Repeated Implementation: A Practical Characterization*

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Abstract

We characterize the social choice functions that are repeatedly implementable. The necessary and sufficient condition is formulated in terms of the equilibrium payoff set of an associated repeated game. It follows that the implementability of a function can be tested numerically by approximating the equilibrium payoff set. Additionally, with the help of our characterization, we demonstrate that an efficient function is implementable if and only if it satisfies a weaker version of Maskin monotonicity. As an application, we prove that utilitarian social choice functions are implementable by showing that continuation payoff promises effectively play the role of side-payments, which are needed for implementation in static setups.

Keywords: Repeated Implementation, Dynamic Monotonicity, Efficiency, Repeated Games, Sufficient and Necessary Conditions

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1 Introduction

Implementation theory studies objectives that society can achieve when its members behave strategically. Most of the literature in this field assumes that the implementation of an objective is a one-off event. (The seminal paper here is Maskin (1999).) However, the majority of interactions within society are repeated, with the objectives often remaining the same over time. The repeated nature of interactions can drastically change what can be implemented: objectives that are one-shot implementable might not be repeatedly implementable, and vice versa (see, for example, Examples 1 and 2 in Mezzetti and Renou (2017)). Our goal is to study what can be repeatedly implementable.

Specifically, we consider an infinite horizon problem when a new state of the world is realised in each period (i.i.d.). A social designer wants to select an outcome in each period, which depends on that period's state. However, the realised states are only observed by agents and never by the designer. Therefore, the designer must construct a sequence of mechanisms, referred to as a regime, that would elicit the state of the world from the agents and, at the same time, implement the desired outcome in each period.

Our objective is to characterize the social choice functions, that is, mappings from states of the world into outcomes, that are repeatedly implementable in Nash equilibrium. A function is repeatedly implementable if there exists a regime such that the set of Nash equilibria of the repeated game is non-empty and, for any sequence of realized states, the sequence of outcomes in any Nash equilibrium is such that in each period, the outcome coincides with the socially desired one.

This problem has been recently studied by Lee and Sabourian (2011) and Mezzetti and Renou (2017).[1] Lee and Sabourian (2011) show in their Theorem 1 that if a social choice function is not weakly efficient in its range, then the function is not repeatedly implementable for sufficiently high discount factors. They also show in their Theorem 2 that strict efficiency in the range, together with some additional assumptions on the preferences (their Assumption A and Condition ω), are sufficient for outcome implementation from period 2 onwards. In turn, Mezzetti and Renou (2017) show in their Theorem 1 that a complex set of dynamic incentive constraints, called dynamic monotonicity (DM), must hold if the function is repeatedly implementable. Furthermore, in their Theorem 2, they show that dynamic monotonicity plus no-veto power (or their Assumption A) are sufficient for repeated implementation both in finite and infinite horizon problems, irrespective of the magnitude of the discount factor.

While it is easy to check the efficiency of a function, there are many functions

[1] Repeated implementation has also been studied by Kalai and Ledyard (1998) and Chambers (2004), but their setup is different: the socially desired outcome is allowed to change over time and according to the state of the world but the latter is drawn only once and is kept fixed for all periods. Repeated implementation has also been studied under incomplete information by Renou and Tomala (2015) and Lee and Sabourian (2013).

that are repeatedly implementable for a fixed discount factor, but are not efficient, and vice versa. On the other hand, due to its inherent complexity, checking DM can be a daunting task. Moreover, there are important functions, which do not satisfy no-veto power but are repeatedly implementable. To summarize, up to now we do not have a complete characterization of the repeatedly implementable functions. Furthermore, we want a practical and systematic way to verify any necessary and sufficient conditions for repeated implementation.

We now describe our main contributions. First, in Theorem 1, we provide an alternative characterization to the necessary DM condition.[2] Namely, given the implementation environment, we construct an associated repeated game and show that a social choice function is DM if and only if: (a) it satisfies a simpler dynamic monotonicity condition, which we call Maskin monotonicity* (MM*); and (b) the associated repeated game has a *unique efficient* equilibrium payoff, which is equal to the value of the function. In some respects, this result provides a connection between the results of Lee and Sabourian (2011) and Mezzetti and Renou (2017), as we show that an appropriately defined efficiency of social choice function is a necessary condition for repeated implementation for all values of the discount factor.

MM* requires that whenever agents jointly lie about the state of the world in a certain period only, and then from the next period on they report honestly, there exists an agent who has incentives to deviate. Hence, MM* is described by only finitely many incentive constraints. It is also implied by both Maskin monotonicity and DM. At the same time, the equilibrium payoff set of the associated repeated game can be calculated using the dynamic programming technique that has been developed by Abreu, Pearce, and Stacchetti (1990). Therefore, the characterization of DM in Theorem 1 offers a practical and systematic way to test DM.

Second, we prove in Theorem 2 that DM is not only necessary but also sufficient for repeated implementation when there are at least three agents.[3] When proving sufficiency, we face the following problem: we want an agent to report when others lie about the state of the world, but we do not want the resulting outcome to be an equilibrium. To ensure this, we need to slightly perturb the outcomes that this agent can induce with his claim. However, because DM involves the infinity of incentive constraints (as the lie can take place over many periods), it is not obvious that such a perturbation is always possible without violating some of these incentive constraints. To show that it is always possible, we use the characterization of DM in Theorem 1, and the fact that the equilibrium

[2]Mezzetti and Renou (2017) state DM for both finite and infinite horizon problems. Our alternative characterization of DM only applies when the horizon is infinite.

[3]We assume that the designer can use random stage mechanisms. Hence, the result of Theorem 2 parallels that established by Bochet (2007) and Benoît and Ok (2008) for one-shot implementation; namely, that with random mechanisms, only monotonicity matters for implementation.

payoff correspondence of the associated repeated game is upper-semicontinuous. Intuitively, a small change in the outcomes that the dissenting agent can induce, does not affect the unique efficient equilibrium payoff of the repeated game.

Third, Theorems 1 and 2 have an immediate implication for the social choice functions that are efficient in the sense of Lee and Sabourian (2011). They imply that an efficient function is repeatedly implementable if and only if it is MM* (see our Proposition 1). This result improves on Theorem 2 of Lee and Sabourian (2011) since we replace their unnecessary domain restrictions with the necessary MM* condition.[4,5] We illustrate the practical importance of this result by applying it to utilitarian social choice functions, which are efficient, in the well-known economic environment of Laffont and Maskin (1982). These functions need not be Maskin monotonic, however we prove in Proposition 2 that they are MM* because the continuation payoffs effectively play the role of side-payments, which are usually needed for implementation in static setups. Furthermore, if all but one agent can have identical preferences, a utilitarian social choice function will typically not satisfy no-veto power. Consequently, one cannot apply the result of Mezzetti and Renou (2017) where no-veto power is assumed. This provides an example when closing the gap, between necessary and sufficient conditions, matters in practice.

Finally, we consider several extensions to Theorem 2. It has been established by assuming that the number of agents is at least three, that agents have strict preferences over alternatives, that the solution concept is pure-strategy Nash equilibrium and, most importantly, that the designer can use random stage mechanisms. We provide additional conditions under which our results extend to the case of two agents and improve upon the corresponding results of Lee and Sabourian (2011) and Mezzetti and Renou (2017). We discuss the case of weak preferences and also argue that our results extend to both mixed-strategy Nash implementation and pure-strategy subgame perfect implementation. We conjecture, however, that the necessary and sufficient condition for mixed-strategy subgame perfect implementation requires more than DM if the designer can only use public communication channels. We briefly discuss how the possibility of private communication can help to implement any function that is DM in mixed-strategy perfect Bayesian equilibrium. Finally, we show that DM is not sufficient for repeated implementation when the stage mechanisms are required to be deterministic, and we introduce an additional condition that together with DM is both necessary and sufficient for repeated implementation in this case. This additional condition is reminiscent of part (ii) of Condition μ in Moore and Repullo (1990).

[4] Note, however, Lee and Sabourian (2011) work with fully deterministic regimes, while we allow for stochastic transitions and stochastic stage mechanisms.

[5] It also improves on Remark 4 in Mezzetti and Renou (2017), which states that Maskin monotonicity and efficiency imply DM. Maskin monotonicity, however, is not implied by DM and, hence, unlike MM*, is not necessary for repeated implementation.

The rest of the paper is organized as follows. In Section 2, we introduce the model and basic notation. In Section 3, we provide the definitions of MM* and DM. In Section 4, we introduce the associated repeated game, state Theorem 1, and demonstrate through an elaborate example how the DM of a function can be checked numerically. In Section 5, we state Theorem 2 and provide a regime that implements any social choice function, which is DM. In Section 6, we relate our results to Lee and Sabourian (2011) and show that efficiency in the range and the necessary condition of MM* are sufficient for implementation. As an application, we next prove that the utilitarian social choice functions are repeatedly implementable in the environment of Laffont and Maskin (1982). In Section 7, we discuss various extensions to Theorem 2. The proofs of Theorems 1 and 2 can be found, respectively, in Sections A and B of the Appendix. In Section C of the Appendix, we study in detail the case when only deterministic stage mechanisms are allowed. Section D contains the regime for repeated implementation in mixed-strategies. Finally, the detailed analysis of the two agent case, as well as the analysis of repeated implementation of efficient functions from the second period onwards, can be found in the Supplementary Material.

2 The Model

2.1 Preferences

There is a finite set of agents, $I = \{1, 2, \dots, n\}$, a finite set of alternatives, A, a finite set of states of the world, Θ, and an infinity of periods, $T = \{0, 1, 2, \dots\}$.[6] In each period $t \in T$, a state of the world $\theta \in \Theta$ is independently and identically realized with probability $p(\theta)$. We assume that $p(\theta) > 0$ for all $\theta \in \Theta$. Let $\tilde{a} \in \Delta A$ denote a random alternative and let $\tilde{a}(a)$ denote the probability that the deterministic alternative $a \in A$ is selected. When we want to emphasize that the selected alternative depends on the state, we will write accordingly $\tilde{a}(\theta)$ and $\tilde{a}(\theta)(a)$. Throughout, we will use superscripts for variables to indicate a time period and subscripts to indicate an agent.

The preferences of the agents are represented by the discounting criterion. Given a sequence of random alternatives, $((\tilde{a}^\tau(\theta))_{\theta\in\Theta})_{\tau\in T}$, the period t continuation payoff of agent i before he has learnt the state of the world of that period, is given by:

$$v_i^t = (1-\delta)\sum_{\tau=t}^{\infty}\sum_{\theta\in\Theta}\sum_{a\in A}\delta^{\tau-t}p(\theta)\tilde{a}^\tau(\theta)(a)u_i(a,\theta).$$

Let $v^t = (v_1^t, \dots, v_n^t)$ be a continuation payoff profile in period t, and let V denote the set of feasible continuation payoff profiles. Note that the set V is

[6] All proofs can be modified to accommodate infinite A as well. In fact, in Section 6.1, we provide an application of our results, where $A = [0, 1]$.

convex and is the same for all t. We will write $u_i(\tilde{a}, \theta)$ for $\sum_{a \in A} \tilde{a}(a) u_i(a, \theta)$. Once agent i learns that the state of the world in period t is θ, his period t payoff is $(1 - \delta) u_i(\tilde{a}^t, \theta) + \delta v_i^{t+1}$ if random alternative $\tilde{a}^t$ is selected in that period and the continuation payoff is v_i^{t+1}.

Agent i's preferences over $A \times V$ in any state θ are completely described by $U_i(a, v, \theta) = (1 - \delta) u_i(a, \theta) + \delta v_i$. We assume that $U_i(\cdot, \cdot, \theta)$ is a Bernoulli utility function for all i, determining the preferences over $\Delta(A \times V)$ as expected utilities. Since U_i is linear in v for all i and V is convex, we can write $\Delta A \times V$ instead of $\Delta(A \times V)$. We also assume that in each state, agents have strict preferences over the set of non-random alternatives, i.e., $u_i(a, \theta) \neq u_i(b, \theta)$ for all i, θ, $a \in A$, and $b \in A$ such that $b \neq a$.[7] Finally, let $\overline{a}_i(\theta) = \arg\max_{a \in A} u_i(a, \theta)$, $\underline{a}_i(\theta) = \arg\min_{a \in A} u_i(a, \theta)$, $\overline{v}_i := \sum_{\theta \in \Theta} p(\theta) u_i(\overline{a}_i(\theta), \theta)$, and $\underline{v}_i := \sum_{\theta \in \Theta} p(\theta) u_i(\underline{a}_i(\theta), \theta)$.

2.1.1 "Preferences" of the Designer

A social choice function maps states of the world into alternatives, $f : \Theta \to A$. The objective of the designer is to select an alternative $f(\theta^t)$ in period t if the state of that period is θ^t. However, the designer never observes the realized state of the world, while all agents observe θ^t at the beginning of period t. Note that for simplicity, we only consider the implementation of deterministic, time independent choice functions.[8]

2.2 Repeated Implementation

2.2.1 Stage Mechanisms

Let Γ be a set of mechanisms or game forms. A mechanism $\gamma \in \Gamma$ is a pair $((\mathcal{M}_i)_{i \in I}, g)$ where $\mathcal{M}_i$ denotes a message space of agent i, and $g : \times_{i \in I} \mathcal{M}_i \to \Delta A$ is a stochastic allocation rule.[9] Let $\mathcal{M} = \times_{i \in I} \mathcal{M}_i$ be the space of message profiles. Let m_i and $m = (m_1, \ldots, m_n)$ be typical elements of $\mathcal{M}_i$ and $\mathcal{M}$, respectively. We write $g(m)(a)$ to denote the probability that the mechanism selects deterministic alternative a when the messages are m.

2.2.2 Histories

The designer chooses, possibly randomly, the current period's mechanism. A state of the world is realized. All agents are informed about the state and the selected

[7] We will discuss in Section 7.1 how our results extend to the case of weak preferences.

[8] The case of social choice correspondences can be dealt with as in Lee and Sabourian (2011) and Mezzetti and Renou (2017). By appropriately redefining the set of alternatives, the analysis also covers stochastic choice functions. At the cost of additional notation, the analysis can also be extended to time-dependent choice functions; see the discussion in Mezzetti and Renou (2017).

[9] We discuss the case when the mechanisms are restricted to be deterministic, in Section 7.2 and in Section C of Appendix.

mechanism. That is, even if the mechanism is chosen randomly, the agents are informed about the realized mechanism. The agents send public, simultaneous messages to the designer.[10] Given these messages, the designer implements a (possibly random) alternative according to the allocation rule of the selected mechanism. Then the process is repeated in the next period, and so on.

Let period 0 history be $h^0 = \emptyset$. The history that is observed by all agents in the beginning of period $t > 0$ is $h^t = (\theta^0, \gamma^0, m^0, a^0, \ldots, \theta^{t-1}, \gamma^{t-1}, m^{t-1}, a^{t-1})$, where $\theta^\tau \in \Theta$, $\gamma^\tau = (\mathcal{M}^\tau, g^\tau) \in \Gamma$, $m^\tau \in \mathcal{M}^\tau$, and $a^\tau \in A$ is the realization of $g^\tau(m^\tau)$. Hence, the period t history does not contain the period t state of the world and the mechanism which will be used by the designer in that period. Let H^t be the space of all possible period t histories that are observed by the agents, with $H^0 = \{\emptyset\}$. The space of all possible histories is $H = \cup_{t=0}^{\infty} H^t$. The designer cannot distinguish between any two period t histories that only differ in the realized states of the world.

