)}$. Therefore, passing again to a subsequence if necessary, $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ weakly in $L^6(\mathbb{R}^3;\mathbb{R}^4)$ for some $\boldsymbol{u} \in L^6(\mathbb{R}^3;\mathbb{R}^4)$. Furthermore, by Rellich's theorem we have the strong convergence $\boldsymbol{u}_n \to \boldsymbol{u}$ in $L^2_{loc}(\mathbb{R}^3;\mathbb{R}^4)$. This implies that necessarily $\boldsymbol{v} = \nabla\boldsymbol{u}$. In particular we deduce that $\|\mathbb{1}_\Omega\boldsymbol{u}\| = 1$ so $\boldsymbol{u} \neq 0$. Passing to the limit in (4.3), $\boldsymbol{u}$ satisfies

$$-\boldsymbol{\Delta}\boldsymbol{u} + \boldsymbol{\mathcal{K}}^0\boldsymbol{u} = 0$$

in the sense of distributions, and by elliptic regularity $\boldsymbol{u} \in H^2_{loc}(\mathbb{R}^3;\mathbb{R}^4)$. In fact $\boldsymbol{u}$ is an L^6-eigenfunction of $\widetilde{\boldsymbol{\mathcal{A}}}_4^0$ with eigenvalue 0, which contradicts our assumptions. This proves the claim.

Step 1.2. *$\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda$ is invertible for all small enough $\lambda > 0$.* For any $\lambda > 0$, 0 does not lie in the essential spectrum of $\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda$, so it is either an eigenvalue or in the resolvent set. Let $\lambda > 0$ be small enough so that (4.2) holds; then any eigenfunction $\boldsymbol{u}$ of 0 satisfies all the assumptions of the claim above, and hence $\|\mathbb{1}_\Omega \boldsymbol{u}\| \le C^{-1}\|\mathbb{1}_\Omega \widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda \boldsymbol{u}\| = 0$ so that $\boldsymbol{u} = 0$ inside Ω. Clearly this implies that $\boldsymbol{u} = 0$ in $\mathbb{R}^3$, which is a contradiction. In the same way we deduce a uniform bound C from below for the operator $\boldsymbol{\mathcal{P}}(\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda)^{-1}\boldsymbol{\mathcal{P}}$ for such small $\lambda > 0$.

Step 1.3. *$\boldsymbol{\mathcal{P}}(\widetilde{\boldsymbol{\mathcal{A}}}_4^0)^{-1}\boldsymbol{\mathcal{P}}$ is well defined and bounded.* Finally, we give a meaning to $\boldsymbol{\mathcal{P}}(\widetilde{\boldsymbol{\mathcal{A}}}_4^0)^{-1}\boldsymbol{\mathcal{P}}$ (which is required as $\widetilde{\boldsymbol{\mathcal{A}}}_4^0$ is not invertible on the whole space). We define it to be the strong operator limit of $\boldsymbol{\mathcal{P}}(\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda)^{-1}\boldsymbol{\mathcal{P}}$ as $\lambda \to 0$. Indeed, suppose that $\boldsymbol{f}$ is fixed with support in Ω and $\lambda_n \to 0$. Then we wish to compute the limit of $\boldsymbol{\mathcal{P}}\boldsymbol{u}_n$ for $\boldsymbol{u}_n = (\widetilde{\boldsymbol{\mathcal{A}}}_4^{\lambda_n})^{-1}\boldsymbol{\mathcal{P}}\boldsymbol{f}$ as $n \to \infty$ and show that it is independent of the sequence $\lambda_n \to 0$. Indeed, $\boldsymbol{u}_n$ will satisfy

$$\widetilde{\boldsymbol{\mathcal{A}}}_4^{\lambda_n}\boldsymbol{u}_n = \lambda_n^2\boldsymbol{u}_n - \boldsymbol{\Delta}\boldsymbol{u}_n + \boldsymbol{\mathcal{K}}^{\lambda_n}\boldsymbol{u}_n = \boldsymbol{f}.$$

By the same argument as before, we can extract a subsequence and limit $\boldsymbol{u} \in L^6(\mathbb{R}^3;\mathbb{R}^4)$ with convergences as in Step 1.1. In particular, $\boldsymbol{\mathcal{P}}\boldsymbol{u}_n \to \boldsymbol{\mathcal{P}}\boldsymbol{u}$. We claim that the limit $\boldsymbol{u}$ is independent of the limiting sequence $\lambda_n \to 0$. Indeed, if two different limits $\boldsymbol{u}$ and $\boldsymbol{v}$ existed, then their difference $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v} \in L^6(\mathbb{R}^3;\mathbb{R}^4)$ would solve $\widetilde{\boldsymbol{\mathcal{A}}}_4^0\boldsymbol{w} = 0$, i.e., would be an L^6-eigenfunction with eigenvalue 0, which we assumed impossible.

Finally, the uniform bound (4.2) implies that the approximations $\boldsymbol{\mathcal{P}}(\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda)^{-1}\boldsymbol{\mathcal{P}}$ are uniformly bounded in operator norm for all sufficiently small positive λ. The convergence, for all $\boldsymbol{u} \in L^2(\mathbb{R}^3;\mathbb{R}^4)$, $\boldsymbol{\mathcal{P}}(\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda)^{-1}\boldsymbol{\mathcal{P}}\boldsymbol{u} \to \boldsymbol{\mathcal{P}}(\widetilde{\boldsymbol{\mathcal{A}}}_4^0)^{-1}\boldsymbol{\mathcal{P}}\boldsymbol{u}$ as $\lambda \to 0$, implies that the limiting operator has the same bound in operator norm.

Step 2. $\operatorname{neg}(\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda) = \operatorname{neg}(\widetilde{\boldsymbol{\mathcal{A}}}_4^0)$ *for all* $\lambda \in [0, \lambda_*]$. $\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda$ is norm resolvent continuous in $\lambda \ge 0$, so the only way the number of negative eigenvalues could change is for an eigenvalue to be absorbed into the essential spectrum at 0 as $\lambda \to 0$. Assume this happens; then we have a sequence $\lambda_n \to 0$, a sequence of negative eigenvalues $\sigma_n \to 0$, and eigenfunctions $\boldsymbol{u}_n$ which satisfy

$$-\boldsymbol{\Delta}\boldsymbol{u}_n + \lambda_n^2\boldsymbol{u}_n + \boldsymbol{\mathcal{K}}^{\lambda_n}\boldsymbol{u}_n = \sigma_n\boldsymbol{u}_n.$$

By the same argument as in the previous steps, we may take subsequences and obtain a contradiction.

Step 3. $\operatorname{neg}(\widetilde{\mathcal{A}}_2^\lambda + (\widetilde{\boldsymbol{\mathcal{B}}}_4^\lambda)^*(\widetilde{\boldsymbol{\mathcal{A}}}_4^\lambda)^{-1}\widetilde{\boldsymbol{\mathcal{B}}}_4^\lambda) \ge \operatorname{neg}(\widetilde{\mathcal{A}}_2^0 + (\widetilde{\boldsymbol{\mathcal{B}}}_4^0)^*(\widetilde{\boldsymbol{\mathcal{A}}}_4^0)^{-1}\widetilde{\boldsymbol{\mathcal{B}}}_4^0)$ *for all* $\lambda \in [0, \lambda_*]$. This may be proved in the same way as Step 3 of Lemma 4.1.

Step 4. $\operatorname{neg}(\widetilde{\mathcal{A}}_2^0 + (\widetilde{\boldsymbol{\mathcal{B}}}_4^0)^*(\widetilde{\boldsymbol{\mathcal{A}}}_4^0)^{-1}\widetilde{\boldsymbol{\mathcal{B}}}_4^0) > \operatorname{neg}(\widetilde{\boldsymbol{\mathcal{A}}}_4^0)$. As $\widetilde{\mathcal{B}}_2^0 = 0$ and $\widetilde{\mathcal{B}}_3^0 = 0$ we have

$$\begin{aligned}&\operatorname{neg}(\widetilde{\mathcal{A}}_2^0 + (\widetilde{\boldsymbol{\mathcal{B}}}_4^0)^*(\widetilde{\boldsymbol{\mathcal{A}}}_4^0)^{-1}\widetilde{\boldsymbol{\mathcal{B}}}_4^0)\\ &\quad = \operatorname{neg}(\widetilde{\mathcal{A}}_2^0 + (\widetilde{\mathcal{B}}_1^0)^*(\widetilde{\mathcal{A}}_1^0)^{-1}\widetilde{\mathcal{B}}_1^0) > \operatorname{neg}(\widetilde{\mathcal{A}}_1^0) + \operatorname{neg}(\widetilde{\mathcal{A}}_3^0) = \operatorname{neg}(\widetilde{\boldsymbol{\mathcal{A}}}_4^0),\end{aligned}$$

where the inequality is obtained from the assumption of the lemma. ∎

4.2.2. Finding a nontrivial kernel. The next few steps of the proof follow those of the 1.5d case; hence we provide only a short overview.

Truncation. As the domain is unbounded, each Laplacian appearing in the problem contributes an essential spectrum on $[0,\infty)$. We therefore introduce a smooth positive potential function $W:\mathbb{R}^3\to\mathbb{R}$ satisfying $W(x)\to\infty$ as $|x|\to\infty$ and denote by $W^{\otimes n}$ the n-dimensional vector-valued function with n copies of W. Then we define

$$\widetilde{\boldsymbol{\mathcal{M}}}_\varepsilon^\lambda=\begin{bmatrix}\widetilde{\mathcal{A}}_{2,\varepsilon}^\lambda & (\widetilde{\boldsymbol{\mathcal{B}}}_4^\lambda)^*\\ \widetilde{\boldsymbol{\mathcal{B}}}_4^\lambda & -\widetilde{\boldsymbol{\mathcal{A}}}_{4,\varepsilon}^\lambda\end{bmatrix}=\underbrace{\begin{bmatrix}-\boldsymbol{\Delta}+\varepsilon W^{\otimes 3} & 0\\ 0 & \boldsymbol{\Delta}-\varepsilon W^{\otimes 4}\end{bmatrix}+\begin{bmatrix}\lambda^2 & 0\\ 0 & -\lambda^2\end{bmatrix}}_{\widetilde{\boldsymbol{\mathcal{A}}}_\varepsilon^\lambda}-\widetilde{\boldsymbol{\mathcal{J}}}^\lambda.$$

As above, we can naturally define finite dimensional operators $\widetilde{\boldsymbol{\mathcal{M}}}_{\varepsilon,n}^\lambda$, for which we can easily prove the next result.

