Supplementary Material for "Computational Design of Steady 3D Dissection Puzzles"

1 Preliminaries

Mathematical Notation. A puzzle with $n+1$ pieces are denoted as $\{P_1, P_2, ..., P_n, R_n\}$, with d_i $(1 \leq i \leq n)$ represents the extraction direction of the piece P_i. Note that $d_i \in \mathbf{D}$, where $\mathbf{D}$ denotes the six axial directions, i.e., $\mathbf{D} = \{-x, +x, -y, +y, -z, +z\}$.

We call a group of puzzle pieces as a piece group G by considering only the blocking relationship between pieces within G and pieces out of G while ignoring the blocking relationship among pieces within G. In particular, we deonte R_i as a piece group that contains all successive pieces of P_i, i.e., $R_i = \{P_{i+1}, P_{i+2}, ..., P_n, R_n\}$. Note that a piece group is allowed to contain a single piece only.

Lemma 1. If a piece group G_i is blocked by a piece group G_j in direction D_t, then $G_i \cup G_k$ is blocked by G_j in D_t if $G_j \cap G_k = \varnothing$, and G_i is blocked by $G_j \cup G_k$ in D_t if $G_i \cap G_k = \varnothing$.

Lemma 2. If a piece group G_i is blocked by a piece group G_j in l $(1 \leq l \leq 6)$ axial directions, then G_j is blocked by G_i in the l opposite axial directions.

These lemmas appear to be straightforward but as we shall soon see, they are helpful in deriving the formal models below.

2 Proof of Basic Formal Model

2.1 Basic Formal Model

Requirements for P_1. When constructing P_1, it should be immobilized by R_1 such that it is only movable along d_1.

Requirements for P_i $(2 \leq i \leq n)$. When constructing P_i, it should satisfy the following requirements, where $\mathbf{S}_i$ denotes the set of all neighboring pieces of P_i that have been extracted before P_i:

1) P_i should be immobilized by P_{i-1} and R_i such that it is movable only along d_i.
2) R_i should block P_i from moving along d_{i-1}, if $d_{i-1} \neq d_i$.
3) For each $P_j \in \mathbf{S}_i$ and each direction $d^{'} \in \mathbf{D} \setminus \{d_i, d_j\}$, if R_i does not block P_j while P_i blocks P_j from moving along $d^{'}$, then R_i should block P_i from moving along $d^{'}$.

2.2 Proof of Basic Formal Model

Here we prove puzzle pieces constructed by iteratively partitioning R_{i-1} into P_i and R_i according to the requirements of the *Basic Formal Model* are guaranteed to be generalized interlocking. In other words, we are going to prove that an arbitrary group of pieces in the puzzle $\{P_1, P_2, ..., P_n, R_n\}$ is movable at most along one axial direction (i.e., either immobilized or only movable along one axial direction).

To achieve this, we first prove an arbitrary group of pieces in $\{P_1, P_2, ..., P_n\}$ is movable at most along one axial direction. It is realized by proving that for each k $(1 \leq k \leq n)$, an arbitrary piece group in $\{P_1, ..., P_k\}$ with P_k is movable at most along d_k using the method of mathematical induction (see **Statement #1**). This is doable because the set of all piece groups in $\{P_1, P_2, ..., P_n\}$ is equivalent to the union of the sets of all piece groups in $\{P_1\}$ with P_1, $\{P_1, P_2\}$ with P_2, ..., $\{P_1, P_2, ..., P_n\}$ with P_n, if we ignore duplicated elements in the union of the sets. Next, we generalize the proof to an arbitrary piece group in $\{P_1, P_2, ..., P_n, R_n\}$ (see **Statement #2**).

Statement #1: For each k, an arbitrary piece group in $\{P_1, ..., P_k\}$ with P_k is movable at most along d_k.

Proof:

1) When $k = 1$, as R_1 $(R_1 = \{P_2, ..., P_n, R_n\})$ should block P_1 in all the other five directions except d_1, thus P_1 is movable at most along d_1.

2) Suppose when $k = i-1$ $(2 \leq i \leq n)$, an arbitrary piece group in $\{P_1, ..., P_{i-1}\}$ with P_{i-1} is movable at most along d_{i-1}. We are going to prove when $k = i$, an arbitrary piece group G in $\{P_1, ..., P_i\}$ with P_i is movable at most along d_i. To prove this, we classify the statement into two cases:

2.1) $P_{i-1} \notin G$.

According to Requirement 1, P_i should be blocked by P_{i-1} and R_i $(R_i = \{P_{i+1}, ..., P_n, R_n\})$ in all the other five directions except d_i, thus G is blocked by P_{i-1} and R_i in all the other five directions except d_i due to Lemma 1, namely G is movable at most along d_i.