2.2.3 Regimes

A *dynamic mechanism regime* or regime for short is a transition rule $r : H \to \Delta\Gamma$,[11] where $r(\gamma|h^t)$ denotes the probability that mechanism γ is selected after history h^t. Note that $r(\gamma|h^t) = r(\gamma|\tilde{h}^t)$ if the designer cannot distinguish between histories h^t and $\tilde{h}^t$. We assume that the designer commits to the chosen regime and that the agents are informed about this regime.

2.2.4 Strategies and Payoffs

Fix a regime r. In period t, after the state θ^t is realized, the agents learn θ^t and the mechanism γ^t, which will be used in period t by the designer. The randomness of $r(\gamma|h^t)$ is resolved before agents send their messages. Hence, a pure strategy s_i of agent i selects a message $s_i(h^t, \theta^t, \gamma^t) \in \mathcal{M}_i^t$ for each $t \in T$ and each $(h^t, \theta^t, \gamma^t) \in H^t \times \Theta \times \Gamma$. Let $s = (s_1, \ldots, s_n)$ be a profile of strategies.

The strategy profile s and the regime r together with the distribution of states of the world p induce a distribution over histories. Let $q(h^t|s, r)$ denote the probability that history h^t is realized given s and r. Define $q(h^0|s, r) = 1$. Given $q(h^t|s, r)$, $q((h^t, \theta^t, \gamma^t, m^t, a^t)|s, r) = q(h^t|s, r)p(\theta^t)r(\gamma^t|h^t)g^t(m^t)(a^t)$ if $s(h^t, \theta^t, \gamma^t) = m^t$ and $q((h^t, \theta^t, \gamma^t, m^t)|s, r) = 0$ otherwise. Given s and r, the

[10] We briefly discuss the case of private messages in Section 7.3.3.

[11] We can assume that Γ is finite. When deriving the necessary condition, we do not impose any restrictions on Γ and allow the designer to choose anything from $\Delta A \times V$. The only constraint that the designer faces is that he does not know states of the world. Clearly, this necessary condition remains necessary when we restrict the designer in some way. On the other hand, when proving our sufficiency result, we will construct a regime that only employs a finite number of mechanisms. Therefore, it follows that any social choice function that can be implemented with infinite Γ, can also be implemented with finite Γ.

payoff of agent i is:

$$v_i(s|r) = (1-\delta)\sum_{t\in T}\sum_{h^t\in H^t}\sum_{\theta^t\in\Theta}\sum_{\gamma^t\in\Gamma}\delta^t q(h^t|s,r)p(\theta^t)r(\gamma^t|h^t)u_i(g^t(s(h^t,\theta^t,\gamma^t)),\theta^t).$$

2.2.5 Repeated Implementation in Nash Equilibrium

A profile of strategies s is a Nash equilibrium if $v_i(s|r) \geq v_i((s_i', s_{-i})|r)$ for all i and s_i'. A regime r *repeatedly implements* a social choice function f if the set of Nash equilibria is non-empty and for each Nash equilibrium s, we have that $g^t(s(h^t,\theta^t,\gamma^t))(f(\theta^t)) = 1$ for all $t \in T$, $\theta^t \in \Theta$, $h^t \in H$ and $\gamma^t \in \Gamma$ such that $q(h^t|s,r)r(\gamma^t|h^t) > 0$. A social choice function f is *repeatedly implementable* in Nash equilibrium if there exists a regime r that repeatedly implements f.[12] The payoff of agent i if f is repeatedly implemented is $v_i^f := \sum_{\theta\in\Theta} p(\theta)u_i(f(\theta),\theta)$. Let $v^f = (v_1^f,\ldots,v_n^f)$.

To summarize, the setup is exactly the same as in Mezzetti and Renou (2017), except that we allow the stage mechanisms to be stochastic.

3 Definitions of Monotonicity

In this section, we define several notions of monotonicity of f. For an arbitrary set X, let $L_i(x,\theta)$ be the *lower contour set* of agent i at outcome $x \in X$ in state θ, consisting of those outcomes in X that agent i considers weakly worse than outcome x in state θ. The set X can be a subset of $\Delta A \times V$, in which case the agent's preferences are described by $U_i(\cdot,\cdot,\theta)$, or X can be a subset of ΔA, in which case the agent's preferences are described by $u_i(\cdot,\theta)$. To simplify notation, from now on, we denote random (and deterministic) alternatives as a instead of $\tilde{a}$.

A necessary condition for a function to be one-shot implementable is Maskin monotonicity due to Maskin (1999). We present it in a slightly modified form, which allows it to be conveniently compared with other notions of monotonicity that will be defined later.

Definition 1 (Maskin monotonicity)**.** *f satisfies* Maskin monotonicity *with respect to $C = (C_i(\theta))_{i,\theta}$ if for each i and θ, $C_i(\theta) \subseteq L_i(f(\theta),\theta)$ and for all pairs (θ,θ^*), we have that (a) implies (b):*

a. $C_i(\theta) \subseteq L_i(f(\theta),\theta^*)$ holds for all i,

b. $f(\theta) = f(\theta^*)$.

[12] In Section 7.3, we also discuss repeated implementation in pure-strategy subgame perfect Nash equilibrium, mixed-strategy Nash equilibrium, and perfect Bayesian equilibrium, when private messages are also allowed.

Remark 1. *f is Maskin monotonic w.r.t. some C (i.e., there exists such C) if and only if f is Maskin monotonic w.r.t. $C = (L_i(f(\theta), \theta))_{i,\theta}$, which gives Maskin's original definition (Maskin, 1999).*[13]

The reason why Maskin monotonicity is necessary for one-shot implementation is that if all agents act in state θ^* as if the state is θ and no agent has incentives to upset such a deception, then f will not be implemented unless $f(\theta) = f(\theta^*)$. Lee and Sabourian (2011) show that Maskin monotonicity is neither necessary nor sufficient for repeated implementation. One of the reasons is that the notion of deception becomes more complicated as it can take place over many periods. Mezzetti and Renou (2017) provide the right definition of monotonicity, which is necessary for repeated implementation. In the next subsections, we introduce the notion of (dynamic) deception and a slightly modified definition of dynamic monotonicity, which was originally proposed by Mezzetti and Renou (2017).

3.1 Deceptions

We consider the following class of deceptions. Suppose that $t \geq 1$. Let $\theta^{\to t} = (\theta^0, \ldots, \theta^{t-1})$ and $\Theta^{\to t}$ be the set of all such sequences. Let $\pi^t : \Theta \times \Theta^{\to t} \to \Theta$ be a deception in period t. That is, π^t specifies a state $\theta' = \pi^t(\theta, \theta^{\to t})$ after any $\theta^{\to t}$ given that the period t state is θ.[14] One can think of it as if after $\theta^{\to t}$ all the agents pretend that the period t state of the world is θ' while in fact it is θ. Let $\pi = (\pi^t)_{t \geq 0}$ be a *deception*, where $\pi^0 : \Theta \to \Theta$. We will refer to π^0 as a static deception. Also, let $\theta^{\to 0} = \emptyset$ and $\Theta^{\to 0} = \{\emptyset\}$. The payoff of agent i, when the agents deceive according to π and the designer selects an alternative according to f, is:

$$v_i^f(\pi) = (1-\delta) \sum_{t \in T} \sum_{\theta^{\to t} \in \Theta^{\to t}} \sum_{\theta^t \in \Theta} \delta^t p(\theta^{\to t}) p(\theta^t) u_i(f(\pi^t(\theta^t, \theta^{\to t}), \theta^t),$$

where $p(\theta^{\to t}) = p(\theta^0) \cdot \ldots \cdot p(\theta^{t-1})$ and $p(\theta^{\to 0}) = 1$. With some abuse of notation, we will denote the payoff of agent i by $v_i^f(\pi^0)$ if the agents use a stationary deception that is obtained by applying a static deception π^0 in every period. Finally, given a deception π and some $\theta^{\to t}$ and θ, we denote the continuation

[13]See, for example, the discussion in Moore and Repullo (1990) on page 1089.

[14]In principle, the agents could condition their deceptions on anything that they observe, namely, on the entire past history. We assume for simplicity that the agents do not have access to any exogenous public randomization device. If they had access to such a device, then we should consider a larger class of deceptions where the current period deception also depends on the past realizations of this device (with the consequence that strictly less functions can be implemented). Note that the agents could use the randomization of the regime itself to substitute for such a device. However, the regimes that we construct, do not involve any randomization by the designer on the equilibrium path. Therefore, it is sufficient if we only consider deceptions, which are functions of the realized current and past states.

deception after period t by $\pi(\theta, \theta^{\to t})$, which is derived in the obvious way from π.[15] The period $t+1$ continuation payoff of agent i, corresponding to $\pi(\theta, \theta^{\to t})$, is $v_i^f(\pi(\theta, \theta^{\to t}))$.

3.2 Dynamic Monotonicity

To gain better intuition, we start by introducing a weaker version of DM, which is also implied by Maskin monotonicity, and that plays a useful role on its own in the sequel.

Definition 2 (Maskin monotonicity*)**.** *f satisfies* Maskin monotonicity* *with respect to $C = (C_i(\theta))_{i,\theta}$ if for each i and θ, $C_i(\theta) \subseteq L_i((f(\theta), v^f), \theta)$ and for all pairs (θ, θ^*), we have that (a) implies (b):*

a. $C_i(\theta) \subseteq L_i((f(\theta), v^f), \theta^*)$ holds for all i,

b. $f(\theta) = f(\theta^*)$.

Remark 2. *f is Maskin monotonic* (MM*) w.r.t. some C if and only if f is MM* w.r.t. $C = (L_i((f(\theta), v^f), \theta))_{i,\theta}$. Therefore, we will sometimes suppress the sets w.r.t. which f is MM*. Also, f is MM* if and only if the function $(f(\cdot), v^f)$ is Maskin monotonic. Hence, MM* is implied by Maskin monotonicity of f, but the converse is not true.*

Remark 3. *MM* is necessary for repeated implementation (as argued below).*

One can think of the necessity of MM* as follows. Suppose that there is a regime, which repeatedly implements f. Fix a Nash equilibrium of this regime. Let $C = (C_i(\theta))_{i,\theta}$ be defined as follows. For all i and θ, let $C_i(\theta)$ represent the set of alternative and continuation payoff pairs that agent i can obtain by deviating from the equilibrium play in the initial period when the state is θ, given that all other agents play according to the fixed Nash equilibrium. Now, by contradiction, suppose that f does not satisfy Definition 2 for this C, that is, part (a) of Definition 2 holds for a pair (θ, θ^*) but we have that $f(\theta^*) \neq f(\theta)$. Consider now the following simple deception: in the initial period, and only in this period, all agents pretend that the state is θ when the true state is actually θ^*, and they continue to play the Nash equilibrium strategies in the following periods as if they had not pretended in period 0. Given these new strategies, the alternative $f(\theta)$ is implemented in period 0 if the state is θ^* and from the next period on, the agents expect v^f since they play the original Nash equilibrium. It is another Nash equilibrium: no agent has incentives to deviate in period 0 because, according to part (a) of Definition 2, each agent i prefers what he gets from the deception, to anything in $C_i(\theta)$. No agent also has incentives to deviate

[15] Note the distinction between a deception in period t, $\pi^t(\theta, \theta^{\to t})$ and a deception from period $t+1$ *onwards*, $\pi(\theta, \theta^{\to t})$.

in the following periods since, by construction, they follow the original Nash equilibrium strategies from period 1 on. Hence, f is not repeatedly implemented.

MM* is, however, far from being a sufficient condition. The period 0 deception described above might be maintained by promising something better than v^f from period 1 on, which in turn can be obtained through future deceptions which support themselves. Hence the following definition:

Definition 3 (Dynamic Monotonicity)**.** *f is* dynamically monotonic *with respect to $C = (C_i(\theta))_{i,\theta}$ if for each i and θ, we have $C_i(\theta) \subseteq L_i((f(\theta), v^f), \theta)$ and for any deception π, we have that (a) implies (b):*

a. $C_i(\theta) \subseteq L_i((f(\theta), v^f(\pi(\theta^*, \theta^{\to t}))), \theta^*)$ holds for all $i \in I$, all $t \in T$, all $\theta^{\to t} \in \Theta^{\to t}$, and all pairs $(\theta, \theta^*) \in \Theta \times \Theta$ for which $\pi^t(\theta^*, \theta^{\to t}) = \theta$,

b. $f(\pi^t(\cdot, \theta^{\to t})) = f(\cdot)$ holds for all $t \in T$ and all $\theta^{\to t} \in \Theta^{\to t}$.

Definition 3 says that if there exists a deception π such that $f(\pi^{t'}(\theta^{t'}, \theta^{\to t'})) \neq f(\theta^{t'})$ for some $t', \theta^{t'}, \theta^{\to t'}$, then there must exist $i, t, \theta^*, \theta^{\to t}, a, v$ such that $(a, v) \in C_i(\theta)$ but $(a, v) \notin L_i((f(\theta), v^f(\pi(\theta^*, \theta^{\to t}))), \theta^*)$ where $\theta = \pi^t(\theta^*, \theta^{\to t})$. Thus, agent i can be thought of as a whistle-blower who informs the designer about the ongoing deception.

Remark 4. *f is dynamically monotonic (DM) w.r.t. some C if and only if f is DM w.r.t. $C = (L_i((f(\theta), v^f), \theta))_{i,\theta}$, which is the definition of DM in Mezzetti and Renou (2017). For this reason, we sometimes suppress the collection w.r.t. which f is DM. Obviously, if f is DM, then it is MM*.*

The reason why, in general, we allow C to differ from $(L_i((f(\theta), v^f), \theta))_{i,\theta}$ is because later, when we show that the DM of f is sufficient for repeated implementation of f, we want to restrict the alternative-continuation payoff pairs a whistle-blower can demand when deviating from a deception. This permits us to rule out certain undesirable equilibria in the game induced by a regime.

We finish this section with an example where we verify DM and MM* of a social choice function. We will return to this example again in the next section.

Example 1

Let $I = \{1, 2\}$, $A = \{a, b, c\}$, $\Theta = \{\theta, \theta'\}$, and $p(\theta) = \frac{1}{2}$. The payoffs of the agents are summarized in the following table:

	$u_1(\cdot, \theta)$	$u_2(\cdot, \theta)$	$u_1(\cdot, \theta')$	$u_2(\cdot, \theta')$
a	12	12	10	18
b	18	10	12	12
c	10	11	11	10

Suppose $f(\theta) = a$ and $f(\theta') = b$. Thus, $v^f = (12, 12)$. Let C be defined as follows: $C_1(\theta) = \{(c, 13, 13)\}$, $C_1(\theta') = \{(b, 12, 12)\}$, $C_2(\theta) = \{(a, 12, 12)\}$, and

$C_2(\theta') = \{(c, 13, 13)\}$. (Each agent obtains a payoff of 13 if alternatives a and b are selected with equal probabilities in every period.)

