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SOME NEW RESULTS ON LIPSCHITZ REGULARIZATION FOR PARABOLIC EQUATIONS

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ABSTRACT. It is well-known that the bounded solution $u(t,x)$ of the heat equation posed in $\mathbb{R}^N \times (0,T)$ for any continuous initial condition becomes Lipschitz continuous as soon as $t > 0$, even if the initial datum is not Lipschitz continuous. We investigate this Lipschitz regularization for both strictly and degenerate parabolic equations of Hamilton-Jacobi type. We give proofs avoiding Bernstein's method which leads to new, less restrictive conditions on the Hamiltonian, i.e. the first order term. We discuss also whether the Lipschitz constant depends on the oscillation for the initial datum or not. Finally, some important applications of this Lipschitz regularization are presented.

1. INTRODUCTION

It is well-known that the bounded solution $u(t,x)$ of the heat equation posed in $\mathbb{R}^N \times (0,T)$ for any continuous initial condition becomes Lipschitz continuous as soon as $t > 0$. The precise statement is: there exists a positive continuous function $\alpha : (0,T] \to [0,\infty)$ which depends only on the oscillation of the initial datum (not on its Lipschitz constant which may not even exist) such that

$$(1.1) \qquad\qquad |Du|_\infty \leq \alpha(t).$$

In this article, we study the above question for the following nonlinear parabolic equation

$$(1.2) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{trace}(A(x,Du)D^2u) + K(x,t,Du) = 0, & (x,t) \in \mathbb{T}^N \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{T}^N, \end{cases}$$

where $A(x,p) = \sigma(x,p)\sigma(x,p)^T$ with $\sigma \in W^{1,\infty}(\mathbb{T}^N; \mathcal{M}_N)$. K will in general be coercive and convex in a carefully quantified way. Depending on the nature of these additional assumptions on the Hamiltonian, the Lipschitz constant at time $t > 0$ may depend on t and the oscillation of the initial datum, or, if the growth of the Hamiltonian is super-quadratic, it may even be independent of the initial datum. The latter behaviour is of course different from what to expect for the heat equation and is related to the fact that the first order part becomes dominant.

In the sequel we will make our assumption precise and compare them with the literature.

In this paper, we always assume that the Hamiltonian K is continuous and $\mathbb{T}^N$-periodic in x. The diffusion matrix A is always assumed to be non-negative semi-definite, i.e.,

$$(1.3) \qquad A(x,p) \geq 0, \quad (x,p) \in \mathbb{T}^N \times \mathbb{R}^N.$$

The diffusion matrix is said to be strictly elliptic if

$$(1.4) \qquad \text{there exists } \nu > 0 \text{ such that } A(x,p) \geq \nu I, \quad (x,p) \in \mathbb{T}^N \times \mathbb{R}^N.$$

More precise structures on diffusion matrices $A(x,p)$ are presented in Section 3.

Comparison with results in the literature: There is a vast literature on this subject. An exhaustible overview would be more appropriate for a monograph than for this brief note, so the following remarks should be rather seen as an effort to put this paper into a context rather than as a definitive overview.

A bound like in (1.1) was obtained in [7] for viscous and non-viscous Hamilton-Jacobi equations using Bernstein's method under quite restrictive structures on the first order. Bernstein's method, however, requires formally to differentiate the equation, which yields to quite restrictive assumptions on the x-dependence of the Hamiltonian K. More precisely, they consider the class of parabolic equations satisfying two main structural assumptions as follows:

- There exists $R_0 > 0$ and a strictly increasing Φ with $\Phi(0) = 0$ such that for some $\delta > 0$, $G(r) = \frac{\Phi(r)}{r^{1+\delta}}$ is increasing. G is coercive and for all $(x,p) \in \mathbb{T}^N \times \mathbb{R}^N$ with $|p| \geq R_0$.

$$(1.5) \qquad D_p K(x,p)p - K(x,p) \geq \Phi(|p|).$$

- There exists a positive constant C such that for all $(x,p) \in \mathbb{T}^N \times \mathbb{R}^N$, $|p| \geq R_0$

$$(1.6) \qquad -D_x K(x,p)p \leq C(D_p K(x,p)p - K(x,p)).$$

The assumption (1.5) is quite natural for super-linear convex-type Hamiltonians but the assumption (1.6) is quite restrictive. The typical example satisfying (1.5) and (1.6) is:

$$(1.7) \quad K(x,p) = |p|^k - f(x) \ k > 1 \text{ is a constant, } f(x) \text{ is a continuous function.}$$

Note that in general $a(x)|p|^k$ does not satisfy (1.6) because the left hand side has no sign and grows like $|p|^{k+1}$, while the right-hand side grows only like $|p|^k$.

The typical example is:

$$(1.8) K(x,p) = a(x)|p|^k - f(x) \ k \leq 2 \text{ is a constant, } a, f \text{ are continuous functions.}$$

In this paper we study this type of result for Hamiltonians of any growth order in gradient variable and even for degenerate parabolic equations in periodic domain. We obtain results for strictly elliptic and degenerate diffusion matrices with quite general structural conditions on the Hamiltonians, avoiding the use of Bernstein's method. As a particular case, we obtain a result inspired by the theory of Mean Field Games which will be presented in Theorem 3.5 and explained in Section 4.

In Priola and Porretta [14], the authors studied (1.1) for both linear and nonlinear parabolic equations posed in $\mathbb{R}^N$. Their main goal is to obtain the gradient estimate independently of the infinity norms of the coefficients of the operator but only depending on their modulus of continuity. Since the authors consider the equations in $\mathbb{R}^N$, which require them to take care the behavior of solutions at infinity, their results mainly apply for strictly elliptic equations and sub-quadratic Hamiltonians.

Gradient bounds for superlinear Hamiltonians were studied in Lions [11], Barles [3], see also Lions and Souganidis [12] and references therein. These ideas were used in Barles and Souganidis [4] and after that extended by Ley and one of the author [13] to study the large time behavior of parabolic equations and systems. Let us mention a work of Cardaliaguet and Sylvestre [6] where oscillations and Hölder bounds for nonlinear degenerate parabolic equations were proved but the bounds depends on the L^∞ norm of the solution.

