Starshapedness and Convexity in Carnot Groups and Geometry of Hörmander Vector Fields

Doaa Filali

Advisor: Dr. Federica Dragoni

A thesis presented for the degree of
Doctor of Philosophy

Cardiff University
Mathematic Department
2018

Contents

Introduction.

Sub-Riemannian geometries are very important, not only for a pure mathematics point of view but also for the many applications in physics (see [73]), in economics (see e.g [76]), in biology and image processing (for example visual vertex model by Citti-Sarti [33]).
Sub-Riemannian geometries are manifolds where the Riemannian metric is defined only on subbundle of the tangent bundle, and this leads to constrains on the allowed directions when moving on the manifold. In particular, we look at the case when the distribution generating the subbundle satisfies the bracket generating condition (see Definition 2.1.1). This implies that, even if some directions are forbidden, we can still move everywhere on the manifold. Unlike the Riemannian case, these geometries are not equivalent to the Euclidean space at any scaling, and presents substantial differences w.r.t. more known Riemannian case (see Section 2.2).
In this thesis, we first study sub-Riemannian manifolds, focusing on geodesics and the relations between the geodesic distance and the bracket generating condition (see Section 2.3). In particular, an important subcase of sub-Riemannian manifolds is given by the case of Carnot groups. Carnot groups are non-commutative stratified nilpotent groups where a much richer structure is available. Then many notions, that can be introduced and studied in Carnot groups, cannot be extended to more general sub-Riemannian geometries, for example: the homogenous distance (see Definition 3.3), the strongly $\mathbb{G}-$starshapedness (see Definition 5.2.1) and many others. The most important example of a Carnot group is the Heisenberg group. We study this geometry in detail, and use it as a reference model to show many results explicitly.
To work on more general sub-Riemannian manifolds, we use the vector field structure, which is deeply connected to a control theory point of view; they allow us to study several results also in sub-Riemannian manifolds.

The aim is to study geometric properties in Carnot groups and extend them whenever possible to more general sub-Riemannian structures. Geometrical properties for sets are an important tool in many areas of mathematics. In particular, they have important applications to the study of PDE's problems. In this thesis, we focus on two specific geometrical properties of sets: convexity and starshapedness.
In the Euclidean space, it is very well-known that starshapedness and convexity are deeply connected. Several generalisations of these notions in this geometrical context are possible, and they are not always equivalent to each other. For example, geodesic convexity is very used in the context of Riemannian manifolds, but cannot be applied to Carnot groups and sub-Riemannian geometries (see Monti-Rickly [74]).
Danielli-Garofalo-Nhieu [39] and Manfredi-Stroffolini-Lu-Juutinen [61] introduced the concept of $\mathcal{H}-$convexity (also named horizontal convexity) for functions defined in Carnot groups. Later, Bardi-Dragoni [10, 11, 12] developed a more general definition for convex functions in the setting of sub-Riemannian manifolds by using the idea of convexity along vector fields. In this framework, we apply the notion of convexity along vector fields to sets. We also generalise some relationships between convex sets and convex functions, that were already known in the case of Carnot groups (see [39]) to more general geometries of vector fields: the properties for level sets and for indicator functions (see Theorems 4.3.3 and 4.3.5), and the characterisation by the epi-graph (see Theorem 4.3.4).
Moreover, we investigate the notion of starshaped sets, also known as starlike sets: we can say that starshaped sets satisfy the same geometric characterisation of convex sets, like level sets and starshaped hull but not w.r.t. all its interior points. Starshaped sets are not yet completely understood in Carnot groups and sub-Riemannian manifolds. So we first consider the nature of starshapedness in Carnot groups. We consider two different notions of starshaped sets in Carnot groups: the first one is called strongly $\mathbb{G}-$starshapedness and the second one is called weakly $\mathbb{G}-$starshapedness (see Definitions 5.2.1 and 5.2.4), by considering, respectively, the anisotropic dilations associated to Carnot groups for the first and the concept of curves with constant horizontal velocity w.r.t. given vector fields for the second one; the second definition thus working also in general sub-Riemannian geometries. More precisely, in the definition of strongly $\mathbb{G}-$starshapedness, we have two different cases: the first

one when the set Ω is strongly $\mathbb{G}-$starshaped w.r.t. the origin $0 \in \Omega$ which means that Ω satisfies the following condition:

$$\delta_t(\Omega) \;\subseteq\; \Omega, \;\forall\, t \in [0,1], \tag{1}$$

where $\delta_t(\Omega) \;=\; \Big\{\delta_t(P) \in \mathbb{G} \,: P \in \Omega\Big\}$. In the second case when Ω is strongly $\mathbb{G}-$starshaped w.r.t. any generic point $P_0 \in \mathbb{G}$, this means that the left-translated set $\Omega' \;:=\; L_{P_0}(\Omega) \;=\; \{Q \in \mathbb{G} : Q = (-P_0) \circ P,\ \text{for some}\ \ P \in \Omega\}$ is strongly starshaped w.r.t. 0, i.e. (1) holds for Ω'.

As an example, we apply these two notions on the Heisenberg group, where the dilations are given by:

$$\delta_t(\Omega) \;=\; \Big\{\delta_t(P) = (tx_1, ..., tx_{2n}, t^2 x_{2n+1}) : P \in \Omega \subseteq \mathbb{R}^{2n+1}\Big\},$$

i.e. they are Euclidean in all the components except the last one, where instead they scale as t^2.

We say that the open set Ω is weakly $\mathbb{G}-$starshaped w.r.t. P_0 (see Definition 5.2.4) if and only if, for all Q belonging to the $\mathcal{X}-$plane, denoted by $\mathbb{V}_{P_0}$ (see Definition 4.3.2), the $\mathcal{X}-$line segment joining P_0 and Q (see Definition 4.3.1) all belongs to Ω.

Note that the definition of weakly $\mathbb{G}$-starshaped is a weaker notion than that of strongly $\mathbb{G}-$starshaped (see Example 5.3.3). Moreover, $\mathcal{X}-$lines in the sub-Riemannian manifolds can turn out to be (Euclidean) straight lines as in the case of the Heisenberg group $\mathbb{H}^1$ (see Example 4.3.1), or more general smooth curves: for example in the Grušin plane they are parabolas (see Example 4.3.3).

We study the mutual relations between these two new notions in Carnot groups, and their relations w.r.t. the standard Euclidean starshapedness. We find counterexamples to prove that, unlike the standard Euclidean case, strongly $\mathbb{G}-$starshapedness and weakly $\mathbb{G}-$starshapedness are not equivalent, and both differ from the standard Euclidean notion (see Section 5.3).

In particular, strongly $\mathbb{G}-$starshapedness is a very interesting geometrical property connected with the natural rescaling defined on Carnot groups but, as down side, when it holds w.r.t. any internal points, it is not equivalent to $\mathcal{H}-$convexity. On the other hand weakly $\mathbb{G}-$starshapedness w.r.t. each internal points is indeed equivalent to the notion of convexity introduced by Bardi-Dragoni, and so to $\mathcal{H}-$convexity too, but it is a very week property

that for example seems to be not preserved by the level sets of solutions of PDEs defined on starshaped rings, while instead strongly $\mathbb{G}-$starshapedness is preserved in this kind of PDE problems (see [46, 38]).
In Chapter 1 we recall some basic notions of topological manifolds. Then we introduce the Riemannian manifolds, and give a brief description of their geodesics and some examples.
In Chapter 2 we introduce sub-Riemannian manifolds and give a definition of vector fields satisfying the Hörmander condition. We also look at some very important examples like: the Heisenberg group, the Grušin plane, the Engel group and others. We spotlight some important differences between Riemannian and sub-Riemannian manifolds: in particular we focus on geodesics and the relations between minimizing geodescis and solutions of the equation of geodesics. We solve, in detail, the geodesic equations in the 1-dimensional Heisenberg group $\mathbb{H}^1$. At the end of the chapter, we study a geometry which does not satisfy the Hörmander condition, but still has a finite geodesics distance (i.e. it is always possible to connect any two points by admissible curves).
In Chapter 3 we give a short introduction about Lie groups, Lie algebras and Carnot groups. Also, we explicitly study the connection between the structures of Lie groups and the manifold structure, in both the Heisenberg group and the Engel group.
In Chapter 4 we start by recalling the notion of convexity in the Euclidean space, and then how the notion has been generalized to Carnot groups, and also to general sub-Riemannian manifolds and geometries of vector fields on $\mathbb{R}^n$. Also, we study the properties for level sets and the characterisation by the epi-graph.
In Chapter 5 we introduce the notion of starshaped sets in the Euclidean setting then in Carnot groups and in the more general case of geometry of vector fields (that include the Hörmander case). Next we investigate the relations between the different notions that we have introduced and explain that by using many examples. Furthermore we study how all these notions of starshaped sets are related to convex sets in the Euclidean space and Carnot groups.
Finally, in Chapter 6 we give a short summary of the all results we obtain in this thesis and were they will be published.

Part I

Riemannian and Sub-Riemannian Manifolds

Chapter 1

Riemannian manifolds.

In this chapter we introduce some important basic notions that we will need later. The literature of Riemannian manifolds is extremely large. We in particular, refer the reader to [1, 23, 65, 66, 67, 68, 90].

1.1 Topological Manifolds.

Definition 1.1.1. Let M be a nonempty set. A ***topology on*** M, or ***usual topology if*** $M = \mathbb{R}^n$, is a collection τ of subsets of M, called ***open subsets***, satisfying:

1. $\emptyset$ and $M \in \tau$,
2. $\bigcup\limits_{i \in I} U_i \in \tau, \quad \forall\ U_i \in \tau, \ \forall\ I \subseteq \mathbb{R}.$
3. $\bigcap\limits_{i=1}^{n} U_i \in \tau, \quad \forall\ U_i \in \tau$ for $i = 1, 2, .., n.$

We call the pair (M, τ) a ***topological space,*** or simply said M is a ***topological space***.

Example 1.1.1.

1. $M_1 = \mathbb{R}^n$ with the usual open sets is a topological space.
2. $M_2 = \{1, 2, 3\}$, with:

(i) $\tau_1 = \{M_2, \emptyset\}$ is always a trivial topological space.

(ii) $\tau_2 = \{M_2, \emptyset, \{1\}\}$ is a topological space.

(iii) $\tau_3 = \{M_2, \emptyset, \{1\}, \{2\}\}$ is not a topology on M_2, because $\{1\} \cup \{2\} = \{1, 2\} \notin \tau_3$.

(iv) $\tau_4 = \{M_2, \emptyset, \{1\}, \{1, 2\}\}$ is a topological space since $\{1\} \cup \{1, 2\} = \{1, 2\} \in \tau_4$ and $\{1\} \cap \{1, 2\} = \{1\} \in \tau_4$.

Definition 1.1.2. Let M_1, M_2 be two topological spaces and define the **pre-image function** ϕ^{-1} of $\phi : M_1 \to M_2$ as $\phi^{-1} : M_2 \to M_1$ with $\phi^{-1}(U_2) = U_1$, where U_1, U_2 are open subsets in M_1, M_2 respectively.
ϕ is called ***continuous*** if for each open subset U_2 in M_2 the pre-image $\phi^{-1}(U_2) = U_1$ is open in M_1.

Example 1.1.2. Consider the sets $M_1 = \{1, 2, 3\}$ and $M_2 = \{1, 2\}$ both with the topology $\tau_i = \Big\{M_i, \emptyset, \{1\}\Big\}$, for $i = 1, 2$, given in Example 1.1.1. Define $\phi : M_1 \to M_2$ then ϕ is always continuous as long as $1 \mapsto 1$.

Definition 1.1.3. Let M and M_2 be two topological spaces. A continuous bijective function $\phi : M_1 \to M_2$ with continuous inverse is called ***homeomorphism*** and we say that M_1, M_2 are ***homeomorphic.***

Example 1.1.3.

1. Consider $(\mathbb{R}, \tau)$, where the topology $\tau = \Big\{(-a, a) : a > 0\Big\} \cup \Big\{\emptyset, \mathbb{R}\Big\}$. If we take:

   (i) $\phi_1 : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ with $x \mapsto x+1$, then for an open set $U = (-a, a)$ in $\mathbb{R}$ we can see that the pre-image of $\phi_1^{-1}(U) = (-a - 1, a - 1)$ is not open, which means ϕ_1 is NOT continuous. In fact if we take $a = \frac{1}{2} > 0$ then:

   $$\phi_1^{-1}\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right) \;=\; \left(-\frac{3}{2}, -\frac{1}{2}\right) \;\notin\; \tau.$$

   (ii) $\phi_2 : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ with $x \mapsto x^2$, then for the open set $U = (-a, a)$ in $\mathbb{R}$ we can see that the pre-image of $\phi_2^{-1}(U) = \Big(-\sqrt{a}, \sqrt{a}\Big)$ is open, which means ϕ_2 is continuous.

2. Let $M = \{1, 2, 3\}$ with the topology $\tau_4 = \Big\{M, \emptyset, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}\Big\}$. If we take $\phi : (M, \tau_2) \to (M, \tau_2)$ with

$$\phi(1) \quad = \quad 2, \quad \phi(2) \quad = \quad 1, \quad \text{and} \quad \phi(3) \quad = \quad 1.$$

Then we can see that the all pre-images $\{1\}$ and $\{2\}$ of ϕ are open, which means ϕ is continuous.

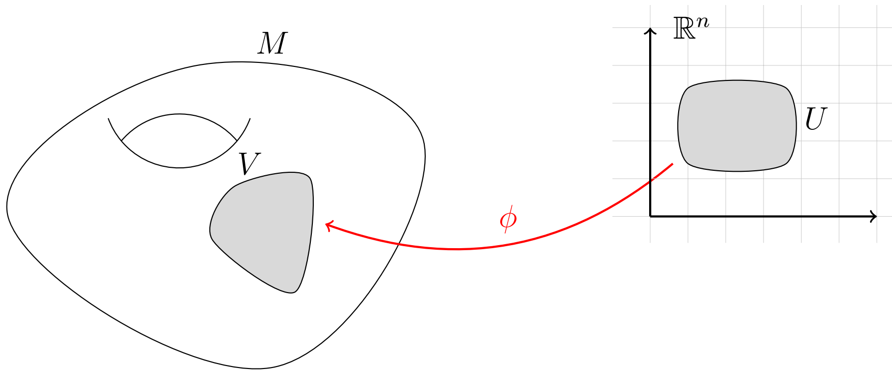

Figure 1.1: `Chart function.`

Definition 1.1.4. Let M be a topological space. A ***chart***, or n***-dimensional chart,*** for M is an homeomorphism function $\phi : U \to V$ where U is an open subset in $\mathbb{R}^n$ with the usual topology on $\mathbb{R}^n$ and $V \subseteq M$.
See Figure 1.1.

Example 1.1.4. Consider the unit circle $S^1 = \Big\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\Big\}$. Let $U = \Big\{(\sin 2\pi t, \cos 2\pi t) : 0 < t < 1\Big\} \subset S^1$ and define the homeomorphism function ϕ as follows:

$$\phi : (0, 1) \subset \mathbb{R} \to U \quad \text{with} \quad t \mapsto \Big(\sin 2\pi t, \cos 2\pi t\Big).$$

So we can see that ϕ is a 1-dimensional chart for S^1.

Formally, $(x_1, x_2, ..., x_n) = \phi^{-1}(P)$ then we call the n-real numbers $x_1, x_2, \ldots, x_n$, the **coordinates** of point $P \in V \subset M$. For this reason charts for M can be called a **coordinate system** of M.

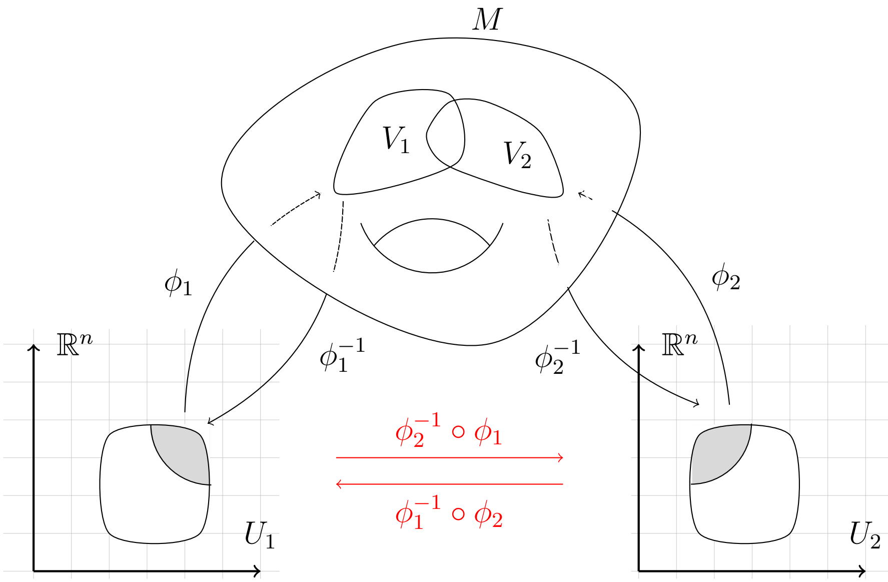

Figure 1.2: `Transition functions.`

Definition 1.1.5. Let M be a topological space. Define two charts $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ for M such that $V_1 \cap V_2 \neq \emptyset$. Then we define the ***transition functions*** $F_{1,2}$ and $F_{2,1}$ as

$$\begin{aligned} F_{1,2} &: \phi_2^{-1}(V_1 \cap V_2) \to \phi_1^{-1}(V_1 \cap V_2) \\ F_{2,1} &: \phi_1^{-1}(V_1 \cap V_2) \to \phi_2^{-1}(V_1 \cap V_2), \end{aligned}$$

given respectively by

$$F_{1,2}(P) := \phi_1^{-1} \circ \phi_2(P) \quad \text{and} \quad F_{2,1}(P) := \phi_2^{-1} \circ \phi_1(P).$$

The transition functions can also be called ***change of coordinates***.

Remark 1.1.1. Not that to make the charts compatible we require the transition functions to be homeomorphisms.

Remark 1.1.2. To require that the transition function $F_{1,2}$ is a homeomorphism is equivalent to requiring that the transition function $F_{2,1}$ is a homeomorphism. In fact we can write $F_{1,2} = F_{2,1}^{-1}$.

Example 1.1.5 (Stereographic coordinate)**.** Consider the unit circle

$$S^1 = \Big\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\Big\}.$$

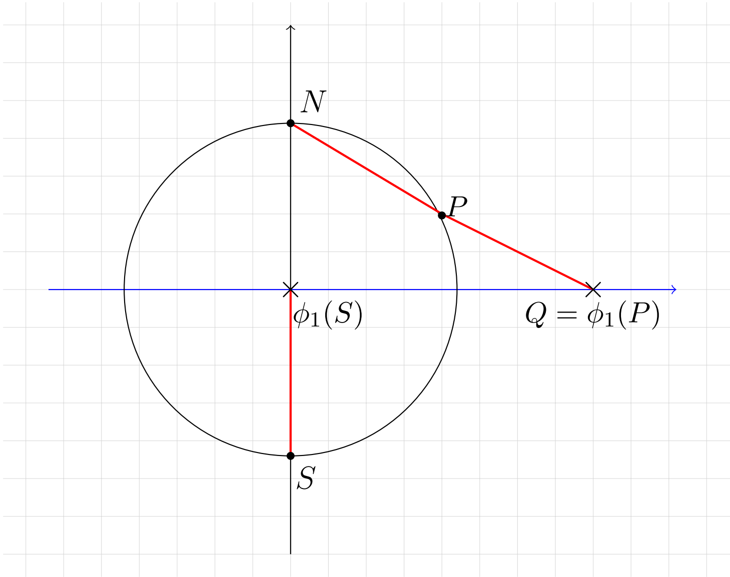

Figure 1.3: `Stereographic projection.`

We now construct the function

$$\phi_1^{-1} : S^1 \setminus \{N\} \to \mathbb{R} \quad \text{with} \quad P \mapsto \phi_1^{-1}(P) \;=\; Q,$$

where Q is defined at the intersection of the line $x_2 = 0$ (x_1-axis) with the unique straight line passing from the North Pole $N = (0, 1)$ and the point $P = (x_1^P, x_2^P) \in S^1 \setminus \{N\}$, see Figure 1.3.
Now we compute the coordinate the coordinates of Q explicitly. Recall that the line between N and P is given by

$$x_2 \;=\; 1 + \frac{x_2^P - 1}{x_1^P} x_1.$$

So we need to find the solution of

$$\begin{cases} x_2 \;=\; 0, \\ x_2 \;=\; 1 + \dfrac{x_2^P - 1}{x_1^P} x_1, \end{cases} \tag{1.1}$$

for all $(x_1^P, x_2^P) \in S^1 \setminus \{N\}$ which implies

$$x_1 \;=\; \frac{x_1^P}{1 - x_2^P}.$$

With abuse of notation we now call the coordinates of P simply (x_1, x_2) and so we can define

$$u = \phi_1^{-1}(P) = \frac{x_1}{1-x_2}. \tag{1.2}$$

Since $P \in S^1$ then

$$x_1^2 + x_2^2 = 1. \tag{1.3}$$

From (1.2) we can find

$$x_2 = \frac{x_1}{u}. \tag{1.4}$$

Equation (1.4) together with equation (1.3) gives

$$x_1^2 + 1 - \frac{2x_1}{u} + \frac{x_1^2}{u^2} = 1.$$

So

$$x_1 = 0 \quad \text{or} x_1 = \frac{2u}{1+u^2}. \tag{1.5}$$

Note $x_1 = 0$ implies $x_2 = \pm 1$ and $P = N$ or $P = S$ so in our case $P = S$. Summing up we can write

$$(x_1, x_2) = \left(\frac{2u}{u^2+1}, \frac{u^2-1}{u^2+1}\right).$$

This coordinates are good for any point on S^1 except the North Pole $N = (0, 1)$. To find chart for N we consider the map

$$\phi_2 : \mathbb{R} \to S^1 \setminus \{S\} \quad \text{with} \quad \tilde{u} \mapsto \left(\frac{2\tilde{u}}{\tilde{u}^2+1}, \frac{1-\tilde{u}^2}{\tilde{u}^2+1}\right),$$

defined as

$$(x_1, x_2) = \left(\frac{2\tilde{u}}{\tilde{u}^2+1}, \frac{\tilde{u}^2-1}{\tilde{u}^2+1}\right).$$

Again, this coordinate is good for any point on S^1 except the South Pole. Now let $V_1 = S^1 \setminus \big\{(0,1)\big\}$ and $V_2 = S^1 \setminus \big\{(0,-1)\big\}$. Define ϕ_1 and ϕ_2 to be two charts for S^1 as follows:

$$\phi_1 : \mathbb{R} \to V_1 \quad \text{with} \quad u \mapsto \left(\frac{2u}{u^2+1}, \frac{u^2-1}{u^2+1}\right)$$

$$\phi_2 : \mathbb{R} \to V_2 \quad \text{with} \quad \tilde{u} \mapsto \left(\frac{2\tilde{u}}{\tilde{u}^2+1}, \frac{1-\tilde{u}^2}{\tilde{u}^2+1}\right).$$

Obviously ϕ_1^{-1}, ϕ_2^{-1} are given by

$$\begin{aligned} \phi_1^{-1}(x_1, x_2) &= u = \frac{x_1}{1-x_2}, \\ \phi_2^{-1}(x_1, x_2) &= \tilde{u} = \frac{x_1}{1+x_2}. \end{aligned}$$

Next we want to check that the transition functions are homeomorphism. In our case

$$V_1 \cap V_2 = \Big(S^1 \setminus \{N\}\Big) \cap \Big(S^1 \setminus \{S\}\Big) = S^1 \setminus \{N, S\}.$$

So the transition functions for these charts are

$$F_{1,2} : \phi_2^{-1}(U) \to \phi_1^{-1}(U) \text{ and } F_{2,1} : \phi_1^{-1}(U) \to \phi_2^{-1}(U),$$

(where $U = S^1 \setminus \{N, S\}$) with

$$\begin{aligned} F_{1,2}(\tilde{u}) := \phi_1^{-1} \circ \phi_2(\tilde{u}) &= \phi_1^{-1}\left(\frac{2\tilde{u}}{\tilde{u}^2+1}, \frac{1-\tilde{u}^2}{\tilde{u}^2+1}\right) \\ &= \frac{\dfrac{2\tilde{u}}{\tilde{u}^2+1}}{1-\dfrac{1-\tilde{u}^2}{\tilde{u}^2+1}} = \frac{1}{\tilde{u}}, \end{aligned}$$

and

$$\begin{aligned} F_{2,1}(u) : \phi_2^{-1} \circ \phi_2(u) &= \phi_2^{-1}\left(\frac{2u}{u^2+1}, \frac{u^2-1}{u^2+1}\right) \\ &= \frac{\dfrac{2u}{u^2+1}}{1+\dfrac{u^2-1}{u^2+1}} = u. \end{aligned}$$

We first need to show that they are homeomorphism so it is enough to show that both $F_{1,2}$ and $F_{2,1}$ are continuous to get that they are homeomorphism. In fact $F_{1,2}(\tilde{u}) = \dfrac{1}{\tilde{u}}$ is invertible whenever it is well-defined, i.e. for $\tilde{u} \neq 0$ and $\phi_2^{-1}(U) = \mathbb{R} \setminus \{0\}$. We notice that $F_{1,2} = F_{2,1}^{-1}$.

Example 1.1.6. Consider $M = \mathbb{R}^2$ and let $V_1 = \mathbb{R}^2$ and $V_2 = \mathbb{R}^2 \setminus \{(x_1, x_2) \in$

$\mathbb{R}^2 : x_1 < 0, x_2 = 0\}$. Define two charts on $\mathbb{R}^2$ as follows:

$$\phi_1 : M \to V_1 \quad \text{with} \quad (x_1, x_2) \mapsto (x_1, x_2),$$
$$\phi_2 : M \to V_2 \quad \text{with} \quad (x_1, x_2) \mapsto \left(r = \sqrt{x_1^2 + x_2^2}, t = \tan^{-1}\left(\frac{x_2}{x_1}\right) \right)$$

where r and t are the polar coordinates.
We can check that both ϕ_1 and ϕ_2 are homeomorphism easily, since both are continuous functions and their inverses are continuous also. Now we obtain the transition functions:

$$\begin{aligned} F_{1,2} : \phi_1^{-1} \circ \phi_2 \quad &\text{with} \quad (r, t) \quad \mapsto \Big(x_1 = r\cos t, x_2 = r\sin t \Big), \\ F_{2,1} : \phi_2^{-1} \circ \phi_1 \quad &\text{with} \quad (x_1, x_2) \mapsto (r, t). \end{aligned}$$

According to Remark 1.1.2 we can easily see that $F_{1,2} = F_{2,1}^{-1}$, in fact:

$$\begin{aligned} F_{2,1}^{-1} &= (\phi_2^{-1} \circ \phi_1)^{-1}(x_1, x_2) \\ &= \phi_1^{-1}(\phi_2(x_1, x_2)) \\ &= \phi_1^{-1}(r, t) \\ &= (r, t) \\ &= F_{1,2}. \end{aligned}$$

Definition 1.1.6. Consider $\phi : U_1 \to U_2$ a continuous function, where $U_1, U_2 \subseteq \mathbb{R}^n$. Then:

1. ϕ is called ***smooth (or infinitely differentiable)***, denoted by C^∞, if it is infinitely differentiable.

2. ϕ is called a ***diffeomorphism*** if it is bijective smooth function and also its inverse is smooth.

Definition 1.1.7. Let M be a topological space. Two charts $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ for M are ***compatible*** if either:

1. $U_1 \cap U_2 = \emptyset$ or

2. the transition functions $F_{1,2}$ and $F_{2,1}$ are diffeomorphisms.

Definition 1.1.8. Let M be a topological space. An ***atlas***, or n***-dimensional atlas,*** for M is a collection $\mathcal{A}$ of compatible charts $\phi_i : U_i \to V_i$, $i = 1, 2, ...m$, such that the union of all the images cover M, i.e.

$$M = \bigcup_i V_i,$$

where $V_i = \phi_i\Big(U_i\Big)$.

Example 1.1.7. In Example 1.1.5, we can see that ϕ_1 and ϕ_2 are both homeomorphisms and we have $\mathcal{A} = \Big\{V_1, V_2\Big\}$ is a 1-dimensional atlas for S^1, where $S^1 = \bigcup_{i=1}^{2} V_i$.

Definition 1.1.9. Let M be a topological space then an atlas $\mathcal{A}$ for M is called ***smooth*** or ***differentiable*** if it is consists of a compatible charts, that means that all the sets $\phi^{-1}(V_i \bigcap V_j)$ are open in M and the transition functions $F_{i,j}$ are smooth.

Example 1.1.8. According to Example 1.1.7 we defined an atlas $\mathcal{A}$ for the unit circle S^1. We can notice that the same atlas $\mathcal{A}$ is a smooth atlas by checking the charts ϕ_1 and ϕ_2, see Example 1.1.7, are compatible, so:

1. First, by computing the inverse of ϕ_1, we get:

$$\phi_1^{-1} : \mathbb{R}^2 \to V_1, \text{ with } (x_1, x_2) \mapsto \frac{x_1}{1 - x_2}.$$

2. Secondly, by computing the transition functions $F_{1,2}(u) = \dfrac{1}{u}$ and $F_{2,1}(u) = u$ are diffeomorphism.

Thus, $\mathcal{A}$ is 1- dimensional smooth atlas for S^1.

Definition 1.1.10. We say that the topological space M is n***-dimensional smooth manifold*** if M is endowed with a smooth n-dimensional atlas and the number n is called the ***dimension*** of M.

Example 1.1.9. In Example 1.1.5 we can see that the transition functions are smooth. We conclude that S^1 with the atlas defined in Example 1.1.7 is a smooth manifold of dimension 1.

Example 1.1.10. Consider the unit sphere

$$S^2 = \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\Big\}.$$

Define the homeomorphisms

$$\begin{aligned}
\phi_1 : \mathbb{R}^2 \to V_1 \quad \text{where} \quad & V_1 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0\right\} \\
\text{with} \quad & (x_1, x_2) \mapsto \left(\sqrt{1 - x_1^2 - x_2^2}, x_1, x_2\right), \\
\phi_2 : \mathbb{R}^2 \to V_2 \quad \text{where} \quad & V_2 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 < 0\right\} \\
\text{with} \quad & (x_1, x_2) \mapsto \left(-\sqrt{1 - x_1^2 - x_2^2}, x_1, x_2\right), \\
\phi_3 : \mathbb{R}^2 \to V_3 \quad \text{where} \quad & V_3 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 > 0\right\} \\
\text{with} \quad & (x_1, x_2) \mapsto \left(\sqrt{1 - x_1^2 - x_2^2}, x_1, x_2\right)
\end{aligned}$$

and

$$\begin{aligned}
\phi_4 : \mathbb{R}^2 \to V_4 \quad \text{where} \quad & V_4 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 < 0\right\} \\
\text{with} \quad & (x_1, x_2) \mapsto \left(-\sqrt{1 - x_1^2 - x_2^2}, x_1, x_2\right).
\end{aligned}$$

Now by evaluating the inverses we have:

$$\begin{aligned}
&\phi_1^{-1} : V_1 \to \mathbb{R}^2 \quad \text{with} \quad (x_1, x_2, x_3) \mapsto (x_1, x_3), \\
&\phi_2^{-1} : V_2 \to \mathbb{R}^2 \quad \text{with} \quad (x_1, x_2, x_3) \mapsto -(x_1, x_3), \\
&\phi_3^{-1} : V_3 \to \mathbb{R}^2 \quad \text{with} \quad (x_1, x_2, x_3) \mapsto (x_2, x_3) \ \text{ and} \\
&\phi_1^{-1} : V_1 \to \mathbb{R}^2 \quad \text{with} \quad (x_1, x_2, x_3) \mapsto -(x_2, x_3).
\end{aligned}$$

Let us compute the transition functions (we do one of them and the others can be computed similarly in the same way) $F_{1,3} : \phi_1^{-1}(V) \to \mathbb{R}$ with $V = V_1 \cap V_3$ defined by

$$\begin{aligned}
F_{1,3}(x_1, x_2) \;=\; \phi_1^{-1} \circ \phi_3(x_1, x_2) \;&=\; \phi_1^{-1}\left(\sqrt{1 - x_1^2 - x_2^2}, x_1, x_2\right) \\
&=\; \left(\sqrt{1 - x_1^2 - x_2^2}, x_2\right).
\end{aligned}$$

Since $\phi_1^{-1} \circ \phi_2$ is a diffeomorphism and we can notice that the co-domain (or the images) $V_i, i = 1, 2, 3, 4,$ cover the unite sphere S^2 then the collection $\mathcal{A} = \{V_1, V_2, V_3, V_4\}$ is a smooth atlas for S^2. Thus S^2 is a 2-dimension smooth manifold.

Definition 1.1.11. Let M be a manifold, any open subset $U \subseteq M$ which itself has the structure of a manifold is called a **submanifold** of M.

Proposition 1.1.1. *Let M be a n-dimensional smooth manifold and $U \subseteq M$, non-empty open subset, then U is itself a n-dimensional smooth manifold. We called any open subset a **submanifold** of M.*
For a proof see [66, Example 1.26].

Example 1.1.11. [General linear group] Consider the general linear group is the set of invertible $n \times n$ matrices with real entries, defined as $GL(n, \mathbb{R}) = \Big\{A\ n \times n\,\text{matrix}\ : \det(A) \neq 0\Big\}$. As the previous example $M(n \times m, \mathbb{R})$ is a $m \times n$-dimensional smooth manifold, and define the function

$$\phi : \mathbb{R} \setminus \{0\} \quad \to \quad M(n \times n, \mathbb{R}) \ \text{ by } \ A \mapsto \det(A)$$

so ϕ is continuous function (it is a polynomial in the entries of the matrix). Then, $GL(n, \mathbb{R}) = \phi^{-1}\Big(\mathbb{R} \setminus \{0\}\Big)$ is an open subset of $M(m \times n, \mathbb{R})$. Thus, by using the proposition 1.1.1 $GL(n, \mathbb{R})$ is a n^2-dimensional smooth manifold.

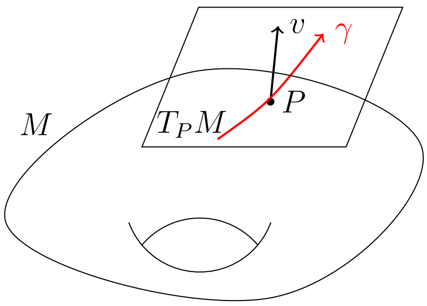

Figure 1.4: `Tangent space` $T_P M$.

Definition 1.1.12. Let M be $n-$smooth manifold and fix any point $P \in M$. A smooth function $\gamma : (-\epsilon, \epsilon) \to M$ is called a ***curve*** on M. Suppose that $\gamma(0) = P$ and let $\mathcal{D}_P$ be the set of all functions on M that are differentiable at P. The ***tangent vector*** to the curve γ at $t = 0$ is a function:

$$\dot{\gamma}(0) : \mathcal{D}_P \to \mathbb{R} \quad \text{with} \quad \dot{\gamma}(0)\, f := \left.\frac{d(f \circ \gamma)}{dt}\right|_{t=0}, \quad f \in \mathcal{D}_P. \tag{1.6}$$

The set of tangent vectors of M at the point P is called the ***tangent space*** of M at P , i.e.

$$T_P M \ := \ \Big\{\dot{\gamma}(0) | \gamma : (-\epsilon, \epsilon) \to M \text{ is smooth}, \ \gamma(0) = P\Big\}.$$

See Figure 1.4.

Remark 1.1.3. If we have a tangent vector at $P \in M \subseteq \mathbb{R}^m$ satisfying Definition 1.1.12 and $Q \in \mathbb{R}^m$, then $Q \in T_P M$ if and only if $\exists\ \epsilon > 0\ \exists\ \gamma : (-\epsilon, \epsilon) \to M$ such that γ is smooth with $\gamma(0) = P$ and $\dot{\gamma}(0) = Q$.

Definition 1.1.13. Let M be a smooth manifold. We define the ***tangent bundle of*** M, denoted by TM, as follows:

$$TM \quad := \quad \Big\{(P, Q) : P \in M, Q \in T_P M\Big\}.$$

Definition 1.1.14. Let M be a $n-$smooth manifold. A ***(smooth) vector field*** X on M is a collection of tangent vectors $X(P) \in T_P M$, one for each point $P \in M$, such that:

$$M \quad \to \quad TM \;\text{ by }\; P \mapsto \Big(P, X(P)\Big) \;\text{ is smooth.}$$

The set of all smooth vector fields on M is denoted by $\mathfrak{X}(M)$.

Associated to a vector field is a smooth map $M \to TM$ whose composition with the projection $\pi : TM \to M$ is the identity map on M. It is very common practice to denote the map from M to TM by X. Thus a vector field can be defined as a smooth map $X : M \to TM$ such that:

$$\pi \circ X \quad = \quad id : M \to M.$$

Such a map is also called a (global) ***section of the tangent bundle***.
A local section of TM can be defined on an open set $V \subseteq M$ and is just the same thing as a vector field on V considered as a submanifold of M. If we consider the chart (V, ϕ) on a $n-$smooth manifold M, then we write $\phi = (\phi_1, ..., \phi_n)$ and we have vector fields defined on V as following:

$$\frac{\partial}{\partial \phi_i} : P \quad \mapsto \quad \left.\frac{\partial}{\partial \phi_i}\right|_P.$$

The ordered set of fields $\left(\dfrac{\partial}{\partial \phi_1}, ..., \dfrac{\partial}{\partial \phi_n}\right)$ is called a ***coordinate frame field***.
In case the smooth vector field X defined on some set is defined on this chart

codomain V, then for some smooth functions X_i defined on V we have:

$$X(P) \;=\; \sum_{i=1}^{n} X_i(P) \left.\frac{\partial}{\partial \phi_i}\right|_P ,$$

or simply just:

$$X(P) \;=\; \sum_{i=1}^{n} X_i \frac{\partial}{\partial \phi_i},$$

and sometimes we will use the standard Einstein summation and simply write

$$X(P) \;=\; X_i \frac{\partial}{\partial \phi_i},$$

omitting the summatary.

Remark 1.1.4. Obviously, if $Q \in T_P M$, then there exists a vector field X where $X(P) = Q$.

Consider the vector fields X, Y on $n-$smooth manifold. We define the addition of vector fields and scaling by real numbers, respectively, by:

$$(X+Y)(P) \;:=\; X(P) + Y(P),$$

$$(\alpha X)(P) \;:=\; \alpha X(P).$$

Then the set $\mathfrak{X}(M)$ is a real vector space. In addition we can define multiplication of a smooth vector field X by a smooth function as follows:

$$(fX)(P) \;:=\; f(P)X(P),$$

commutative algebraic properties hold.
We can explain the set of all vector fields as derivations: the following definitions explain that.

Definition 1.1.15. Let M be a $n-$smooth manifold. A ***derivation*** on $C^\infty(M)$ is a linear map $\mathcal{D} : C^\infty(M) \to C^\infty(M)$ (note that by $C^\infty(M)$ we indicate all the smooth functions $f, g : M \to \mathbb{R}$) such that:

$$\mathcal{D}(fg) \;=\; \mathcal{D}(fg) \;=\; \mathcal{D}(f)g + f\mathcal{D}(g).$$

The set of all derivation of $C^\infty(M)$ by $\mathrm{Der}(C^\infty(M))$.

In this following definition we can notice the difference between a derivation in this sense and a derivation at a point.

Definition 1.1.16. Let X be a vector field on a n-smooth manifold M. The associate map $\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$ given by

$$(\mathcal{L}_X f)(P) \;\; := \;\; X_P f,$$

is called ***Lie derivative***.

It is important to notice that $(\mathcal{L}_X f)(P) = X_P \cdot f = df(X_P)$ for any P and so $\mathcal{L}_X f = df \circ X$. In the case that we have the vector field on an open set U and if f is a function on a domain $V \subseteq U$, then we take $\mathcal{L}_X f$ to be the function defined on V by $P \mapsto X_P f, \;\; \forall\, P \in V$.
We can easily prove that $\mathcal{L}_{\alpha X + \beta Y} = \alpha L_X + \beta \mathcal{L}_Y$ for $\alpha, \beta \in \mathbb{R}$ and $X, Y \in \mathfrak{X}(M)$. The coming result is a significant characterization of smooth vector fields. In particular, it paves the way for the definition of the bracket of any vector fields, which plays an important role in analysis and differential geometry.

Theorem 1.1.1. *For $X \in \mathfrak{X}(M)$, we have $L_X \in \mathrm{Der}(C^\infty(M))$ conversely, if $\mathcal{D} \in \mathrm{Der}(C^\infty(M))$, then $\mathcal{D} = \mathcal{L}_X$ for a uniquely determined $X \in \mathfrak{X}(M)$.*
For a proof see [68, Theorem 2.72].

Because of the previous theorem we can identify

$$\mathrm{Der}(C^\infty(M))$$

with $\mathfrak{X}(M)$ and very often we can write Xf in place of $\mathcal{L}_X f$, i.e.:

$$Xf \;\; = \;\; \mathcal{L}_X f.$$

This allows us to rewrite the derivation law, which is also called Leibniz law, $\mathcal{L}(fg) = g\, \mathcal{L}_X f + f\, \mathcal{L}_X g$ simply as following:

$$X(fg) \;\; = \;\; gXf + fXg.$$

Theorem 1.1.2. *If $\mathcal{D}_1, \mathcal{D}_2 \in \mathrm{Der}(C^\infty(M))$, then $[\mathcal{D}_1, \mathcal{D}_2] \in Der\Big(C^\infty(M)\Big)$ defined by*

$$[\mathcal{D}_1, \mathcal{D}_2] \;\; := \;\; \mathcal{D}_1 \circ \mathcal{D}_2 - \mathcal{D}_2 \circ \mathcal{D}_1,$$

lies in $\mathrm{Der}(C^\infty(M))$.

For a proof see [68, Theorem 2.73].

