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The impact of stochastic lead times on the bullwhip effect under correlated demand and moving average forecasts

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Abstract

We quantify the bullwhip effect (which measures how the variance of replenishment orders is amplified as the orders move up the supply chain) when both random demands and random lead times are estimated using the industrially popular moving average forecasting method. We assume that the lead times constitute a sequence of independent identically distributed random variables and the correlated demands are described by a first-order autoregressive process.

We obtain an expression that reveals the impact of demand and lead time forecasting on the bullwhip effect. We draw a number of conclusions on the bullwhip behaviour with respect to the demand auto-correlation and the number of past lead times and demands used in the forecasts. We find the maxima and minima in the bullwhip measure as a function of the demand auto-correlation.

Keywords: Supply chain, Bullwhip effect, Order-up-to replenishment policy, AR(1) demand, Stochastic lead time, Moving average forecasting method.

1 Introduction

The variability of replenishment orders often increases as they flow upstream in supply chains. This phenomenon is known as the bullwhip effect and has been discussed in the economics and

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operations management literature for 100 and 50 years respectively–see Mitchell [31] and Forrester [21]. The celebrated works of Lee et al., [28] and [29] promoted this problem to the forefront of the supply chain and operations management field. Wang and Disney [42] provide a recent literature review of the bullwhip field, categorising contributions according to the five causes of bullwhip of Lee et al., [28] namely: demand forecasting, non-zero lead time, supply shortage, order batching, and price fluctuation. Of particular importance to this paper are the results of Alwan et al., [2], Chen et al., [14], [15] and Dejonckheere et al., [16]. These contributions investigate the bullwhip consequences of using the moving average method to forecast demand inside the order-up-to (OUT) replenishment policy.

Recently Michna and Nielsen [32] identified another critical cause of the bullwhip–the forecasting of lead times. While the issue of stochastic lead times in bullwhip studies has not been intensively investigated, Michna and Nielsen [32], Michna et al., [35], [34], and Nielsen and Michna [36] provide a recent literature review of this problem. Of particular importance is the work of Duc et al., [20] and Kim et al., [26] where the impact of stochastic lead times on bullwhip is quantified. These works characterise the impact of random lead times on the bullwhip effect via mean values and variances. However, they do not consider the consequences of having to estimate the lead time distribution (a.k.a. lead time forecasting). As identified by Michna and Nielsen [32] and Michna et al., [35] this can be a significant cause of the bullwhip effect. In Duc et al., [20] lead times are assumed to be stochastic, drawn from a known distribution, and are not forecasted when placing an order as the last observed lead time is used to set the OUT levels. Kim et al., [26] used the moving average technique to forecast lead time demand, as did Michna et al., [33].

The influence of stochastic lead time on inventory is a more established field and we refer to the work of Bagchi et al., [3], Chaharsooghi et al., [9], Song [39] and [40], and Zipkin [47]. Stochastic lead time inventory research can be classified into two general streams: those with order crossovers and those without. An order crossover happens when replenishments are received in a different sequence from which the orders were placed (see e.g. Bischak et al., [4], Bradley and Robinson [8], Disney et al., [19] and Wang and Disney [43]). Disney et al., [19] consider the safety stock and inventory cost consequences of using the OUT and proportional order-up-to (POUT) replenishment policies under i.i.d. demand. They show the linear POUT policy is always more economical than the linear OUT policy when order-crossover is present. Wang and Disney [43] show the POUT policy outperforms the OUT policy in the presence of order crossovers in the sense of minimizing inventory variance when demand is an Auto-Regressive Moving Average process with p auto-regressive terms and q moving average terms, ARMA(p,q).

The papers of Boute et al., [5], [6], [7] investigate endogenous lead times in supply chains. Endogenous lead times are dependent on the state of the system as they are function of the previous orders. Here the supplier is modelled as a queue and orders are processed on a first come, first served basis, hence there is no order-crossover. However, as the sojourn time in the queue increases in the variance of the demand placed on the manufacturer, a lead time reduction can be obtained by smoothing the replenishment orders. This lead time reduction can potentially reduce safety

stock requirements. Hum and Parlar [23] also model lead times using queueing theory, analyzing the proportion of demand that can be met within a specific lead time.

1.1 Practical relevance

We have observed that stochastic lead times and order-crossovers are quite common within factories, see Fig. 1. The data represents a single, high volume, product from a supplier of industrial measuring and testing equipment. The distribution of the lead times is discrete and aggregated into weekly buckets to reflect the practice of creating weekly production plans using the OUT policy (for more information on why this is so, we refer to the assumptions and modelling choices discussed later in this section). Fig. 1 also highlights the number of queue positions each production batch gained or lost between the two lists of date sorted production releases and production completions. As this manufacturer manually moved totes of products between process steps within its job shop, a large number of order-crossovers is present. Disney et al., [19] present similar empirical findings in global supply chains (see Figs. 1 and 2 of [19]), where stochastic lead times and order crossovers could be observed in global shipping lanes. Here containers could also gain or lose positions in the date ordered list of dispatches and receipts. We also observe differences in quoted (at the time of shipping) and actual (realised when the container arrives) lead times in global shipping lanes (see Fig. 2). This implies that we are unsure of an order's lead time until an order arrives.

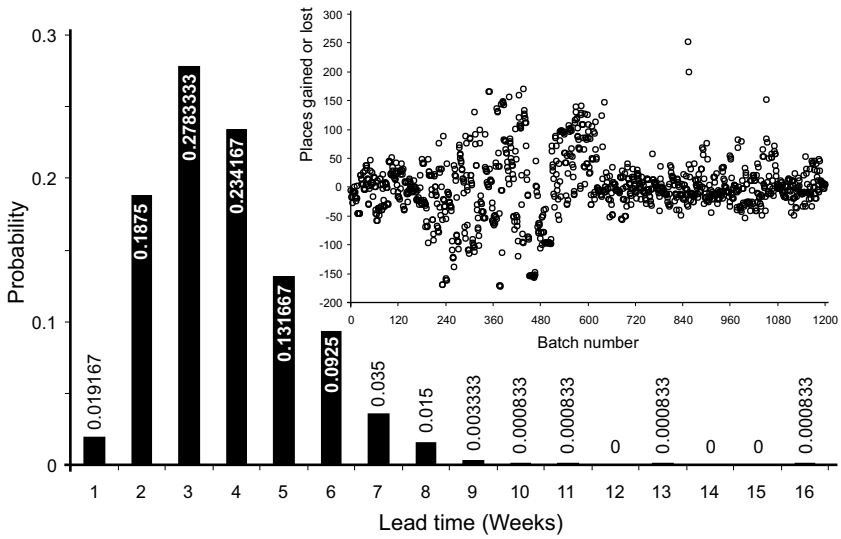

Figure 1: Stochastic lead times and order-crossovers observed in a measuring equipment supplier

We consider a model where a supply chain member (who could be either a retailer, manufacturer, or supplier, but we call a manufacturer for convenience) observes both random demands from his customer and random lead times from his supplier which we assume to be exogenous (that is, they are independent of all other system states). The manufacturer generates replenishment orders to maintain inventory levels by projecting his customers' future demands over his supplier's lead time,

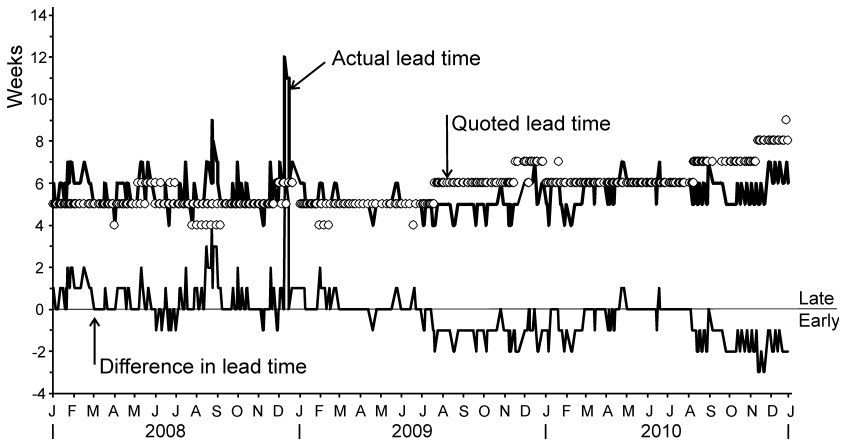

Figure 2: The quoted and actual lead times in a global shipping lane

accounting for both the available inventory and the open orders in the replenishment pipeline.