We would like to mention that the most far-reaching consequence of Bernstein's method to date is the powerful Lions-Souganidis condition, [12], see also [1], which yields Lipschitz estimates for Hamilton-Jacobi equations under very mild structural assumptions. For parabolic equations, however, the application would require a bound on $|\partial_t u(x,t)|$, which requires the initial datum to be more regular than just bounded.

Notation:

We denote $\mathcal{S}(N)$ the class of symmetric $N \times N$ matrices.

$|M|_\infty$ denotes the supremum norm of matrix M.

Remark:

Our methods are based on some change of variables of exponential type, which requires the solutions to be bounded away from zero. Note that we can assume this without loss of generality for bounded time intervals $[0, T]$ on the torus: By adding a big constant to the initial datum u_0 (depending on $\sup_{\mathbb{T}^n \times (0,T)} K(x,t,0)$ and $\inf_{\mathbb{T}^n} u_0(x)$) neither the Lipschitz constant of the solution nor the oscillation of the initial datum is changed and we can from now on assume that

$$(1.9) \qquad\qquad u(x,t) \geq 1 \ \text{ for all } (x,t) \in \mathbb{T}^N \times \mathbb{R}^N.$$

Structure of the paper

We consider separately the degenerate and strictly parabolic case, each leading to separate assumptions and results. Roughly speaking, in the degenerate case the regularization comes from the the Hamiltonian, while in the strictly parabolic case it comes from the second order term.

In section 2, we show that for super-quadratic Hamiltonians the oscillation of solutions does not depend on the initial datum after a certain time, because there is a super-solution which is infinite at time $t = 0$, so the oscillation at any later time is bounded by this supersolution.

In Section 3, we obtain estimates on the Lipschitz constant via an exponential change of variables and the classical doubling-of-variables method. We avoid differentiating the equation as it would be the case with Bernstein's method. Our method yields a bound depending on the oscillation of the initial datum. With the result from Section 2, we can

conclude that the Lipschitz constant of solutions for super-quadratic Hamiltonians does not depend on the initial datum after a certain time.

(Actually, the bound depends on the supremum over time of the oscillation in space of the solution, but the comparison principle allows to bound this by a constant depending on $K(x, t, 0)$ and the oscillation of the initial datum.)

In section 4 we present some examples satisfying our structures, and we explain an important application of Lipschitz regularization which is independent of the oscillation of the initial datum: The long-time behaviour of a class of nonlinear stochastic partial differential equations studied by P.E. Souganidis and one of the authors in [9]

2. Oscillation of solutions with super-quadratic Hamiltonians

We consider the following structural conditions on the Hamiltonians.

(2.1) $\quad \begin{cases} \text{there exists } q > 2, \, M > 0, \text{ there is a function } f \\ \quad \text{depending only on } M, q \text{ such that for all } x \in \mathbb{T}^N, p \in \mathbb{R}^N, t \in (0, T), \text{ then} \\ K(x, t, p) \geq M^{-1} |p|^q - f(M). \end{cases}$

We assume the following condition on the diffusion

(2.2) $\qquad\qquad\qquad |A(x, p)| \leq C_0, \text{ for any } x \in \mathbb{T}^N, p \in \mathbb{R}^N.$

We extend a result proved in [9] for the L^∞ norm of solutions for super-quadratic Hamiltonians. The proof is the same as the proof of [Lemma 4.2, [9]], where the authors obtained a result when A does not depend on p, and will be presented in Appendix for the reader's convenience.

Proposition 2.1. *Assume that A, K are continuous functions, and that* (2.1) *and* (2.2) *hold where $q > r$, then for continuous viscosity solution u of* (1.2)*, we have*

$$|u(x, t) - \min u_0| \leq C(t), \; C \text{ does not depend on initial datum.}$$

It can be seen from [Lemma 4.2, [9]] or from Proposition (2.1) that under (2.1), the oscillation of solutions of equation (1.2) is bounded independently of initial conditions. The question now is that: what happens if (2.1) is violated? We can easily see that the claim is not true for heat equations.

Example 2.2. Consider the heat equation

(2.3) $\qquad\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = 0, & (x, t) \in \mathbb{R} \times (0, T), \\ u(x, 0) = A sin(x), & x \in \mathbb{R}, A > 0 \end{cases}$

The solution of the equation (2.3) is given by

$$u(x, t) = Ae^{-t} sin(x),$$

it is clear that $osc(u(., t)) := \max_{x \in \mathbb{R}} u(x, t) - \min_{x \in \mathbb{R}} u(x, t) = 2Ae^{-t}$ depends on the oscillation of the initial datum.

3. Gradient bounds for parabolic equations

We first recall the following lemma which is proved in [13] when $A(x, p)$ does not depend on x. An almost identical proof for $A(x, p)$ is presented in Appendix for the reader's convenience. We need some structures on $A(x, p)$ as follows:

$$(3.1) \qquad |\sigma(x, p)| \leq |\sigma|_\infty < +\infty, \forall(x, p) \in \mathbb{T}^N \times \mathbb{R}^N.$$

and

$$(3.2) \qquad |\sigma(x, p) - \sigma(y, p)| \leq |\sigma_x|_\infty |x - y| <, |\sigma_x|_\infty < +\infty \; \forall(x, p) \in \mathbb{T}^N \times \mathbb{R}^N.$$

Lemma 3.1. *Suppose* (3.1) *and* (3.2) *hold,* $\phi \in C(\mathbb{T}^N \times [0, T])$. *Let* $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ *be an increasing concave function such that* $\Psi(0) = 0$ *and the maximum of*

$$\max_{x, y \in \mathbb{T}^N, t \in [0,T]} \{\phi(x, t) - \phi(y, t) - \Psi(|x - y|)\},$$

is achieved at $(\bar{x}, \bar{y}, \bar{t})$. *If* $\bar{x} \neq \bar{y}$ *and* $\bar{t} > 0$, *then for every* $\varrho > 0$, *there exists* $(a, p, X) \in \overline{J}^{2,+}\phi(\bar{x}, \bar{t}), (a, p, Y) \in \overline{J}^{2,-}\phi(\bar{y}, \bar{t})$ *such that*

$$(3.3) \qquad \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq M + \varrho M^2,$$

with

$$(3.4) \qquad p = \Psi'(|\bar{x} - \bar{y}|)q, \quad q = \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|}, \quad B = \frac{1}{|\bar{x} - \bar{y}|}(I - q \otimes q),$$

$$(3.5) \qquad M = \Psi'(|\bar{x} - \bar{y}|) \begin{pmatrix} B & -B \\ -B & B \end{pmatrix} + \Psi''(|\bar{x} - \bar{y}|) \begin{pmatrix} q \otimes q & -q \otimes q \\ -q \otimes q & q \otimes q \end{pmatrix}$$

and the following estimate holds

$$(3.6) \qquad -\operatorname{trace}(A(\bar{x}, p)X - A(\bar{y}, p)Y) \geq -N|\sigma_x|_\infty^2 |\bar{x} - \bar{y}|\Psi'(|\bar{x} - \bar{y}|) + O(\varrho).$$