Lemma 1.1.1. *Let* $X, Y \in \mathfrak{X}(M)$*, then there exists a unique vector field* $[X, Y]$ *such that*

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X.$$

Since $\mathcal{L}_X f$ is also can be written as Xf, we can have:

$$[X, Y]f = X(Yf) - Y(Xf) \quad \text{or} \quad [X, Y] = XY - YX.$$

Definition 1.1.17. The vector field $[X, Y]$ defined as in Lemma 1.1.1, is called the ***Lie bracket*** (or ***commutator***). Notice that since X and Y are smooth then obviously $[X, Y]$ is also smooth.

Proposition 1.1.2 (Properties of the Lie bracket)**.** *The Lie bracket map* $[X, Y] : C^\infty(M) \to C^\infty(M)$ *satisfies the following identities for any* $X, Y, Z \in \mathfrak{X}(M)$*:*

1. *Bilinearity :*
$$[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z],$$
$$[Z, \alpha X + \beta Y] = \alpha[Z, X] + \beta[Z, Y].$$

2. *Antisymmetry:*
$$[X, Y] = -[Y, X].$$

3. *Jacobi Identity:*
$$\Big[X, [Y, Z]\Big] + \Big[Y, [Z, X]\Big] + \Big[Z, [X, Y]\Big] = 0.$$

4. *For* $f, g \in C^\infty(M)$*,*
$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X, \quad \forall\, f, g \in C^\infty(M).$$

Proof. The proof comes from a direct calculation using the fact that:

$$\mathcal{L}_{\alpha X+\beta Y} = \alpha\, \mathcal{L}_X + \beta\, \mathcal{L}_Y, \quad \forall\, \alpha, \beta \in \mathbb{R} \text{ and } X, Y \in \mathfrak{X}(M).$$

$\blacksquare$

We ought to notice what the local formula for the Lie derivation looks like in conventional index notation, which is expressed by the following proposition.

Proposition 1.1.3. *Suppose that we have* $X = X_i \dfrac{\partial}{\partial \phi_i}$ *and* $Y = Y_i \dfrac{\partial}{\partial \phi_i}$*. Then we can write the local formula:*

$$[X, Y] = \left(X_i \frac{\partial Y_j}{\partial \phi_i} - Y_i \frac{\partial X_j}{\partial \phi_i} \right) \frac{\partial}{\partial \phi_j}.$$

Proof. Since we know that $[X, Y]$ is a smooth vector field, so it is suffices to evaluate in a single smooth chart, where for a smooth function $f : M \to \mathbb{R}$ we have:

$$\begin{aligned} [X, Y]f &= X_i \frac{\partial}{\partial \phi_i} \left(Y_j \frac{\partial f}{\partial \phi_j} \right) - Y_j \frac{\partial}{\partial \phi_j} \left(X_i \frac{\partial f}{\partial \phi_i} \right) \\ &= X_i \frac{\partial Y_i}{\partial \phi_i} \frac{\partial f}{\partial \phi_j} + X_i Y_j \frac{\partial^2 f}{\partial \phi_i \partial \phi_j} - Y_i \frac{\partial X_i}{\partial \phi_j} \frac{\partial f}{\partial \phi_i} - X_i Y_j \frac{\partial^2 f}{\partial \phi_i \partial \phi_j} \\ &= X_i \frac{\partial Y_j}{\partial \phi_i} \frac{\partial f}{\partial \phi_j} - Y_j \frac{\partial X_i}{\partial \phi_j} \frac{\partial f}{\partial \phi_i}, \end{aligned}$$

where we have used the fact that the mixed partial derivatives of a smooth functions can be written in any order. If we exchange the indices i and j in the second term of the last step we obtain the wanted formula. ∎

Example 1.1.12. Define two vector fields X and Y in $\mathbb{R}^2$ as following:

$$X(P) = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \quad \text{and} \quad Y(P) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad \forall\ P = (x_1, x_2) \in \mathbb{R}^2.$$

The bracket $[X, Y]$ can be evaluated as following with a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$[X, Y]f = XYf - YXf$$

then,

$$\begin{aligned}
X(Yf) &= \left(x_2\,\frac{\partial}{\partial x_1} + x_1\,\frac{\partial}{\partial x_2}\right)\left(-x_2\,\frac{\partial f}{\partial x_1} + x_1\,\frac{\partial f}{\partial x_2}\right) \\
&= -x_2^2\,f_{x_1x_1} + x_2\,f_{x_2} - x_1\,f_{x_1} + x_1^2\,f_{x_2x_2} \quad \text{and} \\
Y(Xf) &= \left(-x_2\,\frac{\partial}{\partial x_2} + x_1\,\frac{\partial}{\partial x_2}\right)\left(x_2\,\frac{\partial f}{\partial x_1} + x_1\,\frac{\partial}{\partial x_2}\right) \\
&= -x_2^2\,f_{x_1x_1} - x_2\,f_{x_2} + x_1\,f_{x_1} + x_1^2\,f_{x_2x_2}.
\end{aligned}$$

Now, we obtain:

$$[X,Y]f = \left(-2x_1\frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_2}\right)f,$$

similarly if we write:

$$[X,Y] = \begin{pmatrix} -2x_1 \\ 2x_2 \end{pmatrix}.$$

The $\mathbb{R}-$ vector space $\mathfrak{X}(M)$ together with the $\mathbb{R}-$bilinear map $(X,Y)\mapsto [X,Y]$ is an example of a very significant and important abstract algebraic structure which is Lie algebra. We discuss this term in the next section.

1.2 Riemannian metric and geodesics.

A very common example of Riemannian geometry is the geometry of surface. In this section we introduce the notions of a n-dimensional Riemannian manifold (M,g) and their geodesics. For more details see [42, 68, 66].

Definition 1.2.1. Let M be an $n-$smooth manifold. We define a Riemannian metric g which associates to every point $P\in M$ an inner product $g_P : T_PM\times T_PM\to\mathbb{R}$ by $(Q_1,Q_2)\mapsto\langle Q_1,Q_2\rangle_P$. In other words, for each $P\in M$, the metric g_P satisfies the following conditions:

1. $g_P(a\,Q_1{+}b\,Q_2,Q) = a\,g_P(Q_1,Q){+}b\,g_P(Q_2,Q),\quad \forall\; Q_1,Q_2,Q\in T_PM$ and

$a, b \in \mathbb{R}$,

2. $g_P(Q, a\,Q_1 + b\,Q_2) = a\,g_P(Q, Q_1) + b\,g_P(Q, Q_2), \quad \forall\, Q, Q_1, Q_2 \in T_P M$ and $a, b \in \mathbb{R}$,

3. $g_P(Q_1, Q_2) = g_P(Q_2, Q_1), \quad \forall\, Q_1, Q_2 \in T_P M$,

4. $g_P(Q, Q) \geq 0 \;\; \forall\, Q \in T_P M \quad$ and

5. $g_P(Q, Q) = 0 \iff Q = 0$.

Furthermore, g_P is smooth in the sense that for any smooth vector fields $X, Y \in \mathfrak{X}(M)$, the function $P \mapsto g_P(XP, YP)$ is smooth.

Locally, a Riemannian metric can be described in terms of its coefficients in a local chart, which are defined by $g_{ij} = g_P\left(\frac{\partial}{\partial \phi_i}, \frac{\partial}{\partial \phi_j}\right)_P$. The smoothness of g_P is equivalent to the smoothness of all the coefficient functions g_{ij} in some chart.

Proposition 1.2.1. *Every smooth manifold M carries a Riemannian metric.*

Proof. Consider $M = \bigcup_\alpha \phi_\alpha(U_\alpha)$, surly be a covering of M by co-domains of charts (V_α, ϕ). For each α, let us consider the Riemannian metric g_α in U_α whose local expression $(g_\alpha)_{ij}$ is the identity matrix. Let ρ_α be a smooth partition of unity of M subordinate to the covering U_α and define the metric:

$$g \;=\; \sum_\alpha \rho_\alpha g_\alpha.$$

Since the family of supports of the ρ_α is locally finite, the above sum is locally finite, and we have that g is well defined and smooth, and it is bilinear and symmetric at each point. from our assumption we obtain that $\sum_\alpha \rho_\alpha = 1$. Since $\rho_\alpha \geq 0$ for all α and we have that g is positive definite and so, it is a Riemannian metric in M. ∎

Definition 1.2.2. Let M be a smooth manifold and g be a Riemannian metric on M. The pair (M, g) is called a ***Riemannian manifold***.

Definition 1.2.3. Let $(M_1, g_1), (M_2, g_2)$ be two Riemannian manifolds. An ***isometry*** between M_1 and M_2 is a diffeomorphism $\phi : M_1 \to M_2$ whose differential is a linear isometry between the corresponding tangent spaces, i.e.

$$g_P(Q_1, Q_2) \;=\; \langle Q_1, Q_2 \rangle_P = \Big\langle d\phi_P(Q_1), d\phi_P(Q_2) \Big\rangle_{\phi_P},$$

for all $P \in M_1$ and $Q_1, Q_2 \in T_P M_1$. We say that $(M_1, g_1), (M_2, g_2)$ are ***isometric Riemannian manifolds*** if there exists an isometry between them. A ***local isometry from*** (M_1, g_1) into (M_2, g_2) is a smooth map $\phi : M_2 \to M_2$ satisfying the condition that every point $P \in M_1$ admits a neighborhood U such that the restriction of ϕ to U is an isometry onto its image. In particular, ϕ is a local diffeomorphism.

Example 1.2.1. The first example we can have is the most trivial one which is the Euclidean space. Take $M = \mathbb{R}^n$ with local coordinate $\dfrac{\partial}{\partial \phi_i}$ and the basis (which are called the orthonormal basis) $e_i = (0, .., 1, ..., 0)$, the standard metric over an open set $\phi(U) \subseteq \mathbb{R}^n$ defined as:

$$g_P : T_P U \times T_P U \to \mathbb{R} \;\text{ with }\; \left\langle \sum_i \alpha_i \frac{\partial}{\partial \phi_i}, \sum_j \beta_j \frac{\partial}{\partial \phi_j} \right\rangle_P \mapsto \sum_i \alpha_i \beta i,$$

then:

$$g_{ij} \;=\; \langle e_i, e_j \rangle \;=\; \delta_{ij}.$$

In this case g is a Riemannian metric and it is called a (canonical) Euclidean metric.

One of the main notions in the study of Riemannian geometry is the study of the notion of their geodesics, which generalises the idea of straight lines in the Euclidean space to curves in Riemannian manifolds. A geodesic is a smooth curve on the manifold, that is locally the shortest curve connecting two points with each other.

We like to give a brief idea about a geodesics in Riemannian manifolds. First, let us recall the standard (Euclidean) length of any smooth curve $\gamma : [a, b] \to \mathbb{R}^n$ to be the number that is given by:

$$l(\gamma) \;:=\; \int_a^b \Big|\dot{\gamma}(t)\Big| dt.$$

In addition, we need to recall the definition of locally absolutely continuous function.

Definition 1.2.4. Let $I \subseteq \mathbb{R}$ be an interval (or an open set). A function $f : I \to \mathbb{R}^n$ is said to be ***absolutely continuous on*** I if for all $\epsilon > 0$ there exists $\delta > 0$, such that for any $m \in \mathbb{N}$ and any selection of disjoint intervals $\{(a_i, b_i)\}_{i=1}^m$ with $[a_i, b_i] \subseteq I$, whose overall length is:

$$\sum_{i=1}^{m} |b_i - a_i| \;<\; \delta, \tag{1.7}$$

f satisfies:

$$\sum_{i=1}^{m} \Big| f(b_i) - f(a_i) \Big| \;<\; \epsilon,$$

where $|.|$ denotes the standard Euclidean norm on $\mathbb{R}^n$.
If $f : I \to \mathbb{R}^n$ is absolutely continuous on all closed subintervals $[a, b] \subseteq I$, then it is called ***locally absolutely continuous on*** I.

Example 1.2.2. The function $u : [0, a] \to \mathbb{R}$ with $u(P) = \sqrt{P}$ is an absolutely continuous function on its domain (by simply choosing $\delta = \epsilon^2$ in the above definition).

Definition 1.2.5. Let $[a, b] \subseteq \mathbb{R}$ be a closed subset and M be a connected manifold. A curve $\gamma : [a, b] \to M$ is called ***absolutely continuous*** on M if for any chart (ϕ, U) of M the composition

$$\phi^{-1} \circ \gamma : \gamma^{-1}\Big(\gamma\big([a, b]\big) \cap U\Big) \;\to\; \phi^{-1}(U) \;\subseteq\; \mathbb{R}^n$$

is locally absolutely continuous.

Remark 1.2.1. Definitions 1.2.4 and 1.2.5 coincide in $\mathbb{R}^n$.

Remark 1.2.2. For any absolutely continuous curve $\gamma : [a, b] \to M$ the derivative $\dot{\gamma}(t)$ exists a.e. and $l(\gamma)$ is well-defined.

In the case of Riemannian manifolds we can see that one of the important tools of the Riemannian metric is that we able to define the length of any absolutely continuous curve $\gamma : [a, b] \to M$, which is the number that is given by:

$$l(\gamma) \;:=\; \int_a^b \left| \Big\langle \dot{\gamma}(t), \dot{\gamma}(t) \Big\rangle_{\gamma(t)} \right|^{\frac{1}{2}} dt.$$

With the definite metrics we have in the Riemannian manifolds, and using the above definition we can then define a distance function (or we can say metric in the sense of metric space) as following definition.

Definition 1.2.6. Let (M, g) be a Riemannian manifold with $P, Q \in M$. We define the Riemannian distance from P to Q as in the follows:

$$d(P, Q) = \inf \Big\{ l(\gamma) : \gamma \text{ absolutely continuous curve joining } P \text{ to } Q \Big\}. \tag{1.8}$$

Definition 1.2.7. A piecewise smooth curve in a Riemannian manifold is called minimizing if:

$$l(\gamma_1) \leq l(\gamma_2), \tag{1.9}$$

for any other absolutely continuous curve γ_2 with the same endpoints.

From the previous definition, we can say that the length of a minimizer of the length function is equal to the distance between its endpoints.

Remark 1.2.3. If there are two minimizers γ_1 and γ_2 for the same endpoints, then trivially

$$l(\gamma_1) = l(\gamma_2).$$

Definition 1.2.8. A parameterized, smooth curve $\gamma : I \to M$ is called ***geodesic at*** $t_0 \in I$, if

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0 \tag{1.10}$$

at the point $t_0 \in I$ where by $\frac{D}{dt}$ we mean the covariant derivative (for a precise definition see [42, Chapter 2]). In the case that γ is geodesic at each point $t_0 \in I$, then γ is said to be a ***geodesic***. If we have $\gamma : [a, b] \subseteq I \to M$ then the restriction $\gamma|_{[a,b]}$ is called a ***geodesic segment*** joining $\gamma(a)$ to $\gamma(b)$.

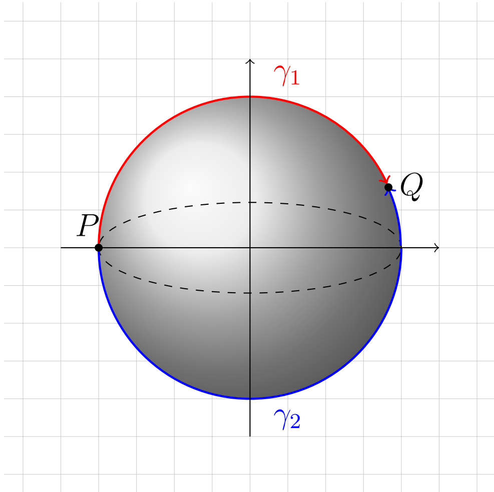

Figure 1.5: γ_1, γ_2 `satisfies (1.10) but only` γ_1 `is a minimizing geodesic.`

Remark 1.2.4. According to the calculus of variations, the minimizer we define in 1.2.8 comes from the solution of Euler-Lagrange equations which satisfy (1.10). The reverse is not true, i.e. if the geodesic satisfies (1.10) that does not mean it must be a minimizing geodesic, see Figure 1.5.

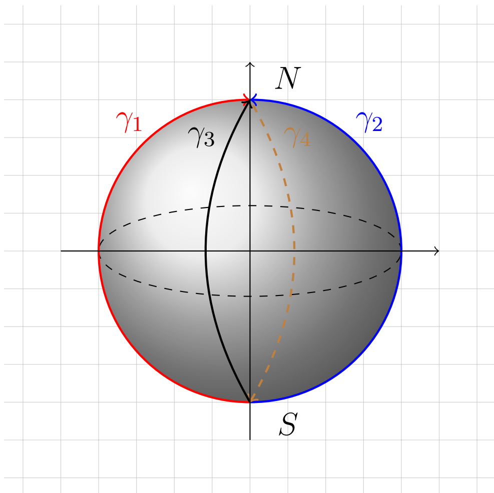

Figure 1.6: `This figure shows that the distance in the sphere between the South and the North pole is` $d(N, S) = l(\gamma_1) = l(\gamma_2)$`, but` $\gamma_1 \neq \gamma_2$`. So, the minimizing geodesic in the Riemannian case is not globally unique.`

Remark 1.2.5. From Figure 1.6, we can see that there are infinitely many geodesics connecting the North pole and the South pole and all of them are minimizing geodesics. Therefore the minimizing geodesic is not globally unique in the Riemannian manifolds.

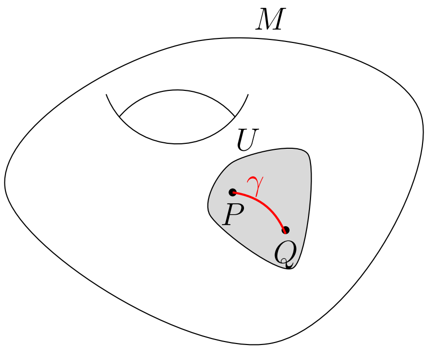

Figure 1.7: `Minimizing geodesics are locally unique.`

Proposition 1.2.2 (Local uniqueness)**.** *Let (M, g) be a Riemannian manifold and $P \in M$, then there exist U a neighborhood of point P such that for each $Q \in U$ we have a unique minimizing geodesic. See Figure 1.7.*
For a proof see [68].

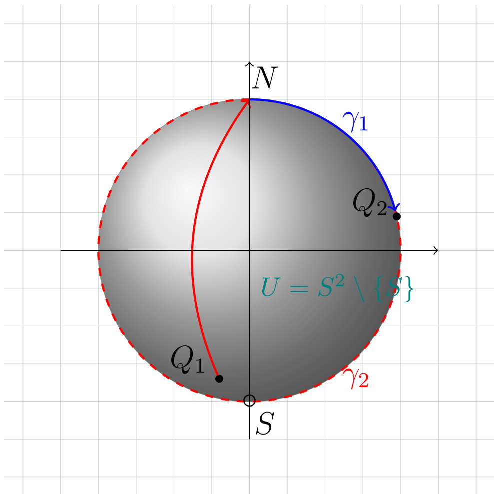

Figure 1.8: `Example of the maximal subset where the minimizing geodesics starting from the North pole are unique.`

Example 1.2.3. Consider the sphere S^2, for each $P \in S^n$ we can have the set $U = S^2 \setminus \{P^A\}$, where P^A is an antipodal point of the sphere S^2, e.g. if we denote the North pole by N and the South pole by S, then if $P = N \Rightarrow P^A = S$. Let $U = S^2 \setminus \{S\}$, then for each $Q \in U$ there exists a unique minimizing geodesic, see Figure 1.8.
Notice that, γ_1 and γ_2 in Figure 1.8 are two geodesics (in the sense that they are both solutions of the Euler-Lagrange equation) but only γ_1 is a minimizer.

Chapter 2

Sub-Riemannian manifolds.

Sub-Riemannian geometry is the study of a smooth manifold equipped with a positive definite inner product on a sub-bundle of the tangent bundle. When the sub-bundle is equal to the tangent bundle we have the case of Riemannian geometry. Thus, we can say that the sub-Riemannian geometry is a kind of generalization of Riemannian geometry. To be more precise, a Sub-Riemannian manifold is a Riemannian manifold together with constraints on the admissible direction of motion. Sub-Riemannian manifolds have many applications in different branches of mathematics, such as in the study of optimal control theory, see [20, 43, 44], calculus of variation, see [4, 8], and the optimal control in laser-induced population transfer, see [80]. We can find some authors prefer to name this geometry with other names like singular Riemannian metrics, e.g. [24, 59], or nonholonomic metrics, see [88]. In this chapter we are going to give an elementary introduction about the Sub-Riemannian geometry starting with the main elements of the structure of this geometry which is the distribution. We defined in the previous chapter the Riemannian metric and that leads us to define the length of any curve in the Riemannian manifolds. Similarly in the sub-Riemannian case, the Riemannian metric defined on the sub-bundle allows us to define the sub-Riemannian distance (called also Carnot-Carathédory distance). The geodesics and minimizing geodesics on sub-Riemannian geometry are also discussed, with some applications to this geometry such as Heisenberg group. For more details on sub-Riemannian manifolds we refer the reader to [5, 13, 15, 26, 47, 73].

2.1 Main definitions and examples.

We start this section defining the most important tool in sub-Riemannian manifold, which is the horizontal distribution, that will correspond to the constrains on the motion.

Definition 2.1.1. Let M be a $n-$smooth manifold. A m-dimensional ***distribution*** (or ***horizontal distribution***), that is denoted by $\mathcal{H}$ with $m \leq n$, is a subbundle of the tangent bundle TM, i.e.

$$\mathcal{H} \;=\; \Big\{(P,Q) : P \in M \;\text{ and }\; Q \in \mathcal{H}_P\Big\},$$

where, $\mathcal{H}_P$ is m-dimensional subspace of the tangent space T_PM.

Example 2.1.1. The first example we can have is the Riemannian geometry case, since we have the distribution is equal the entire tangent bundle, i.e. $\mathcal{H} = TM$.

Definition 2.1.2. Let M be a $n-$smooth manifold and $\mathcal{H}$ a distribution on M. A ***sub-Riemannian metric*** is a smoothly varying positive definite inner product $\langle ., . \rangle$ on $\mathcal{H}$.

Note that in the special case where $\mathcal{H}$ is equal to the tangent bundle, $\langle ., . \rangle$ gives a Riemannian metric.
In the next definition we introduce one of the main properties of smooth vector fields, which is the Hörmander condition. This property allows us to state the main definition in this chapter, that of the sub-Riemannian manifold. The Hörmander condition is connected to regularity of PDEs, in particular hypoellipticity, see [60]. If we consider $\mathcal{X} = \{X_1, \dots, X_m\}$ as a family of a smooth vector fields on a $n-$smooth manifold M, then we know that the Lie algebra (see Definition 3.1.4) generated by $\mathcal{X}$ is the smallest sub-algebra of $\mathfrak{X}(M)$ containing $\mathcal{X}$, namely:

$$\mathcal{L}(\mathcal{X}) \;:=\; span\Big\{[X_1, \dots, \big[X_{i-1}, X_i\big]\big], X_i \in \mathcal{X}, i \in \mathbb{N}\Big\}, \tag{2.1}$$

for more information see [22]

Definition 2.1.3 (The Hörmander condition)**.** Let M be a n-dimensional smooth manifold and let $\mathcal{X} \subseteq \mathfrak{X}(M)$ (see Definition 1.1.14). We say that $\mathcal{X}$

satisfies the ***Hörmander condition*** (also that it is ***bracket-generating***) if the following is satisfied:

$$\mathcal{L}(\mathcal{X})(P) \ := \ T_P M, \ \forall \ P \in M, \tag{2.2}$$

where $\mathcal{L}(\mathcal{X})(P)$ is given by equation (2.1) at the point P. We say that $\mathcal{X}$ satisfies the ***Hörmander condition with step*** $k \in \mathbb{N}$ if for each $P \in M$ we have:

$$\bigcup_{i=1}^{k} \mathcal{L}^k(\mathcal{X})(P) \ = \ T_P M,$$

where

$$\begin{aligned}
\mathcal{L}^1 &= span\Big\{Z = X : X \in \mathcal{X}\Big\} \\
\mathcal{L}^2 &= span\Big\{Z = [X, Y] : X, Y \in \mathcal{L}^1\Big\} \\
&\vdots \\
\mathcal{L}^k &= span\Big\{Z = [X, Y] : X, Y \in \mathcal{L}^{k-1}\Big\}.
\end{aligned}$$

Example 2.1.2. Let $\mathfrak{X} = \Big\{X, Y\Big\}$ where X, Y are the vector fields on $\mathbb{R}^2$ defined in the Example 1.1.12, we can notice that both vector fields X, Y are vanishing at the origin $(0, 0)$ and so the Hörmander condition is not satisfied at the origin.

Example 2.1.3. Define two vector fields X and Y on $\mathbb{R}^2$ as follows:

$$X(P) \ = \ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad Y(P) \ = \ \begin{pmatrix} 0 \\ 1 + x^2 \end{pmatrix},$$

for all $P = (x_1, x_2) \in \mathbb{R}^2$.
We can see that both X and Y are smooth vector fields. Moreover, we need no commutator to say X and Y span the tangent space $\mathbb{R}^2$ for any $P \in \mathbb{R}^2$, i.e.

$$\mathcal{L}^1(P) \ = \ span\Big\{X, Y\Big\} \ = \ \mathbb{R}^2.$$

So, X and Y are satisfying the Hörmander condition with step 1, which means that we are in a Riemannian geometry.

Remark 2.1.1. The Euclidian space $\mathbb{R}^n$ and Riemannian manifold always satisfy the Hörmander condition with step 1, that is we have $\mathcal{H} = TM$, i.e.

$$span\Big(X_1(P), ..., X_n(P)\Big) \;=\; \mathbb{R}^n, \;\; \forall\; P \in \mathbb{R}^n.$$

Now, we are able to introduce the main definition in this chapter; the sub-Riemannian manifold.

Definition 2.1.4. A ***Sub-Riemannian manifold*** is denoted by the triple $\Big(M, \mathcal{H}, \langle .,.\rangle\Big)$, that we have a $n-$smooth manifold M equipped with a sub-Riemannian metric $\langle .,.\rangle$ on a bracket generating distribution $\mathcal{H}$ of rank m such that $m \leq n$.

Example 2.1.4 (The Grušin plane)**.** The Grušin plane, denoted by $\mathbb{G}_2$, induced in $\mathbb{R}^2$ by two smooth vector fields X, Y as follows:

$$X(P) \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\; \text{and} \;\; Y(P) \;=\; \begin{pmatrix} 0 \\ x_1 \end{pmatrix},$$

for each $P = (x_1, x_2) \in \mathbb{R}^2$. In the case $x_1 = 0$, X, Y are not spanning $\mathbb{R}^2$, which means:

$$\mathcal{L}^1(P) \;=\; span\{X, Y\} \neq \mathbb{R}^2.$$

So we need to compute at least a commutator, so given $f : \mathbb{R}^2 \to \mathbb{R}^2$ smooth function then $[X, Y] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and we have:

$$\mathcal{L}^2(P) \;=\; span\Big\{X, Y, [X, Y]\Big\} = \mathbb{R}^2, \;\; \forall\; P = (x_1, x_2) \in \mathbb{R}^2.$$

Therefore, the Grušin plane is a sub-Riemannian manifold which satisfies the Hörmander condition with step 2.

Next, we introduce the simplest example of sub-Riemannian geometry which is the Heisenberg group. This type of group is involved in some mathematical formulations of quantum mechanics. It also appears in the study of several complex variables, Fourier analysis and PDE's. Korányai studied extensively the sub-Riemannian geometry in the Heisenberg group [63, 64]. One of the important applications of the Heisenberg group is the Heisenberg sub-Laplacian which is considered as a significant prototype of sub-Laplacian on non-commutative Carnot groups. As an example we can see Beals-Gaveau-Greiner worked on the subelliptic geometry of Heisenberg groups and how they are related to complex Hamiltonian mechanics [17, 18].

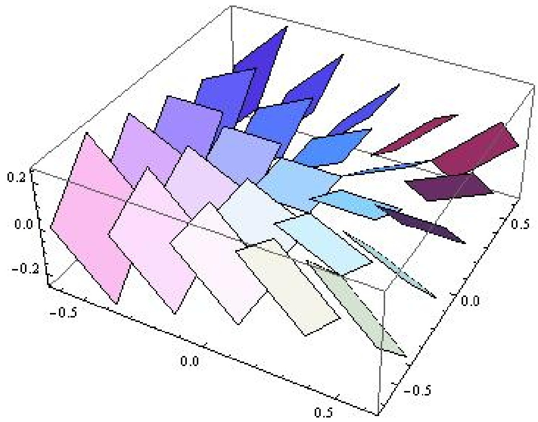

Figure 2.1: The distribution $\mathcal{H}$ in $\mathbb{H}^1$, [71].

Example 2.1.5 (Heisenberg group)**.** The Heisenberg group is denoted by $\mathbb{H}^n \cong \mathbb{R}^{2n+1}$. For simplicity, we study $\mathbb{H}^1 \cong \mathbb{R}^3$, which is called Heisenberg-Weyl group (or also 1-dimension Heisenberg group). The group law for $\mathbb{H}^1$ is given by:

$$P \circ Q = \Big(x_1 + \widetilde{x}_1, x_2 + \widetilde{x}_1, x_3 + \widetilde{x}_3 + \frac{1}{2}\left(x_1\widetilde{x}_2 - \widetilde{x}_1 x_2\right)\Big),$$

for all $P = (x_1, x_2, x_3), Q = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) \in \mathbb{R}^3$.
$\mathbb{H}^1$ has a distribution that is spanned by the following vector fields:

$$X(P) = \begin{pmatrix} 1 \\ 0 \\ -\frac{x_2}{2} \end{pmatrix} \quad \text{and} \quad Y(P) = \begin{pmatrix} 0 \\ 1 \\ \frac{x_1}{2} \end{pmatrix},$$

for all $P = (x_1, x_2, x_3) \in \mathbb{R}^3$.
Then we have $\mathbb{H}^1$ is a sub-Riemannian manifold where Hörmander's condition is satisfied with step 2, since:

$$Z(P) = [X, Y](P) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and so:

$$\mathcal{L}^2(P) \;=\; span\{X,Y,Z\} \;=\; \mathbb{R}^3, \;\; \forall\, P=(x_1,x_2,x_3)\in\mathbb{R}^3.$$

Generally, the $n-$th Heisenberg group, which is denoted by $\mathbb{H}^n \cong \mathbb{R}^{2n+1} = \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$, has the following group law:

$$(P,Q) \;=\; \left(x_1+\tilde{x}_1,\ldots,x_{2n+1}+\tilde{x}_{2n+1}+\frac{1}{2}\sum_{i=1}^{2n} x_i\tilde{x}_{i+n}-x_{i+n}\tilde{x}_i\right),$$

for all $P=(x_1,\ldots,x_{2n+1}), Q=(\tilde{x}_1,\ldots,\tilde{x}_{2n+1})\in\mathbb{R}^{2n+1}$.
Moreover $\mathbb{H}^n$ is generated by the following vector fields:

$$\begin{aligned} X_i(P) &= \frac{\partial}{\partial\, x_i}+\frac{1}{2}\,x_i\,\frac{\partial}{\partial\, x_{2n+1}}, \\ Y_i(P) &= \frac{\partial}{\partial\, x_{i+n}}-\frac{1}{2}\,x_i\,\frac{\partial}{\partial\, x_{2n+1}}, \end{aligned}$$

for all $P=(x_1,\ldots,x_{2n+1})\in\mathbb{R}^{2n+1}$ and $i=1,...,n$.
The only non-vanishing commutation relationships among the generators are

$$[X_i,Y_i](P) \;:=\; -\frac{1}{4}\,Z, \;\; \forall\, i=1,\ldots,n.$$

Again, $\mathbb{H}^n$ is a sub-Riemannian manifold and the Hörmander condition is satisfied with step 2.
See Figure 2.1.
For more properties on the Heisenberg groups see e.g. [29].

Example 2.1.6 (Roto-translation groups)**.** The roto-translation groups which are homeomorphic to $\mathbb{R}^2\times S^1$ with coordinates (x_1,x_2,θ) are sub-Riemannian manifolds defined as $\left(\mathbb{R}^2\times S^1,\mathcal{H},\langle .,.\rangle\right)$. This group is spanned by the following smooth vector fields:

$$X(x_1,x_2,\theta) \;=\; \begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix} \quad\text{and}\quad Y(x_1,x_2,\theta) \;=\; \begin{pmatrix}0\\0\\1\end{pmatrix},$$

for all $P=(x_1,x_2,\theta)\in\mathbb{R}^2\times S^1$.

Computing $[X, Y]$ explicitly we get:

$$
\begin{aligned}
[X,Y]f &= X(Y\,f) - Y(X\,f) \\
&= \left(\cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2}\right)\left(\frac{\partial f}{\partial \theta}\right) - \left(\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial f}{\partial x_1} + \sin\theta\frac{\partial f}{\partial x_2}\right) \\
&= -\sin\theta f_{x_1} + \cos\theta f_{x_2}
\end{aligned}
$$

We have the following:

$$
Z(x_1,x_2,\theta) \;:=\; [X,Y](x_1,x_2,\theta) \;=\; -\sin\theta\partial_{x_1} + \cos\theta\partial_{x_2}.
$$

Here, the Hörmander's condition is satisfied with step 2. This geometry used by Setti-Sarti to model the visual cortex [33].

The main differences between sub-Riemannian manifolds and Riemannian manifolds is that in the sub-Riemannian manifolds we can see some of directions in the tangent spaces are not allowed or forbidden as velocity vectors of curves. The curves whose velocity vectors almost everywhere satisfy the constraints are usually referred to as horizontal (also called admissible) curves.

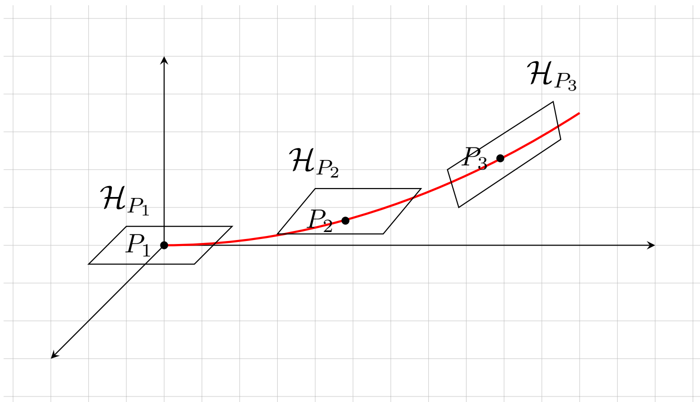

Figure 2.2: `A horizontal curve.`

Definition 2.1.5 (Horizontal curves)**.** Let $\big(M, \mathcal{H}, \langle .,. \rangle\big)$ be a Sub-Riemannian geometry and $\gamma : [a,b] \to M$ be an absolutely continuous function, γ is called ***horizontal curve*** (or ***admissible curve***) if:

$$
\dot\gamma(t) \;\in\; \mathcal{H}_{\gamma(t)}, \quad \text{for a.e. } t \in [a,b].
$$

See Figure 2.2.

Remark 2.1.2. An absolutely continuous curve $\gamma : [a,b] \to M$ is admissible if $\exists\ h : [a,b] \to \mathbb{R}^m$ measurable function such that:

$$\dot{\gamma}(t) \;=\; \sum_{i=1}^{m} h_i(t) X_i\Big(\gamma(t)\Big), \quad \text{for a.e. } t \in [a,b].$$

Thus, we can determine the **length** of the horizontal curve $\gamma : [a,b] \to M$, in the Sub-Riemannian structure $\Big(M, \mathcal{H}, \langle .,. \rangle\Big)$ as follows:

$$l(\gamma) \;=\; \int_a^b \Big\|\dot{\gamma}(t)\Big\|_{\gamma(t)}\, dt,$$

where $\|\dot{\gamma}(t)\|_{\gamma(t)} = \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}}$.

Definition 2.1.6. Let $\Big(M, \mathcal{H}, \langle .,. \rangle\Big)$ be a Sub-Riemannian space, the ***Sub-Riemannian distance*** (or ***Carnot-Carathéodry distance***) between two points $P, Q \in M$ is given by:

$$d(P,Q) \;=\; \inf\Big\{ l(\gamma) : \gamma \text{ is an admissible curve joining } P \text{ to } Q \Big\}.$$

Remark 2.1.3. Every sub-Riemannian manifolds is a metric space with the distance defined in Definition 2.1.6.

In the following we introduce an important result, which can be considered as the most important theorem in the sub-Riemannian geometry, that was demonstrated first by Rashevskii in 1938 [26] and then independently by Chow in 1939 [73]. This theorem asserts that any two points in a connected manifold M endowed with a bracket generating distribution $\mathcal{H}$ can be connected by an admissible curves.

Theorem 2.1.1 (Chow-Rashevskii Theorem)**.** *Let M be a smooth manifold and $\mathcal{H}$ a bracket generating distribution defined on M. If M is connected then there exists a horizontal curve joining two given points of M.*

For a proof see [73]. Therefore, we can conclude from Chow-Rashevskii theorem that in the bracket generating case (i.e. in the case that the Hörmander condition is satisfied) it is guaranteed that the sub-Riemannian distance between two points is finite. This property leads us to introduce the geodesics in sub-Riemannian geometry in the next section. Now, we like to give an example of Chow-Rashevskii theorem.

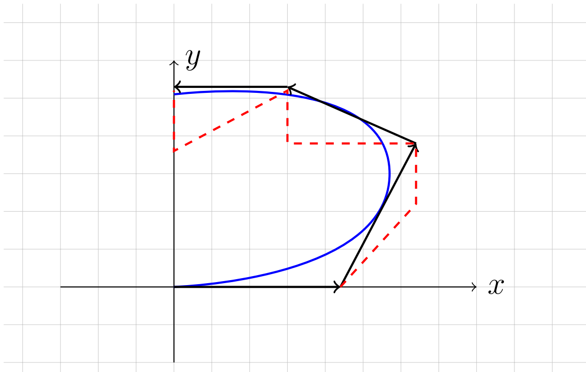

Figure 2.3: `Admissible paths between the origin and` $(0,0,1)$ `in` $\mathbb{H}^1$. `We can see that we start from` $(0,0,0)$ `through the` $x-$`axis in the` x `direction. Then, in the` $-y$ `direction we move from the point` $(0,1,1)$ `till we reach the point` $(1,1,0.5)$. `Next, from the last point we reach in` $-y$ `direction to our last point` $(0,0,1)$. `By the end, we move along` $-y$ `direction till we arrive` $(0,0,1)$.

Example 2.1.7 (The Heisenberg group $\mathbb{H}^1$)**.** As we introduced the Heisenberg group $\mathbb{H}^1$ previously in Example 2.1.5, and we have that the vector fields X and Y generate $\mathbb{H}^1$, the Chow-Rashevskii theorem applies and so, there is an admissible path between any two points in $\mathbb{H}^1$.
See Figure 2.3.

Remark 2.1.4. Chow-Rashevskii theorem is satisfied for all smooth vector fields, that can actually be proved for all Lipschitz vector fields [50]. For more details and example see [73]. We will study a counter-example in Section 2.2.

The reverses of Remark 2.1.4 is not true, since we cannot constract a counterexample with smooth coefficients (even if with analytic cofficients). We will study a geometry in Section 2.3 that does not satisfy the Hörmander condition.

2.2 Geodesics.

From Chapter 1 we know that "most" minimizing geodesics are characterized as solutions to a differential equation of Hamiltonian type. Geodesics in sub-

Riemannian manifolds are different from geodesics in Riemannian manifolds from many sides. We see in this section some of these differences.

Definition 2.2.1 (Geodesic)**.** A ***minimizing geodesic*** between any two points P and Q is any absolutely continuous horizontal curve which realizes the distance defined by (1.8).

In the Riemannian case we have minimizing geodesic γ must satisfy the Euler-Lagrange equations, if they are regular enough (but the reverse is not true). Also in sub-Riemannian case most of the geodesics are the solutions of Euler-Lagrange equations, and they are called **normal geodesics**. Nevertheless, there is a major differences between the geodesics in each geometries from this side, that we can find some geodesics in the sub-Riemannian case do not come from the solutions of Euler-Lagrange equations, which are called **singular geodesics**. In the following we give more details about that and how can we find the geodesic in sub-Riemannian geometries.

Definition 2.2.2. Consider $\gamma : [a, b] \to M$ an absolutely continuous admissible curve, then the ***energy*** of γ is the functional:

$$E(\gamma) \quad = \quad \int_a^b \frac{1}{2} \left\|\dot{\gamma}(t)\right\|^2_{\gamma(t)} dt \quad = \quad \int_a^b \frac{1}{2} \left\langle \dot{\gamma}(t), \dot{\gamma}(t) \right\rangle_{\gamma(t)} dt, \tag{2.3}$$

where $\langle ., . \rangle$ is defined in 2.1.2.

Remark 2.2.1. The minimizer of the energy and the minimizer have the same length in sub-Riemannian manifolds. For more information see [73, Chapter 1.4]

Note that, when the curve γ is parameterized in such a way that $\|\dot{\gamma}(t)\| = c$ for some $c > 0$ and for all $t \in [a, b]$, we say that γ has a **constant speed**. Moreover, in case $c = 1$ we say that γ is parameterized by the length.

Definition 2.2.3. Let M be a $n-$smooth manifold. For each $P \in M$ we define the ***cotangent space at*** P, which is denoted by T^*M, to be the dual space to the tangent space T_PM, i.e.

$$T_P^*M \quad = \quad (T_PM)^*.$$

The elements in T_P^*M are called ***tangent covectors at*** P.

Definition 2.2.4. Let M be a $n-$smooth manifold, the disjoint union:

$$T^*M = \bigsqcup_{P\in M} T^*_P M,$$

is called the ***cotangent bundle*** of M.

Computing the equation of geodesics we now need to introduce the cometric associated to the Riemannian metric.

Definition 2.2.5 ([73])**.** A ***cometric*** $\beta : T^*M \to TM$ on $\Big(M, \mathcal{H}, \langle ., . \rangle\Big)$ a Sub-Riemannian manifold is uniquely defined by the following conditions:

1. $Im(\beta_P) = \mathcal{H}_P$,
2. $p(Q) = \Big\langle \beta_P(p), Q \Big\rangle_P$, $\forall\, p \in T^*_P M \;\; \forall\, Q \in \mathcal{H}_P$ where $P \in M$.

