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Asymptotics of randomly stopped sums in the presence of heavy tails

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We study conditions under which

$$\mathbf{P}\{S_\tau > x\} \sim \mathbf{P}\{M_\tau > x\} \sim \mathbf{E}\tau\mathbf{P}\{\xi_1 > x\} \qquad \text{as } x \to \infty,$$

where S_τ is a sum $\xi_1 + \cdots + \xi_\tau$ of random size τ and M_τ is a maximum of partial sums $M_\tau = \max_{n \le \tau} S_n$. Here, ξ_n, $n = 1, 2, \ldots,$ are independent identically distributed random variables whose common distribution is assumed to be subexponential. We mostly consider the case where τ is independent of the summands; also, in a particular situation, we deal with a stopping time.

We also consider the case where $\mathbf{E}\xi > 0$ and where the tail of τ is comparable with, or heavier than, that of ξ, and obtain the asymptotics

$$\mathbf{P}\{S_\tau > x\} \sim \mathbf{E}\tau\mathbf{P}\{\xi_1 > x\} + \mathbf{P}\{\tau > x/\mathbf{E}\xi\} \qquad \text{as } x \to \infty.$$

This case is of primary interest in branching processes.

In addition, we obtain new uniform (in all x and n) upper bounds for the ratio $\mathbf{P}\{S_n > x\}/\mathbf{P}\{\xi_1 > x\}$ which substantially improve Kesten's bound in the subclass $\mathcal{S}^*$ of subexponential distributions.

Keywords: convolution equivalence; heavy-tailed distribution; random sums of random variables; subexponential distribution; upper bound

1. Introduction

Let $\xi, \xi_1, \xi_2, \ldots$ be independent identically distributed random variables with a finite mean. We assume that their common distribution F is right unbounded, that is, $\overline{F}(x) \equiv \mathbf{P}\{\xi > x\} > 0$ for all x. Moreover, we assume that F has a *heavy* (right) tail. Recall that a random variable η has a *heavy-tailed* distribution if $\mathbf{E}e^{\varepsilon\eta} = \infty$ for all $\varepsilon > 0$ and a *light-tailed* distribution otherwise.

Let $S_0 = 0$ and $S_n = \xi_1 + \cdots + \xi_n$, $n = 1, 2, \ldots,$ and let $M_n = \max_{0 \le i \le n} S_i$ be the partial maxima. Denote by F^{*n} the distribution of S_n.

Let τ be a counting random variable with a finite mean. In this paper, we study the asymptotics for the tail probabilities $\mathbf{P}\{S_\tau > x\}$ and $\mathbf{P}\{M_\tau > x\}$ as $x \to \infty$.

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It is known that for *any* distribution F on $\mathbf{R}^+$ and *any* counting random variable τ which is independent of the sequence $\{\xi_n\}$,

$$\liminf_{x\to\infty}\frac{\mathbf{P}\{S_\tau>x\}}{\overline{F}(x)}\geq\mathbf{E}\tau;$$

see, for example, [13,39]. It was proven in the series of papers [12,13,17] that if F is a heavy-tailed distribution on $\mathbf{R}^+$ with finite mean and if $\mathbf{P}\{c\tau>x\}=\mathrm{o}(\overline{F}(x))$ as $x\to\infty$, for some $c>\mathbf{E}\xi$, then

$$\liminf_{x\to\infty}\frac{\mathbf{P}\{S_\tau>x\}}{\overline{F}(x)}=\mathbf{E}\tau. \tag{1}$$

This gives us an idea of what asymptotic behaviour of $\mathbf{P}\{S_\tau>x\}$ should be expected, at least if the tail of τ is lighter than that of ξ. In particular, by considering the case $\tau=2$, we conclude that if F is a heavy-tailed distribution on $\mathbf{R}^+$ and if $\mathbf{P}\{S_2>x\}\sim c\overline{F}(x)$ as $x\to\infty$, for some c, then necessarily $c=2$ (see [17]). By the latter observation, we restrict our attention to subexponential distributions only.

A distribution F on $\mathbf{R}^+$ with unbounded support is called *subexponential*, $F\in\mathcal{S}$, if $\overline{F*F}(x)\sim 2\overline{F}(x)$ as $x\to\infty$. A distribution F on $\mathbf{R}$ is called subexponential if its conditional distribution on $\mathbf{R}^+$ is subexponential. It is well known that any subexponential distribution is heavy-tailed and, furthermore, is long-tailed. A distribution F with right-unbounded support is called *long-tailed* if $\overline{F}(x+y)\sim\overline{F}(x)$ as $x\to\infty$, for any fixed y.

The key result in the theory of subexponential distributions is the following: if F is subexponential and if τ does not depend on the summands and is light-tailed, then

$$\mathbf{P}\{S_\tau>x\}\sim\mathbf{E}\tau\overline{F}(x)\qquad\text{as } x\to\infty; \tag{2}$$

see, for example, [2,15] and references therein. A converse result also holds: if, for a distribution F on $\mathbf{R}^+$ and for an independent counting random variable $\tau\geq 2$, $\mathbf{P}\{S_\tau>x\}\sim\mathbf{E}\tau\overline{F}(x)$ as $x\to\infty$, then F is subexponential (see, e.g., [14]).

The intuition behind relation (2) is the *principle of one big jump*: in the case of heavy tails, for x large, the most probable way that the event $\{S_n>x\}$ arises is that one of the n summands $\xi_1,\dots,\xi_n$ is large while all others are relatively small. Asymptotically, this gives the probability $n\overline{F}(x)$ and conditioning on τ yields the multiplier $\mathbf{E}\tau$. The keystone of the proof is Kesten's bound: for any subexponential distribution F and any $\varepsilon>0$, there exists $K=K(F,\varepsilon)$ such that the inequality

$$\overline{F^{*n}}(x)\leq K(1+\varepsilon)^n\overline{F}(x)$$

holds for all x and n; see, for example, [3], Section IV.4, [2,15]. Clearly, this estimate does not help to prove (2) if the distribution of τ is heavy-tailed. So the important question here is the following: if we fix a subexponential distribution F, what are the weakest natural conditions on τ which still guarantee relation (2) to hold? Intuitively, the light-tailedness assumption seems to be very strong. The study of this problem is one of the main topics of the present paper.

In order to state our first result, we need to introduce the notion of an $\mathcal{S}^*$-distribution. A distribution F on $\mathbf{R}$ with a finite mean belongs to the class $\mathcal{S}^*$ if

$$\int_0^x \overline{F}(x-y)\overline{F}(y)\,\mathrm{d}y \sim 2a\overline{F}(x) \qquad \text{as } x \to \infty,$$

where $a = 2\int_0^\infty \overline{F}(y)\,\mathrm{d}y$. It is known (see [22]) that any distribution from the class $\mathcal{S}^*$ is subexponential. Although these two classes, $\mathcal{S}^*$ and $\mathcal{S}$, are considered to be rather similar, there exist subexponential distributions which are not in $\mathcal{S}^*$; see, for example, [11] and the discussion in Section 2. Classical examples of distributions from the class $\mathcal{S}^*$ are Pareto, log-normal and Weibull with parameter $\beta \in (0, 1)$.

Theorem 1. *Assume that a counting random variable τ is independent of $\{\xi_n\}$. Let $F \in \mathcal{S}^*$.*

(i) *If $\mathbf{E}\xi < 0$, then*

$$\mathbf{P}\{S_\tau > x\} \sim \mathbf{P}\{M_\tau > x\} \sim \mathbf{E}\tau\overline{F}(x) \qquad \textit{as } x \to \infty. \tag{3}$$

(ii) *If $\mathbf{E}\xi \geq 0$ and there exists $c > \mathbf{E}\xi$ such that*

$$\mathbf{P}\{c\tau > x\} = \mathrm{o}(\overline{F}(x)) \qquad \textit{as } x \to \infty, \tag{4}$$

then asymptotics (3) again hold.

The latter theorem shows that if we restrict our attention from the class of all heavy-tailed distributions to the class $\mathcal{S}^*$, then we obtain equivalence (3) which is stronger than assertion (1) for the '$\liminf$'. We should definitely assume the subexponentiality of F in order to obtain (3). At the end of Section 4, we construct an example demonstrating that the stronger condition $F \in \mathcal{S}^*$ is essential for the statement to hold in its full generality and cannot be replaced by the condition $F \in \mathcal{S}$.

The proof of Theorem 1 is carried out in Section 4. Statement (i) can be found in [19]; in Section 4, we give an alternative proof of (i). Note that these two cases, the negative and positive means of ξ, are substantially different in nature.

Condition (4) seems to be essential, since, for any $c < \mathbf{E}\xi$,

$$\begin{aligned}\mathbf{P}\{S_\tau > x\} &= \mathbf{P}\{S_\tau > x, c\tau \leq x\} + \mathbf{P}\{S_\tau > x, c\tau > x\} \\ &\geq \bigl(\mathbf{E}\tau + \mathrm{o}(1)\bigr)\overline{F}(x) + \bigl(1 + \mathrm{o}(1)\bigr)\mathbf{P}\{c\tau > x\}\end{aligned}$$

as $x \to \infty$, due to the convergence $\mathbf{P}\{S_\tau > x | c\tau > x\} \to 1$, by the law of large numbers. In particular, for τ with a regularly varying tail distribution, condition (4) is necessary for the asymptotic relation (3) to hold. Further discussion of condition (4) can be found in Section 4.

Stam in [41], Theorem 5.1, and Borovkov and Borovkov in [4], Section 7.1, obtained asymptotics (3) under condition (4) for regularly varying F. Some results from [41] have been proven again in [16]. The case where F is a dominated varying distribution was studied in [34] and [9]. A subclass of the so-called semi-exponential F was considered in [4], Section 7.2.