LEMMA 4.7. *Fix $\varepsilon>0$, $n\in\mathbb{N}$. Suppose that there exist $0<\lambda_*<\lambda^*<\infty$ such that* $\operatorname{neg}(\widetilde{\boldsymbol{\mathcal{M}}}_{\varepsilon,n}^{\lambda_*})\neq\operatorname{neg}(\widetilde{\boldsymbol{\mathcal{M}}}_{\varepsilon,n}^{\lambda^*})$. *Then there exists $\lambda_{\varepsilon,n}\in(\lambda_*,\lambda^*)$ for which* $\ker(\widetilde{\boldsymbol{\mathcal{M}}}_{\varepsilon,n}^{\lambda})$ *is nontrivial.*

The spectrum for large λ. This is again similar to the $1.5d$ case, in particular due to the appearance of the λ^2 terms. We have the following result.

LEMMA 4.8. *There is a number $\lambda^*>0$ such that for all $\lambda\geq\lambda^*$, $\varepsilon>0$, and $n\in\mathbb{N}$ the truncated operator $\widetilde{\boldsymbol{\mathcal{M}}}_{\varepsilon,n}^{\lambda}$ has spectrum composed of exactly n positive and n negative eigenvalues. In particular,* $\operatorname{neg}(\widetilde{\boldsymbol{\mathcal{M}}}_{\varepsilon,n}^{\lambda})=n$.

The spectrum for small λ. Again this is similar to the $1.5d$ case.

LEMMA 4.9. *Assume that* (1.10) *holds and that zero is neither an eigenvalue of $\widetilde{\mathcal{A}}_1^0$ nor an L^6-eigenvalue of $\widetilde{\mathcal{A}}_3^0$. Then there exist $\lambda_*,\varepsilon_*>0$ such that for all $\varepsilon\in(0,\varepsilon_*)$ there is $N>0$ such that for all $n>N$ the operator $\widetilde{\boldsymbol{\mathcal{M}}}_{\varepsilon,n}^{\lambda_*}$ satisfies*

$$\operatorname{neg}(\widetilde{\boldsymbol{\mathcal{M}}}_{\varepsilon,n}^{\lambda_*})\geq\operatorname{neg}\left(\widetilde{\mathcal{A}}_2^0+\left(\widetilde{\mathcal{B}}_1^0\right)^*\left(\widetilde{\mathcal{A}}_1^0\right)^{-1}\widetilde{\mathcal{B}}_1^0\right)+n-\operatorname{neg}\left(\widetilde{\mathcal{A}}_1^0\right)-\operatorname{neg}\left(\widetilde{\mathcal{A}}_3^0\right).\tag{4.5}$$

5. Proofs of the main theorems. In this section we complete the proofs of Theorems 1.1 and 1.2. In both settings—the $1.5d$ and the cylindrically symmetric—we first show that the results of section 4 imply that there exists some $\lambda>0$ such that the equivalent problems (3.5) and (3.14) have a nontrivial solution (the λ need not be the same in both cases, of course). Then we show that these nontrivial solutions lead to genuine nontrivial solutions of the linearized RVM in either case.

5.1. The $1.5d$ case.

5.1.1. Existence of a nontrivial kernel of the equivalent problem. By Lemmas 4.3 and 4.5 we have $0<\lambda_*<\lambda^*<\infty$ and $\varepsilon_*>0$ such that for any $\varepsilon<\varepsilon_*$ there is an N_ε such that for $n>N_\varepsilon$ we have

$$\begin{aligned}\operatorname{neg}(\boldsymbol{\mathcal{M}}_{\varepsilon,n}^{\lambda_*})&\geq\operatorname{neg}(\mathcal{A}_2^0+(\mathcal{B}^0)^*(\mathcal{A}_1^0)^{-1}\mathcal{B}^0)+n-\operatorname{neg}(\mathcal{A}_1^0)\\&>n=\operatorname{neg}(\boldsymbol{\mathcal{M}}_{\varepsilon,n}^{\lambda^*}),\end{aligned}$$

where the strict inequality is due to the assumption (1.9). Fix $\varepsilon\in(0,\varepsilon_*)$. By Lemma 4.2 for each $n>N_\varepsilon$ there exists $\lambda_{\varepsilon,n}\in(\lambda_*,\lambda^*)$ such that $0\in\operatorname{sp}(\boldsymbol{\mathcal{M}}_{\varepsilon,n}^{\lambda_{\varepsilon,n}})$. By compactness of the interval $[\lambda_*,\lambda^*]$ we may pass to a subsequence where $\lambda_{\varepsilon,n_k}\to\lambda_\varepsilon$ as $k\to\infty$ for some $\lambda_\varepsilon\in[\lambda_*,\lambda^*]$. By Theorem 2.1 we have $\widetilde{\mu}^\eta_{\lambda_{\varepsilon,n_k},\varepsilon,n_k}\rightharpoonup\mu^\eta_{\lambda_\varepsilon,\varepsilon}$ as $k\to$

∞, where $\widetilde{\mu}^{\eta}_{\lambda_{\varepsilon,n_k},\varepsilon,n_k}$ is the measure generated by the spectra of the approximations $\boldsymbol{\mathcal{M}}^{\lambda_{\varepsilon,n_k}}_{\varepsilon,n_k}$ and where $\mu^{\eta}_{\lambda_\varepsilon,\varepsilon}$ is the measure generated by the spectra of $\boldsymbol{\mathcal{M}}^{\lambda_\varepsilon}_{\varepsilon}$. In order to avoid the continuous spectrum tending to $+\infty$ and the discrete spectrum tending to $-\infty$, the cutoff function ϕ_η must be chosen so that its support lies within $[-\frac{K}{2}, \frac{\lambda_*^2}{2}]$, where $K > 0$ is the spectral gap of the Neumann Laplacian ∂_x^2 on $L^2_0(\Omega)$. Since 0 lies in the support of all $\widetilde{\mu}^{\eta}_{\lambda_{\varepsilon,n_k},\varepsilon,n_k}$ it must also lie in the support of $\mu^{\eta}_{\lambda_\varepsilon,\varepsilon}$. Furthermore, since $\phi_\eta(0) = 1$ we have that $0 \in \mathrm{sp}(\boldsymbol{\mathcal{M}}^{\lambda_\varepsilon}_{\varepsilon})$. We now repeat this argument to send $\varepsilon \downarrow 0$, obtaining $\lambda \in [\lambda_*, \lambda^*]$ with $0 \in \mathrm{sp}(\boldsymbol{\mathcal{M}}^{\lambda})$. Finally, the discreteness of the spectrum of $\boldsymbol{\mathcal{M}}^{\lambda}$ in $(-\infty, \lambda^2)$ (see Lemma 6.4) ensures that 0 is an eigenvalue of $\boldsymbol{\mathcal{M}}^{\lambda}$; i.e., $\boldsymbol{\mathcal{M}}^{\lambda}$ has a nontrivial kernel.

5.1.2. Existence of a growing mode. Now that we know that there exist some $\lambda \in (0,\infty)$ and some $\boldsymbol{u} = \begin{bmatrix}\psi & \phi\end{bmatrix}^T \in H^2(\mathbb{R}) \times H^2_{0,n}(\Omega)$ that solve (3.5), we show that a genuine growing mode as defined in (1.8) really exists. To this end, we use ϕ, ψ, and λ to define

$$E_1 = -\partial_x \phi, \qquad E_2 = -\lambda\psi, \qquad B = \partial_x \psi$$

(which lie in $H^1(\Omega)$, $H^2(\mathbb{R})$, and $H^1(\mathbb{R})$, respectively) and to define $f^\pm(x,v)$ as in (3.2):

$$f^\pm = \pm\mu_e^\pm \phi \pm \mu_p^\pm \psi \pm \lambda(\lambda + \mathcal{D}_\pm)^{-1}[\mu_e^\pm(-\phi + \hat{v}_2\psi)].$$

Observe that $f^\pm$ are both in $L^2(\mathbb{R}\times\mathbb{R}^2)$ since $\mu_e^\pm$ and $\mu_p^\pm$ are continuous functions that are compactly supported in the spatial variable which satisfy the integrability condition (1.6). In fact, $f^\pm$ are in the domains of $\mathcal{D}_\pm$, respectively, since $e^\pm$ and $p^\pm$ are constant along trajectories and ϕ and ψ are twice differentiable.

LEMMA 5.1. *The functions $f^\pm$ solve the linearized Vlasov equations* (3.1).

