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# On rate of convergence in non-central limit theorems

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The main result of this paper is the rate of convergence to Hermite-type distributions in non-central limit theorems. To the best of our knowledge, this is the first result in the literature on rates of convergence of functionals of random fields to Hermite-type distributions with ranks greater than 2. The results were obtained under rather general assumptions on the spectral densities of random fields. These assumptions are even weaker than in the known convergence results for the case of Rosenblatt distributions. Additionally, Lévy concentration functions for Hermite-type distributions were investigated.

*Keywords:* Rate of convergence, Non-central limit theorems, Random field, Long-range dependence, Hermite-type distribution.

## 1. Introduction

This research will focus on the rate of convergence of local functionals of real-valued homogeneous random fields with long-range dependence. Non-linear integral functionals on bounded sets of  $\mathbb{R}^d$  are studied. These functionals are important statistical tools in various fields of application, for example, image analysis, cosmology, finance, and geology. It was shown in [12], [39] and [40] that these functionals can produce non-Gaussian limits and require normalizing coefficients different from those in central limit theorems.

Since many modern statistical models are now designed to deal with non-Gaussian data, non-central limit theory is gaining more and more popularity. Some novel results using different models and asymptotic distributions were obtained during the past few years, see [1], [4], [7], [25], [34], [39] and references therein. Despite such developments of the asymptotic theory, only a few of the existing studies are concerned with rate of convergence, especially in the non-central case.

There are two popular approaches to investigate the rate of convergence in the literature: the direct probability approach [1], [20], and the Stein-Malliavin method introduced in [28].

As the name suggests, the Stein-Malliavin method combines Malliavin calculus and Stein's method. The main strength of this approach is that it does not impose any restrictions on the moments of order higher than four (see, for example, [28]) and even three in some cases (see [26]). For a more detailed description of the method, the reader is referred to [28]. At this moment, the Stein-Malliavin approach is well developed for stochastic processes. However, many problems concerning non-central limit theorems for random fields remain unsolved. The full list of the already solved problems can be found in [42].

One of the first papers which obtained the rate of convergence in the central limit theorem using the Stein-Malliavin approach was [28]. The case of stochastic processes was considered. Further refinement of these results can be found in [29], where optimal Berry-Esseen bounds for the normal approximation of functionals of Gaussian fields are shown. However, it is known that numerous functionals do not converge to the Gaussian distribution. The conditions to obtain the Gaussian asymptotics can be found in so-called Breuer-Major theorems, see [2] and [14]. These results are based on the method of cumulants and diagram formulae. Using the Stein-Malliavin approach, [30] derived a version of a quantitative Breuer-Major theorem that contains a stronger version of the results in [2] and [14]. The rate of convergence for Wasserstein topology was found and an upper bound for the Kolmogorov distance was given as a relationship between the Kolmogorov and Wasserstein distances. In [19] the authors directly derived the upper-bound for the Kolmogorov distance in the same quantitative Breuer-Major theorem as in [30] and showed that this bound is better than the known bounds in the literature, since it converges to zero faster. The results described above are the most general results currently known concerning the rate of convergence in the central limit theorem using the Stein-Malliavin approach.

Related to [30] is the work [36] where, using the same arguments, the author found the rate of convergence for the central limit theorem of sojourn times of Gaussian fields. Similar results for the Kolmogorov distance were obtained in [19].

Concerning non-central limit theorems, only partial results have been found. It is known from [9], [14] and [39] that, depending on the value of the Hurst parameter, functionals of fractional Brownian motion can converge either to the standard Gaussian distribution or a Hermite-type distribution. This idea was used in [7] and [8] to obtain the first rates of convergence in non-central limit theorems using the Stein-Malliavin method. Similar to the case of central limit theorems, these results were obtained for stochastic processes. In [8] fractional Brownian motion was considered, and rates of convergence for both Gaussian and Hermite-type asymptotic distributions were given. Furthermore all the results of [8] were refined in [7] for the case of the fractional Brownian sheet as an initial random element. For the case of random fields with long-range dependence, [8] is the only known work that uses the Stein-Malliavin method to provide the rate of convergence.

Separately stands [3]. This work followed a new approach based only on Stein's method

without Malliavin calculus. The authors worked with Wasserstein-2 metrics and showed the rate of convergence of quadratic functionals of i.i.d. Gaussian variables. It is one of the convergence results which cannot be obtained using the regular Stein-Malliavin method [3]. However, we are not aware of extensions of these results to the multi-dimensional and non-Gaussian cases.

The classical probability approach employs direct probability methods to find the rate of convergence. Its main advantage over the other methods is that it directly uses the correlation functions and spectral densities of the involved random fields. Therefore, asymptotic results can be explicitly obtained for wide classes of random fields using slowly varying functions. Using this approach, the first rate of convergence in the central limit theorem for Gaussian fields was obtained in [20]. In the following years, some other results were obtained, but all of them studied the convergence to the Gaussian distribution.

As for convergence to non-Gaussian distributions, the only known result using the classical probability approach is [1]. For functionals of Hermite rank-2 polynomials of long-range dependent Gaussian fields, it investigated the rate of convergence in the Kolmogorov metric of these functionals to the Rosenblatt-type distribution. In this paper, we generalize these results to Hermite-type distributions.

The main result is given in Theorem 5. It establishes an upper bound of the form

$$\rho(K_r/C(r), X_\kappa(\Delta)) = o(r^{-\varkappa}), \quad r \rightarrow \infty,$$

for the Kolmogorov distance  $\rho(\cdot, \cdot)$  between non-linear functionals  $K_r$  of random fields and Hermite-type random variables  $X_\kappa(\Delta)$  from the  $\kappa$ th order Wiener chaos. Explicit expressions for the normalizing factor  $C(r)$  and the power  $\varkappa$  in the rate of convergence are provided.

It is worth mentioning that these results are obtained under more natural and much weaker assumptions on the spectral densities than those in [1]. These quite general assumptions allow to consider various new asymptotic scenarios even for the Rosenblatt-type case in [1].

It is also worth mentioning that in the known Stein-Malliavin results, the rate of convergence was obtained only for a leading term or a fixed number of chaoses in the Wiener chaos expansion. However, while other expansion terms in higher level Wiener chaoses do not change the asymptotic distribution, they can substantially contribute to the rate of convergence. The method proposed in this manuscript takes into account all terms in the Wiener chaos expansion to derive rates of convergence.

It is well known, see [9, 27, 37], that the probability distributions of Hermite-type random variables are absolutely continuous. In this paper we investigate some fine properties of these distributions which are required to derive rates of convergence. Specifically, we discuss the cases of bounded probability density functions of Hermite-type random variables. Using the method proposed in [31], we derive the anti-concentration inequality that can be applied to estimate the Lévy concentration function of Hermite-type random variables.

The article is organized as follows. In Section 2 we recall some basic definitions and formulae of the spectral theory of random fields. The main assumptions and auxiliary

results are stated in Section 3. In Section 4 we discuss some fine properties of Hermite-type distributions. Section 5 provides the results concerning the rate of convergence. Some conclusions are drawn in Section 6.

## 2. Notations

In what follows  $|\cdot|$  and  $\|\cdot\|$  denote the Lebesgue measure and the Euclidean distance in  $\mathbb{R}^d$ , respectively. We use the symbols  $C$  and  $\delta$  to denote constants which are not important for our exposition. Moreover, the same symbol may be used for different constants appearing in the same proof. It is assumed that all random variables are defined on a fixed probability space  $(\Omega, \mathcal{F}, P)$ .

We consider a measurable mean-square continuous zero-mean homogeneous isotropic real-valued Gaussian random field  $\eta(x)$ ,  $x \in \mathbb{R}^d$ , with the covariance function, see [18],

$$B(r) := \text{Cov}(\eta(x), \eta(y)) = \int_0^\infty Y_d(rz) d\Phi(z), \quad x, y \in \mathbb{R}^d, \quad (1)$$

where  $r := \|x - y\|$ ,  $\Phi(\cdot)$  is the isotropic spectral measure, the function  $Y_d(\cdot)$  is defined by

$$Y_d(z) := 2^{(d-2)/2} \Gamma\left(\frac{d}{2}\right) J_{(d-2)/2}(z) z^{(2-d)/2}, \quad z \geq 0, \quad (2)$$

and  $J_{(d-2)/2}(\cdot)$  is the Bessel function of the first kind of order  $(d-2)/2$ .

**Definition 1.** [18] The random field  $\eta(x)$ ,  $x \in \mathbb{R}^d$ , as defined above is said to possess an absolutely continuous spectrum if there exists a function  $f(\cdot)$  such that

$$\Phi(z) = 2\pi^{d/2} \Gamma^{-1}(d/2) \int_0^z u^{d-1} f(u) du, \quad z \geq 0, \quad u^{d-1} f(u) \in L_1(\mathbb{R}_+). \quad (3)$$

The function  $f(\cdot)$  is called the isotropic spectral density function of the field  $\eta$ . In this case, the field  $\eta$  with an absolutely continuous spectrum has the isonormal spectral representation

$$\eta(x) = \int_{\mathbb{R}^d} e^{i(\lambda, x)} \sqrt{f(\|\lambda\|)} W(d\lambda),$$

where  $W(\cdot)$  is the complex Gaussian random measure on  $\mathbb{R}^d$ .

Consider a Jordan-measurable bounded set  $\Delta \subset \mathbb{R}^d$  such that  $|\Delta| > 0$  and  $\Delta$  contains the origin in its interior. Let  $\Delta(r)$ ,  $r > 0$ , be the homothetic image of the set  $\Delta$ , with the centre of homothety at the origin and the coefficient  $r > 0$ , that is  $|\Delta(r)| = r^d |\Delta|$ .

Consider the uniform distribution on  $\Delta(r)$  with the probability density function (pdf)  $r^{-d} |\Delta|^{-1} \chi_{\Delta(r)}(x)$ ,  $x \in \mathbb{R}^d$ , where  $\chi_A(\cdot)$  is the indicator function of a set  $A$ .

**Definition 2.** Let  $U$  and  $V$  be two random vectors which are independent and uniformly distributed inside the set  $\Delta(r)$ . We denote by  $\psi_{\Delta(r)}(z)$ ,  $z \geq 0$ , the pdf of the distance  $\|U - V\|$  between  $U$  and  $V$ .

Note that  $\psi_{\Delta(r)}(z) = 0$  if  $z > \text{diam}\{\Delta(r)\}$ . Using the above notations, one can obtain the representation

$$\begin{aligned} \int_{\Delta(r)} \int_{\Delta(r)} \Upsilon(\|x - y\|) dx dy &= |\Delta|^2 r^{2d} \mathbb{E} \Upsilon(\|U - V\|) \\ &= |\Delta|^2 r^{2d} \int_0^{\text{diam}\{\Delta(r)\}} \Upsilon(z) \psi_{\Delta(r)}(z) dz, \end{aligned} \quad (4)$$

where  $\Upsilon(\cdot)$  is an integrable Borel function.

**Remark 1.** If  $\Delta(r)$  is the ball  $v(r) := \{x \in \mathbb{R}^d : \|x\| < r\}$ , then

$$\psi_{v(r)}(z) = d r^{-d} z^{d-1} I_{1-(z/2r)^2} \left( \frac{d+1}{2}, \frac{1}{2} \right), \quad 0 \leq z \leq 2r,$$

where

$$I_\mu(p, q) := \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^\mu u^{p-1} (1-u)^{q-1} du, \quad \mu \in (0, 1], \quad p > 0, \quad q > 0,$$

is the incomplete beta function, see [18].

Let  $H_k(u)$ ,  $k \geq 0$ ,  $u \in \mathbb{R}$ , be the Hermite polynomials, see [34]. The Hermite polynomials form a complete orthogonal system in the Hilbert space

$$L_2(\mathbb{R}, \phi(w) dw) = \left\{ G : \int_{\mathbb{R}} G^2(w) \phi(w) dw < \infty \right\}, \quad \phi(w) := \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}.$$

An arbitrary function  $G(w) \in L_2(\mathbb{R}, \phi(w) dw)$  admits the mean-square convergent expansion

$$G(w) = \sum_{j=0}^{\infty} \frac{C_j H_j(w)}{j!}, \quad C_j := \int_{\mathbb{R}} G(w) H_j(w) \phi(w) dw. \quad (5)$$

By Parseval's identity

$$\sum_{j=0}^{\infty} \frac{C_j^2}{j!} = \int_{\mathbb{R}} G^2(w) \phi(w) dw. \quad (6)$$

**Definition 3.** [39] Let  $G(\cdot) \in L_2(\mathbb{R}, \phi(w) dw)$  and assume there exists an integer  $\kappa \in \mathbb{N}$  such that  $C_j = 0$ , for all  $0 \leq j \leq \kappa - 1$ , but  $C_\kappa \neq 0$ . Then  $\kappa$  is called the Hermite rank of  $G(\cdot)$  and is denoted by  $H\text{rank } G$ .