We verify for what values of δ, f is DM w.r.t. C. Since the problem is symmetric, we focus on agent 1. First, we must ensure that $(c, v) \equiv (c, 13, 13) \in L_1((f(\theta), v^f), \theta)$ or, equivalently:

$$(1-\delta)u_1(c,\theta) + \delta v_1 \leq (1-\delta)u_1(a,\theta) + \delta v_1^f,$$

which is satisfied for $\delta \leq \frac{2}{3}$.

Second, we must check if we can eliminate a stationary deception that is obtained by applying a static deception $\pi^0(\theta) = \theta'$ and $\pi^0(\theta') = \theta$ in each period, since it results in the highest joint payoffs $v^f(\pi^0) = (14, 14)$. If we can eliminate this deception, then we can also eliminate any non-stationary deception, where the agents sometimes report the true state, since such a deception would result in a lower payoff for at least one agent. The stationary deception will be eliminated if $(c, v) \notin L_1((f(\theta), v^f(\pi^0)), \theta')$ or, equivalently:

$$(1-\delta)u_1(c,\theta') + \delta v_1 > (1-\delta)u_1(a,\theta') + \delta v_1^f(\pi^0),$$

which says that agent 1 has incentives to demand (c, v) when the other agent claims that the state is θ, although it is actually θ'. This inequality is satisfied for $\delta < \frac{1}{2}$. By symmetry, when agent 1 claims that the state is θ', although it is actually θ, agent 2 has the incentive to demand (c, v) when $\delta < \frac{1}{2}$. Thus, f is DM w.r.t. C for $\delta < \frac{1}{2}$. To check the MM* of f, we only need to replace $v_1^f(\pi^0)$ with v_1^f in the last inequality. In this case, the inequality is satisfied for all δ. Hence, it follows that f is MM* w.r.t. C for $\delta \leq \frac{2}{3}$.[16] ■

4 Alternative Representation of Dynamic Monotonicity

Given the repeated implementation environment as described in Section 2, we now construct an associated repeated game with discounting, perfect monitoring, and random states. We will use this game to characterize the DM of f in terms of its equilibrium payoff set. Specifically, we will show in Theorem 1 that f is DM if and only if it is MM* and v^f is the unique efficient equilibrium payoff vector of the repeated game. This characterization has several advantages.

First, it offers a practical and systematic way to check the DM of f. Unlike DM, to verify the MM* of f only requires checking finitely many inequalities,

[16]By appropriately modifying C, f can be made DM for $\delta < \frac{3}{5}$ and MM* for all δ. The reason why C depends on δ is because the lower contour sets depend on it and we must ensure that (1) $C_1(\theta) \subseteq L_1((f(\theta), v^f), \theta)$ and (2) $C_1(\theta) \not\subseteq L_1((f(\theta), v^f(\pi^0)), \theta')$ for DM or $C_1(\theta) \not\subseteq L_1((f(\theta), v^f), \theta')$ for MM*.

while the equilibrium payoff set of the repeated game can be approximated numerically.

Second, with the help of our characterization, we will show in Lemma 1 in Section 5 that it is always possible to modify the collection of sets $(C_i(\theta))_{i,\theta}$ while preserving the DM of f w.r.t. the new, modified collection. This, in turn, will allow us to prove Theorem 2, namely, that DM is not only necessary, but also a sufficient condition for f to be repeatedly implementable when there are at least three agents.

Finally, the characterization almost immediately implies that DM is equivalent to MM* if f is an efficient function; one only needs to show that the efficient equilibrium payoffs of the repeated game are unique. This will be done in Proposition 1 in Section 6.

We now proceed with defining the repeated game. Without loss of generality, we can assume that $u_i(a,\theta) > 0$ for all i, a, and θ, as well as $(1-\delta)u_i(f(\theta),\theta) + \delta v_i^f \geq \delta \overline{v}_i$ for all i and θ in the implementation problem.[17] Let $C = (C_i(\theta))_{i,\theta}$ be an arbitrary collection of sets such that $C_i(\theta)$ is a nonempty and closed subset of $L_i((f(\theta), v^f), \theta)$ for all i and θ. Let $M_i(C_i(\theta), \theta^*) = \arg\max_{(a,v)\in C_i(\theta)} U_i(a, v, \theta^*)$. Given C, for any pair of states (θ, θ^*), let $\mu_i(\theta,\theta^*) := U_i(a, v, \theta^*)/(1-\delta) = u_i(a, \theta^*) + \delta v_i/(1-\delta)$ for some $(a, v) \in M_i(C_i(\theta), \theta^*)$. Note that $\mu_i(\theta, \theta^*)$ is the same for all $(a, v) \in M_i(C_i(\theta), \theta^*)$. In particular, $\mu_i(\theta,\theta) \leq U_i(f(\theta), v^f, \theta)/(1-\delta)$.

The repeated game G^C is as follows. A state $\theta \in \Theta$ is drawn each period i.i.d. according to p, each player i corresponds to agent i in the implementation problem, and the action sets of the stage game are $\mathcal{A}_i = \mathcal{A} = \Theta \cup \{\omega, o\}$. Then, for an action profile $x \in \mathcal{A}^n$ in state θ^*, the stage game payoffs of the players are defined as follows:

1. $\nu_i(x, \theta^*) = u_i(f(\theta), \theta^*)$ if $x_j = \theta$ for all j. (In the implementation problem, it corresponds to the situation when everyone claims that the state is θ, while the true state is θ^*.)

2. $\nu_i(x, \theta^*) = \mu_i(\theta, \theta^*)$ if $x_j = \theta$ for all $j \in I\backslash\{i\}$ and $x_i = \omega$. (It corresponds to the situation when all but agent i claim that the state is θ but the true state is θ^*, while agent i deviates by demanding an element in $M_i(C_i(\theta), \theta^*)$.)

3. $\nu_i(x, \theta^*) = 0$ if $x_i = o$.

4. $\nu_i(x, \theta^*) \ll 0$ for any other $x \in \mathcal{A}^n$, which does not fall under any of the above points. (When we write that a payoff is $\ll 0$, we mean that it is so negative that the corresponding action profile can never be played on a Nash equilibrium path of the repeated game.) Note that $\nu_i(x, \theta^*) \ll 0$ if $x_j = o$ for some $j \neq i$, but $x_i \neq o$.

[17] If the latter inequality is not satisfied, we can add a constant ρ to all payoffs $(u_i(a,\theta))_{i,a,\theta}$ such that $\rho \geq (\delta\overline{v}_i - ((1-\delta)u_i(f(\theta),\theta) + \delta v_i^f))/(1-\delta)$ holds for all i and θ.

The following example illustrates how to calculate the stage game payoffs of the associated repeated game.

Example 1 continued

Since the problem is symmetric, we only show how to calculate the payoffs of player 1 in the repeated game. Thus:

$$
\begin{aligned}
\mu_1(\theta,\theta) &= \frac{U_1((c,13,13),\theta)}{1-\delta} = 10+\frac{13\delta}{1-\delta}, \\
\mu_1(\theta',\theta) &= \frac{U_1((b,12,12),\theta)}{1-\delta} = 18+\frac{12\delta}{1-\delta}, \\
\mu_1(\theta',\theta') &= \frac{U_1((b,12,12),\theta')}{1-\delta} = \frac{12}{1-\delta}, \\
\mu_1(\theta,\theta') &= \frac{U_1((c,13,13),\theta')}{1-\delta} = 11+\frac{13\delta}{1-\delta}.
\end{aligned}
$$

The stage game payoffs for each of the two states are given in the following matrices. Since $\overline{v}=(15,15)$, we have set $\nu_1(x,\theta^*) = -\frac{15\delta}{1-\delta}$ for all $\theta^*\in\Theta$ if x falls under point 4 in the above definition of the stage game payoffs.

Player 2 (columns); Player 1 (rows)

$\nu(\cdot,\theta)$	θ	θ'	ω	o
θ	$12,12$	$-\frac{15\delta}{1-\delta},-\frac{15\delta}{1-\delta}$	$-\frac{15\delta}{1-\delta},\frac{12}{1-\delta}$	$-\frac{15\delta}{1-\delta},0$
θ'	$-\frac{15\delta}{1-\delta},-\frac{15\delta}{1-\delta}$	$18,10$	$-\frac{15\delta}{1-\delta},11+\frac{13\delta}{1-\delta}$	$-\frac{15\delta}{1-\delta},0$
ω	$10+\frac{13\delta}{1-\delta},-\frac{15\delta}{1-\delta}$	$18+\frac{12\delta}{1-\delta},-\frac{15\delta}{1-\delta}$	$-\frac{15\delta}{1-\delta},-\frac{15\delta}{1-\delta}$	$-\frac{15\delta}{1-\delta},0$
o	$0,-\frac{15\delta}{1-\delta}$	$0,-\frac{15\delta}{1-\delta}$	$0,-\frac{15\delta}{1-\delta}$	$0,0$

Player 2 (columns); Player 1 (rows)

$\nu(\cdot,\theta')$	θ	θ'	ω	o
θ	$10,18$	$-\frac{15\delta}{1-\delta},-\frac{15\delta}{1-\delta}$	$-\frac{15\delta}{1-\delta},18+\frac{12\delta}{1-\delta}$	$-\frac{15\delta}{1-\delta},0$
θ'	$-\frac{15\delta}{1-\delta},-\frac{15\delta}{1-\delta}$	$12,12$	$-\frac{15\delta}{1-\delta},10+\frac{13\delta}{1-\delta}$	$-\frac{15\delta}{1-\delta},0$
ω	$11+\frac{13\delta}{1-\delta},-\frac{15\delta}{1-\delta}$	$\frac{12}{1-\delta},-\frac{15\delta}{1-\delta}$	$-\frac{15\delta}{1-\delta},-\frac{15\delta}{1-\delta}$	$-\frac{15\delta}{1-\delta},0$
o	$0,-\frac{15\delta}{1-\delta}$	$0,-\frac{15\delta}{1-\delta}$	$0,-\frac{15\delta}{1-\delta}$	$0,0$

■

Given a sequence of actions, $((x^t(\theta))_{\theta\in\Theta})_{t\in T}$, the payoff of player i in the repeated game is given by $(1-\delta)\sum_{t\in T}\sum_{\theta\in\Theta}\delta^t p(\theta)\nu_i(x^t(\theta),\theta)$. Let $\mathcal{E}(G^C)$ be the set of pure-strategy subgame perfect Nash equilibrium (SPNE) payoffs of the repeated game G^C, which in our case coincides with the set of Nash equilibrium payoffs of G^C because everyone playing o is a Nash equilibrium of the stage game that results in the minmax payoffs. We say that the equilibrium payoff vector v

is *efficient* (resp., *weakly efficient*) if there is no $w \in \mathcal{E}(G^C)\backslash\{v\}$ such that $w \geq v$ (resp., $w > v$).[18]

The link between the DM of f and the equilibrium payoff set of the repeated game is described in the following theorem.

Theorem 1. *Let $C = (C_i(\theta))_{i,\theta}$ be a collection of non-empty and closed sets with $C_i(\theta) \subseteq L_i((f(\theta), v^f), \theta)$ for all i and θ. f is dynamically monotonic w.r.t. C if and only if:*

1. *f is Maskin monotonic* w.r.t. C,*
2. *v^f is the unique efficient equilibrium payoff vector of G^C.*[19]

The proof is available in Section A of the Appendix.

The theorem says that if f is DM, then v^f is the unique efficient equilibrium payoff vector of G^C. The intuition is simple. Every deception π corresponds to a certain play in the repeated game. When f is DM, there exists a whistle-blower for every deception that results in payoffs different from v^f. By construction of the repeated game, the corresponding play in the repeated game cannot be supported as an equilibrium outcome. The only reason why v^f is not the unique equilibrium payoff vector of G^C is because there are also equilibria in which the players choose o on the equilibrium path. Conversely, if v^f is the unique efficient equilibrium payoff vector of G^C, then the only deceptions that we need to consider when checking the DM of f, are of the type as described after Remark 3, and they are taken care of by MM*.

4.1 Testing Dynamic Monotonicity

The definition of DM says that to check whether or not f is DM may require considering infinitely many deceptions, which is impossible (unless one is lucky to find a deception, which does not satisfy the condition required for DM). The result of Theorem 1, however, offers a systematic and practical way to check the DM of f. First, verifying the MM* of f is relatively easy as one has to check only finitely many static deceptions. Second, one can find the equilibrium payoff set of the associated repeated game using the method that has been pioneered by Abreu, Pearce, and Stacchetti (1990), and that has been applied to a class of stochastic games by Kitti (2016) and Abreu, Brooks, and Sannikov (2016), of which our repeated game G^C is a special case.[20]

[18]Notation $w \geq v$ means $w_i \geq v_i$ for all i and $w_i > v_i$ for at least one i, while $w > v$ means $w_i > v_i$ for all i.

[19]In fact, if f is DM w.r.t. C, then v^f is also the unique weakly efficient equilibrium payoff vector of G^C.

[20]Abreu, Pearce, and Stacchetti (1990) consider repeated games with imperfect monitoring $((S_i)_{i\in I}, P, \Omega, \Psi, (\Pi_i)_{i\in I})$. If we allowed for public randomization, then G^C could also be rein-

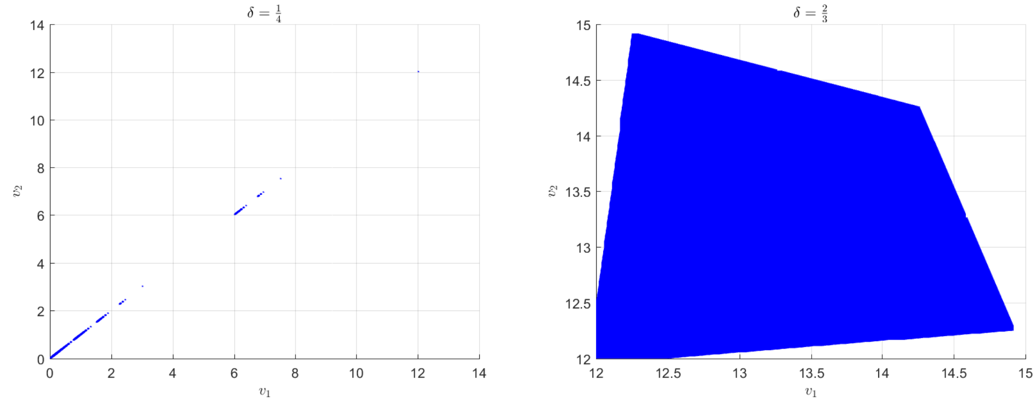

Figure 1: Equilibrium payoffs of the repeated game

Example 1 continued

Figure 1 illustrates equilibrium payoffs for the repeated game when $\delta = \frac{1}{4}$ and $\delta = \frac{2}{3}$, respectively.[21] The figure shows that $v^f = (12, 12)$ is the unique efficient equilibrium payoff vector of $\mathcal{E}(G^C)$ when $\delta = \frac{1}{4}$, which is consistent with our earlier finding that f is DM w.r.t. C for $\delta < \frac{1}{2}$. To better illustrate the frontier of efficient equilibrium payoffs when $\delta = \frac{2}{3}$, we only present the equilibrium payoffs that exceed $v^f = (12, 12)$. Although the approximation of the frontier of efficient equilibrium payoffs is not tight ((14, 14) should be on the frontier), v^f is clearly not an efficient equilibrium payoff vector, which is consistent with the earlier finding that f is not DM w.r.t. C for $\delta \geq \frac{1}{2}$. ■

terpreted as a repeated game with imperfect monitoring. Consequently, the results of Abreu, Pearce, and Stacchetti (1990) would directly apply. Identify the action set S_i of player i with $\mathcal{A}^\Theta$ and let $S = S_1 \times \cdots \times S_n$. P is a public signal, which takes values in $\Omega = \Theta \times \mathcal{A}^n \times [0,1]^{|\Theta|}$ and its distribution $\Psi(\cdot; q)$ for a given action profile $q = (q_1, \ldots, q_n) \in S$ can be calculated as follows. The first coordinate θ is the current period state of the world in G^C and it is drawn from Θ with probability $p(\theta)$. The second coordinate $x \in \mathcal{A}^n$ is the action profile in G^C, which is determined by $q : \Theta \to \mathcal{A}^n$ and by the current state of the world θ as $x = q(\theta)$. The last $|\Theta|$ coordinates serve for the public randomization device of the next period for each possible state of the next period and are drawn i.i.d. according to the uniform distribution on $[0, 1]$. Finally, current period payoffs are $\Pi_i(q) = \sum_{\theta \in \Theta} p(\theta) u_i(q(\theta), \theta)$.