Proof. This is almost a tautology: applying the operators $\lambda + \mathcal{D}_\pm$ to the expressions for $f^\pm$, respectively, one is left precisely with the expressions (3.1). ∎

LEMMA 5.2. *The functions $f^\pm$ belong to $L^1(\mathbb{R}\times\mathbb{R}^2)$.*

Proof. Dropping the $\pm$ for brevity, the first term making up f is estimated as follows:

$$\|\mu_e\phi\|_{L^1(\mathbb{R}^3)} \lesssim \|\mu_e\|_{L^2(\mathbb{R}^3)}\|\phi\|_{L^2(\mathbb{R})} \lesssim \|\mu_e\|^{1/2}_{L^\infty(\mathbb{R}^3)}\|\mu_e\|^{1/2}_{L^1(\mathbb{R}^3)}\|\phi\|_{L^2(\mathbb{R})} < \infty.$$

The other terms are estimated similarly. (For the terms involving the averaging operator this may be seen by writing the ergodic average explicitly (see Remark 3.1) or by using boundedness of the averaging operator on $\mathfrak{L}_\pm$.) This implies that $f^\pm \in L^1(\mathbb{R}\times\mathbb{R}^2)$. ∎

We now define the charge and current densities ρ and j_i by

$$\rho = \int (f^+ - f^-)\, dv, \qquad j_i = \int \hat{v}_i(f^+ - f^-)\, dv, \quad i = 1,2.$$

Integrating $f^\pm$ in the momentum variable v alone, we obtain that $\rho \in L^1(\mathbb{R})$ as well as $j_i \in L^1(\mathbb{R})$ since $|\hat{v}_i| \le 1$. In particular ρ, j_i are distributions on $\mathbb{R}$.

LEMMA 5.3. *The continuity equation $\lambda\rho + \partial_x j_1 = 0$ holds in the sense of distributions.*

Proof. This follows from integrating the linearized Vlasov equations in the momentum variable. Indeed, we informally have

$$\begin{aligned}\int (\lambda + \mathcal{D}_\pm) f^\pm \, dv &= \pm \int \left[(\lambda + \mathcal{D}_\pm)(\mu_e^\pm \phi + \mu_p^\pm \psi) + \lambda \mu_e^\pm (-\phi + \hat{v}_2 \psi) \right] \, dv \\ &= \pm \int \lambda \psi \left(\mu_p^\pm + \mu_e^\pm \hat{v}_2 \right) \, dv \pm \int \mathcal{D}_\pm \left(\mu_e^\pm \phi + \mu_p^\pm \psi \right) \, dv = 0,\end{aligned}$$

where the first term on the right-hand side vanishes due to the identity (1.24) and the second term vanishes since $\mu^\pm$ are even in $\hat{v}_1$, whereas $\mathcal{D}_\pm = \hat{v}_1 \partial_x$ when applied to functions of x alone (recall that $\mu^\pm$ are constant along trajectories of $\mathcal{D}_\pm$, as are $\mu_e^\pm$ and $\mu_p^\pm$). We obtain the continuity equation by subtracting the "$-$" expression above from the "$+$" expression. Owing to the low regularity of $f^\pm$, ρ, and j_1, this is true in a weak sense. ∎

LEMMA 5.4. *Maxwell's equations* (1.21) *hold.*

Proof. Equations (1.21b) and (1.21c) hold due to (3.5) and the definitions of the operators (3.8). Indeed, from the second line of (3.5), we have

$$\begin{aligned}0 &= \left(\int \sum_\pm \mu_p^\pm \, d\boldsymbol{v} \right) \psi + \int \sum_\pm \mu_e^\pm \mathcal{Q}_\pm^\lambda \, [\hat{v}_2 \psi] \, d\boldsymbol{v} + \partial_x^2 \phi - \int \sum_\pm \mu_e^\pm (\mathcal{Q}_\pm^\lambda - 1) \phi \, d\boldsymbol{v} \\ &= \partial_x^2 \phi + \int \sum_\pm \left(\mu_p^\pm \psi + \mu_e^\pm \mathcal{Q}_\pm^\lambda \, [\hat{v}_2 \psi] - \mu_e^\pm (\mathcal{Q}_\pm^\lambda - 1) \phi \right) \, d\boldsymbol{v} \\ &\overset{(3.2)}{=} \partial_x^2 \phi + \int (f^+ - f^-) \, d\boldsymbol{v},\end{aligned}$$

which is (1.21c). Similarly, (1.21b) is obtained from the first line of (3.5).

We therefore just need to show that (1.21a) holds. However, this is a simple consequence of (1.21c) and the continuity equation. Indeed, we may first write

$$-\lambda \partial_x E_1 = \lambda \partial_x^2 \phi \overset{(1.21c)}{=} -\lambda \rho \overset{\text{cont. eq.}}{=} \partial_x j_1,$$

which is the derivative of (1.21a). Next, as $\phi \in H^2_{n,0}(\Omega)$, its derivative E_1 vanishes on $\partial\Omega$, and j_1 also vanishes there due to the compact support of the equilibrium in Ω. Thus, $-\lambda E_1$ and j_1 have the same derivative inside Ω and the same values on $\partial\Omega$, which means they must be equal. ∎

LEMMA 5.5. *The charge and current densities ρ, j_1, and j_2 are elements in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.*

Proof. This follows from Maxwell's equations and the regularity of ψ and ϕ, which are in $H^2(\mathbb{R})$ and $H^2_{0,n}(\Omega)$, respectively. ∎

This concludes the proof of Theorem 1.1.

5.2. The cylindrically symmetric case.

5.2.1. Existence of a nontrivial kernel of the equivalent problem. The proof of the existence of a nontrivial kernel in the cylindrically symmetric case is in complete analogy to that in the 1.5*d* case presented in subsection 5.1.1 and is therefore omitted.

5.2.2. Existence of a growing mode. Let $\lambda > 0$ and $\boldsymbol{u} = \begin{bmatrix}\mathbf{A}_\theta & \varphi & \mathbf{A}_{rz}\end{bmatrix}^T \in H^2_\theta(\mathbb{R}^3;\mathbb{R}^3) \times H^2_{cyl}(\mathbb{R}^3) \times H^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)$ be such that (3.14) is satisfied, i.e., $\widetilde{\boldsymbol{\mathcal{M}}}^\lambda \boldsymbol{u} = \mathbf{0}$. Let $H^2_{cyl}(\mathbb{R}^3;\mathbb{R}^3) \ni \mathbf{A} = \mathbf{A}_\theta + \mathbf{A}_{rz}$ as in (1.32), and define

$$\mathbf{E} = -\nabla\varphi, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

(which each lie in $H^1_{cyl}(\mathbb{R}^3;\mathbb{R}^3) \subseteq H^1(\mathbb{R}^3;\mathbb{R}^3)$). Furthermore, define

$$f^\pm = \pm\mu_e^\pm \varphi \pm r\mu_p^\pm(\mathbf{A}\cdot\boldsymbol{e}_\theta) \pm \mu_e^\pm \lambda(\lambda + \widetilde{\mathcal{D}}_\pm)^{-1}(-\varphi + \mathbf{A}\cdot\hat{\boldsymbol{v}}).$$

As in the 1.5d case we begin by establishing that $f^\pm$ are integrable and satisfy the linearized Vlasov and continuity equations. The proof of this result is analogous to the corresponding results in the 1.5d case, so it is omitted.

LEMMA 5.6. *The functions $f^\pm$ solve the linearized Vlasov equations* (1.29a) *in the sense of distributions and belong to $L^1(\mathbb{R}^3 \times \mathbb{R}^3)$. Furthermore, the charge and current densities ρ and $\mathbf{j}$, defined by*

$$\rho = \int (f^+ - f^-)\, d\boldsymbol{v}, \qquad \mathbf{j} = \int \hat{\boldsymbol{v}}(f^+ - f^-)\, d\boldsymbol{v},$$

belong to $L^1(\mathbb{R}^3)$ and $L^1(\mathbb{R}^3;\mathbb{R}^3)$, respectively, and satisfy the continuity equation $\lambda\rho + \nabla\cdot\mathbf{j} = 0$ in the sense of distributions.

Next we recover Maxwell's equations from (3.14) and the continuity equation.

LEMMA 5.7. *Both the Lorenz gauge condition $\lambda\varphi + \nabla\cdot\mathbf{A} = 0$ (see* (1.31)*) and Maxwell's equations* (1.25) *are satisfied.*

Proof. In the same way as the 1.5d case, (1.25a) is obtained from the second line of (3.14). Similarly, (1.25b) is obtained from the first and third lines of (3.14).

It remains to show that the Lorenz gauge condition holds. Using (1.25a) and (1.25b) in the continuity equation, we have, in the sense of distributions,

$$\begin{aligned} 0 &= \lambda(-\Delta + \lambda^2)\varphi + \nabla\cdot[(-\boldsymbol{\Delta} + \lambda^2)\mathbf{A}] \\ &= (-\Delta + \lambda^2)[\lambda\varphi + \nabla\cdot\mathbf{A}]. \end{aligned}$$

As $-\Delta + \lambda^2$ is invertible, this implies that $\lambda\varphi + \nabla\cdot\mathbf{A} = 0$. ∎

This concludes the proof of Theorem 1.2.

6. Properties of the operators. Here we gather all important properties of the operators defined in section 3, as well as the operators defined in (1.23) and (1.33).

6.1. The 1.5*d* case. As the only dependence on λ is through the operators $\mathcal{Q}^\lambda_\pm$, we start with them, as follows.

LEMMA 6.1. *In the respective spaces $\mathfrak{L}_\pm$, $\mathcal{Q}^\lambda_\pm$ satisfy the following:*

(a) $\left\|\mathcal{Q}^\lambda_\pm\right\|_{\mathfrak{B}(\mathfrak{L}_\pm)} = 1$.

(b) *$\mathcal{Q}^\lambda_\pm$ can be extended from $\lambda > 0$ to $\operatorname{Re}\lambda > 0$ as holomorphic operator-valued functions. In particular, they are continuous for $\lambda > 0$ in operator norm topology.*

(c) *As $\mathbb{R} \ni \lambda \to \infty$, $\mathcal{Q}^\lambda_\pm \xrightarrow{s} 1$, and for $u \in \mathfrak{D}(\mathcal{D}_\pm)$, $\left\|(\mathcal{Q}^\lambda_\pm - 1)u\right\|_{\mathfrak{L}_\pm} \le \left\|\mathcal{D}_\pm u\right\|_{\mathfrak{L}_\pm}/\lambda$.*

(d) *As $\lambda \to 0$, $\mathcal{Q}^\lambda_\pm$ converge strongly to the projection operators $\mathcal{Q}^0_\pm$ defined in Definition* 1.4.