1.2 Contribution

Our contribution differs from previous research in several ways. Most importantly we show that lead time forecasting is a major cause of bullwhip when demands are auto-correlated. This confirms and extends the results of Michna and Nielsen [32] who studied the i.i.d. demand case. We also quantify the impact of the stochastic auto-correlated demands and stochastic lead times on the bullwhip effect under the practically plausible assumption that demands and lead times are forecasted separately using moving averages. Furthermore, we investigate the bullwhip effect as a function of the demand auto-correlation, the characteristics of the lead time distribution, and the number of past demands used in the moving average lead time forecasts. The bullwhip conclusions differ depending on how the parameters are combined. We find maxima and minima in the bullwhip metric as a function of the demand auto-correlation.

Our main result contains as special cases: the bullwhip formulas of Chen et al., [14] (a constant lead time), Theorem 1 in Michna and Nielsen [32] (mutually independent demands), and Wang and Disney [43] (known lead time distribution). The formulation presented in this research involves more parameters, is more general, and allows us to understand more intricate supply chain settings. Our contribution is important as it gives guidance on the consequences of estimating lead times in practical settings; when a short history of lead time data is used to estimate lead times, bullwhip can be introduced by the lead time forecasting mechanism. This is a little discussed source of the bullwhip effect.

1.3 Major assumptions and modelling choices

Our major assumptions and modelling choices are summarised as follows:

a) The supply chain consists of two stages – a manufacturer who receives client's demands and

deliveries from a supplier (or manufacturing process).

b) A periodic replenishment system exists where the demands, D_t, are satisfied and previous orders placed are received during a time period, indexed by the subscript t. At the end of the period, the inventory levels, demand, and lead times of received orders are observed and a new replenishment order, q_t, is placed. The length of the period could be an hour, day, week, or month, but in our experience it is often a week in manufacturing contexts. Note, the receipt of an order is observed only at the end of the period and the lead time is a non-negative integer. An order with zero lead time would be received instantaneously after the order was placed, but its receipt would only be incorporated into the order made at the end of the next period due to the sequence of events.

c) The demand constitutes a first order autoregressive model, AR(1). We have elected to use the AR(1) model as it is the simplest demand process with autocorrelation, a feature commonly observed in real demand patterns, Lee et al., [30]. Ali et al., [1] find, in a European grocer, that 30.3% of 1798 SKU's were AR(1). A further 16.3% were deemed to be i.i.d. and 2.1% were integrated, ARIMA(0,1,0), demands. As all three of these demand processes can be modelled with the AR(1) model, 48.7% of a European grocer's SKUs were represented by this assumption. AR(1) demand is also a frequently adopted assumption in the bullwhip literature (e.g. in Chen et al., [14] and [15], Duc et al., [20] and Lee et al., [30]), allowing comparison of our new results to established theory.

d) The lead times $L_t \in \mathbb{N}_0$ constitute a sequence of independent identically distributed (i.i.d.) random variables which are independent of all system states, including the manufacturer's demand. Moreover, we assume that lead times are bounded and non-negative (e.g. $0 \leq L_t \leq L^+$ periods). This implies that all the moments of the lead time distribution are finite. Furthermore, the lead time forecasts are based on lead time information that is at least L^+ periods old. This allows us to create lead time forecasts that are unbiased. For example, if we based our lead time forecasts on the most recent lead time information (which we observe when we receive orders), some of the orders placed would still be open (not yet received) and our lead time estimates would be biased towards those orders with short lead times. Basing our lead time estimates on data that is at least L^+ periods old is possible as lead times are assumed to be temporally independent and thus constitute a valid dataset for forecasting all future lead times. Practically this approach has the desirable characteristic that we can base our lead time estimates on realised lead times, rather than quoted lead times from the supplier or shipper, see Fig. 2. Furthermore, for ease of data organisation (and modelling) we can retrospectively assign the lead time of an order to the period the order was generated in our simulation.

e) The OUT policy is used to generate the orders placed onto the supplier. The OUT policy is industrially popular as it is available native in many ERP/MRP systems. It has also been

studied extensively in the academic literature (see e.g. Alwan et al., [2], Bishak et al., [4], Chen et al., [14] and [15], Dejonckheere et al., [16] and [17], Duc et al., [20], Kim et al., [26], and Wang et al., [44]). The OUT policy is also the optimal linear replenishment policy for minimizing inventory holding and backlog costs if orders do not cross (see Kaplan [25]).

f) The manufacturer predicts the future demands over future lead times based on predictions generated using the moving average forecasts of past demand and of the lead times of previously received orders. Thus, the forecast of lead time demand is as follows

$$\widehat{D_t^L} = \widehat{L_t}\widehat{D}_{t+i}\,, \tag{1}$$

where $\widehat{L_t}$ is the forecast of the lead time of the next order made at the beginning of period t (defined later in (7)) and $\widehat{D}_{t+i}$ denotes the forecast for a demand for the period $t+i$ made at the beginning of a period t.

1.4 Position within the literature

As in Michna and Nielsen [32], the novel aspect of our approach is the last point f) and differs from much of the previous literature. For example, Duc et al., [20] assumes the lead time of the order placed at time t is known when placing order. This leads to

$$\widehat{D_t^L} = \sum_{i=0}^{L_t-1} \widehat{D}_{t+i}\,. \tag{2}$$

The approach of Duc et al., [20] is to forecast lead time demand using only the last instance of the lead time and a current estimate of demand. Note, we assume the manufacturer does not know the value of L_t until that order has arrived (received), whereas Duc et al., [20] assumes the lead time is known when placing the order. Duc et al., [20] assumes that lead time is stochastic and demand process is either a first-order autoregressive AR(1), or a first-autoregressive moving average ARMA(1,1) process. Minimum mean squared error (MMSE) forecasts are used inside the order-up-to policy to determine the replenishment order. The main difference between the model of Duc et al., [20] and ours is that Duc et al., [20] needs a concrete lead time distribution, whereas our model requires only the first two moments of the lead time distribution.

Chatfield et al., [12] simulate a four echelon supply chain with gamma distributed lead-times in an OUT policy based supply chain with different levels of inventory sharing. A simulation in the SISCO software was used to quantify the level of bullwhip experienced. Increasing levels of information sharing where found to lead to less bullwhip at upstream supply chain levels. Chatfield et al., [12] estimated the parameters (mean and standard deviation) of the lead time demand distribution. Predictions of lead time demand were not created, but instead the mean and variance of the lead time demand distribution are estimated to allow updating of the order-up-to level. Moreover, they define four different levels of information quality (IQL 0, 1, 2, 3) and the estimation

approach differs based on the quality type of the information. At level IQL0 there is no updating of demand or lead-time information used (that is, the demand and lead time distributions are known, as was the case in Bradley and Robinson [8] and Wang and Disney [43]). IQL1 and IQL2 use separate demand and lead time observations to estimate the characteristics of the lead time demand distribution. In these cases, the demand mean and variance are estimated with moving average and moving variance methods while the lead time mean and variance are estimated with a running average and a running variance. Thus, simple forecasts of the lead time are used to identify the lead time distribution parameters. For IQL3, moving average and variance is applied to the actual lead time demand observations. No separate lead time estimates are created when using IQL3. This same approach is used in Chatfield [10].