In [19], Corollary 2, asymptotics (3) were obtained in the case $\mathbf{E}\xi \geq 0$ under the extra assumption $\mathbf{P}\{\tau > h(x)\} = \mathrm{o}(\overline{F}(x))$ for some function $h(x) \to \infty$ such that $\overline{F}(x \pm h(x)) \sim \overline{F}(x)$.

In Section 2, we derive simple new uniform upper bounds for the ratio $\overline{F^{*n}}(x)/\overline{F}(x)$ which generalize Kesten's bound for $\mathcal{S}^*$-distributions. We prove the following theorem.

Theorem 2. *Assume that* $F \in \mathcal{S}^*$. *If* $\mathbf{E}\xi < 0$, *then there exists a constant* K *such that*

$$\frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \leq Kn \qquad \text{for all } n \text{ and } x.$$

If $\mathbf{E}\xi \in [0, \infty)$, *then, for any* $c > \mathbf{E}\xi$, *there exists* K *such that*

$$\frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \leq \frac{K}{\overline{F}(cn)} \qquad \text{for all } n \text{ and } x.$$

The latter estimates are also of interest in their own right. They substantially improve similar bounds in [40], Theorems 1 and 2 (see also [9], Theorem 3). In Theorem 4, Section 2, we show that the condition $F \in \mathcal{S}^*$ is essential for the statement of Theorem 2 to hold; more precisely, we construct a distribution $F \in \mathcal{S} \setminus \mathcal{S}^*$ with negative mean such that $\sup_{n,x} \frac{\overline{F^{*n}}(x)}{n\overline{F}(x)} = \infty$.

A closely related topic involves the asymptotics of the type $\mathbf{P}\{S_n > x\} \sim n\overline{F}(x)$ as $n, x \to \infty$, these having been extensively studied since the 1960s. The earliest works in this area are the remarkable papers [25,29,30] (in the latter, in a special case, the asymptotics are stated, but the key relation (10.10) on page 303 is not supported by a proof) and, later on, [8,27,28], where, in particular, the regularly varying distributions were considered. Namely, if F is regularly varying with the parameter $\alpha > 2$ and $\mathbf{E}\xi_1 = 0$, $\mathbf{E}\xi_1^2 = 1$, then, under mild technical conditions (see [27], [32], Theorem 1.9, or [36], Theorem 6), the following asymptotics hold:

$$\mathbf{P}\{S_n > x\} \sim \overline{\Phi}\big(x/\sqrt{n}\big) + n\overline{F}(x) \qquad \text{as } x \to \infty \text{ uniformly in } n \leq x^2;$$

here, $\overline{\Phi}$ is the tail function of the standard normal law. Further, it follows that if $x \leq \sqrt{(\alpha-2-\varepsilon)n\ln n}$, then the asymptotics follow the central limit theorem, while if $x > \sqrt{(\alpha-2+\varepsilon)n\ln n}$, then the probability of a single big jump dominates. For Weibull-type distributions, the situation is more complicated; see, for example, [28,31,37,38]. Detailed overviews of results in the theory of large deviations for random walks with subexponential increments are given in [32] and [26]. There is still ongoing research in this area; see the recent works [4,5,10] and references therein. In Section 3 of this paper, for an *arbitrary* distribution $F \in \mathcal{S}^*$, we find a range for $n = n(x)$ where the asymptotics $\mathbf{P}\{S_n > x\} \sim n\overline{F}(x)$ hold. The corresponding proof is surprisingly short.

In Section 5, we study the case where the tail distributions of τ and ξ are asymptotically comparable and, for a subclass of subexponential distributions, we obtain the asymptotics for $\mathbf{P}\{S_\tau > x\}$ which differ from (3); see Theorem 8. This generalizes results in [4] and [41]; see Section 5 for further comments. As a corollary, in Section 6, we obtain new tail asymptotics for Galton–Watson branching processes.

In Section 7, we study the case where τ may depend on $\{\xi_n\}$ and, in particular, where τ is a stopping time. First, we prove Theorem 9, where we obtain equivalence (3) for bounded τ. In the proof, we adapt the approach developed in [20] and generalize Greenwood's result to the whole class of subexponential distributions. We then consider an unbounded τ and prove Theorem 10, which states that equivalence (3) holds under a stronger assumption than (4) (see condition (37)). Theorem 10 generalizes earlier results from [21] and [6]; see Corollary 3 and the comments which follow it. Concerning the asymptotics for the maximum, it was shown in [19] (see also [18]) that the equivalence $\mathbf{P}\{M_\tau > x\} \sim \mathbf{E}\tau\overline{F}(x)$ holds without any further assumptions on the tail distribution of τ if $\mathbf{E}\xi < 0$ and under condition (37) otherwise.

2. Uniform upper bounds for tails; proof of Theorem 2

In this section, for the ratios $\overline{F^{*n}}(x)/\overline{F}(x)$, we derive upper bounds more precise than Kesten's bound, which are again uniform in x. We consider two cases, $\mathbf{E}\xi < 0$ and $\mathbf{E}\xi \geq 0$, separately. We need the following result.

Theorem 3 ([24] and [11], Corollary 4). *Assume that $F \in \mathcal{S}^*$ and $\mathbf{E}\xi < 0$. Then, as $x \to \infty$ and uniformly in $n \geq 1$,*

$$\mathbf{P}\{M_n > x\} \sim \frac{1}{|\mathbf{E}\xi|} \int_x^{x+n|\mathbf{E}\xi|} \overline{F}(y)\,\mathrm{d}y.$$

Proof of Theorem 2. First, we consider the case (i) of negative mean. Taking into account the inequality $S_n \leq M_n$, Theorem 3 and the inequality

$$\frac{1}{|\mathbf{E}\xi|} \int_x^{x+n|\mathbf{E}\xi|} \overline{F}(y)\,\mathrm{d}y \leq n\overline{F}(x), \tag{5}$$

we obtain statement (i) of the theorem.

Now, consider the case (ii) where $\mathbf{E}\xi \geq 0$. Take $c > \mathbf{E}\xi$. Put $\widetilde{\xi}_i = \xi_i - c$ and $\widetilde{S}_n = \widetilde{\xi}_1 + \cdots + \widetilde{\xi}_n$. Then $\mathbf{E}\widetilde{\xi} = \mathbf{E}\xi - c < 0$ and we can again apply Theorem 3. Thus, there exists a constant K_1 such that, for all x and n,

$$\overline{\widetilde{F}^{*n}}(x) \leq K_1 \int_0^{n|\mathbf{E}\widetilde{\xi}|} \overline{\widetilde{F}}(x+y)\,\mathrm{d}y,$$

where $\widetilde{F}$ in the distribution of $\widetilde{\xi}$. Therefore,

$$\begin{aligned}\mathbf{P}\{S_n > x\} = \mathbf{P}\{\widetilde{S}_n > x - nc\} &\leq K_1 \int_0^{nc} \overline{\widetilde{F}}(x - nc + y)\,\mathrm{d}y \\ &= K_1 \int_0^{nc} \overline{\widetilde{F}}(x-y)\,\mathrm{d}y.\end{aligned}$$

Since $F \in \mathcal{S}^*$, the distribution F is long-tailed and, hence, $\overline{\overline{F}}(x) \sim \overline{F}(x)$ as $x \to \infty$. It then follows that

$$\mathbf{P}\{S_n > x\} \le K_2 \int_0^{nc} \overline{F}(x-y)\,\mathrm{d}y \tag{6}$$

for some constant K_2 and all $x \ge 0$. If $x \ge nc$, then

$$\begin{aligned}\int_0^{nc} \overline{F}(x-y)\,\mathrm{d}y &\le \int_0^{nc} \overline{F}(x-y)\frac{\overline{F}(y)}{\overline{F}(nc)}\,\mathrm{d}y \\ &\le \int_0^{x} \overline{F}(x-y)\frac{\overline{F}(y)}{\overline{F}(nc)}\,\mathrm{d}y \le K_3\frac{\overline{F}(x)}{\overline{F}(nc)},\end{aligned}$$

where

$$K_3 = \sup_{x\ge 0}\frac{1}{\overline{F}(x)}\int_0^x \overline{F}(x-y)\overline{F}(y)\,\mathrm{d}y$$

is finite, owing to the fact that $F \in \mathcal{S}^*$. If $x < nc$, then

$$\overline{F^{*n}}(x) \le 1 \le \frac{\overline{F}(x)}{\overline{F}(nc)}.$$

These two bounds together with (6) complete the proof of the second assertion of Theorem 2. □

From Theorem 2 and the dominated convergence theorem, we deduce the following corollary.

Corollary 1. *Tail equivalence* (3) *holds if* $F \in \mathcal{S}^*$ *and* $\mathbf{E}\xi \ge 0$, *provided that*

$$\sum_{n=1}^{\infty}\frac{\mathbf{P}\{\tau = n\}}{\overline{F}(cn)} < \infty \qquad \textit{for some } c > \mathbf{E}\xi.$$

The latter condition is stronger than condition (4) because

$$\frac{\mathbf{P}\{\tau > k\}}{\overline{F}(ck)} \le \sum_{n>k}\frac{\mathbf{P}\{\tau = n\}}{\overline{F}(cn)}.$$

Let us now discuss the importance of the condition $F \in \mathcal{S}^*$ in Theorem 2. The following observation shows the essence of the difference between two classes of distributions, $\mathcal{S}$ and $\mathcal{S}^*$. Let a long-tailed distribution F be absolutely continuous with density f. For any function $h(x) > 0$,

$$\int_{h(x)}^{x-h(x)} \overline{F}(x-y)F(\mathrm{d}y) = \int_{h(x)}^{x-h(x)} \overline{F}(x-y)f(y)\,\mathrm{d}y.$$

F is then subexponential if and only if

$$\int_{h(x)}^{x-h(x)} \overline{F}(x-y)f(y)\,\mathrm{d}y = \mathrm{o}(\overline{F}(x)) \qquad \text{as } x \to \infty$$

holds for any function $h(x) \to \infty$ or, equivalently, if it holds for some function $h(x) \to \infty$ such that $\overline{F}(x - h(x)) \sim \overline{F}(x)$. On the other hand, $F \in \mathcal{S}^*$ if and only if

$$\int_{h(x)}^{x-h(x)} \overline{F}(x-y)\overline{F}(y)\,\mathrm{d}y = \mathrm{o}(\overline{F}(x)) \qquad \text{as } x \to \infty.$$

In typical cases, $f(x) = \mathrm{o}(\overline{F}(x))$ and, hence,

$$\int_{h(x)}^{x-h(x)} \overline{F}(x-y)f(y)\,\mathrm{d}y = \mathrm{o}\biggl(\int_{h(x)}^{x-h(x)} \overline{F}(x-y)\overline{F}(y)\,\mathrm{d}y\biggr) \qquad \text{as } x \to \infty.$$

This means that the subexponentiality of F is more likely than $F \in \mathcal{S}^*$. The latter observation provides an idea of how to show that the condition $F \in \mathcal{S}^*$ in Theorem 2 cannot be extended to the subexponentiality of F.