**Remark 2.** Note, that  $E(H_m(\eta(x))) = 0$  and

$$E(H_{m_1}(\eta(x))H_{m_2}(\eta(y))) = \delta_{m_1}^{m_2} m_1! B^{m_1}(\|x - y\|), \quad x, y \in \mathbb{R}^d,$$

where  $\delta_{m_1}^{m_2}$  is the Kronecker delta function.

**Definition 4.** [5] A measurable function  $L : (0, \infty) \rightarrow (0, \infty)$  is said to be slowly varying at infinity if for all  $t > 0$ ,

$$\lim_{r \rightarrow \infty} \frac{L(rt)}{L(r)} = 1.$$

By the representation theorem [5, Theorem 1.3.1], there exists  $C > 0$  such that for all  $r \geq C$  the function  $L(\cdot)$  can be written in the form

$$L(r) = \exp \left( \zeta_1(r) + \int_C^r \frac{\zeta_2(u)}{u} du \right),$$

where  $\zeta_1(\cdot)$  and  $\zeta_2(\cdot)$  are such measurable and bounded functions that  $\zeta_2(r) \rightarrow 0$  and  $\zeta_1(r) \rightarrow C_0$  ( $|C_0| < \infty$ ), when  $r \rightarrow \infty$ .

**Remark 3.** By Proposition 1.3.6 in [5], if  $L(\cdot)$  varies slowly, then for an arbitrary  $\delta > 0$

$$r^\delta L(r) \rightarrow \infty, \quad \text{and} \quad r^{-\delta} L(r) \rightarrow 0 \quad \text{when} \quad r \rightarrow \infty. \quad (7)$$

**Definition 5.** [5] A measurable function  $g : (0, \infty) \rightarrow (0, \infty)$  is said to be regularly varying at infinity, denoted  $g(\cdot) \in R_\tau$ , if there exists  $\tau$  such that, for all  $t > 0$ , it holds that

$$\lim_{r \rightarrow \infty} \frac{g(rt)}{g(r)} = t^\tau.$$

**Definition 6.** [5] Let  $g : (0, \infty) \rightarrow (0, \infty)$  be a measurable function and  $g(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Then a slowly varying function  $L(\cdot)$  is said to be slowly varying with remainder of type 2, or that it belongs to the class  $SR2$ , if

$$\forall x > 1, \quad \frac{L(rx)}{L(r)} - 1 \sim k(x)g(r), \quad r \rightarrow \infty,$$

for some function  $k(\cdot)$ .

If there exists  $x$  such that  $k(x) \neq 0$  and  $k(x\mu) \neq k(\mu)$  for all  $\mu$ , then  $g(\cdot) \in R_\tau$  for some  $\tau \leq 0$  and  $k(x) = Ch_\tau(x)$ , where

$$h_\tau(x) = \begin{cases} \ln(x), & \text{if } \tau = 0, \\ \frac{x^\tau - 1}{\tau}, & \text{if } \tau \neq 0. \end{cases} \quad (8)$$

### 3. Assumptions and auxiliary results

In this section, we list the main assumptions and some auxiliary results from [23] which will be used to obtain the rate of convergence in non-central limit theorems.

**Assumption 1.** *Let  $\eta(x)$ ,  $x \in \mathbb{R}^d$ , be a homogeneous isotropic Gaussian random field with  $E\eta(x) = 0$  and a covariance function  $B(x)$  such that*

$$B(0) = 1, \quad B(x) = E\eta(0)\eta(x) = \|x\|^{-\alpha} L_0(\|x\|),$$

where  $L_0(\cdot)$  is a function slowly varying at infinity.

In this paper we restrict our consideration to  $\alpha \in (0, d/\kappa)$ , where  $\kappa$  is the Hermite rank in Definition 3. For such  $\alpha$  the covariance function  $B(\cdot)$  satisfying Assumption 1 is not integrable, which corresponds to the case of long-range dependence.

Let us denote

$$K_r := \int_{\Delta(r)} G(\eta(x)) dx \quad \text{and} \quad K_{r,\kappa} := \frac{C_\kappa}{\kappa!} \int_{\Delta(r)} H_\kappa(\eta(x)) dx,$$

where  $C_\kappa$  is defined by (5).

**Theorem 1.** [23] *Suppose that  $\eta(x)$ ,  $x \in \mathbb{R}^d$ , satisfies Assumption 1 and  $\text{Hrank } G = \kappa \in \mathbb{N}$ . If at least one of the following random variables*

$$\frac{K_r}{\sqrt{\text{Var } K_r}}, \quad \frac{K_r}{\sqrt{\text{Var } K_{r,\kappa}}} \quad \text{and} \quad \frac{K_{r,\kappa}}{\sqrt{\text{Var } K_{r,\kappa}}},$$

*has a limit distribution, then the limit distributions of the other random variables also exist and they coincide when  $r \rightarrow \infty$ .*

**Assumption 2.** *The random field  $\eta(x)$ ,  $x \in \mathbb{R}^d$ , has the spectral density*

$$f(\|\lambda\|) = c_2(d, \alpha) \|\lambda\|^{\alpha-d} L\left(\frac{1}{\|\lambda\|}\right),$$

where

$$c_2(d, \alpha) := \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^\alpha \pi^{d/2} \Gamma\left(\frac{\alpha}{2}\right)},$$

and  $L$  is a locally bounded function which is slowly varying at infinity and satisfies for sufficiently large  $r$  the condition

$$\left|1 - \frac{L(tr)}{L(r)}\right| \leq C g(r) h_\tau(t), \quad t \geq 1, \quad (9)$$

where  $g(\cdot) \in R_\tau$ ,  $\tau \leq 0$ , such that  $g(x) \rightarrow 0$ ,  $x \rightarrow \infty$ , and  $h_\tau(t)$  is defined by (8).



**Remark 4.** For  $L_0(\cdot)$  and  $L(\cdot)$  given in Assumptions 1 and 2, by Tauberian and Abelian theorems [22], it holds  $L_0(r) \sim L(r)$ ,  $r \rightarrow +\infty$ .

**Remark 5.** In applied statistical analysis of long-range dependent models researchers often assume an equivalence of Assumptions 1 and 2. However, this claim is not true in general, see [15, 22]. This is the main reason for using both assumptions to formulate the most general result in Theorem 5. However, in various specific cases just one of the assumptions may be sufficient. For example, if  $f(\cdot)$  is decreasing in a neighborhood of zero and continuous for all  $\lambda \neq 0$ , then by Tauberian Theorem 4 in [22] both assumptions are simultaneously satisfied. A detailed discussion of relations between Assumption 1 and 2 and various examples can be found in [22, 32]. Some important models used in spatial data analysis and geostatistics that simultaneously satisfy Assumptions 1 and 2 are Cauchy's and Linnik's fields, see [1]. Their covariance functions are of the form  $B(x) = (1 + \|x\|^\sigma)^{-\theta}$ ,  $\sigma \in (0, 2]$ ,  $\theta > 0$ . Exact expressions for their spectral densities in the form required by Assumption 2 are provided in [1, Section 5].

Two simple examples of covariance functions and spectral densities of random fields that satisfy Assumption 2 are given below.

**Example 1.** Let  $\tau = 0$  and  $L(x) = \begin{cases} 0, & x < 1, \\ 4 \ln(x), & x \geq 1. \end{cases}$

Then for  $t > 1$  and  $r > 1$

$$\left| 1 - \frac{L(tr)}{L(r)} \right| = \left| 1 - \frac{4 \ln(tr)}{4 \ln(r)} \right| = \frac{\ln(t)}{\ln(r)}.$$

Thus,  $L(\cdot)$  satisfies condition (9) with  $g(r) = 1/\ln(r)$ ,  $h_\tau(t) = \ln(t)$ , and  $\tau = 0$ .

Let  $d = 3$ ,  $\kappa = 2$ , and  $\alpha = 2$ . In this case

$$f(\|\lambda\|) = c_2(3, 2) \|\lambda\|^{-1} L\left(\frac{1}{\|\lambda\|}\right) = -\frac{1}{\pi} \frac{\ln(\|\lambda\|)}{\|\lambda\|} \chi_{(0, 1]}(\|\lambda\|).$$

By (1), (2) and (3)

$$\begin{aligned} B(r) &= \int_0^\infty Y_3(rz) d\Phi(z) = 16\pi \int_0^\infty \frac{\sin(rz)}{rz} z^2 f(z) dz = -\frac{4}{r} \int_0^1 \sin(rz) \ln(z) dz \\ &= 4 \frac{\ln(r) + \gamma - \text{Ci}(r)}{r^2}, \end{aligned}$$

where  $\text{Ci}(r) = -\int_r^\infty \frac{\cos(z)}{z} dz$  is the cosine integral, see (6.2.11) [33], and  $\gamma$  is Euler's constant.

By (6.2.13) [33] we have  $\ln(r) + \gamma - \text{Ci}(r) = \int_0^r \frac{1 - \cos(z)}{z} dz$ . Therefore, by the L'Hospital rule we get

$$B(0) = 4 \lim_{r \rightarrow 0} \frac{\ln(r) + \gamma - \text{Ci}(r)}{r^2} = 4 \lim_{r \rightarrow 0} \frac{\int_0^r \frac{1 - \cos(z)}{z} dz}{r^2} = 1.$$

Also,

$$B(r) = 4r^{-2}(\ln(r) + \gamma - \text{Ci}(r)) = r^{-2}L_0(r),$$

where  $L_0(r) = 4(\ln(r) + \gamma - \text{Ci}(r)) \sim 4 \ln(r) = L(r)$ , as by (6.2.14) [33] it holds  $\text{Ci}(r) \rightarrow 0$ , when  $r \rightarrow \infty$ .

**Example 2.** Now, we consider the case of  $\tau < 0$ . Let  $L(x) = \begin{cases} 0, & x < 1, \\ 6\left(1 - \frac{1}{x}\right), & x \geq 1. \end{cases}$

For  $t > 1$  and  $r > 1$  it holds

$$\left|1 - \frac{L(tr)}{L(r)}\right| = \left|1 - \frac{\frac{tr-1}{tr}}{\frac{r-1}{r}}\right| = \frac{1}{r-1} \frac{t-1}{t}.$$

Thus,  $L(\cdot)$  satisfies condition (9) with  $g(r) = \frac{1}{r-1}$ ,  $h_\tau(t) = \frac{t-1}{t}$ , and  $\tau = -1$ .

Let  $d = 3$ ,  $\kappa = 2$ , and  $\alpha = 2$ . In this case  $f(\|\lambda\|) = \frac{3}{2\pi} \frac{1 - \|\lambda\|}{\|\lambda\|^2} \chi_{(0,1]}(\|\lambda\|)$ .

Therefore

$$B(r) = \frac{6}{r} \int_0^1 \sin(rz)(1-z) dz = 6 \frac{r - \sin(r)}{r^3}.$$

Note, that by the L'Hospital rule we get

$$B(0) = 6 \lim_{r \rightarrow 0} \frac{r - \sin(r)}{r^3} = 1.$$

Also,

$$B(r) = 6 \frac{r - \sin(r)}{r^3} = r^{-2}L_0(r),$$

where  $L_0(r) = 6 \frac{r - \sin(r)}{r} \sim 6 \frac{r-1}{r} = L(r)$ .

The remarks below clarify condition (9) and compare it with the assumptions used in [1].

**Remark 6.** Assumption 2 implies weaker restrictions on the spectral density than the ones used in [1]. Slowly varying functions in Assumption 2 can tend to infinity or zero. This is an improvement over [1] where slowly varying functions were assumed to converge to a constant. For example, a function that satisfies this assumption, but would not fit that of [1], is  $\ln(\cdot)$ .