[21] Equilibrium payoffs are computed using a modified version of the Octave code that has been kindly provided to us by Mitri Kitti. The code is based on Kitti (2016), where, similar to our model, it is assumed that public randomization is not available to the players. Judd, Yeltekin, and Conklin (2003), Abreu and Sannikov (2014), and Abreu, Brooks, and Sannikov (2016) have developed numerical methods that give better approximation of the equilibrium payoff sets in different repeated games, but only the last paper considers stochastic repeated games while all these papers assume the existence of a public randomization device.

5 Characterization of Repeatedly Implementable Functions

In this section, we prove that DM is not only necessary, but also sufficient for a function to be repeatedly implementable whenever there are at least three agents. The regime that we use to prove the sufficiency result is a modification of the regime in Mezzetti and Renou (2017). In their regime, if there has been no disagreement in the agents' messages in the past, they face a mechanism that is similar to the canonical mechanism that is used in the one-shot implementation. If the agents' messages ever differ, then one of the agents randomly becomes a dictator forever. In Mezzetti and Renou (2017), there can be equilibria in which the agents' messages differ. Therefore, they invoke the assumption of no-veto power (or their Assumption A) to ensure that any such equilibrium results in desirable outcomes. We modify their regime in a way that ensures that no such equilibrium, in which agents' messages differ, even exists. It is done by giving each agent incentives to become the dictator.

We now discuss how it is achieved and the role Theorem 1 plays in it. Unwanted equilibria, in which agents' messages are unanimous, are taken care of by DM. Namely, if the agents follow a deception that results in an undesirable outcome, DM ensures that there exists a whistle-blower, say, agent i who has incentives to deviate by demanding some $(a, v) \in C_i(\theta)$ when the other agents claim that the state is θ, although it is not. If the continuation payoff v_i that the whistle-blower demands for himself is such that $\underline{v}_i < v_i < \overline{v}_i$, then the designer can always introduce a lottery with the expected payoff v_i for agent i that picks every alternative forever with a strictly positive probability. Due to strict preferences, it follows that the agents do not get their highest continuation payoffs with certainty, giving them the incentives to trigger the so-called integer game, in which the agent announcing the highest integer can become the dictator and choose his preferred alternatives forever. The probability of becoming the dictator is, however, strictly less than 1, although by announcing higher and higher integers, the agent can increase this probability. With the remaining probability, again a constant alternative is uniformly chosen forever. As a result, the agents still do not get their highest continuation payoffs with certainty, giving them the incentives to announce even higher integers. This eliminates any equilibrium in which the agents send non-unanimous messages on the equilibrium path.[22]

The designer, however, cannot introduce a lottery over alternatives if the whistle-blower demands (a, v), such that either $v_i = \underline{v}_i$ or $v_i = \overline{v}_i$. Therefore, we

[22]Implementation literature is often criticized for the use of integer games and other similar "tail-chasing" constructs to rule out certain undesirable equilibria. Since our goal is to characterize what in general can be implemented, this criticism is less relevant here. Furthermore, Lee and Sabourian (2015) show that repeated implementation using only simple and finite mechanisms is also possible if the agents have a preference for less complexity. See also Footnote 24.

first show in the following lemma that any such alternative-continuation payoff pair can be replaced with a nearby pair (b, w) where $\underline{v}_i < w_i < \overline{v}_i$, without violating the DM of f. Since DM requires satisfying infinitely many incentive constraints, it is not obvious that we can replace one alternative-continuation payoff pair with another and maintain the DM of f. To show that we can indeed do so, we use the characterization of DM in Theorem 1. Because the equilibrium payoff correspondence of repeated games is upper-semicontinuous, we can show that a slight change in C and, hence, in the equilibrium payoffs of the associated repeated game, $\mathcal{E}(G^C)$, preserves the DM of f.

Lemma 1. *If f is dynamically monotonic w.r.t. $C = (C_i(\theta))_{i,\theta}$, then f is also dynamically monotonic w.r.t. some $D = (D_i(\theta))_{i,\theta}$ such that $\underline{v}_i < v_i < \overline{v}_i$ for all i, θ, $D_i(\theta)$, and $(a, v) \in D_i(\theta)$.*

The proof of lemma is in Section B of the Appendix.

Theorem 2. *When $n > 2$, f is repeatedly implementable if and only if f is dynamically monotonic with respect to some collection $C = (C_i(\theta))_{i,\theta}$.*

Proof. The only-if-part.[23] Suppose that a social choice function f is repeatedly implementable using a regime r. Take a strategy profile s that is a Nash equilibrium of the game induced by regime r. Consider a history h^t and a mechanism γ^t for some t such that $q(h^t|s, r) > 0$ and $r(\gamma^t|h^t) > 0$. Let $C_i(h^t, \theta, \gamma^t)$ be the set of alternative and continuation payoff pairs that agent i can attain by deviating in period t given that the period t state is θ and the other agents follow s_{-i}. Let $C_i(\theta) = \cup_t \cup_{\{h^t|q(h^t|s,r)>0\}} \cup_{\{\gamma^t|r(\gamma^t|h^t)>0\}} C_i(h^t, \theta, \gamma^t)$. The necessity of DM w.r.t. $C = (C_i(\theta))_{i,\theta}$ follows from the proof of Theorem 1 in Mezzetti and Renou (2017), once we replace $L_i((f(\theta), v^f), \theta)$ with $C_i(\theta)$ everywhere in that proof.
The if-part. By Lemma 1, we can select a collection $C = (C_i(\theta))_{i,\theta}$ w.r.t. which f is DM and $\underline{v}_i < v_i < \overline{v}_i$ for all i, θ, $C_i(\theta)$, and $(a, v) \in C_i(\theta)$. We now define a regime and show that it implements f. In the regime, we will use the following stage mechanisms.

Mechanism $\hat{\gamma}$. For each agent $i \in I$, let the message space of agent i be $\mathcal{M}_i = \{(\theta, b, v, z) \in \Theta \times \Delta A \times V \times \mathbb{Z}_+\}$, where $\mathbb{Z}_+$ denotes the set of nonnegative integers. For a message profile m, let i_z denote an agent who sends the highest integer z, and let $m_{i_z} = (\theta_z, b_z, v_z, z_z)$. Given $C = (C_i(\theta))_{i,\theta}$, define the allocation rule g as follows:

I. If there exists $(\theta, b, v, z) \in \Theta \times \Delta A \times V \times \mathbb{Z}_+$ such that $m_i = (\theta, b, v, z)$ for all $i \in I$, then $g(m) = f(\theta)$.

II. If there exists $(\theta, b, v, z) \in \Theta \times \Delta A \times V \times \mathbb{Z}_+$ and $i^* \in I$ such that $m_i = (\theta, b, v, z)$ for all $i \neq i^*$ and $m_{i^*} = (\theta', b', v', z') \neq (\theta, b, v, z)$, then:

[23] This part also applies for $n = 2$.

(a) $g(m) = b'$ if $(b', v') \in C_{i^*}(\theta)$,

(b) $g(m) = f(\theta)$ otherwise.

III. If neither (I) nor (II) applies, then $g(m) = b_z$.

The second mechanism is a dictatorial one, in which some agent i picks an alternative from a subset of A:

Mechanism $\tilde{\gamma}_i(\mathcal{M}_i)$. Let $\mathcal{M}_i \subseteq A$, while $\mathcal{M}_j = \{\emptyset\}$ for each $j \in I \backslash \{i\}$. Let $g(m) = m_i$.

In case $\mathcal{M}_i = \{a\}$ for some $a \in A$ in the dictatorial mechanism, we refer to it as a constant mechanism.

Let $\hat{v}_i := \frac{\sum_{a \in A} \sum_{\theta \in \Theta} p(\theta) u_i(a, \theta)}{|A|}$ for all $i \in I$, where $|A|$ denotes the cardinality of the set A. Also, let $\underline{A}_i = \{\underline{a}_i(\theta) | \theta \in \Theta\}$. The regime is defined as follows:

Regime r.

1. $r(\hat{\gamma}|h^0) = 1$.

2. For $t \geq 1$, if $r(\hat{\gamma}|h^{t-1}) = 1$ and $m^{t-1} = (m_i)_{i \in I}$ is such that:

   (a) Parts (I) or (IIb) of $\hat{\gamma}$ applies, then $r(\hat{\gamma}|h^t) = 1$,

   (b) Part (IIa) of $\hat{\gamma}$ applies with $m_{i^*} = (\theta', b', v', z')$, then

      i. If $\underline{v}_{i^*} < v'_{i^*} \leq \hat{v}_{i^*}$, then

$$\begin{aligned} r(\tilde{\gamma}_{i^*}(\{a\})|h^t) &= \lambda/|A| \text{ for all } a \in A, \\ r(\tilde{\gamma}_{i^*+1}(\underline{A}_{i^*})|h^t) &= 1 - \lambda, \end{aligned}$$

      where

$$\lambda = \frac{v'_{i^*} - \underline{v}_{i^*}}{\hat{v}_{i^*} - \underline{v}_{i^*}},$$

      ii. If $\hat{v}_{i^*} < v'_{i^*} < \overline{v}_{i^*}$, then

$$\begin{aligned} r(\tilde{\gamma}_{i^*}(A)|h^t) &= \lambda, \\ r(\tilde{\gamma}_{i^*}(\{a\})|h^t) &= (1 - \lambda)/|A| \text{ for all } a \in A, \end{aligned}$$

      where

$$\lambda = \frac{v'_{i^*} - \hat{v}_{i^*}}{\overline{v}_{i^*} - \hat{v}_{i^*}},$$

   (c) Part (III) of $\hat{\gamma}$ applies, then $r(\tilde{\gamma}_{i_z}(A)|h^t) = \frac{z_z}{1+z_z}$ and $r(\tilde{\gamma}_{i_z}(\{a\})|h^t) = \frac{1}{(1+z_z)|A|}$ for all $a \in A$,

3. For $t \geq 2$, if $r(\tilde{\gamma}_i(\mathcal{M}_i)|h^{t-1}) = 1$ for some i and $\mathcal{M}_i$, then $r(\tilde{\gamma}_i(\mathcal{M}_i)|h^t) = 1$.

In words, suppose that the agents face mechanism $\hat{\gamma}$ in period $t-1$. Depending on the agents' messages in that period, period t mechanism is determined as follows. If their reports are unanimous or there is a single agent i^* who sends a message $m_{i^*} = (\theta', b', v', z')$ different from (θ, b, v, z), which is sent by all the other agents, and $(b', v') \notin C_{i^*}(\theta)$, then the mechanism $\hat{\gamma}$ is again selected in period t. If instead the message of agent i^* is such that $(b', v') \in C_{i^*}(\theta)$, then the demanded alternative b' is implemented, and from the next period on either agent i^* or $i^* + 1$ is given the right to choose alternatives from set A (in case of i^*) or $\underline{A}_{i^*}$ (in case of $i^* + 1$) forever, or one of the constant mechanisms is applied forever. The probabilities of these scenarios are chosen so as to ensure that agent i^* expects his announced continuation payoff v'_{i^*}, assuming that, on the one hand, agent i^*+1 will always choose agent i^*'s worst alternative from $\underline{A}_{i^*}$ if agent i^*+1 becomes a dictator and, on the other hand, agent i^* plays optimally if he becomes the dictator. Finally, for all other message profiles, either the agent with the highest announced integer is given the right to choose alternatives from A (with probability $z_z/(1 + z_z)$) or one of the constant mechanisms is applied forever (with probability $1/(|A|(1 + z_z))$).[24]

We complete the proof in Section B of the Appendix where we show that the defined regime implements f. Namely, Lemma 6 establishes that there exists an equilibrium that selects the desirable alternative in each period, while Lemma 7 establishes that in any equilibrium, only the desirable alternative is selected in each period. $\square$

The regime that is used to implement f in the above theorem, requires the knowledge of collection C w.r.t. which f is DM. We note that the proof of Lemma 1 suggests an algorithm to construct one such possible C. First, w.l.o.g., $C_i(\theta)$ can be taken to be finite for all i and θ with the property that it contains an alternative-continuation payoff pair in $M_i(L_i((f(\theta), v^f), \theta), \theta^*)$ for every θ^*. Next, if for some i and θ, there exists $(a, v) \in C_i(\theta)$, such that either $v_i = \underline{v}_i$ or $v_i = \overline{v}_i$, then according to Lemma 1 we can replace (a, v) with another alternative-continuation payoff pair (b, w), such that $\underline{v}_i < w_i < \overline{v}_i$.

6 Implementing Efficient Functions

Here we relate Lee and Sabourian (2011)'s notion of efficiency in the range to dynamic monotonicity (see Proposition 1 below), and then we provide an appli-

[24]We could also design a regime without the unbounded (or open) "integer game" in part (III) of $\hat{\gamma}$ by instead allowing the agents to announce numbers from the compact $[0, 1]$ interval. For example, the agent who announces the largest number becomes a dictator with the probability equal to his number if this number is strictly less than 1, and with the remaining probability the constant mechanisms are played. If anyone announces 1, then a constant mechanism is selected with equal probabilities. Hence, the regime (resp., mechanism) is discontinuous in strategies (resp., messages).

cation of this result by studying the implementation of utilitarian social choice functions.