Theorem 4. *There exists a subexponential distribution F on* $\mathbf{R}$ *with a negative mean such that*

$$\overline{F^{*n_k}}(x_k) \ge c\frac{n_k^2}{\ln n_k}\overline{F}(x_k)$$

for some $c > 0$ and for some sequences $n_k, x_k \to \infty$.

The latter theorem yields that, for some distribution $F \in \mathcal{S} \setminus \mathcal{S}^*$ with negative mean, the first estimate of Theorem 2 fails, that is, $\sup_{n,x} \frac{\overline{F^{*n}}(x)}{n\overline{F}(x)} = \infty$.

Proof of Theorem 4. We start with a construction of a specific subexponential distribution G on the positive half-line. Let $R_0 = 0$, $R_1 = 1$ and $R_{k+1} = \mathrm{e}^{R_k}/R_k$ for $k \ge 1$. Since e^x/x is increasing for $x \ge 1$, the sequence R_k is increasing and

$$R_k = \mathrm{o}(R_{k+1}) \qquad \text{as } k \to \infty. \tag{7}$$

Set $t_k = R_k^2$. Define the hazard function $R(x) \equiv -\ln\overline{G}(x)$ as

$$R(x) = R_k + r_k(x - t_k) \qquad \text{for } x \in (t_k, t_{k+1}],$$

where

$$r_k = \frac{R_{k+1} - R_k}{t_{k+1} - t_k} = \frac{1}{R_{k+1} + R_k} \sim \frac{1}{R_{k+1}} \tag{8}$$

by (7). In other words, the hazard rate $r(x) = R'(x)$ is defined as $r(x) = r_k$ for $x \in (t_k, t_{k+1}]$, where r_k is given by (8). By the construction, we have $\overline{G}(t_k) = \mathrm{e}^{-\sqrt{t_k}}$ so that at points t_k, the tail of G behaves like the Weibull tail with parameter $1/2$. Between these points, the tail decays exponentially with indices r_k.

We now prove that G has finite mean and is subexponential. Since, by (8),

$$\int_{t_k}^{t_{k+1}} \mathrm{e}^{-R(y)}\,\mathrm{d}y = r_k^{-1}(\mathrm{e}^{-R_k} - \mathrm{e}^{-R_{k+1}}) \sim r_k^{-1}\mathrm{e}^{-R_k} \sim R_{k+1}\mathrm{e}^{-R_k} = 1/R_k,$$

the mean of G,

$$\int_0^\infty \overline{G}(y)\,\mathrm{d}y = \sum_{k=0}^\infty \int_{t_k}^{t_{k+1}} \overline{G}(y)\,\mathrm{d}y$$

is finite.

It follows from the definition that $r(x)$ decreases to 0. We can then apply Pitman's criterion [35] which states that G is subexponential if the function $\mathrm{e}^{yr(y)-R(y)}r(y)$ is integrable over $[0,\infty)$. In order to estimate the integral of this function, let

$$I_k = \int_{t_k}^{t_{k+1}} \mathrm{e}^{yr(y)-R(y)} r(y)\,\mathrm{d}y.$$

We then have

$$I_k = r_k \int_{t_k}^{t_{k+1}} \mathrm{e}^{yr_k-(R_k+r_k(y-t_k))}\,\mathrm{d}y \le r_k \mathrm{e}^{-R_k+r_kt_k} t_{k+1}.$$

Since

$$r_k t_{k+1} = r_k R_{k+1}^2 \sim R_{k+1} \tag{9}$$

by (8) and

$$r_k t_k = r_k R_k^2 \sim R_k^2/R_{k+1} = R_k^3 \mathrm{e}^{-R_k} \to 0,$$

we get $I_k \le 2R_{k+1}\mathrm{e}^{-R_k} \sim 2/R_k$ for k sufficiently large. Therefore,

$$\int_0^\infty \mathrm{e}^{yr(y)-R(y)} r(y)\,\mathrm{d}y = \sum_{k=0}^\infty I_k < \infty$$

and G is indeed subexponential.

In the sequel, we need to know the asymptotic behaviour of the following internal part of the convolution integral at point t_k:

$$J_k = \int_{t_k/4}^{3t_k/4} \overline{G}(t_k-y)G(\mathrm{d}y) = \int_{t_k/4}^{3t_k/4} \mathrm{e}^{-R(t_k-y)}\mathrm{e}^{-R(y)} r(y)\,\mathrm{d}y.$$

Owing to (7), $t_{k-1} = \mathrm{o}(t_k)$. Thus, $(t_k/4, 3t_k/4] \subset (t_{k-1}, t_k - t_{k-1}]$ for all sufficiently large k. For those values of k, we have

$$\begin{aligned} J_k &= \overline{G}(t_k) \int_{t_k/4}^{3t_k/4} \mathrm{e}^{-(-r_{k-1}y)} \mathrm{e}^{-(R_{k-1}+r_{k-1}(y-t_{k-1}))} r_{k-1}\,\mathrm{d}y \\ &\ge \overline{G}(t_k)(t_k/2)\mathrm{e}^{-R_{k-1}} r_{k-1}. \end{aligned}$$

Applying (9) and the equality $\mathrm{e}^{R_{k-1}} = R_k R_{k-1}$, we obtain, for all sufficiently large k,

$$J_k \ge \overline{G}(t_k)\mathrm{e}^{-R_{k-1}} R_k/3 = \overline{G}(t_k)/3R_{k-1}. \tag{10}$$

Let $\eta_1, \eta_2, \ldots$ be independent random variables with common distribution G and put $T_n = \eta_1 + \cdots + \eta_n$. For any n, we have

$$\begin{aligned}\mathbf{P}\{T_n > x\} &\geq \sum_{1\leq i<j\leq n} \mathbf{P}\{T_n > x, \eta_i > n, \eta_j > n, \eta_l \leq n \text{ for all } l \neq i, j\}\\ &= \frac{n(n-1)}{2}\mathbf{P}\{T_n > x, \eta_1 > n, \eta_2 > n, \eta_3 \leq n, \ldots, \eta_n \leq n\}.\end{aligned}$$

Since the η's are positive, the latter probability is not smaller than

$$\mathbf{P}\{\eta_1 + \eta_2 > x, \eta_1 > n, \eta_2 > n\}\mathbf{P}\{\eta_3 \leq n, \ldots, \eta_n \leq n\}.$$

The mean of η is finite, thus $\overline{G}(n) = \mathrm{o}(1/n)$ as $n \to \infty$ and

$$\mathbf{P}\{\eta_3 \leq n, \ldots, \eta_n \leq n\} = \big(1 - \overline{G}(n)\big)^{n-2} \to 1.$$

Putting all of this together, we get, for all sufficiently large n, the following estimate from below:

$$\mathbf{P}\{T_n > x\} \geq \frac{n^2}{3}\mathbf{P}\{\eta_1 + \eta_2 > x, \eta_1 > n, \eta_2 > n\}. \tag{11}$$

Now take $n = n_k = [\sqrt{t_k}] = [R_k]$. Then, for all sufficiently large k (at least for those k where $n_k < t_k/4$),

$$\mathbf{P}\{\eta_1 + \eta_2 > t_k, \eta_1 > n_k, \eta_2 > n_k\} \geq J_k.$$

Therefore, by (11) and (10), for all sufficiently large k,

$$\mathbf{P}\{T_{n_k} > t_k\} \geq n_k^2\overline{G}(t_k)/9R_{k-1} \sim n_k^2\overline{G}(t_k)/9\ln n_k,$$

due to $R_{k-1} \sim \ln R_k \sim \ln n_k$.

Let $b = \mathbf{E}\eta_1$. If we set $\xi_i = \eta_i - 2b$, then the ξ's have negative mean and $S_n = T_n - 2nb$. Denote by F the distribution of ξ_1; it is subexponential because G is.