**Remark 7.** Slowly varying functions that satisfy condition (9) belong to the class SR2 from Definition 6.

**Remark 8.** By Corollary 3.12.3 [5] for  $\tau \neq 0$  the slowly varying function  $L(\cdot)$  in Assumption 2 can be represented as

$$L(x) = C \left( 1 + c\tau^{-1}g(x) + o(g(x)) \right).$$

As we can see  $L(\cdot)$  converges to some constant as  $x$  goes to infinity. This makes the case  $\tau = 0$  particularly interesting as this is the only case when a slowly varying function with remainder can tend to infinity or zero.

**Lemma 1.** *If  $L$  satisfies (9), then for any  $k \in \mathbb{N}$ ,  $\delta > 0$ , and sufficiently large  $r$*

$$\left| 1 - \frac{L^{k/2}(tr)}{L^{k/2}(r)} \right| \leq C g(r) h_\tau(t) t^\delta, \quad t \geq 1.$$

**Proof.** Applying the mean value theorem to the function  $f(u) = u^n$ ,  $n \in \mathbb{R}$ , on  $\mathbf{A} = [\min(1, u), \max(1, u)]$  we obtain the inequality

$$|1 - x^n| = n\theta^{n-1}|1 - x| \leq n|1 - x| \max(1, x^{n-1}), \quad \theta \in \mathbf{A}.$$

Now, using this inequality for  $x = \frac{L(tr)}{L(r)}$  and  $n = k/2$  we get

$$\left| 1 - \frac{L^{k/2}(tr)}{L^{k/2}(r)} \right| \leq \frac{\kappa}{2} \left| 1 - \frac{L(tr)}{L(r)} \right| \max \left( 1, \left( \frac{L(tr)}{L(r)} \right)^{\frac{k}{2}-1} \right). \quad (10)$$

By Theorem 1.5.6 in [5] we know that for large enough  $r$  there exists  $C > 0$  such that for any  $\delta_1 > 0$

$$\frac{L(tr)}{L(r)} \leq C \cdot t^{\delta_1}, \quad t \geq 1.$$

Applying this result and condition (9) to (10), and by choosing  $\delta = \delta_1(\frac{k}{2} - 1)$ , we get

$$\left| 1 - \frac{L^{k/2}(tr)}{L^{k/2}(r)} \right| \leq C g(r) h_\tau(t) \max \left( 1, t^{\delta_1(\frac{k}{2}-1)} \right) \leq C g(r) h_\tau(t) t^\delta, \quad t \geq 1.$$

□

Let us denote the Fourier transform of the indicator function of the set  $\Delta$  by

$$K_\Delta(x) := \int_{\Delta} e^{i(x,u)} du, \quad x \in \mathbb{R}^d.$$

**Lemma 2.** [23] *If  $t_1, \dots, t_\kappa, \kappa \geq 1$ , are positive constants such that it holds  $\sum_{i=1}^\kappa t_i < d$ , then*

$$\int_{\mathbb{R}^{d\kappa}} |K_\Delta(\lambda_1 + \dots + \lambda_\kappa)|^2 \frac{d\lambda_1 \dots d\lambda_\kappa}{\|\lambda_1\|^{d-t_1} \dots \|\lambda_\kappa\|^{d-t_\kappa}} < \infty.$$

**Theorem 2.** [23] *Let  $\eta(x)$ ,  $x \in \mathbb{R}^d$ , be a homogeneous isotropic Gaussian random field with  $E\eta(x) = 0$ . If Assumptions 1 and 2 hold, then for  $r \rightarrow \infty$  the random variables*

$$X_{r,\kappa} := r^{(\kappa\alpha)/2-d} L^{-\kappa/2}(r) \int_{\Delta(r)} H_\kappa(\eta(x)) dx$$

*converge weakly to*

$$X_\kappa(\Delta) := c_2^{\kappa/2}(d, \alpha) \int_{\mathbb{R}^{d\kappa}}' K_\Delta(\lambda_1 + \dots + \lambda_\kappa) \frac{W(d\lambda_1) \dots W(d\lambda_\kappa)}{\|\lambda_1\|^{(d-\alpha)/2} \dots \|\lambda_\kappa\|^{(d-\alpha)/2}}, \quad (11)$$

*where  $\int_{\mathbb{R}^{d\kappa}}'$  denotes the multiple Wiener-Itô integral.*

**Remark 9.** If  $\kappa = 1$  the limit  $X_\kappa(\Delta)$  is Gaussian. However, for the case  $\kappa > 1$  distributional properties of  $X_\kappa(\Delta)$  are almost unknown. It was shown that the integrals in (11) possess absolutely continuous densities, see [9, 37]. The article [1] proved that these densities are bounded if  $\kappa = 2$ . Also, for the Rosenblatt distribution, i.e.  $\kappa = 2$  and a rectangular  $\Delta$ , the density and cumulative distribution functions of  $X_\kappa(\Delta)$  were studied in [41]. An approach to investigate the boundedness of densities of multiple Wiener-Itô integrals was suggested in [9]. However, it is difficult to apply this approach to the case  $\kappa > 2$  as it requires a classification of the peculiarities of general  $n$ th degree forms.

**Definition 7.** *Let  $Y_1$  and  $Y_2$  be arbitrary random variables. The uniform (Kolmogorov) metric for the distributions of  $Y_1$  and  $Y_2$  is defined by the formula*

$$\rho(Y_1, Y_2) = \sup_{z \in \mathbb{R}} |P(Y_1 \leq z) - P(Y_2 \leq z)|.$$

The following result follows from Lemma 1.8 in [35].

**Lemma 3.** *If  $X, Y$  and  $Z$  are arbitrary random variables, then for any  $\varepsilon > 0$ :*

$$\rho(X + Y, Z) \leq \rho(X, Z) + \rho(Z + \varepsilon, Z) + P(|Y| \geq \varepsilon).$$

## 4. Lévy concentration functions for $X_\kappa(\Delta)$

In this section, we will investigate some fine properties of probability distributions of Hermite-type random variables. These results will be used to derive upper bounds of  $\rho(X_\kappa(\Delta) + \varepsilon, X_\kappa(\Delta))$  in the next section. The following function from Section 1.5 in [35] will be used in this section.

**Definition 8.** The Lévy concentration function of a random variable  $X$  is defined by

$$Q(X, \varepsilon) := \sup_{z \in \mathbb{R}} \mathbb{P}(z < X \leq z + \varepsilon), \quad \varepsilon \geq 0.$$

**Remark 10.** By Definitions 7 and 8

$$\begin{aligned} Q(X_\kappa(\Delta), \varepsilon) &= \sup_{z \in \mathbb{R}} (\mathbb{P}(X_\kappa(\Delta) \leq z + \varepsilon) - \mathbb{P}(X_\kappa(\Delta) \leq z)) \\ &= \sup_{z \in \mathbb{R}} |\mathbb{P}(X_\kappa(\Delta) \leq z) - \mathbb{P}(X_\kappa(\Delta) + \varepsilon \leq z)| = \rho(X_\kappa(\Delta) + \varepsilon, X_\kappa(\Delta)). \end{aligned}$$

We will discuss two important cases, and show how to estimate the Lévy concentration function in each of them.

If  $X_k(\Delta)$  has a bounded probability density function  $p_{X_\kappa(\Delta)}(\cdot)$ , then it holds

$$Q(X_\kappa(\Delta), \varepsilon) = \sup_{z \in \mathbb{R}} \int_z^{z+\varepsilon} p_{X_\kappa(\Delta)}(t) dt \leq \varepsilon \sup_{z \in \mathbb{R}} p_{X_\kappa(\Delta)}(z) \leq \varepsilon C. \quad (12)$$

This inequality is probably the sharpest known upper bound of the Lévy concentration function of  $X_k(\Delta)$ . It is discussed in case 1.

**Case 1.** If the Hermite rank of  $G(\cdot)$  is equal to  $\kappa = 2$  we are dealing with the so-called Rosenblatt-type random variable. It is known that the probability density function of this variable is bounded, consult [1, 9, 10, 21, 24] for proofs by different methods. Thus, one can use estimate (12).

**Case 2.** When there is no information about boundedness of the probability density function, anti-concentration inequalities can be used to obtain estimates of the Lévy concentration function, see Theorem 3 below.

Let us denote by  $I_\kappa(\cdot)$  a multiple Wiener-Itô stochastic integral of order  $d\kappa$ , i.e.  $I_\kappa(f) = \int_{\mathbb{R}^{d\kappa}} f(\lambda_1, \dots, \lambda_\kappa) W(d\lambda_1) \dots W(d\lambda_\kappa)$ , where  $f(\cdot) \in L_2^s(\mathbb{R}^{d\kappa})$ . Here  $L_2^s(\mathbb{R}^{d\kappa})$  denotes the space of symmetrical functions in  $L_2(\mathbb{R}^{d\kappa})$ . Note, that any  $F \in L_2(\Omega)$  can be represented as  $F = \mathbb{E}(F) + \sum_{q=1}^{\infty} I_q(f_q)$ , where the functions  $f_q$  are determined by  $F$ . The multiple Wiener-Itô integral  $I_q(f_q)$  coincides with the orthogonal projection of  $F$  on the  $q$ -th Wiener chaos associated with  $X$ .

The following lemma uses the approach suggested in [31].

**Lemma 4.** For any  $\kappa \in \mathbb{N}$ ,  $t \in \mathbb{R}$ , and  $\hat{\varepsilon} > 0$  it holds

$$\mathbb{P}(|X_\kappa(\Delta) - t| \leq \hat{\varepsilon}) \leq \frac{c_\kappa \hat{\varepsilon}^{1/\kappa}}{\left(C \|\hat{K}_\Delta\|_{L_2(\mathbb{R}^{d\kappa})}^2 + t^2\right)^{1/\kappa}},$$

where  $\hat{K}_\Delta(x_1, \dots, x_\kappa) := \frac{K_\Delta(x_1 + \dots + x_\kappa)}{\|x_1\|^{(d-\alpha)/2} \dots \|x_\kappa\|^{(d-\alpha)/2}}$  and  $c_\kappa$  is a constant that depends on  $\kappa$ .

**Proof.** Let  $\{e_i\}_{i \in \mathbb{N}}$  be an orthogonal basis of  $L_2(\mathbb{R}^d)$ . Then,  $\hat{K}_\Delta \in L_2(\mathbb{R}^{d\kappa})$  can be represented as

$$\hat{K}_\Delta = \sum_{(i_1, \dots, i_\kappa) \in \mathbb{N}^\kappa} c_{i_1, \dots, i_\kappa} e_{i_1} \otimes \cdots \otimes e_{i_\kappa}.$$

For each  $n \in \mathbb{N}$ , set

$$\hat{K}_\Delta^n = \sum_{(i_1, \dots, i_\kappa) \in \{1, \dots, n\}^\kappa} c_{i_1, \dots, i_\kappa} e_{i_1} \otimes \cdots \otimes e_{i_\kappa}.$$

Note, that both  $\hat{K}_\Delta$  and  $\hat{K}_\Delta^n$  belong to the space  $L_2^s(\mathbb{R}^{d\kappa})$ .

By (11) it follows that  $X_\kappa(\Delta) = c_2^{\kappa/2}(d, \alpha) I_\kappa(\hat{K}_\Delta)$ . Let  $X_\kappa^n(\Delta) := c_2^{\kappa/2}(d, \alpha) I_\kappa(\hat{K}_\Delta^n)$ . As  $n \rightarrow \infty$ ,  $\hat{K}_\Delta^n \rightarrow \hat{K}_\Delta$  in  $L_2(\mathbb{R}^{d\kappa})$ . Thus,  $X_\kappa^n(\Delta) \rightarrow X_\kappa(\Delta)$  in  $L_2(\Omega, \mathcal{F}, P)$ . Hence, there exists a strictly increasing sequence  $n_j$  for which  $X_\kappa^{n_j}(\Delta) \rightarrow X_\kappa(\Delta)$  almost surely as  $j \rightarrow \infty$ .