(e) *For any* $\lambda \geq 0$, $\mathcal{Q}^\lambda_\pm$ *are symmetric.*

Proof. $\|\mathcal{Q}^\lambda_\pm\|_{\mathfrak{B}(\mathfrak{L}_\pm)} \leq 1$ follows from $\|(\mathcal{D}_\pm + \lambda)^{-1}\|_{\mathfrak{B}(\mathfrak{L}_\pm)} \leq \frac{1}{|\lambda|}$ as $i\mathcal{D}_\pm$ is self-adjoint and the nearest point of the spectrum of $\mathcal{D}_\pm$ is 0. That $\|\mathcal{Q}^\lambda_\pm\|_{\mathfrak{B}(\mathfrak{L}_\pm)} = 1$ is proved by observing that $\mathcal{Q}^\lambda_\pm 1 = 1$. Part (b) follows from the analyticity of resolvents as functions of λ. For (c) we compute, using functional calculus for $u \in \mathfrak{D}(\mathcal{D}_\pm)$,

$$\begin{aligned}
\left\|\mathcal{Q}^\lambda_\pm u - u\right\|_{\mathfrak{L}_\pm} &= \left\|\left(\frac{\lambda^2}{\lambda^2 - \mathcal{D}^2_\pm} - 1\right) u\right\|_{\mathfrak{L}_\pm} = \left\|\frac{\mathcal{D}^2_\pm}{\lambda^2 - \mathcal{D}^2_\pm} u\right\|_{\mathfrak{L}_\pm} \\
&\leq \left\|\frac{\mathcal{D}_\pm}{\lambda + \mathcal{D}_\pm}\right\|_{\mathfrak{B}(\mathfrak{L}_\pm)} \left\|\frac{1}{\lambda - \mathcal{D}_\pm}\right\|_{\mathfrak{B}(\mathfrak{L}_\pm)} \|\mathcal{D}_\pm u\|_{\mathfrak{L}_\pm} \\
&\leq 1 \cdot \frac{1}{\lambda} \cdot \|\mathcal{D}_\pm u\|_{\mathfrak{L}_\pm} \to 0 \qquad \text{as } \lambda \to \infty
\end{aligned}$$

and deduce the strong convergence $\mathcal{Q}^\lambda_\pm \xrightarrow{s} 1$ by the density of $\mathfrak{D}(\mathcal{D}_\pm)$ in $\mathfrak{L}_\pm$.

For (d) we introduce the spectral measure (resolution of the identity) of the self-adjoint operator $-i\mathcal{D}_\pm$, which we denote by $M_\pm(\alpha)$, where $\alpha \in \mathbb{R}$. The projection onto $\ker(\mathcal{D}_\pm)$ is then $\mathcal{Q}^0_\pm = M_\pm(\{0\}) = \int_{\mathbb{R}} \chi(\alpha)\, dM_\pm(\alpha)$, where $\chi(0) = 1$ and $\chi(\alpha) = 0$ when $\alpha \neq 0$. Recall that $\lambda(\lambda + \mathcal{D}_\pm)^{-1} = \int_{\mathbb{R}} \frac{\lambda}{\lambda + i\alpha}\, dM_\pm(\alpha)$. We compute for $u \in \mathfrak{L}_\pm$

$$\begin{aligned}
\left\|\lambda(\lambda + \mathcal{D}_\pm)^{-1} u - M_\pm(\{0\}) u\right\|^2_{\mathfrak{L}_\pm} &= \left\|\int_{\mathbb{R}} \left(\frac{\lambda}{\lambda + i\alpha} - \chi(\alpha)\right) dM_\pm(\alpha) u\right\|^2_{\mathfrak{L}_\pm} \\
&= \int_{\mathbb{R}} \left|\frac{\lambda}{\lambda + i\alpha} - \chi(\alpha)\right|^2 d\left\|M_\pm(\alpha) u\right\|^2_{\mathfrak{L}_\pm},
\end{aligned}$$

the last equality being due to the orthogonality of spectral projections. This now tends to 0 as $\lambda \to 0$ by the dominated convergence theorem. Replacing $\mathcal{D}_\pm$ with $-\mathcal{D}_\pm$, which has the same kernel, we deduce that $\lambda(\lambda - \mathcal{D}_\pm)^{-1} \xrightarrow{s} \mathcal{Q}^0_\pm$. Finally, we have $\mathcal{Q}^\lambda_\pm = \lambda(\lambda - \mathcal{D}_\pm)^{-1}\lambda(\lambda + \mathcal{D}_\pm)^{-1} \xrightarrow{s} (\mathcal{Q}^0_\pm)^2 = \mathcal{Q}^0_\pm$ by the composition of strong operator convergence. To show (e) for $\lambda > 0$ we simply note that $\mathcal{D}^2_\pm$ are self-adjoint, and extend to $\lambda = 0$ by the strong operator convergence. ∎

These results carry through to the other operators.

LEMMA 6.2. *The operators* $\boldsymbol{\mathcal{J}}^\lambda$ *and* $\mathcal{B}^\lambda$ *have the following properties:*

(a) *For all* $\lambda \in [0, \infty)$, $\mathcal{B}^\lambda$ *maps* $L^2(\mathbb{R})$ *into* $L^2_0(\Omega)$ *and* $\boldsymbol{\mathcal{J}}^\lambda$ *maps* $L^2(\mathbb{R}) \times L^2_0(\Omega) \to L^2(\mathbb{R}) \times L^2_0(\Omega)$.

(b) *The families* $\{\boldsymbol{\mathcal{J}}^\lambda\}_{\lambda \in [0,\infty)}$ *and* $\{\mathcal{B}^\lambda\}_{\lambda \in [0,\infty)}$ *are both uniformly bounded in the operator norm.*

(c) *Both* $(0, \infty) \ni \lambda \mapsto \boldsymbol{\mathcal{J}}^\lambda$ *and* $(0, \infty) \ni \lambda \mapsto \mathcal{B}^\lambda$ *are continuous in the operator norm topology.*

(d) *As* $\lambda \to 0$, $\boldsymbol{\mathcal{J}}^\lambda \to \boldsymbol{\mathcal{J}}^0$ *and* $\mathcal{B}^\lambda \to \mathcal{B}^0$ *in the strong operator topology.*

(e) *For any* $\lambda \geq 0$ *the operator* $\boldsymbol{\mathcal{J}}^\lambda$ *is symmetric.*

(f) *Let* $\boldsymbol{\mathcal{P}}$ *be the multiplication operator acting in* $L^2(\mathbb{R}) \times L^2_0(\Omega)$ *defined by*

$$\boldsymbol{\mathcal{P}} = \begin{bmatrix} \mathbb{1}_\Omega & 0 \\ 0 & \mathbb{1}_\Omega \end{bmatrix},$$

where $\mathbb{1}_\Omega$ *is the indicator function of the set* Ω. *Then* $\boldsymbol{\mathcal{J}}^\lambda = \boldsymbol{\mathcal{J}}^\lambda \boldsymbol{\mathcal{P}}$.

Proof. Part (a) is easily verifiable. We note that due to the relation

$$\mathcal{B}^\lambda = -\begin{bmatrix}0 & 1\end{bmatrix} \boldsymbol{\mathcal{J}}^\lambda \begin{bmatrix}1\\0\end{bmatrix}$$

it is sufficient to prove the results for $\boldsymbol{\mathcal{J}}^\lambda$. We observe that due to the decay assumptions (1.6) on $\mu^\pm$, the moment

$$-\sum_\pm \int \mu^\pm \frac{1+v_1^2}{\langle v\rangle^3}\, d\boldsymbol{v}$$

is bounded in $L^\infty(\mathbb{R})$ and is real-valued, so it is a bounded symmetric multiplication operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. Next we decompose the second part of $\boldsymbol{\mathcal{J}}^\lambda$ as

$$\sum_\pm \int \mu_e^\pm \mathcal{T}_\pm(\mathcal{Q}_\pm^\lambda - 1)\mathcal{T}_\pm^* \begin{bmatrix}\psi\\ \phi\end{bmatrix} d\boldsymbol{v}, \tag{6.1}$$

where $\mathcal{T}_\pm : \mathfrak{L}_\pm \times \mathfrak{L}_\pm \to \mathfrak{L}_\pm$ is multiplication by the vector $\begin{bmatrix}\hat{v}_2 & -1\end{bmatrix}$, and we have used the natural (and bounded) inclusions from $L^2(\mathbb{R})$ and $L_0^2(\Omega)$ into $\mathfrak{L}_\pm$. Clearly $\mathcal{T}_\pm$ are bounded, and we know that $\mathcal{Q}_\pm^\lambda$ have bound 1 by Lemma 6.1. Finally, we note that due to the decay assumptions on $\mu_e^\pm$ and its compact support in x, multiplication by $\mu_e^\pm$ followed by integration $d\boldsymbol{v}$ is bounded from $\mathfrak{L}_\pm$ to $L^2(\mathbb{R})$ and $L^2(\Omega)$. Therefore $\boldsymbol{\mathcal{J}}^\lambda$ has a uniform bound in operator norm. Parts (c) and (d) then follow from the corresponding results for $\mathcal{Q}_\pm^\lambda$ in Lemma 6.1 using (6.1). (e) is clear from the symmetry of $\mathcal{Q}_\pm^\lambda$ and (6.1). Finally, (f) follows from the compact spatial support of $\mu^\pm, \mu_e^\pm, \mu_p^\pm$ inside Ω. ∎

LEMMA 6.3 (properties of $\mathcal{A}_1^\lambda$ and $\mathcal{A}_2^\lambda$). *Let $0 \le \lambda < \infty$.*

(a) *The operator $\mathcal{A}_1^\lambda$ is self-adjoint on $L_0^2(\Omega)$, and the operator $\mathcal{A}_2^\lambda$ is self-adjoint on $L^2(\mathbb{R})$ with the respective domains $H_{0,n}^2(\Omega)$ and $H^2(\mathbb{R})$.*

(b) *Both $[0,\infty) \ni \lambda \mapsto \mathcal{A}_1^\lambda$ and $[0,\infty) \ni \lambda \mapsto \mathcal{A}_2^\lambda$ are continuous in the norm resolvent topology.*

(c) *The spectrum of $\mathcal{A}_1^\lambda$ is purely discrete. The spectrum of $\mathcal{A}_2^\lambda$ in $(-\infty, \lambda^2)$ is discrete and made up of finitely many eigenvalues. It is continuous (possibly with embedded eigenvalues) in $[\lambda^2, \infty)$.*

(d) *There exist constants $\gamma > 0$ and $\Lambda > 0$ such that for all $\lambda \ge \Lambda$, $\mathcal{A}_i^\lambda > \gamma$, $i = 1, 2$.*

Proof. Clearly $-\partial_x^2$ is symmetric. The perturbative terms are symmetric as well since $\mathcal{Q}_\pm^\lambda$ are symmetric; see Lemma 6.1. Self-adjointness is guaranteed by standard arguments, such as the Kato–Rellich theorem.