Definition 4 (Efficiency in the range)**.** *Let* $V^f = \{v \in V | \exists \pi^0 : \Theta \to \Delta\Theta : v = \sum_{\theta \in \Theta} p(\theta)\pi^0(\theta)(\theta')u(f(\theta'), \theta)\}$ *where* $\pi^0(\theta)(\theta')$ *is the probability of deceiving in state* θ *that the state is* θ'*.* f *is* efficient *(resp.,* weakly efficient*)* in the range *if there is no* $v \in V^f$ *such that* $v \geq v^f$ *(resp.,* $v > v^f$*). That is,* f *is efficient in the range when* v^f *is Pareto efficient within the set* V^f*.* f *is* strictly efficient in the range *if it is efficient in the range and there does not exist* $\pi^0 : \Theta \to \Delta\Theta$*, with* π^0 *being different from the identity map, such that* $v^f = \sum_{\theta \in \Theta} p(\theta) \sum_{\theta' \in \Theta} \pi^0(\theta)(\theta')u(f(\theta'), \theta)$*.*

Lee and Sabourian (2011) in their Theorem 1 show that if f is not weakly efficient in the range, then f is not implementable for δ sufficiently large (see also our Example 1 where f is not efficient in the range). To understand this result, one can think that the agents use a stationary random deception, which strictly Pareto dominates v^f.[25] On the other hand, for their sufficiency result, Lee and Sabourian (2011) in their Theorem 2 require that f is strictly efficient in the range in order to obtain outcome implementation. Additionally, they invoke their Assumption A and Condition ω. Furthermore, Mezzetti and Renou (2017) in their Remark 4 show that efficiency in the range and Maskin monotonicity imply DM. In the following proposition, we improve on these results by requiring only efficiency in the range and MM*, where the latter is a necessary condition for implementation.

Proposition 1. *If* f *is efficient in the range and Maskin monotonic*, then* f *is dynamically monotonic and, hence, it is repeatedly implementable when* $n > 2$*.*

Proof. Suppose f is MM* w.r.t. $C = (L_i((f(\theta), v^f), \theta))_{i,\theta}$. It is enough to prove that point 2 of Theorem 1 holds. Theorem 2 then completes the proof.

We know from the proof of Lemma 3 (in the Appendix) that in any efficient equilibrium of G^C, in every period and state, all players must choose a common action in Θ on the equilibrium path. Thus, if f is efficient in the range, then v^f is clearly an efficient equilibrium payoff vector of G^C. We need to prove that it is the unique efficient payoff vector.

Take an efficient equilibrium of G^C. Suppose that in this equilibrium, when period 0 state is θ^*, the players choose action θ (possibly equal to θ^*) and expect continuation payoffs v from period 1 onwards. If $v \neq v^f$, there exists a player i such that $v_i < v_i^f$. This player, by choosing ω in state θ^* of period 0, receives at least a payoff of $(1-\delta)u_i(f(\theta), \theta^*) + \delta v_i^f$, which is strictly more than the payoff obtained by choosing θ. Thus, it must be that $v = v^f$. On the other hand, MM* ensures that if in any period 0 state θ^*, the players choose θ such that

[25] Lee and Sabourian (2011) dispense with public randomization by invoking the result of Fudenberg and Maskin (1991).

$f(\theta) \neq f(\theta^*)$, there exists a player who again prefers to deviate by choosing ω. Therefore, in any efficient equilibrium of G^C, the players expect payoffs v^f. $\square$

Since verifying efficiency in the range and MM* is relatively easy, the result of Proposition 1 offers a simple way to confirm the DM of f.

The proof of Proposition 1 suggests that if f is only efficient in the range, but not MM*, we might still repeatedly implement it from period 1 onwards. However, we show via an example in the Supplementary Material that it is not true in general. Intuitively, there can be deceptions that result in the continuation payoffs v^f and nobody has incentives to deviate when MM* is not satisfied. But if f is strictly efficient in the range, then indeed it is repeatedly implementable from period 1 onwards.

6.1 An Application

As an application of Proposition 1, we study the repeated implementation of (generalized) utilitarian social choice functions.

Definition 5. *A social choice function, f^u, is* utilitarian *if there exists $(\beta_i)_{i\in I}$ such that for each $\theta \in \Theta$:*

$$f^u(\theta) \in \arg\max_{a\in A} \sum_{i\in I} \beta_i u_i(a,\theta),$$

and $\beta_i \geq 0$ for all i, and $\beta_j > 0$ for some j.

Remark 5. *A utilitarian social choice function, f^u, is weakly efficient in the range. If $f^u(\theta) = \arg\max_{a\in A} \sum_{i\in I} \beta_i u_i(a,\theta)$ for all θ, then f^u is (strictly) efficient in the range.*

One-shot implementation of the utilitarian social choice functions, especially in dominant strategies, has been extensively studied in the literature (see, for example, Groves (1973)). We show that they are repeatedly implementable in an environment that has been adapted from Laffont and Maskin (1982). A utilitarian social choice function does not need to be Maskin monotonic, however Laffont and Maskin (1982) have shown that a social choice *rule* consisting of utilitarian social choice function f^u and, for example, a constant private transfer function, is Maskin monotonic (see Theorem 5 and its proof in Laffont and Maskin (1982)). The following proposition, in essence, establishes that in the repeated setup, the continuation payoffs can play the role of monetary transfers. We only need to ensure that the transfers can be chosen to be arbitrarily small.

This result can be applied, for example, to study the efficient provision of public good when monetary transfers are ruled out, as in the case of clean air, or at least, they are not directly linked to the level of public good. That is, we can view $a \in A$ as a level of public good, agent 1 as a producer and the rest of the agents as consumers of the public good, and set $\beta_i = 1$ for all i. (Thus, $u_1(a, \theta)$ stands for a negative of the cost function.) The objective of the designer (which is different from the producer) is to provide the efficient level of public good repeatedly, which changes from one period to another randomly due to changes in the prices of other goods, which in turn affect the production costs or the demand for the public good.

Proposition 2. *Assume that $A = [0, 1]$, for each i and θ, $u_i(a, \theta)$ is strictly concave and differentiable function in the first argument, and in each state θ, $f^u(\theta) \in (0, 1)$. Then, f^u is Maskin monotonic*.*

Proof. Note that the assumptions imply that $v_i^{f^u} > \underline{v}_i$ for all i, and $f^u(\theta)$ is a solution to $\sum_{i \in I} \beta_i u_i'(f^u(\theta), \theta) = 0$ (where the derivative is taken w.r.t. the first argument).

Suppose, first, that $u_i'(f^u(\theta), \theta) = u_i'(f^u(\theta), \theta')$ for all i with $\beta_i > 0$. It follows that $\sum_{i \in I} \beta_i u_i'(f^u(\theta), \theta') = 0$ holds, implying that $f^u(\theta') = f^u(\theta)$. Therefore, suppose $u_i'(f^u(\theta), \theta) \neq u_i'(f^u(\theta), \theta')$ for some i with $\beta_i > 0$. We will argue that there exists (a, v_i) such that $u_i(f^u(\theta), \theta) + v_i^{f^u} \geq u_i(a, \theta) + v_i$ and $u_i(f^u(\theta), \theta') + v_i^{f^u} < u_i(a, \theta') + v_i$. Because of the strict concavity of $u_i(\cdot, \theta)$, we have that:

$$u_i(f^u(\theta), \theta) > u_i(f^u(\theta) + \eta, \theta) - u_i'(f^u(\theta), \theta)\eta.$$

Likewise, by taking the Taylor expansion of $u_i(\cdot, \theta')$ around $f^u(\theta)$, we have:

$$u_i(f^u(\theta), \theta') = u_i(f^u(\theta) + \eta, \theta') - u_i'(f^u(\theta), \theta')\eta + h(\eta)\eta,$$

where $\lim_{\eta \to 0} h(\eta) = 0$. By choosing $\eta > 0$ when $u_i'(f^u(\theta), \theta') > u_i'(f^u(\theta), \theta)$ and $\eta < 0$ when $u'(f^u(\theta), \theta') < u'(f^u(\theta), \theta)$, we have that:

$$u_i(f^u(\theta), \theta') < u_i(f^u(\theta) + \eta, \theta') - u_i'(f^u(\theta), \theta)\eta,$$

since the second order effect of $h(\eta)\eta$ can be ignored if $|\eta|$ is small enough. To summarize, we can set $a = f^u(\theta) + \eta$ and $v_i = v_i^{f^u} - u_i'(f^u(\theta), \theta)\eta$ where the sign of η is determined as before. Note that v_i can be chosen sufficiently close to $v_i^{f^u}$ to ensure $\underline{v}_i < v_i < \overline{v}_i$ as long as $v_i^{f^u} < \overline{v}_i$ holds. If $v_i^{f^u} = \overline{v}_i$, then one can set $a = f^u(\theta')$ and $v_i = v_i^{f^u}$ since $f^u(\cdot) = \overline{a}_i(\cdot)$ and, consequently, $u_i(f^u(\theta), \theta) > u_i(f^u(\theta'), \theta)$ and $u_i(f^u(\theta), \theta') < u_i(f^u(\theta'), \theta')$ due to strict concavity must hold. Hence, whenever part (b) of Definition 2 does not hold, part (a) also does not hold for some i. □

If the assumptions of Proposition 2 hold, then it follows from Propositions 1 and 2 and Remark 5 that f^u is DM and is repeatedly implementable when $n > 2$. On the other hand, f^u does not need to satisfy no-veto power, in which case one cannot apply the results of Mezzetti and Renou (2017) to show that f^u is repeatedly implementable. Also, while f^u is strictly efficient in its range, Condition ω of Lee and Sabourian (2011) does not need to hold. Therefore, their Theorem 2 cannot be invoked either to establish that f^u is implementable.

7 Discussion

7.1 Weak Preferences

The assumption of strict preferences over deterministic alternatives can be relaxed quite substantially. We only use this assumption in two places to prove the sufficiency part of Theorem 2; see Footnotes 30 and 32. From these footnotes, it follows that it is enough to assume that every agent has a unique worst alternative in every state. However, Theorem 2 can also be established with other assumptions about preferences. For example, one can assume the *top coincidence condition*, introduced in Benoît and Ok (2008). It requires that for any θ and any $J \subseteq I$ with $|J| = n-1$, $\cap_{j\in J}\overline{a}_j(\theta)$ is at most a singleton (where $\overline{a}_j(\theta)$ now is a set of best alternatives for agent j in state θ).[26] To see how this helps, note first that we may not be able to pick a collection of sets C, as we do in Lemma 1, such that $v_i > \underline{v}_i$ for all i, θ, $C_i(\theta)$ and $(a, v) \in C_i(\theta)$ if $\underline{a}_i(\theta)$ is not a singleton for some i and θ, and $v_i^f = \underline{v}_i$ (see Footnote 30). Nonetheless, we still want to ensure that there is no equilibrium when in some state θ^*, the agents claim that the state is θ and agent i is an odd-man-out who demands $(a, v) \in C_i(\theta)$ such that $v_i = \underline{v}_i = v_i^f$. To achieve that, we can replace (a, v) with a lottery (b, w) that randomizes between (a, v) and $(f(\theta), v^f)$ without losing the DM of f. Because of the top coincidence condition, it must be that $b \notin \cap_{j\neq i}\overline{a}_j(\theta^*)$. Since there exists an agent $j \neq i$ for whom b is not the best alternative, he has incentives to trigger the integer game. This eliminates such an equilibrium in which an agent is an odd-man-out who demands his lowest continuation payoff.

7.2 Regimes with Deterministic Stage Mechanisms

Mezzetti and Renou (2017) restrict attention to regimes with deterministic stage mechanisms and prove that DM together with either no-veto power or their Assumption A is sufficient for the repeated implementation of f when there are at least three agents. However, neither no-veto power nor Assumption A is necessary for implementation. Here, we discuss where the stochastic stage mechanisms play a role, and provide the intuition for a condition that together with DM is both necessary and sufficient for the repeated implementation of f when the stage mechanisms need to be deterministic (and $n > 2$). This condition, which we call Condition $\lambda 0$, is formally stated in Section C of the Appendix, where we also prove a result akin to Theorem 2 but for the regimes with deterministic stage mechanisms.

There are two instances where the stochastic stage mechanisms play a role in the result of Theorem 2. First, in the definition of DM, we assume that the

[26] The top coincidence condition is much weaker than the no-veto power assumption of Mezzetti and Renou (2017): when implementing social choice *functions*, no-veto power implies top coincidence.

sets in the collection $C = (C_i(\theta))_{i,\theta}$ belong to $\Delta A \times V$. Clearly, DM remains a necessary condition if we require the sets in C to be subsets of $A \times V$. However, because a function that is DM when the sets belong to $\Delta A \times V$ is not necessarily DM when the sets belong to $A \times V$, we can implement strictly less functions in the latter case.

Second, we need stochastic mechanisms in the proof of Lemma 1 when we replace $(a, v) \in C_i(\theta)$ such that $v_i = \underline{v}_i$ for some i and θ with another alternative-continuation payoff pair (b, w) such that $w_i > \underline{v}_i$ and b is a random alternative. Thus, we might not be able to replace (a, v) if we are only allowed to use deterministic mechanisms, and this introduces an additional necessary condition that f must satisfy. This extra condition, Condition $\lambda 0$, is similar to part (ii) of Condition μ in Moore and Repullo (1990). (Condition μ is necessary and sufficient for implementation in a static setup.) Namely, if in some state θ^*, (a, v) with $v_i = \underline{v}_i$ is the best that agent i can demand in $C_i(\theta)$ and if his demand also gives the best outcome in $A \times V$ for the other agents, then there exists an equilibrium in which agent i makes this demand. (Note, though, that unlike the static implementation problem, the agent can get more than he demands because the agents can follow a deception in the continuation.) Since socially desirable alternatives must also be selected in this equilibrium, it defines a condition on f.

In Section C, we also provide an example of f that is DM, but does not satisfy Condition $\lambda 0$. Hence, this f is repeatedly implementable only if the designer can use stochastic mechanisms.

7.3 Alternative Solution Concepts

7.3.1 Subgame Perfection

Theorem 2 remains valid if, instead of pure-strategy Nash equilibrium, we consider pure-strategy SPNE as a solution concept.[27] Mezzetti and Renou (2017) have already argued that DM remains necessary for subgame perfect implementation. To prove its sufficiency, we must slightly modify our regime r that is given in the proof of Theorem 2. To see why, note that the equilibrium constructed in Lemma 6 (see Section B) requires agent $i + 1$ to punish agent i if the latter unilaterally deviates in the mechanism $\hat{\gamma}$. Such punishment, however, might not be credible. To ensure its credibility, similar to Mezzetti and Renou (2017), we can require that the implemented alternative is the one chosen by a qualified majority. If such a qualified majority does not exist, only then is the alternative chosen by agent $i + 1$ implemented.

[27]The problem considered here should not be confused with the one-shot implementation in SPNE as studied, for example, in Moore and Repullo (1988) and Abreu and Sen (1990).

7.3.2 Mixed Strategies

So far, we have only considered implementation in pure strategies. Mezzetti and Renou (2017) show that DM and no-veto power are sufficient for implementation in mixed strategies once the designer can use stochastic stage mechanisms. We can similarly extend the result of our Theorem 2 to mixed strategies.