It also follows that

$$\begin{aligned} X_\kappa^n(\Delta) &= c_2^{\kappa/2}(d, \alpha) I_\kappa \left( \sum_{(i_1, \dots, i_\kappa) \in \{1, \dots, n\}^\kappa} c_{i_1, \dots, i_\kappa} e_{i_1} \otimes \cdots \otimes e_{i_\kappa} \right) \\ &= c_2^{\kappa/2}(d, \alpha) \sum_{m=1}^{\kappa} \sum_{\substack{1 \leq i'_1 < \dots < i'_m \leq n \\ \kappa_1 + \dots + \kappa_m = \kappa}} c_{i'_1, \dots, i'_m}^{\kappa_1, \dots, \kappa_m} I_\kappa(e_{i'_1}^{\otimes \kappa_1} \otimes \cdots \otimes e_{i'_m}^{\otimes \kappa_m}), \end{aligned}$$

where  $\kappa_i \in \mathbb{N}$ ,  $i = 1, \dots, m$ ,  $c_{i'_1, \dots, i'_m}^{\kappa_1, \dots, \kappa_m} = \sum_{(i_1, \dots, i_\kappa) \in A_{i'_1, \dots, i'_m}^{\kappa_1, \dots, \kappa_m}} c_{i_1, \dots, i_\kappa}$ , and  $A_{i'_1, \dots, i'_m}^{\kappa_1, \dots, \kappa_m} := \{(i_1, \dots, i_\kappa) \in \{1, \dots, n\}^\kappa : \kappa_1 \text{ indices } i_l = i'_1, \dots, \kappa_m \text{ indices } i_l = i'_m, l = 1, \dots, \kappa\}$ .

By the Itô isometry [18]:

$$I_{\kappa_1 + \dots + \kappa_m}(e_{i'_1}^{\otimes \kappa_1} \otimes \cdots \otimes e_{i'_m}^{\otimes \kappa_m}) = \prod_{j=1}^m H_{\kappa_j} \left( \int_{\mathbb{R}^d} e_j(\lambda) W(d\lambda) \right) = \prod_{j=1}^m H_{\kappa_j}(\xi_j),$$

where  $\xi_j \sim \mathcal{N}(0, 1)$ .

Thus,  $X_\kappa^n(\Delta)$  can be represented as  $X_\kappa^n(\Delta) = U_{n, \kappa}(\xi_1, \dots, \xi_n)$ , where  $U_{n, \kappa}(\cdot)$  is a polynomial of degree at most  $\kappa$ . Furthermore,  $X_\kappa^n(\Delta) - t$  is also a polynomial of degree at most  $\kappa$ .

Now, applying Carbery-Wright inequality, see Theorem 2.5 in [31], one obtains that there exists a constant  $\hat{c}_\kappa$  such that for any  $n \in \mathbb{N}$  and  $\hat{\varepsilon} > 0$

$$P \left( |X_\kappa^n(\Delta) - t| \leq \hat{\varepsilon} \left( E(X_\kappa^n(\Delta) - t)^2 \right)^{\frac{1}{2}} \right) \leq \hat{c}_\kappa \hat{\varepsilon}^{1/\kappa}.$$

Analogously to [31], using Fatou's lemma and the correspondingly selected subsequence  $\{n_j\}$  we get

$$\mathbb{P} \left( |X_\kappa(\Delta) - t| \leq \hat{\varepsilon} \left( \mathbb{E} (X_\kappa(\Delta) - t)^2 \right)^{\frac{1}{2}} \right) \leq \hat{c}_\kappa 2^{1/\kappa} \hat{\varepsilon}^{1/\kappa} = c_\kappa \hat{\varepsilon}^{1/\kappa}.$$

It is known, see (1.3) and (1.5) in [16], that  $\mathbb{E} X_\kappa(\Delta) = 0$  and  $\mathbb{E} X_\kappa^2(\Delta) = C \|\hat{K}_\Delta\|_{L_2(\mathbb{R}^{d\kappa})}^2$ . Thus, the above inequality can be rewritten as

$$\mathbb{P} (|X_\kappa(\Delta) - t| \leq \hat{\varepsilon}) \leq \frac{c_\kappa \hat{\varepsilon}^{1/\kappa}}{\left( \mathbb{E} (X_\kappa(\Delta) - t)^2 \right)^{\frac{1}{2\kappa}}} = \frac{c_\kappa \hat{\varepsilon}^{1/\kappa}}{\left( C \|\hat{K}_\Delta\|_{L_2(\mathbb{R}^{d\kappa})}^2 + t^2 \right)^{1/\kappa}}.$$

□

The following theorem combines the two cases above and provides an upper-bound estimate of the Lévy concentration function.

**Theorem 3.** *For any  $\kappa \in \mathbb{N}$  and an arbitrary positive  $\varepsilon$  it holds*

$$Q(X_\kappa(\Delta), \varepsilon) \leq C \varepsilon^a,$$

where the constant  $a$  equals 1 if  $\kappa = 2$  and  $1/\kappa$  if  $\kappa > 2$ .

**Proof.** If  $\kappa = 2$ , following the discussion of case 1 above, the result of the theorem is an immediate corollary of (12) and the boundedness of  $p_{X_\kappa(\Delta)}(\cdot)$ .

If  $\kappa > 2$ , applying Lemma 4 with  $t = z + \frac{\varepsilon}{2}$  and  $\hat{\varepsilon} = \frac{\varepsilon}{2}$  we get

$$\begin{aligned} Q(X_\kappa(\Delta), \varepsilon) &= \sup_{z \in \mathbb{R}} \mathbb{P} \left( \left| X_\kappa(\Delta) - \left( z + \frac{\varepsilon}{2} \right) \right| \leq \frac{\varepsilon}{2} \right) \\ &\leq \sup_{z \in \mathbb{R}} \left( \frac{c_\kappa \left( \frac{\varepsilon}{2} \right)^{1/\kappa}}{\left( C \|\hat{K}_\Delta\|_{L_2(\mathbb{R}^{d\kappa})}^2 + \left( z + \frac{\varepsilon}{2} \right)^2 \right)^{\frac{1}{2\kappa}}} \right) \leq \frac{c_\kappa \varepsilon^{1/\kappa}}{\left( 2C \|\hat{K}_\Delta\|_{L_2(\mathbb{R}^{d\kappa})} \right)^{\frac{1}{\kappa}}} = C \varepsilon^{1/\kappa}. \end{aligned}$$

□

**Remark 11.** There are other cases when the constant  $a$  also is equal to 1. For example, some interesting results about boundedness of probability density functions of Hermite-type random variables were obtained in [17] by Malliavin calculus. To recapture a key result of [17] we recall some definitions from Malliavin calculus.

Let  $X = \{X(h), h \in L^2(\mathbb{R}^d)\}$  be an isonormal Gaussian process defined on a complete probability space  $(\Omega, \mathcal{F}, P)$ . Let  $\mathbf{S}$  denote the class of smooth random variables of the form  $F = f(X(h_1), \dots, X(h_n))$ ,  $n \in \mathbb{N}$ , where  $h_1, \dots, h_n$  are in  $L^2(\mathbb{R}^d)$ , and  $f$  is a function, such that  $f$  itself and all its partial derivatives have at most polynomial growth.

The Malliavin derivative  $DF$  of  $F = f(X(h_1), \dots, X(h_n))$  is the  $L^2(\mathbb{R}^d)$  valued random variable given by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X(h_1), \dots, X(h_n))h_i.$$

The operator  $D$  is a closable operator on  $L_2(\Omega)$  taking values in  $L_2(\Omega; L^2(\mathbb{R}^d))$ . By iteration one can define higher order derivatives  $D^k F \in L_2(\Omega; L^2(\mathbb{R}^d)^{\odot k})$ , where  $\odot$  denotes the symmetric tensor product. For any integer  $k \geq 0$  and any  $p \geq 1$  we denote by  $\mathbb{D}^{k,p}$  the closure of  $\mathbf{S}$  with respect to the norm  $\|\cdot\|_{k,p}$  given by

$$\|F\|_{k,p}^p = \sum_{i=0}^k \mathbb{E} \left( \|D^i F\|_{L^2(\mathbb{R}^d)^{\otimes i}}^p \right).$$

Let us denote by  $\delta$  the adjoint operator of  $D$  from a domain in  $L_2(\Omega; L^2(\mathbb{R}^d))$  to  $L_2(\Omega)$ . An element  $u \in L_2(\Omega; L^2(\mathbb{R}^d))$  belongs to the domain of  $\delta$  if and only if for any  $F \in \mathbb{D}^{1,2}$  it holds

$$\mathbb{E}[\langle DF, u \rangle] \leq c_u \sqrt{\mathbb{E}[F^2]},$$

where  $c_u$  is a constant depending only on  $u$ .

The following theorem gives sufficient conditions to guarantee boundedness of Hermite-type densities.

**Theorem 4.** [17] *Let  $F \in \mathbb{D}^{2,s}$  such that  $\mathbb{E}[|F|^{2q}] < \infty$  and*

$$\mathbb{E} \left[ \|DF\|_{L^2(\mathbb{R}^d)}^{-2r} \right] < \infty, \quad (13)$$

where  $q, r, s > 1$  satisfy  $\frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 1$ .

Denote  $w = \|DF\|_{L^2(\mathbb{R}^d)}^2$  and  $u = w^{-1}DF$ . Then  $u \in \mathbb{D}^{1,q'}$  with  $q' = \frac{q}{q-1}$  and  $F$  has a density given by  $p_F(x) = \mathbb{E}[\mathbf{1}_{F>x}\delta(u)]$ . Furthermore,  $p_F(x)$  is bounded and  $p_F(x) \leq C_q \|w^{-1}\|_r \|F\|_{2,s} \min(1, |x|^{-2} \|F\|_{2q}^2)$ , for any  $x \in \mathbb{R}$ , where  $C_q$  is a constant depending only on  $q$ .

Note, that the Hermite-type random variable  $X_\kappa(\Delta)$  does belong to the space  $\mathbb{D}^{2,s}$ ,  $s > 1$ , and  $\mathbb{E}[|X_\kappa(\Delta)|^{2q}] < \infty$  by the hypercontractivity property, see (2.11) in [17]. Thus, if the condition (13) holds, then  $X_\kappa(\cdot)$  possess a bounded probability density function. In general, it is very difficult to verify the condition (13).

## 5. Rate of convergence

In this section we consider the case of Hermite-type limit distributions in Theorem 2. The main result describes the rate of convergence of  $\frac{\kappa!}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)} \int_{\Delta(r)} G(\eta(x)) dx$  to

$X_\kappa(\Delta) = c_2^{\kappa/2}(d, \alpha) \int_{\mathbb{R}^{d_\kappa}} K_\Delta(\lambda_1 + \dots + \lambda_\kappa) \frac{W(d\lambda_1) \dots W(d\lambda_\kappa)}{\|\lambda_1\|^{(d-\alpha)/2} \dots \|\lambda_\kappa\|^{(d-\alpha)/2}},$  when  $r \rightarrow \infty$ . To prove it we use some techniques and facts from [6, 23, 21].



**Theorem 5.** *Let Assumptions 1 and 2 hold and  $\text{Hrank } G = \kappa \in \mathbb{N}$ .*

*If  $\tau \in (-\frac{d-\kappa\alpha}{2}, 0)$  then for any  $\varkappa < \frac{a}{2+a} \min\left(\frac{\alpha(d-\kappa\alpha)}{d-(\kappa-1)\alpha}, \varkappa_1\right)$*

$$\rho\left(\frac{\kappa! K_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_\kappa(\Delta)\right) = o(r^{-\varkappa}), \quad r \rightarrow \infty,$$

*where  $a$  is the constant from Theorem 3,  $C_\kappa$  is defined by (5), and*

$$\varkappa_1 := \min\left(-2\tau, \frac{1}{\frac{1}{d-2\alpha} + \dots + \frac{1}{d-\kappa\alpha} + \frac{1}{d+1-\kappa\alpha}}\right).$$

*If  $\tau = 0$  then*

$$\rho\left(\frac{\kappa! K_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_\kappa(\Delta)\right) = O\left(g^{\frac{2a}{2+a}}(r)\right), \quad r \rightarrow \infty,$$

*where  $g(\cdot)$  is from Assumption 2.*

**Remark 12.** For  $\kappa = 1$  the result of Theorem 5 holds and  $\varkappa_1 = \min(-2\tau, d+1-\alpha)$ . Theorem 5 generalizes the result for the Rosenblatt-type case ( $\kappa = 2$ ) in [1] to Hermite-type asymptotics ( $\kappa > 2$ ). It also relaxes the assumptions on the spectral density used in [1], see Remarks 6–8.

**Proof.** The value of  $\tau$  in the proof is in  $(-\frac{d-\kappa\alpha}{2}, 0]$ . The situations where special considerations are required for the case  $\tau = 0$  are emphasised and corresponding derivations are provided in the proof.