Let us prove (b), considering first $\mathcal{A}_2^\lambda$. It is sufficient to prove that $(\mathcal{A}_2^\lambda - i)^{-1} \to (\mathcal{A}_2^\sigma - i)^{-1}$ in operator norm as $\lambda \to \sigma$ (with $\lambda, \sigma \ge 0$). We use the second resolvent identity to obtain

$$\begin{aligned}(\mathcal{A}_2^\lambda - i)^{-1} - (\mathcal{A}_2^\sigma - i)^{-1} &= (\mathcal{A}_2^\lambda - i)^{-1}(\mathcal{A}_2^\sigma - \mathcal{A}_2^\lambda)(\mathcal{A}_2^\sigma - i)^{-1}\\ &= (\mathcal{A}_2^\lambda - i)^{-1}((\sigma^2 - \lambda^2) - (\mathcal{J}_{11}^\sigma - \mathcal{J}_{11}^\lambda))(\mathcal{A}_2^\sigma - i)^{-1},\end{aligned}$$

where $\mathcal{J}_{11}^\lambda$ is the upper left component of $\boldsymbol{\mathcal{J}}^\lambda$ written in block matrix form. Hence, as the resolvents are each bounded in operator norm by 1,

$$\left\|(\mathcal{A}_2^\lambda - i)^{-1} - (\mathcal{A}_2^\sigma - i)^{-1}\right\|_{\mathfrak{B}(L^2(\mathbb{R}))} \le |\sigma^2 - \lambda^2| + \left\|(\mathcal{J}_{11}^\sigma - \mathcal{J}_{11}^\lambda)(\mathcal{A}_2^\sigma - i)^{-1}\right\|_{\mathfrak{B}(L^2(\mathbb{R}))}.$$

It thus suffices using Lemma 6.2(f) to show that $(\mathcal{J}_{11}^{\sigma} - \mathcal{J}_{11}^{\lambda})\mathcal{P}(\mathcal{A}_2^{\sigma} - i)^{-1} \to 0$ in operator norm, where $\mathcal{P}$ is the multiplication operator on $L^2(\mathbb{R})$ given by the indicator function of the set Ω. $\mathcal{P}$ is relatively compact with respect to $-\partial_x^2$ by the Rellich theorem, and hence also relatively compact with respect to $\mathcal{A}_2^{\sigma}$ as it also has the domain $H^2(\mathbb{R})$ by part (a). Hence $\mathcal{P}(\mathcal{A}_2^{\sigma} - i)^{-1}$ is compact, which allows us to upgrade the strong convergence $\mathcal{J}_{11}^{\lambda} \xrightarrow{s} \mathcal{J}_{11}^{\sigma}$ given by Lemma 6.2 to operator norm convergence. The norm resolvent continuity of $\mathcal{A}_2^{\lambda}$ follows. The proof for $\mathcal{A}_1^{\lambda}$ is analogous but lacking the $|\sigma^2 - \lambda^2|$ term.

Part (c) is simple: both operators have a differential part (Laplacian) and a relatively compact perturbation. Hence both conclusions follow from Weyl's theorem [10, Chapter IV, Theorem 5.35]. The finiteness of the discrete spectrum below the essential part in the case of $\mathcal{A}_2^{\lambda}$ is a consequence of Lemma 2.2. For part (d), we begin with $\mathcal{A}_1^{\lambda}$. Fix an arbitrary $h \in H_{0,n}^2(\Omega)$; then we compute

$$\left\langle \mathcal{A}_1^{\lambda} h, h \right\rangle_{L_0^2(\Omega)} = \|\partial_x h\|_{L_0^2(\Omega)}^2 - \sum_{\pm} \iint \overline{h} \mu_e^{\pm} (1 - \mathcal{Q}_{\pm}^{\lambda}) h \, d\boldsymbol{v} dx.$$

Now we note as $h \in H_{0,n}^2(\Omega)$, h is in $\mathfrak{D}(\mathcal{D}_{\pm})$ when interpreted in $\mathfrak{L}_{\pm}$. We now use Lemma 6.1(c) to estimate

$$\begin{aligned}
\left\langle \mathcal{A}_1^{\lambda} h, h \right\rangle_{L_0^2(\Omega)} &\geq \|\partial_x h\|_{L_0^2(\Omega)}^2 - \frac{1}{\lambda} \sum_{\pm} \left\| \frac{\mu_e^{\pm}}{w^{\pm}} \right\|_{L^{\infty}(\Omega \times \mathbb{R}^3)} \|\mathcal{D}_{\pm} h\|_{\mathfrak{L}_{\pm}} \|h\|_{\mathfrak{L}_{\pm}} \\
&\geq \|\partial_x h\|_{L_0^2(\Omega)}^2 - \frac{C}{\sqrt{K}\lambda} \|\partial_x h\|_{L_0^2(\Omega)}^2 \\
&\geq K \|h\|_{L_0^2(\Omega)}^2 \left(1 - \frac{C}{\sqrt{K}\lambda}\right),
\end{aligned}$$

where K is the spectral gap of the Laplacian on the bounded domain Ω, and we have used

$$\|\mathcal{D}_{\pm} h\|_{\mathfrak{L}_{\pm}}^2 = \iint w^{\pm} |\hat{v}_1 \partial_x h|^2 \, d\boldsymbol{v} dx \leq \|\partial_x h\|_{L_0^2(\Omega)}^2 \sup_{x \in \Omega} \int w^{\pm} |\hat{v}_1|^2 \, d\boldsymbol{v}$$

and the natural bounded inclusions from $L_0^2(\Omega)$ into $\mathfrak{L}_{\pm}$. We now just take $\Lambda > C/\sqrt{K}$.

For $\mathcal{A}_2^{\lambda}$ the proof is easier due to the λ^2 term. For $h \in H^2(\mathbb{R})$ we compute, using the formulation (3.8a),

$$\begin{aligned}
\left\langle \mathcal{A}_2^{\lambda} h, h \right\rangle_{L^2(\mathbb{R})} &= \|\partial_x h\|_{L^2(\mathbb{R})}^2 + \lambda^2 \|h\|_{L^2(\mathbb{R})}^2 - \left\langle \boldsymbol{\mathcal{J}}^{\lambda} \begin{bmatrix} h \\ 0 \end{bmatrix}, \begin{bmatrix} h \\ 0 \end{bmatrix} \right\rangle_{L^2(\mathbb{R}) \times L_0^2(\Omega)} \\
&\geq (\lambda^2 - C') \|h\|_{L^2(\mathbb{R})}^2,
\end{aligned}$$

where we have used the uniform bound in operator norm of $\boldsymbol{\mathcal{J}}^{\lambda}$ given by Lemma 6.2. We now take $\Lambda > \sqrt{C'}$. ∎

LEMMA 6.4 (properties of $\boldsymbol{\mathcal{M}}^{\lambda}$). *For each $\lambda \in [0, \infty)$, the operator $\boldsymbol{\mathcal{M}}^{\lambda}$ is self-adjoint on $L^2(\mathbb{R}) \times L_0^2(\Omega)$ with domain $H^2(\mathbb{R}) \times H_{0,n}^2(\Omega)$. For any $\lambda \geq 0$, the operator $\boldsymbol{\mathcal{M}}^{\lambda}$ has essential spectrum $[\lambda^2, \infty)$. The family $\{\boldsymbol{\mathcal{M}}^{\lambda}\}_{\lambda \in [0,\infty)}$ is continuous in the norm resolvent topology.*

Proof. The proof essentially mimics (and uses) the preceding proofs and is therefore left for the reader. ∎

6.2. The cylindrically symmetric case. As many of the proofs are the same as in the $1.5d$ case above, we give the details only where they differ.

LEMMA 6.5. *In the respective spaces $\mathfrak{N}_\pm$, $\widetilde{\mathcal{Q}}^\lambda_{\pm,sym}$ and $\widetilde{\mathcal{Q}}^\lambda_{\pm,skew}$ satisfy the following:*

(a) $\|\widetilde{\mathcal{Q}}^\lambda_{\pm,sym}\|_{\mathfrak{B}(\mathfrak{N}_\pm)} = 1$ *and* $\|\widetilde{\mathcal{Q}}^\lambda_{\pm,skew}\|_{\mathfrak{B}(\mathfrak{N}_\pm)} \le \frac{1}{2}$.