Definition 2.2.6. Given a cometric $(., .)_P$ on the cotangent bundle $T^*_P M$, we can define a ***sub-Riemannian Hamiltonian*** as follows:

$$H(P, Q) = \frac{1}{2} \langle P, P \rangle_Q, \text{ where } P \in M \text{ and } Q \in T^*M. \tag{2.4}$$

Remark 2.2.2. Consider we have the admissible curve $\gamma : [a, b] \to M$, i.e.:

$$\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)} \quad \text{a.e. } t \in [a, b],$$

then we can write:

$$\begin{aligned} \frac{1}{2} \|\dot{\gamma}(t)\|^2_{\gamma(t)} &:= \frac{1}{2} \Big\langle \dot{\gamma}(t), \dot{\gamma}(t) \Big\rangle_{\gamma(t)} \\ &= \frac{1}{2} \langle P, P \rangle_{\gamma(t)}, \quad \text{with } P = \dot{\gamma}(t) \\ &= H(Q, P). \end{aligned}$$

Proof. The proof of the previous relation is easily comes from the definition of the cometric as follows:

$$\begin{aligned} \gamma(t) \text{ is admissible } &\Leftrightarrow \dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)} = Im(\beta_{\gamma(t)}) \text{ (by Definition 2.2.5)}, \\ &\Leftrightarrow \exists\, Q \in T^*_{\gamma(t)} M \text{ s.t. } \beta_{\gamma(t)}(P) = \dot{\gamma}(t). \end{aligned}$$

From Definition 2.2.5 we know that:

$$p(\widetilde{P}) = \left\langle \beta_{\gamma(t)}(P), \widetilde{P} \right\rangle_Q, \quad \forall\ Q \in T_P^*M \ \ \forall\ \widetilde{P} \in \mathcal{H}_Q.$$

Take $v := \gamma(t)$, then:

$$\begin{aligned} p(\widetilde{P}) &= \left\langle \beta_{\gamma(t)}(Q), \dot{\gamma}(t) \right\rangle_{\gamma(t)} \\ &= \left\langle \dot{\gamma}(t)(Q), \dot{\gamma}(t) \right\rangle_{\gamma(t)} \\ &= \left\| \dot{\gamma}(t) \right\|^2. \end{aligned}$$

Thus, we have:

$$\frac{1}{2}\left\| \dot{\gamma}(t) \right\|^2 = H(P,Q).$$

■

Definition 2.2.7. ***Geodesics*** in Sub-Riemannian manifolds are curves $\gamma : [a,b] \to M$ where the following are satisfied:

1. $\gamma(t)$ is an admissible curve, i.e. γ is absolutely continuous and

$$\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}, \quad \text{a.e. } t \in [a,b].$$

2. $l(\gamma) = d(P,Q)$, where $\gamma(a) = P$ and $\gamma(b) = Q$.

Definition 2.2.8. Let M be $n-$smooth manifold and X_a be a vector field on M, we define a linear function $P_{X_a} := P_a$ on the cotangent bundle, where:

$$P_a : T^*M \to \mathbb{R} \text{ with } (P,Q) \mapsto p\Big(X_a(P)\Big), \ \forall\ P \in M, \ Q \in T_P^*M. \tag{2.5}$$

This function P_a is called a ***momentum function***.

If we have the expression for the vector field X_a in coordinate as:

$$X_a(P) = \sum_i X_a^i(P)\left(\frac{\partial}{\partial x_i}\right),$$

then, we can write the following expression:

$$p_a(x,p) = \sum_i X_a^i(P) p_i,$$

where $P_i = P_{\frac{\partial}{\partial x_i}}$ are the momentum functions for the coordinate vector fields. Note that x_i and p_i from the coordinate system on the tangent bundle T^*M are called **canonical coordinates**.

Let us define

$$g^{ab}(P) \;=\; \Big\langle X_a(P), X_b(P) \Big\rangle_P \tag{2.6}$$

to be the matrix of inner products defined by our distribution frame $\mathcal{H}$. Consider $g^{ab}(P)$ to be the inverse matrix of g_{ab}. We can see that g^{ab} is a $n \times n$ matrix-valued function defined in some open set of M.

Proposition 2.2.1. *Let P_a and g^{ab} be the functions on the cotangent bundle T^*M defined respectively by* (2.5) *and* (2.6)*, which are induced by the local distribution $\{X_a\}$, then we have:*

$$H(P,Q) \;=\; \frac{1}{2}\sum_{a,b} g^{ab}(P)\, P_a(P,Q)\, P_b(P,Q).$$

Lemma 2.2.1. *In the particular case when the X_a form an orthonormal frame, which means:*

1. $\Big\langle X_a(P), X_b(P) \Big\rangle_P \;=\; 0 \quad \forall\; a \neq b$ *and*
2. $\Big\langle X_a(P), X_a(P) \Big\rangle_P \;=\; \|X_a(P)\|_P^2 \;=\; 1,$

then,

$$H(P,Q) \;=\; \frac{1}{2}\sum P_a^2.$$

Proof.

$$H(P,Q) \;=\; \frac{1}{2}\sum_{a,b} g^{ab}(Q) P_a(Q,P) P_b(P,Q)$$

$$g^{ab}(P) \;=\; \Big\langle X_a(P), X_b(P) \Big\rangle_P \;=\; \begin{cases} 0, & \text{if} \quad a \neq b, \\ 1, & \text{if} \quad a = b. \end{cases}$$

Then,

$$\begin{aligned} H(P,Q) \;&=\; \frac{1}{2}\sum_{a\neq b} 0 + \frac{1}{2}\sum_{a=b} 1 \cdot P_a(P,Q) P_a(P,Q) \\ &=\; \frac{1}{2}\sum_{a} P_a^2(P,Q). \end{aligned}$$

■

In order to introduce the geodesic equations associated with the Hamiltonian differential equations using the canonical coordinate (x_i, p_i) we can write:

$$\dot{x}_i \;=\; \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i \;=\; -\frac{\partial H}{\partial x_i}. \tag{2.7}$$

As we mentioned earlier, in a sub-Riemannian geometry there are two types of geodesics, the Euler-Lagrange equations are the equations for one type of geodesics, which are the equations of **normal geodesics**.

Definition 2.2.9. The Hamiltonian differential Equations (2.7) are called ***normal geodesic equations***.

Theorem 2.2.1 (Normal geodesics)**.** *Let $\zeta(t) = \big(\gamma(t), p(t)\big)$ be a solution of the Hamiltonian differential equations on the cotangent bundle T^*M for a sub-Riemannian Hamiltonian H and consider $\gamma(t)$ be its projection to M. Then, every sufficiently short length of γ is a minimizing sub-Riemannian geodesic. Moreover, γ can be considered as the unique minimizing geodesic that joins the endpoints.*
For a detailed proof see [73].

We now come to one of the major differences between Riemannian and sub-Riemannian manifolds. As we know, if the distribution is the entire tangent space then sub-Riemannian geometry becomes Riemannian. We can say that all geodesics in a Riemannian manifolds are normal. Whereas, in a sub-Riemannian manifolds there are minimizers which are not the projections of integral curves for the Hamiltonian vector field of H which are called **singular geodesics**. We can see this type of geodesics, for example in the Martinet distribution.

Example 2.2.1 (The Martinet distribution)**.** The Martinet distribution is the distribution on $\mathbb{R}^3$ that is spanned by the following vector fields:

$$X(P) \;=\; \begin{pmatrix} 1 \\ 0 \\ -x_2^2 \end{pmatrix} \quad \text{and} \quad Y(P) \;=\; \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

for all $P = (x_1, x_2, x_3) \in \mathbb{R}^3$. This a sub-Riemannian distribution with step 3. In fact, we have the Lie brackets:

$$\begin{aligned} [X, Y](P) &= -2\, x_2 \frac{\partial}{\partial x_3}, \\ \Big[[X, Y], Y\Big](P) &= -2 \frac{\partial}{\partial x_3}, \end{aligned}$$

then

$$span\Big\{X_1, X_2, \Big[[X, Y], Y\Big]\Big\}(P) \;=\; \mathbb{R}^3, \quad \forall\; P = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Hence, in this case any two points are connected by an horizontal curve, see Definition 2.1.5, but there are singular geodesics, i.e. geodesics which do not satisfy the Hörmander equation.
For more details see [73].

The next theorem shows the existence of minimizing geodesics in sub-Riemannian manifolds.

Theorem 2.2.2. *If $\Big(M, \mathcal{H}, \langle ., . \rangle\Big)$ is a sub-Riemannian manifold, we have:*

Local existence *For any point $P \in M$ there is $U \subseteq M$ neighborhood of P such that for any $Q \in U$ we can find that P and Q are connected to each other by a minimizing geodesic.*

Global existence *If M is connected and complete, then any two points $P, Q \in M$ are connected to each other by a minimizing geodesic.*

For a proof see [73, Chapter 1.6].

Now, we show how to obtain a (normal) geodesic by solving the Hamiltonian differential equation, as we can see in the following example.

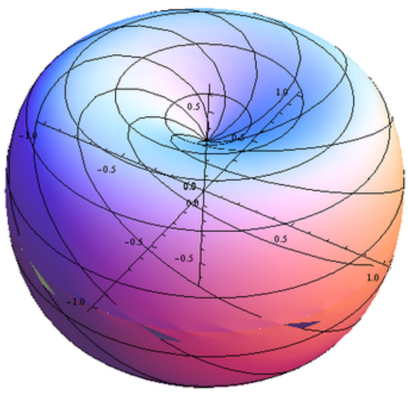

Figure 2.4: Canonical Heisenberg ball, which is the set of points of geodesics starting from $(0, 0, 0)$ parameterized by unit length at time 1, [71].

Example 2.2.2 (Canonical Heisenberg groups)**.** Consider the Heisenberg group $\mathbb{H}^1$ with canonical orthonormal vector fields defined in Example 2.1.5, see Figure 2.4. In this case the Hamiltonian is given by:

$$
\begin{aligned}
H(x_1, x_2, x_3, p_1, p_2, p_3) &= \frac{1}{2} \sum_{i=1,2} P_i^2 \\
&= \frac{1}{2} \left(P_{X_1}^2 + P_{X_2}^2 \right),
\end{aligned}
$$

where

$$
\begin{aligned}
P_X(x, p) &= \sum_{i=1}^{3} X^i(x_1, x_2, x_3)\, P_i \\
&= P_1 - \frac{x_2}{2}\, P_3 \\
&= P_{x_1} - \frac{x_2}{2}\, P_{x_3},
\end{aligned}
$$

and

$$
\begin{aligned}
P_Y(x, p) &= \sum_{i=1}^{3} Y^i(x_1, x_2, x_3)\, P_i \\
&= P_2 + \frac{x_1}{2}\, P_3 \\
&= P_{x_2} + \frac{x_1}{2}\, P_{x_3}.
\end{aligned}
$$

Now, we can write the Hamiltonian that associated to $\mathbb{H}^1$ as follows:

$$H(x_1,x_2,x_3,p_1,p_2,p_3) = \frac{1}{2}\left(p_1 - \frac{x_2}{2}\,p_3\right)^2 + \frac{1}{2}\left(p_2 + \frac{x_1}{2}\,p_3\right)^2.$$

From the Hamiltonian equation we can obtain the equations of geodesics as following:

$$\begin{cases} \dot{x_1}(t) = \dfrac{\partial H}{\partial p_1} = p_1 - \dfrac{x_2}{2}\,p_3, \\ \dot{x_2}(t) = \dfrac{\partial H}{\partial p_2} = p_2 + \dfrac{x_1}{2}\,p_3, \\ \dot{x_3}(t) = \dfrac{\partial H}{\partial p_3} = \left(-\dfrac{x_2}{2}\right)\left(p_1 - \dfrac{x_2}{2}\,p_3\right) + \left(\dfrac{x_1}{2}\right)\left(p_2 + \dfrac{x_1}{2}\,p_3\right), \end{cases} \tag{2.8}$$

and

$$\begin{cases} \dot{p_1}(t) = -\dfrac{\partial H}{\partial x_1} = -\dfrac{p_3}{2}\left(p_2 + \dfrac{x_1}{2}\,p_3\right), \\ \dot{p_2}(t) = -\dfrac{\partial H}{\partial x_2} = \dfrac{p_3}{2}\left(p_1 - \dfrac{x_2}{2}\,p_3\right), \\ \dot{p_3}(t) = -\dfrac{\partial H}{\partial x_3} = 0. \end{cases} \tag{2.9}$$

We can see directly from the last system that $p_3 = C$, where $C \neq 0$ is any non-vanishing constant (the case $C = 0$ is considered later). By substituting what we have from (2.8) in (2.9), we can rewrite (2.9) system as follows:

$$\begin{cases} \dot{p_1}(t) = -\dfrac{C}{2}\,\dot{x}_2(t), \\ \dot{p_2}(t) = \dfrac{C}{2}\,\dot{x}_1(t). \end{cases}$$

Hence, by using (2.8) we have:

$$\begin{cases} \dot{x}_1(t) = p_1(t) - \dfrac{C}{2}\,x_2(t), \\ \dot{x}_2(t) = p_2(t) + \dfrac{C}{2}\,x_1(t). \end{cases} \tag{2.10}$$

Differentiating that last equations and using (2.9) we have:

$$\begin{cases} \ddot{x}_1(t) + C\,\dot{x}_2(t) = 0, \\ \ddot{x}_2(t) - C\,\dot{x}_1(t) = 0. \end{cases} \tag{2.11}$$

In order to solve (2.11) we can set that $p = \dot{x}_1$ and $q = \dot{x}_2$, so from (2.11) we obtain the following second order ODE system:

$$\begin{cases} \dot{p}(t) + C\,q(t) &= 0, \\ \dot{q}(t) - C\,p(t) &= 0. \end{cases} \Rightarrow \begin{cases} \dot{p} &= -C\,\dot{q}, \\ \dot{q} &= C\,p \end{cases}.$$

By deriving and substituting in the previous equations, we find:

$$\begin{cases} \ddot{p}(t) + C^2\,\dot{p}(t) &= 0, \\ \ddot{q}(t) + C^2\,\dot{q}(t) &= 0. \end{cases}$$

Now, the characteristic equation for the 1st ODE equation in the last system can be given by:

$$r^2 + C^2 = 0 \Rightarrow r = \pm i\,C.$$

Thus, the general solution for this ODE is:

$$\dot{x}_1(t) = p(t) = c_1 \cos(C\,t) + c_2 \sin(C\,t). \tag{2.12}$$

Since we have $q = -\dfrac{\dot{p}}{C}$, then:

$$\dot{x}_2(t) = q(t) = c_1 \sin(C\,t) - c_2 \cos(C\,t). \tag{2.13}$$

Note that c_1 and c_2 in (2.12) and (2.13) are any suitable constants. To compute these two constant c_1 and c_2 we need to use the initial data, so consider the initial data:

$$\begin{cases} x_1(0) &= 0, \\ x_2(0) &= 0, \\ x_3(0) &= 0, \end{cases} \quad \text{and} \quad \begin{cases} p_1(0) &= a, \\ p_2(0) &= b, \\ p_3(0) &= c, \end{cases},$$

where $a, b, c \in \mathbb{R}$. Then we have:

$$\begin{aligned} a &= p_1(0) = \dot{x}_1(0) = p(0) \quad \text{and} \\ b &= p_2(0) = \dot{x}_2(0) = q(0). \end{aligned}$$

So $c_1 = a$ and $c_2 = -b$. Thus we obtain the solution of the system (2.8) as follows:

$$\begin{cases} \dot{x}_1(t) &= a\cos(C\,t) - b\sin(C\,t), \\ \dot{x}_2(t) &= a\sin(C\,t) - b\cos(C\,t). \end{cases} \tag{2.14}$$

In order to evaluate (2.14), we just need to integrate it, so we have:

$$\begin{cases} x_1(t) &= \dfrac{a}{C}\sin(C\,t) + \dfrac{b}{C}\cos(C\,t) + C_1, \\ x_2(t) &= -\dfrac{a}{C}\cos(C\,t) + \dfrac{b}{C}\sin(C\,t) + C_2, \end{cases}$$

where $C \neq 0$ by assumption.
Let us choose $C_1 = -\dfrac{b}{C}, C_2 = \dfrac{a}{C}$ and $C \neq 0$, so we rewrite the last equation as follows

$$\begin{cases} x_1(t) &= \dfrac{a}{C}\sin(C\,t) + \dfrac{b}{C}\cos(Ct) - \dfrac{b}{C}, \\ x_2(t) &= -\dfrac{a}{C}\cos(C\,t) + \dfrac{b}{C}\sin(Ct) - \dfrac{a}{C}. \end{cases}$$

Then,

$$\begin{cases} x_1(t) &= \dfrac{a}{C}\sin(C\,t) - \dfrac{b}{C}\Big(1 - \cos(C\,t)\Big), \\ x_2(t) &= \dfrac{b}{C}\sin(C\,t) + \dfrac{a}{C}\Big(1 - \cos(C\,t)\Big). \end{cases} \tag{2.15}$$

Now, we have to compute $x_3(t)$, from the last equation in the system (2.8). Using from (2.10) and (2.12) we have the following:

$$\begin{aligned} \dot{x}_3(t) &= -\frac{x_2}{2}\,p_1 + \frac{x_1}{2}\,p_2 + \frac{C}{4}(x_1^2 + x_2^2) \\ &= \frac{C}{4}(x_1^2 + x_2^2) \qquad \text{(using (2.15))} \\ &= \frac{C}{4}\left(\left(\frac{a}{C}\sin(C\,t) - \frac{b}{C}(1 - \cos(C\,t))\right)^2 + \right. \\ &\qquad\qquad \left.\left(\frac{b}{C}\sin(C\,t) - \frac{a}{C}(1 - \cos(C\,t))\right)^2\right) \\ &= \frac{C}{4}\left(\frac{a^2 + b^2}{C^2}\right)\left(\sin^2(C\,t) + \Big(1 - \cos(C\,t)\Big)^2\right) \end{aligned}$$

$$\begin{aligned} &= \frac{a^2+b^2}{4C}\Big(\sin^2(C\,t)+1-2\cos(C\,t)+\cos^2(C\,t)\Big) \\ &= \frac{a^2+b^2}{4C}\Big(1+1-2\cos(C\,t)\Big) \\ &= \frac{a^2+b^2}{2C}\Big(1-\cos(C\,t)\Big). \end{aligned}$$

Integrate $\dot{x}_3(t)$ and using the initial condition we considered earlier, we obtain:

$$x_3(t) = \frac{a^2+b^2}{2C^2}\Big(C\,t-\sin(C\,t)\Big).$$

At the end, we can write the geodesic equations for the canonical Heisenberg groups as follows:

$$\begin{cases} \gamma_1(t) = \dfrac{a}{C}\sin(C\,t) - \dfrac{b}{C}\Big(1-\cos(C\,t)\Big), \\ \gamma_2(t) = \dfrac{b}{C}\sin(C\,t) + \dfrac{a}{C}\Big(1-\cos(C\,t)\Big), \\ \gamma_3(t) = \dfrac{a^2+b^2}{2C^2}\Big(C\,t-\sin(C\,t)\Big), \end{cases}$$

where $C \neq 0$.
In case $C = 0$ the geodesic equations system (2.14) becomes:

$$\begin{cases} \dot{x}_1(t) = a, \\ \dot{x}_2(t) = b. \end{cases}$$

Integrate the previous system, we have:

$$\begin{cases} x_1(t) = a\,t, \\ x_2(t) = b\,t. \end{cases}$$

Now, use (2.8) and (2.9) to compute $\dot{x}_3(t)$ as following:

$$\begin{aligned}\dot{x}_3(t) &= \frac{-x_2(t)}{2}\,p_1(t) + \frac{x_1}{2}\,p_2(t)\\ &= \frac{-x_2(t)}{2}\,\dot{x}(t) + \frac{x_1}{2}\,\dot{x}_2(t)\\ &= -\frac{bt}{2}\,a + \frac{at}{2}\,b\\ &= 0.\end{aligned}$$

In order to compute $\dot{x}_3(t) = 0$, using the initial condition $x_3(0) = 0$, we obtain:

$$x_3(t) = 0.$$

So the geodesic equations under the condition $C = 0$ are given by:

$$\begin{cases}\gamma_1(t) &= a\,t,\\ \gamma_2(t) &= b\,t,\\ \gamma_3(t) &= 0,\end{cases}$$

where $a, b \in \mathbb{R}$. Notice that in this case all normal geodesic we compute turn to regular Euclidean lines.

In the following we like to show how the geodesics in the Heisenberg groups $\mathbb{H}^1$ look like. Calin, Chang and Greiner discussed this object extensively in their paper [26]. Here, we like to highlight the geodesics between the origin $P = (0, 0, 0)$ and a point Q which will be as follows:

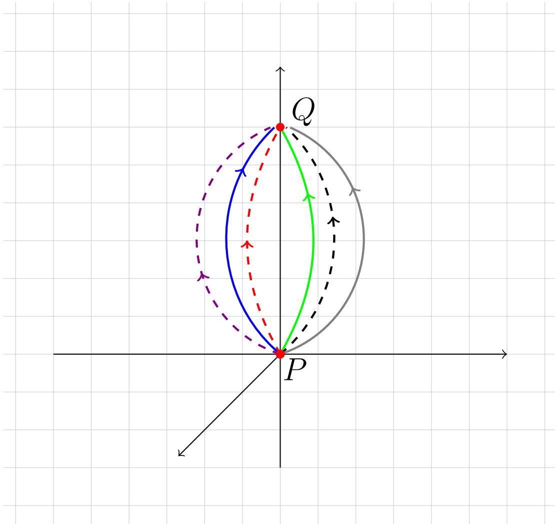

Figure 2.5: `In the picture we show some minimizing geodesics from the origin to the point` Q`, whenevr` Q `lies on the` $z-$`axis. As we can see minimizing geodesic in the sub-Riemannian case are not even locally unique.`

When $Q = (0, 0, 1)$: in this case we have many geodesics as solution of (2.8) parameterized by the circle $\mathbb{S}^1$ and they are all minimizing geodesics. See Figure 2.5.

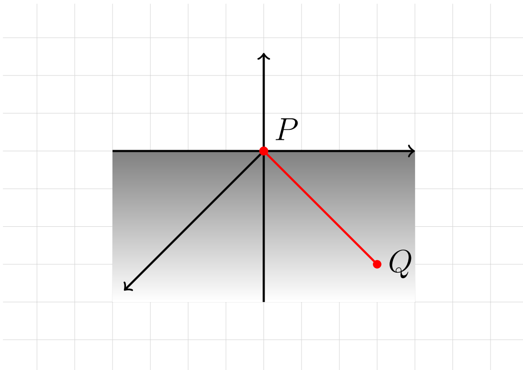

Figure 2.6: `A unique geodesics appears from the origin to` $Q = (1, 1, 0)$.

When $Q = (1, 1, 0)$: in this case the geodesics is a unique straight line, which lies in the $z = 0$ plane (it is obviously also the unique minimizing geodesics). See Figure 2.6.

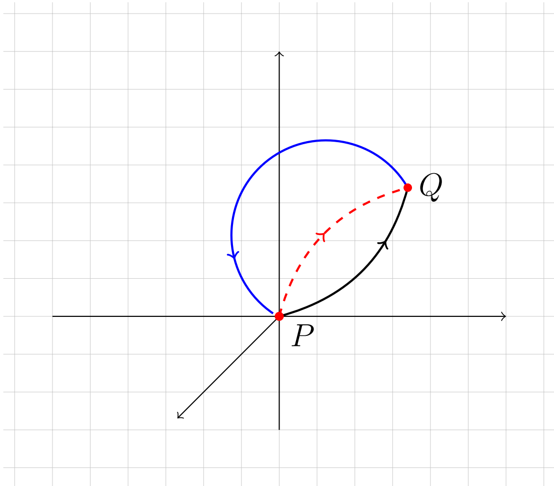

Figure 2.7: `Only one of the arcs should be a minimizing geodesic. To find it we need to compute their length.`

When $Q = (1, 1, 1)$: in this last case we have finitely many geodesics as a solution of (2.8) which join the origin to the endpoint $(1, 1, 1)$ (since $x_1, x_2 \neq 0$). These geodesics are arcs of circles. See Figure 2.7.

For explicit computation for the geodesics in $\mathbb{H}^1$ we refer the reader to e.g. [47].

2.3 An example of geometry not satisfying the Hörmander condition.

In this section we study a special kind of vector fields inducing a degenerate Riemannian manifold which do not satisfy the Hörmander condition.
Look at the following vector fields on $\mathbb{R}^2$:

$$X(P) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad Y(P) = \begin{pmatrix} 0 \\ a(x_1) \end{pmatrix}, \quad \forall\ P = (x_1, x_2) \in \mathbb{R}^2,$$

where $a(x_1) \;=\; \begin{cases} 0, & \text{if } x_1 < 0, \\ 1, & \text{if } x_1 \geq 0. \end{cases}$

We notice that the vector field Y is not smooth, since the second component is discontinuous on its domain, whereas X is smooth. Nevertheless, up to small modification we could find a smooth Y that generate a manifold with the same characterisations. So for sake of simplicity we study this non smooth case. First, let us compute the distance and the length of each geodesic we have between the points $P = (x_1, x_2), Q = (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^2$. Therefore, we have to study 3 different cases as follows:

1. $x_1, \tilde{x}_1 < 0$,
2. $x_1, \tilde{x}_1 \geq 0$,
3. $x_1 > 0$ and $\tilde{x}_1 \leq 0$.

Let us study them case by case:

When $x_1, \tilde{x}_1 < 0$: there are two different situations:

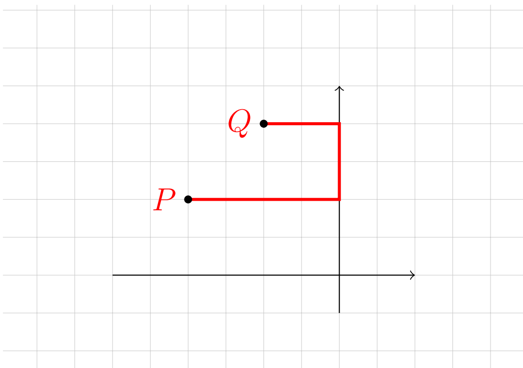

Figure 2.8: `Geodesic when` $x_1, \tilde{x}_1 < 0$ `and` $x_2 \neq \tilde{x}_2$.

1. If $x_2 \neq \tilde{x}_2$, then the distance is given by:

$$d(P, Q) \quad = \quad |x_1| + |\tilde{x}_1| + |x_2 - \tilde{x}_2|.$$

See Figure 2.8.
In this case the geodesic between P and Q is a union of 3 line segments. Thus, we write $\gamma : [0, 1] \to \mathbb{R}^2$, which has 3 different segments and can be parameterized as follows:
(i) $\gamma_1 : [0, a] \to \mathbb{R}^2$, $a \in (0, 1)$, given by:

$$\gamma_1(t) = \big(x_1(t), x_2(t)\big) \text{ where } \begin{aligned} x_1(t) &= \frac{(a-t)}{a} x_1, \\ x_2(t) &= x_2, \end{aligned}$$

(ii) $\gamma_2 : [a, b] \to \mathbb{R}^2$, $0 < a < b < 1$, given by:

$$\gamma_2(t) = (x_1(t), x_2(t)) \text{ where } \begin{aligned} x_1(t) &= 0, \\ x_2(t) &= \frac{a-t}{a-b}\,\widetilde{x}_2 + \frac{(b-t)}{b-a}\,x_2, \end{aligned}$$

(iii) $\gamma_3 : [b, 1] \to \mathbb{R}^2$, where

$$\gamma_3(t) = \big(x_1(t), x_2(t)\big) \text{ where } \begin{aligned} x_1(t) &= \frac{b-t}{b-1}\,\widetilde{x}_1 \\ x_2(t) &= \widetilde{x}_2. \end{aligned}$$

Then we obtain the geodesic

$$\gamma(t) = \begin{cases} \gamma_1(t), & \text{if } t \in [0, a], \\ \gamma_2(t), & \text{if } t \in [a, b], \\ \gamma_3(t), & \text{if } t \in [b, 1]. \end{cases}$$

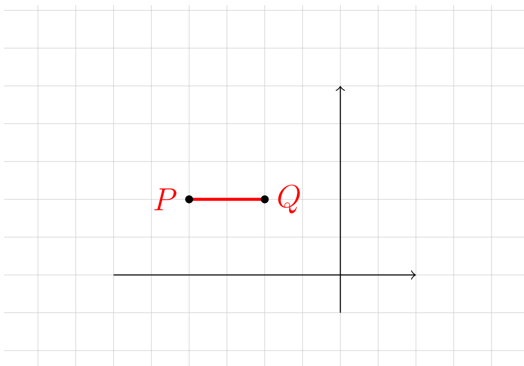

Figure 2.9: `Geodesic when` $x_1, \widetilde{x}_1 < 0$ `and` $x_2 \neq \widetilde{x}_2$.

2. If $x_2 = \widetilde{x}_2$ then, the distance is given by:

$$d(P, Q) = |x_1 - \widetilde{x}_1|.$$

See Figure 2.9.

The geodesic in this case is $\gamma : [0,1] \to \mathbb{R}^2$ where:

$$\gamma(t) = \big(x_1(t), x_2(t)\big) \text{ such that } \begin{aligned} x_1(t) &= t\,\widetilde{x}_1 + (1-t)x_1, \\ x_2(t) &= t\,\widetilde{x}_2 + (1-t)x_2, \end{aligned}$$

where $x_2 = \widetilde{x}_2 \;\Rightarrow\; x_2(t) = x_2$.

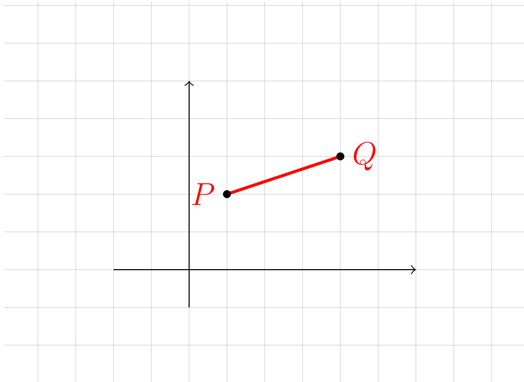

Figure 2.10: `Geodesic when` $x_1, \widetilde{x}_1 \geq 0$.

When $x_1, \widetilde{x}_1 \geq 0$: in this case the distance between p and Q is the standard Euclidian distance which is defined as:

$$d(P,Q) = \Big((x_1 - \widetilde{x}_1)^2 + (x_2 - \widetilde{x}_2)^2\Big)^{\frac{1}{2}}.$$

See Figure 2.10.

We have here that the geodesic $\gamma : [0,1] \to \mathbb{R}^2$ is given by:

$$\gamma(t) = \big(x_1(t), x_2(t)\big) \text{ such that } \begin{aligned} x_1(t) &= t\widetilde{x}_1 + (1-t)x_1, \\ x_2(t) &= t\widetilde{x}_2 + (1-t)x_2. \end{aligned}$$

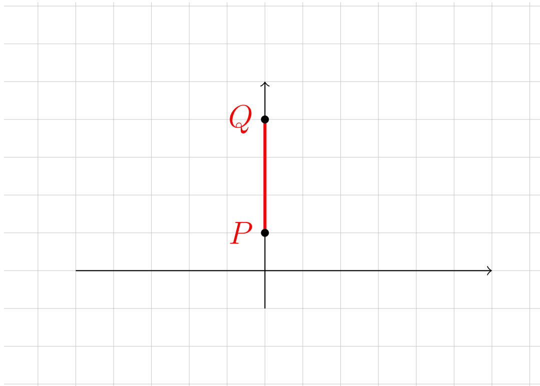

Figure 2.11: Geodesic in case $x_1 = \widetilde{x}_1 = 0$.

Remark 2.3.1. In case of $x_1 = \widetilde{x}_1 = 0$ the distance is given by:

$$d(P,Q) \quad = \quad |x_2 - \widetilde{x}_2|.$$

See Figure 2.11.
The geodesic in this case is $\gamma : [0,1] \to \mathbb{R}^2$ where

$$\begin{aligned} \gamma(t) = \big(x_1(t), x_2(t)\big) \text{ such that } \; x_1(t) &= 0, \\ x_2(t) &= t\widetilde{x}_2 + (1-t)x_2. \end{aligned}$$

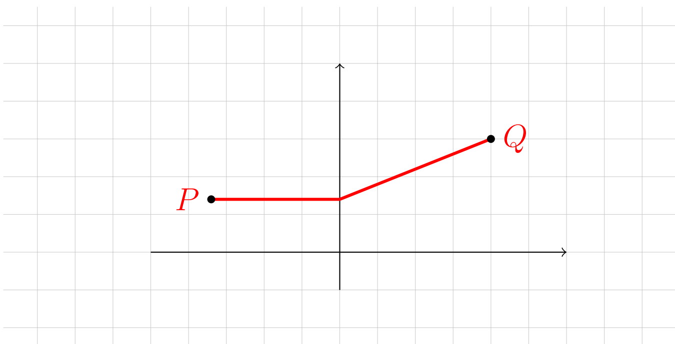

Figure 2.12: Geodesic when $x_1 > 0$ and $\widetilde{x}_1 \leq 0$.

When $x_1 > 0$ and $\widetilde{x}_1 \leq 0$: the distance is given by:

$$d(P,Q) \quad = \quad |x_1| + \left(\widetilde{x}_1^2 + (x_2 - \widetilde{x}_2)^2\right)^{\frac{1}{2}}.$$

See Figure 2.12. Hence, the geodesic in this case is a union of 2 line segments which are given by:

1. $\gamma_1 : [0, a] \to \mathbb{R}^2$, $a \in (0, 1)$, where

$$\gamma_1(t) = \Big(x_1(t), x_2(t)\Big) \text{ such that } \begin{aligned} x_1(t) &= \frac{(a-t)}{a}\, x_1, \\ x_2(t) &= x_2, \end{aligned}$$

2. $\gamma_2 : [a, 1] \to \mathbb{R}^2$, where

$$\gamma_2(t) = \Big(x_1(t), x_2(t)\Big) \text{ such that } \begin{aligned} x_1(t) &= \frac{t-a}{1-a}\,\widetilde{x}_1 + \frac{1-t}{1-a}\, x_1, \\ x_2(t) &= \frac{t-a}{1-a}\,\widetilde{x}_2 + \frac{1-t}{1-a}\, x_2. \end{aligned}$$

So we can write the geodesic $\gamma(t) = \begin{cases} \gamma_1(t), & \text{if } t \in [0, a], \\ \gamma_2(t), & \text{if } t \in [a, 1]. \end{cases}$

2.3.1 Equation of geodesics.

In order to obtain the geodesic equations for this example, let us write the Hamiltonian, which is given by:

$$\begin{aligned} H(x_1, x_2, p_1, p_2) &= \frac{1}{2} \sum_i P_i^2 \\ &= \frac{1}{2}\left(P_X^2 + P_Y^2\right), \end{aligned}$$

where:

$$\begin{aligned} P_1 &= \sum_{i=1}^{2} X(x_1, x_2) p_i \\ &= p_1, \end{aligned}$$

and

$$P_2 = \sum_{i=1}^{2} Y(x_1, x_2) p_i.$$

This implies

$$P_2 = 0 \text{ or } P_2 = p_2,$$

since $a(x_1)$ is a piecewise function. Note that the vector field X_2 is not smooth, that means H is a discontinuous function and so H is not differentiable. Therefore, we can find a decomposition for the domain, which allows us to differentiate H. For that, let us consider the following sets:

$$\begin{aligned}\Omega_1 &= (-\infty, 0) \times \mathbb{R}^3 \quad \text{and} \\ \Omega_2 &= (0, \infty) \times \mathbb{R}^3.\end{aligned}$$

where $\overline{\Omega}_1 \cup \overline{\Omega}_2 = \Omega = \mathbb{R}^4$, restricted to Ω_1 and Ω_2. Now, we can redefine the Hamiltonian:

1. The Hamiltonian defined on Ω_1, i.e. $H : \Omega_1 \to \mathbb{R}$, which is given by:

$$H(x_1, x_2, p_1, p_2) = \frac{1}{2} p_1^2.$$

2. The Hamiltonian define on Ω_2, i.e. $H : \Omega_2 \to \mathbb{R}$, which is given by:

$$H(x_1, x_2, p_1, p_2) = \frac{1}{2}\left(p_1^2 + p_2^2\right).$$

Recall the general formula for the Hamiltonian system:

$$\begin{cases} \dot{x}_i(t) = \dfrac{\partial H}{\partial p_i}, \\ \dot{p}_i(t) = -\dfrac{\partial H}{\partial x_i}. \end{cases}$$

First, in Ω_1 we have the following system:

$$\begin{cases} \dot{x}_1(t) = p_1(t), \\ \dot{x}_2(t) = 0. \end{cases} \tag{2.16}$$

$$\begin{cases} \dot{p}_1(t) = 0, \\ \dot{p}_2(t) = 0. \end{cases} \tag{2.17}$$

Consider the following initial conditions:

$$\begin{cases} x_1(0) = x_1^0, \\ x_2(0) = x_2^0, \end{cases} \tag{2.18}$$

and

$$\begin{cases} p_1(0) &= r, \\ p_2(0) &= s. \end{cases} \tag{2.19}$$

where $(x_1^0, x_2^0, r, s) \in \Omega_1$. From the system (2.17) we can immediately obtain the solutions by integrating the system and using the initial conditions in (2.19):

$$\begin{cases} p_1(t) &= r, \\ p_2(t) &= s. \end{cases}$$

Using the last system in (2.17), we obtain:

$$\begin{cases} \dot{x}_1(t) &= r, \\ \dot{x}_2(t) &= 0. \end{cases}$$

Integrate the above system:

$$\begin{cases} x_1(t) &= r\,t + C_1, \\ x_2(t) &= C_2. \end{cases}$$

where C_1, C_2 are constant in $\mathbb{R}$. To evaluate C_1 and C_2 we use the initial conditions in (2.19):

$$x_1(0) \;=\; C_1 \;\Rightarrow\; x_1^0 \;=\; C_1.$$

Hence

$$x_1(t) \;=\; r\,t + x_1^0.$$

Now to evaluate $x_2(t)$, we have:

$$x_2(0) \;=\; C_2 \;\Rightarrow\; C \;=\; x_2^0,$$

So, the geodesic on $(-\infty, 0)$ can be parameterized as:

$$\begin{cases} \gamma_1(t) &= rt + x_1^0, \\ \gamma_2(t) &= x_2^0. \end{cases} .$$

In Ω_2 we have the following system:

$$\begin{cases} \dot{x}_1(t) &=& p_1(t), \\ \dot{x}_2(t) &=& p_2(t) \end{cases} \tag{2.20}$$

and

$$\begin{cases} \dot{p}_1(t) &=& 0, \\ \dot{p}_2(t) &=& 0. \end{cases} . \tag{2.21}$$

Consider the following initial conditions:

$$\begin{cases} x_1(0) &=& x_1^0, \\ x_2(0) &=& x_2^0. \end{cases} \tag{2.22}$$

and

$$\begin{cases} p_1(0) &=& r, \\ p_2(0) &=& s. \end{cases} \tag{2.23}$$

where $(x_1^0, x_2^0, r, s) \in \Omega_2$. From the system (2.21) we can immediately obtain the solution by integrating the system and using the initial conditions in (2.22) and (2.23):

$$\begin{cases} p_1(t) &=& r, \\ p_2(t) &=& s. \end{cases}$$

Using last system in (2.21), we obtain:

$$\begin{cases} \dot{x}_1(t) &=& r, \\ \dot{x}_2(t) &=& s. \end{cases}$$

Integrate the above system:

$$\begin{cases} x_1(t) &=& r\,t + C_1, \\ x_2(t) &=& s\,t + C_2. \end{cases}$$

where C_1, C_2 is any constant. To evaluate C_1, C_2 we use the initial conditions in (2.19):

$$x_1(0) \;=\; C_1 \;\Rightarrow\; x_1^0 \;=\; C_1.$$

Hence

$$x_1(t) \;=\; r\,t + x_1^0.$$

Now, evaluate $y(t)$ we have:

$$x_2(0) \;=\; C_2 \;\Rightarrow\; C_2 \;=\; x_2^0,$$

Hence

$$x_2(t) \;=\; s\,t + x_2^0.$$

So, the geodesic on Ω_2 can be parameterized as:

$$\begin{cases} \gamma_1(t) &= r\,t + x_1^0, \\ \gamma_2(t) &= s\,t + x_2^0. \end{cases}$$

Then the geodesics found before can be found as solutions of the Hamiltonian in Ω_1 and Ω_2 but not as solutions of a Hamiltonian on Ω since they are not regular enough.

2.3.2 The Hörmander condition is not satisfied.

Here we want to show that the Hörmander condition is not satisfied. As we know X is a smooth vector field but Y is discontinuous vector field, so we cannot check the Hörmander condition on X and Y generally. But we are able to check it on the half plane only, i.e. on $(0,\infty)\times\mathbb{R}$ or on $(-\infty,0)\times\mathbb{R}$ separately, where both the vector fields are smooth. Firstly, we check it on $\Omega_1 = (0,\infty)\times\mathbb{R}$ where we have:

$$span\Big(\mathcal{L}(X,Y)\Big)(P) \;=\; \mathbb{R}^2, \quad \forall\; P=(x_1,x_2)\in\Omega_1.$$

Therefore, in this case X, Y satisfy the Hörmander's condition with step 1, i.e. the geometry is Riemannian.
Secondly, look at the negative plane $\Omega_2 = (-\infty,0)\times\mathbb{R}$, where the vector field Y is equal to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, Then we obtain nothing to commute with, then obviously

the Hörmander condition is not satisfied, in fact:

$$\dim\Big(span\ \mathcal{L}(X,Y)\Big)(P) \;=\; \dim(span\ X(P)) \;=\; 1, \quad \forall\ P=(x_1,x_2)\in\Omega_2.$$

This means that the Hörmander condition is not satisfied in this geometry.