Since  $\text{Hrank } G = \kappa$ , it follows that  $K_r$  can be represented in the space of squared-integrable random variables  $L_2(\Omega)$  as

$$K_r = K_{r,\kappa} + S_r := \frac{C_\kappa}{\kappa!} \int_{\Delta(r)} H_\kappa(\eta(x)) dx + \sum_{j \geq \kappa+1} \frac{C_j}{j!} \int_{\Delta(r)} H_j(\eta(x)) dx,$$

where  $C_j$  are coefficients of the Hermite series (5) of the function  $G(\cdot)$ .

Notice that  $\mathbb{E}K_{r,\kappa} = \mathbb{E}S_r = \mathbb{E}X_\kappa(\Delta) = 0$ , and

$$X_{r,\kappa} = \frac{\kappa! K_{r,\kappa}}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}.$$

Since the weak limit  $X_\kappa(\Delta)$  and the random variables  $X_{r,\kappa}$  are not necessarily defined on the same probability space, let us consider the distributional couples  $X_{r,\kappa}^*$  of  $X_{r,\kappa}$  that share the same random measure as  $X_\kappa(\Delta)$ .

A short scheme of the proof is as follows. First, we show how to estimate  $\text{Var } S_r$ . Then, we apply Lemma 3 to  $X = X_{r,\kappa}$ ,  $Y = \frac{\kappa! S_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}$ , and  $Z = X_\kappa(\Delta)$ . Thus, the upper bound can be given as

$$\rho(X_{r,\kappa}^*, X_\kappa(\Delta)) + \rho(X_{r,\kappa}(\Delta) + \varepsilon, X_\kappa(\Delta)) + P\left(\left|\frac{\kappa! S_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}\right| \geq \varepsilon\right),$$

for any positive  $\varepsilon$ . The second summand is the Lévy concentration function of  $X_\kappa(\cdot)$  and can be estimated using the results of Section 4. The third summand can be bounded by applying Chebyshev's inequality and the estimate of  $\text{Var } S_r$ . To estimate  $\rho(X_{r,\kappa}^*, X_\kappa(\Delta))$ , we apply Lemma 3 once more. The obtained bound is  $\rho(X_\kappa(\Delta) + \varepsilon, X_\kappa(\Delta)) + \varepsilon^{-2} \text{Var}(X_{r,\kappa}^* - X_\kappa(\Delta))$ . The rest of the proof shows how to estimate  $\text{Var}(X_{r,\kappa}^* - X_\kappa(\Delta))$ .

$S_r$  used in this paper is a particular case of  $V_r$  in [23, p. 1470] when the random field is scalar-valued. By the estimate of  $\text{Var } V_r$  in [23, p. 1471],

$$\begin{aligned} \text{Var } S_r &\leq |\Delta|^2 r^{2d-(\kappa+1)\alpha} \sum_{j \geq \kappa+1} \frac{C_j^2}{j!} \int_0^{\text{diam}\{\Delta\}} z^{-(\kappa+1)\alpha} L_0^{\kappa+1}(rz) \psi_\Delta(z) dz \\ &\leq |\Delta|^2 r^{2d-\kappa\alpha} L_0^\kappa(r) \sum_{j \geq \kappa+1} \frac{C_j^2}{j!} \int_0^{\text{diam}\{\Delta\}} z^{-\kappa\alpha} \frac{L_0^\kappa(rz)}{L_0^\kappa(r)} \frac{L_0(rz)}{(rz)^\alpha} \psi_\Delta(z) dz. \end{aligned} \quad (14)$$

We represent the integral in (14) as the sum of two integrals  $I_1$  and  $I_2$  with the ranges of integration  $[0, r^{-\beta_1}]$  and  $(r^{-\beta_1}, \text{diam}\{\Delta\})$  respectively, where  $\beta_1 \in (0, 1)$ .

It follows from Assumption 1 that  $|L_0(u)/u^\alpha| = |B(u)| \leq B(0) = 1$  and for  $r > 1$  we can estimate the first integral as

$$I_1 \leq \int_0^{r^{-\beta_1}} z^{-\kappa\alpha} \frac{L_0^\kappa(rz)}{L_0^\kappa(r)} \psi_\Delta(z) dz \leq \left( \frac{\sup_{0 \leq s \leq r} s^{\delta/\kappa} L_0(s)}{r^{\delta/\kappa} L_0(r)} \right)^\kappa \int_0^{r^{-\beta_1}} z^{-\delta} z^{-\kappa\alpha} \psi_\Delta(z) dz,$$

where  $\delta$  is an arbitrary number in  $(0, \min(\alpha, d - \kappa\alpha))$ .

By Assumption 1 the function  $L_0(\cdot)$  is locally bounded. By Theorem 1.5.3 in [5], there exists  $r_0 > 0$  and  $C > 0$  such that for all  $r \geq r_0$

$$\frac{\sup_{0 \leq s \leq r} s^{\delta/\kappa} L_0(s)}{r^{\delta/\kappa} L_0(r)} \leq C.$$

Using (4) with  $r = 1$  we obtain

$$\begin{aligned} \int_0^{r^{-\beta_1}} z^{-\delta} z^{-\kappa\alpha} \psi_\Delta(z) dz &= \frac{1}{|\Delta|^2} \int_\Delta \int_\Delta \|x - y\|^{-\delta-\kappa\alpha} \chi_{[0, r^{-\beta_1}]}(\|x - y\|) dx dy \\ &\leq \sup_{y \in \Delta} \frac{1}{|\Delta|} \int_{\Delta-y} \|u\|^{-\delta-\kappa\alpha} \chi_{[0, r^{-\beta_1}]}(\|u\|) du = \sup_{y \in \Delta} \frac{1}{|\Delta|} \int_{\Delta-y \cap v(r^{-\beta_1})} \|u\|^{-\delta-\kappa\alpha} du \\ &\leq \frac{C}{|\Delta|} \int_0^{r^{-\beta_1}} \tau^{d-\kappa\alpha-1-\delta} d\tau = \frac{C r^{-\beta_1(d-\kappa\alpha-\delta)}}{(d-\kappa\alpha-\delta) |\Delta|}, \end{aligned}$$

where  $\Delta - y = \{x \in \mathbb{R}^d : x + y \in \Delta\}$  and  $v(r)$  is a ball with center 0 and radius  $r$ .

Notice that

$$I_2 \leq \frac{\sup_{r^{1-\beta_1} \leq s \leq r \cdot \text{diam}\{\Delta\}} s^\delta L_0^\kappa(s)}{r^\delta L_0^\kappa(r)} \cdot \sup_{r^{1-\beta_1} \leq s \leq r \cdot \text{diam}\{\Delta\}} \frac{L_0(s)}{s^\alpha} \int_0^{\text{diam}\{\Delta\}} z^{-(\delta+\kappa\alpha)} \psi_\Delta(z) dz$$

Applying Theorem 1.5.3 in [5], and property (7) we get

$$I_2 = o(r^{-(\alpha-\delta)(1-\beta_1)}).$$

According to (6)

$$\sum_{j \geq \kappa+1} \frac{C_j^2}{j!} \leq \int_{\mathbb{R}} G^2(w) \phi(w) dw < +\infty.$$

Hence, for sufficiently large  $r$

$$\text{Var } S_r \leq C r^{2d-\kappa\alpha} L_0^\kappa(r) \left( r^{-\beta_1(d-\kappa\alpha-\delta)} + o\left(r^{-(\alpha-\delta)(1-\beta_1)}\right) \right).$$

Since, by Remark 4,  $L_0(\cdot) \sim L(\cdot)$ , we can replace  $L_0(\cdot)$  by  $L(\cdot)$  in the above estimate. Also, by choosing  $\beta_1 = \frac{\alpha}{d-(\kappa-1)\alpha}$  to minimize the upper bound we get

$$\text{Var } S_r \leq C r^{2d-\kappa\alpha} L^\kappa(r) r^{-\frac{\alpha(d-\kappa\alpha)}{d-(\kappa-1)\alpha} + \delta}.$$

It follows from Theorem 3 with Remark 10 that

$$\rho(X_\kappa(\Delta) + \varepsilon, X_\kappa(\Delta)) \leq C\varepsilon^a.$$

Applying Chebyshev's inequality and Lemma 3 to  $X = X_{r,\kappa}$ ,  $Y = \frac{\kappa! S_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}$ , and  $Z = X_\kappa(\Delta)$ , we get

$$\begin{aligned} \rho\left(\frac{\kappa! K_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_\kappa(\Delta)\right) &= \rho\left(X_{r,\kappa} + \frac{\kappa! S_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_\kappa(\Delta)\right) \\ &\leq \rho(X_{r,\kappa}^*, X_\kappa(\Delta)) + C\left(\varepsilon^a + \varepsilon^{-2} r^{-\frac{\alpha(d-\kappa\alpha)}{d-(\kappa-1)\alpha} + \delta}\right), \end{aligned}$$

for a sufficiently large  $r$ .

Choosing  $\varepsilon := r^{-\frac{\alpha(d-\kappa\alpha)}{(2+a)(d-(\kappa-1)\alpha)}}$  to minimize the second term we obtain

$$\rho\left(\frac{\kappa! K_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_\kappa(\Delta)\right) \leq \rho(X_{r,\kappa}^*, X_\kappa(\Delta)) + C r^{\frac{-a\alpha(d-\kappa\alpha)}{(2+a)(d-(\kappa-1)\alpha)} + \delta}. \quad (15)$$

Applying Lemma 3 once again to  $X = X_\kappa(\Delta)$ ,  $Y = X_{r,\kappa}^* - X_\kappa(\Delta)$ , and  $Z = X_\kappa(\Delta)$  we obtain for  $\varepsilon_1 > 0$

$$\begin{aligned} \rho(X_{r,\kappa}^*, X_\kappa(\Delta)) &\leq \varepsilon_1^a C + P\{|X_{r,\kappa}^* - X_\kappa(\Delta)| \geq \varepsilon_1\} \\ &\leq \varepsilon_1^a C + \varepsilon_1^{-2} \text{Var}(X_{r,\kappa}^* - X_\kappa(\Delta)). \end{aligned} \quad (16)$$

Now we show how to estimate  $\text{Var}(X_{r,\kappa}^* - X_\kappa(\Delta))$ .

By the self-similarity of Gaussian white noise and formula (2.1) in [12]

$$X_{r,\kappa}^* \stackrel{\mathcal{D}}{=} c_2^{\frac{\kappa}{2}}(d, \alpha) \int_{\mathbb{R}^{d\kappa}}' K_\Delta(\lambda_1 + \dots + \lambda_\kappa) Q_r(\lambda_1, \dots, \lambda_\kappa) \frac{W(d\lambda_1) \dots W(d\lambda_\kappa)}{\|\lambda_1\|^{(d-\alpha)/2} \dots \|\lambda_\kappa\|^{(d-\alpha)/2}},$$

where

$$Q_r(\lambda_1, \dots, \lambda_\kappa) := r^{\frac{\kappa}{2}(\alpha-d)} L^{-\frac{\kappa}{2}}(r) c_2^{-\frac{\kappa}{2}}(d, \alpha) \left[ \prod_{i=1}^{\kappa} \|\lambda_i\|^{d-\alpha} f\left(\frac{\|\lambda_i\|}{r}\right) \right]^{1/2}.$$

Notice that

$$X_\kappa(\Delta) = c_2^{\frac{\kappa}{2}}(d, \alpha) \int_{\mathbb{R}^{d\kappa}}' K_\Delta(\lambda_1 + \dots + \lambda_\kappa) \frac{W(d\lambda_1) \dots W(d\lambda_\kappa)}{\|\lambda_1\|^{(d-\alpha)/2} \dots \|\lambda_\kappa\|^{(d-\alpha)/2}}.$$

By the isometry property of multiple stochastic integrals

$$R_r := \frac{\mathbb{E} |X_{r,\kappa}^* - X_\kappa(\Delta)|^2}{c_2^{\frac{\kappa}{2}}(d, \alpha)} = \int_{\mathbb{R}^{d\kappa}} \frac{|K_\Delta(\lambda_1 + \dots + \lambda_\kappa)|^2 (Q_r(\lambda_1, \dots, \lambda_\kappa) - 1)^2}{\|\lambda_1\|^{d-\alpha} \dots \|\lambda_\kappa\|^{d-\alpha}} d\lambda_1 \dots d\lambda_\kappa.$$

Let us rewrite the integral  $R_r$  as the sum of two integrals  $I_3$  and  $I_4$  with the integration regions  $A(r) := \{(\lambda_1, \dots, \lambda_\kappa) \in \mathbb{R}^{d\kappa} : \max_{i=1, \dots, \kappa} (\|\lambda_i\|) \leq r^\gamma\}$  and  $\mathbb{R}^{d\kappa} \setminus A(r)$  respectively, where  $\gamma \in (0, 1)$ . Our intention is to use the monotone equivalence property of regularly varying functions in the regions  $A(r)$ .