In Kim et al., [26] the lead time demand is predicted with

$$\widehat{D_t^L} = \frac{1}{n} \sum_{j=1}^{L_t-1} \sum_{i=1}^{n} D_{t-i+j} \,, \tag{3}$$

which under a constant lead-time, has the property of producing bullwhip as a linear function, rather than a quadratic function, of the lead-time. They then utilized knowledge of the variance of the random sum of random variables to find the variance of lead time demand under a stochastic i.i.d. lead-time. Notably, this approach was extendable to a multi-echelon setting, with and without information sharing, under AR(1) demand. Kim et al., [26] also measure bullwhip using their lead time demand moving average forecasting approach under AR(1) demand for a constant lead time.

A different approach was taken by Bradley and Robinson [8], Disney et al., [19], Wang and Disney [43], where it is assumed beforehand that the lead time distribution is known. That is, the complete lead time distribution can be observed from previous realisations of the lead time and does not need to be forecasted. The consequences of this assumption are that stochastic lead times have no influence on the order variance under i.i.d. demand, Disney et al., [19].

Herein, we study the estimation of lead time based on the product of a moving average of past demand observations and a moving average of past lead time observations. This is important as a moving average forecast of lead times is a practically intuitive and feasible approach but little is known of its consequences. We will show that little is gained from forecasting lead times. We find lead time forecasting introduces bullwhip, but that increasing the history used in the moving average forecast reduces the bullwhip effect. We also show the bullwhip measure contains new components depending on the lead time forecasting parameter, and the demand auto-correlation coefficient. This was not quantified in Michna and Nielsen [32], neither was it included in the study of ARMA(p,q) demand in Wang and Disney [43]. These new terms amplify the bullwhip measures and are evidence that lead time estimation in itself is a significant cause of the bullwhip effect, perhaps equally as important as demand forecasting.

2 Supply chain model

We want to consider temporally dependent demands and the simplest way to achieve this is to model a manufacturer observing periodic customer demands, D_t, constituting of a stationary first-order autoregressive, AR(1), process,

$$D_t = \mu_D + \rho(D_{t-1} - \mu_D) + \epsilon_t \,, \tag{4}$$

where $|\rho| < 1$ ensures a stationary demand process and $\{\epsilon_t\}_{t=-\infty}^{\infty}$ is a sequence of independent identically distributed random variables such that $\mathbb{E}(\epsilon_t) = 0$ and $\mathbf{Var}(\epsilon_t) = \sigma^2$. Under the stationarity assumption it can be easily found that $\mathbb{E}(D_t) = \mu_D$, $\mathbf{Var}(D_t) = \sigma_D^2 = \frac{\sigma^2}{1-\rho^2}$ and $\mathbf{Corr}(D_t, D_{t-k}) = \rho^k$ (see for example, Chen et al., [14] and Duc et al., [20]). The distribution of ϵ can be arbitrary but its second moment must be finite.

A random lead time L_t is assigned to each order at the beginning of time t. It is observed and used to predict future lead time when the order is received. The random lead times $\{L_t\}_{t=-\infty}^{\infty}$ are mutually i.i.d. random variables that were also assumed in Duc et al., [20], Kim et al., [26], Robinson et al., [37] and Disney et al., [19]. The expected value of the discrete lead times is $\mathbb{E}L_t = \sum_{i=0}^{L^+} ip_i = \mu_L$ where p_i is the probability that the lead time is i periods long, $\mathbf{Var}(L_t) = \sum_{i=0}^{L^+} p_i(i-\mu_L)^2 = \sigma_L^2$. The sequences $\{D_t\}_{t=-\infty}^{\infty}$ and $\{L_t\}_{t=-\infty}^{\infty}$ are mutually independent.

The lead time demand at the beginning of a period t is defined as

$$D_t^L = D_t + D_{t+1} + \ldots\ldots + D_{t+L_t-1} = \sum_{i=0}^{L_t-1} D_{t+i} \,, \tag{5}$$

which reflects the demand over the lead time. At the beginning of period t the manufacturer does not know this value of L_t so he must forecast its value before calculating his replenishment order (see (1)).

Let us notice that there is a dependency between $\widehat{D_t^L}$ and $\widehat{L_t}$ due to (1). That is, the lead time demand forecast is a function of past lead times. Employing the moving average forecast method with the integer delay parameter $n \geq 1$ for demand forecasting we get

$$\widehat{D}_{t+j} = \frac{1}{n}\sum_{i=1}^{n} D_{t-i} \,, \tag{6}$$

where $j = 0, 1, \ldots$ is the j-periods ahead forecats and D_{t-i} $i = 1, 2, \ldots, n$ are previous demands which have been observed at the beginning of period t. The j-period ahead forecast of demand is a moving average of previous demands. Note, all future forecasts, regardless of j, are straight line predictions of the current forecast. Clearly this is not an optimal, MMSE forecast of AR(1) demand. However, it does reflect common industrial practice as the moving average forecast is available in many commercial ERP systems and can be readily incorporated into spreadsheets by analysts. It has also been studied from a theoretical basis (see Chatfield et al., [12], Chatfield and

Hayya [11], Chen et al., [14], Dejonckheere et al., [16] and Kim and Ryan [27]). Yan et al., [45] verify the results of Chen et al., [14], providing insights into how different lead times in different supply chain echelons influence the bullwhip effect.

The manufacturer also predicts the lead time but has to be careful because the previous orders cannot be completely observed. Precisely, using the moving average forecast method with $m \geq 1$ for lead time forecasting we obtain

$$\widehat{L_t} = \frac{1}{m} \sum_{i=1}^{m} L_{t-i-L^+} \,, \tag{7}$$

where L_{t-i-L^+} are lead times which are guaranteed to have been observed by the manufacturer at the beginning of a period $t-i$ (or earlier) as they are at least L^+ periods old (see item d of our discussion of assumptions in § 1). Knowing the average lead time (in practice estimating it) we are able to find the average unrealized orders (see Robinson et al., [37] and Disney et al., [19]). However, our procedure of collecting lead times avoids bias resulting from the open orders with long lead times that have not been received when we make the lead time forecast. Thus by (1), (6), and (7) we propose the following forecast for a lead time demand (see also Michna and Nielsen [32]).

$$\widehat{D_t^L} = \widehat{L_t}\widehat{D_t} = \frac{1}{mn} \sum_{i=1}^{n} D_{t-i} \left(\sum_{i=1}^{m} L_{t-i-L^+} \right). \tag{8}$$

The motivation for the lead time demand forecasting mechanism given in (8) is that $\mathbb{E}(D_t^L) = \mathbb{E}(L)\mathbb{E}(D)$ (under the assumption that demands and lead times are mutually independent) and employing the natural estimators of $\mathbb{E}(L)$ and $\mathbb{E}(D)$ we arrive at (8).

We assume the manufacturer uses the OUT policy to generate replenishment orders. Let S_t be the desired inventory position at the beginning of a period t,

$$S_t = \widehat{D_t^L} + TNS, \tag{9}$$

where TNS is a constant, time invariant, target net stock (safety stock), set to achieve a desired level of availability or to minimize a set of unit inventory holding (h) and unit backlog (b) costs via the newsvendor principle, Silver et al., [38]. It is often assumed in constant lead time scenarios that the demand and the inventory levels are normally distributed and thus

$$TNS = z\widehat{\sigma_t};\ z = \Phi^{-1}\left(\frac{b}{b+h}\right)$$

holds, where $\Phi^{-1}(\cdot)$ is the cumulative probability density function (cdf) of the standard normal distribution and

$$\widehat{\sigma_t}^2 = \mathbf{Var}(D_t^L - \widehat{D_t^L})$$

is the variance of the lead time demand forecast error. In some articles (for example Chen et al., [14]) $\widehat{\sigma_t}^2$ is defined more practically. That is, instead of the variance, one takes the sample variance

of $D_t^L - \widehat{D_t^L}$. This complicates the theoretical calculations somewhat and the estimation of $\widehat{\sigma_t}^2$ increases the bullwhip effect[1].