2.3.3 Conclusion.

We study this example to show that the Hörmander condition is not a necessary condition for getting a finite distance, while it is a sufficient condition, see Chow's Theorem 2.1.1. Instead the example seems to suggest that it is necessary for the C-C distance under some additional regularity for the vector fields, to be continuous w.r.t. the original topology, that in our example is the standard Euclidean topology on $\mathbb{R}^2$. In fact, one could indeed show that the Hörmander condition is equivalent to this continuity property for the C-C distance, see e.g [73]. More precisely the previous example shows a geometry where the Hörmander condition is not satisfied, while the explicit computations prove that the associated distance is finite. The same distance is instead not continuous (w.r.t. the Euclidean topology): in fact

$$\lim_{P\to Q} d(P,Q) \;=\; 2|x_Q|,$$

for all $Q=(x_Q,y_Q)$ with $x_Q<0$, then clearly

$$\lim_{P\to Q} d(P,Q) \;\neq\; 0.$$

Chapter 3

Lie algebras and Carnot groups.

Lie groups and Lie algebras play an important role in physics and mathematics. In mathematics it can be related to many research areas, such as Riemannian and sub-Riemannian manifolds, algebraic geometry and symplectic geometry. Lie groups and Lie algebra are the base structures of Carnot groups. The latter are distinguished spaces that are rich of structure: they are those Lie groups equipped with a path distance that is invariant by left-translations of the group and admit automorphisms that are dilations with respect to the distance. In this chapter we present the basic notions about Lie algebras and Carnot groups together with some examples. Moreover, we study the connection between the Lie algebras and manifolds structures. For further details on Lie groups and Lie algebra we refer the reader to [22, 47, 52, 56, 57, 66, 68, 87]. The reader can find a short and clear reference on the Carnot groups in the paper of J. Heinonen [58].

3.1 Lie groups and Lie algebras.

Definition 3.1.1. A Group $\mathbb{G}$ is called a ***Lie group*** if it is an n-dimensional smooth manifold such that the group operations

$$\begin{array}{llcccl} mult : \mathbb{G}\times\mathbb{G}\to\mathbb{G} & \text{by} & (P,Q) & \mapsto & P\circ Q, & \forall\ P,Q\in\mathbb{G}, \\ inv\ \ : \mathbb{G}\to\mathbb{G} & \text{by} & P & \mapsto & P^{-1}, & \forall\ P\in\mathbb{G}, \end{array}$$

are smooth, see Definition 1.1.10.

Example 3.1.1. $\mathbb{R}$ is $1-$dimensional (abelian) Lie group, where the group multiplication is the usual addition. Similarly, any real $n-$dimensional vector space is a $n-$dimensional Lie group under vector addition.

Example 3.1.2. The general linear group $GL(n,\mathbb{R})$ is a Lie group, in fact:

1. $GL(n,\mathbb{R})$ is a smooth manifold, see Example 1.1.11, and

2. If $A, B \in GL(n,\mathbb{R})$, then we know that:

   (i) The multiplication is smooth because the matrix entries of any product matrix AB are polynomials in the entries of A and B.

   (ii) The inversion of the matrix A with entries of it is inverse are a polynomials divided by $det(A)$ and we know that $\dfrac{1}{\det(A)}$ is smooth.

Therefore $GL(n,\mathbb{R})$ is a Lie group of dimension n^2.

Definition 3.1.2. Let $\mathbb{G}$ be a Lie group. Then for $P \in \mathbb{G}$, the ***left translation***, denoted by L_P, and the ***right translation***, denoted by R_P, are respectively given by:

$$\begin{aligned} L_P &: \mathbb{G} \to \mathbb{G} \quad \text{with} \quad L_P(Q) := P \circ Q, \\ R_P &: \mathbb{G} \to \mathbb{G} \quad \text{with} \quad R_P(Q) := Q \circ P, \end{aligned}$$

where $Q \in \mathbb{G}$.

Because we can write L_P as the composition of smooth maps:

$$\mathbb{G} \xrightarrow{\imath_P} \mathbb{G} \times \mathbb{G} \xrightarrow{m} \mathbb{G},$$

where $\imath_P(Q) = (P, Q)$ and m is the left multiplication, it implies that L_P is smooth. In fact L_P is a diffeomorphism of $\mathbb{G}$, since $L_{P^{-1}}$ is the smooth inverse of L_P. The same is true for the right translation $R_P : \mathbb{G} \to \mathbb{G}$. Note that if $\mathbb{G}$ is not abelian in general $L_P \neq R_P$ so we will usually use L_P.

Remark 3.1.1. Note that if $\mathbb{G}_1, \mathbb{G}_2$ are smooth manifolds, $f : \mathbb{G}_1 \to \mathbb{G}_2$ is a diffeomorphism and $df : T\mathbb{G}_1 \to T\mathbb{G}_2$ is the differential of f, then

$$df\Big([X,Y]\Big) \;=\; \Big[df(X), df(Y)\Big]. \tag{3.1}$$

Definition 3.1.3. Let $\mathbb{G}$ be a Lie group, a vector field X on $\mathbb{G}$ is called ***left-invariant*** if it is invariant under all left translations, i.e. for any $P, Q \in \mathbb{G}$:

$$dL_P\Big(X(Q)\Big) = X(P \circ Q). \tag{3.2}$$

Similarly, a vector field X is called ***right invariant*** if for any $P \in \mathbb{G}$ we have:

$$dR_P\Big(X(Q)\Big) = X(Q \circ P).$$

Proposition 3.1.1. *Let $\mathbb{G}$ be a Lie group and suppose that $X, Y \in \mathfrak{X}(\mathbb{G})$ (see Definition 1.1.14) are two left-invariant vector fields. Then $[X, Y]$ is also a left-invariant vector field.*

Proof. It follows directly from using the fact that the Lie bracket is preserved under diffeomorphisms. Indeed, by using (3.1) for any $P, Q \in \mathbb{G}$ we obtain:

$$\begin{aligned} dL_P\Big([X,Y](Q)\Big) &= \Big[dL_P\Big(X(Q)\Big), dL_P\Big(Y(Q)\Big)\Big] \\ &= \Big[X\Big(P \circ Q\Big), Y\Big(P \circ Q\Big)\Big] \\ &= [X,Y](P \circ Q). \end{aligned}$$

That means $[X, Y]$ is left-invariant vector field. ■

The coming definition is considered one of the most important application of Lie bracket.

Definition 3.1.4 (Abstract Lie algebra)**.** A ***Lie algebra***, denoted by $\mathfrak{g}$, is a real vector space endowed with a Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by $(X, Y) \mapsto [X, Y]$, that satisfies the following properties for all $X, Y, Z \in \mathfrak{g}$:

1. Bilinearity :

$$\begin{aligned} [\alpha X + \beta Y, Z] &= \alpha[X,Z] + \beta[Y,Z], \\ [Z, \alpha X + \beta Y] &= \alpha[Z,X] + \beta[Z,Y]. \end{aligned}$$

2. Antisymmetry:

$$[X,Y] = -[Y,X].$$

3. Jacobi Identity:

$$\Big[X,[Y,Z]\Big]+\Big[Y,[Z,X]\Big]+\Big[Z,[X,Y]\Big] \;=\; 0.$$

Example 3.1.3.

1. The Euclidean space $\mathbb{R}^3$ endowed with the standard vector products is a Lie algebra.

2. Any vector space $\mathbb{V}$ with the trivial commutator $[X,Y]=0$ is Lie algebra.

Definition 3.1.5 (The Lie algebra of a Lie group)**.** The space $\mathfrak{g}$ of all left-invariant vector fields on a Lie group $\mathbb{G}$ endowed with the standard Lie bracket is called ***Lie algebra of the Lie group*** $\mathbb{G}$.

Example 3.1.4. The vector space $M(n,\mathbb{R})$ of $n\times n$ real matrices becomes an n^2-dimensional Lie algebra under the bracket: $[A,B]=AB-BA$. The first two conditions are obvious. We obtain the Jacobi identity by straightforward calculation for any $n\times n$ matrix A,B,C as follows:

$$\begin{aligned}\Big[A,[B,C]\Big]+\Big[B,[C,A]\Big]+\Big[C,[A,B]\Big] &= [A,BC-CB]+[B,CA-AC]\\ &\quad +[C,AB-BA]\\ &= ABC-ACB-BCA+CBA\\ &\quad +BCA-BAC-CAB+ACB\\ &\quad +CAB-CBA-ABC+BAC\\ &= 0.\end{aligned}$$

When we are regarding $M(n,\mathbb{R})$ as a Lie algebra with this bracket, we denoted it by $\mathfrak{gl}(n,\mathbb{R})$.

Proposition 3.1.2. *Let $\mathbb{G}$ be a Lie group and $X\in\mathfrak{g}$, then there exists a unique $1-$parameter subgroup $\gamma_X(t)$ defined on $\mathbb{G}$, which is called **vector flux**, such that for all $t\in\mathbb{R}$ it satisfies the following:*

$$\begin{cases} \dfrac{d}{dt}\gamma_X(t)\Big|_{t=0} &= X,\\ \gamma_X(0) &= e.\end{cases}$$

Definition 3.1.6. Let $\mathbb{G}$ be a Lie group, the ***exponential map*** is the smooth function defined by

$$\begin{aligned} \exp \ : \ & \mathfrak{g} \to \mathbb{G} \\ & X \mapsto \gamma_X(1). \end{aligned}$$

For more properties on the exponential map we refer to [66, Chapter 20].

3.2 Carnot groups.

3.2.1 Some basic algebraic topology notions.

We like first to introduce the concept of the fundamental group in terms of loops in topological spaces, that will allow us to introduce the definition of Carnot groups.

Definition 3.2.1. Let M be a topological space with $P, Q \in M$. We define a ***path*** in M from P to Q to be a continuous function $\gamma : [0,1] \to M$ such that $\gamma(0) = P$ and $\gamma(0) = Q$. The points P, Q are called the ***endpoints***.

Definition 3.2.2. Let M be a topological space and $\gamma : [0,1] \to M$ is a path in M, the ***inverse path*** of γ is defined as $\gamma^{-1} : [0,1] \to M$ with

$$\gamma^{-1}(t) \ := \ \gamma(1-t), \quad \forall \ t \in [0,1].$$

From the previous two definitions we can think there are so many paths, even on a simple topological spaces. We desire a way that keeping the end-points fixed and this is done by continuously deforming one path into another. In another words, we consider continuous deformations of paths with the same end-points such that the end-points remain fixed during the transformation. This idea is made more precise in the next definition.

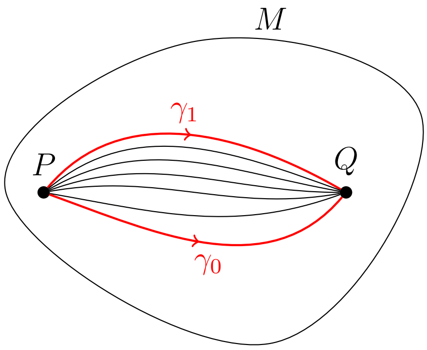

Figure 3.1: `Homotopic of two pathes in a manifold M.`

Definition 3.2.3. Let M be a topological space with two paths γ_1 and γ_2 that have endpoints $P, Q \in M$. A ***homotopy*** from γ_1 to γ_2 is a family of paths $\gamma_t : [0,1] \to M$ such that for all $t \in [0,1]$ we have that γ_t satisfies the following conditions:

1. The endpoints $\gamma_t(0) = P$ and $\gamma_t(1) = Q$ are independent of t.
2. The associated map $\Gamma : [0,1] \times [0,1] \to M$ defined by $\Gamma_{t_1}(t_2) = \gamma(t_2, t_1)$ is continuous.

When there exist a homotopy between two paths γ_1 and γ_2 they are said to be ***homotopic*** and we write $\gamma_1 \simeq \gamma_2$.
The ***homotopy class*** of γ, denoted by $[\gamma]$, is the equivalence class of path γ under the equivalence relation of homotopy.
See Figure 3.1.

Proposition 3.2.1. *Let M be a topological space with two endpoints $P, Q \in M$. A path homotopy is an equivalence relation on the set of all paths from P to Q.*

Example 3.2.1 (Linear homotopy)**.** Any two pathes γ_1 and γ_2 in $\mathbb{R}^n$ such that they have the same endpoints are homotopic via the linear homotopy, which is defined by:

$$\gamma_{t_1}(t_2) \quad = \quad (1 - t_1)\gamma_1(t_2) + t_1\, \gamma_2(t_2).$$

That means that the path $\gamma_1(t_1)$ travels along the line segment to $\gamma_2(t_1)$ with constant speed.

In the following definition we define a notion for paths where only one point is fixed.

Definition 3.2.4. Let M be a topological space, a ***loop*** in M is a path such that $\gamma(0) = P = \gamma(1)$, for some $P \in M$, i.e. the start and the end points coincide. P is called the ***basepoint*** of the loop.

Definition 3.2.5. Let M be a topological space with the usual topology and $\gamma_1, \gamma_2 : [0,1] \to M$ be two pathes in M such that $\gamma_1(1) = \gamma_2(0)$. The product path $\gamma_1 \cdot \gamma_2$ is defined as follows:

$$\gamma_1 \cdot \gamma_2(t) \;\; := \;\; \begin{cases} \gamma_1(2t), & 0 \geq t \geq \frac{1}{2} \\ \gamma_2(2t-1), & \frac{1}{2} \geq t \geq 1. \end{cases}$$

Theorem 3.2.1. *Let M be a topological space. The set of homotopy classes $[\gamma]$ of loops $\gamma : [0,1] \to M$ at the basepoint p forms a group under the product $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$.*

The group, mention in the Theorem 3.2.1, is called the **fundamental group** and we denote it by $\pi_1(M, p)$.

Definition 3.2.6. Let M be a topological space, M is ***path-connected*** if for every $P, Q \in M$, there exists a continuous path γ such that $\gamma(0) = P$ and $\gamma(1) = Q$.

Definition 3.2.7. Let M be a topological space, M is ***simply-connected*** if it is path-connected and has a trivial fundamental group.

3.2.2 Carnot groups.

Before giving the definition of Carnot group we introduce the concept of nilpotent Lie algebra.

Definition 3.2.8 (Nilpotent Lie algebra)**.** A Lie algebra $\mathfrak{g}$ of a Lie group $\mathbb{G}$ (see Definition 3.1.1) is called ***nilpotent of step*** k if there exists $k \in \mathbb{N} \setminus \{0\}$ and a decomposition

$$\mathfrak{g} \;\; = \;\; \mathfrak{g}^{(1)} \oplus \ldots \mathfrak{g}^{(k)} \tag{3.3}$$

where

$$\begin{aligned}\mathfrak{g}^{(1)} &= \mathfrak{g}_1 \leq \mathfrak{g},\\ \mathfrak{g}^{(n+1)} &= \Big[\mathfrak{g}_1, \mathfrak{g}^{(n)}\Big], \quad n \in \mathbb{N}\setminus\{0\},\end{aligned}$$

and

$$\Big[\mathfrak{g}_1, \mathfrak{g}^{(n)}\Big] := \Big\{[X,Y] : X \in \mathfrak{g}_1,\ Y \in \mathfrak{g}^{(n)}\Big\},$$

for $n = 1, \dots, k$ and $\mathfrak{g}^{(k+1)} = \{0\}$.

Definition 3.2.9. A group $\mathbb{G}$ is called ***Carnot group*** (also called ***stratified group***) ***of step*** k, if it is connected Lie group whose Lie algebra is nilpotent of step k.

Carnot groups are equipped with a family of automorphisms of the group, namely dilations.

Definition 3.2.10 (Dilation in Carnot groups)**.** A family of ***dilation*** in Carnot group $\mathbb{G}$ is a family of a smooth map $\delta_t : \mathbb{G} \to \mathbb{G}$, where $t > 0$, that is defined as follows:

$$\delta_t := \exp \circ \triangle_t \circ \exp^{-1},$$

where exp is the exponential map defined in Definition 3.1.6 and $\triangle_t : \mathfrak{g} \to \mathfrak{g}$ is defined as $\triangle_t(X) = t^i\, X$ for all $X \in \mathfrak{g}_i$ for $i = 1, \dots, k$. By using equation (3.3) for all $X \in \mathfrak{g}$ we have

$$\triangle_t(X) = \lambda\, X_1 + \cdots + \lambda^k\, X_k, \text{ for } X = X_1 + \cdots + X_k \text{ and } X_i \in \mathfrak{g}_i.$$

By the definition of dilations the following properties hold.

Lemma 3.2.1 (Properties of dilations)**.** *Given a Carnot group* $(\mathbb{G}, \circ)$ *endowed with a family of dilations* δ_t*, then for all* $P, Q \in \mathbb{G}$ *and* $t, t_1, t_2 \in \mathbb{R}$ *the following properties hold:*

1. $\delta_1(P) = P$*, for all* $P \in \mathbb{G}$*,*
2. $\delta_t(P) = \Big(\delta_{-t}\Big)^{-1}(P)$*, for all* $P \in \mathbb{G}, t \in \mathbb{R}$*,*

3. $\delta_{t_1}\Big(\delta_{t_2}(P)\Big) = \delta_{t_1 t_2}(P)$, *for all* $P \in \mathbb{G}$ *and* $t_1, t_2 \in \mathbb{R}$,

4. $\delta_t(P) \circ \delta_t(Q) = \delta_t(P \circ Q)$, *for all* $P, Q \in \mathbb{G}$ *and* $t \in \mathbb{R}$ *and*

5. *whenever* $k \geq 2$, *then* $\delta_{t_1}(P) \circ \delta_{t_2}(P) \neq \delta_{t_1+t_2}(P)$, *for all* $P \in \mathbb{G}$ *and* $t_1, t_2 \in \mathbb{R}$.

Example 3.2.2 (The Euclidian space)**.** $\mathbb{R}^n$ spaces can be considered as a trivial abelian Carnot group of step 1 with respect to its vector space structure. Hence the group law is the standard sum $P+Q$, where $P = (x_1, \dots, x_n), Q = (\widetilde{x}_1, \dots, \widetilde{x}_n) \in \mathbb{R}^n$.
The standard base of Lie algebra $\mathfrak{r}$ of $\mathbb{R}^n$ is given by following the left-invariant vector fields:

$$X_i(P) \;=\; \frac{\partial}{\partial x_i}, \quad \forall\; P = (x_1, \dots, x_n) \in \mathbb{R}^n, i = 1, ..., n.$$

In this case the dilations are defined as follows:

$$\delta(P) \;=\; \Big(tx_1, \dots, tx_n\Big), \quad \forall\; P = (x_1, \dots, x_n) \in \mathbb{R}^n,\; t > 0.$$

Example 3.2.3 (Heisenberg group $\mathbb{H}^n$)**.** The $n-$th Heisenberg group $\mathbb{H}^n \cong \mathbb{R}^{2n+1}$ (see Example 2.1.5) is a popular example of a non-commutative nilpotent Lie group with stratified algebra

$$\mathfrak{h} \;=\; \mathfrak{h}_1 \oplus \mathfrak{h}_2;$$

here $\mathfrak{h}_1$ is $2n$-dimensional and generated by the vectors $X_1, ..., X_n, Y_1, ..., Y_n$. Whereas, $\dim \mathfrak{h}_2 = 1$ and $\mathfrak{h}_2 = span\{Z\}$.
Observe that the group identity is $(0, \dots, 0) \in \mathbb{R}^{2n+1}$.
Moreover the dilations are given by the family of non-isotropic maps:

$$\delta_t(P) \;=\; \Big(tx_1, tx_2, ..., t^2 x_{2n+1}\Big), \quad \forall\; P = (x_1, \dots, x_{2n+1}) \in \mathbb{H}^n,\; t > 0.$$

In the following example we like to study the left-invariant vector fields of a very important type of a step 3 sub-Riemannian geometry of dimension 4, which is the **Engel group**. This geometry is endowed with a non-commutative Lie group law. Engel group is an interesting group since when certain constants are set to be zero the geodesics can be considered lifting to either Heisenberg group geodesics or Martin geodesics, see [73]. Many authors study the Engel groups from different points of view, we refer the reader to, e.g. [2, 21, 85, 91].

Example 3.2.4 (Engel group)**.** The **Engel group** (briefly $E4$) is a Carnot group of step 3 on $\mathbb{R}^4$. Let us recall the group law:

$$P \circ Q \;=\; \left(x_1 + \widetilde{x}_1, x_2 + \widetilde{x}_2, x_3 + \widetilde{x}_3 + x_1\widetilde{x}_2, x_4 + \widetilde{x}_4 + x_1\widetilde{x}_3 + \frac{x_1^2}{2}\,\widetilde{x}_2\right), \quad (3.4)$$

where $P = (x_1, x_2, x_3, x_4), Q = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4) \in \mathbb{R}^4$, with group identity is $(0,0,0,0) \in \mathbb{R}^4$.
We deal here with the Lie algebra $\mathfrak{e}$ generated by the following vector fields:

$$\begin{aligned} X_1(P) &= \frac{\partial}{\partial x_1}, \\ X_2(P) &= \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + \frac{x_1^2}{2}\frac{\partial}{\partial x_4}, \\ X_3(P) &= \frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4}, \\ X_4(P) &= \frac{\partial}{\partial x_4} \end{aligned}$$

where $P = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. So it is easy to check that $E4$ is a step 3 Carnot group, since the associated stratified algebra is

$$\mathfrak{e} \;=\; \mathfrak{e}_1 \oplus \mathfrak{e}_2 \oplus \mathfrak{e}_3,$$

where

$$\mathfrak{e}_1 \;=\; span\{X_1, X_2\}, \;\; \mathfrak{e}_2 \;=\; span\{X_3\} \text{ and } \mathfrak{e}_3 \;=\; span\{X_4\}.$$

The only non-vanishing commutation relationships among the generators are given by

$$[X_1, X_2] \;=\; X_3, \;\; [X_1, X_3] \;=\; X_4.$$

In addition, the group dilation is given by:

$$\delta_t(P) \;=\; \left(tx_1, tx_2, t^2x_3, t^3x_4\right), \;\; \forall\, P = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4, \;\; t > 0.$$

Lemma 3.2.2 ([22], Proposition 2.2.22)**.** *If $\mathbb{G}$ is a simply connected nilpotent*

Lie group then it is isomorphic to $\mathbb{R}^n$ *(so we indicate any element* P *by the corresponding point* Q *in* $\mathbb{R}^n$*) with a polynomial multiplication law* $(P,Q) \to P \circ Q$ *whose identity is* 0 *and inverse is* $P^{-1} = -P$.

Remark 3.2.1. According to Lemma 3.2.2, we can identify $\mathbb{G}$ with the triple $(\mathbb{R}^n, \circ, \delta_t)$.

3.3 The homogenous distance.

Carnot groups can be endowed by a distance and a norm defined in line with the stratification of the Lie algebra.

Definition 3.3.1. Let $\mathbb{G}$ be a Carnot group with stratifications $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$. The ***homogenous norm*** $\|.\|_h$ is a continuous function from $\mathbb{G} = \Big(\mathbb{R}^n, \circ, \delta_\lambda\Big)$ to $[0, +\infty)$ and it is defined as

$$\|P\|_h \;:=\; \left(\sum_{i=1}^{k} |x_i|^{\frac{2k!}{i}}\right)^{\frac{1}{2k!}}, \tag{3.5}$$

where $|x_i|$ is the usual $n-$dimentional Euclidean norm that defined on the vector space $\mathfrak{g}_i$ with $n = \dim \mathfrak{g}_i$ (where we have used the identification given in Lemma 3.2.2).

Example 3.3.1 (The Heisenberg group $\mathbb{H}^n$)**.** As we know $\mathbb{H}^n$ is a Carnot group with step 2, look at Example 2.1.5. Applying (3.5) we obtain:

$$\|(P,Q)\|_h \;=\; \left(\left(\sum_{i=1}^{2n} x_i^2\right)^2 + Q^2\right)^{\frac{1}{4}}, \;\; P = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}, Q \in \mathbb{R}.$$

Remark 3.3.1. The homogenous norm is satisfying the following properties:

1. $\|P\|_h \;=\; 0 \quad \Longleftrightarrow \quad P = e$, where e is the identity of the group,
2. $\|P^{-1}\|_h \;=\; \|P\|_h, \;\; \forall\; P \in \mathbb{R}^n$
3. $\|\delta_t(P)\|_h \;=\; t\|P\|_h, \;\; \forall\; P \in \mathbb{R}^n, t > 0,$
4. $\|P \circ Q\|_h \;\leq\; \|P\|_h + \|Q\|_h, \;\; \forall\; P, Q \in \mathbb{R}^n.$

Definition 3.3.2. The induced homogenous distance between two given points $P, Q \in \mathbb{G}$ is defined by:

$$d_h(P,Q) \;=\; \|Q^{-1} \circ P\|_h.$$

where we have indicate by P^{-1} the inverse of P w.r.t. the law group, i.e. $P \circ P^{-1} = P^{-1} \circ P = e$ and e is the identity in $\mathbb{G}$.

Recall that by Lemma 3.2.2 we can identify $P \equiv x \in \mathbb{R}^N$,$Q \equiv y \in \mathbb{R}^n$ and then $d_h(x-y) \;=\; \| - y \circ x\|_h$ with $\| \cdot \|_h$ given in (3.5).

3.4 Manifold structure vs Lie algebra structure.

From the previous section we know that the Lie algebra of a Lie groups is the space of all left-invariant vector fields defined on $\mathbb{G}$. Since that space is naturally isomorphic to $T_e\mathbb{G}$, one usually identifies the Lie algebra $\mathfrak{g}$ with $T_e\mathbb{G}$. The following theorem explains that precisely.

Theorem 3.4.1 ([22])**.** *Let $\mathbb{G}$ be a Lie group and $\mathfrak{g}$ be its Lie algebra. Then the following statements are satisfied:*

1. *$\mathfrak{g}$ is a vector space and the function*

$$\begin{aligned} \phi : \mathfrak{g} &\;\to\; T_e\mathbb{G}, \\ X &\;\to\; \phi(X) := X(e) \end{aligned}$$

   is isomorphism between $\mathfrak{g}$ and the tangent space $T_e\mathbb{G}$ (see Definition 1.1.12) to $\mathbb{G}$ at the identity e of $\mathbb{G}$. As a consequence, $\dim \mathfrak{g} = \dim T_e\mathbb{G} = \dim \mathbb{G}$.

2. *$\mathfrak{g}$ with the commutation operation is a Lie algebra.*

For a proof see [22].

Theorem 3.4.1 together with the function left-translation allows us to compute the left-invariant vector fields for any $X(P) \in \mathbb{G}$. Take $X(e) \in T_e\mathbb{G}$, then we can define a corresponding vector $X(P) \in T_P\mathbb{G}$ as

$$X(P) \ := \ \Big(dL_P\Big)\Big(X(e)\Big), \quad P \in \mathbb{G}, \tag{3.6}$$

where L_P is the left-translations, see Definition 3.1.2.

The following examples illustrate how to compute explicitly the vector fields given in Example 2.1.5.

Example 3.4.1 (The Heisenberg group $\mathbb{H}^1$)**.** As the left-invariant vector fields for $T_e\mathbb{H}^1$ we choose e_1, e_2, e_3 standard Euclidean 3-dimensional at basis. In order to compute the left-invariant vectors for $\mathbb{H}^1$, let us fix a point $P = (x_1, x_2, x_3) \in \mathbb{H}^1$ and consider the curve $\tilde{\gamma} : [0,1] \to \mathbb{H}^1$ defined as

$$\tilde{\gamma}(t) \ = \ f\Big(\gamma(t)\Big),$$

where $\gamma : [0,1] \to \mathbb{H}^1$ satisfies

$$\begin{aligned} \gamma(0) &= e = (0,0,0), \\ \dot{\gamma}(0) &= X(e), \end{aligned}$$

with $X(e)$ respectively equal to e_1, e_2 and then e_3, given $f : \mathbb{H}^1 \to \mathbb{H}^1$ then

$$(df)_e\Big(X(e)\Big) \ := \ \frac{d}{dt}\,\tilde{\gamma}(t)\Big|_{t=0}.$$

Choose as function f the left-translations $L_P : \mathbb{G} \to \mathbb{G}$ with $L_P(Q) = P \circ Q$ and consider $X(e) = e_i$, for $i = 1,2,3$, we have

$$\begin{aligned} \tilde{\gamma}_i(t) &= L_P\Big(\gamma_i(t)\Big), \\ \dot{\tilde{\gamma}}_i(0) &= (dL_P)(e_i), \end{aligned}$$

for $i = 1,2,3$.

Now, let us find the left translations with $P = (x_1, x_2, x_3)$

$$\begin{aligned} L_P\Big(\gamma(t)\Big) &= (x_1, x_2, x_3) \circ \Big(\gamma_1(t), \gamma_2(t), \gamma_3(t)\Big) \\ &= \left(x_1 + \gamma_1(t), x_2 + \gamma_2(t), x_3 + \gamma_3(t) + \frac{1}{2}\Big(x_1\gamma_2(t) - x_2\gamma_1(t)\Big)\right). \end{aligned}$$

To obtain the left-invariant vector fields we need to differentiate the left translations

$$\Big(dL_P\Big)_e(X) \;=\; \Big(\dot\gamma_1(0), \dot\gamma_2(0), \dot\gamma_3(0) + \frac{1}{2}\Big(x_1\dot\gamma_2(0) - x_2\dot\gamma_1(0)\Big)\Big),$$

where $\dot\gamma(0) = X$. Hence, the left-invariant vector field $X_1(P)$ corresponding to $e_1 = (1,0,0)$ is

$$\begin{aligned} X_1(P) \;=\; \dot{\tilde\gamma}_1(0) \;&=\; (dL_P)(1,0,0) \\ &=\; \Big(1, 0, -\frac{x_2}{2}\Big) \\ &=\; \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3}. \end{aligned}$$

Similarly, the left-invariant vector field $X_2(P)$ corresponding to $e_2 = (0,1,0)$ in $T_e\mathbb{H}^1$ is:

$$\begin{aligned} X_2(P) \;=\; \dot{\tilde\gamma}_2(0) \;&=\; (dL_P)(0,1,0) \\ &=\; \Big(0, 1, \frac{x_1}{2}\Big) \\ &=\; \frac{\partial}{\partial x_2} + \frac{x_2}{2}\frac{\partial}{\partial x_3}. \end{aligned}$$

Finally, the left-invariant vector field $X_3(P)$ corresponding to $e_3 = (0,0,1)$ in $T_e\mathbb{H}^1$ is:

$$\begin{aligned} X_3(P) \;=\; \dot{\tilde\gamma}_3(0) \;&=\; (dL_P)(0,0,1) \\ &=\; (0,0,1) \\ &=\; \frac{\partial}{\partial x_3}. \end{aligned}$$

Example 3.4.2 (Engel group $E4$)**.** In this example we compute the left-invariant vector fields in the Engel group, see Example 3.2.4. As left-invariant vector fields for T_eE4 we choose e_1, e_2, e_3 and e_4 standard Euclidean 4-dimensional basis. In order to compute the left invariant vector fields for the Engel group, let us fix a point $P = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4E4$ (see Lemma 3.2.2) and consider the curve $\widetilde{\gamma} : [0,1] \to \mathbb{R}^4$ defined as

$$\widetilde{\gamma}(t) \;=\; f\Big(\gamma(t)\Big),$$

where $\gamma : [0.1] \to \mathbb{R}^4$ satisfies

$$\begin{cases} \gamma(0) & = e = (0,0,0,0), \\ \dot{\gamma}(0) & = X(e), \end{cases}$$

with $X(e)$ respectively equal e_1, e_2, e_3 and e_4. Given $f : E4 \to E4$ then

$$(df)_e\Big(X(e)\Big) \;:=\; \frac{d}{dt}\,\widetilde{\gamma}(t)\Big|_{t=0}.$$

As the previous example, choose as function f the left-translations $L_P : \mathbb{G} \to \mathbb{G}$ with $L_P(Q) = P \circ Q$ and consider $X(e) = e_i$, for $i = 1,2,3,4$, we find

$$\begin{aligned} \widetilde{\gamma}_i(t) &= L_P\Big(\gamma_i(t)\Big), \\ \dot{\widetilde{\gamma}}_i(0) &= (dL_P)(e_i), \end{aligned}$$

for $i = 1,2,3$.

Now, let us compute the left-translations with $P = (x_1, x_2, x_3, x_4)$

$$\begin{aligned} L_P\Big(\gamma(t)\Big) \quad &= (x_1, x_2, x_3, x_4) \circ \Big(\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t)\Big) \\ &= \Bigg(x_1 + \gamma_1(t), x_2 + \gamma_2(t), x_3 + \gamma_3(t) + x_1\gamma_2(t), \\ &\qquad\qquad x_4 + \gamma_4(t) + x_1\gamma_3(t) + \frac{x_2^2}{2}\,\gamma_2(t)\Bigg). \end{aligned}$$

To obtain the left-invariant vector fields we need to differentiate the left translations at e_i.

$$(dL_P)_e(X) \;=\; \Bigg(\dot{\gamma}_1(0), \dot{\gamma}_2(0), \dot{\gamma}_3(0) + x_1\dot{\gamma}_2(t), \dot{\gamma}_4(0) + x_1\dot{\gamma}_3(0) + \frac{x_2^2}{2}\,\dot{\gamma}_2(0)\Bigg),$$

where $\dot{\gamma}(0) = e_i$ for $i = 1,2,3,4$. Hence, the left-invariant vector field $X_1(P)$

corresponding to $e_1 = (1,0,0,0)$ is

$$
\begin{aligned}
X_1(P) &= (dL_P)(1,0,0,0) \\
&= (1,0,0,0) \\
&= \frac{\partial}{\partial x_1}.
\end{aligned}
$$

Similarly, the left-invariant vector field $X_2(P)$ corresponding to $e_2 = (0,1,0,0)$ is

$$
\begin{aligned}
X_2(P) &= (dL_P)(0,1,0,0) \\
&= \left(0,1,x_1,\frac{x_2^2}{2}\right) \\
&= \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + \frac{x_2^2}{2}\frac{\partial}{\partial x_4}.
\end{aligned}
$$

Also, the left-invariant vector field $X_3(P)$ corresponding to $e_3 = (0,1,0,0)$ is

$$
\begin{aligned}
X_3(P) &= (dL_P)(0,0,1,0) \\
&= (0,1,1,x_1) \\
&= \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4}.
\end{aligned}
$$

Finally, the left-invariant vector field $X_4(P)$ corresponding to $e_4 = (0,0,0,1)$ is

$$
\begin{aligned}
X_4(P) &= (dL_P)(0,0,1,0) \\
&= (0,0,0,1) \\
&= \frac{\partial}{\partial x_4}.
\end{aligned}
$$

Recall that the first layer of the stratified Lie algebra is spanned by X_1 and X_2 only.

Part II

Starshapedness and Convexity

Chapter 4

Convexity in the Euclidean space vs. Carnot groups and sub-Riemannian geometries.

In this chapter we start recalling convex sets and convex functions in the Euclidean case, and some mutual relations. Rockafellar studied all these notions extensively in his book [81]. The idea of presenting these basic concepts of convex analysis is to remind the reader about some core notions in this setting and so more easily to highlight later the analogies and differences with the corresponding notions in the Carnot groups and the sub-Riemannian manifolds. Later, we study the case of the Carnot groups, looking at both properties of convex sets and convex functions. Some of these properties have been studied by Danielli-Garofalo-Nhieu in Carnot groups in [39]. Our goal is to generalize these results in general the geometry of vector fields, which apply in particular to a very large clam of sub-Riemannian geometries as an example the Grushin space. At this purpose we use the notion of $\mathcal{X}-$lines and investigate some relations between $\mathcal{X}-$convex sets and $\mathcal{X}-$convex functions.

4.1 Euclidean convexity.

We start with recalling the notion of (Euclidean) convex functions and convex sets. Then we remind the reader of some well-known properties that connect convex functions with convex sets, such as level sets and the epigraph. The literature on this subject is very vast, e.g. [14, 34, 36, 40, 49, 51, 54, 81].

4.1.1 Convex functions.

Definition 4.1.1. Let $\Omega \subseteq \mathbb{R}^n$ be open, a function $u : \Omega \to \mathbb{R}$ is said to be ***convex function*** in Ω if for each $P, Q \in \Omega$ the following condition is satisfied:

$$u\Big((1-t)P + t\,Q\Big) \;\leq\; (1-t)u(P) + t\,u(Q), \quad \forall\; t \in (0,1). \tag{4.1}$$

The function u is said to be ***strictly convex*** if (4.1) is strict for any $P \neq Q$ and $t \in [0,1]$.
We say $u : \Omega \to \mathbb{R}$ is ***concave*** if the function $-u : \Omega \to \mathbb{R}$ is convex.

Remark 4.1.1. Definition 4.1.1 requires that the function u to be defined at $(1-t)P + t\,Q$, for each $P, Q \in \Omega$ and $t \in [0,1]$, which is equivalent to requiring the domain to be convex (see Definition 4.1.2 below).

Example 4.1.1. The function $u(P) = |P|$ is convex over $\mathbb{R}$.

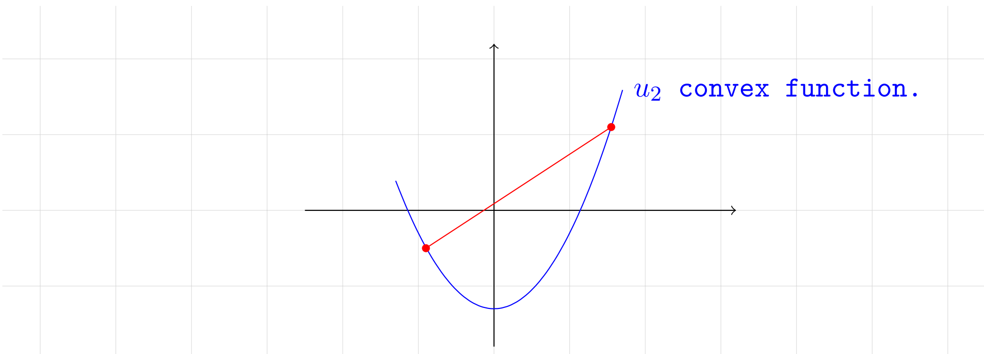

Figure 4.1: `A graphical representation of convex functions in` $\mathbb{R}$.

Example 4.1.2. In Figure 4.1 we show the geometrical meaning of convexity in the Euclidean space.

4.1.2 Convex sets.

For any two distinct points $P_1, P_2 \in \mathbb{R}^n$, that points of the form

$$Q \quad = \quad (1-t)P_1 + t\,P_2, \quad \text{where } t \in \mathbb{R}^n,$$

forms the straight line segment passing through P_1 and P_2. The parameter value $t = 0$ corresponds to $Q = P_2$, and the parameter value $t = 1$ corresponds to $Q = P_1$. The values of the parameter t between 0 and 1 correspond to all the points of the (closed) straight line segment between P_1 and P_2.

Figure 4.2: `Graphical meaning of convex sets.`

Definition 4.1.2. Let $\Omega \subseteq \mathbb{R}^n$, Ω is said to be ***convex*** if for each $P, Q \in \Omega$ the following condition is satisfied:

$$(1-t)P + t\,Q \quad \in \quad \Omega, \qquad \forall \; t \in [0,1]. \tag{4.2}$$

See Figure 4.2.

Example 4.1.3. A cone is a convex set. See Figure 4.3.

Example 4.1.4. The Euclidean ball

$$B_R(P) \quad = \quad \left\{ Q \in \mathbb{R}^n : (\tilde{x}_1 - x_1)^2 + \cdots + (\tilde{x}_n - x_n)^2 \leq R^2 \right\},$$

such that $P = (x_1, \ldots, x_n), Q = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \mathbb{R}^n$, is convex in $\mathbb{R}^n$ for all $R > 0$ and all centers $P \in \mathbb{R}^n$.

Example 4.1.5. One trivial example of convex sets are **half spaces**. So, if we consider the associated linear map $\gamma : \mathbb{R}^n \to \mathbb{R}$ and $t \in \mathbb{R}$, then the following

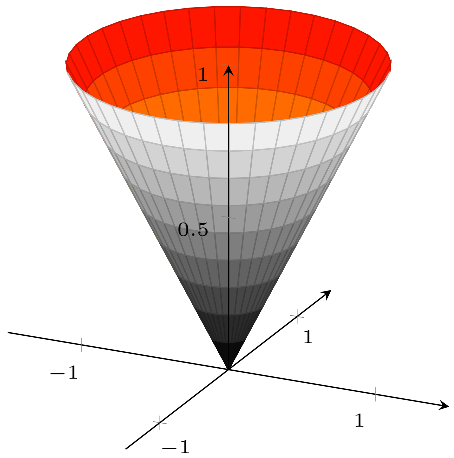

Figure 4.3: The cone in $\mathbb{R}^3$ is an example of a convex set.

sets:

$$\Big\{P \in \mathbb{R}^n : \gamma(P) < t\Big\}, \qquad \Big\{P \in \mathbb{R}^n : \gamma(P) > t\Big\},$$
$$\Big\{P \in \mathbb{R}^n : \gamma(P) \leq t\Big\}, \qquad \Big\{P \in \mathbb{R}^n : \gamma \geq t\Big\},$$

are convex.

Note that, the **hyperplane** that associated to these half spaces sets is the set:

$$\mathcal{P} \ := \ \Big\{P \in \mathbb{R}^n : \gamma(P) = t\Big\},$$

which is again a convex set.

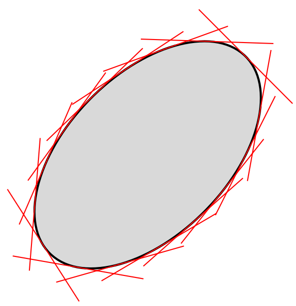

Figure 4.4: Illustration of the fact that intersection of infinitely many half spaces.

Remark 4.1.2. The intersection of convex sets is convex; in particular, the intersection of any number of half spaces is convex. See Figure 4.4.

Definition 4.1.3. Let $\Omega \subseteq \mathbb{R}^n$ be open, a ***convex combination*** of finitely many points $P_1, ..., P_m \in \Omega$, where $m \geq 2$, means any element $P \in \Omega$ of the form

$$P \;=\; \sum_{i=1}^{m} t_i\, P_i,$$

such that the coefficients t_i of the convex combination are nonnegative numbers summing to one, i.e. $t_i \geq 0$ and $\sum\limits_{i=1}^{m} t_i = 1$.