Take $x = x_k = t_k - 2n_k b$ so that $x_k \sim n_k^2$. By the latter inequality, we have

$$\mathbf{P}\{S_{n_k} > x_k\} = \mathbf{P}\{T_{n_k} > t_k\} \geq n_k^2\overline{G}(t_k)/10\ln n_k.$$

Also, note that

$$\overline{F}(x_k) = \overline{G}(t_k - 2n_k b) = \overline{G}(t_k)\mathrm{e}^{r_{k-1}2n_k b} \leq \overline{G}(t_k)\mathrm{e}^{2b}$$

because $r_{k-1}n_k \leq r_{k-1}R_k \leq 1$ by (8). Therefore, the inequality

$$\mathbf{P}\{S_{n_k} > x_k\} \geq n_k^2\overline{F}(x_k)\mathrm{e}^{-2b}/10\ln n_k$$

holds which yields the conclusion of the theorem. □

The subexponential distribution G constructed in the latter proof cannot belong to the class $\mathcal{S}^*$ because otherwise the theorem's conclusion fails, as follows from Theorem 2. The fact that $G \notin \mathcal{S}^*$ can also be proven directly. Klüppelberg's criterion [22] states that $G \in \mathcal{S}^*$ if and only if

$$\int_0^x \mathrm{e}^{yr(x)-R(y)}\,\mathrm{d}y \to \int_0^\infty \overline{G}(y)\,\mathrm{d}y \qquad \text{as } x \to \infty.$$

In our construction,

$$\begin{aligned}\int_0^{t_k-0} \mathrm{e}^{yr(t_k-0)-R(y)}\,\mathrm{d}y &\ge \int_{t_{k-1}}^{t_k} \mathrm{e}^{yr_{k-1}-R(y)}\,\mathrm{d}y \\ &\ge (t_k - t_{k-1})\mathrm{e}^{-R_{k-1}} \\ &\sim R_k^2 \mathrm{e}^{-R_{k-1}} = \mathrm{e}^{R_{k-1}}/R_{k-1}^2 \to \infty\end{aligned}$$

as $k \to \infty$. Hence, $G \notin \mathcal{S}^*$.

3. On the asymptotics $\mathbf{P}\{S_n > x\} \sim n\overline{F}(x)$

As before, we assume $\mathbf{E}\xi$ to be finite. Then, by the strong law of large numbers,

$$\mathbf{P}\{S_n > -An\} \to 1 \qquad \text{as } A \to \infty \text{ uniformly in } n \ge 1 \tag{12}$$

and, by Chebyshev's inequality,

$$\mathbf{P}\{\xi_1 > An\} \le \mathbf{E}|\xi_1|/An \qquad \text{for all } A > 0 \text{ and } n \ge 1. \tag{13}$$

Theorem 5. *Let $F \in \mathcal{S}^*$ and let an increasing function $h(x) > 0$ be such that $\overline{F}(x \pm h(x)) \sim \overline{F}(x)$. Then $\mathbf{P}\{S_n > x\} \sim n\overline{F}(x)$ as $x \to \infty$ uniformly in $n \le h(x)$.*

Proof. Proof of the lower bound is similar to that in [10], Section 4. Fix $A > 0$. We use the following inequalities:

$$\begin{aligned}\mathbf{P}\{S_n > x\} &\ge \sum_{i=1}^n \mathbf{P}\{S_n > x, \xi_i > x + An, \xi_j \le An \text{ for all } j \ne i\} \\ &\ge n\mathbf{P}\{S_n - \xi_1 > -An, \xi_1 > x + An, \xi_2 \le An, \dots, \xi_n \le An\} \\ &= n\overline{F}(x + An)\mathbf{P}\{S_{n-1} > -An, \xi_1 \le An, \dots, \xi_{n-1} \le An\}.\end{aligned}$$

We have $\overline{F}(x + An) \sim \overline{F}(x)$ as $x \to \infty$ uniformly in $n \le h(x)$. Also taking into account the fact that

$$\mathbf{P}\{S_{n-1} > -An, \xi_1 \le An, \dots, \xi_{n-1} \le An\} \ge \mathbf{P}\{S_{n-1} > -An\} - (n-1)\mathbf{P}\{\xi_1 > An\},$$

we get, for any fixed $A > 0$,

$$\liminf_{x\to\infty} \inf_{n\le h(x)} \frac{\mathbf{P}\{S_n > x\}}{n\overline{F}(x)} \ge \inf_n \bigl(\mathbf{P}\{S_{n-1} > -An\} - (n-1)\mathbf{P}\{\xi_1 > An\}\bigr).$$

Since the infimum on the right goes to 1 as $A \to \infty$ owing to (12) and (13), we arrive at the following lower bound:

$$\liminf_{x\to\infty} \inf_{n\le h(x)} \frac{\mathbf{P}\{S_n > x\}}{n\overline{F}(x)} \ge 1.$$

To prove the upper bound, we apply Theorem 3 to the random variables $\widetilde{\xi}_i = \xi_i - \mathbf{E}\xi_1 - 1$ with negative mean $\mathbf{E}\widetilde{\xi}_i = -1$ and to $\widetilde{S}_n = S_n - n(\mathbf{E}\xi_1 + 1)$. Thus,

$$\begin{aligned}\mathbf{P}\{S_n > x\} &= \mathbf{P}\{\widetilde{S}_n > x - n(\mathbf{E}\xi_1 + 1)\} \\ &\le \bigl(1 + \mathrm{o}(1)\bigr) \int_{x-n(\mathbf{E}\xi_1+1)}^{x-n\mathbf{E}\xi_1} \overline{\widetilde{F}}(x+u)\,\mathrm{d}u \\ &\le \bigl(1 + \mathrm{o}(1)\bigr) n \overline{\widetilde{F}}\bigl(x - n(\mathbf{E}\xi_1 + 1)\bigr)\end{aligned}$$

as $x \to \infty$, where $\widetilde{F}$ is the distribution of $\widetilde{\xi}$. If $n \le h(x)$, then $\overline{\widetilde{F}}(x - n(\mathbf{E}\xi_1 + 1)) \sim \overline{F}(x)$ as $x \to \infty$ and the proof is complete. □

The range $n \le h(x)$ is usually more narrow than one could expect. For instance, for regularly varying distributions (more generally, for intermediate regularly varying distributions – see the definition in Section 5), we can take $h(x) = \mathrm{o}(x)$. We then get the range $n = \mathrm{o}(x)$, while (if the mean is zero and the second moment is finite) the standard range is $x^2 > cn\ln n$; in the class of distributions with finite mean, the relation $\mathbf{P}\{S_n > x\} \sim n\overline{F}(x)$ holds in the range $x > (\mathbf{E}\xi + \varepsilon)n$, $\varepsilon > 0$; see [33]. The advantage of the result in Theorem 5 is its simplicity and universality since it is valid for all distributions from $\mathcal{S}^*$ without any further moment or regularity assumptions (cf. the series of results in [4,5,10] where the hazard rate is assumed to be sufficiently smooth).

As follows from [10], if the mean is zero and the second moment is finite, then the right range should be $n \le h^2(x)$, roughly speaking. Our technique allows the lower bound for this range to be proven.

Theorem 6. *Let $\mathbf{E}\xi = 0$ and $\mathbf{E}\xi^2 < \infty$. Let F be a long-tailed distribution and let an increasing function $h(x) > 0$ be such that $\overline{F}(x \pm h(x)) \sim \overline{F}(x)$. Then $\mathbf{P}\{S_n > x\} \ge (1 + \mathrm{o}(1))n\overline{F}(x)$ as $x \to \infty$ uniformly in $n \le h^2(x)$.*

Proof. Fix $A > 0$. By Chebyshev's inequality,

$$\mathbf{P}\bigl\{\xi_1 > A\sqrt{n}\bigr\} \le \mathbf{E}\xi^2/A^2 n \quad \text{and} \quad \mathbf{P}\bigl\{S_n > -A\sqrt{n}\bigr\} \ge 1 - \mathbf{E}\xi^2/A^2. \tag{14}$$

In this proof, we use a slightly different inequality than in the previous theorem:

$$\begin{aligned}\mathbf{P}\{S_n > x\} &\geq \sum_{i=1}^{n} \mathbf{P}\big\{S_n > x, \xi_i > x + A\sqrt{n}, \xi_j \leq A\sqrt{n} \text{ for all } j \neq i\big\}\\ &\geq n\mathbf{P}\big\{S_n - \xi_1 > -A\sqrt{n}, \xi_1 > x + A\sqrt{n}, \xi_2 \leq A\sqrt{n}, \ldots, \xi_n \leq A\sqrt{n}\big\}\\ &= n\overline{F}\big(x + A\sqrt{n}\big)\mathbf{P}\big\{S_{n-1} > -A\sqrt{n}, \xi_1 \leq A\sqrt{n}, \ldots, \xi_{n-1} \leq A\sqrt{n}\big\}.\end{aligned}$$

Since $n \leq h^2(x)$, $\overline{F}(x + A\sqrt{n}) \sim \overline{F}(x)$ as $x \to \infty$. Applying (14), we obtain

$$\begin{aligned}&\mathbf{P}\big\{S_{n-1} > -A\sqrt{n}, \xi_1 \leq A\sqrt{n}, \ldots, \xi_{n-1} \leq A\sqrt{n}\big\}\\ &\quad\geq \mathbf{P}\big\{S_{n-1} > -A\sqrt{n}\big\} - (n-1)\mathbf{P}\big\{\xi_1 > A\sqrt{n}\big\}\\ &\quad\geq 1 - 2\mathbf{E}\xi^2/A^2 \to 1 \qquad \text{as } A \to \infty.\end{aligned}$$

The lower bound for $\mathbf{P}\{S_n > x\}$ now follows. □

4. Proof of Theorem 1

Since τ is independent of the ξ's, we can use the following decomposition:

$$\mathbf{P}\{S_\tau > x\} = \sum_{n=0}^{\infty} \mathbf{P}\{\tau = n\}\overline{F^{*n}}(x).$$

By the subexponentiality, the nth term here is equivalent to $n\mathbf{P}\{\tau = n\}\overline{F}(x)$ as $x \to \infty$. In particular, by Fatou's lemma,

$$\liminf_{x\to\infty} \frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \geq \sum_{n=0}^{\infty} n\mathbf{P}\{\tau = n\} = \mathbf{E}\tau, \tag{15}$$

without any condition on the sign of $\mathbf{E}\xi$. In the case of negative mean, the nth term is bounded from above by $n\overline{F}(x)$; see (5). The dominated convergence for series then yields statement (i) of the theorem.