If, in addition, (1.4) *holds, then there exists* $\tilde{C} = \tilde{C}(N, \nu, |\sigma|_\infty, |\sigma_x|_\infty)$ *such that*

$$(3.7) \quad -\operatorname{trace}(A(\bar{x}, p)X - A(\bar{y}, p)Y) \geq -4\nu\Psi''(|\bar{x} - \bar{y}|) - \tilde{C}\Psi'(|\bar{x} - \bar{y}|)|\bar{x} - \bar{y}| + O(\varrho)$$

The trace estimates can be found in [10, 8, 2, 4].

3.1. Strictly parabolic equations.
We consider the following structure in the next theorem

$$(3.8) \qquad \begin{cases} \text{For any } L > 1, \text{ there exists } C \in \mathbb{R}, \text{ such that} \\ \text{for all } x, y \in \mathbb{T}^N, t \in [0, T], |p| \geq L, \text{ and } \mu \geq 1 + L|x - y|, \text{ we have} \\ K(x, t, p) - \mu K(y, t, \frac{p}{\mu}) \geq -C|p| - C. \end{cases}$$

Theorem 3.2. *Assume* (3.1), (3.2) (1.4) *and* (3.8) *hold. Let u be a continuous viscosity solution of* (1.2)*satisfying* (1.9)*, and define*

$$\exp(w(x,t)) = u(x,t), \quad \text{for any } x \in \mathbb{T}^N, t \in [0,T]$$

Then, there exists a positive continuous function $\alpha : (0,T] \to [0,\infty)$, which only depends on the oscillation of the initial datum, such that $|Dw|_\infty \leq \alpha(t)$.

If (2.1) *also holds, $\alpha(t)$ can be chosen independently of the oscillation of the initial datum.*

Remark 3.3. The second part of the theorem follows directly from Proposition 2.1 which claims that under (2.1), the oscillation of solutions of equation (1.2) is bounded independently of initial conditions.

The following proof is inspired by the proofs of [Lemma 2.6 , [4]], [Lemma 2.5 , [13]] and [Theorem 3.3 , [14]].

Proof of Theorem 3.2. First of all, we have

$$\exp(w)w_t = u_t(x,t), \exp(w)Dw = Du(x,t), \exp(w)Dw \otimes Dw + \exp(w)D^2w = D^2u.$$

So the function w solves the new equation

$$(3.9) \qquad\qquad \frac{\partial w}{\partial t} - \operatorname{tr}(AD^2w) + G(x,t,w,Dw) = 0,$$

where

$$G(x,t,w,p) = e^{-w}K(x,t,e^wp) - |\sigma(x,p)^Tp|^2.$$

Step 1. Appropriate test functions.

Fix any $t_0 \in (0,T)$, we define the open set

$$\Delta = \Delta(t_0) = \left\{ (t,x,y) \in (0,T) \times \mathbb{T}^N \times \mathbb{T}^N : \frac{t_0}{2} < t < (T \wedge \frac{3}{2}t_0) \right\},$$

Let us set

$$\omega_{t_0}(v) = \operatorname{osc}_{(\frac{t_0}{2}, T \wedge \frac{3}{2}t_0)}(v) = \sup\left\{ v(x,t) - v(y,t), (x,y) \in \mathbb{T}^N \times \mathbb{T}^N, \frac{t_0}{2} < t < (T \wedge \frac{3}{2}t_0) \right\},$$

Consider the function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$(3.10) \qquad\qquad \Psi(s) = \frac{A_1}{A_2}(1 - e^{-A_2 s})$$

where $A_1, A_2 > 0$ (depends on t_0) will be chosen later. It is straightforward to see that Ψ is a C^∞ concave increasing function satisfying $\Psi(0) = 0$ and, for all $r > 0$ and $s \in [0,r]$,

$$(3.11) \qquad \Psi'' + A_2\Psi' = 0, \quad A_1 e^{-A_2 r} = \Psi'(r) \leq \Psi'(s) \leq \Psi'(0) = A_1.$$

We define the functions

$$(3.12) \qquad \Phi_\epsilon(t,x,y) = w(x,t) - w(y,t) - \Psi(|x-y|) - C_0(t-t_0)^2 - \frac{\epsilon}{T-t},$$

$$z(x, y, t) = \Psi(|x - y|) + C_0(t - t_0)^2 + \frac{\epsilon}{T - t},$$

In case where the supremum over $(t, x, y) \in \overline{\Delta}$ of Φ_ϵ defined in (3.12) is non-positive for small ϵ, we let $\epsilon \to 0$ and $t = t_0$ and having for all $x, y \in \mathbb{T}^N$,

$$w(x, t_0) - w(y, t_0) \leq \Psi(|x - y|) \leq A_1(t_0)|x - y|,$$

where the latter inequality follows from the concavity of Ψ. This yields the desired Lipschitz bound.

We argue by contradiction assuming that the supremum over $(t, x, y) \in \overline{\Delta}$ of Φ_ϵ defined in (3.12) is positive. We define Ψ as in (3.10) and we will show later that we can choose A_1, A_2 such that

$$(3.13) \qquad \Psi(\sqrt{N}) > \omega_{t_0}(w).$$

We make the following straightforward observations:
By choosing

$$(3.14) \qquad C_0 = \frac{4\omega_{t_0}(w)}{t_0^2},$$

we have $\Phi_\epsilon < 0$ if either $t = \frac{t_0}{2}$ or $t = \frac{3t_0}{2}$. Moreover, since $\lim_{t \to T^-} \frac{\epsilon}{T - t} = +\infty$, we conclude that Φ_ϵ cannot have a maximum if $t = \frac{t_0}{2}$ or $t = (T \wedge \frac{3}{2}t_0)$.