Corollary 4.1.1. *Ω is a convex set if and only if any convex combination of points in Ω belongs to Ω.*

Next we like to recall an important notion in the study of convex analysis.

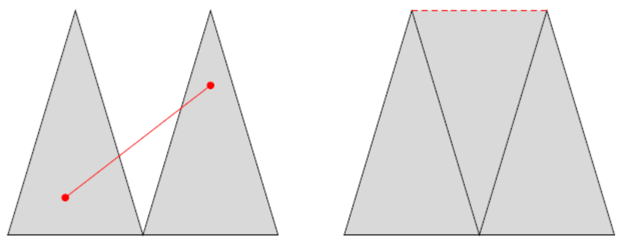

Figure 4.5: `The convex hull is always convex even for a non-convex set.`

Definition 4.1.4 (Convex hull)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open, the ***convex hull*** of Ω, denoted by $co\ \Omega$, is the smallest convex set containing Ω. In another words, the convex hull of Ω is the intersection of all convex subsets in $\mathbb{R}^n$ that contain Ω. See Figure 4.5

Remark 4.1.3. Obviously Ω is convex $\iff$ $co\ \Omega = \Omega$.

Definition 4.1.4 is well posed since there is always at least one convex set containing the set Ω, which is the space $\mathbb{R}^n$.

Corollary 4.1.2. *The convex hull of open set $\Omega \subseteq \mathbb{R}^n$ can be described as the set of all convex combinations generated by points in Ω, i.e.*

$$co\ \Omega \;:=\; \left\{ \sum_{i=1}^{m} t_i\, P_i : P_i \in \Omega,\;\; t_i \geq 0,\;\; \sum_{i=0}^{m} t_i = 1,\;\; m \geq 2 \right\}.$$

In the following we recall some important and very well-known relations of standard convex sets and convex functions.

Definition 4.1.5 (Level sets)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open and consider $u : \Omega \to \mathbb{R}$ and $a \in \mathbb{R}$, the set

$$\Omega_a \;:=\; \Big\{P \in \Omega : u(P) \le a\Big\}$$

is called a ***level set*** of the function u.

Lemma 4.1.1. *The level sets Ω_a of the convex function $u : \Omega \to \mathbb{R}$ are always convex sets for all $a \in \mathbb{R}$.*

Proof. Assume that $P, Q \in \Omega_a$, we need to prove that:

$$(1-t)P + t\,Q \;\in\; \Omega_a, \qquad \forall\, t \in [0,1].$$

Since u is a convex function

$$u\Big((1-t)P + t\,Q\Big) \;\le\; (1-t)u(P) + t\,u(Q).$$

Using the fact that $P, Q \in \Omega_a$ implies $u(P) \le a$ and $u(Q) \le a$, we have:

$$(1-t)u(P) + t\,u(Q) \;\le\; (1-t)a + t\,a \;=\; a.$$

Thus $(1-t)P + t\,Q \in \Omega_a$, i.e. Ω_a is a convex set for all $a \in \mathbb{R}$. ∎

Remark 4.1.4. The reverse of Lemma 4.1.1 is not true. Indeed, there exists a non-convex function whose level sets are convex. For example, $u(P) = P^3$ over $\mathbb{R}$ is not convex and its level sets are convex.

We now introduce the notion of quasiconvex functions which is related with the convexity of level sets.

Definition 4.1.6. Let $\Omega \subseteq \mathbb{R}^n$ be open, a function $u : \Omega \to \mathbb{R}$ is said to be ***quasiconvex*** in Ω if for all $P, Q \in \Omega$ the following condition is satisfied:

$$u\Big((1-t)P + t\,Q\Big) \;\le\; \max\Big\{u(P), u(Q)\Big\}, \qquad \forall\, t \in (0,1). \tag{4.3}$$

The function u is said to be ***strictly quasiconvex*** if (4.3) is strict for any $P \neq Q$ and $t \in [0,1]$.
We say $u : \Omega \to \mathbb{R}$ is ***quasiconcave*** if $-u : \Omega \to \mathbb{R}$ is quasiconvex.

Example 4.1.6. The function $u(P) = P^3$ is quasiconvex over $\mathbb{R}$. In general, any monotonic function is quasiconvex.

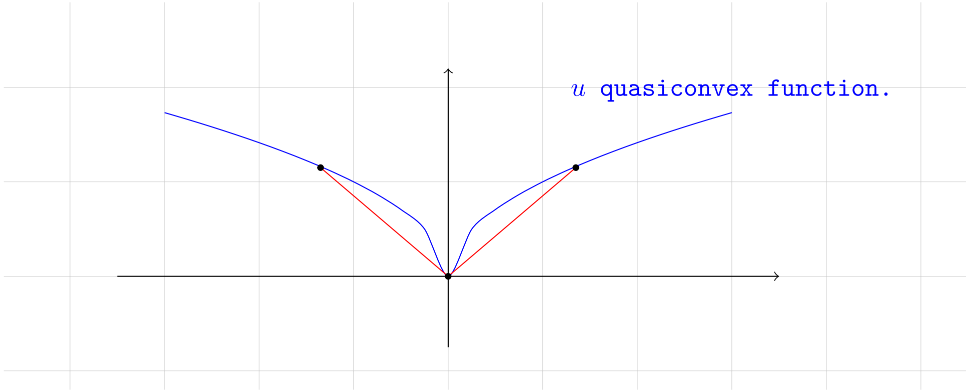

Figure 4.6: The function $u(P) = \sqrt{|P|}$ is clearly not convex.

Lemma 4.1.2. *(Euclidean) convexity always implies quasiconvexity.*

Proof. Let $\Omega \subseteq \mathbb{R}^n$ be open and consider the function $u : \Omega \to \mathbb{R}^n$ is convex on Ω, i.e. for each $P, Q \in \Omega$ and for every $t \in [0,1]$ we can write:

$$\begin{aligned} u\Big((1-t)P + t\,Q\Big) &\leq (1-t)u(P) + t\,u(Q) \\ &\leq (1-t)\ \max\Big\{u(P), u(Q)\Big\} + t\ \max\Big\{u(P), u(Q)\Big\} \\ &= \max\Big\{u(P), u(Q)\Big\}. \end{aligned}$$

Thus u is quasiconvex. We conclude that convexity implies quasiconvexity. ■

Remark 4.1.5. The notion of quasiconvexity is weaker than the one of convexity in the Euclidean setting, therefore the reverse of Lemma 4.1.2 is not true. As an example: the function $u(P) = \sqrt{|P|}$ is quasiconvex but not convex over $\mathbb{R}$.
See Figure 4.6.

Theorem 4.1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and consider the function $u : \Omega \to \mathbb{R}$. Then u is a quasiconvex function in Ω if and only if all the level sets Ω_a are convex sets for all $a \in \mathbb{R}$.*

Proof. ($\Rightarrow$) Assume that u is a quasiconvex and consider the level set Ω_a on u. We want to prove that Ω_a is convex.
Let $P, Q \in \Omega_a$ and $t \in [0,1]$ and by using the fact that u is a quasiconvex

function we can write:

$$u\Big((1-t)P + t\,Q\Big) \;\leq\; \max\Big\{u(P), u(Q)\Big\} \;\leq\; a,$$

since $u(P) \leq a$ and $u(Q) \leq a$. Thus $(1-t)P + t\,Q \in \Omega_a$, i.e. Ω_a is a convex set.

($\Leftarrow$) In order to prove the opposite statement, assume, without loss of generality, that $\max\Big\{u(P), u(Q)\Big\} = u(P)$ and consider the level set Ω_a with $a = u(P)$, that is

$$\Omega_a \;=\; \Big\{Q \in \Omega : u(Q) \leq u(P)\Big\};$$

Obviously $P \in \Omega_a$.

Since Ω_a for all $a \in \mathbb{R}$ is convex, we have:

$$(1-t)P + t\,Q \;\in\; \Omega_a, \qquad \forall\; t \in [0,1],$$

i.e.

$$u\Big((1-t)P + t\,Q\Big) \;\leq\; u(P) \;=\; \max\Big\{u(P), u(Q)\Big\}.$$

Thus u is a quasiconvex function on its domain. ∎

Next, we recall an important link between convex sets and convex functions by characterising the epigraph.

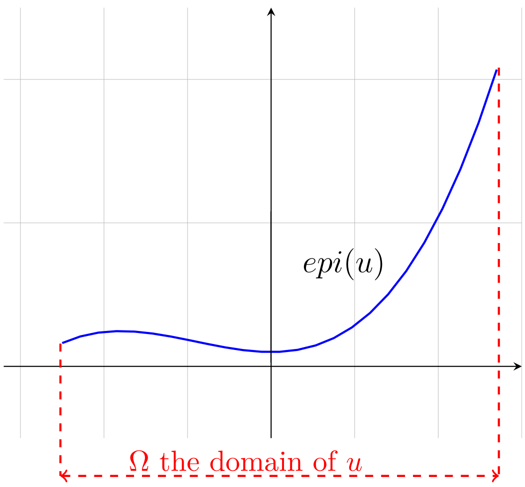

Figure 4.7: `Epigraph of a function` u `in` $\mathbb{R}$.

Definition 4.1.7 (Epigraph of a function)**.** Let Ω be an open set on $\mathbb{R}^n$ and define the function $u : \Omega \to \mathbb{R}$. The set

$$epi(u) \;\; := \;\; \Big\{(P,a) \in \Omega \times \mathbb{R} : u(P) \leq a\Big\}$$

is called the ***epigraph*** of u. See Figure 4.7.

Proposition 4.1.1. *Consider the open subset $\Omega \subseteq \mathbb{R}^n$. The function $u : \Omega \to \mathbb{R}$ is convex if and only if its epigraph is a convex subset of $\Omega \times \mathbb{R}$.*

Proof. ($\Rightarrow$) Assume that the function u is convex. Let $(P,a),(Q,b) \in epi(u)$, we need to prove that $epi\ u$ is convex, i.e.

$$\Big((1-t)P + t\,Q, (1-t)a + t\,b\Big) \;\; \in \;\; epi(u), \quad \forall\, t \in [0,1].$$

Since $(P,a),(Q,b) \in epi(u)$ that implies $u(P) \leq a$ and $u(Q) \leq b$. As u is a convex function we can write:

$$u\Big((1-t)P + t\,Q\Big) \;\; \leq \;\; (1-t)u(P) + t\,u(Q) \;\; \leq \;\; (1-t)a + t\,b.$$

We conclude

$$\Big((1-t)P + t\,Q, (1-t)a + t\,b\Big) \;\; \in \;\; epi(u),$$

i.e. $epi(u)$ is a convex subset of $\Omega \times \mathbb{R}$.

($\Leftarrow$) Assume that $epi(u)$ is a convex subset of $\Omega \times \mathbb{R}$ and we need to prove that u is a convex function on its domain, i.e. for all $P, Q \in \Omega$ we have

$$u\Big((1-t)P + t\,Q\Big) \;\; \leq \;\; (1-t)u(P) + t\,u(Q), \;\; \forall\, t \in [0,1]. \tag{4.4}$$

Since $epi(u)$ is convex then for all $(P,a)(Q,b) \in epi(u)$ the segment:

$$\Big((1-t)P + t\,Q, (1-t)a + b\,t\Big) \;\; \in \;\; epi(u), \quad t \in [0,1].$$

Then by using the definition of $epi(u)$ we have:

$$u\Big((1-t)P + t\,Q\Big) \;\; \leq \;\; (1-t)a + t\,b, \;\; \forall\, t \in [0,1]. \tag{4.5}$$

Note that $\Big(P, u(P)\Big), \Big(Q, u(Q)\Big) \in epi(u)$ is trivial. So we rewrite (4.5) with $a = u(P)$ and $b = u(Q)$ which implies (4.4). Hence u is a convex function. $\blacksquare$

In the following we recall another relation between convex functions and convex sets which is the convex envelope of a function u. This property is an important one since it is involved in the study of the solution of PDEs. For example it is often used to prove that the level sets of solutions of elliptic capacitary remain convex, see e.g. [84].

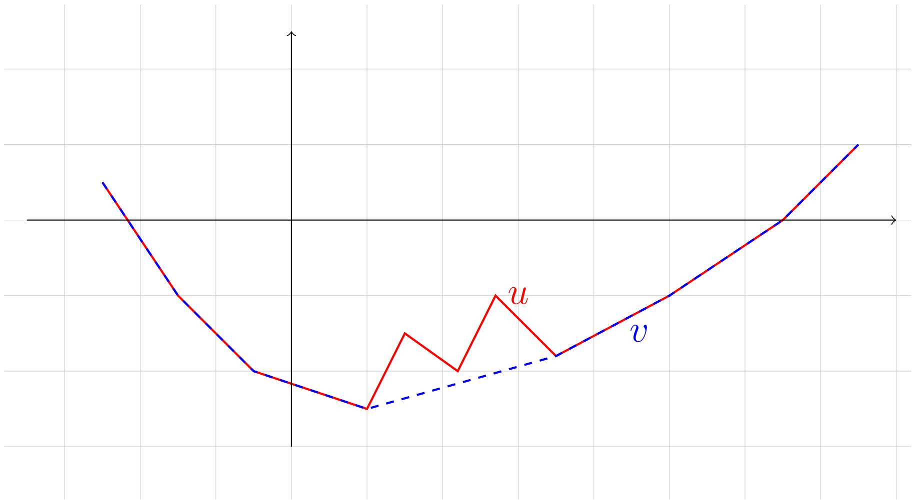

Figure 4.8: `Convex envelope of a function` u.

Definition 4.1.8. Let $\Omega \subset \mathbb{R}^n$ be open convex and define the function $u : \Omega \to \mathbb{R}^n$, the ***convex envelope*** of u over its convex domain Ω, denoted by u^*, is the largest possible convex of u over Ω, i.e.

$$u^*(P) \;\; := \;\; \sup \Big\{ v(P) : v \text{ is convex and } \; v \leq u \Big\}.$$

See Figure 4.8.

Last connection we like to recall between the convexity of functions and the convexity of sets in the Euclidean space is related to the next notion.

Definition 4.1.9. Let $\Omega \subseteq \mathbb{R}^n$ be open, we define the ***indicator function*** on Ω as $1_\Omega : \mathbb{R}^n \to \mathbb{R} \bigcup \{\infty\}$ which is given by:

$$1_\Omega(P) \;\; := \begin{cases} 0, & \text{if } P \in \Omega, \\ +\infty, & \text{if } P \notin \Omega. \end{cases}$$

Lemma 4.1.3. *A set Ω is convex if and only if its indicator function 1_Ω is a convex function.*

Proof. ($\Rightarrow$) Assume that Ω is convex, and let $P, Q \in \mathbb{R}^n$, and $t \in [0,1]$. If either P or Q are not in Ω then

$$(1-t)1_\Omega(P) + t\,1_\Omega(Q) \;=\; +\infty.$$

Therefore

$$1_\Omega\Big((1-t)P + t\,Q\Big) \;\leq\; +\infty \;=\; (1-t)1_\Omega(P) + t\,1_\Omega(Q).$$

On the other hand, if $P, Q \in \Omega$, then also

$$(1-t)P + t\,Q \;\in\; \Omega,$$

and therefore

$$1_\Omega\Big((1-t)P + t\,Q\Big) \;=\; (1-t)1_\Omega(P) + t\,1_\Omega(Q) \;=\; 0.$$

Thus 1_Ω is convex.
($\Leftarrow$) Now assume that 1_Ω is convex, and let $P, Q \in \Omega$ with $t \in [0,1]$. The convexity of 1_Ω implies that

$$1_\Omega\Big((1-t)P + t\,Q\Big) \;\leq\; (1-t)1_\Omega(P) + t\,1_\Omega(Q) \;=\; 0.$$

Since 1_Ω only takes the values 0 and ∞ this shows that,

$$1_\Omega\Big((1-t)P + t\,Q\Big) \;=\; 0,$$

i.e.

$$(1-t)P + t\,Q \;\in\; \Omega.$$

Thus Ω is convex. ∎

4.2 Convexity in Carnot groups.

Danielli-Garofalo-Nhieu [39] and Juutinen-Lu-Manfredi-Stroffolini [61] have introduced the concept of $\mathcal{H}-$convexity (also called horizontal convexity) for functions defined on Carnot groups. In this section we study the idea of

$\mathcal{H}-$convexity and introduce some properties of $\mathcal{H}-$convex functions and $\mathcal{H}-$convex sets.

4.2.1 Convex functions in Carnot groups.

We introduce two different notions of $\mathcal{H}-$convex functions; a strongly $\mathcal{H}-$convex notion and a weakly $\mathcal{H}-$convex notion. These two types of functions do not require their domains to be $\mathcal{H}-$convex, unlike standard (Euclidean) convex functions.
We first introduce an important notion in the study of the convexity in Carnot groups which play an important role on this area; it is given in the following definition.

Definition 4.2.1 (Horizontal plane in Carnot groups)**.** Let $\mathbb{G}$ be a Carnot group and define its Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ where $m = \dim\, \mathfrak{g}_1$, (see Definition 3.2.9). Fix a point $P \in \mathbb{G}$ we define the ***horizontal plane*** through P, denoted by $\mathcal{H}_P$, as the $m-$dimensional submanifold of $\mathbb{G}$ which is given by

$$\mathcal{H}_P \;:=\; L_P\Bigg(\exp\Big(\mathfrak{g}_1 \times \{0\}\Big)\Bigg), \tag{4.6}$$

where L_P is the left translation (see Definition 3.1.2), and 0 is the $n-m-$dimensional zero vector in $\mathfrak{g}$, with $n = \dim\, \mathfrak{g}_1 + \cdots + \dim\, \mathfrak{g}_r$.

Definition 4.2.2. Let $P, Q \in \mathbb{G}$, for $t \in [0,1]$ we define the ***convex combination*** of P and Q based on P, denoted by $P_t^{\mathbb{G}}$, as follows:

$$P_t^{\mathbb{G}} \;:=\; P \circ \delta_t\left(P^{-1} \circ Q\right), \tag{4.7}$$

where δ_t is the dilation on Carnot groups, see Definition 3.2.10.

Example 4.2.1. Let $P = (x_1, x_2, x_3), Q = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) \in \mathbb{H}^1$ and for $t \in [0,1]$

we write the convex combination of P and Q as follows:

$$\begin{aligned}
P_t^{\mathbb{H}^1} &:= P \circ \delta_t(P^{-1} \circ Q) \\
&= (x_1, x_2, x_3) \circ \delta_t\Big((-x_1, -x_2, -x_3) \circ (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)\Big) \\
&= (x_1, x_2, x_3) \circ \Bigg(t(\widetilde{x}_1 - x_1), t(\widetilde{x}_2 - x_2), \\
&\qquad\qquad t^2\left(\widetilde{x}_3 - x_3 + \frac{1}{2}\left(-x_1\widetilde{x}_2 + x_2\widetilde{x}_1\right)\right)\Bigg) \\
&= \Bigg(x_1 + t(\widetilde{x}_1 - x_1), x_2 + t(\widetilde{x}_2 - x_2), \\
&\qquad x_3 + \frac{t}{2}\left(x_1\widetilde{x}_2 - x_2\widetilde{x}_1\right) + \frac{t^2}{2}\left(\widetilde{x}_3 - x_3 + \frac{1}{2}\left(-x_1\widetilde{x}_2 + x_2\widetilde{x}_1\right)\right)\Bigg).
\end{aligned}$$

Definition 4.2.3. Let $\mathbb{G}$ be a Carnot group and $\Omega \subseteq \mathbb{G}$, a real-valued function $u : \mathbb{G} \to (-\infty, \infty)$ is called ***strongly*** $\mathcal{H}-$***convex*** if the convex combination $P_t^{\mathbb{G}}$ for any $P, Q \in \Omega$ satisfies the following condition:

$$u\left(P_t^{\mathbb{G}}\right) \;\leq\; (1-t)u(P) + t\,u(Q), \qquad \forall\, t \in [0,1]. \tag{4.8}$$

Remark 4.2.1. Definition 4.2.3 is equivalent to the (Euclidean) convexity in case $\mathbb{G} = \mathbb{R}^n$.

Definition 4.2.4. Let $\Omega \subseteq \mathbb{G}$, a real-valued function $u : \Omega \to \mathbb{R}$ is called ***weakly*** $\mathcal{H}-$***convex*** if the convex combination $P_t^{\mathbb{G}}$ for any $P \in \Omega$ and for each $Q \in \mathcal{H}_P \cap \Omega$, where $P \neq Q$, satisfies the following condition:

$$u\left(P_t^{\mathbb{G}}\right) \;\leq\; (1-t)u(P) + t\,u(Q), \qquad \forall\, t \in (0,1). \tag{4.9}$$

The function u is said to be ***strictly weakly*** $\mathcal{H}-$***convex*** if (4.9) is strict for all $t \in (0,1)$.

Regularity for $\mathcal{H}-$convex function.

First order regularity. It is very well-known that (standard) convex functions on $\mathbb{R}^n$ are locally Lipschitz which is equivalent to requiring that (at least in convex sets) the $L^\infty-$norm of the gradient is locally bounded, i.e. it is

bounded on any compact subset of $\mathbb{R}^n$.
The same result is still true in the case of Carnot groups, indeed Magnani [72] and Rickly [78] proved that any $\mathcal{H}-$convex function is locally Lipschitz continuous w.r.t. homogenous distance. It it noticeable that the $L^\infty-$norm of the total gradient may not be locally bounded but the horizontal gradient is.

Second order regularity. It is also very well-known that in the 1-dimensional Euclidean settings $\mathbb{R}$ a C^2 function u is convex if and only if $u'' \geq 0$. Similarly in $\mathbb{R}^n$, where

$$D^2 u \geq 0 \tag{4.10}$$

has to be interpreted in the sense of non-negative definite matrices. In another words (4.10) means that $\forall\ a \in \mathbb{R}^n$, $a^T\, D^2 u\, a \geq 0$, which is equivalent to requiring that all eigenvalues of $D^2 u$ are non-negative (recall $D^2 u$ is always symmetric for C^2 functions).
It is also known that if the convex function is not C^2, then the same characterisation for the second derivatives given in (4.10) is still true but it needs to be interpreted in the viscosity sense (for more details on viscosity solutions, see e.g. [9, 25, 35]). The characterisation in the viscosity sense is non-trivial and it has been proved by Alvarez-Lasry-Lions in [7].
This important characterisation of convex functions has been successfully generalised on Heisenberg group by Lu-Manfredi-Stroffolini [70] and later to more general on Carnot groups by Juutinen-Lu-Manfredi-Stroffolini [61]. For similar results see also Balogh-Rickly [79]. In this case (4.10) becomes $D^2_{\mathcal{H}} u \geq 0$, where $D^2_{\mathcal{H}} u$ is defined as

$$\left(D^2_{\mathcal{H}}\, u\right)_{ij=1}^{m} \;=\; \left(X_i X_j u\right)_{ij=1}^{m} \tag{4.11}$$

and X_i, X_j are the left-invariant vector fields (see Definition 3.1.3).
Since $D^2_{\mathcal{H}} u$ is not symmetric even for smooth functions often it is replaced by $\left(D^2_{\mathcal{H}} u\right)^*$ which is the symmetric part of $D^2_{\mathcal{H}}\, u$, i.e

$$\left(D^2_{\mathcal{H}}\, u\right)^*_{i,j} \;=\; \left(\frac{X_i X_j u + X_j X_i u}{2}\right)_{i,j=1,\dots,m}. \tag{4.12}$$

For more related results we refer the reader to e.g. [30, 55, 89].

4.2.2 Convexity of sets in Carnot groups.

In this section we introduce the notion of $\mathcal{H}-$convex sets and study some properties which show how they are related to weakly $\mathcal{H}-$convex functions.

Definition 4.2.5. Let $\mathbb{G}$ be a Carnot group and $\Omega \subseteq \mathbb{G}$, Ω is called $\mathcal{H}-$convex if for any $P \in \Omega$ and any $Q \in \Omega \bigcap \mathcal{H}_P$ the following condition is satisfied:

$$(1-t)P + t\,Q \in \Omega, \quad \forall\; t \in [0,1]. \tag{4.13}$$

Next we like to introduce two characteristics of weakly $\mathcal{H}-$convex functions.

Proposition 4.2.1. *Consider the weakly $\mathcal{H}-$convex function u and $a \in \mathbb{R}$, the level sets $\Omega_a := \Big\{P \in \mathbb{G} : u(P) \leq a\Big\}$ are $\mathcal{H}-$convex.*
For the proof see [39].

Proposition 4.2.2. *Let Ω be a $\mathcal{H}-$convex set in a Carnot group $\mathbb{G}$, a function $u : \Omega \to \mathbb{R}$ is weakly $\mathcal{H}-$convex if and only if epi(u) is a $\mathcal{H}-$convex set in $\mathbb{G}$.*
For the proof see [39].

4.3 Convexity in Hörmander vector fields geometries.

In this section we introduce a notion of convexity depending on the geometry of vector fields, denoted by $\mathcal{X}-$convexity. First of all, in this geometry we need $\mathcal{X}-$lines. It is worth mentioning that $\mathcal{X}-$convexity extended the meaning of the $\mathcal{H}-$convexity to general Riemannian geometries. Full details on $\mathcal{X}-$convexity and its applications can be found in [10, 11, 12].

4.3.1 Convex functions in Hörmander vector fields geometries.

As we mentioned earlier, in Chapter 2, in Sub-Riemannian geometry there are a special type of admissible paths; which are those curves whose velocities are

admissible and they are called horizontal curves (see Definition 2.1.5). We now introduce a special type of horizontal curves: the $\mathcal{X}-$lines.
Throughout our study we consider a family of at least C^2-vector fields $\mathcal{X} = \Big\{X_1(P), ..., X_m(P)\Big\}$, $X_i : \Omega \to \mathbb{R}^n, i = 1, 2, .., m,\ m \le n$, where Ω is open in $\mathbb{R}^n$.

Definition 4.3.1 ($\mathcal{X}-$lines)**.** Any absolutely continuous curve $x_\alpha : [0,1] \to \mathbb{R}^n$ (see Definition 1.2.4) is called $\mathcal{X}-$***line*** if it solves the following ODE:

$$\dot{x}_\alpha(t) \;=\; \sum_{i=1}^{m} \alpha_i X_i\Big(x_\alpha(t)\Big) \;=\; \sigma\Big(x_\alpha(t)\Big)\alpha, \quad t \in [0,1], \tag{4.14}$$

for some $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$, where $\sigma(x)$ is the $n \times m-$matrix having the vectors $X_1, \dots, X_m$ as columns.

Thus $\dot{x}_\alpha(t)$ is not constant but its coordinates w.r.t. the given vector fields $X_1, \dots, X_m$ are constant, we recall that the vector, α is also called horizontal velocity; thus $\mathcal{X}-$lines are horizontal curves with constant horizontal velocity.

Remark 4.3.1. The $\mathcal{X}-$lines have at least the same regularity of the vector fields.

Note that in the case of the Euclidean setting $\mathcal{X}-$lines turn to a standard Euclidean lines. In the following examples we compute the $\mathcal{X}-$lines in the Heisenberg group $\mathbb{H}^1$ and Engel group.

Example 4.3.1 ($\mathcal{X}-$line in $\mathbb{H}^1$)**.** First let us write the matrix $\sigma(x)$ associated to the vector fields of $\mathbb{H}^1$ (see Example 2.1.5),

$$\sigma(x) \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\dfrac{x_2}{2} & \dfrac{x_1}{2} \end{pmatrix},$$

for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ we can write:

$$\sigma(x)\alpha \;=\; \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ -\alpha_1 \dfrac{x_2}{2} & \alpha_2 \dfrac{x_1}{2} \end{pmatrix}.$$

Hence the $\mathcal{X}-$lines in $\mathbb{H}^1$ are the solutions of the following ODE's:

$$\begin{cases} \dot{x}_1(t) &=& \alpha_1, \\ \dot{x}_2(t) &=& \alpha_2, \\ \dot{x}_3(t) &=& \dfrac{-\alpha_1\, x_2(t) + \alpha_2\, x_1(t)}{2}, \end{cases} \tag{4.15}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$.
The solution of (4.15) becomes:

$$\begin{cases} x_1(t) &=& \alpha_1\, t + C_1, \\ x_2(t) &=& \alpha_2\, t + C_2, \end{cases}$$

and

$$\begin{aligned} \dot{x}_3(t) &= \frac{-\alpha_1(\alpha_2 t + C_2) + \alpha_2(\alpha_1 t + C_1)}{2}, \\ &= \frac{-\alpha_1\, C_2 + \alpha_2\, C_1}{2}, \end{aligned}$$

where C_1, C_2, C_3 are real constants to determine by using the following initial condition at $P_0 = (x_1^0, x_2^0, x_3^0) \in \mathbb{H}^1$, i.e.

$$\begin{cases} x_1(0) &=& x_1^0, \\ x_2(0) &=& x_2^0, \\ x_3(0) &=& x_3^0. \end{cases} \tag{4.16}$$

So

$$x_3(t) = \frac{-\alpha_1 C_2 + \alpha_2 C_2}{2}\, t + C_3. \tag{4.17}$$

Then (4.16) and (4.17) together becomes:

$$\begin{cases} x_1(t) &=& \alpha_1\, t + x_1^0, \\ x_2(t) &=& \alpha_2\, t + x_1^0, \\ x_3(t) &=& \dfrac{x_1^0\, \alpha_2 - x_2^0\, \alpha_1}{2}\, t + x_3^0. \end{cases} \tag{4.18}$$

We conclude that the $\mathcal{X}-$line starting from P_0 are Euclidean lines from P_0 but depending only on two parameters (α_1, α_2) instead of 3 parameters like in the Euclidean case. Hence the Euclidean convexity implies the $\mathcal{H}-$convexity in $\mathbb{H}^1$ whereas the reverse is false, for an example see [45].

Example 4.3.2 ($\mathcal{X}-$lines in Engel group)**.** First let us write the matrix $\sigma(x)$ associated to the vector fields X_1, X_2 of the Engel group (see Example 3.2.4),

$$\sigma(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & x_1 \\ 0 & \frac{x_1^2}{2} \end{pmatrix},$$

for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ we can write:

$$\sigma(x)\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & \alpha_2\, x_1 \\ 0 & \frac{\alpha_2}{2}\, x_1^2 \end{pmatrix}.$$

Hence the $\mathcal{X}-$lines in $\mathbb{H}^1$ are the solutions of the following ODE's:

$$\begin{cases} \dot{x}_1(t) = \alpha_1, \\ \dot{x}_2(t) = \alpha_2, \\ \dot{x}_3(t) = \alpha_2\, x_1, \\ \dot{x}_4(t) = \frac{\alpha_2}{2}\, x_1^2, \end{cases} \tag{4.19}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$.
The solution of (4.19) becomes:

$$\begin{cases} x_1(t) = \alpha_1\, t + C_1, \\ x_2(t) = \alpha_2\, t + C_2, \end{cases}$$

where C_1, C_2 are real constants. Then

$$\begin{cases} \dot{x}_3(t) &= \alpha_1\alpha_2\, t + \alpha_2 C_1, \\ \dot{x}_4(t) &= \dfrac{\alpha_2}{2}\,(\alpha_1 t + C_1)^2 \end{cases}$$

which implies

$$\begin{cases} x_3(t) &= \alpha_1\alpha_2, \\ x_4(t) &= \alpha_2\alpha_1(\alpha_1 t + C_1). \end{cases}$$

In order to solve (4.19), let us consider the following initial condition at $P_0 = (x_1^0, x_2^0, x_3^0, x_4^0) \in E4$,

$$\begin{cases} x_1(0) &= x_1^0, \\ x_2(0) &= x_2^0, \\ x_3(0) &= x_3^0, \\ x_4(0) &= x_4^0. \end{cases} \tag{4.20}$$

Then (4.20) becomes:

$$\begin{cases} x_1(t) &= \alpha_1\, t + x_1^0, \\ x_2(t) &= \alpha_2\, t + x_2^0, \\ x_3(t) &= \alpha_2\Big(\alpha_1\, t + x_1^0\Big)t + x_3^0, \\ x_4(t) &= \dfrac{\alpha_2\Big(\alpha_1\, t + x_1^0\Big)^2}{2}\, t + x_4^0. \end{cases} \tag{4.21}$$

Therefore $\mathcal{X}-$lines in $E4$ are parabolas.

$\mathcal{X}-$lines can be applied not only to Carnot groups as we see in Example 4.3.1 and 4.3.2 but also to sub-Riemannian geometries that are not Carnot group like Grušin plane.

Example 4.3.3 ($\mathcal{X}-$line in the Grušin plane)**.** Let us write the matrix $\sigma(x)$

associated to the vector fields of the Grušin plane (see Example 2.1.4),

$$\sigma(x) \;=\; \begin{pmatrix} 1 & 0 \\ 0 & x_1 \end{pmatrix}.$$

for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ we can write:

$$\sigma(x)\alpha \;=\; \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2\, x_1 \end{pmatrix}.$$

Similarly to Example 4.3.1, we solve the following ODE:

$$\begin{cases} \dot{x}_1(t) \;=\; \alpha_1, \\ \dot{x}_2(t) \;=\; \alpha_2\, x_1, \end{cases} \tag{4.22}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$.
By solving (4.22) we obtain the following $\mathcal{X}-$lines:

$$\begin{cases} x_1(t) \;=\; x_1^0 + \alpha_1 t, \\ x_2(t) \;=\; x_2^0 + \alpha_2 x_1^0 t + \dfrac{\alpha_1\alpha_2}{2}\, t^2. \end{cases}$$

Unlike $\mathbb{H}^1$ we can see that the $\mathcal{X}-$lines in the Grušin plane are parabolas. Obviously the $\mathcal{X}-$convexity $\not\Rightarrow$ the Euclidean convexity in the Grušin plane.

Next we define the set of all $\mathcal{X}-$lines passing through a point.

Definition 4.3.2. Let $\Omega \subseteq \mathbb{R}^n$ be open, the $\mathcal{X}-$***plane associated to any point*** $P \in \Omega$, denoted by $\mathbb{V}_P$, is the set of all points that one can reach from P through a $\mathcal{X}-$line, i.e.

$$\mathbb{V}_P \;:=\; \Big\{ Q \in \mathbb{R}^n : \exists\; \alpha \in \mathbb{R}^m \text{ s.t. } x_\alpha(0) = P, x_\alpha(1) = Q \Big\}. \tag{4.23}$$

Note that $\dim\, \mathbb{V}_P = m \leq n$.

Remark 4.3.2.

1. We can say that the $\mathcal{X}-$plane associated to any point P is the union of all $\mathcal{X}-$lines starting from the point P.

2. $Q \notin \mathbb{V}_P$ means $\nexists\, x_\alpha$ joining P and Q and visa versa.

Example 4.3.4. In case that the vector fields generate a Carnot group, $\mathbb{V}_P$ becomes the horizontal plane $\mathcal{H}_P$ (see Definition 4.2.1).

For additional information about $\mathcal{X}-$plane, we refer the reader to [10, 12]. Now we introduce the notion of $\mathcal{X}-$convex function. The $\mathcal{X}-$plane will a critical notion for our later study.

Definition 4.3.3. Let $\Omega \subseteq \mathbb{R}^n$ be open, the function $u : \Omega \to \mathbb{R}$ is called $\mathcal{X}-$***convex*** if for any $\alpha \in \mathbb{R}^m$ we have $u \circ x_\alpha$ is convex in I, where x_α is the $\mathcal{X}-$line defined by (4.14).

We can assume that $I = [0, 1]$ with $x_\alpha(0) = P$ and $x_\alpha(1) = Q$, for $P, Q \in \Omega$, which implies $u \circ x_\alpha$ is convex on $[0, 1]$.

Remark 4.3.3. As we know the (Euclidean) convex functions require the convexity for their domains (see Remark 4.1.1). Instead the domain of $\mathcal{X}-$convex function is not required to be $\mathcal{X}-$convex. Since Definition 4.3.3 requires $u \circ x_\alpha$ to be convex in all connected components of the pre-image $x_\alpha^{-1}(\Omega)$. All of these connected components are disjoint open intervals. Therefore, the condition of convexity for the domain is not necessary in the case of $\mathcal{X}-$convex functions.

Remark 4.3.4. The $\mathcal{X}-$convexity and (Euclidean) convexity for functions are not related, for examples see [10]. However there are some cases where a standard convex function is a $\mathcal{X}-$convex at the same time, e.g. in $\mathbb{H}^1$; since all the $\mathcal{X}-$lines are a selection of Euclidean lines, see Example 4.3.1.

In the next we state the relation between the $\mathcal{X}-$convexity and the horizontal convexity in each of Carnot groups and the Heisenberg group.

Theorem 4.3.1. *[22, Proposition 8.3.17] Let $(\mathbb{G}, *)$ be a Carnot group on $\mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$ open and connected and $u : \Omega \to \mathbb{R}$ upper semicontinuous. Then u is convex along the generator $\mathcal{X}$ of $\mathbb{G}$ (see Definition 4.3.3) if and only if u is $\mathcal{H}-$convex (see Definition 4.2.4).*

Remark 4.3.5. $\mathcal{X}-$convexity and $\mathcal{H}-$convexity in $\mathbb{H}^1$ are not only equivalent but they turn out to be indeed exactly the same notion. In fact since the $\mathcal{X}-$lines are Euclidean straight lines then the curve ($\mathcal{X}-$convexity) coincides at any points with its tangent line ($\mathcal{H}-$convexity). For more details on this remark see [22].

More properties of $\mathcal{X}-$convex functions and the connection between $\mathcal{X}-$convex functions and the subdifferential has been studied by Bardi-Dragoni [12].

Regularity for $\mathcal{X}-$convexity.

First order regularity. Now we want to mention some of the regularity results explained in Subsection 4.2.1 for $\mathcal{H}-$convex functions in Carnot groups are still true for $\mathcal{X}-$convex functions in the case of C^1-vector fields satisfying the Hörmander condition. Bardi-Dragoni have proved that the property of Lipschitz continuity w.r.t. the Carnot Carathédory distance is still true in the case of $\mathcal{X}-$convexity and the corresponding L^∞ bound for the gradient.

Proposition 4.3.1. *[10, Proposition 6.1, Theorem 6.1] Let $u : [0,1] \to \Omega$ be a $\mathcal{X}-$convex function and bounded, with $\Omega \subseteq \mathbb{R}^n$ open and bounded. Then*

1. *For any open $\Omega_1 \subset\subset \Omega$, $\exists\ L = L(\Omega_1)$ such that, for any $\Omega_1 \subset\subset \Omega$*

$$\Big|D_{\mathcal{X}}u(P)\Big| \ := \ \Big|\sigma^T(P)Du(P)\Big| \ \leq \ L \ \text{ in the viscosity sense in } \Omega_1,$$

   for some $L = L\Big(\|u\|_\infty, \delta\Big) < +\infty$, where $\delta = d(\Omega_1, \partial\Omega) = \inf\{d(P,Q)\,|\,P \in \Omega_1, Q \in \partial\Omega\}$ and where $d(P,Q)$ is the distance induced by the family of vector fields $\mathcal{X}$.

2. *u is locally $L-$Lipschitz w.r.t. the distance $d(P,Q)$, i.e. $\Big|u(P)-u(Q)\Big| \leq L\, d(P,Q)$ in any $\Omega_1 \subset\subset \Omega$.*

3. *If, moreover we assume that the vector fields are smooth and they satisfy the Hörmander condition, the distributional horizontal derivatives X_iu exist almost surely for all $i = 1,\dots,m$ and $\|D_{\mathcal{X}}u\|_\infty$ is locally bounded in Ω.*

Second order regularity. Also the regularity results explained in Subsection 4.2.1 for $\mathcal{H}-$convex functions in Carnot groups are still true for $\mathcal{X}-$convex functions in the case of C^2-vector fields.

Theorem 4.3.2. *[10, Theorem 3.1] Let $\Omega \subseteq \mathbb{R}^n$ be open and connected and $\mathcal{X} = \{X_1, \dots, X_m\}$ be a family of C^2 vector fields on $\mathbb{R}^n$. Then an upper semicontinuous function $u : \Omega \to \mathbb{R}$ is $\mathcal{X}-$convex (see Definition 4.3.3) if and only if it satisfies the following:*

$$D^2_{\mathcal{X}} u \;\; \geq \;\; 0$$

where $\left(D^2_{\mathcal{X}} u\right)_{ij} = \left(X_i X_j u\right)_{ij}$ for $i, j = 1, \dots, m$ in the viscosity sense.

Remark 4.3.6. In Carnot groups $D^2_{\mathcal{X}} u = D^2_{\mathcal{H}} u$.

4.3.2 Convexity of sets in Hörmander vector field geometries.

While convex functions w.r.t. vector fields have already been studied, this idea so far has never been applied to sets.

Definition 4.3.4. Let $\Omega \subseteq \mathbb{R}^n$ be open, we define ***the set of all $\mathcal{X}-$segments*** between any $P \in \Omega$ and any $Q \in \mathbb{V}_P \bigcap \Omega$ and for $\alpha \in \mathbb{R}^m, m \leq n$, denoted by $\mathcal{Y}^{P,Q}_\alpha$, where $\mathbb{V}_P$ is the $\mathcal{X}-$plane (see Definition 4.3.2) as follows:

$$\mathcal{Y}^{P,Q}_\alpha \;\; := \;\; \Big\{ x_\alpha : [0,1] \to \mathbb{R}^n : x_\alpha(0) = P,\; x_\alpha(1) = Q \;\text{ and } x_\alpha \text{ solves } (4.14) \Big\}. \tag{4.24}$$

Definition 4.3.5. Let $\Omega \subseteq \mathbb{R}^n$ be open, Ω is called $\mathcal{X}-$**convex** if for each $P \in \Omega$, for each $Q \in \mathbb{V}_P \cap \Omega$ and for any $\alpha \in \mathbb{R}^m$ we have the following condition is satisfied:

$$\mathcal{Y}^{P,Q}_\alpha(t) \;\; \subseteq \;\; \Omega, \;\; \forall\, t \in [0,1], \tag{4.25}$$

where $\mathcal{Y}^{P,Q}_\alpha(t)$ is the $\mathcal{X}-$segment. In another words, we ask that any $\mathcal{X}-$segment that joins P and Q is contained in Ω, exactly as in the Euclidean case.