We only need to prove that DM remains sufficient for repeated implementation in mixed-strategy Nash equilibrium. The added difficulty when implementing in mixed strategies can be described as follows. Agent i might randomize in equilibrium between sending the same message that is announced by everyone else, and sending a different message. Now, it is no longer possible to argue that another agent, say agent j, has incentives to deviate because he can only trigger the integer game with a probability less than one. With the remaining probability, agent j himself becomes the odd-man-out and can be punished in the continuation for his deviation.

Therefore, to prove the sufficiency part, we again modify regime r. The modified regime, which is provided in Section D of the Appendix, ensures that such an equilibrium (as described in the previous paragraph), will not exist. We apply the same method that we use to eliminate unwanted equilibria in the integer game, to the case when an agent is the odd-man-out. Namely, the integer announced by the odd-man-out now determines his probability of being a dictator in the continuation. Because this probability is strictly increasing in the announced integer, there is no best response in the continuation game. This eliminates equilibria (by eliminating best responses), in which an agent becomes the odd-man-out with a positive probability on the equilibrium path.[28]

7.3.3 Perfect Bayesian Equilibrium in Mixed Strategies

The construction that achieves implementation in mixed strategies, cannot be used if one also requires subgame perfection, because there exist subgames that do not have Nash equilibria. We now argue that sequential rationality can be made compatible with mixed strategies, in the sense that DM remains sufficient for implementation if private (but still simultaneous) messages are allowed. Hence, the appropriate equilibrium concept is the mixed-strategy perfect Bayesian equilibrium (instead of SPNE).

Take the regime that implements social choice functions in pure-strategy SPNE, but with private messages, and modify it as follows. If there exists an odd-man-out who sends a message different from the common message sent by everyone else, then the odd-man-out now receives what he demands with a probability less than one. (This probability can be decided by the odd-man-out himself.) With the remaining probability, the regime proceeds as if there has been no

[28]We are grateful to Ludovic Renou who pointed out to us in private communication that our method could be applied to eliminate mixed strategy equilibria.

odd-man-out. Note that in the latter case, since the messages are now assumed to be private, the other agents do not observe that there has been an odd-man-out.

Consider again the possible equilibrium discussed in the previous subsection, in which agent i randomizes between sending the message m, which is sent by everyone else as well, and sending a different message. We argue that agent $j \neq i$ now has incentives to deviate from m. With private messages, the other agents will only learn that agent j has deviated and, hence, he will be punished if agent j is given what he demands, which only happens with some probability. This probability can always be made small enough so that the threat of being punished is outweighed by the gain from becoming a dictator when agent i also sends a message different from m. Note that this argument does not rely on the non-existence of best responses to eliminate mixed-strategy equilibria.

Also, note that the incentives to deviate from joint deceptions are preserved. If there exists an agent who has strict incentives to deviate from a deception when he is given for sure what he demands, then he still has incentives to deviate when he is given what he demands with a probability less than one. The reason is that the others will not detect his deviation if he is not given what he demands and, in the worst case, he will earn the payoff promised by the ongoing deception.

Finally, it is easy to see that DM also remains necessary under private communication: if for some deception π, part (a) of Definition 3 applies, then no agent has the incentive to be a whistle-blower whether the messages are public or private.

7.4 The Two Agents Case

Any necessary and sufficient conditions for the two agent case must clearly include the necessary and sufficient conditions for the case of more than two agents. However, when $n = 2$, the designer must additionally face the problem that the agents are sending different messages about the state of the world but the "deviator" cannot be identified. To overcome this problem, we need to introduce additional conditions. While we have not identified sufficient and necessary conditions for repeated implementation when $n = 2$, we offer sufficient conditions that improve on the existing results in literature in the Supplementary Material.

First, we establish a result similar to the one in Proposition 1 for $n = 2$. Besides Maskin monotonicity* and efficiency (not just efficiency in the range) of social choice function, we also assume that the function satisfies a version of (static) self-selection condition. The self-selection condition is necessary for the static implementation as shown by Moore and Repullo (1990) and Dutta and Sen (1991). It is also used by Lee and Sabourian (2011) in their Theorem 3, which states that strict efficiency in the range, self-selection, and their usual domain restrictions are sufficient for repeated implementation of social choice function from period 2 on when $n = 2$. However, we argue in the Supplementary Material that either self-selection or (strict) efficiency in the range must be strengthened to obtain repeated implementation. Therefore, to obtain repeated implementation, we assume efficiency. Alternatively, we could assume efficiency in the range but then self-selection must be replaced with *self-selection in the range*.

Second, we show that the result of Theorem 2 carries over to the $n = 2$ case if we assume that there exists a bad outcome. Now, whenever the agents send different messages and the deviator cannot be identified, the designer can simply implement the bad outcome forever. This is sufficient to rule out contradictory messages in the equilibrium.

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Appendix

A Proof of Theorem 1

In what follows, let $C = (C_i(\theta))_{i,\theta}$ be a collection of non-empty and closed sets with $C_i(\theta) \subseteq L_i((f(\theta), v^f), \theta)$ for all i and θ. Let W be the set of the feasible payoff profiles of the associated repeated game G^C. For all pairs of states (θ, θ^*), let $Q_{G^C}(\theta,\theta^*) = \{v \in W | \forall i : \mu_i(\theta,\theta^*) \leq u_i(f(\theta),\theta^*) + \delta v_i/(1-\delta)\}$ be the compact set of continuation payoffs for which it is incentive compatible for all players to play action θ in the stage game of G^C when the state is θ^*. Note that W and $\mu_i(\theta,\theta^*)$ depend on C and, hence, $Q_{G^C}(\theta,\theta^*)$ also depends on C. It must be obvious that the players can play θ in state θ^* on an equilibrium path of G^C if and only if $Q_{G^C}(\theta,\theta^*) \cap \mathcal{E}(G^C) \neq \emptyset$.

To prove Theorem 1, we first prove a couple of lemmas.

Lemma 2. *For all C, $v^f \in Q_{G^C}(\theta,\theta) \cap \mathcal{E}(G^C)$.*

Proof. Note, first, that all players playing o in every period and every state is an equilibrium of G^C. Now, v^f is attained if in each period, all players play the action that coincides with the state of the world of that period. This can be supported as an equilibrium if after any deviation, all players play o forever. Therefore, the deviator gets at most the same payoff that he would obtain if he followed the original strategy. □

Lemma 3. *f is dynamically monotonic w.r.t. C if and only if for all θ,θ^* with $f(\theta) \neq f(\theta^*)$, $Q_{G^C}(\theta,\theta^*) \cap \mathcal{E}(G^C) = \emptyset$.*

Proof. The if-part. Suppose f is not DM w.r.t. C. Then there exists a deception π such that part (a) of Definition 3 holds, but part (b) does not. Given this π, we can define the following trigger strategy of the repeated game, common to all players: choose actions according to π as long as everyone has done it in all previous periods; if any player ever deviates, then choose o forever. From the fact that π satisfies part (a) of Definition 3, it follows that $(1-\delta)\mu_i(\theta,\theta^*) \leq (1-\delta)u_i(f(\theta),\theta^*) + \delta v_i^f(\pi(\theta^*,\theta^{\to t}))$ for all i, t, $\theta^{\to t}$, (θ,θ^*) such that $\pi^t(\theta^*;\theta^{\to t}) = \theta$. Therefore, no player will ever deviate from the strategy by choosing ω. Also, no player will deviate on the equilibrium path by choosing o or θ', which is different from the one that is prescribed by π. Therefore, it follows that this common trigger strategy is an equilibrium of the repeated game and $v^f(\pi(\theta^*,\theta^{\to t})) \in Q_{G^C}(\theta,\theta^*) \cap \mathcal{E}(G^C)$ for all t, $\theta^{\to t}$, (θ,θ^*) such that $\pi^t(\theta^*;\theta^{\to t}) = \theta$. But since π does not satisfy part (b) of Definition 3, there exists i, t, $\theta^{\to t}$, (θ,θ^*) such that $\pi^t(\theta^*;\theta^{\to t}) = \theta$ and $f(\theta) \neq f(\theta^*)$. This completes the proof of the if-part.

The only-if-part. Conversely, suppose that for some (θ,θ^*) with $f(\theta) \neq f(\theta^*)$, $Q_{G^C}(\theta,\theta^*) \cap \mathcal{E}(G^C) \neq \emptyset$. We will show that f is not DM w.r.t. C. We start by stating some auxiliary results. First, by the construction of the repeated game, we know that in any equilibrium of the repeated game, in any period and state, all players must choose the same action in $\Theta \cup \{o\}$. Second, it is enough to consider equilibrium strategies, in which after any deviation from the equilibrium path, all players choose o forever since then the continuation payoffs are equal to the minmax value of 0, which is the most severe possible punishment. Third, for any equilibrium such that o is chosen by the players in some period and state on the equilibrium path, we can always construct another equilibrium that never involves playing o and that results in higher payoffs for all players. To see it, consider any history, call it h, on the equilibrium path at which the players are choosing o but they are not choosing o for any sub-history of h. Instead of playing o, let now all players to choose the action that coincides with the true state at h and also do the same in the continuation along the new equilibrium path. If any player deviates from the new strategy at h or after, then all players play o forever. No further changes in the original strategy are made. Note that the payoff of each player i is at most $\delta\overline{v}_i$ at h under the original strategy profile, while it is $(1-\delta)u_i(f(\theta),\theta)+\delta v_i^f \geq \delta\overline{v}_i$ under the new strategy profile, where θ represents the state at h. Given Lemma 2, we only need to verify that no player has incentives to deviate at any sub-history of h. It is indeed true because the payoffs from deviating at any such sub-history have not changed, but the payoff from the equilibrium strategy has increased. To sum up, given that $Q_{G^C}(\theta,\theta^*)\cap\mathcal{E}(G^C)\neq\emptyset$, there exists an equilibrium with payoffs $v\in Q_{G^C}(\theta,\theta^*)$ such that in every period and state, all players are choosing a common action in Θ on the equilibrium path, while any deviation is punished by playing o forever.

Now, consider the following deception π: if $\theta^0=\theta^*$, set $\pi^0(\theta^*)=\theta$ and, in the continuation, deceive according to the above (common) equilibrium strategy that results in the payoffs v; if $\theta^0\neq\theta^*$, then in period 0 and in the continuation, announce the state truthfully. This deception satisfies part (a) of Definition 3, but not part (b) because $f(\pi^0(\theta^*))\neq f(\theta^*)$. Part (a) is clearly satisfied for all $t\geq 0$ if $\theta^0\neq\theta^*$. It is also satisfied for all $t\geq 0$ when $\theta^0=\theta^*$ because no agent had incentives to deviate in the repeated game by choosing ω either in period 0 or in the continuation when they follow the common equilibrium strategy. (Note that choosing ω by player i corresponds to choosing an element in $M_i(C_i(\theta''),\theta')$ when in state θ', the other agents pretend that the state is θ''.) Thus, f is not DM w.r.t. C, which completes the proof of the only-if-part. □

Proof of Theorem 1. Suppose f is DM w.r.t. C. Then it is trivially Maskin monotonic* w.r.t. C. From the proof of Lemma 3, it follows that in any equilibrium of G^C, in each period, either all players must choose o or θ such that $f(\theta)=f(\theta^*)$ where θ^* is the state of the world in that period. For any equilibrium, where all players play θ such that $f(\theta)=f(\theta^*)$ in some period with state θ^*, there exists a payoff-equivalent equilibrium where the players instead play θ^* in the same period with state θ^*. On the other hand, any equilibrium strategy that involves playing o by all players in some period and state, will result in equilibrium payoffs $v<v^f$. This establishes point 2 in the lemma.

On the other hand, suppose that points 1 and 2 are satisfied but f is not DM w.r.t. C. By Lemma 3, this means that there is a pair (θ,θ^*) such that $f(\theta)\neq f(\theta^*)$ and $Q_{G^C}(\theta,\theta^*)\cap\mathcal{E}(G^C)\neq\emptyset$. However, because $v\leq v^f$ for all $v\in\mathcal{E}(G^C)\backslash\{v^f\}$ and $v^f\in\mathcal{E}(G^C)$ by Lemma 2, it follows that $v^f\in Q_{G^C}(\theta,\theta^*)\cap\mathcal{E}(G^C)$, which in turn contradicts Maskin monotonicity* of f w.r.t. C. □

B Proofs of the Results in Section 5

We introduce some additional notation. For any two vectors k, k' from some finite l dimensional vector space $\mathbb{R}^l$, denote the distance between k and k' by $d(k,k') = \max_{i=1,\ldots,l} |k_i - k'_i|$. Then, for a compact set X in this vector space, the ε-fattening of X is $X^\varepsilon = \cup_{k\in X}\{k' \in \mathbb{R}^l | d(k,k') \leq \varepsilon\}$. Given $C = (C_i(\theta))_{i,\theta}$ and $D = (D_i(\theta))_{i,\theta}$, let G^C and G^D be the corresponding repeated games. ($C_i(\theta)$ and $D_i(\theta)$ are assumed to be non-empty and closed, but their cardinalities can differ.) Let $d(G^C, G^D)$ be the distance between the stage game payoffs of the two games across all players, states, and action profiles. This distance is well-defined because both games have the same dimensions. We say that D is $\xi > 0$ close to C if $d(G^C, G^D) \leq \xi$.

Lemma 1 is proven with the help of the following two lemmas.

Lemma 4. *Suppose f is DM w.r.t. C. Then, there exists $\varepsilon > 0$ such that for all θ, θ^* with $f(\theta) \neq f(\theta^*)$, it is true that $Q_{G^C}(\theta,\theta^*) \cap \mathcal{E}(G^C)^\varepsilon = \emptyset$. Moreover, there is $\xi > 0$ such that if D is $\xi-$close to C, then $Q_{G^D}(\theta,\theta^*) \cap \mathcal{E}(G^C)^\varepsilon = \emptyset$.*

Proof. Due to finiteness of the action and state spaces of the stage game, we can find $\varepsilon > 0$ such that for all θ, θ^* with $f(\theta) \neq f(\theta^*)$, we have that $\{v^f\}^\varepsilon \cap Q_{G^C}(\theta,\theta^*) = \emptyset$. We know from point 2 of Theorem 1 that $v \leq v^f$ for all $v \in \mathcal{E}(G^C)\backslash\{v^f\}$. This implies that $\{v\}^\varepsilon \cap Q_{G^C}(\theta,\theta^*) = \emptyset$ for all $v \in \mathcal{E}(G^C)$. This establishes the first claim of the lemma. Finally, by compactness of $\mathcal{E}(G^C)$ and $Q_{G^D}(\theta,\theta^*)$, there is $\xi > 0$ such that $Q_{G^D}(\theta,\theta^*) \cap \mathcal{E}(G^C)^\varepsilon = \emptyset$ if D is $\xi-$close to C. □

Lemma 5. *Suppose f is DM w.r.t. C. Then, there exists $\xi > 0$ such that f is DM w.r.t. any D, which is ξ-close to C.*

Proof. Due to finiteness of the state and action spaces, it follows from Berge's maximum theorem that $\mathcal{E}(\cdot)$ is upper-semicontinuous.[29] That is, for any $\varepsilon > 0$, there is $\xi > 0$ such that $\mathcal{E}(G^D) \subset \mathcal{E}(G^C)^\varepsilon$ if $d(G^C, G^D) \leq \xi$. By Lemma 4, if $\varepsilon > 0$ and $\xi > 0$ are small enough, we have that for all θ, θ^* with $f(\theta) \neq f(\theta^*)$, $Q_{G^D}(\theta,\theta^*) \cap \mathcal{E}(G^C)^\varepsilon = \emptyset$ and, hence, $Q_{G^D}(\theta,\theta^*) \cap \mathcal{E}(G^D) = \emptyset$. Lemma 3 completes the proof. □

Proof of Lemma 1. Note if f is DM w.r.t. $(L_i((f(\theta), v^f), \theta))_{i,\theta}$, then f is also DM w.r.t. $C = (C_i(\theta))_{i,\theta}$ such that $C_i(\theta)$ is finite for all i and θ, and it contains a pair $(a, v) \in M_i(L_i((f(\theta), v^f), \theta), \theta^*)$ for every θ^*. W.l.o.g., we assume that f is DM w.r.t. such C. We will construct $D_i(\theta)$ by replacing a finite number of elements in $C_i(\theta)$ for all i and θ.