We then deduce that this supremum is a maximum achieved at some $(\bar{x}, \bar{y}, \bar{t}) \in \Delta$. Notice that $\bar{x} \neq \bar{y}$ because of the continuity of w.

Step 2. Viscosity inequalities for (3.9)**.** Writing the viscosity inequalities at $(\bar{x}, \bar{y}, \bar{t})$, we obtain

$$a - \operatorname{trace}(A(\bar{x}, p)X) + G(\bar{x}, \bar{t}, w(\bar{x}, \bar{t}), p) \leq 0,$$
$$b - \operatorname{trace}(A(\bar{y}, p)Y) + G(\bar{y}, \bar{t}, w(\bar{y}, \bar{t}), p) \geq 0.$$

where $a - b = \frac{\partial z}{\partial t}(\bar{x}, \bar{y}, \bar{t}) = 2C_0(t - t_0) + \frac{\epsilon}{(T-t)^2}, p = \Psi'(|\bar{x} - \bar{y}|)\frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|}$.

Therefore,

$$2C_0(t - t_0) - \operatorname{trace}(A(\bar{x}, p)X - A(\bar{y}, p)Y)$$
$$(3.15) \qquad + \quad G(\bar{x}, \bar{t}, w(\bar{x}, \bar{t}), p) - G(\bar{y}, \bar{t}, w(\bar{y}, \bar{t}), p) \leq 0.$$

Step 3: Obtaining a contradiction from (3.15)**.**

From Lemma (3.1), we have

$$(3.16) \quad -\operatorname{trace}(A(\bar{x}, p)X - A(\bar{y}, p)Y) \geq -4\nu\Psi''(|\bar{x} - \bar{y}|) - \tilde{C}\Psi'(|\bar{x} - \bar{y}|)|\bar{x} - \bar{y}| + O(\varrho).$$

We now estimate $G(\bar{x}, \bar{t}, w(\bar{x}, \bar{t}), p) - G(\bar{y}, \bar{t}, w(\bar{y}, \bar{t}), p)$ using (3.8). Set $P := e^{w(\bar{x}, \bar{t})}p$ and $\mu := e^{w(\bar{x}, \bar{t}) - w(\bar{y}, \bar{t})}$, we have

$$G(\bar{x}, \bar{t}, w(\bar{x}, \bar{t}), p) - G(\bar{y}, \bar{t}, w(\bar{y}, \bar{t}), p)$$

(3.17)
$$= e^{-w(\tilde{x})}\left(K(\bar{x}, \bar{t}, P) - \mu K(\bar{y}, \bar{t}, \frac{P}{\mu})\right) - |\sigma(\bar{x}, p)^T p|^2 + |\sigma(\bar{y}, p)^T p|^2.$$

Fix any $L > 1$, there is always a constant C depending on $K, |\sigma|_\infty, |\sigma_x|_\infty, L$ from (3.8). We can always enlarge C such that

(3.18)
$$L \geq \frac{4\omega_{t_0}(u)}{Ct_0} + 1,$$

We will prove in Step 4 that it is possible to choose A_1, A_2, r in (3.13) such that

(3.19)
$$|P| = |e^{w(\bar{x}, \bar{t})}p| \geq |p| = \Psi'(|\bar{x} - \bar{y}|) \geq \Psi'(r) = A_1 e^{-A_2 r} \geq L.$$

Since the maximum in (3.12) is positive and Ψ is concave, we get

$$\mu \geq 1 + w(\bar{x}, \bar{t}) - w(\bar{y}, \bar{t}) > 1 + \Psi(|\bar{x} - \bar{y}|)$$
$$\geq 1 + \Psi'(|\bar{x} - \bar{y}|)|\bar{x} - \bar{y}| \geq 1 + L|\bar{x} - \bar{y}|.$$

We will show in Step 4 that we can choose

$$|p| \geq L.$$

It follows that we can apply (3.8) to (3.17) to get

(3.20) $\quad G(\bar{x}, w(\bar{x}), p) - G(\bar{y}, w(\bar{y}), p)$
$$\geq -Ce^{-w(\tilde{x})}|P| - C - 2C^2|\bar{x} - \bar{y}|(\Psi')^2 \geq -C\Psi'(|\bar{x} - \bar{y}|) - C - 2C^2|\bar{x} - \bar{y}|(\Psi')^2.$$

Plugging (3.16),
and (3.20) in (3.15) and by letting $\varrho \to 0$, we obtain

$$-4\nu\Psi''(|\bar{x} - \bar{y}|) - (\tilde{C}|\bar{x} - \bar{y}| + C)\Psi'(|\bar{x} - \bar{y}|) - 2C^2|\bar{x} - \bar{y}|(\Psi')^2(|\bar{x} - \bar{y}|) - C$$
$$< 2C_0(t_0 - t) < C_0 t_0 = \frac{4\omega_{t_0}(w)}{t_0}.$$

Using the fact that for all $s \in [0, \sqrt{N}]$, $\Psi''(s) + A_2\Psi'(s) = 0$ and $s\Psi'(s) \leq \Psi(s) \leq \Psi(\sqrt{N})$, we can rewrite the above inequality as

(3.21) $\quad \left[4\nu A_2 - (\tilde{C}\sqrt{N} + C + 2C^2\Psi(|\bar{x} - \bar{y}|))\right]\Psi'(|\bar{x} - \bar{y}|) - C < \frac{4\omega_{t_0}(w)}{t_0}.$

Step 4. Choosing appropriate constants satisfying all the conditions. We now choose the constants A_1, A_2 fulfilling (3.13), (3.19) and hence obtaining a contradiction in (3.21).
We set

$$A_2 = \frac{1}{4\nu}(\tilde{C}\sqrt{N} + 2C + 2C^2\omega_{t_0}(w)) \quad \text{and} \quad A_1 = (L + \omega_{t_0}(w))\exp(A_2\sqrt{N}).$$

From the choice of A_1, we have for all $r \in [0, \sqrt{N}]$

$$\Psi'(r) = A_1 e^{-A_2 r} = (L + \omega_{t_0}(w)) e^{A_2(\sqrt{N} - r)} \geq L$$

and (3.19) holds.