(b) $\widetilde{\mathcal{Q}}^\lambda_{\pm,sym}$ *and* $\widetilde{\mathcal{Q}}^\lambda_{\pm,skew}$ *can be extended from* $\lambda > 0$ *to* $\operatorname{Re}\lambda > 0$ *as holomorphic operator-valued functions. In particular they are continuous for* $\lambda > 0$ *in the operator norm topology.*

(c) *As* $\mathbb{R} \ni \lambda \to \infty$, $\widetilde{\mathcal{Q}}^\lambda_{\pm,sym} \xrightarrow{s} 1$, *and for* $u \in \mathfrak{D}(\widetilde{\mathcal{D}}_\pm)$ *we have the bound* $\|(\widetilde{\mathcal{Q}}^\lambda_{\pm,sym} - 1)u\|_{\mathfrak{N}_\pm} \le \|\widetilde{\mathcal{D}}_\pm u\|_{\mathfrak{N}_\pm}/\lambda$.

(d) *As* $\mathbb{R} \ni \lambda \to \infty$, $\widetilde{\mathcal{Q}}^\lambda_{\pm,skew} \xrightarrow{s} 0$, *and for* $u \in \mathfrak{D}(\widetilde{\mathcal{D}}_\pm)$ *we have the bound* $\|\widetilde{\mathcal{Q}}^\lambda_{\pm,sym} u\|_{\mathfrak{N}_\pm} \le \|\widetilde{\mathcal{D}}_\pm u\|_{\mathfrak{N}_\pm}/\lambda$.

(e) *As* $0 < \lambda \to 0$, $\widetilde{\mathcal{Q}}^\lambda_{\pm,sym} \xrightarrow{s} \widetilde{\mathcal{Q}}^0_\pm$, *where* $\widetilde{\mathcal{Q}}^0_\pm$ *are defined in Definition* 1.5.

(f) *As* $0 < \lambda \to 0$, $\widetilde{\mathcal{Q}}^\lambda_{\pm,skew} \xrightarrow{s} 0$.

(g) *For any* $\lambda \ge 0$, $\widetilde{\mathcal{Q}}^\lambda_{\pm,sym}$ *are symmetric and* $\widetilde{\mathcal{Q}}^\lambda_{\pm,skew}$ *are skew-symmetric.*

Proof. The claims about $\widetilde{\mathcal{Q}}^\lambda_{\pm,sym}$ may be proved in the same way as those in Lemma 6.1. For (a), the spectral theorem applied to the self-adjoint operators $-i\widetilde{\mathcal{D}}_\pm$ implies that $\widetilde{\mathcal{Q}}^\lambda_{\pm,skew}$ are unitarily equivalent to a multiplication operator, so that

$$\left\|\widetilde{\mathcal{Q}}^\lambda_{\pm,skew}\right\|_{\mathfrak{B}(\mathfrak{N}_\pm)} = \left\|\frac{-i\alpha\lambda}{\lambda^2+\alpha^2}\right\|_{L^\infty_\alpha(\operatorname{sp}(i\widetilde{\mathcal{D}}_\pm))} \le \left\|\frac{-i\alpha\lambda}{\lambda^2+\alpha^2}\right\|_{L^\infty_\alpha(\mathbb{R})} = \frac{1}{2}.$$

The proof of (b) follows, as in the proof of Lemma 6.1, from the holomorphicity of the resolvent. For (d), we let $u \in \mathfrak{D}(\widetilde{\mathcal{D}}_\pm)$, and then for $\lambda > 0$ we have

$$\begin{aligned}\left\|\widetilde{\mathcal{Q}}^\lambda_{\pm,skew} u\right\|_{\mathfrak{N}_\pm} &\le \lambda \left\|(\lambda^2 + \widetilde{\mathcal{D}}^2_\pm)^{-1}\right\|_{\mathfrak{B}(\mathfrak{N}_\pm)} \left\|\widetilde{\mathcal{D}}_\pm u\right\|_{\mathfrak{N}_\pm} \\ &\le \frac{1}{\lambda}\left\|\widetilde{\mathcal{D}}_\pm u\right\|_{\mathfrak{N}_\pm} \to 0 \qquad \text{as } \lambda \to \infty.\end{aligned}$$

The strong convergence to 0 then follows from the density of $\mathfrak{D}(\widetilde{\mathcal{D}}_\pm)$. For (f), we repeat the proof of Lemma 6.1, noting that it is shown that $\lambda(\lambda + \widetilde{\mathcal{D}}_\pm)^{-1} \xrightarrow{s} \widetilde{\mathcal{Q}}^0_\pm$ and $\widetilde{\mathcal{Q}}^\lambda_{\pm,sym} \xrightarrow{s} \widetilde{\mathcal{Q}}^0_\pm$ as $\lambda \to 0$. That $\widetilde{\mathcal{Q}}^\lambda_{\pm,skew} \xrightarrow{s} 0$ as $\lambda \to 0$ now follows from the identity, valid for all $\lambda > 0$,

$$\lambda(\lambda + \widetilde{\mathcal{D}}_\pm)^{-1} = \widetilde{\mathcal{Q}}^\lambda_{\pm,sym} + \widetilde{\mathcal{Q}}^\lambda_{\pm,skew}.$$

Finally, (g) is a consequence of Lemma 3.2. ∎

LEMMA 6.6. *The operators $\widetilde{\boldsymbol{\mathcal{J}}}^\lambda$ and $\widetilde{\mathcal{B}}^\lambda_i$ for $i = 1,2,3,4$ have the following properties:*

(a) *For all* $\lambda \in [0,\infty)$, $\widetilde{\mathcal{B}}^\lambda_1 \in \mathfrak{B}\big(L^2_\theta(\mathbb{R}^3;\mathbb{R}^3), L^2_{cyl}(\mathbb{R}^3)\big)$, $\widetilde{\mathcal{B}}^\lambda_2 \in \big(L^2_\theta(\mathbb{R}^3;\mathbb{R}^3), L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)\big)$, $\widetilde{\mathcal{B}}^\lambda_3 \in \mathfrak{B}\big(L^2_{cyl}(\mathbb{R}^3), L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)\big)$, *and* $\widetilde{\boldsymbol{\mathcal{B}}}^\lambda_4 \in \mathfrak{B}\big(L^2_\theta(\mathbb{R}^3;\mathbb{R}^3), L^2_{cyl}(\mathbb{R}^3) \times L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)\big)$ *with bounds uniform in* λ.

(b) *Each of* $(0,\infty) \ni \lambda \mapsto \widetilde{\boldsymbol{\mathcal{J}}}^\lambda$ *and* $(0,\infty) \ni \lambda \mapsto \widetilde{\mathcal{B}}^\lambda_i$, $i = 1,2,3,4$, *is continuous in the operator norm topology.*

(c) *As* $\lambda \to 0$, $\widetilde{\boldsymbol{\mathcal{J}}}^\lambda \to \widetilde{\boldsymbol{\mathcal{J}}}^0$, $\widetilde{\mathcal{B}}^\lambda_1 \to \widetilde{\mathcal{B}}^0_1$, $\widetilde{\mathcal{B}}^\lambda_2 \to 0$, *and* $\widetilde{\mathcal{B}}^\lambda_3 \to 0$ *in the strong topology.*

(d) *For any* $\lambda \ge 0$ *the operator* $\widetilde{\boldsymbol{\mathcal{J}}}^\lambda$ *is symmetric.*

(e) *Let* $\widetilde{\boldsymbol{\mathcal{P}}}$ *be the multiplication operator acting in* $L^2_{cyl}(\mathbb{R}^3)\times L^2_\theta(\mathbb{R}^3;\mathbb{R}^3)\times L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)$ *defined by*

$$\widetilde{\boldsymbol{\mathcal{P}}} = \begin{bmatrix} \mathbb{1}_\Omega & 0 & 0 \\ 0 & \mathbb{1}_\Omega & 0 \\ 0 & 0 & \mathbb{1}_\Omega \end{bmatrix},$$

where $\mathbb{1}_\Omega$ *is the indicator function of the set* Ω. *Then* $\widetilde{\boldsymbol{\mathcal{J}}}^\lambda = \widetilde{\boldsymbol{\mathcal{J}}}^\lambda\widetilde{\boldsymbol{\mathcal{P}}}$.

Proof. That the operators map the corresponding spaces to each other may be verified directly from (3.16), noting in particular the notation $\hat{\boldsymbol{v}}_\theta = \boldsymbol{e}_\theta\hat{v}_\theta$ and $\hat{\boldsymbol{v}}_{rz} = \boldsymbol{e}_r\hat{v}_r + \boldsymbol{e}_z\hat{v}_z$. As in the proof of Lemma 6.2, the uniform (in λ) bound on the operator norms may be obtained using the decay assumptions on the equilibrium and the uniform bound on the norms of $\widetilde{\mathcal{Q}}^\lambda_{\pm,sym}$ and $\widetilde{\mathcal{Q}}^\lambda_{\pm,skew}$ given by Lemma 6.5. In the same way (c) and (d) follow from the corresponding results in Lemma 6.5.

To show the symmetry of $\widetilde{\boldsymbol{\mathcal{J}}}^\lambda$ for $\lambda > 0$ we use the block matrix form, noting that $\widetilde{\mathcal{Q}}^\lambda_{\pm,sym}$ appears on the diagonal and that the off diagonal entries have $\widetilde{\mathcal{B}}^\lambda_i$ and their adjoints in the appropriate configuration. Then we extend to $\lambda = 0$ by strong convergence. As in Lemma 6.2, (e) follows from the spatial support properties of the equilibrium. ∎

LEMMA 6.7 (properties of $\widetilde{\mathcal{A}}^\lambda_1$, $\widetilde{\mathcal{A}}^\lambda_2$, $\widetilde{\mathcal{A}}^\lambda_3$, and $\widetilde{\boldsymbol{\mathcal{A}}}^\lambda_4$). *Let* $0 \le \lambda < \infty$.