First we consider the case of  $(\lambda_1, \dots, \lambda_\kappa) \in A(r)$ . By Assumption 2 and the inequality

$$\left| \sqrt{\prod_{i=1}^{\kappa} x_i} - 1 \right| \leq \sum_{i=1}^{\kappa} \left| x_i^{\frac{\kappa}{2}} - 1 \right|$$

we obtain

$$|Q_r(\lambda_1, \dots, \lambda_\kappa) - 1| = \left| \sqrt{\prod_{j=1}^{\kappa} \frac{L\left(\frac{r}{\|\lambda_j\|}\right)}{L(r)}} - 1 \right| \leq \sum_{j=1}^{\kappa} \left| \frac{L^{\frac{\kappa}{2}}\left(\frac{r}{\|\lambda_j\|}\right)}{L^{\frac{\kappa}{2}}(r)} - 1 \right|.$$

By Lemma 1, if  $\|\lambda_j\| \in (1, r^\gamma)$ ,  $j = 1, \dots, \kappa$ , then for arbitrary  $\delta_1 > 0$  and sufficiently large  $r$  we get

$$\left| 1 - \frac{L^{\frac{\kappa}{2}}\left(\frac{r}{\|\lambda_j\|}\right)}{L^{\frac{\kappa}{2}}(r)} \right| = \frac{L^{\frac{\kappa}{2}}\left(\frac{r}{\|\lambda_j\|}\right)}{L^{\frac{\kappa}{2}}(r)} \cdot \left| 1 - \frac{L^{\frac{\kappa}{2}}(r)}{L^{\frac{\kappa}{2}}\left(\frac{r}{\|\lambda_j\|}\right)} \right| \leq C \frac{L^{\frac{\kappa}{2}}\left(\frac{r}{\|\lambda_j\|}\right)}{L^{\frac{\kappa}{2}}(r)} g\left(\frac{r}{\|\lambda_j\|}\right)$$

$$\times \|\lambda_j\|^{\delta_1} h_\tau(\|\lambda_j\|) = C \|\lambda_j\|^{\delta_1} h_\tau(\|\lambda_j\|) g(r) \frac{g\left(\frac{r}{\|\lambda_j\|}\right)}{g(r)} \left( \frac{L\left(\frac{r}{\|\lambda_j\|}\right)}{L(r)} \right)^{\frac{\kappa}{2}}.$$

For any positive  $\beta_2$  and  $\beta_3$ , applying Theorem 1.5.6 in [5] to  $g(\cdot)$  and  $L(\cdot)$  and using the fact that  $h_\tau\left(\frac{1}{t}\right) = -\frac{1}{t^\tau} h_\tau(t)$  we obtain

$$\begin{aligned} \left| 1 - \frac{L^{\frac{\kappa}{2}}\left(\frac{r}{\|\lambda_j\|}\right)}{L^{\frac{\kappa}{2}}(r)} \right| &\leq C \|\lambda_j\|^{\delta_1 + \frac{\kappa\beta_2}{2} + \beta_3} \|\lambda_j\|^{-\tau} h_\tau(\|\lambda_j\|) g(r) \\ &= C \|\lambda_j\|^\delta \left| h_\tau\left(\frac{1}{\|\lambda_j\|}\right) \right| g(r). \end{aligned} \quad (17)$$

By Lemma 1 for  $\|\lambda_j\| \leq 1$ ,  $j = 1, \dots, \kappa$ , and arbitrary  $\delta > 0$ , we obtain

$$\left| 1 - \frac{L^{\frac{\kappa}{2}}\left(\frac{r}{\|\lambda_j\|}\right)}{L^{\frac{\kappa}{2}}(r)} \right| \leq C \|\lambda_j\|^{-\delta} h_\tau\left(\frac{1}{\|\lambda_j\|}\right) g(r). \quad (18)$$

Hence, by (17) and (18)

$$\begin{aligned} |Q_r(\lambda_1, \dots, \lambda_\kappa) - 1|^2 &\leq \kappa \sum_{j=1}^{\kappa} \left| \frac{L^{\frac{\kappa}{2}}\left(\frac{r}{\|\lambda_j\|}\right)}{L^{\frac{\kappa}{2}}(r)} - 1 \right|^2 \\ &\leq C \sum_{j=1}^{\kappa} h_\tau^2\left(\frac{1}{\|\lambda_j\|}\right) g^2(r) \max\left(\|\lambda_j\|^{-\delta}, \|\lambda_j\|^\delta\right), \end{aligned}$$

for  $(\lambda_1, \dots, \lambda_\kappa) \in A(r)$ .

Notice that in the case  $\tau = 0$  for any  $\delta > 0$  there exists  $C > 0$  such that  $h_0(x) = \ln(x) < Cx^\delta$ ,  $x \geq 1$ , and  $h_0(x) = \ln(x) < Cx^{-\delta}$ ,  $x < 1$ . Hence, by Lemma 2 for  $-\tau \leq \frac{d-\kappa\alpha}{2}$  we get

$$\int_{A(r) \cap \mathbb{R}^{\kappa d}} \frac{h_\tau^2\left(\frac{1}{\|\lambda_j\|}\right) \max\left(\|\lambda_j\|^{-\delta}, \|\lambda_j\|^\delta\right) \left| K_\Delta\left(\sum_{i=1}^{\kappa} \lambda_i\right) \right|^2 d\lambda_1 \dots d\lambda_\kappa}{\|\lambda_1\|^{d-\alpha} \dots \|\lambda_\kappa\|^{d-\alpha}} < \infty.$$

Therefore, we obtain for sufficiently large  $r$

$$I_3 \leq C g^2(r) \sum_{j=1}^{\kappa} \int_{A(r) \cap \mathbb{R}^{\kappa d}} \frac{h_\tau^2\left(\frac{1}{\|\lambda_j\|}\right) \cdot \max\left(\|\lambda_j\|^{-\delta}, \|\lambda_j\|^\delta\right)}{\|\lambda_1\|^{d-\alpha} \dots \|\lambda_\kappa\|^{d-\alpha}}$$

$$\begin{aligned}
& \times |K_{\Delta}(\lambda_1 + \dots + \lambda_{\kappa})|^2 d\lambda_1 \dots d\lambda_{\kappa} \leq C g^2(r) \int_{A(r) \cap \mathbb{R}^{\kappa d}} \frac{h_{\tau}^2\left(\frac{1}{\|\lambda_1\|}\right)}{\|\lambda_1\|^{d-\alpha} \dots \|\lambda_{\kappa}\|^{d-\alpha}} \\
& \times \max\left(\|\lambda_1\|^{-\delta}, \|\lambda_1\|^{\delta}\right) |K_{\Delta}(\lambda_1 + \dots + \lambda_{\kappa})|^2 d\lambda_1 \dots d\lambda_{\kappa} \leq C g^2(r). \quad (19)
\end{aligned}$$

It follows from Assumption 2 and the specification of the estimate (23) in the proof of Theorem 5 in [23] that for each positive  $\delta$  there exists  $r_0 > 0$  such that for all  $r \geq r_0$ ,  $(\lambda_1, \dots, \lambda_{\kappa}) \in B_{(1, \mu_2, \dots, \mu_{\kappa})} = \{(\lambda_1, \dots, \lambda_{\kappa}) \in \mathbb{R}^{\kappa d} : \|\lambda_j\| \leq 1, \text{ if } \mu_j = -1, \text{ and } \|\lambda_j\| > 1, \text{ if } \mu_j = 1, j = \overline{1, \kappa}\}, \text{ and } \mu_j \in \{-1, 1\}, \text{ it holds}$

$$\begin{aligned}
& \frac{|K_{\Delta}(\lambda_1 + \dots + \lambda_{\kappa})|^2 (Q_r(\lambda_1, \dots, \lambda_{\kappa}) - 1)^2}{\|\lambda_1\|^{d-\alpha} \dots \|\lambda_{\kappa}\|^{d-\alpha}} \leq \frac{C |K_{\Delta}(\lambda_1 + \dots + \lambda_{\kappa})|^2}{\|\lambda_1\|^{d-\alpha} \dots \|\lambda_{\kappa}\|^{d-\alpha}} \\
& + C \frac{|K_{\Delta}(\lambda_1 + \dots + \lambda_{\kappa})|^2}{\|\lambda_1\|^{d-\alpha-\delta} \|\lambda_2\|^{d-\alpha-\mu_2\delta} \dots \|\lambda_{\kappa}\|^{d-\alpha-\mu_{\kappa}\delta}}.
\end{aligned}$$

Since the integrands are non-negative, we can estimate  $I_4$  as it is shown below

$$\begin{aligned}
I_4 & \leq \kappa \int_{\mathbb{R}^{(\kappa-1)d}} \int_{\|\lambda_1\| > r^{\gamma}} \frac{|K_{\Delta}(\lambda_1 + \dots + \lambda_{\kappa})|^2 (Q_r(\lambda_1, \dots, \lambda_{\kappa}) - 1)^2 d\lambda_1 \dots d\lambda_{\kappa}}{\|\lambda_1\|^{d-\alpha} \dots \|\lambda_{\kappa}\|^{d-\alpha}} \\
& \leq C \int_{\mathbb{R}^{(\kappa-1)d}} \int_{\|\lambda_1\| > r^{\gamma}} \frac{|K_{\Delta}(\lambda_1 + \dots + \lambda_2)|^2 d\lambda_1 \dots d\lambda_{\kappa}}{\|\lambda_1\|^{d-\alpha} \dots \|\lambda_{\kappa}\|^{d-\alpha}} \\
& + C \sum_{\substack{\mu_i \in \{0, 1, -1\} \\ i \in \overline{2, \dots, \kappa}}} \int_{\mathbb{R}^{(\kappa-1)d}} \int_{\|\lambda_1\| > r^{\gamma}} \frac{|K_{\Delta}(\lambda_1 + \dots + \lambda_{\kappa})|^2 d\lambda_1 \dots d\lambda_{\kappa}}{\|\lambda_1\|^{d-\alpha-\delta} \|\lambda_2\|^{d-\alpha-\mu_2\delta} \dots \|\lambda_{\kappa}\|^{d-\alpha-\mu_{\kappa}\delta}} \\
& \leq C \max_{\substack{\mu_i \in \{0, 1, -1\} \\ i \in \overline{2, \dots, \kappa}}} \left[ \int_{\mathbb{R}^{(\kappa-1)d}} \int_{\|\lambda_1\| > r^{\gamma}} \frac{|K_{\Delta}(\lambda_1 + \dots + \lambda_{\kappa})|^2 d\lambda_1 \dots d\lambda_{\kappa}}{\|\lambda_1\|^{d-\alpha-\delta} \|\lambda_2\|^{d-\alpha-\mu_2\delta} \dots \|\lambda_{\kappa}\|^{d-\alpha-\mu_{\kappa}\delta}} \right]. \quad (20)
\end{aligned}$$

**Lemma 5.** *Let  $2 \leq m \leq \kappa$  and*

$$I(m, \gamma) := \max_{\substack{\mu_i \in \{0, 1, -1\} \\ i \in \overline{m, \dots, \kappa}}} \left[ \int_{\mathbb{R}^{(\kappa-m+1)d}} \int_{\|v\| > r^{\gamma}} \frac{|K_{\Delta}(v + \lambda_m + \dots + \lambda_{\kappa})|^2 dv d\lambda_m \dots d\lambda_{\kappa}}{\|v\|^{d-(m-1)(\alpha+\delta)} \|\lambda_m\|^{d-\alpha-\mu_m\delta} \dots \|\lambda_{\kappa}\|^{d-\alpha-\mu_{\kappa}\delta}} \right].$$

*Then, for any  $\gamma_0 \in (0, \gamma)$  and sufficiently large  $r$  it holds*

$$I(m, \gamma) \leq C \left( r^{-(\gamma-\gamma_0)(d-m\alpha-m\delta)} + I(m+1, \gamma_0) \right), \quad \text{if } m < \kappa,$$