Note, in our modelling setting, even when demand is normally distributed, neither the inventory levels, nor the orders, are normally distributed. Rather the stochastic lead times create a multi-modal inventory distribution (as it did in Disney et al., [19]) and the lead time forecasting mechanism creates a multi-modal order distribution (which was not present in the setting considered by Disney et al., [19] as the lead time distribution was assumed to be known beforehand). Thus, the TNS must be set with

$$TNS = F^{-1}\left(\frac{b}{b+h}\right)$$

where $F^{-1}(\cdot)$ is the cdf of the inventory levels (arbitrary distribution).

The order quantity q_t placed at the beginning of a period t by the OUT policy is

$$q_t = S_t - S_{t-1} + D_{t-1}\,. \tag{10}$$

Note that by (8), (9), and (10), the quantity of the order placed by the manufacturer to the supplier depends upon the supplier's lead time.

Our quest is to find $\mathbf{Var}(q_t)$ and then to calculate the following bullwhip ratio

$$BM = \frac{\mathbf{Var}(q_t)}{\mathbf{Var}(D_t)}\,.$$

This is one of the typical supply chain performance measurements (see e.g. Towill et al., [41]).

Proposition 1 *The variance of the forecast error over the lead time demand does not depend on t. That is, $\widehat{\sigma_t}^2 = \widehat{\sigma}^2$.*

Proof: The variance of the forecast error is the expected value of a function of $D_{t-n}, \ldots, D_{t-1}, D_t, D_{t+1}, \ldots$ and $L_{t-m-L^+}, L_{t-m+1-L^+}, \ldots, L_{t-1-L^+}, L_t$ whose distribution is independent of t. The stationarity of the sequences $\{D_t\}_{t=-\infty}^{\infty}$ and $\{L_t\}_{t=-\infty}^{\infty}$ and their mutual independence yield the assertion. $\square$

Since the variance of the forecast error for the lead time demand is independent of t, from (9) and (10) we get,

$$q_t = \widehat{D_t^L} - \widehat{D_{t-1}^L} + D_{t-1}\,, \tag{11}$$

allowing us to derive the exact bullwhip expression.

[1]This can be deduced from the fact that Chen et al.'s [14] formula is a lower bound for the bullwhip measure whereas we get an equality.

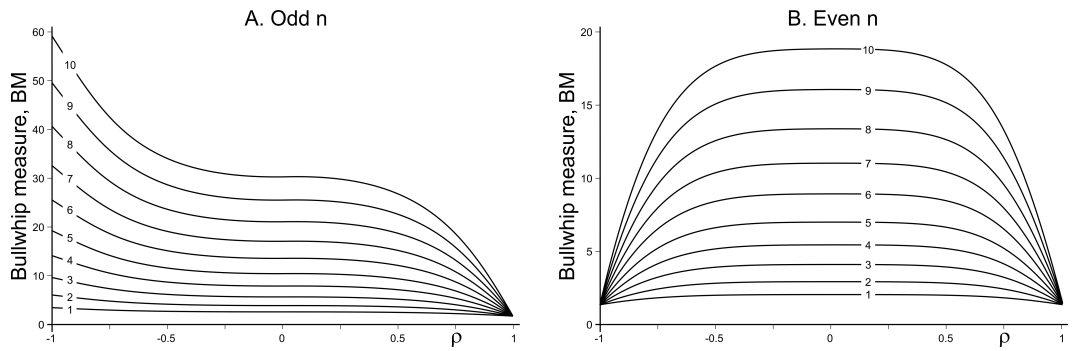

Figure 3: The impact of the mean lead time, μ_L on bullwhip. Parameters: Panel A, $\{n = 3, m = 4, \sigma_D = 50, \sigma_L = 1, \mu_D = 80\}$, Panel B, $\{n = 4, m = 4, \sigma_D = 50, \sigma_L = 1, \mu_D = 80\}$

Theorem 1 *The measure of the bullwhip effect has the following form,*

$$BM = \frac{2\sigma_L^2}{n^2 m^2}\left(m(1-\rho^n) + \frac{n(1+\rho)}{1-\rho} - \frac{(1+\rho^2)(1-\rho^n)}{(1-\rho)^2}\right) + \frac{2\sigma_L^2\mu_D^2}{\sigma_D^2 m^2} + \left(\frac{2\mu_L^2}{n^2} + \frac{2\mu_L}{n}\right)(1-\rho^n) + 1. \tag{12}$$

Proof: The proof of Theorem 1 is given in Appendix 1. □

Remarks on Theorem 1. Notice, due to the ρ^n terms, there is an odd-even effect created by n when $\rho < 0$ that we will explore numerically later in § 3. The first summand (12) describes the impact of lead time variability, demand and lead time forecasting, and the demand correlation and is responsible for the complex bullwhip behaviour (its maxima and minima) that we will highlight later. The second summand shows an impact of lead time forecasting, demand mean and variance, and lead time variance on the bullwhip effect. The first two summands are not present in the constant lead time case. The third term gives the amplification of the variance by the demand forecasting mechanism, the demand correlation, and the mean lead time.

If lead times are deterministic (that is $L_t = L = const.$) then the bullwhip effect is described by

$$BM_{L=const.} = \left(\frac{2L^2}{n^2} + \frac{2L}{n}\right)(1-\rho^n) + 1 \tag{13}$$

which coincides with Eq. 5 in Chen et al., [14]. Eq. (12) (and (13)) is increasing in L (μ_L), see Fig. 3.

Note that Duc et al., [20] also obtained the result of Chen et al., [14] in a special case and as an exact value (not a lower bound). Chen et al., [14] obtain this expression as a lower bound because they defined the error $\widehat{\sigma}_t$ (see (9)) as the sample variance of $D_t^L - \widehat{D_t^L}$, indicating that the estimation of the variance of $D_t^L - \widehat{D_t^L}$ amplifies the bullwhip effect.

The following limits of (12) exist:

$$\lim_{n\to\infty} BM = 1 + \frac{2\mu_D^2\sigma_L^2}{m^2\sigma_D^2}, \tag{14}$$

$$\lim_{m\to\infty} BM = 1 + (1-\rho^n)\left(\frac{2\mu_L^2}{n^2} + \frac{2\mu_L}{n}\right), \tag{15}$$

$$\lim_{\{n,m\}\to\infty} BM = 1. \tag{16}$$

The last result, (16) was also reported by Wang and Disney, [42]. It is easy to see from (12) that bullwhip is strictly decreasing in m, but this is not true for n as there is an odd-even effect in n for negative ρ, see Fig. 3. When $n=1$ then the BM is a linear function in ρ as

$$BM_{n=1} = \frac{\rho 2\sigma_D^4\left(\sigma_L^2 - m\left(m\mu_L\left(\mu_L+1\right)+\sigma_L^2\right)\right)}{m^2} + \frac{2\mu_D^2\sigma_D^2\sigma_L^2 + m\sigma_D^4\left(2m\mu_L\left(\mu_L+1\right)+2\sigma_L^2+m\right)}{m^2}, \tag{17}$$

which always has a negative gradient in ρ (unless $\mu_L = 0$ and $m=1$, in which case the gradient is zero).