(a) *The operator* $\widetilde{\mathcal{A}}^\lambda_1$ *is self-adjoint on* $L^2_{cyl}(\mathbb{R}^3)$, *the operator* $\widetilde{\mathcal{A}}^\lambda_2$ *is self-adjoint on* $L^2_\theta(\mathbb{R}^3;\mathbb{R}^3)$, $\widetilde{\mathcal{A}}^\lambda_3$ *is self-adjoint on* $L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)$, *and* $\widetilde{\boldsymbol{\mathcal{A}}}^\lambda_4$ *is self-adjoint on* $L^2_{cyl}(\mathbb{R}^3)\times L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)$ *with the respective domains* $H^2_{cyl}(\mathbb{R}^3)$, $H^2_\theta(\mathbb{R}^3;\mathbb{R}^3)$, $H^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)$, *and* $H^2_{cyl}(\mathbb{R}^3)\times H^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)$.

(b) *The mappings* $[0,\infty)\ni\lambda\mapsto\widetilde{\mathcal{A}}^\lambda_1$, $[0,\infty)\ni\lambda\mapsto\widetilde{\mathcal{A}}^\lambda_2$, $[0,\infty)\ni\lambda\mapsto\widetilde{\mathcal{A}}^\lambda_3$, *and* $[0,\infty)\ni\lambda\mapsto\widetilde{\boldsymbol{\mathcal{A}}}^\lambda_4$ *are continuous in the norm resolvent topology.*

(c) *The spectra of* $\widetilde{\mathcal{A}}^\lambda_1$, $\widetilde{\mathcal{A}}^\lambda_2$, $\widetilde{\mathcal{A}}^\lambda_1$, *and* $\widetilde{\boldsymbol{\mathcal{A}}}^\lambda_4$ *in* $(-\infty,\lambda^2)$ *are discrete and finite. In* $[\lambda^2,\infty)$ *their spectra are continuous (possibly with embedded eigenvalues).*

(d) *There exist constants* $\gamma > 0$ *and* $\Lambda > 0$ *such that for all* $\lambda \ge \Lambda$, $\widetilde{\mathcal{A}}^\lambda_i > \gamma$, $i = 1,2,3,4$.

Proof. The proof for each of $\widetilde{\mathcal{A}}^\lambda_i$, $i = 1,2,3,4$, is analogous to that of Lemma 6.3 for $\mathcal{A}^\lambda_2$. We omit the details. ∎

LEMMA 6.8 (properties of $\widetilde{\boldsymbol{\mathcal{M}}}^\lambda$). *For each* $\lambda\in[0,\infty)$, *the operator* $\widetilde{\boldsymbol{\mathcal{M}}}^\lambda$ *is self-adjoint on* $L^2_\theta(\mathbb{R}^3;\mathbb{R}^3)\times L^2_{cyl}(\mathbb{R}^3)\times L^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)$ *with domain* $H^2_\theta(\mathbb{R}^3;\mathbb{R}^3)\times H^2_{cyl}(\mathbb{R}^3)\times H^2_{rz}(\mathbb{R}^3;\mathbb{R}^3)$. *For any* $\lambda\ge 0$, *the operator* $\widetilde{\boldsymbol{\mathcal{M}}}^\lambda$ *has essential spectrum* $(-\infty,-\lambda^2]\cup[\lambda^2,\infty)$. *The family* $\{\widetilde{\boldsymbol{\mathcal{M}}}^\lambda\}_{\lambda\in[0,\infty)}$ *is continuous in the norm resolvent topology.*

Proof. This is again analogous to the previous proofs and is left to the reader. ∎

7. Existence of equilibria. In this section we prove that there exist compactly supported equilibria of the $1.5d$ system which can be written in the form (1.16)–(1.17). Existence in the $3d$ case was already provided in [12]. We note that providing explicit *examples* of equilibria is a much more challenging task, which we do not pursue here. The construction below utilizes the physically relevant idea of *magnetic confinement*. We mention in this context the recent result [16] where global-in-time existence and uniqueness of solutions was established in a similar setting, though with a singular magnetic potential.

PROPOSITION 7.1 (existence of confined equilibria). *Let $R > 0$, $\alpha > 2$, and $A^\pm \subset \mathbb{R}^2$ be bounded domains. Then there are constants $c, C > 0$ such that if two functions $\mu^\pm(e^\pm, p^\pm) \in C_0^1(\mathbb{R}^2)$ with support in $A^\pm$ satisfy*

$$|\mu^\pm|, |\mu_e^\pm|, |\mu_p^\pm| \le c(1 + |e^\pm|)^{-\alpha}$$

and a function $\psi^{ext} \in H^2_{loc}(\mathbb{R})$ satisfies

$$|\psi^{ext}(x)| \ge C(1 + |x|^2) \quad \text{for } |x| \ge R,$$

then there are potentials $\phi^0, \psi^0 \in H^2_{loc}(\mathbb{R})$ such that $(\mu^\pm(e^\pm, p^\pm), \phi^0, \psi^0, \psi^{ext}, \phi^{ext} = 0)$ is an equilibrium of the 1.5d relativistic Vlasov–Maxwell equations (1.15) with spatial support in $[-R, R]$, where the relationship between (x, v_1, v_2) and $(e^\pm, p^\pm)$ is as defined in (1.17).

Remark 7.1 (trivial solutions). Of course, the result does not say that the obtained equilibrium is not everywhere zero. This may be ruled out by choosing $\mu^\pm$ and ψ^{ext} in such a way that (for example) $\mu^\pm(x = 0, v = 0) > 0$ if $\phi^0, \psi^0 \equiv 0$. Let us sketch the argument. Recall that we write $f^{0,\pm}(x, v) = \mu^\pm(\langle \boldsymbol{v} \rangle \pm \phi^0(x), v_2 \pm \psi^0(x) \pm \psi^{ext}(x)) = \mu^\pm(e^\pm, p^\pm)$. If $f^{0,\pm}(x, \boldsymbol{v}) = 0$ for all $(x, \boldsymbol{v})$, then $\rho, j_i = 0$ and $\phi^0, \psi^0 = 0$ for all x. Therefore $e^\pm = \langle \boldsymbol{v} \rangle$ and $p^\pm = v_2 \pm \psi^{ext}(x)$, and

$$f^{0,\pm}(0, 0) = \mu^\pm(1, \pm\psi^{ext}(0)).$$

The right-hand side is something we can ensure is positive by choosing $A^\pm$, $\mu^\pm$, and ψ^{ext} appropriately. Under this appropriate choice one obtains a contradiction.

Proof of Proposition 7.1. Given two elements $\rho, j_2 \in L^2(\mathbb{R})$ with compact support, we define

$$\phi^0 = G * \rho, \qquad \psi^0 = G * j_2,$$

where $G(x) = -|x|/2$ is the fundamental solution of the Laplacian in one dimension. (We note that one expects j_1 to vanish for an equilibrium, due to parity in v_1; hence it does not appear in the setup.) Thus we define $e^\pm = e^\pm(x, v_1, v_2)$ and $p^\pm = p^\pm(x, v_1, v_2)$ via the usual relations (1.17), which we recall for the reader's convenience:

$$e^\pm(x, \boldsymbol{v}) = \langle \boldsymbol{v} \rangle \pm \phi^0(x), \qquad p^\pm(x, \boldsymbol{v}) = v_2 \pm \psi^0(x) \pm \psi^{ext}(x)$$

(ϕ^{ext} is zero). We let $\mathcal{F} : L^2(\mathbb{R})^2 \to L^2(\mathbb{R})^2$ be the (nonlinear) map defined by

$$\mathcal{F}\begin{bmatrix} \rho \\ j_2 \end{bmatrix} = \int \begin{bmatrix} 1 \\ \hat{v}_2 \end{bmatrix} (\mu^+(e^+, p^+) - \mu^-(e^-, p^-))\, d\boldsymbol{v}. \tag{7.1}$$

A fixed point of $\mathcal{F}$ is the charge and current densities of an equilibrium $(\mu^\pm(e^\pm, p^\pm), \phi^0, \psi^0, \psi^{ext}, \phi^{ext} = 0)$. We define $X \subseteq L^2(\mathbb{R})^2$ as

$$X = \{(\rho, j_2) \in L^2(\mathbb{R})^2 : \text{ both supported in } [-R, R] \text{ and bounded by } C'\}$$

for a positive constant C' to be chosen. This set is clearly convex. We will show that for $c > 0$ sufficiently small and $C > 0$ sufficiently large, $\mathcal{F}$ is a compact continuous map $X \hookrightarrow X$ and thus, by the Schauder fixed point theorem, has a fixed point.

Step 1: *Compact support.* We show that C' and C can be chosen so that $\mathcal{F}$ maps X into functions supported in $[-R, R]$.

For $(\rho, j_2) \in X$ and $|x| > R$ we have

$$|\phi^0(x)| = |(G * \rho)(x)| \le C' \int_{-R}^{R} |G(x-y)|\, dy = \frac{C'}{2} \int_{-R}^{R} |x-y|\, dy = C'R|x|,$$

and the same bound holds for ψ^0. This allows us to control v_2 using $e^\pm$ and x. Indeed,

$$|v_2| \le \langle \boldsymbol{v} \rangle = e^\pm \mp \phi^0(x) \le |e^\pm| + |\phi^0(x)| \le |e^\pm| + C'R|x|,$$

which gives the following lower bound for $|p^\pm| + |e^\pm|$ in terms of x:

$$\begin{aligned} |p^\pm| + |e^\pm| &= |v_2 \pm \psi^0(x) \pm \psi^{ext}(x)| + |e^\pm| \ge \psi^{ext}(x) - |v_2| - |\psi^0(x)| + |e^\pm| \\ &\ge \psi^{ext}(x) - |e^\pm| - 2C'R|x| + |e^\pm| \\ &\ge C(1+|x|^2) - 2C'R|x|. \end{aligned}$$

By taking C' small enough and C large enough, we can ensure that if $|x| > R$, then $(e^\pm, p^\pm)$ lie outside any disc in $\mathbb{R}^2$, and in particular outside $A^\pm$, where $\mu^\pm$ are supported. This proves the claim.