4.3.3 $\mathcal{X}-$convex functions vs $\mathcal{X}-$convex sets.

In this section we will generalise three important mutual relations of $\mathcal{X}-$convex sets and $\mathcal{X}-$convex functions; level sets, epigraph and the indicator function.

Level sets.

We start with extending Proposition 4.2.1 to the geometry of vector fields and in particular to sub-Riemannian geometry.

Theorem 4.3.3. *Let $\Omega \subseteq \mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}$ be a $\mathcal{X}-$convex function, then for any $a \in \mathbb{R}$ the level set $\Omega_a = \Big\{P \in \Omega : u(P) \leq a\Big\}$ is an $\mathcal{X}-$convex set.*

Proof. Let us consider $P \in \Omega_a$ and $Q \in \mathbb{V}_P \cap \Omega_a$, where $\mathbb{V}_P$ is the $\mathcal{X}-$plane at P (see Definition 4.3.2).
In order to prove that Ω_a is a $\mathcal{X}-$convex set we need to show that for any $\alpha \in \mathbb{R}^m$ the following condition is satisfied:

$$\mathcal{Y}_\alpha^{P,Q}(t) \;\; \subseteq \;\; \Omega_a, \quad \forall\, t \in [0,1], \tag{4.26}$$

where $\mathcal{Y}_\alpha^{P,Q}(t)$ is the set of all $\mathcal{X}-$line segments from P to Q (see Definition 4.3.4).
Since $Q \in \mathbb{V}_P$ then there exists $\mathcal{X}-$ line segment joining P and Q, i.e. there exists $x_\alpha(t) \in \mathcal{Y}_\alpha^{P,Q}(t)$, where $t \in [0,1]$, such that

$$\begin{cases} x_\alpha(0) \;=\; P, \\ x_\alpha(1) \;=\; Q, \\ \dot{x}_\alpha \;=\; \sigma\Big(x_\alpha(t)\Big)\alpha. \end{cases}$$

By using the fact that the function u is $\mathcal{X}-$convex we obtain:

$$\begin{aligned}
u\Big(x_\alpha(t)\Big) &= u\bigg(x_\alpha\Big((1-t)0+t\Big)\bigg) \\
&\leq (1-t)\,u\Big(x_\alpha(0)\Big)+t\,u\Big(x_\alpha(1)\Big) \\
&= (1-t)u(P)+t\,u(Q) \qquad\qquad (\text{Since} \quad P,Q\in\Omega_a) \\
&\leq (1-t)a+t\,a \\
&= a.
\end{aligned}$$

Thus $x^\alpha(t)\in\Omega_a,\ \ \forall\, t\in[0,1]$, i.e. the level sets Ω_a are $\mathcal{X}-$convex in Ω. ■

Epi-graph.

Now we extend Proposition 4.2.2 to the geometry of vector fields and in particular to sub-Riemannian manifolds as the Grušin space. Recall that given $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}$ then $epi(u)$ is defined in Definition 4.1.7.

Theorem 4.3.4. *Let $\mathcal{X}=\{X_1,\dots,X_m\}$ be a family of vector fields on $\mathbb{R}^n$ and $\Omega\subseteq\mathbb{R}^n$ be an open and bounded set, a function $u:\Omega\to\mathbb{R}$ is $\mathcal{X}-$convex if and only if*

$$epi(u) \ = \ \{(P,a)\in\Omega\times\mathbb{R} : u(P)\leq a\}$$

is $\widetilde{\mathcal{X}}-$convex, where $\widetilde{\mathcal{X}}$ is a family of vector fields on $\mathbb{R}^{n+1}$ defined as $\widetilde{\mathcal{X}}=\Big\{\widetilde{X}_1,\dots,\widetilde{X}_m,\widetilde{X}_{m+1}\Big\}$ with

$$\widetilde{\mathcal{X}_i}(P,a) \ := \ \begin{pmatrix} X_i(P)\\ 0\end{pmatrix} \ \textit{for } i=1,\dots,m \ \textit{ and } \widetilde{\mathcal{X}}_{m+1}(P,a) \ := \ \begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}.$$

Proof. ($\Rightarrow$) Let us consider a point $(P,a)\in epi(u)\subseteq\Omega\times\mathbb{R}$ and fix $(Q,b)\in\mathbb{V}^{\widetilde{\mathcal{X}}}_{(P,a)}\bigcap epi(u)$. Note that

$$(Q,b)\in\mathbb{V}^{\widetilde{\mathcal{X}}}_{(P,a)} \quad\Longleftrightarrow\quad P\in\mathbb{V}^{\mathcal{X}}_P \ \text{ and } \ b\in\mathbb{R}, \tag{4.27}$$

with $\mathbb{V}^{\mathcal{X}}_P$ the $\mathcal{X}-$plane defined in Definition 4.3.2 w.r.t. the family of vector fields $\mathcal{X}=\{X_1,\dots,X_m\}$. Then let us consider any $\widetilde{\mathcal{X}}-$line segment $\xi:[0,1]\to$

$\Omega \times \mathbb{R}$ joining $(P, a) \in epi(u)$ to $(Q, b) \in \mathbb{V}^{\widetilde{\mathcal{X}}}_{(P,a)} \cap epi(u)$.
To prove that $epi(u)$ is $\mathcal{X}-$convex, we need to show that

$$\xi(t) \ \subseteq \ epi(u), \quad \forall\, t \in [0, 1], \tag{4.28}$$

where $\xi(t) = \Big(x_\alpha(t), (b - a)t + a\Big)$ with $x_\alpha(t)$ any $\mathcal{X}-$line joining P to $Q \in \mathbb{V}^{\mathcal{X}}_P \cap \Omega$.
Recall that (4.14) is equivalent to requiring

$$u\Big(x_\alpha(t)\Big) \ \leq \ (b - a)t + a, \quad \forall\, t \in [0, 1]. \tag{4.29}$$

To prove (4.29) we use the fact that the function u is a $\mathcal{X}-$convex function, which gives:

$$\begin{aligned} u\Big(x_\alpha(t)\Big) &= u\bigg(x_\alpha\Big((1-t)0 + t\Big)\bigg) \\ &\leq (1-t)u\Big(x_\alpha(0)\Big) + (1-t)u\Big(x_\alpha(1)\Big) \\ &= (1-t)u(P) + t\,u(Q) \qquad\qquad \text{(by using (4.27))} \\ &\leq (1-t)a + t\,b, \end{aligned}$$

where we have used that $(P, a), (Q, b) \in epi(u)$. Thus (4.29) is proved, i.e. $epi(u)$ is a $\widetilde{\mathcal{X}}-$convex set.
($\Leftarrow$) Now let (4.29) hold and for each $\mathcal{X}-$line x_α, where $x_\alpha(0) = P$ and $x_\alpha(1) = Q$ and for all $a, b \in \mathbb{R}$ such that:

$$\begin{cases} (P, a) \in epi(u), \\ (Q, b) \in epi(u). \end{cases}$$

We want to prove that u is a $\mathcal{X}-$convex function, i.e. that the following inequality holds

$$u\bigg(x_\alpha\Big((1-t)t_1 + t\,t_2\Big)\bigg) \ \leq \ (1-t)u\Big(x_\alpha(t_1)\Big) + t\,u\Big(x_\alpha(t_2)\Big), \tag{4.30}$$

holds for all $t, t_1, t_2 \in [0, 1]$ and for each $\mathcal{X}-$line x_α such that x_α joining P to Q in time $t = 1$. We define $f_\alpha(t) := u\Big(x_\alpha(t)\Big)$, for all $x_\alpha(.)$ fixed with $x_\alpha(0) = P$, i.e. for all $\alpha \in \mathbb{R}^m$ and for all $P \in \Omega$.

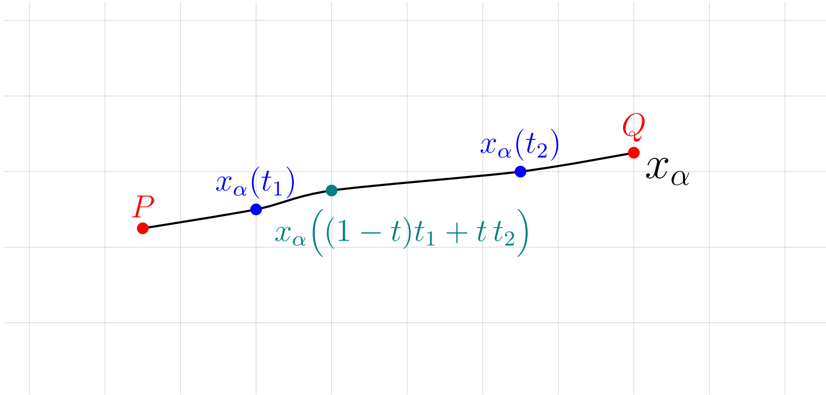

Figure 4.9

Now, (4.30) can be rewritten as:

$$f_\alpha\big((1-t)t_1 + t\,t_2\big) \;\leq\; (1-t)f_\alpha(t_1) + t\,f_\alpha(t_2).$$

So (4.30) is equivalent to proving $f_\alpha : [0,1] \to \mathbb{R}$ is an Euclidean convex 1-dimensional function for all $\alpha \in \mathbb{R}^m$. If we can prove that $epi(f_\alpha)$ is a convex set in $\mathbb{R} \times \mathbb{R}$, then we can use Proposition 4.1.1 in the 1-dimensional case to conclude. In fact $epi(f_\alpha)$ convex implies the function f_α is convex. Now remark that $(t_1, a), (t_2, b) \in epi(f_\alpha)$ is equivalent to $(x_\alpha(t_1), a)(u)$ and $(x_\alpha(t_2), b)(u)$. Then since $epi(u)$ is $\widetilde{X}-$convex then

$$\Big((1-t)x_\alpha(t)_1 + t\,x_\alpha(t_2), (1-t)a + t\,b\Big) \;\in\; epi(u), \quad \forall\, t \in [0,1]. \tag{4.31}$$

Recall that if $\forall\ (t_1, a), (t_2, b) \in epi(f_\alpha)$ then $f_\alpha(t_1) \;\leq\; a$ and $f_\alpha(t_2) \;\leq\; b$, and for each $t \in [0,1]$ then we have:

$$\Big((1-t)t_1 + t\,t_2, (1-t)a + t\,b\Big) \;\in\; epi(f_\alpha),$$

i.e.

$$f_\alpha\big((1-t)t_1 + t\,t_2\big) \;\leq\; (1-t)a + t\,b. \tag{4.32}$$

See Figure 4.9.

Thus $epi(f_\alpha)$ is (Euclidean) convex $\to f_\alpha$ is a convex function. which gives (4.32) and then (4.30). This conclude the proof. ■

Indicator function.

The final connection between $\mathcal{X}-$convex sets and $\mathcal{X}-$convex functions we want to investigate is the link with the indicator function. Recall that given $1_\Omega : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ then 1_Ω is defined in 4.1.9.

Theorem 4.3.5. *An open subset $\Omega \subseteq \mathbb{R}^n$ is a $\mathcal{X}-$convex set if and only if the indicator function $1_\Omega : \Omega \subset \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ of Ω is a $\mathcal{X}-$convex function.*

Proof. ($\Rightarrow$) Let assume that Ω is a $\mathcal{X}-$convex set and we want to prove that 1_Ω is a $\mathcal{X}-$convex function, i.e. for any $\alpha \in \mathbb{R}^m$ and for any corresponding $\mathcal{X}-$line x_α the following condition holds:

$$1_\Omega\Bigg(x_\alpha\Big((1-t)\,t_1 + t\,t_2\Big)\Bigg) \;\leq\; (1-t)\,1_\Omega\Big(x_\alpha(t_1)\Big) + t\,1_\Omega\Big(x_\alpha(t_2)\Big), \qquad (4.33)$$

where $t, t_1, t_2 \in [0,1]$ and $t_1 \leq t \leq t_2$. Now set $x_\alpha(t_1) = P$ and $x_\alpha(t_2) = Q$ and note that $Q \in \mathbb{V}_P$ (up to a rescaling in time of x_α). Use that Ω is a $\mathcal{X}-$convex set, which implies that

$$P = x_\alpha(t_1),\; Q = x_\alpha(t_2),\; x_\alpha\Big((1-t)\,t_1 + t\,t_2\Big) \;\in\; \Omega, \quad \forall\, t, t_1, t_2 \in [0,1]. \quad (4.34)$$

Now by using the definition of the indicator function and (4.34) we obtain:

$$1_\Omega\Bigg(x_\alpha\Big((1-t)t_1 + t\,t_2\Big)\Bigg) \;=\; 0, \qquad \forall\, t, t_1, t_2 \in [0,1]. \qquad (4.35)$$

Also using the fact that $P, Q \in \Omega$, i.e. $1_\Omega(P) = 1_\Omega\Big(x_\alpha(t_1)\Big) = 0$ and $1_\Omega(Q) = 1_\Omega\Big(x_\alpha(t_2)\Big) = 0$, we obtain:

$$(1-t)1_\Omega\Big(x_\alpha(t_1)\Big) + t\,1_\Omega\Big(x_\alpha(t_2)\Big) \;=\; 0. \qquad (4.36)$$

Hence from (4.35) and (4.36) we have trivially that

$$1_\Omega\Bigg(x_\alpha\Big((1-t)t_1 + t\,t_2\Big)\Bigg) \;=\; (1-t)1_\Omega\Big(x_\alpha(t_1)\Big) + t\,1_\Omega\Big(x_\alpha(t_2)\Big),$$

i.e. (4.33) holds, that means 1_Ω is a $\mathcal{X}-$convex function.
($\Leftarrow$) Let assume that 1_Ω is a $\mathcal{X}-$convex function and we want to show that Ω is a $\mathcal{X}-$convex set, i.e. for all $P \in \Omega$ and all $Q \in \mathbb{V}_P \cap \Omega$ and for any $\alpha \in \mathbb{R}^m$

the following condition is satisfied:

$$x_\alpha^{P,Q}\Big([0,1]\Big) \;\subset\; \Omega,$$

where $x_\alpha\Big([0,1]\Big)$ is a $\mathcal{X}-$line joining P to Q in time $t=1$. Note that since $Q \in \mathbb{V}_P$ then there always exists $\mathcal{X}-$ line joining P and Q in time 1 (and if $X_1,\dots,X_m$ are left-i that $\mathcal{X}-$line is unique), that means that

$$\begin{cases} x_\alpha(0) &= P, \\ x_\alpha(1) &= Q, \\ \dot{x}_\alpha(t) &= \sigma\Big(x_\alpha(t)\Big)\alpha, \quad t \in (0,1). \end{cases}$$

By using the fact that the function 1_Ω is $\mathcal{X}-$convex, we obtain that for any $t \in (0,1)$

$$\begin{aligned} 1_\Omega\Big(x_\alpha(t)\Big) &= 1_\Omega\bigg(x_\alpha\Big((1-t)0+t\Big)\bigg) \\ &\le (1-t)\,1_\Omega\Big(x_\alpha(0)\Big) + t\,1_\Omega\Big(x_\alpha(1)\Big) \\ &= (1-t)1_\Omega(P) + t\,1_\Omega(Q) \\ &= 0 \qquad\qquad\qquad\qquad (\text{Since} \quad P,Q \in \Omega). \end{aligned}$$

Thus by the definition of the indicator function $x_\alpha(t)$ belongs to Ω for all $t \in [0,1]$, i.e. Ω is a $\mathcal{X}-$convex set. $\blacksquare$

Chapter 5

Starshaped sets.

In this chapter we study starshaped sets and how they are related to the convex sets and then convex functions in different geometries. In the Euclidean space all convex sets are starshaped with respect to each interior point. Furthermore, we have in the Euclidean space different equivalent definitions for starshapedness. We generalize these notions to Carnot groups and more general geometries of vector fields (e.g. sub-Riemannian manifolds on $\mathbb{R}^n$). We will show that in Carnot groups the different definitions of starshapedness that we introduce are not equivalent. We will then generalise these notions to general geometries of vector fields and apply these ideas to the study of convex sets along vector fields.

5.1 Starshaped sets in the Euclidean $\mathbb{R}^n$.

Definition 5.1.1. The set $\Omega \subseteq \mathbb{R}^n$ is called starshaped (or also starlike) w.r.t a generic interior point $P_0 \in \Omega$ if and only if the line segment starting from P_0 and joining to any point $P \in \Omega$ is all contained in Ω, i.e.

$$(1-t)P_0 + tP \in \Omega, \quad t \in [0,1].$$

In Figures 5.1, 5.2, 5.3, 5.4, 5.5, 5.6 and 5.7 we give some examples of starshaped and non-starshaped sets.

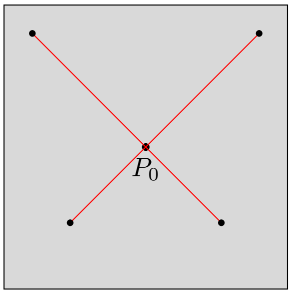

Figure 5.1: A starshaped set w.r.t. all its interior points which is clearly convex at the same time.

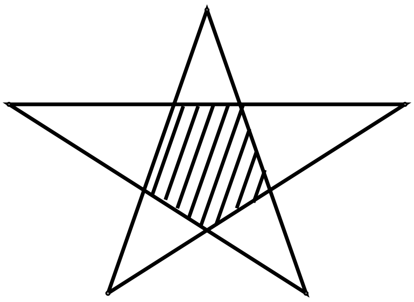

Figure 5.2: We can see that a star is starshaped w.r.t. any P_0 is in the marked region. Note also that it is not starshaped w.r.t. all other internal points and so it is not a convex set.

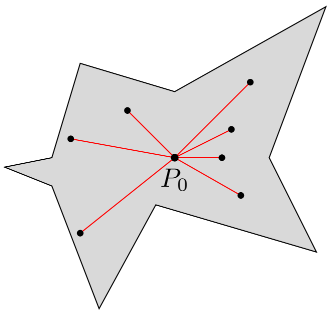

Figure 5.3: A starshaped set w.r.t. P_0 which is clearly not convex.

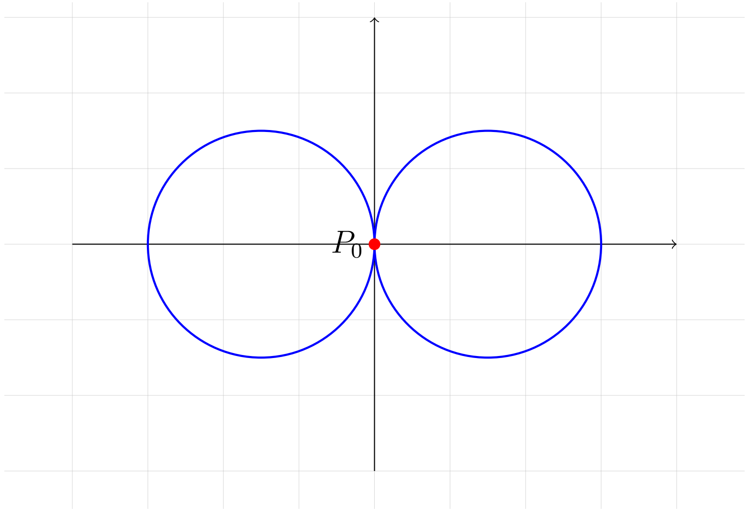

Figure 5.4: The butterfly set is starshaped w.r.t. only one point.

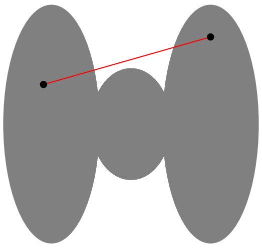

Figure 5.5: The set in the picture is non-starshaped w.r.t. any point.

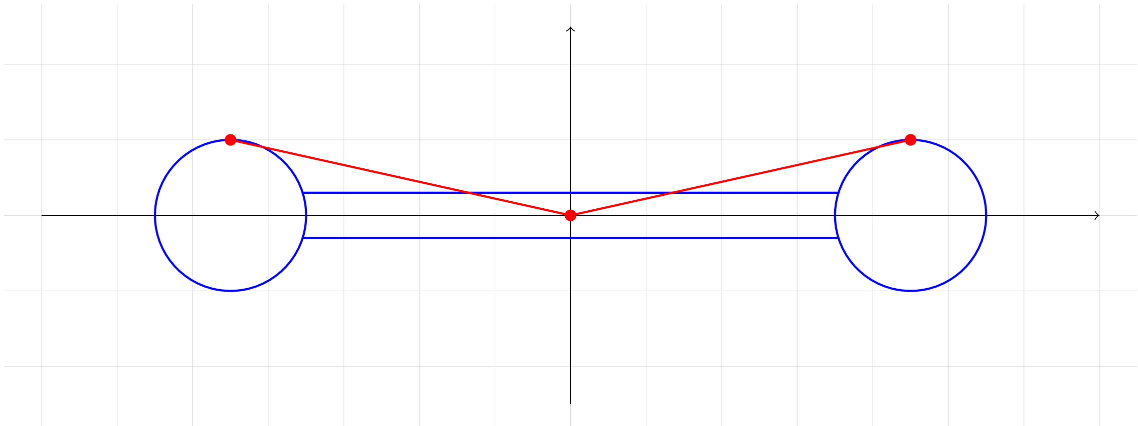

Figure 5.6: The dumbbell is not a starshaped set w.r.t. any of its internal points, in the figure we show that it is not a starshaped set w.r.t. $(0,0)$.

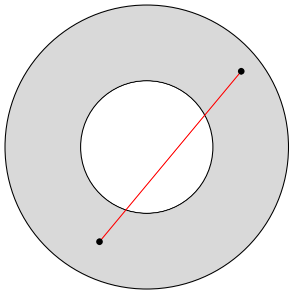

Figure 5.7: `The annular set is non-starshaped w.r.t. any point.`

There are many equivalent ways to define starshaped sets in the Euclidean space. The next lemma gives one definition equivalent to Definition 5.1.1 and which depends on the Euclidean dilations of the set.
Let us recall the Euclidean dilation of $\Omega \subseteq \mathbb{R}^n$, knowing also as product by scalar:

$$t\,\Omega \;\; := \;\; \Big\{t\,P = (tx_1, \dots, tx_n) \;:\; P = (x_1, \dots, x_n) \in \Omega\Big\}.$$

Note that if $\Omega \subseteq \mathbb{R}^n$ then for any generic point $P_0 \in \mathbb{R}^n$ we can define the following set:

$$P_0 + \Omega \;\; := \;\; \Big\{Q \in \mathbb{R}^n \;:\; \exists\, P \in \Omega \;\, s.t. \;\, Q = P_0 + P\Big\}.$$

Lemma 5.1.1. $\Omega \subseteq \mathbb{R}^n$ *is Euclidean starshaped w.r.t. a generic point* $P_0 \in \Omega$ *if and only if:*

$$(1-t)P_0 + t\,\Omega \;\; \subseteq \;\; \Omega, \qquad \forall\; 0 \leq t \leq 1. \tag{5.1}$$

In the case $P_0 = 0$ *this takes the form*

$$t\,\Omega \;\; \subseteq \;\; \Omega, \quad \forall\; t \in [0,1]. \tag{5.2}$$

Proof. It is sufficient to remark that the line segment between 0 and $P = (x_1, ..., x_n)$ at time 1 is parameterised as $\gamma(t) = t\,P$, while for a generic P_0 the line segment is parameterised as $\gamma(t) = tP + P_0$. ∎

Next, we like to point out some properties of the relations between star-shapedness and convexity of sets in the Euclidean settings.

Proposition 5.1.1. *$\Omega \subseteq \mathbb{R}^n$ is (Euclidean) convex if and only if it is (Euclidean) starshaped w.r.t. all its internal points.*

Proof. The proof is trivial by using Definition 5.1.1. ∎

In the next example we apply Lemma 5.1.1, even though the set in the example is trivially starshaped to introduce the approach used in the general to Carnot groups.

Example 5.1.1. Consider the Euclidean ball $B_R(0)$ in $\mathbb{R}^3$, which is given by:

$$B_R(0) \;=\; \left\{P = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq R^2\right\}.$$

Compute the Euclidean dilation of $B_R(0)$ for some $t \in [0,1]$, we obtain:

$$t\,B_R(0) \;=\; \left\{t\,P = (tx_1, tx_2, tx_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq t^2R^2\right\}.$$

Denote by $t\,x_1 := \widetilde{x}_1$, $t\,x_2 := \widetilde{x}_2$ and $t\,x_3 := \widetilde{x}_3$, we have

$$\begin{aligned} t\,B_R(0) \;&=\; \left\{(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) \in \mathbb{R}^3 : \widetilde{x}^2 + \widetilde{x}_2^2 + \widetilde{x}_3^2 \leq (t\,R)^2\right\} \\ &=\; B_{t\,R}(0), \qquad \forall\, t \in [0,1]. \end{aligned}$$

Also we have $t \in [0,1]$, so for any $R > 0$ we know that $t\,R \leq R$.
Therefore $t\,B_R(0) \subseteq B_R(0)$, which means $B_R(0)$ is starshaped w.r.t. 0 in $\mathbb{R}^3$.

Remark 5.1.1. All Euclidean balls are starshaped w.r.t all their internal points since they are convex.

Example 5.1.2 (The Euclidean box)**.** The Euclidean box of radius R in $\mathbb{R}^3$ given by:

$$Box_R \;=\; \left\{P = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1| + |x_2| + |x_3| < R\right\},$$

is (Euclidean) starshaped w.r.t all its interior points since it is obviously convex. In fact if we assume that two points $P, Q \in Box_R$ and $t \in [0,1]$ then

$$(1-t)P + t\,Q \;\leq\; (1-t)R + t\,R \;=\; R,$$

that implies

$$(1-t)P + tQ \quad \in \quad Box_R,$$

i.e. Box_R is convex and according to Proposition 5.1.1 Box_R is starshaped w.r.t. all its internal points.

In the next proposition we study the union of convex and starshaped sets.

Lemma 5.1.2.

1. *The union of two starshaped sets w.r.t. the same point P_0 is still starshaped w.r.t. P_0.*

2. *The union of a starshaped set w.r.t. some P_0 with a convex set is starshaped whenever P_0 belongs to the intersection.*

3. *The union of two convex sets is in general not convex but it is starshaped w.r.t. all the points in the intersection.*

Proof. The proof is trivial so we omit it. ∎

We illustrate all the previous statements geometrically in Figures 5.8, 5.9 and 5.10.

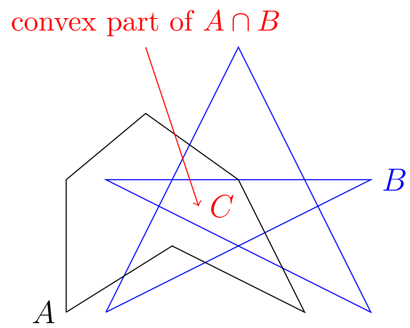

Figure 5.8: $C = A \cup B$ `is a starshaped set w.r.t. all` $P_0 \in C$`, where` C `is the convex part of` $A \cap B$`.`

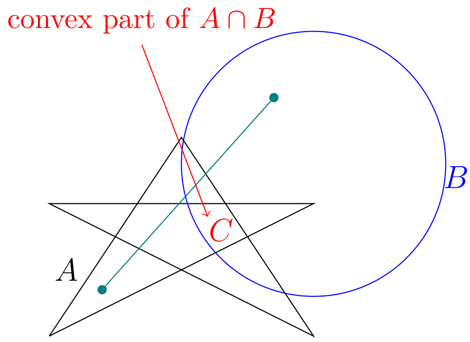

Figure 5.9: `Here` $C = B\cap$ `convex part of` A `which is the part that containing all` P_0 `such that` A `is a starshaped set w.r.t.` P_0`;` `thus` $A \cup B$ `is a starshaped set w.r.t each` $P_0 \in C$`.`

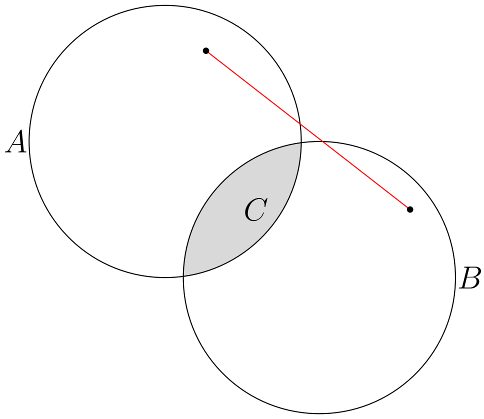

Figure 5.10: `The union of two convex sets is NOT necessarily convex but it is starshaped w.r.t. all points` $P_0 \in C$`.`

5.1.1 Starshapedness for sets with boundary.

The next lemma gives a characterisation of starshaped sets in the Euclidean setting, related to the normal at the boundary.
We need first to recall an important definition in topological spaces.

Definition 5.1.2. Let M be a topological space and Ω be an open set in M, Ω is called a ***regular open*** set in M if it is equal to the interior of its closure.

Example 5.1.3. Consider the real line $\mathbb{R}$ with the usual topology generated by the open intervals. We have the following two examples:

1. The open interval (a, b) is regular whenever $-\infty < a \leq b < \infty$.

2. $(a,b)\bigcup(b,c)$ is not regular for $-\infty \le a < b < c \le +\infty$. The interior of the closure of $(a,b)\bigcup(b,c)$ is (a,c).

Remark 5.1.2. Given Ω open regular set in $\mathbb{R}^n$ with $\partial\,\Omega \in C^1$ then

$$\exists\; u : \mathbb{R}^n \to \mathbb{R}, u \in C^2 \;\; s.t. \;\; \Omega = \Big\{P \in \mathbb{R}^n : u \le 0\Big\}.$$

For more details see [48].

Theorem 5.1.1. *Given an open regular bounded set $\Omega \subseteq \mathbb{R}^n$ with C^1 boundary.*

1. *If Ω is a starshaped w.r.t. $P_0 \in \Omega$ then*

$$\Big\langle n(P), P - P_0 \Big\rangle \;\; \ge \;\; 0, \quad \forall\, P \in \partial\Omega, \tag{5.3}$$

where $n(P)$ is the normal pointing outside of Ω at the point P.

2. *Assume that the strict inequality in* (5.3) *holds true, then Ω is starshaped w.r.t. P_0.*

Proof. The result is well-known but we include a proof for completeness and for later generalisations. Fix a point $P \in \partial\Omega$ and indicate by $\gamma_{P_0,P}$ the segment line joining P_0 to P in time $t = 1$, i.e. $\gamma_{P_0,P} : [0,1] \to \mathbb{R}^n$ defined as

$$\gamma_{P_0,P}(t) \;\; = \;\; (P - P_0)t + P_0.$$

Note that, without loos of generality, we can assume that there exist some C^1 function $u : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\partial\Omega \;\; = \;\; \Big\{P \in \mathbb{R}^n : u(P) < 0\Big\} \quad \text{and} \quad \partial\Omega \;\; = \;\; \Big\{P \in \mathbb{R}^n : u(P) = 0\Big\}$$

(otherwise we can argue locally). In this case we obtain

$$n(P) \;\; = \;\; \frac{\nabla u}{|\nabla u|} \quad \text{and} \quad \dot\gamma_{P_0,P}(1) \;\; = \;\; P - P_0. \tag{5.4}$$

Now let us assume that Ω is a starshaped w.r.t. P_0, this means that $\gamma_{P_0,P}(t) \in$

Ω for all $t \in [0,1)$, i.e.

$$\begin{cases} u\Big(\gamma_{P_0,P}(t)\Big) \ < \ 0, & t \in [0,1), \\ u\Big(\gamma_{P_0,P}(t)\Big) \ = \ u(P) \ = \ 0. \end{cases} \tag{5.5}$$

Then

$$\frac{d}{dt}\, u\Big(\gamma_{P_0,P}(t)\Big)\Big|_{t=1} \ = \ \lim_{t\to 1^{-1}} \frac{u\Big(\gamma_{P_0,P}(t)\Big) - u\Big(\gamma_{P_0,P}(1)\Big)}{t-1} \ \geq \ 0. \tag{5.6}$$

On the other hand, using (5.4) we find

$$\frac{d}{dt}\, u\Big(\gamma_{P_0,P}(t)\Big)\Big|_{t=1} \ = \ \langle \nabla u(P), \gamma_{P_0,P}(1)\rangle \tag{5.7}$$

$$= \ \Big| \nabla u(P)\Big| \Big\langle n(P), P - P_0\Big\rangle. \tag{5.8}$$

Compining (5.6) and (5.7) we have (5.3).
To prove the onther hand we assume the strict inequality in (5.3), then by using (5.7) we conclude that there exisit some $\eta \in (0,1]$ such that

$$u\Big(\gamma_{P_0,P}(t)\Big) \ \leq \ 0 \quad \forall\, t \in (1-\eta, 1),$$

which means $\gamma_{P_0,P}(t) \in \Omega$, for all $t \in (1-\eta,1)$.
Let us define

$$\bar{t} \ := \ \min\Big\{t \in [0,1] : \gamma_{P_0,P}(t) \in \Omega, \ \ \forall\ (t,1)\Big\}.$$

Note that such a minimum exists by continuaty and also $\bar{t} \neq 1$ since $\bar{t} \leq 1-\eta < 1$. If $\bar{t} = 0$ we can conclude that Ω is starshaped w.r.t. P_0. So let us assume that $\bar{t} \in (0,1)$. In this case obviously

$$\bar{P} \ := \ \gamma_{P_0,P}(t)(\bar{t}) \ \in \ \partial\Omega,$$

then we can apply the strict inequality in (5.3) at $\bar{P}$. Arguing as above we can find $\tilde{\eta} \in (0,1]$ such that

$$\gamma_{P_0,\bar{P}}(t) \in \Omega, \quad \forall\, t \in (1-\tilde{\eta}, 1).$$

On the other hand by using the structure of (Euclidean) lines and the fact that Ω is a regular open set, this means that

$$\gamma_{P_0,P}(t) \ \in \ \Omega, \quad \text{for } \ t \in (\bar{t} - \tilde{\eta}, 1),$$

which contradicts the minimality of $\bar{t}$. ■

The following remark is essential to understand the large generality of the proof above.

Remark 5.1.3. We can actually replace in the above proof, the segment line $\gamma_{P_0,P}(t)$ with any other family of C^1 curves joining P_0 to P in time $t = 1$ as soon as following rescaling property is satisfied

$$\dot{\gamma}_{P_0,P}(\tilde{t}) \ = \ C\,\dot{\gamma}_{P_0,\bar{P}}(1), \ \text{ with } \bar{P} \ = \ \gamma_{P_0,P}(\bar{t}) \tag{5.9}$$

for some $C = C(\bar{t}) > 0$.
In fact assumption (5.9) ensures that

$$\Big\langle n(\bar{P}), \dot{\gamma}_{P_0,\bar{P}}(1) \Big\rangle \ \geq \ 0 \ \Rightarrow \ \frac{d}{dt} u\Big(\gamma_{P_0,\bar{P}}(t)\Big)\bigg|_{t=\bar{t}} \ < \ 0.$$

Obviously (5.9) is trivially true for (Euclidean) lines since

$$\dot{\gamma}_{P_0,\bar{P}}(1) \ = \ (\bar{P} - P_0) \ = \ \bar{t}(P - P_0) \ = \ \bar{t}\,\dot{\gamma}_{P_0,P}(\bar{t})$$

but this rescaling property is easy to check in other family of curves which will be crucial in our later generalisation.

5.1.2 Characterisations of Euclidean starshaped sets.

In this subsection we illustrate the meaning of a starshaped hull and starshaped envelope in the Euclidean setting and give some examples and properties.

5.1.3 Starshaped hull.

Definition 5.1.3 (Starshaped hull)**.** Given an open subset $\Omega \subseteq \mathbb{R}^n$ and a point $P_0 \in \Omega$, we define the ***starshaped hull of the set*** Ω ***w.r.t. the point*** P_0, we indicate it by $\Omega^*_{P_0}$, as the intersection of all the starshaped sets w.r.t. P_0 containing the set Ω.
If $P_0 = 0$, then we simply write Ω^* instead of Ω^*_0.

Proposition 5.1.2. *Given an open subset* $\Omega \subseteq \mathbb{R}^n$*,* Ω *is starshaped w.r.t.* P_0 *if and only if*

$$\Omega \;=\; \Omega^*_{P_0}.$$

Proof. The proof is trivial. We will write the proof directly for the more general case of Carnot groups (see Proof 5.2.3). ∎

In the case $P_0 = 0$, one can easily rewrite Ω^* as

$$\Omega^* \;=\; \bigcup_{t\in[0,1]} t\,\Omega, \tag{5.10}$$

and for a generic P_0 we have

$$\Omega^*_{P_0} \;=\; (1-t)P_0 + \bigcup_{t\in[0,1]} t\,\Omega. \tag{5.11}$$

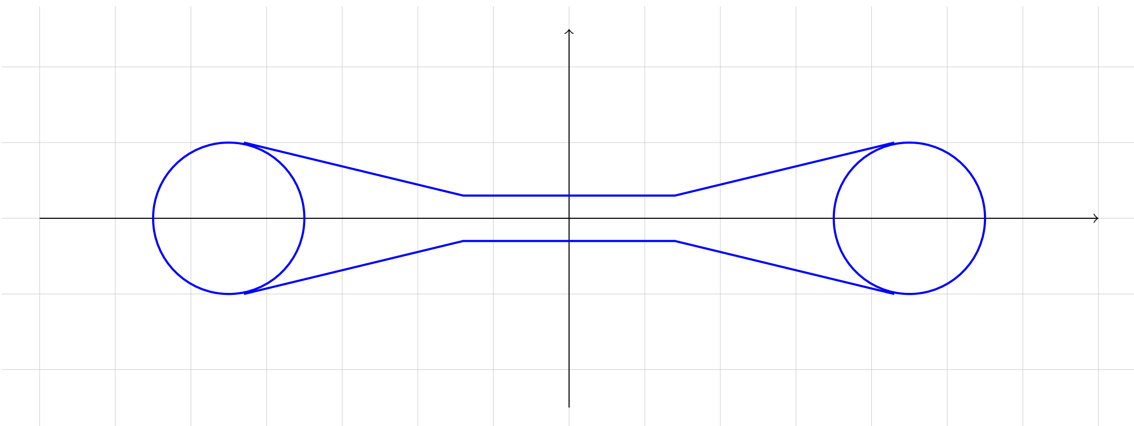

Figure 5.11: `The starshaped hull w.r.t.` $(0,0)$ `of the dumbbell in Figure 5.6.`

Example 5.1.4 (Dumbbell)**.** The dumbbell in the Euclidean $\mathbb{R}^2$ is taken here as the union of the two balls

$$\begin{aligned} B_1 &= \left\{P = (x_1, x_2) \in \mathbb{R}^2 : (x_1 + 5)^2 + x_2^2 < 1\right\}, \\ B_2 &= \left\{P = (x_1, x_2) \in \mathbb{R}^2 : (x_1 - 5)^2 + x_2^2 < 1\right\} \end{aligned}$$

and the finite cylinder

$$C = \left\{P = (x_1, x_2) \in \mathbb{R}^2 : -5 < x_1 < 5 \text{ and } -\frac{1}{4} < x_2 < \frac{1}{4}\right\}.$$

See Figure 5.6.

To find the starshaped hull w.r.t. the origin in this example is easy. One needs only to write a generic tangent line to any point of each of B_1 and B_2, and then impose for those tangent lines to pass through the origin $P_0 = (x_1^0, x_2^0)$.

We know that the tangent line to any circle is parameterized as

$$(x_1^0 - h)(x_1 - h) + (x_2^0 + k)(x_2 - k) = R^2.$$

Consider B_2, we have:

$$h = -5, \quad k = 0 \text{ and } R = 1.$$

So we obtain:

$$(x_1^0 + 5)(x_1 + 5) + x_2^0 x_2 = 1,$$

which implies

$$x_1^0 x_1 + 5x_1^0 + 5x_1 + 25 + x_2^0 x_2 = 1.$$

The last equation we obtain is the tangent line to ∂B_1 at the point P_0.

We want now to compute the tangent line that pass through the origin. So:

$$Q = -5x_1^0 - 25 + 1 = 0,$$

Evaluating the coordinates as following

$$\overline{x}_1^0 = -\frac{24}{5} = -4.8$$

and

$$\begin{aligned}\overline{x}_2^0 &= \pm\sqrt{1-(\overline{x}_1^0+5)^2} = \pm\sqrt{1-\left(-\frac{24}{5}+5\right)^2}\\ &= \pm\sqrt{1-\frac{1}{25}} = \pm\frac{\sqrt{24}}{5}.\end{aligned}$$

Hence the straight lines to consider are the ones connecting the origin to the point $\left(-\frac{24}{5}, \frac{\sqrt{24}}{5}\right)$ and to the other 3 symmetric points (that can be found similarly). In order to find the point Q_i we compute the intersection of:

$$t\,\overline{P}_1 = \begin{pmatrix} \frac{-24}{5}t \\ \frac{\sqrt{24}}{5}t \end{pmatrix}$$

with $x_2 = \frac{1}{4}$, i.e. $t = \frac{1}{4}\frac{5}{\sqrt{24}}$, for $i = 1$ (and similarly for $i = 2, 3, 4$).

Hence, we find the point $Q_1 = \left(-\frac{\sqrt{24}}{4}, \frac{1}{4}\right) \simeq (1.225, 0.25)$.

Therefore it is enough to intersect those 4 lines with the lines $Q = \frac{1}{4}$ and $Q = -\frac{1}{4}$, respectively, to determine the starshaped hull shown in Figure 5.11.

5.2 Carnot groups and general sub-Riemannian manifolds.

In the case of Carnot groups we define two different notions of starshaped sets: the first one is called strongly $\mathbb{G}$-starshaped (or just $\mathbb{G}-$starshaped) and the second one is called weakly $\mathbb{G}$-starshaped. In the following we illustrate each definition with examples and some properties that show the mutual relations and the relations w.r.t. the (Euclidean) starshaped notion.