For i.i.d. demand, the following bullwhip measure exists

$$\begin{aligned} BM_{iid} &= 1 + \frac{2\mu_L^2}{n^2} + \frac{2\mu_L}{n} + \frac{2\mu_D^2\sigma_L^2}{m^2\sigma_D^2} - \frac{2\sigma_L^2}{m^2n^2} + \frac{2\sigma_L^2}{m^2n} + \frac{2\sigma_L^2}{mn^2} \\ &= 1 + \frac{2\mu_D^2\sigma_L^2}{m^2\sigma_D^2} + \frac{2\sigma_L^2(m+n-1)}{m^2n^2} + \frac{2\mu_L\left(\mu_L+n\right)}{n^2} \end{aligned} \tag{18}$$

which is strictly decreasing in n and m and the result is consistent with Michna and Nielsen [32].

Note, we do not recover the results of Duc et al., [20] when $m=1$ as we have a different forecasting mechanism. However, for i.i.d. demand, setting $n\to\infty$ (to create an MMSE forecast of i.i.d. demand with the moving average forecasting method) and $m=1$ (such that only one lead time, the lead time associated to the order L^+ periods ago is used to create a lead time forecast) reveals that Duc et al.'s [20] bullwhip measure and BM measure are equal under these settings. Letting $\rho\to 1$ produces the Integrated (ARIMA(0,1,0)) demand process. The MMSE forecast of the Integrated demand process is accessed by letting $n=1$. Further setting $m=1$ in our model results in the same bullwhip measure as Duc et al., [20] as $\rho\to 1$. For all other ρ the results of Duc et al., [20] are slightly different to ours due to the different forecasting mechanism. However, for odd n, the bullwhip measure of Duc et al., [20] is qualitatively similar to ours.

Consider now the results of Kim et al., [26], where the lead time demand is forecasted directly, rather than by multiplying the demand forecast with the lead time forecast. A set of numerical results are given in Table 1. Kim et al.'s [26] policy only requires one forecasting parameter, whereas we require two, n and m. Selecting $n=m$ to make a comparison reveals that Kim et al.'s [26] policy produces less bullwhip than ours. However, bullwhip reduces in n; when $n=2m$ our BM is slightly below Kim et al.'s [26], when $m\to\infty$, coinciding with the MMSE forecast of the demand, we reach a lower bound in the BM measure.

The derivative of the bullwhip measure in (12) at $\rho=0$ for $n>1$ is

$$\left.\frac{\mathrm{d}BM}{\mathrm{d}\rho}\right|_{\rho=0} = \frac{4(n-1)\sigma_L^2}{m^2n^2} \tag{19}$$

m	1	2	3	4	5	6	7	8	9	10
Kim et al., [26]	31.3	9.075	4.81111	3.26875	2.532	2.11944	1.86327	1.69219	1.5716	1.483
BM, $n = m$	37.5	11.0137	5.8516	3.9593	3.04291	2.52276	2.19551	1.97421	1.81637	1.69906
BM, $n = 2m$	29.695	8.60281	4.58031	3.12959	2.43769	2.05057	1.81031	1.64989	1.53684	1.45376
BM, $n = \infty$	25.5	7.125	3.72222	2.53125	1.98	1.68056	1.5	1.38281	1.30247	1.245

Table 1: BM comparison of our approach to lead time demand forecasting method of Kim et al., [26] under i.i.d. demands. Parameters: $\{\mu_D = 10, \sigma_D = 2, \mu_L = 1.9, \sigma_L = 0.7\}$

which is always positive (for $n = 1$, the derivative is negative for all $-1 < \rho < 1$).

When $\rho \to 1$ the following expression defines the bullwhip measure

$$BM_{\rho\to 1} = 1 + \frac{2\sigma_L^2\left(\mu_D^2 + \sigma_D^2\right)}{m^2\sigma_D^2} \tag{20}$$

which is independent of n and decreasing in m. Notice $BM_{iid} \geq BM_{\rho\to 1}$ if

$$n \leq \frac{\sigma_L^2 + m^2\mu_L + \sqrt{(\sigma_L^2 + m^2\mu_L)^2 + 4\sigma_L^2(\sigma_L^2(m-1) + m^2\mu_L^2)}}{2\sigma_L^2}\,. \tag{21}$$

Note, if the lead time is a constant then $BM_{iid} > BM_{\rho\to 1}$, except when $m = 0$ as then $BM_{iid} = BM_{p\to 1}$.

If $\rho \to -1$ then

$$BM_{\rho\to -1} = 1 + \frac{2\mu_D^2\sigma_L^2}{m^2\sigma_D^2} - \frac{(2m-1)\left((-1)^n - 1\right)\sigma_L^2}{m^2n^2} - \frac{2\left((-1)^n - 1\right)\mu_L\left(\mu_L + n\right)}{n^2} \tag{22}$$

which is decreasing in m, but the odd-even impact of n can be clearly seen in the exponent of the (-1); when n is odd then

$$BM_{\rho\to -1,\text{ odd } n} = 1 + \frac{2\mu_D^2\sigma_L^2}{m^2\sigma_D^2} + \frac{2\sigma_L^2(2m-1)}{m^2n^2} + \frac{4\mu_L\left(\mu_L + n\right)}{n^2}\,, \tag{23}$$

and when n is even then

$$BM_{\rho\to -1,\text{ even } n} = 1 + \frac{2\mu_D^2\sigma_L^2}{m^2\sigma_D^2}\,. \tag{24}$$

Numerical investigations (see the $\{n = 6, n = 16\}$ cases in Fig. 5 and the $\{n = 6, n = 22\}$ cases in Fig. 6) seem to suggest that there are no stationary points in the region $-1 < \rho < 0$ when n is even, but we remain unable to prove so. However, this is congruent with our previous results as $BM_{iid} > BM_{\rho\to -1,\text{ even } n}$ and $\frac{\mathrm{d}BM}{\mathrm{d}\rho}\big|_{\rho=0,n>1} > 0$.

$BM_{iid} \leq BM_{\rho\to -1,\text{ odd } n}$ if

$$m \geq \frac{\sigma_L\sqrt{\sigma_L^2 + 4n\mu_L(\mu_L + n)} - \sigma_L^2}{2\mu_L(\mu_L + n)}\,. \tag{25}$$

When (25) holds and $n > 1$ there must be at least one stationary point between $-1 < \rho < 0$ because of the positive derivative at $\rho = 0$, see (19). Note this is a sufficient, but not a necessary condition.

The derivative of BM at $\rho = -1$

$$\frac{\mathrm{d}BM}{\mathrm{d}\rho}|_{\rho\to -1} = \frac{\sigma_L^2 + (-1)^n(2m^2\mu_L(n+\mu_L) + (2m-1)\sigma_L^2)}{nm^2} \tag{26}$$

is positive for even n and negative for odd n. This implies, together with the positive derivative at $\rho = 0$, (19), there is at least one stationary point between $-1 < \rho < 0$ when n is odd. Extensive numerical investigations (see Figs. 5 and 6) suggest that only one stationary point exists in this area though we can not prove it. Finally, for large m the derivative at $\rho = 0$ is almost zero, see (19) and Fig. 6. In many of these bullwhip expressions it is remarkable that the mean demand, μ_D is present. This feature is not present in a constant lead time case. The derivative of BM with respect to ρ is

$$\frac{\mathrm{d}BM}{\mathrm{d}\rho} = \frac{p(\rho)}{n^2m^2(1-\rho)^3}\,, \tag{27}$$

where

$$\begin{aligned} p(\rho) =& (2A - 2\sigma_L^2 n)\rho^{n+2} + (-6A + 2\sigma_L^2(n+2))\rho^{n+1} + (6A - 2\sigma_L^2(n-2))\rho^n \\ &+ (-2A + 2\sigma_L^2 n)\rho^{n-1} - 4\sigma_L^2(n+1)\rho + 4\sigma_L^2(n-1)\,, \end{aligned} \tag{28}$$