We now turn to the proof of statement (ii) where $\mathbf{E}\xi \geq 0$. Since $S_\tau \leq M_\tau$, it follows from (15) that it is sufficient to prove that

$$\mathbf{P}\{M_\tau > x\} \sim \mathbf{E}\tau\overline{F}(x) \qquad \text{as } x \to \infty. \tag{16}$$

To prove the latter relation, we start with the following representation: for any N,

$$\begin{aligned}\mathbf{P}\{M_\tau > x\} &= \mathbf{P}\{M_\tau > x, \tau \leq N\} + \mathbf{P}\{M_\tau > x, \tau \in (N, x/c]\} + \mathbf{P}\{M_\tau > x, c\tau > x\}\\ &\equiv P_1 + P_2 + P_3.\end{aligned} \tag{17}$$

Since any $\mathcal{S}^*$-distribution is subexponential and $S_n \le M_n \le \xi_1^+ + \cdots + \xi_n^+$, we have

$$\mathbf{P}\{M_n > x\} \sim n\overline{F}(x)$$

as $x \to \infty$, for any n. Thus, for any fixed N,

$$\mathbf{P}\{M_\tau > x, \tau \le N\} = \sum_{n=1}^{N} \mathbf{P}\{\tau = n\}\mathbf{P}\{M_n > x\} \sim \mathbf{E}\{\tau; \tau \le N\}\overline{F}(x)$$

as $x \to \infty$, which implies the existence of an increasing function $N(x) \to \infty$ such that

$$P_1 = \mathbf{P}\{M_\tau > x, \tau \le N(x)\} \sim \mathbf{E}\tau\overline{F}(x). \tag{18}$$

In what follows, we use representation (17) with $N(x)$ in place of N. We further estimate the second term on the right-hand side of (17). Let $\varepsilon = (c - \mathbf{E}\xi)/2 > 0$ and $b = (\mathbf{E}\xi + c)/2$. Consider $\widetilde{\xi}_n = \xi_n - b$, $\widetilde{S}_n = \widetilde{\xi}_1 + \cdots + \widetilde{\xi}_n$ and $\widetilde{M}_n = \max(\widetilde{S}_1, \ldots, \widetilde{S}_n)$. Then $\mathbf{E}\widetilde{\xi} = -\varepsilon < 0$ and we can apply Theorem 3. Taking into account the fact that $M_n \le \widetilde{M}_n + bn$, we obtain that there exists K such that, for all x and n,

$$\mathbf{P}\{M_n > x\} \le \mathbf{P}\{\widetilde{M}_n > x - bn\} \le K\int_0^{n\varepsilon} \overline{\widetilde{F}}(x - nb + y)\,\mathrm{d}y \le K\int_0^{n\varepsilon} \overline{F}(x - nb + y)\,\mathrm{d}y.$$

Hence,

$$P_2 = \mathbf{P}\big\{M_\tau > x, \tau \in \big(N(x), x/c\big]\big\} \le K \sum_{n=N(x)}^{[x/c]} \mathbf{P}\{\tau = n\} \int_0^{n\varepsilon} \overline{F}(x - nb + y)\,\mathrm{d}y.$$

Since $b - \varepsilon = \mathbf{E}\xi$,

$$\int_0^{n\varepsilon} \overline{F}(x - nb + y)\,\mathrm{d}y = \int_{n\mathbf{E}\xi}^{nb} \overline{F}(x - y)\,\mathrm{d}y.$$

We then have

$$\begin{aligned} P_2 &\le K\int_{N(x)\mathbf{E}\xi}^{b[x/c]} \overline{F}(x - y)\,\mathrm{d}y \sum_{n=\max(N(x),[y/b]+1)}^{[x/c]} \mathbf{P}\{\tau = n\} \\ &\le K\int_{N(x)\mathbf{E}\xi}^{bx/c} \overline{F}(x - y)\mathbf{P}\{\tau > y/b\}\,\mathrm{d}y \qquad (19) \\ &\le K\int_{N(x)\mathbf{E}\xi}^{bx/c} \overline{F}(x - y)\mathbf{P}\{\tau > y/c\}\,\mathrm{d}y \end{aligned}$$

because $b < c$. By condition (4), $\mathbf{P}\{\tau > y/c\} \le K_1\overline{F}(y)$ for some K_1 and all y. Therefore, the inequality

$$P_2 \le KK_1\int_{N(x)\mathbf{E}\xi}^{bx/c} \overline{F}(x - y)\overline{F}(y)\,\mathrm{d}y = \mathrm{o}(\overline{F}(x)) \qquad \text{as } x \to \infty \tag{20}$$

follows from $b/c < 1$ and from $F \in \mathcal{S}^*$. Indeed, for any $\mathcal{S}^*$-distribution,

$$\int_{h(x)}^{x-h(x)} \overline{F}(x-y)\overline{F}(y)\,\mathrm{d}y = \mathrm{o}(\overline{F}(x)) \qquad \text{as } x \to \infty \tag{21}$$

for any function $h(x) \to \infty$ such that $h(x) \le x/2$ (see, e.g., [22]).

We now estimate the third term on the right-hand side of (17) using condition (4):

$$P_3 \le \mathbf{P}\{c\tau > x\} = \mathrm{o}(\overline{F}(x)) \qquad \text{as } x \to \infty. \tag{22}$$

Relations (18), (20) and (22) together complete the proof of Theorem 1.

We now provide an example where

$$\frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \to \infty,$$

given that condition (4) is satisfied only for $c = \mathbf{E}\xi > 0$ and not for any larger c. Assume that F is a Weibull distribution on the positive half-line with parameter $\beta \in (1/2, 1)$, that is, $\overline{F}(x) = \mathrm{e}^{-x^\beta}$. Let τ have a distribution such that $\mathbf{P}\{c\tau > x\} \sim x^{-1}\mathrm{e}^{-x^\beta}$ as $x \to \infty$. We consider the following lower bound:

$$\mathbf{P}\{S_\tau > x\} \ge \mathbf{P}\big\{S_\tau > x | c\tau > x - \sqrt{x}\big\}\mathbf{P}\big\{c\tau > x - \sqrt{x}\big\}.$$

By the central limit theorem,

$$\delta \equiv \liminf_{x\to\infty} \mathbf{P}\big\{S_\tau > x | c\tau > x - \sqrt{x}\big\} \ge \liminf_{x\to\infty} \mathbf{P}\big\{S_{[(x-\sqrt{x})/c]} > x\big\} > 0.$$

Hence,

$$\liminf_{x\to\infty} \frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \ge \delta \liminf_{x\to\infty} \frac{\mathbf{P}\{c\tau > x - \sqrt{x}\}}{\overline{F}(x)} = \delta \liminf_{x\to\infty} \frac{\mathrm{e}^{x^\beta - (x-\sqrt{x})^\beta}}{x - \sqrt{x}} = \infty$$

because $\beta > 1/2$.

We conclude this section with an example showing that the conclusion of Theorem 1 cannot hold for all subexponential distributions. Indeed, take F with negative mean, as described in Theorem 4. Without loss of generality, we assume that the series $\sum_k n_k^{-1} \ln n_k$ converges. Consider τ taking values n_k with probabilities $c \ln^2 n_k / n_k^2$, where c is the normalizing constant. τ then has a finite mean, but

$$\mathbf{P}\{S_\tau > x_k\} \ge \mathbf{P}\{S_{n_k} > x_k\}\mathbf{P}\{\tau = n_k\} \ge c \frac{n_k^2}{\ln n_k} \overline{F}(x_k) \frac{\ln^2 n_k}{n_k^2}$$

so that, as $k \to \infty$,

$$\frac{\mathbf{P}\{S_\tau > x_k\}}{\overline{F}(x_k)} \to \infty.$$

5. The case where ξ and τ may be tail-comparable

In this section, we do not assume condition (4) to hold. Such a situation is of particular importance for branching processes. To begin, we define two important classes of distributions.

A distribution F is called *dominated varying* if there exists c such that $\overline{F}(x) \le c\overline{F}(2x)$ for all x. It is known that any long-tailed and dominated varying distribution with a finite mean belongs to the class $\mathcal{S}^*$; see [22].

We say that a distribution G is *intermediate regularly varying* at infinity (due to [7]) if

$$\lim_{\varepsilon\downarrow 0}\limsup_{x\to\infty}\frac{\overline{G}((1-\varepsilon)x)}{\overline{G}(x)} = 1. \tag{23}$$

In particular, any regularly varying at infinity distribution satisfies the latter relation. Any intermediate regularly varying distribution is long-tailed and dominated varying; in particular, it belongs to the class $\mathcal{S}^*$, provided its mean is finite.

Theorem 7. *Assume that a counting random variable τ is independent of $\{\xi_n\}$. Let $F \in \mathcal{S}^*$, $\mathbf{E}\xi > 0$ and*

$$\overline{F}(x) = \mathrm{O}(\mathbf{P}\{\tau > x\}) \qquad \textit{as } x \to \infty. \tag{24}$$

If the distribution of τ is intermediate regularly varying, then

$$\mathbf{P}\{S_\tau > x\} \sim \mathbf{P}\{M_\tau > x\} \sim \mathbf{E}\tau\overline{F}(x) + \mathbf{P}\{\tau > x/\mathbf{E}\xi\} \qquad \textit{as } x \to \infty. \tag{25}$$

We strongly believe that the statement of the theorem stays valid in a more general setting where the distribution of τ is assumed to be *square root insensitive*, that is, $\mathbf{P}\{\tau > x \pm \sqrt{x}\} \sim \mathbf{P}\{\tau > x\}$, and the variance of ξ is finite. Probably, some further minor regularity assumptions are required. For example, the Weibull distribution $\overline{F}(x) = \mathrm{e}^{-x^\beta}$ with parameter $\beta < 1/2$ is square root insensitive. For a distribution which is *not* square root insensitive, the asymptotics are different and more complicated.