5.2.1 Strongly $\mathbb{G}$-starshaped sets.

We know that if $\Omega \subseteq \mathbb{G}$ then, for any generic point $P_0 \in \mathbb{G}$ and, by using the left-translation, we can define the following set:

$$P_0 \circ \Omega \ := \ \Big\{ Q \in \mathbb{G} \ : \ \exists \ P \in \Omega \ s.t. \ Q = P_0 \circ P \Big\} \ \subseteq \ \mathbb{G}. \tag{5.12}$$

Notice that $P_0 \circ \Omega \neq \Omega \circ P_0$ because Carnot groups are not commutative.
Let us recall the dilation for $\Omega \subseteq \mathbb{G}$ (see Definition 3.2.10):

$$\delta_t(\Omega) \ := \ \Big\{ \delta_t(P) \ : \ P \in \Omega, \ t \in [0,1] \Big\}.$$

Definition 5.2.1. Let $\Omega \subseteq \mathbb{G}$, Ω is called ***strongly*** $\mathbb{G}$***-starshaped*** (or simply **$\mathbb{G}$-starshaped) w.r.t.** $0 \in \mathbb{G}$ if the following condition is satisfied:

$$\delta_t(\Omega) \ \subseteq \ \Omega, \quad \forall \ t \in [0,1]. \tag{5.13}$$

Moreover Ω is called ***strongly*** $\mathbb{G}$***-starshaped*** (or simply $\mathbb{G}$***-starshaped***) ***w.r.t. a generic point*** P_0 ***in*** $\mathbb{G}$ if the left-translated set Ω' at the point $P_0 \in \mathbb{G}$, i.e.

$$\begin{aligned} \Omega' \ := \ L_{-P_0}(\Omega) \ &= \ -P_0 \circ \Omega \\ &= \ \Big\{ P \in \mathbb{G} : \exists Q \in \mathbb{G} \ s.t. \ P = -P_0 \circ Q \Big\}, \end{aligned} \tag{5.14}$$

is strongly $\mathbb{G}$-starshaped w.r.t. 0. In another words Ω is strongly $\mathbb{G}$-starshaped if the inclusion (5.13) holds for Ω'.

Remark 5.2.1. If $P_0 \in \Omega$ then $0 \in \Omega'$.

Remark 5.2.2.

1. Definition 5.2.1 gives back the standard Euclidean notion if we take $\delta_t(P) = t\,P$ and $L_{P_0}(P) = P_0 + P$.

2. We need dilation and left-translation in Definition 5.2.1, therefore this notion cannot be extended to more general geometry than Carnot groups.

In the following we compute first two explicit examples in $\mathbb{H}^1$.

Example 5.2.1 (Homogeneous ball in $\mathbb{H}^1$)**.** The homogenous ball $B^h_R(0) \subseteq \mathbb{H}^1$ is defined as:

$$\begin{aligned} B^h_R(0) &:= \left\{ P = (x_1, x_2, x_3) \in \mathbb{R}^3 : \left(\left(|x_1|^2 + |x_2|^2 \right)^2 + |x_3|^2 \right)^{\frac{1}{4}} \leq R \right\} \\ &= \left\{ P = (x_1, x_2, x_3) \in \mathbb{R}^3 : \left(|x_1|^2 + |x_2|^2 \right)^2 + |x_3|^2 \leq R^4 \right\}. \end{aligned}$$

The dilation of $B^h_R(0)$ can be computed for any $t \in [0,1]$ as follows

$$\delta_t\Big(B^h_R(0)\Big) = \left\{ \delta_t(P) \in \mathbb{R}^3 : \left(|x_1|^2 + |x_2|^2 \right)^2 + |x_3|^2 \leq R^4 \right\}.$$

Denote by $tx_1 := \widetilde{x}_1$, $tx_2 := \widetilde{x}_2$ and $t^2 x_3 := \widetilde{x}_3$, we obtain

$$\begin{aligned} \delta_t(B^h_R(0)) &= \left\{ \widetilde{P} \in \mathbb{R}^3 : \left(\left| \frac{\widetilde{x}_1}{t} \right|^2 + \left| \frac{\widetilde{x}_2}{t} \right|^2 \right)^2 + \left| \frac{\widetilde{x}_3}{t^2} \right|^2 \leq R^4 \right\} \\ &= \left\{ \widetilde{P} \in \mathbb{R}^3 : \left(|\widetilde{x}_1|^2 + |\widetilde{x}_2|^2 \right)^2 + \left| \widetilde{x}_3^2 \right| \leq (tR)^4 \right\} \\ &= B^h_{tR}(0), \qquad \forall\, t \in [0,1]. \end{aligned}$$

Since $t \in [0,1]$ then for any $R > 0$ we have $tR \leq R$, therefore

$$\delta_t\Big(B^h_R(0)\Big) \subseteq B^h_R(0),$$

which means $B^h_R(0)$ is strongly $\mathbb{H}^1$-starshaped w.r.t. 0.

Example 5.2.2 (The Box in $\mathbb{H}^1$)**.** The set

$$Box^{\mathbb{H}^1}_R = \left\{ P = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1| + |x_2| + |x_3|^{\frac{1}{2}} \leq R \right\}$$

is called a box in the Heisenberg group $\mathbb{H}^1$. $Box^{\mathbb{H}^1}_R$ is a strongly $\mathbb{H}^1$-starshaped set w.r.t. 0.
In fact if we assume $\widetilde{P} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \delta_t \left(Box^{\mathbb{H}^1}_R \right)$, this means that for some

$t \in [0,1]$ and for some $P = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\widetilde{P} \; = \; \delta_t(P)$ and

$$\begin{aligned} |\tilde{x}_1| + |\tilde{x}_2| + |\tilde{x}_3|^{\frac{1}{2}} &= |t\tilde{x}_1| + |t\tilde{x}_2| + |t^2\tilde{x}_3|^{\frac{1}{2}} \\ &= t\Big(|x_1| + |x_2| + |x_3|^{\frac{1}{2}}\Big) \\ &\leq |x_1| + |x_2| + |x_3|^{\frac{1}{2}} \\ &\leq R \text{ (since } P \in Box_R). \end{aligned}$$

This implies $\tilde{P} \in Box_R$ and proves the inclusion.

Example 5.2.3 (Homogeneous ball in Carnot groups)**.** Consider the homogenous ball

$$B_R^h(0) \; := \; \Big\{P \in \mathbb{R}^n \, : \, \|P\|_h < R\Big\} \; \subseteq \; \mathbb{G},$$

where $\|P\|_h$ is the homogeneous norm given by (3.5) in Section 3.3. By using

$$\|\delta_t(P)\|_h \; = \; t\,\|P\|_h\,,$$

an easy computation shows that:

$$\delta_t\left(B_R^h(0)\right) \; = \; B_{tR}^h(0),$$

which trivially implies $B_R^h(0)$ is $\mathbb{G}$-starshaped w.r.t. 0.

In the same way one can show that $B_R^h(x_0)$ is $\mathbb{G}$-starshaped w.r.t. a generic centre P_0 in the next proposition.

Proposition 5.2.1. *Any homogeneous ball in Carnot groups is strongly starshaped w.r.t. the centre P_0.*

Proof. Let us compute

$$\begin{aligned} \Omega' &= L_{-P_0}\Big(B_R^h(P_0)\Big) \\ &= \Big\{-P_0 \circ P \in \mathbb{G} : d_{hom}(P_0, P) \leq R\Big\} \\ &= \Big\{-P_0 \circ P \in \mathbb{G} : \|P_0 \circ P\|_{hom} \leq R\Big\} \end{aligned}$$

Denote by $\tilde{P} := -P_0 \circ P$, so we have:

$$\begin{aligned}\Omega' &= L_{-P_0}\Big(B^h_R(P_0)\Big)\\ &= \Big\{\tilde{P} \leq \mathbb{G} : \|\tilde{P}\|_{hom} \leq R\Big\}\\ &= B^h_R(0).\end{aligned}$$

We conclude that:

$$\Omega' \;=\; L_{-P_0}\left(B^h_R\Big(P_0\Big)\right) \;\subseteq\; B^h_R(0).$$

Since $B^h_R(0)$ is a starshaped w.r.t. 0, as we proved in Example 5.2.1, then we have that $B^h_R\Big(P_0\Big)$ is a starshaped set w.r.t. P_0 in the Carnot groups. ■

It is very well-known that in the classic Euclidean space all balls are Euclidean starshaped sets not only w.r.t. their centers but w.r.t. any internal point since they are convex, whereas that is not true in Carnot groups. For example if we consider the simplest case of Carnot group, which is the Heisenberg group $\mathbb{H}^1$, Danielli-Garofalo proved that the homogeneous ball in $\mathbb{H}^1$ is not $\mathbb{H}^1$-starshaped w.r.t. an open set of internal points.

Proposition 5.2.2. *[37, Proposition 3.8] Consider the homogenous ball*

$$B^h_1(0) \;:=\; \Big\{P = (x_1, x_2, x_3) \in \mathbb{R}^3 : \Big(|x_1|^2 + |x_2|^2\Big)^2 + |x_3|^2 < 1\Big\}$$

in $\mathbb{H}^1$. There exists some continuum set of points

$$\left\{P^\epsilon_0 = \left(\left(\frac{\epsilon}{2}\right)^{\frac{1}{4}}, 0, \sqrt{1-\epsilon}\right) \in \mathbb{R}^3 : 0 < \epsilon < \epsilon_0\right\} \subset B^h_1(0),$$

for some sufficient small $\epsilon_0 \in [0, 1]$, such that for every fixed $\epsilon \in (0, \epsilon_0)$, $B^h_1(0)$ is not a strongly $\mathbb{H}^1$-starshaped w.r.t. P^ϵ_0.

In general for a set Ω to be strongly starshaped w.r.t. 0 means to be starshaped along the dilations curves, which are the curves $\gamma : [0, 1] \to \mathbb{G}$ defined by

$$\gamma(t) \;=\; \delta_t(P), \quad \forall\, P \in \Omega.$$

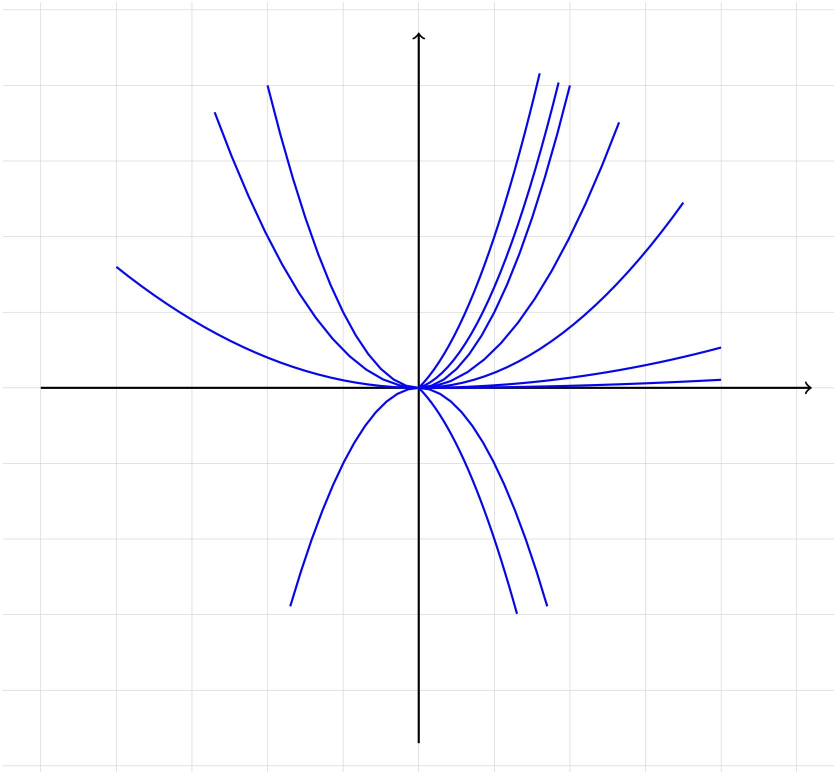

Figure 5.12: `The dilation curves` $\gamma(t;P) = (t\,x_1, t\,x_2, t^2\,x_3)$ `in` $\mathbb{H}^1$, `represented on the projection plane` $P_2 = 0$.

In the next example we write the dilation curves explicitly in the Heisenberg group.

Example 5.2.4 ($\mathbb{G}$-starshaped in the Heisenberg group)**.** In the particular case of the 1-dimensional Heisenberg group $\mathbb{H}^1$, (5.13) means that for $P = (x_1, x_2, x_3) \in \Omega$ the curve $\gamma(t;P) = (t\,x_1, t\,x_2, t^2\,x_3)$ is contained in Ω for all $t \in [0,1]$.
For any fixed P, $\gamma(t;P)$ are parabolas around the z-axis starting at the origin and passing trough the point P, see Figure 5.12.
Note that the parabolas degenerate into Euclidean straight lines on the plane $z = 0$. Moreover, they become straight segment also on the z-axis but with velocity $2x_3\,t$ (instead of constant velocity as in the Euclidean case). Using these parabolic curves, it is easy to build plenty of examples of $\mathbb{G}$-starshaped sets in the Heisenberg group, which we will indicate simply by $\mathbb{H}^1-$starshaped.

5.2.2 A geometrical characterisation at the boundary.

As we see in Section 5.1 that in the classic Euclidean setting, one of the characterisations for starshaped sets is known to be related to the normal at the boundary, see Lemma 5.1.1.

Definition 5.2.2. The ***infinitesimal generator associated to the dilations*** is defined as

$$\Gamma(P) \ := \ \dot{\gamma}_P(0) \ = \ \frac{d}{dt}\Big(\delta_t(P)\Big)\bigg|_{t=1}, \tag{5.15}$$

that is the velocity of the dilation smooth curve $\gamma(t) = \delta_t(P)$ at the point P. More in general for $P_0 \neq 0$ we define

$$\Gamma(P, P_0) \ := \ \frac{d}{dt}\Big(P_0 \circ \delta_t(-P_0 \circ P)\Big)\bigg|_{t=1}. \tag{5.16}$$

Example 5.2.5. In the n-dimensional Heisenberg group $\mathbb{H}^n$:

$$\Gamma(P) \ = \ \Gamma(x_1, \dots, x_{2n}, x_{2n+1}) = (x_1, \dots, x_{2n}, 2\, x_{2n+1}),$$

where $P = (x_1 \dots, x_{2n+1})$, while obviously in the Euclidean case $\Gamma(P) = P$. In general, using the structures of dilations in Carnot groups:

$$\delta_t(P) \ = \ (tx_1, \dots, tx_m, t^{\alpha_{m+1}}x_{m+1}, \dots, t^{\alpha_n}x_n)\,,$$

where $P = (x_1 \dots, x_{2n+1})$, with $\alpha_{m+1}, \dots, \alpha_n \geq 1$ natural numbers, we find

$$\frac{d}{dt}\,\delta_t(P) \ = \ \Big(x_1, \dots, x_m, \alpha_{m+1}t^{\alpha_{m+1}-1}x_{m+1}, \dots, \alpha_n t^{\alpha_n-1}x_n\Big)$$

which implies

$$\Gamma(P) \ = \ (x_1, \dots, x_m, \alpha_{m+1}x_{m+1}, \dots, \alpha_n x_n)\,.$$

Remark 5.2.3. The proof of Lemma 5.1.1 can be generalised to the idea of starshapedness w.r.t. any smooth curve γ if (5.3) is replaced by

$$\Big\langle n(P), \Gamma(P)\Big\rangle \ \geq \ 0, \quad \forall\, P \in \partial\Omega,$$

with $\gamma(0) = P_0$ and $\gamma(1) \;\; = P$.

Theorem 5.2.1. *Let us consider a Carnot group* $\mathbb{G}$ *(identified as usual with* $R^n byLemma 5.1.1) and \Omega \subseteq \mathbb{R}^n$ *open, regular and bounded with* C^1 *boundary and a point* $P_0 \in \Omega$*.*

1. *If* Ω *is strongly* $\mathbb{G}-$*starshaped w.r.t. the point* P_0 *(according to Definition 5.2.1), then*
$$\Big\langle \Gamma(P, P_0), n(P) \Big\rangle, \quad \forall\, P \in \partial\Omega, \tag{5.17}$$
where $\Gamma(P, P_0)$ *is the infinitesimal generator of the dilation defined in* (5.16) *and* $n(P)$ *is the normal of* $\partial\Omega$ *pointing outside at the point* $P \in \partial\Omega$*.*

2. *Assume that the strict inequality in* (5.17) *holds true, then* Ω *is strongly* $\mathbb{G}-$*starshaped w.r.t.* P_0*.*

Proof. A proof of this result can be found in [37, 46]. Here we find back the result simply by checking explicitly the rescaling property (5.9) for the smooth curves
$$\gamma_{P_0,P}(t) \;\; = \;\; P_0 \circ \delta_t(-P_0 \circ P) \;\; = \;\; P_0 \circ \delta_t(-P_0) \circ \delta_t(P);$$
then the proof of the Euclidean result applies (see Theorem 5.1.1 and Remark 5.1.3).
To this purpose, by using
$$\delta_t(P \circ Q) \;\; = \;\; \delta_t(P) \circ \delta_t(Q),$$
we first set
$$\bar{P} \;\; := \;\; \gamma_{P,P_0}(\bar{t}) \;\; = \;\; P_0 \circ \delta_t(-P_0) \circ \delta_{\bar{t}}(P),$$
and we compute
$$\begin{aligned} \gamma_{P_0,\bar{P}}(t) \;\; &= \;\; P_0 \circ \delta_t(-P_0) \circ \delta_t\Big(P_0 \circ \delta_{\bar{t}}(-P_0) \circ \delta_{\bar{t}}(P)\Big) \\ &= \;\; P_0 \circ \delta_{\bar{t}t}(-P_0) \circ \delta_{\bar{t}t}(P). \end{aligned}$$

Thus setting $s = \bar{t}t$, we obtain

$$
\begin{aligned}
\dot{\gamma}_{P_0,\bar{P}}(1) &= P_0 \circ \delta_t(-P_0) \circ \bar{(}P) \\
&= \frac{d}{dt}\Big(P_0 \circ \delta_t(-P_0) \circ \delta_t(\bar{P})\Big)\Big|_{s=\bar{t}} \\
&= \frac{d}{ds}\Big(P_0 \circ \delta_t(-P_0) \circ \delta_s(P)\Big)\Big|_{t=1} \frac{ds}{dt} \\
&= \bar{t}\dot{\gamma}_{P_0,\bar{P}}(\bar{t}).
\end{aligned}
$$

Then the rescaling property in Remark 5.2.3 is satisfied and the proof of the Euclidean result applies and gives our result. ■

As in the standard Euclidean case, one can approximate sets by strongly $\mathbb{G}-$starshaped sets.

5.2.3 $\mathbb{G}$-starshaped hull.

We start introducing the $\mathbb{G}$-starshaped hull.

Definition 5.2.3 ($\mathbb{G}$-starshaped hull)**.** Let $\mathbb{G}$ be a Carnot group and consider an open subset $\Omega \subseteq \mathbb{G}$. We define the ***$\mathbb{G}$-starshaped hull of the set Ω (w.r.t. the origin)***, and we simply indicate it by $\Omega^{*\mathbb{G}}$, as the set given by the intersection of all sets $\mathbb{G}$-starshaped w.r.t. the origin and containing Ω. This can be also found as

$$
\Omega^{*\mathbb{G}} = \bigcup_{t\in[0,1]} \delta_t\,(\Omega)\,. \tag{5.18}
$$

Similarly one could define the $\mathbb{G}$-starshaped hull of the set Ω w.r.t. a generic point P_0, we simply indicate it by $\Omega^{*\mathbb{G}}_{P_0}$, as the set given by the intersection of all sets $\mathbb{G}$-starshaped w.r.t. P_0 and containing Ω, i.e.

$$
\Omega^{*\mathbb{G}}_{P_0} = \bigcup_{t\in[0,1]} \Big(P_0 \circ \delta_t\,(-P_0) \circ \delta_t\,(\Omega)\Big).
$$

Lemma 5.2.1. *Given an open subset $\Omega \subseteq \mathbb{R}^n$, Ω is $\mathbb{G}$-starshaped w.r.t. the origin if and only if $\Omega = \Omega^{*\mathbb{G}}$.*

Proof. Note that trivially $\Omega \subseteq \Omega^{*\mathbb{G}}$ always by definition.
If Ω is $\mathbb{G}$-starshaped w.r.t. the origin then

$$\delta_t\Big(\Omega\Big) \;\subseteq\; \Omega, \quad \forall\, t \in [0,1].$$

So

$$\Omega^{*\mathbb{G}} \;=\; \bigcup_{t\in[0,1]} \delta_t\left(\Omega\right) \;\subseteq\; \Omega,$$

which give the identity.
The reverse implication is similar and we omit it. ∎

As in the standard Euclidean case the union of $\mathbb{G}$-starshaped sets w.r.t. the same point P_0 is still $\mathbb{G}$-starshaped, while in general the union of a $\mathbb{G}$-starshaped set with a $\mathcal{H}-$convex set could no longer be $\mathbb{G}$-starshaped even when the point P_0 belongs to the intersection (see e.g. Proposition 5.2.2). This will be clearer in Section 5.4, when the relations between $\mathbb{G}$-starshapedness and $\mathcal{H}-$convexity will be investigated. In the following example we study how to find explicitly the $\mathbb{H}^1-$starshaped hull of the dumbbell in the Heisenberg group $\mathbb{H}^1$.

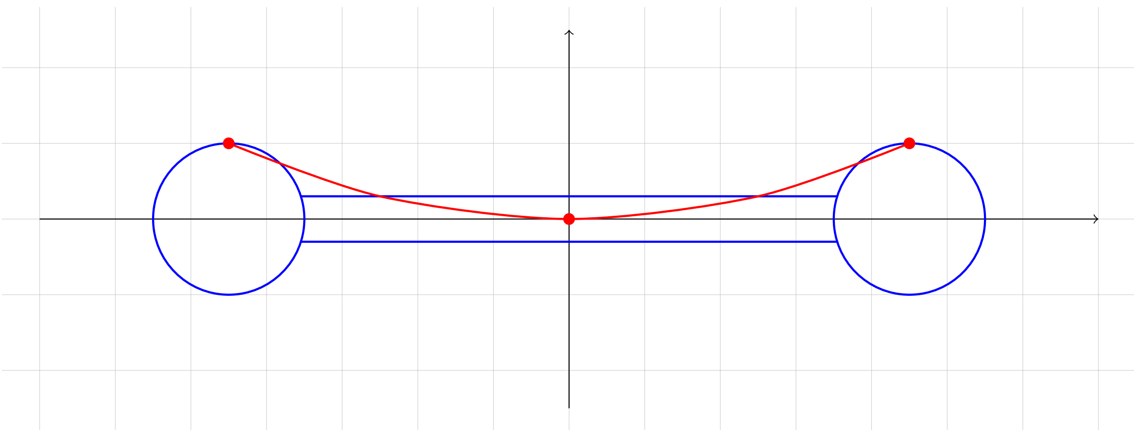

Figure 5.13: `The dumbbell is not` $\mathbb{H}^1$`-starshaped w.r.t. the origin (2D-projection on` $y = 0$`).`

Example 5.2.6 ($\mathbb{H}^1$-starshaped hull of the dumbbell)**.** To keep in line with the Euclidean case, see Example 5.1.4, and make computations slightly easier we consider everything projected on the 2D-plane $y = 0$.
In Figure 5.13 we show that the dumbbell is not $\mathbb{H}^1$-starshaped w.r.t. the origin. In fact, it is evident that the curves $\Big(-5t, t^2\Big)$ and $\Big(5t, t^2\Big)$, joining

the origin to $(-5,1)$ and $(5,1)$ respectively, are not all contained in the set. Hence we can compute the $\mathbb{H}^1$-starshaped hull. Then look at a generic point $P_0 = (x_1^0, x_2^0)$ on the boundary of one of the two balls of the dumbbell:

$$\begin{aligned} B_1 &= \Big\{P = (x_1, x_2) \in \mathbb{R}^2 : (x_1+5)^2 + x_2^2 < 1\Big\}, \\ B_2 &= \Big\{P = (x_1, x_2) \in \mathbb{R}^2 : (x_1-5)^2 + x_2^2 < 1\Big\}. \end{aligned}$$

We focus on

$$B_1 = \Big\{P_0 = (x_1^0, x_2^0) \in \mathbb{R}^2 : (x_1^0+5)^2 + \left(x_2^0\right)^2 < 1\Big\}.$$

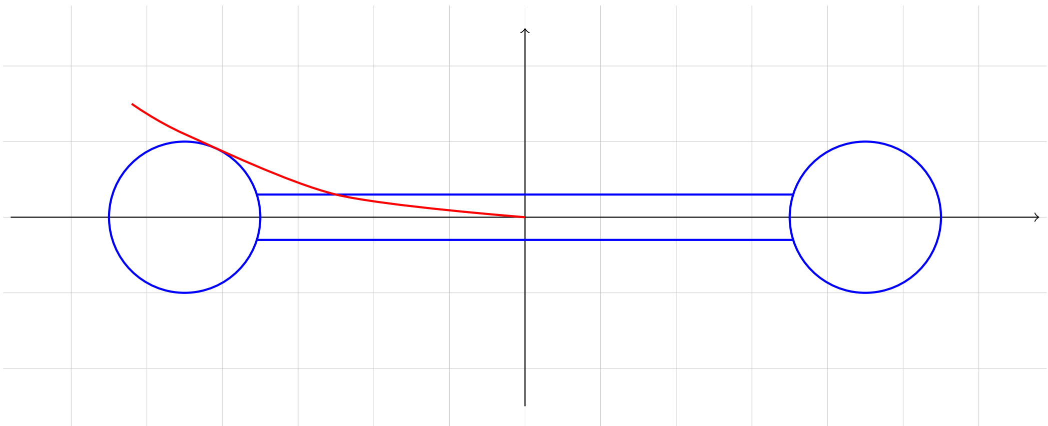

Figure 5.14: `A 2-dimensional` $\mathbb{H}^1$`-dilation curve, tangent to the dumbbell.`

In order to find P_1 and Q_1, we can take the point $(x_1^0, x_2^0) \in B_1$ and compute the parabolic curves

$$\gamma\left(t; x_1^0, x_2^0\right) = \delta_t\left(x_1^0, x_2^0\right) = \left(x_1^0 t, x_2^0 t^2\right), \quad \text{where } t \in [0,1].$$

The tangent vector is given by

$$\gamma'\left(t \to 1; x_1^0, x_2^0\right) = \left(x_1^0, x_2^0\right)\Big|_{t=1} = \left(x_1^0, 2x_2^0\right) = \underline{v}.$$

See Figure 5.14.
Now we want $\gamma'\left(t \to 1; x_1^0, x_2^0\right)$ to be tangent to B_1, i.e.

$$\underline{v} \perp (P_0 - C),$$

where $(P_0 - C) = (x_1^0 + 5, x_2^0)$.
Then

$$\langle \underline{v}, P_0 \rangle \;=\; 0.$$

In fact the normal at ∂B_1 at the point P_0 is given by

$$\frac{P_0 - C}{|P_0 - C|}.$$

This implies

$$x_1^0(x_1^0 + 5) + 2(x_2^0)^2 \;=\; 0.$$

Then

$$\left(x_1^0\right)^2 + 5x_0 \;=\; -2\left(x_2^0\right)^2,$$

so

$$2\left(x_2^0\right)^2 \;=\; \left(-5x_1^0 + x_1^0\right)^2.$$

Using that $(x_1^0, x_2^0) \in \partial B_1$, we have the following:

$$\left(x_1^0 + 5\right)^2 + \left(x_2^0\right)^2 \;=\; 1, \tag{5.19}$$

which implies

$$\left(x_2^0\right)^2 \;=\; 1 - (x_1^0 + 5)^2. \tag{5.20}$$

By substituting (5.19) in (5.20), we obtain:

$$2 - 2x_0^2 - 20x_0 - 50 + 5x_0 + x_0^2 \;=\; 0,$$

which leads to

$$x_0^2 + 15x_0 + 48 \;=\; 0.$$

So we find the two solutions:

$$\begin{aligned} x_1^0 \;&=\; \frac{-15 \pm \sqrt{15^2 - 4.48}}{2} \;=\; \frac{-15 \pm \sqrt{33}}{2} \;\approx\; \frac{-15 \pm 5.74}{2} \\ &\approx\; \begin{cases} -4.63 \text{ or} \\ -10.37 \end{cases} \end{aligned}$$

We can see that in case $x_1^0 = -10.37 < -5$ is not admissible since it cannot correspond to a point on ∂B_1.

So the unique acceptable solution is $x_1^0 = -4.63$. Now we need to compute y_0 as following:

$$\begin{aligned} x_2^0 &= \pm\sqrt{1-(4.63+5)^2} = \pm\sqrt{1-(0.73)^2} = \pm\sqrt{1-0.1369} \\ &= \pm\sqrt{0.8631} \approx \pm 0.93. \end{aligned}$$

That means $x_1^0 > 0$ is the second coordinate of P_1. Note that if $x_1^0 < 0$ then it is the second coordinate of P_4, since we mentioned earlier that all the other points are "symmetric" to P_1. Then one point to consider is

$$P_1 = \left(x_1, \sqrt{1-(x_1+5)^2}\right) \approx (-4.63, 0.93),$$

and the 3 corresponding symmetric points w.r.t. axis.
Next, we need to look at the $\mathbb{H}^1$-starshaped hull of the dumbbell which can be found by looking at the parabola $\overline{\gamma}(t) = \left(\overline{\gamma}_1, \overline{\gamma}_2\right) = \left(-4.63\,t, 0.93\,t^2\right)$. So, to compute the point $\overline{Q}_1$ we look at the following intersection:

$$\begin{cases} \overline{\gamma}_1(t), & t \in [0,1], \\ x_2 = \dfrac{1}{4}. \end{cases}$$

This implies:

$$0.93\,t^2 = \frac{1}{4} \Rightarrow t = \sqrt{\frac{1}{4 \times 0.93}} \approx 0.518.$$

So

$$\overline{Q}_1 = \gamma_1(0.518) = \left(-4.63\,t, 0.93\,t^2\right) = \left(-4.36(0.518), \frac{1}{4}\right) \approx (2.4, 0.25).$$

Thus, the unique intersection point between the parable $\bar{\gamma}_1(t)$ and the line $x_2 = \dfrac{1}{4}$ is $\bar{Q}_1$. This allows us to find the $\mathbb{H}^1-$starshaped hull of the given dumbbell.

5.2.4 Weakly $\mathbb{G}$-starshaped sets.

The notion of strong $\mathbb{G}$-starshapedness that we have studied in Section 5.2.1 can be considered as an algebraic notion, since it uses the structure of Carnot groups as Lie groups. In this section we introduce a second notion of starshapedness related to the more geometrical curves, $\mathcal{X}$-lines, that we have introduced in Section 4.3.2. Thus, this second notion can be considered as a more geometrical notion. This second notion is weaker than the previous one, as we will show in Section 5.3, but can be applied to all sub-Riemannian geometries (e.g. to the Grušin plane) and it characterises $\mathcal{H}-$convexity in Carnot groups (see Section 5.4).

Definition 5.2.4 (Weakly $\mathbb{G}$-starshaped set)**.** Consider an open set $\Omega \subseteq \mathbb{G}$, Ω is called a ***weakly*** $\mathbb{G}$***-starshaped set w.r.t.*** $P_0 \in \Omega$ if for each $Q \in \mathbb{V}_{P_0} \cap \Omega$, where $\mathbb{V}_{P_0}$ is the $\mathcal{X}$-plane w.r.t. P_0 (see Definition 4.3.2), and for all $\alpha \in \mathbb{R}^m$, we have

$$\mathcal{Y}_\alpha^{P_0,Q}\Big([0,1]\Big) \ \subseteq \ \Omega, \tag{5.21}$$

where $\mathcal{Y}_\alpha^{P_0,Q}\left([0,1]\right)$ is the set of all $\mathcal{X}$-segments joining P_0 and Q.

Remark 5.2.4. Euclidean starshaped sets are weakly $\mathbb{G}-$starshaped sets in Heisenberg group $\mathbb{H}^1$ since all $\mathcal{X}-$lines are in particular Euclidean straight lines. See Example 4.3.1 for more details.

Example 5.2.7. Consider the set

$$\Omega \ := \ \left\{P = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3^6\,(x_3+2)^3\,(x_3-1) < \frac{1}{2}\right\} \subseteq \mathbb{H}^1.$$

See Figure 5.15.
Ω is a weakly $\mathbb{H}^1$-starshaped w.r.t 0. In fact, we have

$$\mathbb{V}_0 \ = \ \Big\{(\alpha_1, \alpha_2, 0) \in \mathbb{R}^3 : \alpha_1, \alpha_2 \in \mathbb{R}\Big\}.$$

In addition

$$\forall\ Q \in \mathbb{V}_0 \cap \Omega \ \Rightarrow \ Q \ = \ (\alpha_1, \alpha_2, 0), \ \text{where } \alpha_1, \alpha_2 \in \mathbb{R}.$$

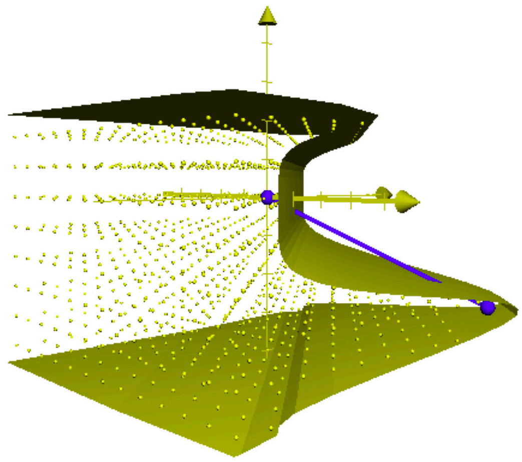

Figure 5.15: A weakly $\mathbb{H}^1$-starshaped set. The set is given by the portion of 3D-space indicated by the small yellow spheres and delimited by the yellow surface shown in the picture.

Now, let us consider all $P \in \mathcal{Y}_\alpha^{0,Q}\big([0,1]\big)$, that means

$$P = x_\alpha(t), \quad \text{for some } t \in [0,1],$$

and for some $\mathcal{X}-$line $x_\alpha(0) = 0$ and $x_\alpha(1) = Q$. In this case we can use the $\mathcal{X}-$lines computed in Example 4.3.1. This we need $P = (\alpha_1 t, \alpha_2 t, 0)$ for some $t \in [0,1]$ and $Q = x_\alpha(1) = (\alpha_1, \alpha_2, 0)$. We have

$$Q \in \Omega \Rightarrow \alpha_1 - \alpha_2 \le \frac{1}{2}.$$

The next step is to check that $P \in \Omega$, in fact:

$$\alpha_1 t - \alpha_2 t = (\alpha_1 - \alpha_2)t \le \frac{1}{2}t \le \frac{1}{2}, \quad \text{where } t \in [0,1].$$

Hence $P \in \Omega$, that implies Ω is a weakly $\mathbb{H}^1$-starshaped set w.r.t. 0. See Figure 5.15.

Note that weak $\mathbb{G}$-starshapedness is a very weakly notion since we can control the set only on a m-dimensional subspace. For example in $\mathbb{H}^1$, taking $P_0 = 0$, we have not any control of the set in the vertical direction (i.e. in the direction of z).

5.2.5 A geometric characterisation at the boundary.

Here we want to give a result to characterise weak starshaped bounded sets on their boundary, in line with Theorem 5.1.1 and Theorem 5.2.1.

Theorem 5.2.2. *Let $\mathcal{X} = \{X_1, \dots, X_m\}$ be a family of C^1 vector fields defined on $\mathbb{R}^n$, with $n \le m$, consider a regular open bounded set $\Omega \subset \mathbb{R}^n$ with C^1 boundary and a point $P_0 \in \Omega$.*

1. *If Ω is a weak starshaped w.r.t. P_0, then*

$$\Big\langle \sigma(P)\alpha_P, n(P) \Big\rangle \;\geq\; 0, \quad \forall\, P \in \mathbb{V}_{P_0} \cap \partial\Omega, \tag{5.22}$$

   where $\sigma(P)$ is the matrix associated to the vector fields (see Definition 4.3.1), $n(P)$ is the standard (Euclidean) normal vector pointing outside at the point $P \in \partial\Omega$ and α_P is the unique constant horizontal velocity such that the corresponding $\mathcal{X}-$line staring from P_0 joins P at time $t = 1$.

2. *Assume that the strict inequality in (5.22) holds true, then Ω is weakly starshaped w.r.t. P_0.*

Proof. Again to prove the result we need only to remark that $\mathcal{X}-$lines are C^1 since the vector fields are assumed to be C^1 and that the $\mathcal{X}-$lines satisfy the rescaling property (5.9). Then the proof of the Euclidean result (see Theorem 5.1.1) applies (see Remark 5.1.3). To check (5.9) we need only to show that, for all $\bar{t} \in [0, 1]$ and $\bar{P} = x^{\alpha_P}_{P_0}(\bar{t})$, we have

$$\gamma_{P_0,\bar{P}}(t) \;=\; x^{\alpha_{\bar{P}}}_{P_0}(\bar{t}) \;=\; x^{\bar{\alpha}}_{P_0}(\bar{t}), \quad \text{with } \bar{\alpha} \;=\; \bar{t}\alpha_P. \tag{5.23}$$

This trivially implies

$$\dot{\gamma}_{P_0,\bar{P}}(1) \;=\; \sigma\Big(\gamma_{P_0,\bar{P}}(1)\Big)(\bar{\alpha}) \tag{5.24}$$

$$\;=\; \bar{t}\sigma(\bar{P})\alpha_P \tag{5.25}$$

$$\;=\; \bar{t}\,\dot{\gamma}_{P_0,P}(\bar{t}). \tag{5.26}$$

To check (5.23) it not hard: in fact, set for simplicity $y(s) := x_{P_0}^{\alpha_{\bar{P}}}(\bar{t}\,s)$, then $y(0) = P_0$ and

$$\dot{y}(s) \;=\; \bar{t}\dot{x}_{P_0}^{\alpha}(\bar{t}\,s) \;=\; \sigma\Big(y(s)\Big)\bar{\alpha}.$$

By uniqueness for ODEs with C^1 coefficients, we deduce (5.24). ■

In the particular case of Carnot groups, using canonical coordinates, we can show that the first $m-$components of $\mathcal{X}-$lines are actually (Euclidean) straight lines (see Lemma 2.2 in [12]), that means

$$\alpha_x \;=\; \pi_m(P - P_0), \quad \forall\, P \in \mathbb{V}_{P_0},$$

where by $\pi_m : \mathbb{R}^n \to \mathbb{R}^m$ is the projection on the first $m-$components. Then we can rewrite the previous result in the following way.

Corollary 5.2.1. *Let $\mathbb{G}$ be a Carnot group and $\mathcal{X}$ be the associated to family of the left-invariant vector fields associated to the first layer of the stratified Lie algebra $\mathfrak{g}$ in canonical coordinates, consider a set $\Omega \in \mathbb{G}$ and apoint $P_0 \in \mathbb{G}$.*

1. *If Ω is weak starshaped w.r.t. P_0, then*

$$\Big\langle \sigma(P)\pi_m\Big(P - P_0\Big), n(P)\Big\rangle \;\geq\; 0, \quad \forall\, P \in \mathbb{V}_{P_0} \cap \partial\Omega, \tag{5.27}$$

   where $n(P)$ is the standard (Euclidean) normal pointing outside at the point P and π_m is the projection on the first $m-$components.

2. *Assume that the strict inequality in* (5.27) *holds true , then Ω is weakly starshaped w.r.t. P_0.*

The proof is trivial so we omit it.

Again we want to point out as Theorem 5.2.2 and Corollary 5.2.1 give a characterisation for the weakly starshaped sets only on a subset of the boundary since they consider only points in $\partial\Omega \cap \mathbb{V}_{P_0}$.

5.3 Relations between the different notions of starshapedness.

After we introduce all these notions of starshapedness in different cases, we want to see how they are related to each others and how they are related to the Euclidean starshapedness. These different relations will also help us to explain the relations between weak and strong Therefore $\mathbb{G}-$starshaped sets and the $\mathcal{H}-$convex sets later in this thesis.

5.3.1 Euclidean starshaped sets vs strongly $\mathbb{G}$-starshaped sets.

As already mentioned. we can say that for a set Ω to be strongly $\mathbb{G}$-starshaped, means that the set has to be "starshaped" along curves, which we call dilation-curves, and are defined as

$$\gamma : [0, 1] \;\rightarrow\; \mathbb{G} \;\text{ with }\; \gamma(t) \;:=\; \delta_t(P),$$

for any fixed $P \in \Omega$. Similarly, the (Euclidean) standard starshapedness means instead to be "starshaped" along (Euclidean) straight lines. With this difference on mind, it is not difficult to build counterexamples to show that the two notions are completely non-equivalent in the sense that no one of the two implies the other.

We build these counterexamples in the 1-dimensional Heisenberg group, where the dilation-curves are parables around the z-axis, see Example 5.2.4.

We first show that Euclidean starshapedness does not imply strongly $\mathbb{H}^1$-starshapedness. There are many possible counterexamples, the easiest being the cone.

Example 5.3.1 (An Euclidean starshaped set that is not strongly $\mathbb{H}^1$-starshaped). Consider the set

$$\Omega \;:=\; \left\{ P = (x_1, x_2, x_3) \,\in\, \mathbb{R}^3 \,:\, \sqrt{(x_1)^2 + (x_2)^2} - 0.1 \,<\, x_3 \right\}.$$

Note that in the cone Ω the origin is an internal point and so is the point

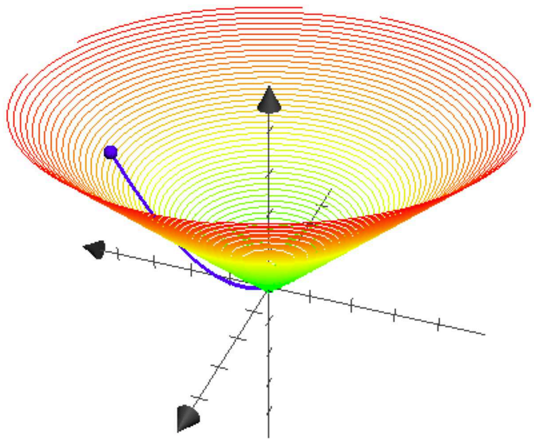

Figure 5.16: This cone is (Euclidean) starshaped w.r.t. all its interior points but it is not strongly $\mathbb{H}^1$-starshaped w.r.t. the origin.