*and*

$$I(\kappa, \gamma) \leq C \left( r^{-(\gamma-\gamma_0)(d-\kappa\alpha-\kappa\delta)} + \int_{\|u\| > r^{\gamma_0}} \frac{|K_{\Delta}(u)|^2 du}{\|u\|^{d-\kappa\alpha-\kappa\delta}} \right).$$

**Proof.** First, let us consider the case  $m < \kappa$ . By replacing  $v + \lambda_m$  by  $u$  in  $I(m, \gamma)$  we obtain

$$\begin{aligned}
I(m, \gamma) &\leq \max_{\substack{\mu_i \in \{0, 1, -1\} \\ i \in m, \dots, \kappa}} \left[ \int_{\mathbb{R}^{(\kappa-m+1)d}} \int_{\|v\| > r^\gamma} \frac{|K_\Delta(u + \lambda_{m+1} + \dots + \lambda_\kappa)|^2}{\|v\|^{d-(m-1)(\alpha+\delta)} \|u - v\|^{d-\alpha-\mu_m\delta}} \right. \\
&\quad \times \left. \frac{dv du d\lambda_{m+1} \dots d\lambda_\kappa}{\|\lambda_{m+1}\|^{d-\alpha-\mu_{m+1}\delta} \dots \|\lambda_\kappa\|^{d-\alpha-\mu_\kappa\delta}} \right] \leq \max_{\substack{\mu_i \in \{0, 1, -1\} \\ i \in m, \dots, \kappa}} \left[ \int_{\mathbb{R}^{(\kappa-m+1)d}} \frac{1}{\|u\|^{d-m\alpha-(\mu_m+m-1)\delta}} \right. \\
&\quad \times \left. \frac{|K_\Delta(u + \lambda_{m+1} + \dots + \lambda_\kappa)|^2}{\|\lambda_{m+1}\|^{d-\alpha-\mu_{m+1}\delta} \dots \|\lambda_\kappa\|^{d-\alpha-\mu_\kappa\delta}} \int_{\|v\| > \frac{r^\gamma}{\|u\|}} \frac{dv du d\lambda_{m+1} \dots d\lambda_\kappa}{\|v\|^{d-(m-1)(\alpha+\delta)} \left\| \frac{u}{\|u\|} - v \right\|^{d-\alpha-\mu_m\delta}} \right].
\end{aligned}$$

Let us show that for  $\delta \in (0, \min(\alpha, d/\kappa - \alpha))$  it holds

$$\begin{aligned}
&\sup_{u \in \mathbb{R}^d \setminus \{0\}} \int_{\mathbb{R}^d} \frac{dv}{\|v\|^{d-(m-1)(\alpha+\delta)} \left\| \frac{u}{\|u\|} - v \right\|^{d-\alpha-\mu_m\delta}} \\
&= \sup_{u: \|u\|=1} \int_{\mathbb{R}^d} \frac{dv}{\|v\|^{d-(m-1)(\alpha+\delta)} \left\| \frac{u}{\|u\|} - v \right\|^{d-\alpha-\mu_m\delta}} < \infty.
\end{aligned}$$

One can split  $\mathbb{R}^d$  into three disjoint regions  $A_1 := \{v \in \mathbb{R}^n : \|v\| < \frac{1}{2}\}$ ,  $A_2 := \{v \in \mathbb{R}^n : \frac{1}{2} \leq \|v\| < \frac{3}{2}\}$  and  $A_3 := \{v \in \mathbb{R}^n : \|v\| \geq \frac{3}{2}\}$ . The integrand has only singularity in each of the first two regions and no singularities but the infinite integration range  $A_3$  in the third case. After proceeding to the spherical coordinates the integral is bounded by the sum of three univariate integrals:  $I_{A_1}$ ,  $I_{A_2}$  and the improper integral  $I_{A_3}$ . Integrals  $I_{A_1}$  and  $I_{A_2}$  are finite since the powers of the integrand at their singular points are  $(m-1)(\alpha+\delta)-1$  and  $\alpha+\mu_m\delta-1$  respectively that are greater than  $-1$ . The integral  $I_{A_3}$  is finite since the powers of the integrand at infinity is  $-(d+1-(m-1)(\alpha+\delta)-(\alpha+\mu_m\delta)) \leq -(d+1-m\alpha-m\delta) < -1$ .

Therefore, we obtain

$$\begin{aligned}
I(m, \gamma) &\leq \max_{\substack{\mu_i \in \{0, 1, -1\} \\ i \in m, \dots, \kappa}} \left[ \int_{\mathbb{R}^{(\kappa-m)d}} \int_{\|u\| \leq r^{\gamma_0}} \frac{1}{\|u\|^{d-m\alpha-(\mu_m+m-1)\delta}} \right. \\
&\quad \times \left. \frac{|K_\Delta(u + \lambda_{m+1} + \dots + \lambda_\kappa)|^2}{\|\lambda_{m+1}\|^{d-\alpha-\mu_{m+1}\delta} \dots \|\lambda_\kappa\|^{d-\alpha-\mu_\kappa\delta}} \int_{\|v\| > r^{\gamma-\gamma_0}} \frac{dv du d\lambda_{m+1} \dots d\lambda_\kappa}{\|v\|^{d-(m-1)(\alpha+\delta)} \left\| \frac{u}{\|u\|} - v \right\|^{d-\alpha-\mu_m\delta}} \right. \\
&\quad \left. + C \int_{\mathbb{R}^{(\kappa-m)d}} \int_{\|u\| > r^{\gamma_0}} \frac{|K_\Delta(u + \lambda_{m+1} + \dots + \lambda_\kappa)|^2 du d\lambda_{m+1} \dots d\lambda_\kappa}{\|u\|^{d-m\alpha-(\mu_m+m-1)\delta} \|\lambda_{m+1}\|^{d-\alpha-\mu_{m+1}\delta} \dots \|\lambda_\kappa\|^{d-\alpha-\mu_\kappa\delta}} \right],
\end{aligned}$$

where  $\gamma_0 \in (0, \gamma)$ .

By Lemma 2, there exists  $r_0 > 1$  such that for all  $r \geq r_0$  the first summand is bounded by

$$\begin{aligned} & \max_{\substack{\mu_i \in \{0, 1, -1\} \\ i \in m, \dots, \kappa}} \left[ \int_{\mathbb{R}^{(\kappa-m)d}} \int_{\|u\| \leq r^{\gamma_0}} \frac{|K_{\Delta}(u + \lambda_{m+1} + \dots + \lambda_{\kappa})|^2 du d\lambda_{m+1} \dots d\lambda_{\kappa}}{\|u\|^{d-m\alpha-(\mu_m+m-1)\delta} \|\lambda_{m+1}\|^{d-\alpha-\mu_{m+1}\delta} \dots \|\lambda_{\kappa}\|^{d-\alpha-\mu_{\kappa}\delta}} \right. \\ & \quad \times \left. \int_{\|v\| > r^{\gamma-\gamma_0}} \frac{C dv}{\|v\|^{2d-m\alpha-(m-1)\delta-\mu_m\delta}} \right] \leq C r^{-(\gamma-\gamma_0)(d-m\alpha-m\delta)}. \end{aligned}$$

When  $\|u\| > r^{\gamma_0}$ ,  $r > 1$ , for any  $\mu_m \in \{0, 1, -1\}$  it holds  $\frac{1}{\|u\|^{d-m\alpha-(\mu_m+m-1)\delta}} \leq \frac{1}{\|u\|^{d-m\alpha-m\delta}}$ . Therefore, for sufficiently large  $r$ ,

$$\begin{aligned} I(m, \gamma) & \leq C r^{-(\gamma-\gamma_0)(d-m\alpha-m\delta)} \\ & + C \max_{\substack{\mu_i \in \{0, 1, -1\} \\ i \in m, \dots, \kappa}} \left[ \int_{\mathbb{R}^{(\kappa-m)d}} \int_{\|u\| > r^{\gamma_0}} \frac{|K_{\Delta}(u + \lambda_{m+1} + \dots + \lambda_{\kappa})|^2 du d\lambda_{m+1} \dots d\lambda_{\kappa}}{\|u\|^{d-m\alpha-m\delta} \|\lambda_{m+1}\|^{d-\alpha-\mu_{m+1}\delta} \dots \|\lambda_{\kappa}\|^{d-\alpha-\mu_{\kappa}\delta}} \right] \end{aligned}$$

Since the second summand does not depend on  $\mu_m$ , it is easy to see that it is equal  $C \cdot I(m+1, \gamma_0)$ . Thus, we obtain

$$I(m, \gamma) \leq C \left( r^{-(\gamma-\gamma_0)(d-m\alpha-m\delta)} + I(m+1, \gamma_0) \right).$$

Following the same arguments as above, it is straightforward to obtain the statement of the lemma in the case  $m = \kappa$ .  $\square$

Therefore, applying Lemma 5 ( $\kappa - 1$ ) times to (20) one obtains

$$\begin{aligned} I_4 & \leq C r^{-(\gamma-\gamma_0)(d-2\alpha-2\delta)} + \dots + C r^{-(\gamma_{\kappa-3}-\gamma_{\kappa-2})(d-\kappa\alpha-\kappa\delta)} \\ & + C \int_{\|u\| > r^{\gamma_{\kappa-2}}} \frac{|K_{\Delta}(u)|^2 du}{\|u\|^{d-\kappa\alpha-\kappa\delta}}, \end{aligned} \tag{21}$$

where  $\gamma > \gamma_0 > \gamma_1 > \dots > \gamma_{\kappa-2} > 0$ .

By the spherical  $L_2$ -average decay rate of the Fourier transform [6] for  $\delta < d + 1 - \kappa\alpha$  and sufficiently large  $r$  we get the following estimate of the integral in (21)

$$\int_{\|u\| > r^{\gamma_{\kappa-2}}} \frac{|K_{\Delta}(u)|^2 du}{\|u\|^{d-\kappa\alpha-\kappa\delta}} \leq C \int_{z > r^{\gamma_{\kappa-2}}} \int_{S^{d-1}} \frac{|K_{\Delta}(z\omega)|^2}{z^{1-\kappa\alpha-\kappa\delta}} d\omega dz$$



$$\leq C \int_{z > r^{\gamma_{\kappa-2}}} \frac{dz}{z^{d+2-\kappa\alpha-\kappa\delta}} = C r^{-\gamma_{\kappa-2}(d+1-\kappa\alpha-\kappa\delta)} = C r^{-(\gamma_{\kappa-2}-\gamma_{\kappa-1})(d+1-\kappa\alpha-\kappa\delta)}, \quad (22)$$

where  $S^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$  is a sphere of radius 1 in  $\mathbb{R}^d$  and  $\gamma_{\kappa-1} = 0$ .

Now let us consider the case  $\tau < 0$ . In this case by Theorem 1.5.6 in [5] for any  $\delta > 0$  we can estimate  $g(r)$  as follows

$$g(r) \leq C r^{\tau+\delta}. \quad (23)$$

Combining estimates (15), (16), (19), (21), (22), (23) and choosing  $\varepsilon_1 := r^{-\beta}$ , we obtain

$$\begin{aligned} \rho \left( \frac{\kappa! K_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_\kappa(\Delta) \right) &\leq C \left( r^{-\frac{a\alpha(d-\kappa\alpha)}{(2+a)(d-(\kappa-1)\alpha)} + \delta} + r^{-a\beta} + r^{2\tau+2\delta+2\beta} \right. \\ &\quad + r^{-(\gamma-\gamma_0)(d-2\alpha-2\delta)+2\beta} + \dots + r^{-(\gamma_{\kappa-3}-\gamma_{\kappa-2})(d-\kappa\alpha-\kappa\delta)+2\beta} \\ &\quad \left. + r^{-(\gamma_{\kappa-2}-\gamma_{\kappa-1})(d+1-\kappa\alpha-\kappa\delta)+2\beta} \right). \end{aligned}$$

Therefore, for any  $\tilde{\varkappa}_1 \in (0, \frac{2+a}{a}\varkappa_0)$  one can choose a sufficiently small  $\delta > 0$  such that

$$\rho \left( \frac{\kappa! K_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_\kappa(\Delta) \right) \leq C r^\delta \left( r^{-\frac{a\alpha(d-\kappa\alpha)}{(2+a)(d-(\kappa-1)\alpha)}} + r^{-\frac{a\tilde{\varkappa}_1}{2+a}} \right), \quad (24)$$

where

$$\begin{aligned} \varkappa_0 := & \sup_{\substack{1 > \gamma > \gamma_0 > \dots > \gamma_{\kappa-1} = 0 \\ \beta > 0}} \min(a\beta, -2\tau - 2\beta, (\gamma - \gamma_0)(d - 2\alpha) - 2\beta, \dots, \\ & (\gamma_{\kappa-3} - \gamma_{\kappa-2})(d - \kappa\alpha) - 2\beta, (\gamma_{\kappa-2} - \gamma_{\kappa-1})(d + 1 - \kappa\alpha) - 2\beta). \end{aligned}$$

**Lemma 6.** Let  $\mathbf{\Gamma} = \{\gamma = (\gamma_1, \dots, \gamma_{n+1}) \mid b = \gamma_0 > \gamma_1 > \dots > \gamma_{n+1} = 0\}$  and  $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{R}_+^{n+1}$  be some fixed vector.