Moreover from the above choice of A_1, A_2, the left hand side of (3.21) is greater than $CL - C = \frac{4\omega_{t_0}(w)}{t_0}$. This explains the choice of L in (3.18).

We then get a contradiction in (3.21).

$\square$

Remark 3.4. (**Alternative Structural Condition**)

Without performing a change of function in the proof of Theorem 3.2, we have a slightly different result if replacing (3.8) by

(3.22)
$$\begin{cases} \text{For any } L > 1, \text{ there exists } C \in \mathbb{R}, \text{ such that} \\ \text{for all } x, y \in \mathbb{T}^N, t \in [0, T], |p| \geq L, \text{ we have} \\ K(x, t, p) - K(y, t, p) \geq -C|p| - C - 2C^2|x - y||p|^2. \end{cases}$$

Theorem 3.5. *Under the same assumptions as in Theorem 3.2 where (3.8) is replaced by (3.22). We have the same conclusion.*

The proof is similar and therefore omitted. Note in particular that (3.22) is always satisfied for a sublinear Hamiltonian, see Section 4 for a brief discussion.

3.2. **Degenerate parabolic equations.** We now consider degenerate diffusion matrices and hence the Hamiltonians are the ones yielding the lipschitz regularity of the solution. This is reflected in the difference between (3.8) and (3.23): Now the required lower bound on the right hand side is much stronger.

We consider the following structure

(3.23)
$$\begin{cases} \text{There exists } L > 1 \text{ such that} \\ \text{for all } x, y \in \mathbb{T}^N, t \in [0, T], |p| \geq L, \text{ and } \mu \geq 1 + L|x - y|, \text{ we have} \\ K(x, t, p) - \mu K(y, t, \frac{p}{\mu}) \geq 2\mathcal{C}|x - y||p|^2 + N|\sigma_x|_\infty^2 |x - y||p|, \end{cases}$$

where $\mathcal{C} = 2|\sigma|_\infty |\sigma_x|_\infty$.

Theorem 3.6. *Assume (3.1), (3.2) (3.23) hold. Let u be a continuous viscosity solution of (1.2) satisfying (1.9). Set*

$$\exp(w(x, t)) = u(x, t), \quad \text{for any } x \in \mathbb{T}^N, t \in [0, T].$$

Call L the constant such that (3.23) holds, then we have

(3.24)
$$|Dw(x, t)| \leq \frac{1}{2|\sigma|_\infty |\sigma_x|_\infty t} + L.$$

Proof of Theorem 3.6. Recall that the function w defined by

$$\exp(w(x,t)) = u(x,t), \quad \text{for any } x \in \mathbb{T}^N, t \in [0,T]$$

solves

$$\frac{\partial w}{\partial t} - \text{tr}(A(x,p)D^2 w) + G(x,t,w,Dw) = 0,$$

where

$$G(x,t,w,p) = e^{-w}K(x,t,e^w p) - |\sigma(x,p)^T p|^2.$$

Step 1. Appropriate test functions.

We set the first conditions on the auxiliary function φ : We require $\varphi \geq 1$ and

$$\varphi(0) = +\infty,$$

We define the function

(3.25) $$\Phi(x,y,t) = w(x,t) - w(y,t) - L|x-y|\varphi(t),$$

and we define for convenience

$$\mathcal{S} = \sup_{(x,y,t) \in \mathbb{T}^N \times \mathbb{T}^N \times [0,T]} \Phi(x,y,t)$$

.

If $\mathcal{S}$ is non-positive we have for all $x,y \in \mathbb{T}^N$,

$$w(x,t) - w(y,t) \leq \varphi(t)L|x-y|,$$

this yields the desired result.

So now we argue by contradiction that $\mathcal{S}$ is positive.

We find that the supremum defined in (3.25) is a maximum and achieved at $(\bar{x}, \bar{y}, \bar{t}) \in \mathbb{T}^N \times \mathbb{T}^N \times (0,T]$. Notice that $\bar{x} \neq \bar{y}$ because the continuity of w.

Step 2. Viscosity inequalities for (3.9)**.** Writing the viscosity inequalities at $(\bar{x}, \bar{y}, \bar{t})$, we obtain

$$a - \text{trace}(A(\bar{x},p)X) + G(\bar{x}, \bar{t}, w(\bar{x}, \bar{t}), p) \leq 0,$$
$$b - \text{trace}(A(\bar{y},p)Y) + G(\bar{y}, \bar{t}, w(\bar{y}, \bar{t}), p) \geq 0.$$

where $a - b = L|\bar{x} - \bar{y}|\varphi'(\bar{t})$, $p = L\varphi(\bar{t})\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}$.

Therefore,

(3.26) $$L|\bar{x} - \bar{y}|\varphi'(\bar{t}) - \text{trace}(A(\bar{x},p)X - A(\bar{y},p)Y)$$
$$+ \quad G(\bar{x}, \bar{t}, w(\bar{x}, \bar{t}), p) - G(\bar{y}, \bar{t}, w(\bar{y}, \bar{t}), p) \leq 0.$$

Step 3. Estimates of the terms in (3.26)**.**

From Lemma (3.1), we have

(3.27) $$-\text{trace}(A(\bar{x},p)X - A(\bar{y},p)Y) \geq -LN|\sigma_x|_\infty^2 |\bar{x} - \bar{y}| + O(\varrho).$$

We now estimate $L|\bar{x} - \bar{y}|\varphi'(\bar{t}) + G(\bar{x}, \bar{t}, w(\bar{x}, \bar{t}), p) - G(\bar{y}, \bar{t}, w(\bar{y}, \bar{t}), p)$ using (3.23). Set $P := e^{w(\bar{x}, \bar{t})}p$ and $\mu := e^{w(\bar{x}, \bar{t}) - w(\bar{y}, \bar{t})}$, we have

$$L|\bar{x} - \bar{y}|\varphi'(\bar{t}) + G(\bar{x}, \bar{t}, w(\bar{x}, \bar{t}), p) - G(\bar{y}, \bar{t}, w(\bar{y}, \bar{t}), p)$$

$$(3.28) \quad = \quad L|\bar{x} - \bar{y}|\varphi'(\bar{t}) + e^{-w(\bar{x})}\left(K(\bar{x}, \bar{t}, P) - \mu K(\bar{y}, \bar{t}, \frac{P}{\mu})\right) - |\sigma(\overline{x})^T p|^2 + |\sigma(\overline{y})^T p|^2.$$