Proof of Theorem 7. Since the distribution of τ is intermediate regularly varying (23), for any fixed $\delta > 0$, we can choose $a < \mathbf{E}\xi$ and $c > \mathbf{E}\xi$ sufficiently close to $\mathbf{E}\xi$ such that

$$1 - \delta/2 \le \liminf_{x\to\infty}\frac{\mathbf{P}\{a\tau > x\}}{\mathbf{P}\{\tau > x/\mathbf{E}\xi\}} \le \limsup_{x\to\infty}\frac{\mathbf{P}\{c\tau > x\}}{\mathbf{P}\{\tau > x/\mathbf{E}\xi\}} \le 1 + \delta/2.$$

Then, due to $S_\tau \le M_\tau$, it is sufficient to prove the lower bound for the sum,

$$\mathbf{P}\{S_\tau > x\} \ge \big(\mathbf{E}\tau + \mathrm{o}(1)\big)\overline{F}(x) + \big(1 + \mathrm{o}(1)\big)\mathbf{P}\{\tau > x/a\}, \tag{26}$$

and the upper bound for the maximum,

$$\mathbf{P}\{M_\tau > x\} \le \big(\mathbf{E}\tau + \mathrm{o}(1)\big)\overline{F}(x) + \big(1 + \mathrm{o}(1)\big)\mathbf{P}\{\tau > x/c\} \qquad \text{as } x \to \infty. \tag{27}$$

We have

$$\mathbf{P}\{S_\tau > x\} = \mathbf{P}\{S_\tau > x, \tau \le x/a\} + \mathbf{P}\{S_\tau > x, \tau > x/a\}.$$

Since $a < \mathbf{E}\xi$, $\mathbf{P}\{S_\tau > x | \tau > x/a\} \to 1$ as $x \to \infty$, by the law of large numbers. The standard arguments now lead to (26).

To prove the upper bound, we use a representation similar to (17) (see the previous proof):

$$\begin{aligned}\mathbf{P}\{M_\tau > x\} &= \mathbf{P}\{M_\tau > x, \tau \le N(x)\} + \mathbf{P}\{M_\tau > x, \tau \in (N(x), x/c]\} + \mathbf{P}\{M_\tau > x, c\tau > x\} \\ &\equiv P_1 + P_2 + P_3.\end{aligned}$$

The first summand, P_1, can be treated as before. The second summand, P_2, can be estimated as follows: if condition (24) holds, then, by estimate (19),

$$P_2 \le K K_2 \int_{N(x)\mathbf{E}\xi}^{bx/c} \mathbf{P}\{\tau > x - y\}\mathbf{P}\{\tau > y\}\,\mathrm{d}y$$

for some K_2. Since the distribution of τ is intermediate regularly varying and, therefore, belongs to $\mathcal{S}^*$, we have

$$P_2 = \mathrm{o}(\mathbf{P}\{\tau > x\}).$$

Also taking into account the fact that $P_3 \le \mathbf{P}\{c\tau > x\}$, we finally get

$$\mathbf{P}\{M_\tau > x\} \le \bigl(\mathbf{E}\tau + \mathrm{o}(1)\bigr)\overline{F}(x) + \mathbf{P}\{\tau > x/c\} + \mathrm{o}(\mathbf{P}\{\tau > x\}) \qquad \text{as } x \to \infty.$$

Since the distribution of τ is (in particular) dominated varying, $\mathbf{P}\{\tau > x\} = \mathrm{O}(\mathbf{P}\{\tau > x/c\})$. Therefore, (27) is proved and the conclusion of Theorem 7 follows. □

Theorem 8. *Assume that a counting random variable τ is independent of $\{\xi_n\}$. Let $\mathbf{E}\xi > 0$ and let τ have an intermediate regularly varying distribution. If the distribution F is long-tailed and dominated varying, then (25) holds.*

A particular corollary is that if both ξ and τ have regularly varying tail distributions, then asymptotics (25) hold; this result was proven in [41], Theorems 1.3 and 1.4, for positive ξ and in [4], Section 7.1, for signed ξ. Also, Theorems 7 and 8 generalize and improve [1], Theorem 1.3.

Proof of Theorem 8. This follows along the lines of the previous proof with only the term P_2 needing a different estimate. From the bound (19), we get

$$[P_2 \le K\overline{F}(x - bx/c) \int_{N(x)\mathbf{E}\xi}^{bx/c} \mathbf{P}\{c\tau > y\}\,\mathrm{d}y.$$

Since F is dominated varying, $\overline{F}(x - bx/c) = \mathrm{O}(\overline{F}(x))$ as $x \to \infty$. Therefore, $P_2 = \mathrm{o}(\overline{F}(x))$ and the proof is complete. □

6. Applications to branching processes

A Galton–Watson process is a stochastic process $\{X_n\}$ which evolves according to the recurrence formula $X_0 = 1$ and

$$X_{n+1} = \sum_{j=1}^{X_n} \xi_j^{(n+1)},$$

where $\{\xi_j^{(n)}\}$ is a family of independent identically distributed non-negative integer-valued random variables with a finite mean and such that their common distribution does not depend on n. Here, X_n is the number of items in the nth generation. Taking into account the fact that any intermediate regularly varying distribution with finite mean belongs to the class $\mathcal{S}^*$, we obtain the following application of Theorem 7 to the branching process.

Corollary 2. *Let the common distribution of the* ξ*'s be intermediate regularly varying. Then, as* $x \to \infty$,

$$\mathbf{P}\{X_2 > x\} \sim \mathbf{E}\xi\mathbf{P}\{\xi > x\} + \mathbf{P}\{\xi > x/\mathbf{E}\xi\}.$$

In particular, if the branching process is *critical*, that is, if $\mathbf{E}\xi = 1$, then

$$\mathbf{P}\{X_2 > x\} \sim 2\mathbf{P}\{\xi > x\} \qquad \text{as } x \to \infty.$$

More generally, by induction arguments, the tail of the distribution of the number of items in the nth generation is asymptotically equivalent to $n\mathbf{P}\{\xi > x\}$. A similar result (for critical processes) was obtained in [42], Theorem 2, in the case of regularly varying distribution of ξ's and for possibly growing n.

7. Equivalences in the case where a counting random variable τ may depend on ξ's

We continue to assume that the random variables $\{\xi_n\}$ are independent and identically distributed. For any family Ξ of random variables, denote by $\sigma(\Xi)$ the σ-algebra generated by Ξ. Traditionally, a counting random variable τ is called a *stopping time* for a sequence $\{\xi_n\}$ if $\{\tau \le n\} \in \sigma(\xi_1, \dots, \xi_n)$ for all n.

We say that a counting random variable τ *does not depend on the future of the sequence* $\{\xi_n\}$ if the family $(\xi_1, \dots, \xi_n, \mathbf{I}\{\tau \le n\})$ is independent of $(\xi_j, j \ge n+1)$ for all n. Dependence of this type goes back to [23], wherein Wald's identity is proved under the condition that the event $\{\tau \le n\}$ is independent of ξ_j for all $n \ge 1$ and $j \ge n+1$.

Provided we have independence of ξ's, any stopping time τ does not depend on the future of the sequence $\{\xi_n\}$. If a counting random variable τ is independent of the ξ's, then it does not depend on the future of the sequence $\{\xi_n\}$.

Let $\mathcal{F}_n$ be a filtration of σ-algebras. A counting random variable τ is called a stopping time for this filtration if $\{\tau \le n\} \in \mathcal{F}_n$ for all n. In this terminology, τ is a stopping time for a sequence $\{\xi_n\}$ if and only if τ is a stopping time for the natural filtration $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$.

Consider a special filtration $\mathcal{F}_n = \sigma(\xi_k, \mathbf{I}\{\tau = k\}, k \le n)$. Then τ is a stopping time for this filtration. In addition, τ does not depend on the future of the sequence $\{\xi_n\}$ if and only if the family $(\xi_j, j \ge n+1)$ is independent of $\mathcal{F}_n$ for all n.

We start with a result for a bounded counting stopping time (recall that a random variable is *bounded* if its distribution has a bounded support).

Theorem 9. *Let ξ have a subexponential distribution F on $\mathbf{R}$ (we do not assume finite mean) and let the counting variable τ not depend on the future. If τ is bounded, then $\mathbf{P}\{S_\tau > x\} \sim \mathbf{E}\tau \overline{F}(x)$ as $x \to \infty$.*

Similar result for M_τ may be found in [18], Theorem 1. Note that one cannot expect the latter asymptotics to hold for any τ with unbounded support, which may depend on $\{\xi_n\}$ – even for a stopping time. Indeed, consider a stopping time $\tau = \min\{n : S_n \le 0\}$. If $\mathbf{E}\xi < 0$, then $\mathbf{E}\tau$ is finite but $\mathbf{P}\{S_\tau > x\} = 0$ for any $x > 0$.