The function  $G(\gamma) = \min_i (\gamma_i - \gamma_{i+1}) x_i$  reaches its maximum at  $\bar{\gamma} = (\bar{\gamma}_0, \dots, \bar{\gamma}_{n+1}) \in \mathbf{\Gamma}$  such that for any  $0 \leq i \leq n$  it holds

$$(\bar{\gamma}_i - \bar{\gamma}_{i+1}) x_i = (\bar{\gamma}_{i+1} - \bar{\gamma}_{i+2}) x_{i+1}. \quad (25)$$

**Proof.** Let us show that any deviation of  $\gamma$  from  $\bar{\gamma}$  leads to a smaller value. Consider a vector  $\hat{\gamma}$  such that for some  $i \in \{1, \dots, n\}$  and some  $\varepsilon > 0$  the following relation is true

$$\hat{\gamma}_i - \hat{\gamma}_{i+1} = \bar{\gamma}_i - \bar{\gamma}_{i+1} + \varepsilon.$$

Since  $\sum_{i=0}^n \hat{\gamma}_i - \hat{\gamma}_{i+1} = \hat{\gamma}_0 - \hat{\gamma}_{n+1} = b$  we can conclude that there exist some  $\varepsilon_1 > 0$  and  $j \neq i, j = 1, \dots, n$ , such that  $\hat{\gamma}_j - \hat{\gamma}_{j+1} = \bar{\gamma}_j - \bar{\gamma}_{j+1} - \varepsilon_1$ .

Obviously, in this case

$$G(\hat{\gamma}) \leq (\hat{\gamma}_j - \hat{\gamma}_{j+1}) x_j = (\bar{\gamma}_j - \bar{\gamma}_{j+1} - \varepsilon_1) x_j = (\bar{\gamma}_j - \bar{\gamma}_{j+1}) x_j - \varepsilon_1 x_j.$$

Since  $\varepsilon_1 > 0$  and  $x_j > 0$  it follows from (25) that

$$G(\hat{\gamma}) \leq (\bar{\gamma}_j - \bar{\gamma}_{j+1}) x_j - \varepsilon_1 x_j < (\bar{\gamma}_j - \bar{\gamma}_{j+1}) x_j = G(\bar{\gamma}).$$

So it is clearly seen that any deviation from  $\bar{\gamma}$  will yield a smaller value.  $\square$

Note, that  $\varkappa_0$  can be rewritten as

$$\begin{aligned} \varkappa_0 = \sup_{\beta > 0} \sup_{\gamma \in (0,1)} \sup_{\gamma > \gamma_0 > \dots > \gamma_{\kappa-1} = 0} \min(a\beta, -2\tau - 2\beta, (\gamma - \gamma_0)(d - 2\alpha) - 2\beta, \\ \dots, (\gamma_{\kappa-3} - \gamma_{\kappa-2})(d - \kappa\alpha) - 2\beta, (\gamma_{\kappa-2} - \gamma_{\kappa-1})(d + 1 - \kappa\alpha) - 2\beta). \end{aligned}$$

Now,

$$\begin{aligned} \min(a\beta, -2\tau - 2\beta, (\gamma - \gamma_0)(d - 2\alpha) - 2\beta, \dots, (\gamma_{\kappa-3} - \gamma_{\kappa-2})(d - \kappa\alpha) - 2\beta, \\ (\gamma_{\kappa-2} - \gamma_{\kappa-1})(d + 1 - \kappa\alpha) - 2\beta) = \min(a\beta, -2\tau - 2\beta, \min((\gamma - \gamma_0)(d - 2\alpha), \dots, \\ (\gamma_{\kappa-3} - \gamma_{\kappa-2})(d - \kappa\alpha), (\gamma_{\kappa-2} - \gamma_{\kappa-1})(d + 1 - \kappa\alpha)) - 2\beta), \end{aligned}$$

and the first two terms  $a\beta$  and  $-2\tau - 2\beta$  do not depend on  $\gamma, \gamma_0, \dots, \gamma_{\kappa-1}$ . Therefore, using the fact that  $\sup_{\gamma} \min(A, B, C(\gamma)) = \min(A, B, \sup_{\gamma} C(\gamma))$ , where  $A$  and  $B$  do not depend on  $\gamma$ , we obtain

$$\begin{aligned} \varkappa_0 = \sup_{\beta > 0} \min \left( a\beta, -2\tau - 2\beta, \sup_{\gamma \in (0,1)} \sup_{\gamma > \gamma_0 > \dots > \gamma_{\kappa-1} = 0} \min((\gamma - \gamma_0)(d - 2\alpha), \right. \\ \left. \dots, (\gamma_{\kappa-3} - \gamma_{\kappa-2})(d - \kappa\alpha), (\gamma_{\kappa-2} - \gamma_{\kappa-1})(d + 1 - \kappa\alpha)) - 2\beta \right). \end{aligned}$$

For fixed  $\gamma \in (0, 1)$  by Lemma 6

$$\begin{aligned} \sup_{\gamma > \gamma_0 > \dots > \gamma_{\kappa-1} = 0} \min((\gamma - \gamma_0)(d - 2\alpha), \dots, (\gamma_{\kappa-3} - \gamma_{\kappa-2})(d - \kappa\alpha), \\ (\gamma_{\kappa-2} - \gamma_{\kappa-1})(d + 1 - \kappa\alpha)) = \frac{\gamma}{\frac{1}{d-2\alpha} + \dots + \frac{1}{d-\kappa\alpha} + \frac{1}{d+1-\kappa\alpha}} \end{aligned}$$

and

$$\sup_{\gamma \in (0,1)} \frac{\gamma}{\frac{1}{d-2\alpha} + \dots + \frac{1}{d-\kappa\alpha} + \frac{1}{d+1-\kappa\alpha}} = \frac{1}{\frac{1}{d-2\alpha} + \dots + \frac{1}{d-\kappa\alpha} + \frac{1}{d+1-\kappa\alpha}}.$$

Thus,  $\varkappa_0 = \sup_{\beta > 0} \min(a\beta, \varkappa_1 - 2\beta) = \frac{a\varkappa_1}{2+a}$ .

Finally, from (24) for  $\tilde{\varkappa}_1 < \varkappa_1$  the first statement of the theorem follows.

Now let us consider the case  $\tau = 0$ . In this case by Theorem 1.5.6 in [5] for any  $s > 0$  and sufficiently large  $r$

$$g(r) > r^{-s}. \quad (26)$$

Combining estimates (15), (16), (19), (21), (22), replacing all powers of  $r$  for  $g^2(r)$  using (26), and choosing  $\varepsilon_1 := g^\beta(r)$ ,  $\beta \in (0, 1)$  we obtain

$$\rho\left(\frac{\kappa! K_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_\kappa(\Delta)\right) \leq C (g^2(r) + g^{a\beta}(r) + g^{2-2\beta}(r)).$$

Since  $\sup_{\beta \in (0,1)} \min(2, a\beta, 2-2\beta) = \frac{2a}{2+a}$ , it follows that

$$\rho\left(\frac{\kappa! K_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_\kappa(\Delta)\right) \leq C g^{\frac{2a}{2+a}}(r).$$

This proves the second statement of the theorem.  $\square$

**Remark 13.** The derived rate does depend on the magnitude of the higher-order terms. Namely, the upper bound  $C r^{2d-\kappa\alpha} L^\kappa(r) r^{-\frac{\alpha(d-\kappa\alpha)}{d-(\kappa-1)\alpha} + \delta}$  for the higher-order terms is given in the estimate of  $\text{Var } S_r$ . Then, this bound appears in the final expression of the rate of convergence as  $\varkappa < \min\left(\frac{a}{2+a} \times \frac{\alpha(d-\kappa\alpha)}{d-(\kappa-1)\alpha}, \frac{a}{2+a} \varkappa_1\right)$ .

For the particular case  $\kappa = 2$ , the importance of the contribution of higher-order terms was illustrated in [1]. In Example 6 [1], it was shown that the contribution of the high-order terms can be larger than the contribution of the leading 2nd order term. The analogous arguments and examples are valid for any  $\kappa \geq 2$ .

**Remark 14.** The bound (15) in the above proof requires to estimate  $\rho(X_{r,\kappa}^*, X_\kappa(\Delta))$  which is the Kolmogorov distance between two Wiener-Itô integrals of the same order. It can be written as  $\rho(I_\kappa(f_r), I_\kappa(f))$ , where  $f_r$  and  $f$  are appropriate functions. This distance is estimated as

$$\rho(I_\kappa(f_r), I_\kappa(f)) \leq C \|f_r - f\|^{\frac{1}{\kappa+1/2}}. \quad (27)$$

Since the Kolmogorov distance  $\rho(\cdot)$  is majorised by the total variation distance  $\rho_{TV}(\cdot)$  ( $\rho(\xi, \eta) \leq \rho_{TV}(\xi, \eta)$ ), the result  $\rho_{TV}(I_\kappa(f_r), I_\kappa(f)) \leq C \|f_r - f\|^{\frac{1}{\kappa}}$  in [11] would be an improvement of our estimate. However, only a sketch of a proof was provided in [11], and [31] questioned the result. Therefore, the new bound  $\rho_{TV}(I_\kappa(f_1), I_\kappa(f_2)) \leq C \|f_1 - f_2\|^{\frac{1}{2\kappa}}$  was proved in [31]. Note, that this result is worse than ours for the Kolmogorov distance. Thus, estimate (27) was used as the best available fully proven self-contained result.

**Remark 15.** It follows from the proof that similar upper bounds exist in  $L_2$ , discrepancy, Lévy, and other equivalent metrics.

**Remark 16.** If  $\tau \in (-\frac{d-\kappa\alpha}{2}, 0)$ , then the convergence rate in Theorem 5 is  $o(r^{-\varkappa})$ , where  $\varkappa$  is strictly smaller than the critical value  $\varkappa_c = \frac{a}{2+a} \min\left(\frac{\alpha(d-\kappa\alpha)}{d-(\kappa-1)\alpha}, \varkappa_1\right)$ . This is due to the presence of slowly varying functions in the assumptions. A slowly varying

function can be unbounded, but, by Remark 3, for an arbitrary positive  $\delta$  it holds  $L(r) < r^\delta$  if  $r$  is large enough. Hence, the inequality for  $\varkappa$  must be strict because Theorem 5 derives the *power* upper bound. In the particular case, when slowly varying functions are replaced by constants, the multiplier  $r^\delta$  in the proof is redundant and the rate of convergence would be of order  $r^{-\varkappa_c}$ .

**Remark 17.** The upper bound on the rate of convergence in Theorem 5 is given by explicit formulae that are easy to evaluate and analyze. For example, for fixed values of  $\alpha$  and  $\kappa$  it is simple to see that the upper bound for  $\varkappa$  approaches  $\frac{a}{2+a} \min(\alpha, -2\tau)$ , when  $d \rightarrow +\infty$ . For fixed values of  $d$  and  $\kappa$  the upper bound for  $\varkappa$  is of the order  $O(d - \kappa\alpha)$ , when  $\alpha \rightarrow d/\kappa$ . This result is expected as the value  $\alpha = d/\kappa$  corresponds to the boundary where a phase transition between short- and long-range dependence occurs.

## 6. Conclusion

The rate of convergence to Hermite-type limit distributions in non-central limit theorems was investigated. The results were obtained under rather general assumptions on the spectral densities of the considered random fields, that weaken the assumptions used in [1]. Similar to [1], the direct probabilistic approach was used, which has, in our view, an independent interest as an alternative to the methods in [7, 28, 29]. Additionally, some fine properties of the probability distributions of Hermite-type random variables were investigated. Some special cases when their probability density functions are bounded were discussed. New anti-concentration inequalities were derived for Lévy concentration functions.

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