$Q = \left(\frac{13}{10}, 1, 2\right)$: in fact

$$\sqrt{\left(\frac{13}{10}, 1, 2\right)^2 + (1)^2} - 0.1 \;\approx\; 0.410 \;<\; 2 \;\Rightarrow\; Q \in \Omega.$$

The dilation-curve starting from to Q is

$$\gamma(t) \;=\; \delta_t(Q) \;=\; \left(\frac{13}{10}\,t, t, 2\,t^2\right)$$

and this does not all belong to Ω for all $t \in [0, 1]$, e.g. if we take $t = \frac{1}{4}$, then we obtain the point $(0.325, 0.25, 0.125)$ which does not belong to Ω. In fact:

$$\sqrt{(0.325)^2 + (0.250)^2} - 0.1 \;\approx\; 0.311 \;\not<\; 0.125.$$

The set Ω is Euclidean starshaped w.r.t. all internal points since it is Euclidean convex. We conclude that Ω is not a strongly $\mathbb{H}^1$-starshaped set. See Figure 5.16.

Next we show that strongly $\mathbb{H}^n$-starshaped does not imply Euclidean starshaped. To this purpose it will be enough to look at any non-convex set defined as complementary of paraboloids. We give a specific example but many others could be easily computed.

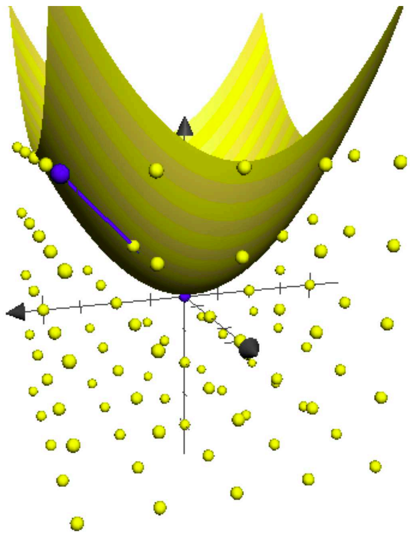

Figure 5.17: A strongly $\mathbb{H}^1$-starshaped that is not Euclidean starshaped. In fact the straight segment-line joining two internal points (the origin with $(1, 1, 1)$) is not all contained in the set. The set is the portion of 3D-space indicated by the small yellow balls and delimitated by the yellow paraboloid surface.

Example 5.3.2 (A strongly $\mathbb{H}^1$-starshaped set that is not Euclidean starshaped)**.** Consider the set

$$\Omega \ := \ \left\{ P = (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1^2 + x_2^2 + \frac{1}{30} > x_3 \right\} \ \subseteq \ \mathbb{H}^1,$$

then the origin is an internal point and so is the point $(1, 1, 1)$. In fact:

$$(1)^2 + (1)^2 + \frac{1}{30} \ \approx \ 2.0333 \ > 1 \ \Rightarrow \ Q = (1, 1, 1) \in \Omega.$$

The set is $\mathbb{H}^1$-starshaped w.r.t. the origin; in fact, set

$$\widetilde{P} \ = \ \delta_t(P) \ = \ (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3),$$

for all $t \leq 1$. Then

$$\begin{aligned}\widetilde{x}_3 &= t^2 x_3 \\ &< t^2\left(x_1^2 + x_2^2 + \frac{1}{30}\right) \\ &= (tx_1)^2 + (tx_2)^2 + \frac{t^2}{30} \\ &< (\widetilde{x}_1)^2 + (\widetilde{x}_2)^2 + \frac{1}{30},\end{aligned}$$

where we have used $t \leq 1$.
This shows that $\delta_t(P) \in \Omega$, for all $t \in [0,1]$ and for all $P \in \Omega$; thus Ω is $\mathbb{H}^1$-starshaped w.r.t. the origin.
Look now at the straight segment-line joining $(0,0,0)$ to $Q = (1,1,1)$, i.e.

$$\gamma(t) \;:=\; t\,Q \;=\; (t,t,t).$$

Then for $t = \frac{1}{3} \in (0,1)$, the corresponding point $\gamma\left(\frac{1}{3}\right) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ does not belong to Ω. In fact:

$$\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \frac{1}{30} \;=\; \frac{69}{270} \not> \frac{1}{3}.$$

Thus Ω is not Euclidean starshaped w.r.t. the origin. See Figure 5.17.

5.3.2 Strongly $\mathbb{G}$-starshaped sets vs weakly $\mathbb{G}$-starshaped sets.

We now justify the names “strongly” and “weakly” starshaped proving that the strong notion implies the weak notion while the reverse is not true.

Proposition 5.3.1. *Given a Carnot group $\mathbb{G}$ identified with $\mathbb{R}^n$ and an open $\Omega \subseteq \mathbb{R}^n$, if we assume that Ω is strongly $\mathbb{G}$-starshaped w.r.t. some point $P_0 \in \Omega$, then Ω is weakly $\mathbb{G}$-starshaped w.r.t. P_0.*

Proof. For sake of simplicity we consider first the case $P_0 = 0$. So assume that

Ω is strongly $\mathbb{G}-$starshaped w.r.t. 0, that means

$$\delta_t(Q) \;\in\; \Omega,\;\; \forall\, t \in [0,1] \;\text{ and }\; \forall\, Q \in \Omega.$$

To show that Ω is weakly $\mathbb{G}-$starshaped w.r.t. 0, we need to consider a generic point $Q \in \mathbb{V}_0 \cap \Omega$, i.e.

$$Q \;=\; x_\alpha(1) \;\text{ for some } \alpha \in \mathbb{R}^m \;\text{ with }\; x_\alpha(\cdot)$$

corresponding $\mathcal{X}$-line starting at the time $t = 0$ at the origin; then we claim that:

$$x_\alpha(t) \;=\; \delta_t(Q). \tag{5.28}$$

Assuming (5.28), it is immediate to conclude: in fact, weakly $\mathbb{G}$-starshaped can be written again as

$$\delta_t(Q) \;\in\; \Omega,\;\; \forall\, t \in [0,1]$$

but only for all $Q \in \mathbb{V}_0 \cap \Omega \subseteq \Omega$. Then the implication from strongly to weakly follows.

We remain to show (5.28). At this purpose we need to recall a non trivial result for the left-invariant vector fields of Carnot groups in exponential coordinates (see [22] Proposition 1.3.5 and Corollary 1.3.19). Set $\sigma(x)$ (see Definition 4.3.1) the smooth $n \times m$-matrix whose columns are the left-invariant vector fields $X_1, \dots, X_m$, then in exponential coordinates $\sigma(\cdot)$ can be written as

$$\sigma(x) \;=\; \begin{pmatrix} Id_{m\times m} \\ \\ A(x) \end{pmatrix}$$

where $Id_{m\times m}$ is the $m \times m$-identity matrix while $A(x) = \big(a_{j\,i}(x)\big)$ for $i = 1, \dots, n$ and $j = 1, \dots, m$ has polynomial coefficients such that, whenever the component $m+i$ rescales as t^k (w.r.t. the associated dilation $\delta_t(\cdot)$) then $a_{j\,i}(x)$ is a polynomial in the variables $x_1, \dots, x_m$ of degree exactly equal to $k-1$.

Using such a result, it is not difficult to show that the $\mathcal{X}$-lines starting at $t = 0$ at the origin can be written as

$$x_1(t) \;=\; \alpha_1\, t, \dots, x_m(t) = \alpha_m\, t$$

and in general

$$x^i_\alpha(t) \;=\; p_i^{k-1}(\alpha_1,\dots,\alpha_m)\,t^k,$$

whenever x_i rescales as t^k in the associated dilation $\delta_t(\cdot)$ and where by $p_i^{k-1}(x_1,\dots,x_m)$ we indicate a generic polynomial of order $k-1$. Taking $t=1$, we find that for all $Q\in\mathbb{V}_0$ then $Q=(\alpha_1,\dots,\alpha_m,\dots,p_i^{k-1}(\alpha_1,\dots,\alpha_m),\dots)$, which proves (5.28).

We now consider the general case $P_0\neq 0$: strongly $\mathbb{G}$-starshaped means that

$$P_0\circ\delta_t(-P_0)\circ\delta_t\Big(\Omega\Big)\;\subset\;\Omega.$$

Recall that by dilation property (See Lemma 3.2.1)

$$\delta_t(-P_0)\circ\delta_t(Q)\;=\;\delta_t(-P_0\circ Q).$$

Therefore for all $Q\in\mathbb{V}_{P_0}$ we need to show that

$$P_0\circ\delta_t(-x_0\circ Q)\;=\;x^{P_0}_\alpha(t), \tag{5.29}$$

where by $x^{P_0}_\alpha(\cdot)$ we indicate the $\mathcal{X}$-line with horizontal velocity α and such that

$$\begin{cases} x^{P_0}_\alpha(0) \;=\; P_0,\\[2ex] x^{P_0}_\alpha(1) \;=\; Q.\end{cases}$$

Since $Q\in\mathbb{V}_{P_0}$ if and only if $-P_0\circ Q\in\mathbb{V}_0$, then there exists some constant $\widetilde{\alpha}$ such that the corresponding $\mathcal{X}$-line $P_{\widetilde{\alpha},0}(\cdot)$ connect 0 to $-P_0\circ Q$ at time 1. For (5.28) we know that $\delta_t(-P_0\circ Q)=x^0_{\widetilde{\alpha}}(t)$.

Define $\tilde{x}(t)=P_0\circ x^0_{\widetilde{\alpha}}(t)$, by property of left-translation, then $\tilde{x}(\cdot)$ is still horizontal with constant velocity $\widetilde{\alpha}$.

Moreover it is easy to check that $\tilde{x}(0)=P_0$ and $\tilde{x}(1)=Q$, so by uniqueness of $\mathcal{X}$-lines joining two given points in Carnot group, $\tilde{x}(t)=x^{P_0}_\alpha(t)$, so (5.29) is proved and we can conclude as in the case $P_0=0$. ■

The reverse implication of the previous result is false. From the proof is obvious that the two notions of starshapedness coincide only whenever $\mathbb{V}_0=\mathbb{R}^n$, but this means only in the commutative case (i.e. step= 1).We next show explicitly a counterexample in the case of the Heisenberg group.

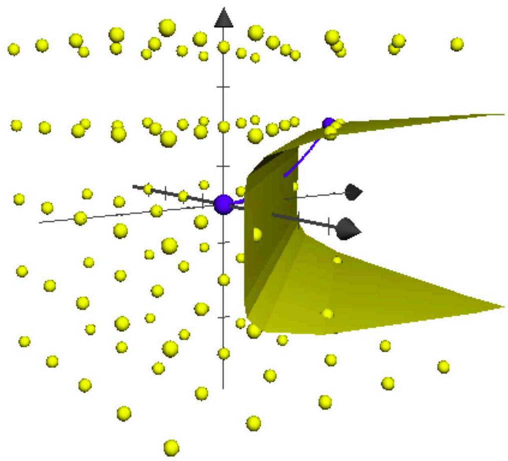

Figure 5.18: A weakly $\mathbb{H}^1$-starshaped set that is not strongly $\mathbb{H}^1$-starshaped. The set is given by the portion of 3D-space indicated by the small yellow spheres and delimited by the yellow surface. The blue curve is the dilation-curve joining the origin to an internal point of the set. Clearly the curve exits the set.

Example 5.3.3 (A weakly $\mathbb{H}^1$-starshaped set that is not strongly $\mathbb{H}^1$-starshaped)**.** Consider the set

$$\Omega \ := \ \left\{P = (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1 + x_2 - x_3^8 < 1\right\} \ \subseteq \ \mathbb{H}^1.$$

Using that

$$\mathbb{V}_0 \ = \ \left\{P = (x_1, x_2, 0) \ : \ x_1, x_2 \in \mathbb{R}\right\},$$

in the 1-dimensional Heisenberg group $\mathbb{H}^1$, then it is trivial to show that Ω is weakly $\mathbb{H}^1$-starshaped w.r.t. the origin: in fact the projection of Ω on $\mathbb{V}_0$ is a plane containing the origin. Instead Ω is not strong $\mathbb{H}^1$-starshaped w.r.t. the origin: in fact, if we look at the point $Q = (0, 1.8, 1)$, then

$$Q \ \in \ \Omega \quad \text{since} \ \ 1.8 - (1)^8 \ = \ 0.8 \ < \ 1.$$

Nevertheless if we choose $t = \dfrac{3}{4} \in [0,1]$, we have that

$$\delta_{\frac{3}{4}}(Q) \;=\; \frac{3}{4}(0, 1.8, 1) \;=\; \left(0, \frac{3}{4}(1.8), \left(\frac{3}{4}\right)^2 (1)\right) \;\approx\; (0, 1.35, 0.56).$$

That implies:

$$\delta_{\frac{3}{4}}(Q) \;\notin\; \Omega, \;\text{ since }\; 1.35 - (0.56)^8 \;\approx\; 1.34 \;\not< \;1.$$

So we conclude that a weakly $\mathbb{G}$-starshaped set is not necessarily a strongly $\mathbb{G}$-starshaped set. See Figure 5.18.
Note that the set in the example is unbounded but one could easily modify the example to obtain a bounded set with a similar behaviour.

5.3.3 Weakly $\mathbb{G}$-starshaped sets vs Euclidean starshaped sets.

In the n-dimensional Heisenberg group it is easy to show that Euclidean starshaped sets are always weakly $\mathbb{H}^n$-starshaped since the $\mathcal{X}$-lines are a selection of Euclidean straight lines (see Example 4.3.1).
The same is true whenever $\mathcal{X}$-lines are Euclidean straight lines while it is false when this is not the case: e.g. in the Grušin plane (see Example 4.22) it is easy to build examples of Euclidean starshaped sets which are not weakly $\mathbb{G}$-starshaped. In fact in that case the $\mathcal{X}$-lines are parables, then the same cone given in Figure 5.16 becomes a counterexample also for this case. Note that the Grušin place is not a Carnot group but similar counterexamples can be found also using Carnot groups (e.g. in the Engel group). We omit further details.
Finding weakly $\mathbb{G}$-starshaped sets which are not Euclidean starshaped is possible in any non commutative Carnot group and in general in any sub-Riemannian geometry with step ≥ 2. We give the following example in the Heisenberg group.

Example 5.3.4. As we see in Example 5.2.7 the set

$$\Omega \;:=\; \left\{P = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3^6(x_3+2)^3(x_3-1) < \frac{1}{2}\right\} \;<\; \mathbb{H}^1,$$

is a weakly $\mathbb{H}^1$-starshaped set w.r.t. 0. Instead Ω is not Euclidean starshaped w.r.t 0, in fact the point $Q = \left(0, \frac{3}{2}, -1\right) \in \Omega$, since:

$$\frac{3}{2} + (-1)^6(-1+2)^3(-1-1) \;=\; \frac{3}{2} - 2 \;=\; -\frac{1}{2} \;<\; \frac{1}{2},$$

whereas for $t = \dfrac{1}{2} \in [0,1]$ we have:

$$P(t) \;=\; t\,Q \;=\; \frac{1}{2}\left(0, \frac{3}{2}, -1\right) \;=\; \left(0, -\frac{3}{4}, -\frac{1}{2}\right) \;\notin\; \Omega,$$

because:

$$\begin{aligned}\left(\frac{3}{4}\right) + \left(-\frac{1}{2}\right)^6\left(-\frac{1}{2}+2\right)^3\left(-\frac{1}{2}+2\right) \;&=\; \frac{3}{4} + \left(\frac{1}{64}\right)\left(\frac{27}{8}\right)\left(-\frac{3}{2}\right)\\ &\approx\; 0.750 + 0.016(3.375)(-1.5)\\ &\approx\; 0.696 \;\not<\; 0.5\ .\end{aligned}$$

See Figure 5.15.

Proposition 5.3.2. *If $\Omega \subseteq \mathbb{R}^{2n+1}$ is an Euclidean starshaped w.r.t. P_0 then Ω is a weakly $\mathbb{H}^n$-starshaped set, whenever the $\mathcal{X}-$lines are (Euclidean) straight lines.*

Proof. We know that the $\mathcal{X}-$lines on $\mathbb{H}^n$ are $2n+1$-parameters Euclidean lines, i.e.

$$\mathcal{Y}^{P,Q}_{\mathbb{H}^n}\left([0,1]\right) \;\subseteq\; \mathcal{Y}^{P,Q}_{\mathbb{R}^{2n+1}}\left([0,1]\right) \;\subseteq\; \Omega.$$

Which means, Ω is a weakly $\mathbb{H}^n$-starshaped set w.r.t. P_0. ∎

Remark 5.3.1. The reverse of the previous proposition is generally false in Carnot groups since $\mathcal{X}-$lines are not necessarily Euclidean lines, see Example 5.3.4.

5.3.4 Conclusion.

In the next remark we summarise all the relations between the different notions of starshapedness that we have proved in this section.

Remark 5.3.2.

1. Euclidean starshaped $\not\Rightarrow$ strongly $\mathbb{G}$-starshaped, see Example 5.3.1.

2. Euclidean starshaped $\Rightarrow$ weakly $\mathbb{G}$-starshaped in the Heisenberg group and whenever the $\mathcal{X}$-lines are (Euclidean) straight lines, see Proposition 5.3.2.

3. Strong $\mathbb{G}$-starshaped $\not\Rightarrow$ Euclidean starshaped, see Example 5.3.2.

4. Strong $\mathbb{G}$-starshaped $\Rightarrow$ weakly $\mathbb{G}$-starshaped in every Carnot group, see Proposition 5.3.1.

5. Weak $\mathbb{G}$-starshaped $\not\Rightarrow$ strongly $\mathbb{G}$-starshaped whenever the group is non-commutative (i.e. step > 1), see Example 5.3.3.

6. Weak $\mathbb{G}$-starshaped $\not\Rightarrow$ Euclidean starshaped, see Example 5.3.4.

5.4 Convex sets and their relations with strongly and weakly $\mathbb{G}$-starshapedness.

In this section we study the connection between the different notions which we have introduced in the previous section and $\mathcal{H}-$convex sets (see Definition 4.2.5) in Carnot groups. We start by the following result: even if this result is trivial to prove, it is very important since it gives a way to connect $\mathcal{H}-$convexity to starshapedness.

Proposition 5.4.1 ($\mathcal{X}-$convex sets and weakly $\mathbb{G}$-starshaped sets)**.** *Given any family $\mathcal{X} = \{X_1, \dots, X_m\}$ of vector fields on $\mathbb{R}^n$, an open set $\Omega \subseteq \mathbb{R}^n$ is $\mathcal{X}-$convex if and only if Ω is weakly $\mathbb{G}$-starshaped w.r.t. every $P_0 \in \Omega$.*

Proof. We recall that a set is said to be $\mathcal{X}$-convex if and only if for $P_0 \in \Omega$ and all $Q \in \mathbb{V}_0 \cap \Omega$, the $\mathcal{X}-$line connecting P_0 to Q in time 1 is all contained in Ω. Then the equivalence is trivial. ∎

By using the equivalence between $\mathcal{X}-$convex sets and $\mathcal{H}-$convex sets (see [10]) we have the following corollary as immediate consequence of the previous proposition.

Corollary 5.4.1 ($\mathcal{H}-$convexity and weakly $\mathbb{G}$-starshaped sets in Carnot groups)**.** *An open set Ω in a Carnot groups $\mathbb{G}$ is $\mathcal{H}$-convex if and only if it is weakly $\mathbb{G}$-starshaped w.r.t. all $P_0 \in \Omega$.*

Obviously since strongly $\mathbb{G}$-starshaped implies weakly $\mathbb{G}$-starshaped then we have the following result.

Proposition 5.4.2. *Given a Carnot group $\mathbb{G}$, and an open set $\Omega \subset \mathbb{G}$, if Ω is strongly $\mathbb{G}$-starshaped w.r.t. all $P_0 \in \Omega$ then it is $\mathcal{H}$-convex.*

In Proposition 5.2.2 (see [37]) is shown that the opposite implication is false we will give two additional explicit examples to show this.

Example 5.4.1 ($\mathcal{H}-$convex set in $\mathbb{H}^1$ that is not strongly $\mathbb{H}^1$-starshaped w.r.t. some internal points)**.** Consider the cone

$$\Omega \ := \ \left\{ P = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > \sqrt{(x_1)^2 + (x_2)^2} - 0.1 \right\},$$

then Ω is convex in the standard (Euclidean) sense so it is also $\mathcal{H}-$convex in the 1-dimensional Heisenberg group $\mathbb{H}^1$ but it is not strongly $\mathbb{G}$-starshaped w.r.t. the origin (that is an internal point), see Example 5.3.1.

The following example is very similar to the previous one but by considering a strictly Euclidean convex, that implies strictly $\mathcal{H}-$convex in $\mathbb{H}^1$. There are many way to define strictly $\mathcal{H}-$convex, e.g. by using the equivalent notion of $\mathcal{X}-$convexity one can require that, given a point $P_0 \in \overline{\Omega}$ and any $Q \in \mathbb{V}_0 \cap \overline{\Omega}$, then the corresponding $\mathcal{X}-$segment $x_\alpha\Big((0,1)\Big) \subset \Omega$. The cone is $\mathcal{H}-$convex in $\mathbb{H}^1$ but not strict $\mathcal{H}-$convex.

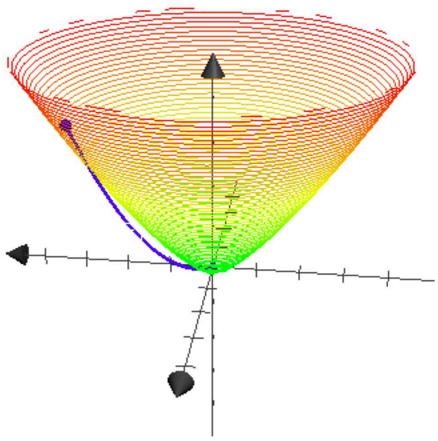

Figure 5.19: A Strictly $\mathbb{X}-$convex set which is not strongly $\mathbb{H}^1$-starshaped w.r.t. all internal points.

Example 5.4.2 (A strictly $\mathcal{H}-$convex set that is not strongly $\mathbb{H}^1$-starshaped w.r.t. all internal points)**.** We consider

$$\Omega \quad := \quad \left\{ P(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > \left(x_1^2 + x_2^2\right)^{\frac{3}{4}} - 0.0001 \right\},$$

then Ω has no corner on the boundary and it is strictly horizontally convex in $\mathbb{H}^1$ but it is not strongly $\mathbb{G}$-starshaped w.r.t. the origin. In fact by using the same point $Q = \left(\frac{13}{10}, 1, 2\right)$ from Example 5.3.1, we obtain

$$\delta_{\frac{1}{4}}\left(\frac{13}{10}, 1, 2\right) \quad \approx \quad (0.325, 0.250, 0.125),$$

but

$$\left((0.325)^2 + (0.250)^2\right)^{\frac{1}{4}} - 0.0001 \quad \approx \quad 0.262 \quad \not< \quad 0.125.$$

Which means that Ω is not strongly $\mathbb{G}-$starshaped w.r.t. the origin.

We like to conclude summing up all the relations between starshapedness and convexity of sets.

Remark 5.4.1.

1. Ω is $\mathcal{X}-$convex set $\iff$ Ω is weakly $\mathbb{G}-$starshaped w.r.t. every $P_0 \in \Omega$, see Proposition 5.4.1.

2. Ω is $\mathcal{H}-$convex set $\iff$ Ω is weakly $\mathbb{G}-$starshaped w.r.t. every $P_0 \in \Omega$, see Corollary 5.4.1.

3. *In general Ω is $\mathcal{X}-$convex set $\nRightarrow$ Ω Euclidean convex.*, since we have: Ω is $\mathcal{X}-$convex set $\iff$ Ω is weakly $\mathbb{G}-$starshaped w.r.t. all $P_0 \in \Omega$ (see Proposition 5.4.1) $\nRightarrow$ Euclidean starshaped (See Example 5.3.4).

4. *Ω is Euclidean starshaped w.r.t. all its interior points $\Rightarrow$ Ω is $\mathcal{X}-$convex in* $\mathbb{H}^1$, since we have:
   Euclidean starshapedness $\Rightarrow$ weakly $\mathbb{G}-$starshapedness in $\mathbb{H}^1$ (see Proposition 5.3.2) $\iff$ Ω is $\mathcal{X}-$convex (see Proposition 5.4.1).

5. *Ω is $\mathcal{X}-$convex in* $\mathbb{H}^1$ $\nRightarrow$ *Ω is Euclidean convex (or equivalently Euclidean starshaped w.r.t. all its interior points),* since we have:
   Ω is $\mathcal{X}-$convex $\iff$ Ω is weakly $\mathbb{G}-$starshaped in $\mathbb{H}^1$ w.r.t. all $P_0 \in \Omega$ (see Proposition 5.4.1) $\nRightarrow$ Ω is Euclidean starshaped w.r.t. all $P_0 \in \Omega$ (see Example 5.3.4).

6. *Ω is strongly $\mathbb{G}-$starshaped w.r.t. all its interior points $\Rightarrow$ Ω is $\mathcal{H}-$convex,* since we have:
   Ω is strongly $\mathbb{G}-$starshaped w.r.t. all $P_0 \in \Omega \Rightarrow \Omega$ is weakly $\mathbb{G}-$starshaped w.r.t. all $P_0 \in \Omega$ (see Proposition 5.3.1) $\iff$ Ω is $\mathcal{X}-$convex that equivalent to $\mathcal{H}-$convex. (see Proposition 5.4.1).

7. *Ω is $\mathcal{H}-$convex set $\nRightarrow$ Ω is strongly $\mathbb{G}-$starshaped w.r.t. all $P_0 \in \Omega$,* since we have:
   Ω is $\mathcal{H}-$convex is equivalent to $\mathcal{X}-$convex $\iff$ Ω is weakly $\mathbb{G}-$starshaped in $\mathbb{H}^1$ w.r.t. all $P_0 \in \Omega$, is (see Proposition 5.4.1) $\nRightarrow$ Ω is strongly $\mathbb{G}-$starshaped in $\mathbb{H}^1$ w.r.t. all $P_0 \in \Omega$ (see Example 5.3.3).

Chapter 6

Conclusion.

The first part of the thesis consisted of reviewing Carnot groups and sub-Riemannian manifolds, focusing on the theory of convexity for Carnot groups and, more generally, for sub-Riemannian manifolds. Then we applied the idea of $\mathcal{X}-$convexity developed by Bardi-Dragoni in [10] to sets. This has never been done before, and allowed us to generalise some connection between convex functions and convex sets already known in the standard Euclidean case and also for $\mathcal{H}-$convexity in Carnot groups, to the more general case of geometries of vector fields (so in particular to the sub-Riemannian setting, too).
The second part of the thesis focused on starshapedness in Carnot groups and, more in general, in the geometry of vector fields. We used two notions: the first one was introduced by Danielli-Garofalo in [38] and recently used by Dragoni-Garofalo-Salani in [46] to study the geometric properties for level sets of potential problems, while the other one is newly introduced in this thesis and it is deeply connected to the idea of convexity along vector fields. We first study all the mutual relations between these two different notions of starshapedness in Carnot groups and between them and the standard (Euclidean) starshapedness (see Remark 5.3.2 for a detailed sum-up). Finally we study the relations between then different notions of starshaped sets and $\mathcal{H}-$convex sets and $\mathcal{X}-$convex sets (see Remark 5.4.1). All the new results on starshaped and convex sets are contained in the following paper: *Starshaped sets in Carnot groups and general sub-Riemannian geometries* by F. Dragoni, D.Filali [45].

Bibliography

[1] Abraham, R., Marsden, J.E. and Ratiu, T., 2012. Manifolds, tensor analysis, and applications (Vol. 75). Springer Science and Business Media.

[2] Adams, M.R. and Tie, J., 2013. On subRiemannian geodesics on the Engel groups: Hamilton's equations. Mathematische Nachrichten, 286(14-15), pp.1381-1406.

[3] Agrachev, A., Barilari, D. and Boscain, U., 2016. Notes on Riemannian and SubRiemanniann geometry. preprint.

[4] Agrachev, A.A. and Gauthier, J.P.A., 1999. On the Dido problem and plane isoperimetric problems. Acta Applicandae Mathematica, 57(3), pp.287-338.

[5] Agrachev, A.A., Barilari, D. and Boscain, U., 2012. Introduction to Riemannian and sub-Riemannian geometry.

[6] Agrachev, A.A., Gentile, A. and Lerario, A., 2015. Geodesics and horizontal-path spaces in Carnot groups. Geometry Topology, 19(3), pp.1569-1630.

[7] Alvarez, O., Lasry, J.M. and Lions, P.L., 1997. Convex viscosity solutions and state constraints. Journal de mathématiques pures et appliquées, 76(3), pp.265-288.

[8] Ambrosio, L. and Rigot, S., 2004. Optimal mass transportation in the Heisenberg group. Journal of Functional Analysis, 208(2), pp.261-301.

[9] Bardi, M. and Capuzzo-Dolcetta, I., 2008. Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations. Springer Science and Business Media.

[10] Bardi, M. and Dragoni, F. 2011. Convexity and semiconvexity along vector fields. Calculus of Variations and Partial Differential Equations, 42(3-4), pp.405-427.

[11] Bardi, M. and Dragoni, F., 2010. Convexity along vector fields and applications to equations of Monge-Ampére type. In Progress in Analysis and Its Applications: Proceedings of the 7th International ISAAC Congress, Imperial College, London, UK, 13-18 July 2009 (p. 455). World Scientific.

[12] Bardi, M. and Dragoni, F., 2014. Subdifferential and properties of convex functions with respect to vector fields. Journal of Convex Analysis, 21(3), pp.785-810

[13] Barilari, D., Boscain, U. and Sigalotti, M., 2016. Geometry, Analysis and Dynamics on sub-Riemannian Manifolds.

[14] Barvinok, A., 2002. A course in convexity (Vol. 54). American Mathematical Soc..

[15] Baudoin, F. and Garofalo, N., 2016. Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries. Journal of the European Mathematical Society, 19(1), pp.151-219.

[16] Baudoin, F., 2004. An introduction to the geometry of stochastic flows. World Scientific.

[17] Beals, R., Gaveau, B. and Greiner, P.C., 1997. Complex Hamiltonian mechanics and parametrices for subelliptic Laplacians, I. Bulletin des sciences mathématiques, 121(1), pp.1-36.

[18] Beals, R., Gaveau, B. and Greiner, P.C., 2000. Hamilton–Jacobi theory and the heat kernel on Heisenberg groups. Journal de Mathématiques Pures et Appliquées, 79(7), pp.633-689.

[19] Beer, G., 1975. Starshaped sets and the Hausdorff metric. Pacific Journal of Mathematics, 61(1), pp.21-27.

[20] Bellaïche, A., 1997. The tangent space in sub-Riemannian geometry. Journal of Mathematical Sciences, 83(4), pp.461-476.

[21] Bellettini, C. and Donne, E.L., 2012. Regularity of sets with constant horizontal normal in the Engel group. arXiv preprint arXiv:1201.6399.

[22] Bonfiglioli, A., Lanconelli, E. and Uguzzoni, F., 2007. Stratified Lie groups and potential theory for their sub-Laplacians. Springer Science and Business Media.

[23] Boothby, W.M., 1986. An introduction to differentiable manifolds and Riemannian geometry (Vol. 120). Academic press.

[24] Brockett, R.W., 1982. Control theory and singular Riemannian geometry. In New directions in applied mathematics (pp. 11-27). Springer, New York, NY.

[25] Caffarelli, L.A. and Cabré, X., 1995. Fully nonlinear elliptic equations (Vol. 43). American Mathematical Soc..

[26] Calin, O., Chang, D. and Greiner, P., 2005. Geometric mechanics on the Heisenberg group. Bulletin-Institute of Mathematics Academia Sinica, 33(3), p.185.

[27] Cannarsa, P. and Rifford, L., 2008, July. Semiconcavity results for optimal control problems admitting no singular minimizing controls. In Annales de l'Institut Henri Poincare (C) Non Linear Analysis (Vol. 25, No. 4, pp. 773-802). Elsevier Masson.

[28] Cannarsa, P. and Sinestrari, C., 2004. Semiconcave functions, Hamilton-Jacobi equations, and optimal control (Vol. 58). Springer Science Business Media.

[29] Capogna, L., Danielli, D., Pauls, S.D. and Tyson, J., 2007. An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem (Vol. 259). Springer Science and Business Media.

[30] Capogna, L., Pauls, S. and Tyson, J., 2010. Convexity and horizontal second fundamental forms for hypersurfaces in Carnot groups. Transactions of the American Mathematical Society, 362(8), pp.4045-4062.

[31] Cassano, F.S., 2016. Some topics of geometric measure theory in Carnot groups. Dynamics, geometry and analysis on sub-Riemannian manifolds, 1.

[32] Chow, W.L., 1940. Über Systeme von linearer partiellen Differentialgleichungen erster Ordnung. Mathematische Annalen, 117(1), pp.98-105.

[33] Citti, G. and Sarti, A., 2006. A cortical based model of perceptual completion in the roto-translation space. Journal of Mathematical Imaging and Vision, 24(3), pp.307-326.

[34] Clarke, F., 2013. Functional analysis, calculus of variations and optimal control (Vol. 264). Springer Science and Business Media.

[35] Crandall, M.G., Ishii, H. and Lions, P.L., 1992. User's guide to viscosity solutions of second order partial differential equations. Bulletin of the American mathematical society, 27(1), pp.1-67.

[36] Dahl, G., 2010. An introduction to convexity. Retrieved from University of Oslo: http://heim. ifi. uio. no/ geird/conv. pdf.

[37] Danielli, D. and Garofalo, N., 2003. Green functions in nonlinear potential theory in Carnot groups and the geometry of their level sets. preprint.

[38] Danielli, D. and Garofalo, N., 1997. Geometric properties of solutions to subelliptic equations in nilpotent Lie groups. Lecture Notes in Pure and Applied Mathematics, pp.89-106.

[39] Danielli, D., Garofalo, N. and Nhieu, D.M., 2003. Notions of convexity in Carnot groups. Communications in Analysis and Geometry, 11(2), pp.263-342.

[40] Dattorro, J., 2010. Convex optimization and Euclidean distance geometry. Lulu. com.

[41] Diaz, J.I. and Kawohl, B., 1993. On convexity and starshapedness of level sets for some nonlinear elliptic and parabolic problems on convex rings. Journal of Mathematical Analysis and Applications, 177(1), pp.263-286.

[42] Do Carmo, M.P., 1992. Riemannian geometry. Mathematics: Theory and Applications. Birkhuser Boston. Inc., Boston, MA.

[43] Dolcetta, I.C. and Ishii, H., Hopf formulas for state dependent Hamilton-Jacobi equations. preprint.

[44] Dolcetta, I.C., 2003. Representations of solutions of Hamilton-Jacobi equations. In Nonlinear Equations: Methods, Models and Applications (pp. 79-90). Birkhäuser, Basel.

[45] Dragoni, F., Filali, D.. Starshaped sets in Carnot groups and general sub-Riemannian geometries. Submited paper 2018.

[46] Dragoni, F., Garofalo, N., Salani, P.. Starshapedness for level sets of non-linear subelliptic PDEs. To appear in Journal of London Mathematical Society.

[47] Dragoni, F., 2005. Carnot-Caratheodory metrics and viscosity solutions, PhD Thesis, Scuola Normale Superiore, Pisa, Italy.

[48] Duarte, P. and Torres, M.J., 2014. Smoothness of boundaries of regular sets. Journal of mathematical imaging and vision, 48(1), pp.106-113.

[49] Eggleston, H.G., 2007. Problems in Euclidean space: Application of convexity. Courier Corporation.

[50] Feleqi, E. and Rampazzo, F., 2017. Iterated Lie brackets for nonsmooth vector fields. Nonlinear Differential Equations and Applications NoDEA, 24(6), p.61.

[51] Giaquinta, M. and Modica, G., 2011. Mathematical analysis: foundations and advanced techniques for functions of several variables. Springer Science and Business Media.

[52] Gilmore, R., 2012. Lie groups, Lie algebras, and some of their applications. Courier Corporation.

[53] Gromov, M., 1996. Carnot-Carathéodory spaces seen from within. In Sub-Riemannian geometry (pp. 79-323). Birkhäuser Basel.

[54] Gursky, M., Ambrosetti, A., Chang, S. and Malchiodi, A. (2009). Geometric analysis and PDEs. 1st ed. Dordrecht: Springer.

[55] Gutiérrez, C.E. and Montanari, A., 2004. On the second order derivatives of convex functions on the Heisenberg group. In Ann. Scuola Norm. Sup. Pisa Cl. Sci.

[56] Hall, B., 2015. Lie groups, Lie algebras, and representations: an elementary introduction (Vol. 222). Springer.

[57] Hausner, M. and Schwartz, J.T., 1968. Lie groups, Lie algebras. CRC Press.

[58] Heinonen, J., 1995. Calculus on Carnot groups. Fall School in Analysis (Jyväskylä, 1994), 68, pp.1-31.

[59] Hermann, R., 1973. Geodesics of singular Riemannian metrics. Bulletin of the American Mathematical Society, 79(4), pp.780-782.

[60] Hörmander, L., 1967. Hypoelliptic second order differential equations. Acta Mathematica, 119(1), pp.147-171.

[61] Juutinen, P., Lu, G., Manfredi, J.J. and Stroffolini, B., 2007. Convex functions on Carnot groups. Revista Matemática Iberoamericana, 23(1), pp.191-200.

[62] Kishimoto, I., 2003. Geodesics and isometries of Carnot groups. Journal of Mathematics of Kyoto University, 43(3), pp.509-522.

[63] Korányi, A., 1983. Geometric aspects of analysis on the Heisenberg group. Topics in modern harmonic analysis, 1, pp.209-258.

[64] Korányi, A., 1985. Geometric properties of Heisenberg-type groups. Advances in Mathematics, 56(1), pp.28-38.

[65] Lang, S., 2006. Introduction to differentiable manifolds. Springer Science and Business Media.

[66] Lee, J.M., 2003. Introduction to Smooth Manifolds. Number 218 in Graduate texts in mathematics.

[67] Lee, J.M., 2006. Riemannian manifolds: an introduction to curvature (Vol. 176). Springer Science and Business Media.

[68] Lee, J.M., 2009. Manifolds and differential geometry (Vol. 107). Providence: American Mathematical Society.

[69] Liu, W. and Sussmann, H.J., 1995. Shortest paths for sub-Riemannian metrics on rank-two distributions (Vol. 564). American Mathematical Soc..

[70] Lu, G., Manfredi, J.J. and Stroffolini, B., 2003. Convex functions on the Heisenberg group. Calculus of Variations and Partial Differential Equations, 19(1), pp.1-22.

[71] Lukyanenko.net. 2018. Anton Lukyanenko. [online] Available at: http://lukyanenko.net/gallery.php?page=1.

[72] Magnani, V., 2006. Lipschitz continuity, Aleksandrov theorem and characterizations for H-convex functions. Mathematische Annalen, 334(1), pp.199-233.

[73] Montgomery, R., 2006. A tour of subriemannian geometries, their geodesics and applications (No. 91). American Mathematical Soc.

[74] Monti, R. and Rickly, M., 2005. Geodetically convex sets in the Heisenberg group. J. Convex Anal, 12(1), pp.187-196.

[75] Oberman, A. and Silvestre, L., 2011. The Dirichlet problem for the convex envelope. Transactions of the American Mathematical Society, 363(11), pp.5871-5886.

[76] Pascucci, A., 2011. PDE and martingale methods in option pricing. Springer Science Business Media.

[77] Rashevsky, P.K., 1938. Any two points of a totally nonholonomic space may be connected by an admissible line. Uch. Zap. Ped. Inst. im. Liebknechta, Ser. Phys. Math, 2, pp.83-94.

[78] Rickly, M., 2006. First-order regularity of convex functions on Carnot groups. Journal of geometric analysis, 16(4), pp.679-702.

[79] Rickly, Z.B.M., 2003. Regularity of convex functions on Heisenberg groups. Journal: Ann. Scuola Norm. Sup. Pisa, 2(5), pp.847-868.

[80] Rifford, L., 2014. Sub-Riemannian geometry and optimal transport. Springer.

[81] Rockafellar, R.T., 2015. Convex analysis. Princeton University Press.

[82] Rubinov, A.M. and Sharikov, E.V., 2006. Star-shaped separability with applications. Journal of Convex Analysis, 13(3/4), p.849-860.

[83] Rubinov, A.M. and Yagubov, A.A., 1986. The space of star-shaped sets and its applications in nonsmooth optimization. Quasidifferential Calculus, pp.176-202.

[84] Salani, P., 2005. Starshapedness of level sets of solutions to elliptic PDEs. Applicable Analysis, 84(12), pp.1185-1197.

[85] Traustason, G., 2011. Engel groups. Groups St Andrews 2009 in Bath, 2, pp.520-550.

[86] Valentine, F.A., 1965. Local convexity and starshaped sets. Israel Journal of Mathematics, 3(1), pp.39-42.

[87] Varadarajan, V.S., 2013. Lie groups, Lie algebras, and their representations (Vol. 102). Springer Science and Business Media.

[88] Vershik, A.M. and Gershkovich, V.Y., 1987. Nonholonomic dynamical systems. Geometry of distributions and variational problems. Itogi Nauki i Tekhniki. Seriya" Sovremennye Problemy Matematiki. Fundamental'nye Napravleniya", 16, pp.5-85.

[89] Wang, C., 2005. Viscosity convex functions on Carnot groups. Proceedings of the American Mathematical Society, pp.1247-1253.

[90] Warner, F.W., 2013. Foundations of differentiable manifolds and Lie groups (Vol. 94). Springer Science and Business Media.

[91] Wu, Q. and Wang, W., 2009. Conformal mappings and CR mappings on the Engel group. Science in China Series A: Mathematics, 52(12), p.2759.