[29] See, for example, Propositions 19 and 20 in Plan (2014). While in that paper, the upper-semicontinuity of $\mathcal{E}(\cdot)$ is established only for games without random states, the proof also works for games with random states. Furthermore, the result holds in pure strategies.

Now, if for all i and θ, we have that $C_i(\theta)$ does not contain the alternative and continuation payoff pairs, in which agent i gets his minimal or maximal continuation payoff, then set $D = C$. Therefore, suppose instead that for some i, θ, and $(a, v) \in C_i(\theta)$, we have that $v_i = \overline{v}_i$. But then v_i can be slightly decreased and, due to Lemma 5, (a, v) can be replaced with (a, v') such that $v'_i < \overline{v}_i$.

Suppose now that for some i, θ, and $(a, v) \in C_i(\theta)$, we have that $v_i = \underline{v}_i$. If $(f(\theta), v^f)$ is strictly preferred by agent i to (a, v) at θ, then v_i can be slightly increased (while ensuring that $(f(\theta), v^f)$ is still the best for agent i under θ) and (a, v) can be replaced with $(a, v') \in L_i((f(\theta), v^f), \theta)$ where $v'_i > \underline{v}_i$. According to Lemma 5, this can be done without losing dynamic monotonicity, but also intuitively, this change cannot hurt DM since we increase the agent's payoff from a deviation. Hence, suppose that agent i is indifferent between $(f(\theta), v^f)$ and (a, v) in state θ. There are two cases to consider. If $\underline{v}_i < v^f_i$, then it cannot be that a is the worst alternative for i at θ. In this case, we replace (a, v) with a close enough alternative and continuation payoff pair $(b, v') \in \Delta A \times V$ by slightly increasing the continuation payoff and by slightly decreasing the expected payoff from the alternative, while ensuring $(b, v') \in L_i((f(\theta), v^f), \theta)$. Again, due to Lemma 5, this can be done without losing dynamic monotonicity.

Finally, if $\underline{v}_i = v^f_i$, then because of strict preferences, it must be that $f(\theta) = \underline{a}_i(\theta)$ and $\{(f(\theta), v^f)\} = C_i(\theta)$ for all θ. But this means that agent i will never have a profitable deviation from any deception since $C_i(\theta) = \{(f(\theta), v^f)\} \subseteq L_i((f(\theta), v^f(\pi(\theta^*, \theta^{\to t}))), \theta^*)$ always holds in part (a) of the definition of DM. Therefore, we can simply set $D_i(\theta) = \emptyset$ for all θ without compromising DM.[30,31]

$\square$

Lemma 6. *There exists a Nash equilibrium s such that $g(s(h^t, \theta^t, \gamma^t))(f(\theta^t)) = 1$ for all $t \in T$, $\theta^t \in \Theta$, $h^t \in H^t$, and $\gamma^t \in \Gamma$ such that $q(h^t|s, r)r(\gamma^t|h^t) > 0$.*

Proof. Let i^* be the agent that is defined in part (2b) of r. For all t, θ^t, and h^t, let s be defined as follows:

1. $s_i(h^t, \hat{\gamma}, \theta^t) = (\theta^t, f(\theta^t), v^f, 0)$ for all $i \in I$.
2. $s_{i^*}(h^t, \tilde{\gamma}_{i^*}(A), \theta^t) = \overline{a}_{i^*}(\theta^t)$ and $s_j(h^t, \tilde{\gamma}_{i^*}(A), \theta^t) = \emptyset$ for all $j \neq i^*$.
3. $s_{i^*+1}(h^t, \tilde{\gamma}_{i^*+1}(\underline{A}_{i^*}), \theta^t) = \underline{a}_{i^*}(\theta^t)$ and $s_j(h^t, \tilde{\gamma}_{i^*+1}(\underline{A}_{i^*}), \theta^t) = \emptyset$ for all $j \neq i^* + 1$.

[30] This is one of the places where we use the assumption of strict preferences. If preferences were not strict, then it can be that agent i is indifferent between $f(\theta)$ and a at θ, and (a, v) is used to eliminate a deception at some θ^*. A simple, though, extra assumption would suffice to circumvent such situations, namely, requiring that there is a unique worst outcome for each agent at each θ.

[31] The argument in this paragraph does not invoke Lemma 5. Therefore, it is fine to set $D_i(\theta) = \emptyset$.

Note that we have left s unspecified when multilateral deviations occur.

Given s, the payoff of agent i in period t when the state of the world is θ is $(1-\delta)u_i(f(\theta),\theta)+\delta v_i^f$. Suppose period t is the first period in which agent i deviates from s_i. The only period t deviations that matter are the ones that fall under part (IIa) of $\hat{\gamma}$, that is, $i=i^*$. However, for any such deviation, $(b,v)\in C_i(\theta)$ and i's payoff is $(1-\delta)u_i(b,\theta)+\delta v_i$, which is weakly smaller than $(1-\delta)u_i(f(\theta),\theta)+\delta v_i^f$. Note that v_i is, indeed, the highest continuation payoff that agent i can expect given the strategy of agent $i+1$ if the mechanism $\tilde{\gamma}_{i+1}(\underline{A}_i)$ is selected in period $t+1$. □

Lemma 7. *In any Nash equilibrium s of r, $g(s(h^t,\theta^t,\gamma^t))(f(\theta^t))=1$ for all $t\in T$, $\theta^t\in\Theta$, $h^t\in H^t$, and $\gamma^t\in\Gamma$ such that $q(h^t|s,r)r(\gamma^t|h^t)>0$.*

Proof. Fix some Nash equilibrium s.

Claim 1 (An odd-man-out)**.** *There is no t, θ^t, h^t such that $q(h^t|s,r)r(\hat{\gamma}|h^t)>0$, but $s_i(h^t,\theta^t,\hat{\gamma})=(\theta,b,v,z)$ for all $i\neq i^*$ and $s_{i^*}(h^t,\theta^t,\hat{\gamma})=(\cdot,b',v',\cdot)\neq(\theta,b,v,z)$ with $(b',v')\in C_{i^*}(\theta)$ (that is, part (IIa) of $\hat{\gamma}$ and part (2b) of r apply).*

Proof of Claim 1. If there is some t, θ^t, h^t such that $q(h^t|s,r)r(\hat{\gamma}|h^t)>0$, but $s_i(h^t,\theta^t,\hat{\gamma})=(\theta,b,v,z)$ for all $i\neq i^*$ and $s_{i^*}(h^t,\theta^t,\hat{\gamma})=(\cdot,b',v',\cdot)\neq(\theta,b,v,z)$ with $(b',v')\in C_{i^*}(\theta)$, then with a positive probability, one of the constant mechanisms is played forever starting period $t+1$. Due to strict preferences,[32] there is an agent $j\neq i^*$ who prefers to play the integer game and to announce high enough integer to decrease the probability with which the constant mechanisms are played. This contradicts the assumption that s is an equilibrium.

□

Claim 2 (Integer game)**.** *There is no t, θ^t, h^t such that $q(h^t|s,r)r(\hat{\gamma}|h^t)>0$ and for some $i,j,k\in I$, $s_i(h^t,\theta^t,\hat{\gamma})\neq s_j(h^t,\theta^t,\hat{\gamma})\neq s_k(h^t,\theta^t,\hat{\gamma})\neq s_i(h^t,\theta^t,\hat{\gamma})$ (that is, part (III) of $\hat{\gamma}$ and part (3) of r apply).*

Proof of Claim 2. If there is t, θ^t, h^t such that $q(h^t|s,r)r(\hat{\gamma}|h^t)>0$ and for some $i,j,k\in I$, $s_i(h^t,\theta^t,\hat{\gamma})\neq s_j(h^t,\theta^t,\hat{\gamma})\neq s_k(h^t,\theta^t,\hat{\gamma})\neq s_i(h^t,\theta^t,\hat{\gamma})$, then again agents want to announce higher and higher integers to decrease the probability of the constant mechanisms. (The observation in Footnote 32 again applies.) This contradicts the assumption that s is an equilibrium. □

It follows from Claims 1 and 2 that in any equilibrium, the mechanism $\hat{\gamma}$ must be played on the equilibrium path in every period.

Claim 3 (Full deception)**.** *$g(s(h^t,\theta^t,\hat{\gamma}))(f(\theta^t))=1$ for all $t\in T$, $\theta^t\in\Theta$, and $h^t\in H^t$ such that $q(h^t|s,r)r(\hat{\gamma}|h^t)>0$.*

[32] In fact, it is enough that there are two agents who are not always indifferent between all alternatives.

Proof of Claim 3. Since $\hat{\gamma}$ is played on the equilibrium path in every period, it must be that for all t, θ^t, h^t such that $q(h^t|s,r)r(\hat{\gamma}|h^t) > 0$, there exists $(\phi^t, b^t, v^t, z^t) \in \Theta \times \Delta A \times V \times \mathbb{Z}_+$ (which is different for different θ^t and h^t) such that either $s_i(h^t, \theta^t, \hat{\gamma}) = (\phi^t, b^t, v^t, z^t)$ for all i (that is, part (I) of $\hat{\gamma}$ and part (2a) of r apply), or $s_i(h^t, \theta^t, \hat{\gamma}) = (\phi^t, b^t, v^t, z^t)$ for all $i \neq i^*$ and $s_{i^*}(h^t, \theta^t, \hat{\gamma}) = (\cdot, a^t, w^t, \cdot) \neq (\phi^t, b^t, v^t, z^t)$ with $(a^t, w^t) \notin C_{i^*}(\phi^t)$ (that is, part (IIb) of $\hat{\gamma}$ and part (2a) of r apply). Note that $g(s(h^t, \theta^t, \hat{\gamma}))(f(\phi^t)) = 1$.

Let a deception π be defined as follows: $\pi^t(\theta^t, \theta^{\to t}) = \phi^t$ for all t, θ^t, and $\theta^{\to t}$, where $\theta^{\to t}$ is the history of states contained in h^t. Since s is a Nash equilibrium, it must be that $C_i(\phi^t) \subseteq L_i((f(\phi^t), v^f(\pi(\theta^t, \theta^{\to t}))), \theta^t)$ holds for all $i \in I$, all $t \in T$, all $\theta^{\to t} \in \Theta^{\to t}$, and all pairs $(\phi^t, \theta^t) \in \Theta \times \Theta$ for which $\pi^t(\theta^t, \theta^{\to t}) = \phi^t$. But then, by DM, $f(\phi^t) = f(\pi^t(\theta^t, \theta^{\to t})) = f(\theta^t)$ for all $t \in T$, all $\theta^{\to t} \in \Theta^{\to t}$, and all pairs $(\phi^t, \theta^t) \in \Theta \times \Theta$ for which $\pi^t(\theta^t, \theta^{\to t}) = \phi^t$. Thus, $g(s(h^t, \theta^t, \hat{\gamma}))(f(\theta^t)) = 1$ for all $t \in T$, $\theta^t \in \Theta$, and $h^t \in H^t$ such that $q(h^t|s,r)r(\hat{\gamma}|h^t) > 0$. $\square$

$\square$

C Regimes with Deterministic Stage Mechanisms

Here, we study repeated implementation of f when the regime only uses deterministic stage mechanisms. We first derive an additional necessary condition and then prove that this condition together with DM is sufficient for repeated implementation. Note that we continue to assume strict preferences.

Suppose that a social choice function f is repeatedly implementable by a regime that only employs deterministic stage mechanisms. Let s be a Nash equilibrium of the game induced by this regime. Let the collection $C = (C_i(\theta))_{i,\theta}$ be defined as in the proof of the only-if-part of Theorem 2. Suppose now that there exist an agent i, a pair of states (θ, θ^*), and an alternative and continuation payoff pair (a, v) and a static deception $\pi^0 : \Theta \to \Theta$ such that:

1. $(a, v) \in C_i(\theta)$, $v_i = \underline{v}_i$ and $a = \overline{a}_j(\theta^*)$ for all $j \in I \backslash \{i\}$,
2. There is no w such that $(a, w) \in C_i(\theta)$ and $w_i > \underline{v}_i$,
3. $\underline{a}_i(\pi^0(\theta')) = \overline{a}_j(\theta')$ for all θ' and $j \in I \backslash \{i\}$, and
4. $C_i(\theta) \subseteq L_i((a, v^{\underline{a}_i}(\pi^0)), \theta^*)$ where $v^{\underline{a}_i}(\pi^0) = \sum_{\theta' \in \Theta} p(\theta') u(\underline{a}_i(\pi^0(\theta')), \theta')$.

Then, one can construct a new Nash equilibrium in the following way. By point (1), there must be a history (h^t, θ, γ^t) for some t such that $q(h^t|s,r) > 0$, $r(\gamma^t|h^t) > 0$, and $(a, v) \in C_i(h^t, \theta, \gamma^t) \subseteq C_i(\theta)$. ($C_i(h^t, \theta, \gamma^t)$ is defined in the proof of the only-if-part of Theorem 2.) Consider now history $(h^t, \theta^*, \gamma^t)$. Let agents $-i$ (i.e., all agents except agent i) play after that history as if the state was θ instead of θ^* and let agent i demand (a, v). In the continuation, agents $-i$

pretend to play according to s_{-i}, using π^0 in each period. This joint deviation from s can be maintained as a Nash equilibrium. First, because of point (4), agent i will be (weakly) worse off if he does not demand (a, v) and agents $-i$ truly play s_{-i} in the continuation. Second, agent i cannot have a continuation strategy, which would increase his continuation payoff above $v_i^{\underline{a}_i}(\pi^0)$, as otherwise he could already obtain a higher continuation payoff than $\underline{v}_i$ if he deviated in the original equilibrium, that is, because, according to point (2), there is no w such that $(a, w) \in C_i(\theta)$ and $w_i > \underline{v}_i$. Finally, by points (1) and (3), agents $-i$ get their best possible payoffs. Therefore, if f is repeatedly implementable, the following condition must necessarily hold:

Condition $\lambda 0$. f satisfies $\lambda 0$ with respect to $C = (C_i(\theta))_{i,\theta}$:

If for some agent i and for some pair of states (θ, θ^*), there exists an alternative and continuation payoff pair $(a, v) \in C_i(\theta)$ such that $v_i = \underline{v}_i$ and $a = \overline{a}_j(\theta^*)$ for all $j \in I \backslash \{i\}$, there is no w such that $(a, w) \in C_i(\theta)$ and $w_i > \underline{v}_i$, there exists a static deception $\pi^0 : \Theta \to \Theta$ such that $\underline{a}_i(\pi^0(\theta')) = \overline{a}_j(\theta')$ for all θ' and $j \in I \backslash \{i\}$, and $C_i(\theta) \subseteq L_i((a, v^{\underline{a}_i}(\pi^0)), \theta^*)$, then $f(\theta') = \overline{a}_j(\theta')$ for all θ' and $j \in I \backslash \{i\}$.