2.2) $P_{i-1} \in G$.

As supposed, piece group $G^{'} = G \setminus \{P_i\}$ with $P_{i-1} \in G^{'}$ is movable at most along d_{i-1}. For each other direction

$D_a \in \mathbf{D} \setminus \{d_{i-1}\}$ of group $G^{'}$, it could also be classified into two cases:

Case 1: D_a of group $G^{'}$ is blocked by R_i or $\{P_1, ..., P_{i-1}\} \setminus G^{'}$, then $G^{'} \cup \{P_i\} = G$ is blocked by R_i or $\{P_1, ..., P_{i-1}\} \setminus G^{'}$ in D_a according to Lemma 1.

Case 2: D_a is enforced by P_i on its neighbour piece in $G^{'}$ such as $P_b \in G^{'}$, and R_i do not block P_b in D_a. Note that $d_b \neq D_a$ as P_b is extracted before P_i, so P_i is impossible to block P_b in d_b. According to Requirement 3, R_i should block P_i in D_a if $D_a \neq d_i$. Therefore, $G^{'} \cup \{P_i\} = G$ is blocked by R_i in D_a if $D_a \neq d_i$ according to Lemma 1.

Since case 1 and case 2 cover all possible values of D_a, we induce G is blocked in $\mathbf{D} \setminus \{d_{i-1}, d_i\}$.

According to Requirement 2, R_i should block P_i in d_{i-1} if $d_{i-1} \neq d_i$, thus G is movable at most along d_i according to Lemma 1.

As 1) and 2) are satisfied, we can infer that for each k, an arbitrary piece group in $\{P_1, ..., P_k\}$ with P_k is movable at most along d_k. ∎

Statement #2: An arbitrary piece group in $\{P_1, P_2, ..., P_n, R_n\}$ is movable at most along one axis direction.

Proof:

According to **Statement #1**, an arbitrary group of pieces G in $\{P_1, P_2, ..., P_n\}$ is movable at most along one axis direction. Therefore, the group $G^{'}$ with all the other pieces should block G in at least five directions, where $G^{'} = \{P_1, P_2, ..., P_n, R_n\} \setminus G$, $G \cap G^{'} = \varnothing$ and $G \cup G^{'} = \{P_1, P_2, ..., P_n, R_n\}$. According to Lemma 2, G should also block $G^{'}$ in at least five directions, namely $G^{'}$ is movable at most along one axis direction. As G and $G^{'}$ cover all the subset of groups in $\{P_1, P_2, ..., P_n, R_n\}$, we can infer that an arbitrary piece group in $\{P_1, P_2, ..., P_n, R_n\}$ is movable at most along one direction. ∎

3 Proof of the Formal Model

3.1 The Formal Model

By choosing an arbitrary $P_t \in \mathbf{S}_i$ $(1 \leq t \leq i-1)$, when constructing P_i, it should satisfy the following requirements, where $\mathbf{S}_i$ denotes the set of all neighboring pieces of P_i that have been extracted before P_i:

1) P_i should be immobilized by P_t and R_i such that it is only movable along d_i .
2) For each $P_j \in \{P_1, ..., P_{i-1}\} \setminus \{P_t\}$, R_i should block P_i from moving along d_j, if R_i does not block P_j or P_t from moving along d_i and $d_i \neq d_j$.
3) For each $P_j \in \mathbf{S}_i$ and each direction $d^{'} \in \mathbf{D} \setminus \{d_i, d_j\}$, if R_i does not block P_j but P_i blocks P_j from moving along $d^{'}$, then R_i should block P_i from moving along $d^{'}$.

Note that in case we could not find any P_t that satisfies the above three requirements, we allow P_i to be immobilized by R_i only such that it is only movable along d_i.

3.2 Proof of the Formal Model

Here we prove puzzle pieces constructed by iteratively partitioning R_{i-1} into P_i and R_i according to the requirements of *the Formal Model* are guaranteed to be generalized interlocking. In other words, we are going to prove that an arbitrary group of pieces in the puzzle $\{P_1, P_2, ..., P_n, R_n\}$ is movable at most along one axial direction (i.e., either immobilized or only movable along one axial direction).