Proof of Theorem 9. We adopt the corresponding proof from [20] where a stopping time and regularly varying tails were considered. Let N be such that $\mathbf{P}\{\tau \le N\} = 1$. The starting point of the proof is the following representation:

$$\begin{aligned}\mathbf{P}\{S_\tau > x\} &= \sum_{n=1}^{N} (\mathbf{P}\{S_n > x, \tau \ge n\} - \mathbf{P}\{S_n > x, \tau \ge n+1\}) \\ &= \mathbf{P}\{S_1 > x, \tau \ge 1\} + \sum_{n=2}^{N} (\mathbf{P}\{S_n > x, \tau \ge n\} - \mathbf{P}\{S_{n-1} > x, \tau \ge n\}).\end{aligned}$$

Therefore,

$$\mathbf{P}\{S_\tau > x\} = \overline{F}(x) + \sum_{n=2}^{N} (\mathbf{P}\{S_{n-1} \le x, S_n > x, \tau \ge n\} - \mathbf{P}\{S_{n-1} > x, S_n \le x, \tau \ge n\}).$$

It now suffices to show that, for each n,

$$P_1 \equiv \mathbf{P}\{S_{n-1} \le x, S_n > x, \tau \ge n\} \sim \overline{F}(x)\mathbf{P}\{\tau \ge n\} \tag{28}$$

and

$$P_2 \equiv \mathbf{P}\{S_{n-1} > x, S_n \le x, \tau \ge n\} = \mathrm{o}(\overline{F}(x)). \tag{29}$$

The subexponentiality of F implies that, for each $n \ge 2$,

$$\mathbf{P}\{S_n > x\} \sim n\overline{F}(x) \qquad \text{as } x \to \infty. \tag{30}$$

In particular, there exists c such that, for all $n = 2, \dots, N$,

$$\mathbf{P}\{S_n > x\} \le c\overline{F}(x) \qquad \text{for all } x. \tag{31}$$

The subexponentiality of F also implies, for any $A(x) \to \infty$ such that $\overline{F}(x + A(x)) \sim \overline{F}(x)$, that

$$\int_{A(x)}^{x+A(x)} \overline{F}(x-y)F(\mathrm{d}y) = \mathrm{o}(\overline{F}(x)) \qquad \text{as } x \to \infty. \tag{32}$$

To establish (28), we first note that $\{\tau \ge n\} = \overline{\{\tau \le n-1\}}$ and thus $\sigma(S_{n-1}, \mathbf{I}\{\tau \ge n\})$ is independent of ξ_n since τ does not depend on the future. This implies that

$$\begin{aligned} P_1 &= \int_0^\infty \mathbf{P}\{S_{n-1} \in (x-y, x], \xi_n \in \mathrm{d}y, \tau \ge n\} \\ &= \int_0^\infty \mathbf{P}\{S_{n-1} \in (x-y, x], \tau \ge n\}F(\mathrm{d}y). \end{aligned}$$

We use the following decomposition, $A > 0$:

$$\begin{aligned} P_1 &= \left(\int_0^A + \int_A^{x+A} + \int_{x+A}^\infty\right)\mathbf{P}\{S_{n-1} \in (x-y, x], \tau \ge n\}F(\mathrm{d}y) \\ &\equiv I_1 + I_2 + I_3. \end{aligned} \tag{33}$$

By (30) and by the long-tailedness of F, for any fixed A,

$$I_1 \le \mathbf{P}\{S_{n-1} \in (x-A, x]\} = \mathrm{o}(\overline{F}(x)) \qquad \text{as } x \to \infty. \tag{34}$$

By (31) and (32), we get, for $A = A(x) \to \infty$,

$$\begin{aligned} I_2 &\le \int_A^{x+A} \mathbf{P}\{S_{n-1} > x-y\}F(\mathrm{d}y) \\ &\le c\int_A^{x+A} \overline{F}(x-y)F(\mathrm{d}y) = \mathrm{o}(\overline{F}(x)) \qquad \text{as } x \to \infty. \end{aligned} \tag{35}$$

Uniformly in $y \ge x + A(x)$, $\mathbf{P}\{S_{n-1} \in (x-y, x], \tau \ge n\} \to \mathbf{P}\{\tau \ge n\}$ as $x \to \infty$. Thus,

$$I_3 \sim \mathbf{P}\{\tau \ge n\}\overline{F}\big(x + A(x)\big) \sim \mathbf{P}\{\tau \ge n\}\overline{F}(x) \qquad \text{as } x \to \infty. \tag{36}$$

Substituting (34)–(36) into (33), we get (28).

To prove (29), we note that

$$P_2 \le \mathbf{P}\{S_{n-1} \in (x, x+A]\} + \mathbf{P}\{S_{n-1} > x+A\}F(-A).$$

As in (34), the first term on the right-hand side is of order $\mathrm{o}(\overline{F}(x))$. Due to (31), the second term is not greater than $c\overline{F}(x)F(-A)$, where $F(-A)$ can be made as small as we please by the choice of sufficiently large A. The proof is thus complete. □

Here is our general result for a counting random variable with, possibly, unbounded support.

Theorem 10. *Let* $\mathbf{E}|\xi| < \infty$ *and let a counting variable* τ *not depend on the future. Assume that* $F \in \mathcal{S}^*$ *and that there exists an increasing function* $h(x)$ *such that*

$$\overline{F}\big(x \pm h(x)\big) \sim \overline{F}(x) \quad \textit{and} \quad \mathbf{P}\{\tau > h(x)\} = \mathrm{o}(\overline{F}(x)) \qquad \textit{as } x \to \infty. \tag{37}$$

Then $\mathbf{P}\{S_\tau > x\} \sim \mathbf{E}\tau \overline{F}(x)$ *as* $x \to \infty$.

Proof. This follows from Lemmas 1 and 2. Condition (37) is stronger than condition (4). At the end of this section, we provide an example of a stopping time which shows that condition (37) is essential and cannot be weakened to (4). □

Lemma 1. *Let* $\mathbf{E}\xi > 0$ *and let a counting variable* τ *not depend on the future. If* F *is long-tailed, then*

$$\liminf_{x\to\infty} \frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \ge \mathbf{E}\tau.$$

If, in addition, $F \in \mathcal{S}^*$ *and condition* (37) *holds, then* $\mathbf{P}\{S_\tau > x\} \sim \mathbf{E}\tau \overline{F}(x)$ *as* $x \to \infty$.

Proof. Fix a positive integer N and a positive A. The following lower bound holds, for $x > A$:

$$\begin{aligned}
\mathbf{P}\{S_\tau > x\} &\ge \sum_{j=1}^{N} \mathbf{P}\{S_1, \dots, S_{j-1} \in [-A, A], \xi_j > x + 2A, S_\tau > x, \tau \ge j\} \\
&\ge \sum_{j=1}^{N} \mathbf{P}\Big\{S_1, \dots, S_{j-1} \in [-A, A], \xi_j > x + 2A, \min_{i>j}(S_i - S_j) > -A, \tau \ge j\Big\}.
\end{aligned}$$

Since $\{\tau \ge j\} = \overline{\{\tau \le j-1\}}$ and since τ does not depend on the future,

$$\begin{aligned}
\mathbf{P}\{S_\tau > x\} &\ge \sum_{j=1}^{N} \mathbf{P}\{S_1, \dots, S_{j-1} \in [-A, A], \tau \ge j\} \mathbf{P}\Big\{\xi_j > x + 2A, \min_{i>j}(S_i - S_j) > -A\Big\} \\
&= \overline{F}(x + 2A) \mathbf{P}\Big\{\min_{i\ge1} S_i > -A\Big\} \sum_{j=1}^{N} \mathbf{P}\{S_1, \dots, S_{j-1} \in [-A, A], \tau \ge j\}.
\end{aligned}$$

By the long-tailedness of F,

$$\liminf_{x\to\infty} \frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \ge \mathbf{P}\Big\{\min_{i\ge1} S_i > -A\Big\} \sum_{j=1}^{N} \mathbf{P}\{S_1, \dots, S_{j-1} \in [-A, A], \tau \ge j\}.$$

Since the mean of ξ is positive, $\mathbf{P}\{\min_{i\geq 1} S_i > -A\} \to 1$ as $A \to \infty$. Hence, for any N,

$$\liminf_{x\to\infty} \frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \geq \sum_{j=1}^{N} \mathbf{P}\{\tau \geq j\}.$$

Now, letting $N \to \infty$ completes the proof of the lower bound.

The upper bound,

$$\limsup_{x\to\infty} \frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \leq \mathbf{E}\tau,$$

follows from [19], Corollary 3, which states that, under the conditions $F \in \mathcal{S}^*$ and (37), $\mathbf{P}\{M_\tau > x\} \sim \overline{F}(x)\mathbf{E}\tau$ as $x \to \infty$. The proof is thus complete. □

Lemma 2. *Let* $\mathbf{E}\xi \leq 0$ *and let a counting variable* τ *not depend on the future. If* $F \in \mathcal{S}^*$, *then*

$$\limsup_{x\to\infty} \frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \leq \mathbf{E}\tau.$$

Under the additional condition (37), $\mathbf{P}\{S_\tau > x\} \sim \mathbf{E}\tau\overline{F}(x)$ *as* $x \to \infty$.

Proof. The upper bound follows from [19], Corollary 3, in the same way as the upper bound in the previous proof. To obtain the lower bound, take any positive ε and consider a random walk $\widetilde{S}_n = S_n + n(|\mathbf{E}\xi| + \varepsilon)$ with a positive drift. We have

$$\mathbf{P}\{S_\tau > x\} = \mathbf{P}\{\widetilde{S}_\tau > x + (|\mathbf{E}\xi| + \varepsilon)\tau\} \geq \mathbf{P}\{\widetilde{S}_\tau > x + (|\mathbf{E}\xi| + \varepsilon)h(x)\} - \mathbf{P}\{\tau > h(x)\}.$$

Here the last term on the right-hand side is $\mathrm{o}(\overline{F}(x))$ and, by Lemma 1, the first term is equivalent to $\mathbf{E}\tau\overline{F}(x + (|\mathbf{E}\xi| + \varepsilon)h(x)) \sim \mathbf{E}\tau\overline{F}(x)$ as $x \to \infty$. This completes the proof. □

For intermediate regularly varying tail distributions, Theorem 10 implies the following result.

Corollary 3. *Let* $\mathbf{E}|\xi| < \infty$ *and let a counting variable* τ *not depend on the future. Assume that* F *is an intermediate regularly varying distribution and that*

$$\mathbf{P}\{\tau > x\} = \mathrm{o}(\overline{F}(x)) \qquad \text{as } x \to \infty. \tag{38}$$

Then $\mathbf{P}\{S_\tau > x\} \sim \mathbf{E}\tau\overline{F}(x)$ *as* $x \to \infty$.

This corollary generalizes the corresponding result, [21], Theorem 1, where a regularly varying F and a stopping time τ were considered. In [6], Theorem 2, an upper bound for the tail distribution of S_τ was obtained, assuming that the tail distributions of ξ_1 and τ are both bounded from above by the same dominated varying distribution.