It is clear that

$$(3.29) \qquad\qquad |P| = |e^{w(\bar{x}, \bar{t})}p| \geq |p| = L\varphi(\bar{t}) \geq L,$$

Since the maximum in (3.25) is positive, we get

$$\mu \geq 1 + w(\bar{x}, \bar{t}) - w(\bar{y}, \bar{t}) > 1 + L|\bar{x} - \bar{y}|\varphi(\bar{t}) \geq 1 + L|\bar{x} - \bar{y}|.$$

So we have

$$|p| \geq L \quad \text{and} \quad \mu \geq 1 + L|\bar{x} - \bar{y}|.$$

Noticing that $e^{-w}|P|^2 = e^w|p|^2 \geq |p|^2 = L^2\varphi(\bar{t})^2$. It follows that we can apply (3.23) to (3.28) to get

$$(3.30) \qquad L|\bar{x} - \bar{y}|\varphi'(\bar{t}) + G(\bar{x}, w(\bar{x}), p) - G(\bar{y}, w(\bar{y}), p)$$

$$\geq \quad L|\bar{x} - \bar{y}|\varphi'(\bar{t}) + e^{-w(\bar{x})}(2\mathcal{C}|\overline{x} - \overline{y}||P|^2 + N|\sigma_x|_\infty^2|\overline{x} - \overline{y}||P|) - \mathcal{C}|\overline{x} - \overline{y}||p|^2,$$

$$> \quad N|\sigma_x|_\infty^2|\overline{x} - \overline{y}|L + L|\bar{x} - \bar{y}|\varphi'(\bar{t}) + L^2\varphi(\bar{t})^2\mathcal{C}|\overline{x} - \overline{y}|,$$

where we recall that $\mathcal{C} = 2|\sigma|_\infty|\sigma_x|_\infty$.

Plugging (3.27), and (3.30) in (3.26), we can rewrite (3.26) as

$$L|\bar{x} - \bar{y}|\varphi'(\bar{t}) + L^2\varphi(\bar{t})^2\mathcal{C}|\overline{x} - \overline{y}| < 0.$$

or,

$$\varphi'(\bar{t}) + L\varphi(\bar{t})^2\mathcal{C} < 0.$$

So if we choose from the beginning $\varphi \geq 1$ such that $\varphi'(\bar{t}) + L\varphi(\bar{t})^2\mathcal{C} \geq 0$, for instance with the choice of φ

$$\varphi(t) = \frac{1}{L\mathcal{C}t} + 1, \mathcal{C} = 2|\sigma|_\infty|\sigma_x|_\infty$$

in the above calculations, we get a contradiction and hence obtain (3.24).

$\square$

Remark 3.7. We would like to make some comments about the bound obtained in Theorems 3.2 and 3.6.

In these two theorems, we proved that $|Dw|_\infty \leq \alpha(t)$, where $\exp(w(x, t)) = u(x, t)$. But we are interested in the bound for the original solutions u. The desired bound for u can be obtained with a similar change of variable of type $\exp(v(x, t)) = u(x, t) - \min_{x \in \mathbb{T}^N} u(x, t) + 1$, we still have $|Dv|_\infty \leq \alpha(t)$ (α depends on the oscillation of v) with an almost identical proof.

It is clear that $\min_{x \in \mathbb{T}^N} v(x,t) = 0$ and hence

$$\exp(osc(v(.,t))) = \exp(\max_{x \in \mathbb{T}^N} v(x,t)) = osc(u(.,t)) + 1.$$

It also implies easily that $osc(v(.,t)) \leq osc(u(.,t))$

Since we have

$$
\begin{aligned}
|Du(x,t)| &= \exp(v(x,t))|Dv(x,t)| \leq (osc(u(.,t)) + 1)|Dv(x,t)| \\
&\leq (osc(u(.,t)) + 1)\alpha(osc(v(.,t))) \leq (osc(u(.,t)) + 1)\alpha(osc(u(.,t))).
\end{aligned}
$$

So we have the bound for the original solutions u from Theorems 3.2 and 3.6.

4. Examples and Applications

4.1. **Examples.** We present an example satisfying (3.23).

(4.1) $$K(x,t,p) = a(x,t)|p|^k - f(x,t),$$

where $k > 1$, $a > 0, f$ are Lipschitz continuous functions with the Lipschitz constant bounded uniformly in $t \in [0,T]$.

We show that K satisfies (3.23). For all $x, y \in \mathbb{T}^N, t \in [0,T]$, $|p| \geq L$, M is some constant depending only on the oscillation of the solution, for $M \geq \mu \geq 1 + L|x-y|$, we need to verify that

(4.2) $$K(x,t,p) - \mu K(y,t,\frac{p}{\mu}) \geq C|x-y||p|^2,$$

for large value of L chosen later.

With $\alpha > 1$, Bernoulli's inequality yields

(4.3) $$\mu^\alpha - \mu \geq (\alpha-1)(\mu-1),$$

We have,

$$K(x,t,p) - \mu K(y,t,\frac{p}{\mu}) = a(x,t)|p|^k - a(y,t)\frac{|p|^k}{\mu^{k-1}} - f(x,t) + \mu f(y,t) := S - f(x,t) + \mu f(y,t).$$

We have

$$
\begin{aligned}
S = a(x,t)|p|^k - a(y,t)\frac{|p|^k}{\mu^{k-1}} &= \frac{|p|^k}{\mu^{k-1}}[a(x,t) - a(y,t)] + a(x,t)[|p|^k - \frac{|p|^k}{\mu^{k-1}}] \\
&\geq -\frac{|p|^k}{\mu^{k-1}}\frac{Lip(a)(\mu-1)}{L} + min(a)[|p|^k - \frac{|p|^k}{\mu^{k-1}}] \quad \text{(and from (4.3), we get)} \\
&\geq \frac{|p|^k(\mu-1)}{\mu^k}[min(a)(k-1) - \frac{Lip(a)\mu}{L}] \\
&\geq \frac{|p|^k(\mu-1)}{\mu^k}[min(a)(k-1) - \frac{Lip(a)M}{L}].
\end{aligned}
$$

By choosing L big enough, it shows the desired inequality.