where

$$A = \mu_L n^2 m^2 + \mu_L^2 n m^2 + \sigma_L^2 n m\,. \tag{29}$$

The polynomial $p(\rho)$ has 4 sign variations for $n \geq 2$ (if A is large enough compared to σ_L^2 see (29), that is, σ_L^2/μ_L cannot be too large). By Descartes' rule of signs, p can have 0 or 2 or 4 positive roots including their multiplicity. That $p(1) = 0$ implies that there must be 1 or 3 different positive roots other than 1 if all the roots are single[2]. Budan's theorem shows if we were able to find sign variations of the polynomial $p(\rho+1)$ and $p(\rho-1)$ we could obtain the number of roots to $p(\rho)$ in the interval $(0,1]$ and $(-1,0]$, respectively. Finding the sign variations of $p(\rho+1)$ or $p(\rho-1)$ is rather a cumbersome task. However, Budan's theorem, together with $p(1) = 0$, implies for $n = 2$, $p(\rho)$ has 1 or 3 different roots in $(0,1)$ if all the roots are single, and no roots in $(-1,0]$. For $n = 3$ under the assumption $A > 5\sigma_L^2$ (which we believe would be very likely see (29)) $p(\rho+1)$ has no sign variations and $p(\rho)$ has 4 sign variations which implies 0, 2, 4 roots in $(0,1]$ and knowing that $p(1) = 0$ we get 1 or 3 different roots in $(0,1)$, if all the roots are single. Under the same assumptions $p(\rho)$ has one root in $(-1,0]$.

Figure 4 provides a illustration of the bullwhip measure, highlighting the different characters

[2]Note the root at $p(1)$ does not imply the derivative is zero as (27) is undefined at this point

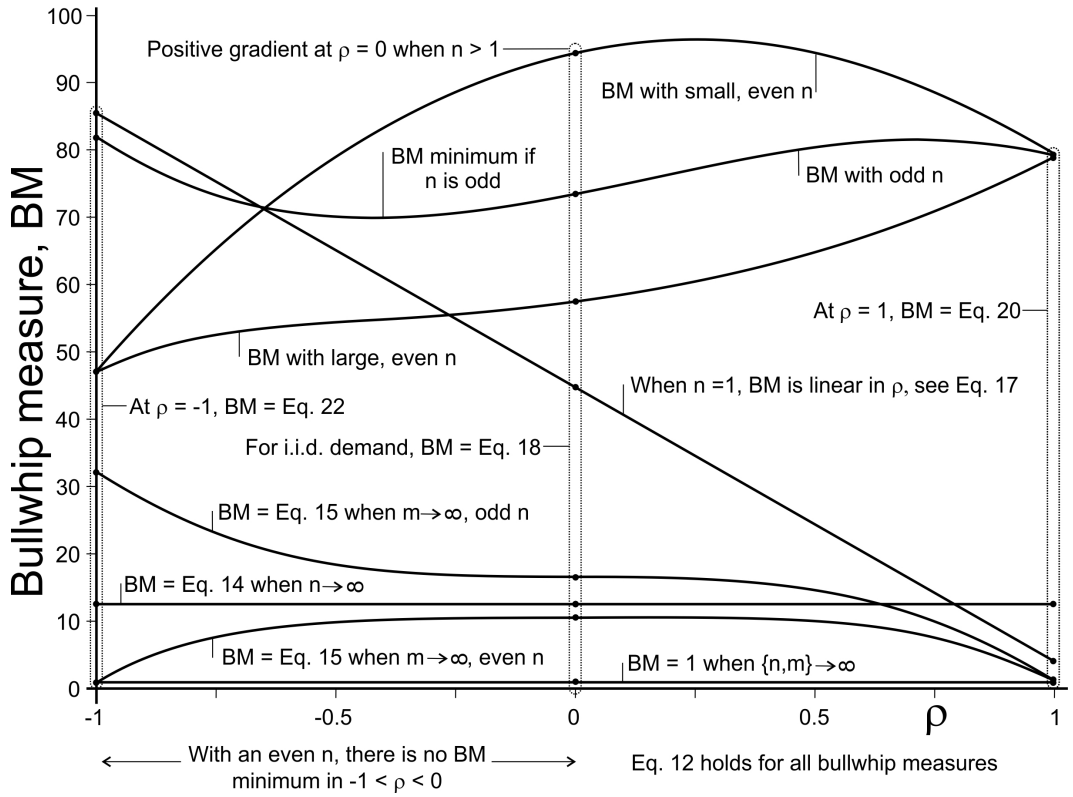

Figure 4: A caricature of the nature of the bullwhip measure

the measure possesses and linking behaviour to our theoretical results.

3 Numerical investigations

Let us further investigate the influence of the demand correlation on the bullwhip effect by analyzing some concrete numerical examples. We plot the bullwhip effect measure as a function of the demand correlation parameter ρ. Fixing $\mu_D = 80$, $\sigma_D = 50$, $\mu_L = 7$, $\sigma_L = 4$ we depict the bullwhip measure in four different scenarios[3]. That is when: both n and m are small, one of them is small and the other is large, and when both are large. These choices allow us to illustrate Eqs. (14)-(17) and to compare special instances of our modelling set-up to known results in the literature. If n is small, we need to distinguish two further cases; that is, whether n is even or odd, because of the influence of n on the bullwhip effect when $\rho < 0$ as highlighted by Eqs. (22)-(24).

If $n = 5$ and $m = 2$ (small and odd n) the bullwhip measure has a minimum at $\rho \approx -0.55$ and $\rho = 1$ and a maximum at $\rho = -1$ and $\rho \approx 0.7$ (see the $n = 5$ case in Fig. 5). The bullwhip measure behavior for $\rho = -1$ and $\rho = 1$ can be predicted by taking the limit as the AR(1) model is well-defined when $-1 < \rho < 1$. In Duc et al., [20] the minimal value of BM is attained for ρ near -0.6 or -0.7 and the maximal value of BM is for ρ around 0.6 or 1. Their results are

[3]We selected $\mu_L = 7$ as this is representative of a shipping lead time between China and the EU and $\sigma_L = 4$ to indicate that approximately 68% of shipments are received with 7 ± 4 weeks.

very close to ours if n is small and odd as their model does not predict the lead time corresponding to $m = 1$ in our model. Recall, Duc et al., [20] forecasts demand with MMSE, whereas we use moving average to forecast demand.

For $n = 6$ and $m = 2$ (small and even n) we observe a different behavior. Specifically, the smallest value of the bullwhip effect is attained for $\rho = -1$ and the largest for $\rho \approx 0.8$ (see the $n = 6$ case in Fig. 5). This concurs with Kahn [24] who revealed positively correlated demands create the bullwhip effect.

We note the bullwhip effect becomes large when n and m are small. That is, when the average age of the data used in the moving average forecast ($(n-1)/2$ or $(m-1)/2$) is small, a large bullwhip effect is produced. Slower forecasts, with a large average of age of data, lead to smaller levels of bullwhip. Under constant lead times, this conclusion also holds true and concurs with the findings of Dejonckheere et al., [16].

The situation changes if n is large and m is small (see the $\{n = 15, n = 16\}$ cases in Fig. 5). Then the bullwhip measure is almost an increasing function of the demand correlation except for odd n and ρ close to -1 where we observe a minimum e.g. for $n = 15$ and $m = 2$ if $\rho \lessapprox -0.8$ the bullwhip measure is decreasing and for $\rho \gtrapprox -0.8$ it is increasing. Moreover, bullwhip increases quite slowly in the region of $-0.8 \lessapprox \rho \lessapprox 0.5$. The odd-even effects in n are now much less noticeable. These observations are consistent with our theoretical analysis in the previous section.

As m, the number of periods used in the lead time moving average forecast increases, the bullwhip effect becomes independent of the demand correlation ρ regardless of the number of periods used in the moving average forecast of demand, n. That is, bullwhip remains almost constant except near $\rho = \{-1, 1\}$. Fig. 6 confirms this independence for the cases when $n = 5$, $n = 6$, $n = 21$, $n = 22$ and $m = 20$. This is caused by the first summand of (12) which vanishes as $m \to \infty$ or $n \to \infty$ and the third summand is rather insensitive to ρ. For ρ close to -1 or 1 the bullwhip effect can dramatically increase or decrease. Moreover, much less bullwhip is generated with large values of n and m. Bullwhip reducing with larger m is congruent with the results of Disney et al., [19] and Wang and Disney [43].