Step 2: *Uniform L^∞ bound.* We show that C' and c can be chosen so that $\mathcal{F}$ maps to functions with L^∞ norm smaller than C'.

Estimating $\phi^0(x)$ for $|x| \le R$,

$$|\phi^0(x)| \le \frac{C'}{2} \int_{-R}^{R} |x-y|\, dy = \frac{C'}{2}(x^2 + R^2),$$

we take C' small enough (recall that it was already taken to be small in the previous step; hence we may require it to be even smaller) so that $|\phi^0(x)| \le 3/4$ for $|x| \le R$. Now the decay assumption on $\mu^\pm$ allows us to show a uniform bound on $|\mathcal{F}_1(\rho, j_2)(x)|$ in $|x| \le R$:

$$\begin{aligned} |\mathcal{F}_1(\rho, j_2)(x)| &\le \sum_\pm \int |\mu^\pm(e^\pm, p^\pm)|\, d\boldsymbol{v} \le \sum_\pm \int \frac{c}{(1+|e^\pm|)^\alpha}\, d\boldsymbol{v} \\ &\le \sum_\pm \int \frac{c}{(1+\langle \boldsymbol{v} \rangle - |\phi^0(x)|)^\alpha}\, d\boldsymbol{v} \le \sum_\pm \int \frac{c}{(1+\langle \boldsymbol{v} \rangle - \frac{3}{4})^\alpha}\, d\boldsymbol{v} \\ &= \int \frac{2c}{(\frac{1}{4} + \langle \boldsymbol{v} \rangle)^\alpha}\, d\boldsymbol{v} = C''c < \infty. \end{aligned} \tag{7.2}$$

We can bound $|\mathcal{F}_2(\rho, j_2)(x)|$ in the same way as $|\hat{v}| \le 1$. Finally, we choose c so that $C''c \le C'$.

Step 3: *Uniform L^∞ bound on the derivative.* We show that there is a constant C''' such that for any $(\rho, j_2) \in X$ we have $\|\partial_x \mathcal{F}_1(\rho, j_2)\|_{L^\infty[-R,R]} \le C'''$ and $\|\partial_x \mathcal{F}_2(\rho, j_2)\|_{L^\infty[-R,R]} \le C'''$.

We compute for $\mathcal{F}_1$ and note that $\mathcal{F}_2$ is analogous. Using the chain rule, we have

$$\begin{aligned} &\partial_x \int \left(\mu^+(e^+, p^+) - \mu^-(e^-, p^-)\right)\, d\boldsymbol{v} \\ &\quad = (\partial_x \phi^0) \int (\mu_e^+(e^+, p^+) + \mu_e^-(e^-, p^-))\, d\boldsymbol{v} \\ &\qquad + (\partial_x \psi^0 + \partial_x \psi^{ext}) \int (\mu_p^+(e^+, p^+) + \mu_p^-(e^-, p^-))\, d\boldsymbol{v}. \end{aligned}$$

The two integrals are bounded uniformly in x by the arguments in Step 2 using the corresponding assumed bounds on $\mu_e^\pm$ and $\mu_p^\pm$, respectively. As the external field ψ^{ext} lies in $H^2_{loc}(\mathbb{R})$, its derivative $\partial_x\psi^{ext}$ lies in $H^1([-R,R])$ and is bounded in $L^\infty([-R,R])$ by Morrey's inequality. It remains to bound $\partial_x\phi^0$ and $\partial_x\psi^0$ uniformly for all $x \in [-R,R]$. These are controlled directly using the Green's function $G(x)$ and uniform bounds of Step 2. Indeed,

$$|(\partial_x\phi^0)(x)| = |((\partial_x G) * \rho)(x)| \leq \frac{C'}{2}\int_{-R}^{R} |\operatorname{sign}(x-y)|\, dy \leq C'R,$$

and the computation for $\partial_x\psi^0$ is identical.

Step 4: $\mathcal{F}$ is a compact continuous map from X to X. Steps 1 and 2 imply that $\mathcal{F}(X) \subseteq X$. Step 3 and the Rellich theorem imply that $\mathcal{F}(X)$ is relatively compact in X. It remains to show that $\mathcal{F}$ is continuous. This may be shown using dominated convergence and the bounds in Step 2. Indeed, suppose that $\{(\rho^n, j_2^n)\}_{n\in\mathbb{N}} \subseteq X$ is a sequence converging to $(\rho, j_2) \in X$ strongly in $L^2(\mathbb{R})^2$. We shall show that $\mathcal{F}_1(\rho^n, j_2^n) \to \mathcal{F}_1(\rho, j_2)$ in L^2; the result for $\mathcal{F}_2$ is analogous. By Step 2 and dominated convergence it is enough to show convergence pointwise, i.e., for each $x \in [-R,R]$. Next, by (7.2) and dominated convergence again, it is sufficient to show that the corresponding densities $\mu^\pm(e^\pm, p^\pm)$ converge pointwise in $(x, \boldsymbol{v})$. Continuity of $\mu^\pm$ reduces this to showing pointwise convergence of the corresponding microscopic energy and momenta $e^\pm$ and $p^\pm$. The definitions of these quantities imply that it is enough to show that the corresponding electric and magnetic potentials $\phi^{0,n}$ and $\psi^{0,n}$ converge pointwise. Finally, as the potentials are (ρ^n, j_2^n) convolved with $G(x) = -|x|/2$, the convergence $(\rho^n, j_2^n) \to (\rho, j_2)$ in $L^2([-R,R])^2$ gives the required pointwise convergence.

This concludes the proof. ∎

REFERENCES

[1] J. Batt, W. Faltenbacher, and E. Horst, *Stationary spherically symmetric models in stellar dynamics*, Arch. Ration. Mech. Anal., 93 (1986), pp. 159–183.

[2] J. Ben-Artzi, *Instability of nonmonotone magnetic equilibria of the relativistic Vlasov-Maxwell system*, Nonlinearity, 24 (2011), pp. 3353–3389.

[3] J. Ben-Artzi, *Instability of nonsymmetric nonmonotone equilibria of the Vlasov-Maxwell system*, J. Math. Phys., 52 (2011), 123703.

[4] J. Ben-Artzi and T. Holding, *Approximations of strongly continuous families of unbounded self-adjoint operators*, Comm. Math. Phys., 345 (2016), pp. 615–630.

[5] A. Choffrut and V. Šverák, *Local structure of the set of steady-state solutions to the 2d incompressible Euler equations*, Geom. Funct. Anal., 22 (2012), pp. 136–201.

[6] R. T. Glassey and J. W. Schaeffer, *Control of velocities generated in a two-dimensional collisionless plasma with symmetry*, Transport Theory Statist. Phys., 17 (1988), pp. 467–560.

[7] R. T. Glassey and J. W. Schaeffer, *On the "one and one-half dimensional" relativistic Vlasov-Maxwell system*, Math. Methods Appl. Sci., 13 (1990), pp. 169–179.

[8] D. D. Holm, J. E. Marsden, T. Ratiu, and A. Weinstein, *Nonlinear stability of fluid and plasma equilibria*, Phys. Rep., 123 (1985), pp. 1–116.

[9] J. H. Jeans, *On the theory of star-streaming and the structure of the universe*, Mon. Not. R. Astron. Soc., 76 (1915), pp. 70–84.

[10] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1995.

[11] M. Lemou, F. Méhats, and P. Raphaël, *Orbital stability of spherical galactic models*, Invent. Math., 187 (2011), pp. 145–194.

[12] Z. Lin and W. A. Strauss, *Linear stability and instability of relativistic Vlasov-Maxwell systems*, Comm. Pure Appl. Math., 60 (2007), pp. 724–787.

[13] Z. Lin and W. A. Strauss, *A sharp stability criterion for the Vlasov-Maxwell system*, Invent. Math., 173 (2008), pp. 497–546.

[14] C. Marchioro and M. Pulvirenti, *A note on the nonlinear stability of a spatially symmetric Vlasov-Poisson flow*, Math. Methods Appl. Sci., 8 (1986), pp. 284–288.

[15] C. Mouhot and C. Villani, *On Landau damping*, Acta Math., 207 (2011), pp. 29–201.

[16] T. T. Nguyen, T. V. Nguyen, and W. A. Strauss, *Global magnetic confinement for the* 1.5*D Vlasov-Maxwell system*, Kinet. Rel. Models, 8 (2015), pp. 153–168.

[17] T. T. Nguyen and W. A. Strauss, *Linear stability analysis of a hot plasma in a solid torus*, Arch. Ration. Mech. Anal., 211 (2014), pp. 619–672.

[18] O. Penrose, *Electrostatic instabilities of a uniform non-Maxwellian plasma*, Phys. Fluids, 3 (1960), pp. 258–265.

[19] J. W. Schaeffer, *A class of counterexamples to Jeans' theorem for the Vlasov-Einstein system*, Comm. Math. Phys., 204 (1999), pp. 313–327.

[20] T. Tao, *The spectral theorem and its converses for unbounded symmetric operators*, from the blog What's New, 20 December 2011, http://terrytao.wordpress.com/2011/12/20/the-spectral-theorem-and-its-converses-for-unbounded-symmetric-operators/.

[21] B. Turkington, *Statistical equilibrium measures and coherent states in two-dimensional turbulence*, Comm. Pure Appl. Math., 52 (1999), pp. 781–809.

[22] S. Wollman, *An existence and uniqueness theorem for the Vlasov-Maxwell system*, Comm. Pure Appl. Math., 37 (1984), pp. 457–462.