Now, we present an example that satisfies (3.8). For any Lipschitz continuous function a with the Lipschitz constant bounded uniformly in $t \in [0,T]$ (if $k \geq 1$, a needs to be non-negative, for $0 \leq k < 1$ no positivity is required), f continuous and $|f(x,p)| \leq C(|p|+1)$

$$(4.4) \qquad K(x,t,p) = a(x,t)|p|^k + f(x,p), \quad k \geq 0 \ .$$

In the theory of Mean Field Games, sublinear Hamiltonians are important. The following is an important example coming from Mean Field Games that satisfies (3.22). For any continuous function K satisfying

$$(4.5) \qquad |K(x,t,p)| \leq C|p| + C,$$

then K satisfies (3.22) and hence Theorem 3.5 applies.

4.2. **Application to the Large-time behavior for Hamilton-Jacobi equations forced by additive noise.** Dirr and Souganidis, [9], consider Hamilton-Jacobi equations driven by additive noise which is white in time and smooth in space (depending only on finitely many independent Brownian motions). They show (Theorem 2.3 in [9]) that if the deterministic equation has an attractor which consists of a uinique solution up to constants, then the attractor of the stochastic version consists of a unique trajectory defined for times on all of $\mathbb{R}$.

A crucial ingredient in the proof are Lipschitz estimates. They are used in the following way:

By elementary probabilistic arguments, there are plenty of intervals of small noise, in which different solutions behave like solutions to the deterministic equation. This means that their distance (modulo constants) decreases, while by the comparison principle this distance cannot increase outside those "good" intervals. Hence also the distance (always uo to constants) of solutions of the stochastic equation decreases as time increases and they pass through sufficiently many small-noise intervals.

However, in order to show that solutions of the stochastically perturbed and unperturbed equation stay close if the noise is small thewy need to control the Lipschitz-constant. As the initial datum ina small-noise interval is the solution at the end of the preceding bi-noise interval, the initial datum and its oscillation is not controlled, so Lipschitz regularization independently of the initial datum is necessary.

The results in this paper allow to strengthen the results of [9] in as much as for superquadratic Hamiltonian and nonzero second-order part our assumptions, which are slightly weaker than those in [9].

5. APPENDIX

5.1. **Proof of Proposition 2.1.**

Proof of Proposition 2.1. **Step 1.** Assume without loss of generality that $\min u_0 = 0$ and

(5.1) $$\beta = q - 2 > 0, \gamma = (1 - \theta)\frac{q-2}{q-1}, \alpha = \gamma - 1 + 2\theta,$$

where $\theta \in (0, \frac{1}{2})$ is chosen so that $\alpha > 0$.

Step 2: General form of super-solutions

For $a, b > 0$, we consider the function $G_{a,b} : \mathbb{R}^N \times (0, \infty) \to \mathbb{R}$

$$G_{a,b}(x, t) = f(M)t + 2bC_0 t^\gamma + at^\alpha + b\gamma |x|^2 t^{\gamma-1},$$

where C_0 is the constant appearing in assumption (2.2).

Note that for any $x \neq 0$, $\lim_{t \to 0^+} G_{a,b}(x, t) = +\infty$.

Since $D^2 |x|^2 = 2I$, we have

$$
\begin{aligned}
\mathcal{M} &= \frac{\partial G}{\partial t} - \mathrm{trace}(A(x, DG)D^2G) + M^{-1}|DG|^q - f(M) \\
&\geq a\alpha t^{\alpha-1} + b(\gamma - 1)\gamma |x|^2 t^{\gamma-2} + M^{-1}|DG|^q \\
&\geq a\alpha t^{\alpha-1} + b(\gamma - 1)\gamma |x|^2 t^{\gamma-2} + \frac{1}{2M}|DG|^q \\
&= a\alpha t^{\alpha-1} + |x|^2 t^{\gamma-2}[\frac{1}{M}|2b\gamma t^{\gamma-1}|^q |x|^{q-2} t^{2-\gamma} - b(1-\gamma)\gamma] \\
&= a\alpha t^{\alpha-1} + |x|^2 t^{\gamma-2}[\frac{1}{M}|2b\gamma|^q (\frac{|x|}{t^\theta})^{q-2} - b(1-\gamma)\gamma].
\end{aligned}
$$

Step 3: Choosing the suitable constants a, b.

We need to choose the suitable constants such that $\mathcal{D} > 0$.

If $|x| \geq t^\theta$, we can choose b depending on q, θ, and K but not on a such that $\mathcal{D} > 0$.

If $|x| < t^\theta$, we choose a such that

$$\mathcal{D} > a\alpha t^{\alpha-1} - b(1-\gamma)\gamma |x|^2 t^{\gamma-2} > 0.$$

This is satisfied if

$$a\alpha t^{\alpha-1} - b(1-\gamma)\gamma t^{2\theta+\gamma-2} = t^{\alpha-1}[a\alpha - b(1-\gamma)\gamma] > 0.$$

Finally, the periodic super-solution $V(x, t)$ is defined as follows

$$V(x, t) = \inf_{z \in \mathbb{T}^N} G(x - z, t).$$

$\square$

5.2. **Proof of Lemma 3.1.** The theory of second order viscosity solutions yields (see [8, Theorem 3.2] for instance), for every $\varrho > 0$, the existence of $(a, p, X) \in \overline{J}^{2,+} \phi(\overline{x}, \overline{t})$, $(a, p, Y) \in \overline{J}^{2,-} \phi(\overline{y}, \overline{t})$ such that (3.3), (3.4), (3.5) hold.