The influence of an odd or even n near $\rho = -1$ can clearly be seen in the Figs. 5 and 6. Even n results in a pooling of period-to-period demand that is oscillating around the mean (when ρ is near -1), and this reduces the amount of bullwhip generated. Odd n leads to demand forecasts that oscillate period-to-period and increases the bullwhip effect. In other words, demand forecast errors can be reduced by aggregating the forecasts into *high and low demand pairs* with even n. For odd n, the demand forecast error can be much bigger than for small, even n when demands are strongly negatively correlated.

4 Conclusions and further research opportunities

We quantified the bullwhip effect when demands and lead times must be forecasted. Demand and lead time forecasting are necessary when placing an order if demands and lead times are stochastic.

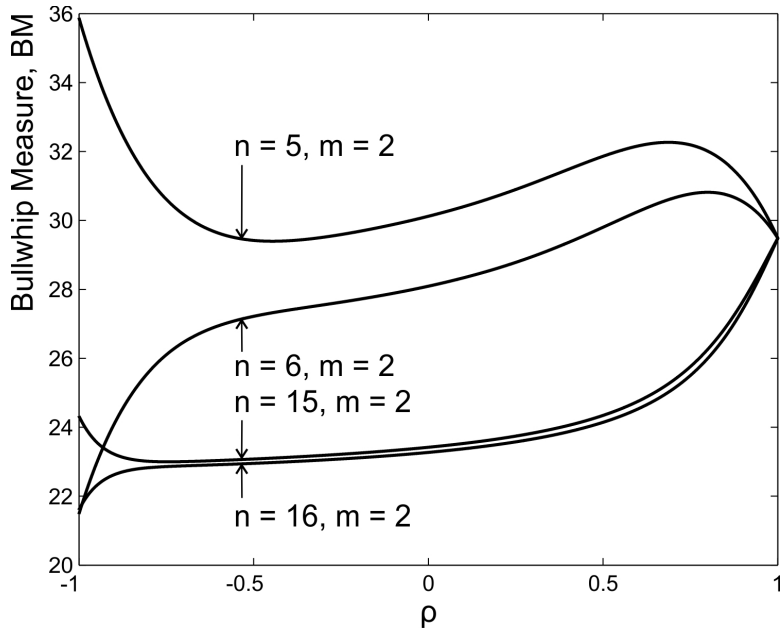

Figure 5: Bullwhip when moving average over $m = 2$ observations is used to forecast stochastic lead-times

We have confirmed, extended, and sharpened the insights of Michna and Nielsen [32] that lead time forecasting is a major cause of the bullwhip effect. We assumed that demands constitute a first order autoregressive process and we obtained quantitative results which link bullwhip and the demand correlation when demands and lead times are to be predicted separately. We conclude that how one goes about forecasting demand and lead time is important as it can cause significant amounts of bullwhip. Moreover, the dependence of the bullwhip measure on the demand correlation parameter is different according to the forecasting parameters used to make lead time and demand predictions. Our main result (12) generalizes the results of Michna and Nielsen [32] who studied i.i.d. demand, by considering the impact of auto-regressive demands. Demand auto-correlation was found to produce some complex bullwhip behaviors (related with the maxima, minima, and the monotonicity which were not present in the uncorrelated case). Moreover, the first term of (12) reveals that bullwhip is exacerbated by the joint action of the lead time forecasting, demand forecasting, and demand correlation. Additionally, this term is responsible for the maxima and minima of bullwhip with respect to the correlation coefficient. Managerially, one should be careful to collect unbiased information by forecasting from realized lead-times only.

As the assumptions on the behavior on L are realistic and limited (all we need is that L must be bounded and non-negative) and the demand models closely match the conditions commonly seen in real settings, there is a good foundation for a broad interpretation of the implications of our work. It is clear that lead time forecasting should be an integral part of supply chain planning and that supply chain operations under stochastic lead times and auto-correlated demands are quite complex. However, we can mitigate the consequences of lead time forecasting with the following actions:

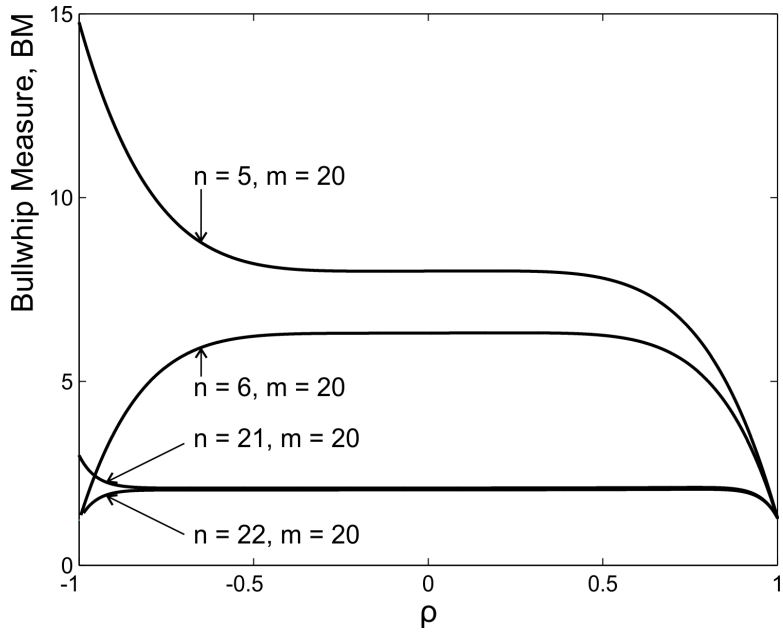

Figure 6: Bullwhip when moving average over $m = 20$ observations is used to forecast stochastic lead-times

- Even with a perfect understanding of the demand process we observe a significant bullwhip effect for systems where stochastic lead times must be forecast.
- When forecasting demand using moving averages, an even number of demand observations work best for negatively correlated AR(1) demand processes.
- Increasing m, the number of data points used to forecast lead times strictly leads to a reduction in the bullwhip measure.
- Under many realistic settings, lead times will be a stochastic variable and, unless many lead time observations are used in the lead-time forecast, its impact on the bullwhip measure should not be ignored.
- In practice, one should be careful to collect unbiased information by forecasting from realized lead-times only and recording these lead times in an accurate manner.

From this it seems clear that supply chain managers who forecast and update lead times information may underestimate the bullwhip that will be created. This can lead to poor decision making which in turn can further increase the bullwhip effect in the chain. One might also conjecture that when demand has a seasonal pattern perhaps n, the number of the observations in the moving average demand forecast, should be a multiple of the number of periods in a season.

Future research could be focused on the impact of lead time forecasting when lead times are correlated, either temporally, or with other system states such as customer demand. For example, if large orders lead to long lead times, there is a correlation between the lead time and the order

size and this dependence should be captured somehow. This seems to be difficult to quantify analytically, although important progress on this matter has been made recently by Hellemans et al., [22]. Another important open question is how the bullwhip effect propagates in multi-echelon supply chains as the AR(1) demand process will have (presumably) changed into a multi-modal ARMA(1,1) process, Alwan et al., [2]. Some progress has been made by Kim et al., [26] and Chatfield et al., [12] to this regard, albeit for a different set of modelling assumptions.

Other opportunities lie in studying the impact of different forecasting methods for lead times and demands (see Zhang [46] for the case of demand forecasting). Another challenge is to quantify bullwhip in the presence of unrealized previous orders when placing an order. More precisely, forecasting the most recent lead times when some orders are not yet received will distort the lead time distribution and have an impact on bullwhip. The quantification of this issue seems to be a difficult task, but it will become important when the lead times are temporally correlated.