To achieve this, we first prove an arbitrary group of pieces in $\{P_1, P_2, ..., P_n\}$ is movable at most along one axial direction. It is realized by proving that for each k $(1 \leq k \leq n)$, an arbitrary piece group G in $\{P_1, ..., P_k\}$ with P_k is movable at most along D^G, where D^G is the extraction direction of one piece in G, using the method of mathematical induction (see **Statement #3**). This is doable because the set of all piece groups in $\{P_1, P_2, ..., P_n\}$ is equivalent to the union of the sets of all piece groups in $\{P_1\}$ with P_1, $\{P_1, P_2\}$ with P_2, ..., $\{P_1, P_2, ..., P_n\}$ with P_n, if we ignore duplicated elements in the union of the sets. Next, we generalize the proof to an arbitrary piece group in $\{P_1, P_2, ..., P_n, R_n\}$ (see **Statement #4**).

Statement #3: For each k, an arbitrary piece group G in $\{P_1, ..., P_k\}$ with P_k is movable at most along D^G, where D^G is the extraction direction of one piece in G.

Proof:

1) When $k = 1$, as R_1 $(R_1 = \{P_2, ..., P_n, R_n\})$ should block P_1 in all the other five directions except d_1, thus P_1 is movable at most along d_1.

2) Suppose when $k \leq i-1$ $(2 \leq i \leq n)$, arbitrary piece group $G^{''}$ in $\{P_1, ..., P_k\}$ with P_k is movable at most along $D^{G^{''}}$, where $D^{G^{''}}$ is the extraction direction of one piece in $G^{''}$. We are going to prove when $k = i$, an arbitrary piece group G in $\{P_1, ..., P_i\}$ with P_i is movable at most along D^G, where D^G is the extraction direction of one piece in G. To prove this, we classify the statement into two cases:

2.1) If P_i does not adopt any P_t to stabilize it, then R_i $(R_i = \{P_{i+1}, ..., P_n, R_n\})$ should block P_i in all the other five directions except d_i, thus G is blocked by R_i in all the other five directions except d_i due to Lemma 1, namely G is movable at most along d_i.

2.2) P_i is constructed by adopting P_t $(t \leq i-1)$ and R_i, which could be classified into two cases:

2.2.1) $P_t \notin G$.

According to Requirement 1, P_i should be blocked by P_t and R_i in all the other five directions except d_i, thus G is blocked by P_t and R_i in all the other five directions except d_i due to Lemma 1, namely G is movable at most along d_i.

2.2.2) $P_t \in G$.

As supposed, piece group $G^{'} = G \setminus \{P_i\}$ with $P_t \in G^{'}$ is movable at most along one direction $D^{G^{'}}$. We denote P_m $(P_m \in G^{'})$ as an arbitrary piece whose extraction direction is d_m and $d_m = D^{G^{'}}$.

For each other direction $D_a \in \mathbf{D} \setminus \{d_m\}$ of group $G^{'}$, it could be classified into two cases:

Case 1: D_a is blocked by R_i or $\{P_1, ..., P_{i-1}\} \setminus G^{'}$, then $G^{'} \cup \{P_i\} = G$ is blocked by R_i or $\{P_1, ..., P_{i-1}\} \setminus G^{'}$ in D_a according to Lemma 1.

Case 2: D_a is applied by P_i on its neighbour piece in $G^{'}$ such as $P_b \in G^{'}$, and R_i do not block P_b in D_a. Note that $d_b \neq D_a$ as P_b is extracted before P_i, so P_i is impossible to block P_b in d_b. According to Requirement 3, R_i should block P_i in D_a if $D_a \neq d_i$. Therefore, $G^{'} \cup \{P_i\} = G$ is blocked by R_i in D_a if $D_a \neq d_i$ according to Lemma 1.

Since case 1 and case 2 cover all possible values of D_a, we induce G is movable at most along $\{d_i, d_m\}$.

For the direction d_i and d_m, it could also be classified into two cases:

Case 3: $d_i = d_m$, then G is movable at most along d_i or d_m.

Case 4: $d_i \neq d_m$, it could be also classified into two subcases:

- Subcase 4.1: R_i block P_t or P_m in d_i.
- Subcase 4.2: R_i do not block P_m or P_t in d_i. Then according to Requirement 2, R_i should block P_i in d_m.

In the above two subcases, $G^{'} \cup \{P_i\} = G$ is blocked by R_i in d_i or d_m according to Lemma 1, namely G is movable at most in d_i or d_m.

Since case 3 and case 4 cover all possible values of d_i and d_m, we induce G is movable at most along d_i or d_m.

As 1) and 2) are satisfied, we can infer that for each k, an arbitrary piece group G in $\{P_1, ..., P_k\}$ with P_k is movable at most along D^G, where D^G is the extraction direction of one piece in G. ■

Statement #4: An arbitrary piece group in $\{P_1, P_2, ..., P_n, R_n\}$ is movable at most along one direction.

Proof:

Similar to the proof of **Statement #2**. ■