Le us prove (3.6) and (3.7). From (3.3), for every $\zeta, \xi \in \mathbb{R}^N$, we have

$$\langle X\zeta, \zeta \rangle - \langle Y\xi, \xi \rangle \leq \Psi'\langle \zeta - \xi, B(\zeta - \xi) \rangle + \Psi''\langle \zeta - \xi, (q \otimes q)(\zeta - \xi) \rangle + O(\varrho).$$

We estimate $\text{trace}(A(\overline{x}, p)X)$ and $\text{trace}(A(\overline{y}, p)Y)$ using two orthonormal bases $(e_1, \cdots, e_N)$ and $(\tilde{e}_1, \cdots, \tilde{e}_N)$ in the following way:

$$
\begin{aligned}
T := \text{trace}(A(\overline{x}, p)X - A(\overline{y}, p)Y) &= \sum_{i=1}^{N} \langle X\sigma(\overline{x}, p)e_i, \sigma(\overline{x}, p)e_i \rangle - \langle Y\sigma(\overline{y}, p)\tilde{e}_i, \sigma(\overline{y}, p)\tilde{e}_i \rangle \\
&\leq \sum_{i=1}^{N} \Psi'\langle \zeta_i, B\zeta_i \rangle + \Psi''\langle \zeta_i, (q \otimes q)\zeta_i \rangle + O(\varrho) \\
(5.2) \qquad &\leq \Psi''\langle \zeta_1, (q \otimes q)\zeta_1 \rangle + \sum_{i=1}^{N} \Psi'\langle \zeta_i, B\zeta_i \rangle + O(\varrho),
\end{aligned}
$$

where we set $\zeta_i = \sigma(\overline{x}, p)e_i - \sigma(\overline{y}, p)\tilde{e}_i$ and noticing that $\Psi''\langle \zeta_i, (q \otimes q)\zeta_i \rangle = \Psi''\langle \zeta_i, q \rangle^2 \leq 0$ since Ψ is concave.

We now build a suitable base to prove (3.6) and another one to prove (3.7).

In the case of (3.6) where σ could be degenerate, we choose any orthonormal basis such that $e_i = \tilde{e}_i$. It follows

$$
\begin{aligned}
T &\leq \sum_{i=1}^{N} \Psi'\langle (\sigma(\overline{x}, p) - \sigma(\overline{y}, p))e_i, B(\sigma(\overline{x}, p) - \sigma(\overline{y}, p))e_i \rangle + O(\varrho) \\
&\leq \Psi' N |\sigma(\overline{x}, p) - \sigma(\overline{y}, p)|^2 |B| + O(\varrho) \\
&\leq \Psi' N |\sigma_x|_{\infty}^2 |\overline{x} - \overline{y}| + O(\varrho)
\end{aligned}
$$

since $|B| \leq 1/|\overline{x} - \overline{y}|$. Thus (3.6) holds.

When (1.4) holds, i.e., $A(x, p) \geq \nu I$ for every $(x, p) \in \mathbb{T}^N \times \mathbb{R}^N$, the matrix $\sigma(x, p)$ is invertible and we can set

$$e_1 = \frac{\sigma(\overline{x}, p)^{-1}q}{|\sigma(\overline{x}, p)^{-1}q|}, \quad \tilde{e}_1 = -\frac{\sigma(\overline{y}, p)^{-1}q}{|\sigma(\overline{y}, p)^{-1}q|}, \quad \text{where } q \text{ is given by (3.4)}.$$

If e_1 and $\tilde{e}_1$ are collinear, then we complete the basis with orthogonal unit vectors $e_i = \tilde{e}_i \in e_1^\perp$, $2 \leq i \leq N$. Otherwise, in the plane $\text{span}\{e_1, \tilde{e}_1\}$, we consider a rotation $\mathcal{R}$ of angle $\frac{\pi}{2}$ and define

$$e_2 = \mathcal{R}e_1, \quad \tilde{e}_2 = -\mathcal{R}\tilde{e}_1.$$

Finally, noticing that $\text{span}\{e_1, e_2\}^\perp = \text{span}\{\tilde{e}_1, \tilde{e}_2\}^\perp$, we can complete the orthonormal basis with unit vectors $e_i = \tilde{e}_i \in \text{span}\{e_1, e_2\}^\perp$, $3 \leq i \leq N$.

From (1.4), we have

(5.3) $$\nu \leq \frac{1}{|\sigma(x,p)^{-1}q|^2} \leq |\sigma|_\infty^2.$$

It follows

$$\langle \zeta_1, (q \otimes q)\zeta_1 \rangle = \left(\frac{1}{|\sigma(\overline{x},p)^{-1}q|} + \frac{1}{|\sigma(\overline{y},p)^{-1}q|} \right)^2 \geq 4\nu.$$

From (3.4), we deduce $Bq = 0$. Therefore

$$\langle \zeta_1, B\zeta_1 \rangle = 0.$$

For $3 \leq i \leq N$, we have

$$\langle \zeta_i, B\zeta_i \rangle = \langle (\sigma(\overline{x},p) - \sigma(\overline{y},p))e_i, B(\sigma(\overline{x},p) - \sigma(\overline{y},p))e_i \rangle \leq |\sigma_x|_\infty^2 |\overline{x} - \overline{y}|.$$

Now, we estimate ζ_2

$$|\zeta_2| = |(\sigma(\overline{x},p) - \sigma(\overline{y},p))\mathcal{R}e_1 + \sigma(\overline{y},p)\mathcal{R}(e_1 + \tilde{e}_1)| \leq |\sigma_x|_\infty |\overline{x} - \overline{y}| + |\sigma|_\infty |e_1 + \tilde{e}_1|.$$

It remains to estimate

$$\begin{aligned} |e_1 + \tilde{e}_1| &\leq \frac{1}{|\sigma(\overline{x},p)^{-1}q|} |\sigma(\overline{x},p)^{-1}q - \sigma(\overline{y},p)^{-1}q| + |\sigma(\overline{y},p)^{-1}q| \left| \frac{1}{|\sigma(\overline{x},p)^{-1}q|} - \frac{1}{|\sigma(\overline{y},p)^{-1}q|} \right| \\ &\leq \frac{2|\sigma|_\infty |\sigma_x|_\infty}{\nu} |\overline{x} - \overline{y}|, \end{aligned}$$

from (5.3) and $|(\sigma^{-1})_x|_\infty \leq |\sigma_x|_\infty / \nu$.

We hence obtain from (5.2) $T \leq 4\nu\Psi'' + \tilde{C}\Psi'|\overline{x} - \overline{y}| + O(\varrho)$ where

(5.4) $$\tilde{C} = \tilde{C}(N, \nu, |\sigma|_\infty, |\sigma_x|_\infty) := |\sigma_x|_\infty^2 (N - 2 + (1 + \frac{2|\sigma|_\infty^2}{\nu})^2).$$

This yields (3.7).

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