Remark 6. *It is w.l.o.g. to assume that $C_i(\theta)$ is a closed set for all i and θ in Condition $\lambda 0$. If it is not for some i and θ, we can take its closure, denoted as $\overline{C}_i(\theta)$. If there is no $(a, v) \in C_i(\theta)$ such that $v_i = \underline{v}_i$, but there exists such $(a, v) \in \overline{C}_i(\theta)$, then there also necessarily exists $(a, w) \in \overline{C}_i(\theta)$ such that $w_i > \underline{v}_i$. Hence, (a, v) cannot be the alternative-continuation pair in the definition of Condition $\lambda 0$.*

The following theorem is a counterpart to Theorem 2 for regimes with deterministic stage mechanisms:

Theorem 3. *When $n > 2$, f is repeatedly implementable with a regime that only uses deterministic stage mechanisms if and only if there is a collection $C = (C_i(\theta))_{i,\theta}$ such that $C_i(\theta) \subseteq A \times V$ for all i and θ, with respect to which f is dynamically monotonic and f satisfies Condition $\lambda 0$.*

Proof. We have already argued about the necessity of Condition $\lambda 0$ above. The proof of the necessity of DM in Theorem 2 still applies, except that now $C_i(\theta) \subseteq A \times V$ for all i and θ. To prove the sufficiency part, let $D = (D_i(\theta))_{i,\theta}$ be a collection w.r.t. which f is DM and satisfies Condition $\lambda 0$. We first define a new collection $C = (C_i(\theta))_{i,\theta}$ such that $C_i(\theta)$ is finite for all i and θ, it contains a pair $(a, v) \in M_i(D_i(\theta), \theta^*)$ for every θ^*, and it does not contain any pair $(a, v) \notin M_i(D_i(\theta), \theta^*)$ for some θ^*. (The last part ensures that if $(a, v) \in C_i(\theta)$ such that $v_i = \underline{v}_i$, then there is no $(a, w) \in C_i(\theta)$ such that $w_i > \underline{v}_i$) It is easy to see that f continues to satisfy DM and Condition $\lambda 0$ w.r.t. C. Next, as in Lemma 1, we modify the collection C by replacing a finite number of elements to ensure that $v_i < \overline{v}_i$ for all i, θ, $C_i(\theta)$, and $(a, v) \in C_i(\theta)$ without violating either DM or Condition $\lambda 0$. (To do that, we do not need random stage mechanisms.)

Now, to prove the sufficiency of DM and Condition $\lambda 0$, we can still use the regime r that is defined in the proof of the if-part of Theorem 2, after replacing ΔA with A everywhere. Also, we now allow $v'_{i^*} = \underline{v}_{i^*}$ in part 2(b)i of r. Therefore, Lemmas 6 and 7 still apply, except in Lemma 7 we need to consider one additional case when the period t messages fall under part 2(b)i of r and $v'_{i^*} = \underline{v}_{i^*}$. (The case with $v'_{i^*} > \underline{v}_{i^*}$ is covered in Claim 1 of Lemma 7.) Thus, suppose there is a Nash equilibrium s such that $s_i(h^t, \theta^t, \hat{\gamma}) = (\theta, b, v, z)$ for all $i \neq i^*$ and $s_{i^*}(h^t, \theta^t, \hat{\gamma}) = (\cdot, b', v', \cdot) \neq (\theta, b, v, z)$ with $(b', v') \in C_{i^*}(\theta)$ and $v'_{i^*} = \underline{v}_{i^*}$ for some t, θ^t, h^t such that $q(h^t|s, r)r(\hat{\gamma}|h^t) > 0$. Note that if this history occurs, the mechanism $\tilde{\gamma}_{i^*+1}(\underline{A}_{i^*})$ is played after period t forever. Since any agent $i \neq i^*$ can trigger the integer game, it must be that $b' = \overline{a}_i(\theta^t)$ for all $i \neq i^*$ and there exists π^0 such that $\underline{a}_{i^*}(\pi^0(\theta')) = \overline{a}_i(\theta')$ for all θ' and $i \neq i^*$. That is, agent $i^* + 1$ must be choosing $\underline{a}_{i^*}(\pi^0(\theta^\tau)) \in \underline{A}_{i^*}$ in state θ^τ in period $\tau > t$. Further, since agent i^* does not want to deviate by announcing something different from $s_{i^*}(h^t, \theta^t, \hat{\gamma}) = (\cdot, b', v', \cdot)$, it must be that $C_{i^*}(\theta) \subseteq L_{i^*}((b', v^{\underline{a}_{i^*}}(\pi^0)), \theta^t)$. Therefore, the premises of Condition $\lambda 0$ apply. But then it immediately follows from the condition that $g(s(h^t, \theta^t, \hat{\gamma}^t)) = b' = f(\theta^t)$ and $g(s(h^\tau, \theta^\tau, \tilde{\gamma}_{i^*+1}(\underline{A}_{i^*}))) = f(\theta^\tau)$ for all $\tau > t$, θ^τ, and h^τ for which h^t is a sub-history. $\square$

Example 2

The example illustrates f that is DM, but does not satisfy Condition $\lambda 0$.

Let $I = \{1, 2, 3\}$, $A = \{a, b, c, d, e\}$, $\Theta = \{\theta, \theta'\}$, $p(\theta) = \frac{1}{2}$, and $\delta = \frac{1}{3}$. The payoffs are summarized in the following table:

	$u_1(\cdot, \theta)$	$u_2(\cdot, \theta) = u_3(\cdot, \theta)$	$u_1(\cdot, \theta')$	$u_2(\cdot, \theta') = u_3(\cdot, \theta')$
a	4	6	2	2
b	0	6	1	6
c	6	6	3	4
d	2	2	1	0
e	2	6	0	6

Let $f(\theta) = d$ and $f(\theta') = a$. Thus, $v^f = (2, 2, 2)$. The only deception that we need to consider is a stationary one that is obtained by applying a static deception $\pi^0(\theta) = \theta'$ and $\pi^0(\theta') = \theta'$ in each period, since it results in the highest payoffs $v^f(\pi^0) = (3, 4, 4)$. We show that this deception can be eliminated and, hence, f can be made DM but only if $C_1(\theta')$ contains $(c, (0, 6, 6))$. (Importantly, there is no $(c, v) \in L_1((f(\theta'), v^f), \theta')$ with $v_1 > \underline{v}_1 = 0$.) For that we need to consider every possible deviation by each agent.

The deception will be eliminated if the following inequalities will hold for some agent i and some pair (x, v):

$$\begin{aligned}(1-\delta)u_i(x, \theta') + \delta v_i &\leq (1-\delta)u_i(a, \theta') + \delta v_i^f, \\ (1-\delta)u_i(x, \theta) + \delta v_i &> (1-\delta)u_1(a, \theta) + \delta v_i^f(\pi^0).\end{aligned}$$

In case of agent 1, these inequalities become

$$\begin{aligned}\frac{2}{3}u_1(x,\theta') + \frac{1}{3}v_1 &\leq 2, \\ \frac{2}{3}u_1(x,\theta) + \frac{1}{3}v_1 &> \frac{11}{3}.\end{aligned}$$

The above inequalities require that

- If $x = a$, then $v_1 \leq 2$ and $v_1 > 3$, which is impossible.
- If $x = b$, then $v_1 \leq 4$ and $v_1 > 11$, which is impossible.
- If $x = c$, then $v_1 \leq 0$ and $v_1 > -1$. Since $\underline{v}_1 = 0$, it follows that $(c,(0,6,6))$ is the unique alternative-continuation payoff pair that agent 1 can announce to eliminate the deception.
- If $x = d$, then $v_1 \leq 4$ and $v_1 > 7$, which is impossible.
- If $x = e$, then $v_1 \leq 6$ and $v_1 > 7$, which is impossible.

In case of agent 2 (and agent 3), these inequalities become

$$\begin{aligned}\frac{2}{3}u_2(x,\theta') + \frac{1}{3}v_2 &\leq 2, \\ \frac{2}{3}u_2(x,\theta) + \frac{1}{3}v_2 &> \frac{16}{3}.\end{aligned}$$

The above inequalities require that

- If $x = a$, then $v_2 \leq 2$ and $v_2 > 4$, which is impossible.
- If $x = b$, then $v_2 \leq -6$ and $v_2 > 4$, which is impossible.
- If $x = c$, then $v_2 \leq -2$ and $v_2 > 4$, which is impossible.
- If $x = d$, then $v_2 \leq 6$ and $v_2 > 12$, which is impossible.
- If $x = e$, then $v_2 \leq -6$ and $v_2 > 4$, which is impossible.

Thus, f is DM only if $(c,(0,6,6)) \in C_1(\theta')$.

We now argue that the regime r that is used to prove Theorem 3 fails to implement f. Thus, consider a strategy profile in which in state θ, agent 1 announces $(c,(0,6,6))$, while agents 2 and 3 report that it is state θ'. In the continuation, agent 2 selects alternative b in state θ and alternative e in state θ'. If agent 1 does not announce $(c,(0,6,6))$ in state θ, then agents 2 and 3 report the state honestly in all future periods. Now, if agent 1 announces $(c,(0,6,6))$ in state θ, he receives a payoff of:

$$\frac{2}{3}6 + \frac{1}{3}\left(\frac{1}{2}0 + \frac{1}{2}0\right) = 4.$$

If agent 1 does not announce $(c,(0,6,6))$ in state θ, his payoff is:

$$\frac{2}{3}4+\frac{1}{3}2=\frac{10}{3}<4.$$

Therefore, agent 1 does not want to deviate from the specified strategy. Since agents 2 and 3 receive their best alternatives, they also do not want to deviate. Hence, we have a Nash equilibrium, in which an undesirable alternative is implemented. The same argument can be obtained for any regime, which does not use stochastic stage mechanisms. If, instead, we can use stochastic mechanisms, then we can replace $(c,(0,6,6))\in C_1(\theta')$ with another alternative-continuation payoff pair that gives a lower current period payoff to agent 1 while we increase his continuation payoff. For example, agent 1 can demand a lottery between a and c with probabilities of $\frac{1}{3}$ and $\frac{2}{3}$, respectively, and a continuation payoff of $v_1=\frac{2}{3}$.

Finally, suppose now $u_1(c,\theta')=2$, while everything else remains as before. It is easy to verify that in order to eliminate the above deception π^0, we can include $(c,(2,2,2))$ instead of $(c,(0,6,6))$ in $C_1(\theta')$. In this case, f is repeatedly implementable with a regime that only uses deterministic stage mechanisms. Note, however, that f does not satisfy no-veto power and is not even weakly efficient in the range. Therefore, one cannot invoke Theorem 2 of Mezzetti and Renou (2017) or Theorem 2 of Lee and Sabourian (2011) to show that f is implementable. ■

D The Regime for Mixed Strategy Implementation

By Lemma 1, we can select a collection $C=(C_i(\theta))_{i,\theta}$ w.r.t. which f is DM and $\underline{v}_i<v_i<\overline{v}_i$ for all i, θ, $C_i(\theta)$, and $(a,v)\in C_i(\theta)$. In the regime, we are going to use the mechanisms $\hat{\gamma}$ and $\tilde{\gamma}_i(\mathcal{M}_i)$ that are defined in the proof of Theorem 2, but additionally we will use the following, slightly modified, dictatorial mechanism for agent i:

Mechanism $\dot{\gamma}_i$. Let $\mathcal{M}_i=\{(b,z)\in A\times\mathbb{Z}_+\}$, while $\mathcal{M}_j=\{\emptyset\}$ for all $j\in I\backslash\{i\}$. Let $g(m)=b$.

The regime is defined as follows:

Regime $\dot{r}$.

1. $\dot{r}(\hat{\gamma}|h^0)=1$.

2. For $t\geq 1$ if $\dot{r}(\hat{\gamma}|h^{t-1})=1$ and $m^{t-1}=(m_i)_{i\in I}$ is such that

   (a) Parts (I) or (IIb) of $\hat{\gamma}$ applies, then $\dot{r}(\hat{\gamma}|h^t)=1$.

(b) Part (IIa) of $\hat{\gamma}$ applies with $m_{i^*} = (\theta', b', v', z')$, then

$$\begin{aligned} \dot{r}(\dot{\gamma}_{i^*}|h^t) &= \lambda, \\ \dot{r}(\tilde{\gamma}_{i^*+1}(\underline{A}_{i^*})|h^t) &= 1 - \lambda, \end{aligned}$$

where

$$\lambda = \frac{v'_{i^*} - \underline{v}_{i^*}}{\overline{v}_{i^*} - \underline{v}_{i^*}}.$$

(c) Part (III) of $\hat{\gamma}$ applies, then $\dot{r}(\dot{\gamma}_{i_z}|h^t) = 1$.

3. For $t \geq 2$, if $\dot{r}(\dot{\gamma}_i|h^{t-1}) = 1$ for some i and $m_i^{t-1} = (b, z)$, then

$$\begin{aligned} \dot{r}(\tilde{\gamma}_i(A)|h^t) &= \frac{z}{1+z}, \\ \dot{r}(\tilde{\gamma}_i(\{a\})|h^t) &= \frac{1}{(1+z)|A|} \text{ for all } a \in A. \end{aligned}$$

4. For $t \geq 2$, if $\dot{r}(\tilde{\gamma}_i(\mathcal{M}_i)|h^{t-1}) = 1$ for some i and $\mathcal{M}_i$, then $r(\tilde{\gamma}_i(\mathcal{M}_i)|h^t) = 1$.

First, note that in any equilibrium, the mechanism $\hat{\gamma}$ must be selected by the regime in every period. Suppose not. Then, with a strictly positive probability, the mechanism $\dot{\gamma}_i$ for some i is selected. But agent i can always increase his payoff by announcing higher integer, which contradicts that it is an equilibrium. Second, since the mechanism $\hat{\gamma}$ must be selected in every period, then for any equilibrium, possibly in mixed strategies, there exists an outcome-equivalent equilibrium in pure strategies, in which all agents send identical messages in each period. But then it follows from DM that a socially desired outcome is implemented in every period. Note that while the odd-man-out i^* in (IIa) of $\hat{\gamma}$ does not receive in expectation the continuation payoff v'_{i^*} exactly, by announcing a sufficiently large integer in the next period if $\dot{\gamma}_{i^*}$ is selected, he can ensure a continuation payoff arbitrarily close to v'_{i^*}. Therefore, his incentives to deviate are unaltered. Finally, along the lines of Lemma 6, we can establish that the regime possesses a Nash equilibrium.