Proof of Corollary 3. From condition (38), for any $\varepsilon > 0$,

$$\mathbf{P}\{\tau > \varepsilon x\} = \mathrm{o}(\overline{F}(\varepsilon x)) = \mathrm{o}(\overline{F}(x)) \qquad \text{as } x \to \infty$$

since F is intermediate regularly varying. Thus, there exists an increasing function $h(x) = o(x)$ such that $\mathbf{P}\{\tau > h(x)\} = o(\overline{F}(x))$ as $x \to \infty$. Again by the intermediate regular variation of F, for any $h(x) = o(x)$, $\overline{F}(x \pm h(x)) \sim \overline{F}(x)$. So, condition (37) is fulfilled and we can conclude the desired asymptotics from Theorem 10.

We conclude with an example of a stopping time τ showing that condition (37) is essential for the conclusion of Theorem 10. Consider a distribution F on $[1, \infty)$. Take an increasing function $H(x) : \mathbf{R} \to \mathbf{Z}^+$ such that $H(x) < x/2$. The counting random variable $\tau = H(2\xi_1) + 1$ is a stopping time. On the event $\xi_1 > x - H(x)$, we have $\tau \ge H(2(x - H(x))) + 1 \ge H(x) + 1$. Hence,

$$\mathbf{P}\{S_\tau > x\} \ge \mathbf{P}\{\xi_1 > x - H(x), \xi_2 + \cdots + \xi_\tau \ge H(x)\} = \mathbf{P}\{\xi_1 > x - H(x)\},$$

due to $\xi \ge 1$. For a Weibull-type distribution, namely $\overline{F}(x) = e^{-x^\beta}$, $0 < \beta < 1$, $x \ge 1$, we can choose $H(x)$ in such a way that $H(x) = o(x)$ and $H(x)/x^{1-\beta} \to \infty$ as $x \to \infty$. Condition (4) then holds, but asymptotics (3) do not because $\overline{F}(x - H(x))/\overline{F}(x) \to \infty$ and

$$\frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \to \infty.$$

In this example, there is no function $h(x)$ such that condition (37) holds. Indeed, if $\overline{F}(x - h(x)) \sim \overline{F}(x)$, then $h(x) = o(x^{1-\beta})$ and $H^{-1}(h(x) - 1) = o(x)$, which implies that

$$\mathbf{P}\{\tau > h(x)\}/\overline{F}(x) = \mathbf{P}\{H(2\xi) > h(x) - 1\}/\overline{F}(x) \to \infty \qquad \text{as } x \to \infty. \qquad \square$$

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References

[1] Aleškevičienė, A., Leipus, R. and Šiaulys, J. (2008). Tail behavior of random sums under consistent variation with applications to the compound renewal risk model. *Extremes* **11** 261–279. MR2429907
[2] Asmussen, S. (2003). *Applied Probability and Queues*, 2nd ed. New York: Springer. MR1978607
[3] Athreya, K.B. and Ney, P.E. (1972). *Branching Processes*. New York: Springer. MR0373040
[4] Borovkov, A.A. and Borovkov, K.A. (2008). *Asymptotic Analysis of Random Walks: Heavy-Tailed Distributions*. Cambridge: Cambridge Univ. Press. MR2424161
[5] Borovkov, A.A. and Mogul'skii, A.A. (2006). Integro-local and integral theorems for sums of random variables with semiexponential distributions. *Siberian Math. J.* **47** 990–1026. MR2302841
[6] Borovkov, A.A. and Utev, S.A. (1993). Estimates for distributions of sums stopped at a Markov time. *Theory Probab. Appl.* **38** 214–225. MR1317979

[7] Cline, D.B.H. (1994). Intermediate regular and π variation. *Proc. London Math. Soc.* **68** 594–616. MR1262310

[8] Cline, D.B.H. and Hsing, T. (1998). Large deviation probabilities for sums of random variables with heavy or subexponential tails. Technical report, Texas A&M University.

[9] Daley, D.J., Omey, E. and Vesilo, R. (2007). The tail behaviour of a random sum of subexponential random variables and vectors. *Extremes* **10** 21–39. MR2397550

[10] Denisov, D., Dieker, A.B. and Shneer, V. (2008). Large deviations for random walks under subexponentiality: the big-jump domain. *Ann. Probab.* **36** 1946–1991. MR2440928

[11] Denisov, D., Foss, S. and Korshunov, D. (2004). Tail asymptotics for the supremum of a random walk when the mean is not finite. *Queueing Systems* **46** 15–33. MR2072274

[12] Denisov, D., Foss, S. and Korshunov, D. (2008). On lower limits and equivalences for distribution tails of randomly stopped sums. *Bernoulli* **14** 391–404. MR2544093

[13] Denisov, D., Foss, S. and Korshunov, D. (2008). Lower limits for distribution tails of randomly stopped sums. *Theory Probab. Appl.* **52** 690–699.

[14] Embrechts, P., Goldie, C.M. and Veraverbeke, N. (1979). Subexponentiality and infinite divisibility. *Z. Wahrsch. Verw. Gebiete* **49** 335–347. MR0547833

[15] Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997). *Modelling Extremal Events for Insurance and Finance*. Berlin: Springer. MR1458613

[16] Faÿ, G., González-Arévalo, B., Mikosch, T. and Samordnitsky, G. (2006). Modeling teletraffic arrivals by a Poisson cluster process. *Queueing Systems* **54** 121–140. MR2268057

[17] Foss, S. and Korshunov, D. (2007). Lower limits and equivalences for convolution tails. *Ann. Probab.* **35** 366–383. MR2303954

[18] Foss, S., Palmowski, Z. and Zachary S. (2005). The probability of exceeding a high boundary on a random time interval for a heavy-tailed random walk. *Ann. Appl. Probab.* **15** 1936–1957. MR2152249

[19] Foss, S. and Zachary, S. (2003). The maximum on a random time interval of a random walk with long-tailed increments and negative drift. *Ann. Appl. Probab.* **13** 37–53. MR1951993

[20] Greenwood, P. (1973). Asymptotics of randomly stopped sequences with independent increments. *Ann. Probab.* **1** 317–321. MR0350846

[21] Greenwood, P. and Monroe, I. (1977). Random stopping preserves regular variation of process distributions. *Ann. Probab.* **5** 42–51. MR0426139

[22] Klüppelberg, C. (1988). Subexponential distributions and integrated tails. *J. Appl. Probab.* **25** 132–141. MR0929511

[23] Kolmogorov, A.N. and Prokhorov, Y.V. (1949). On sums of a random number of random terms (Russian). *Uspehi Matem. Nauk* **IV** 168–172. MR0031215

[24] Korshunov, D. (2002). Large-deviation probabilities for maxima of sums of independent random variables with negative mean and subexponential distribution. *Theory Probab. Appl.* **46** 355–366. MR1968696

[25] Linnik, Y.V. (1961). On the probability of large deviations for the sums of independent variables. In *Proc. Fourth Berkeley Symp. Math. Statist. Probability, Univ. Calif. Press* **2** 289–306. Berkeley, CA: Univ. California Press. MR0137142

[26] Mikosch, T. and Nagaev, A.V. (1998). Large deviations of heavy-tailed sums with applications in insurance. *Extremes* **1** 81–110. MR1652936

[27] Nagaev, A.V. (1969). Limit theorems that take into account large deviations when Cramer's condition is violated (Russian). *Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk* **13** 17–22. MR0282396

[28] Nagaev, A.V. (1969). Integral limit theorems taking large deviations into account when Cramér's condition does not hold I, II. *Theory Probab. Appl.* **14** 51–64, 193–208. MR0247651

[29] Nagaev, S.V. (1960). Local limit theorems for large deviations (Russian). *Theory Probab. Appl.* **5** 259–261.

[30] Nagaev, S.V. (1962). An integral limit theorem for large deviations. (Russian). *Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk* **1962** 37–43. MR0150839
[31] Nagaev, S.V. (1973). Large deviations for sums of independent random variables. In *Trans. Sixth Prague Conf. (1971). Inform. Theory, Statist. Decision Functions, Random Processes* 657–674. Prague: Academia. MR0362460
[32] Nagaev, S.V. (1979). Large deviations of sums of independent random variables. *Ann. Probab.* **7** 745–789. MR0542129
[33] Nagaev, S.V. (1981). On asymptotic behaviour of probabilities of one-sided large deviations. *Theory Probab. Appl.* **26** 362–366. MR0616627
[34] Ng, K.W., Tang, Q.H. and Yang, H. (2002). Maxima of sums of heavy-tailed random variables. *ASTIN Bull.* **32** 43–55. MR1928012
[35] Pitman, E.J.G. (1980). Subexponential distribution functions. *J. Austral. Math. Soc. Ser. A* **29** 337–347. MR0569522
[36] Rosovskii, L.V. (1990). Probabilities of large deviations of sums of independent random variables with common distribution function in the domain of attraction of the normal law. *Theory Probab. Appl.* **34** 625–644.
[37] Rosovskii, L.V. (1993). Probabilities of large deviations on the whole axis. *Theory Probab. Appl.* **38** 53–79. MR1317784
[38] Rosovskii, L.V. (1997). Probabilities of large deviations for sums of independent random variables with a common distribution function from the domain of attraction of an asymmetric stable law. *Theory Probab. Appl.* **42** 454–482. MR1618791
[39] Rudin, W. (1973). Limits of ratios of tails of measures. *Ann. Probab.* **1** 982–994. MR0358919
[40] Shneer, V. (2004). Estimates for distributions of sums of random variables with subexponential distributions. *Siberian Math. J.* **45** 1143–1158. MR2123303
[41] Stam, A.J. (1973). Regular variation of the tail of a subordinated probability distribution. *Adv. in Appl. Probab.* **5** 308–327. MR0339353
[42] Wachtel, V. (2008). Limit theorems for probabilities of large deviations of a critical Galton–Watson process having power tails. *Theory Probab. Appl.* **52** 674–688.

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