Another important challenge is the investigation of the variance amplification of both orders and inventory levels simultaneously because an inappropriate focus on bullwhip reduction can amplify the variability of inventory levels (see Chen and Disney [13], Devika et al., [18] and Disney et al., [19]) which can be as harmful as the bullwhip. Moreover, the proportional OUT replenishment policy, Wang and Disney [43], should be investigated under the assumption of lead time forecasting.

5 Appendix

Proof of Theorem 1. We apply the law of total variance to find variance of q_t. Namely, let us put

$$L = (L_{t-1-L^+}, L_{t-2-L^+}, \ldots, L_{t-1-m-L^+})$$

and then

$$\mathbf{Var}(q_t) = \mathbf{Var}(\mathbb{E}(q_t|L)) + \mathbb{E}(\mathbf{Var}(q_t|L)) \,. \tag{30}$$

Using (11) it can be seen that

$$\begin{aligned} q_t &= \widehat{D_t^L} - \widehat{D_{t-1}^L} + D_{t-1} \\ &= \widehat{L_t}\widehat{D_t} - \widehat{L_{t-1}}\widehat{D_{t-1}} + D_{t-1} \end{aligned} \tag{31}$$

revealing that $\mathbb{E}(q_t) = \mathbb{E}(D_t) = \mu_D$. From the second expression for q_t it follows that

$$\mathbb{E}(q_t|L) = (L_{t-1-L^+} - L_{t-1-m-L^+})\frac{\mu_D}{m} + \mu_D$$

which gives

$$\mathbf{Var}(\mathbb{E}(q_t|L)) = \frac{2\sigma_L^2\mu_D^2}{m^2} \,. \tag{32}$$

To calculate the conditional variance of q_t we express it as a function of D_{t-1-n} and the error

terms $\epsilon_{t-n}, \epsilon_{t-n+1}, \ldots, \epsilon_{t-1}$ which are mutually independent. Thus by (31) and (4) we get

$$\begin{aligned} q_t &= \left(\frac{\widehat{L_t}}{n}+1\right) D_{t-1} + \frac{L_{t-1-L^+} - L_{t-1-m-L^+}}{nm} \sum_{k=2}^{n} D_{k-2} - \frac{\widehat{L_{t-1}}}{n} D_{t-1-n} \\ &= \left(\frac{\widehat{L_t}}{n}+1\right) \mu_D(1-\rho^n) + \frac{(L_{t-1-L^+} - L_{t-1-m-L^+})\mu_D}{nm} \left(n-1-\frac{\rho(1-\rho^{n-1})}{1-\rho}\right) \\ &\quad + \left[\left(\frac{\widehat{L_t}}{n}+1\right)\rho^n + \frac{(L_{t-1-L^+} - L_{t-1-m-L^+})\rho(1-\rho^{n-1})}{nm(1-\rho)} - \frac{\widehat{L_{t-1}}}{n}\right] D_{t-1-n} \\ &\quad + \sum_{k=1}^{n} \left[\left(\frac{\widehat{L_t}}{n}+1\right)\rho^{k-1} + \frac{(L_{t-1-L^+} - L_{t-1-m-L^+})(1-\rho^{k-1})}{nm(1-\rho)}\right] \epsilon_{t-k} \end{aligned}$$

which gives

$$\mathbf{Var}(q_t|L) = \sigma_D^2 C_1^2 + \sigma^2 \sum_{k=1}^{n} C_{2,k}^2 \,, \tag{33}$$

where

$$C_1 = \left(\frac{\widehat{L_t}}{n}+1\right)\rho^n + \frac{(L_{t-1-L^+} - L_{t-1-m-L^+})\rho(1-\rho^{n-1})}{nm(1-\rho)} - \frac{\widehat{L_{t-1}}}{n}$$

and

$$C_{2,k} = \left(\frac{\widehat{L_t}}{n}+1\right)\rho^{k-1} + \frac{(L_{t-1-L^+} - L_{t-1-m-L^+})(1-\rho^{k-1})}{nm(1-\rho)} \,.$$

Thus we get

$$\mathbb{E}(C_1) = \left(\frac{\mu_L}{n}+1\right)\rho^n - \frac{\mu_L}{n}$$

and

$$\mathbb{E}(C_{2,k}) = \left(\frac{\mu_L}{n}+1\right)\rho^{k-1} \,.$$

To calculate $\mathbb{E}(\mathbf{Var}(q_t|L))$ it is necessary to find $\mathbb{E}(C_1^2)$ and $\mathbb{E}(C_{2,k}^2)$ by (33). We compute them by finding variance and adding the square of the first moment. Thus, to obtain the variance of C_1 and $C_{2,k}$, we express them as a sum of independent random variables,

$$C_1 = \frac{\rho^n - 1}{nm} \sum_{k=2}^{m} L_{t-k-L^+} + \frac{\rho(1-\rho^n)}{(1-\rho)nm} L_{t-1-L^+} - \frac{1-\rho^n}{(1-\rho)nm} L_{t-1-m-L^+} + \rho^n$$

and

$$C_{2,k} = \frac{\rho^{k-1}}{nm} \sum_{k=2}^{m} L_{t-k-L^+} + \frac{1-\rho^k}{(1-\rho)nm} L_{t-1-L^+} - \frac{1-\rho^{k-1}}{(1-\rho)nm} L_{t-1-m-L^+} + \rho^{k-1} \,.$$

Hence we obtain

$$\mathbf{Var}(C_1) = \frac{(1-\rho^n)^2 \sigma_L^2}{n^2 m^2} \left(m + \frac{2\rho}{(1-\rho)^2}\right)$$

and

$$\mathbf{Var}(C_{2,k}) = \frac{\sigma_L^2}{n^2m^2}\left[\rho^{2(k-1)}(m-1) + \left(\frac{1-\rho^k}{1-\rho}\right)^2 + \left(\frac{1-\rho^{k-1}}{1-\rho}\right)^2\right].$$

So we get

$$\mathbb{E}(C_1^2) = \frac{(1-\rho^n)^2\sigma_L^2}{n^2m^2}\left(m + \frac{2\rho}{(1-\rho)^2}\right) + \left[\left(\frac{\mu_L}{n}+1\right)\rho^n - \frac{\mu_L}{n}\right]^2 \tag{34}$$

and

$$\begin{aligned}\mathbb{E}(C_{2,k}^2) &= \left(\frac{\sigma_L^2(m-1)}{n^2m^2} + \left(\frac{\mu_L}{n}+1\right)^2 + \frac{\sigma_L^2(\rho^2+1)}{n^2m^2(1-\rho)^2}\right)\rho^{2(k-1)} \\ &\quad -\frac{2\sigma_L^2(\rho+1)}{n^2m^2(1-\rho)^2}\rho^{k-1} + \frac{2\sigma_L^2}{n^2m^2(1-\rho)^2}.\end{aligned}$$

Summing the last expression we obtain

$$\begin{aligned}\sum_{k=1}^{n}\mathbb{E}(C_{2,k}^2) &= \left(\frac{\sigma_L^2(m-1)}{n^2m^2} + \left(\frac{\mu_L}{n}+1\right)^2 + \frac{\sigma_L^2(\rho^2+1)}{n^2m^2(1-\rho)^2}\right)\frac{1-\rho^{2n}}{1-\rho^2} \\ &\quad -\frac{2\sigma_L^2(\rho+1)}{n^2m^2(1-\rho)^2}\frac{1-\rho^n}{1-\rho} + \frac{2\sigma_L^2}{nm^2(1-\rho)^2}.\end{aligned} \tag{35}$$

Plugging (35), (34), (33), (32) into (30) yields the formula from the assertion after some algebra.

Acknowledgments

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