Elementary Iterated Revision and the Levi Identity[*]

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Abstract. Recent work has considered the problem of extending to the case of iterated belief change the so-called 'Harper Identity' (HI), which defines single-shot contraction in terms of single-shot revision. The present paper considers the prospects of providing a similar extension of the Levi Identity (LI), in which the direction of definition runs the other way. We restrict our attention here to the three classic iterated revision operators–natural, restrained and lexicographic, for which we provide here the first collective characterisation in the literature, under the appellation of 'elementary' operators. We consider two prima facie plausible ways of extending (LI). The first proposal involves the use of the rational closure operator to offer a 'reductive' account of iterated revision in terms of iterated contraction. The second, which doesn't commit to reductionism, was put forward some years ago by Nayak *et al*. We establish that, for elementary revision operators and under mild assumptions regarding contraction, Nayak's proposal is equivalent to a new set of postulates formalising the claim that contraction by $\neg A$ should be considered to be a kind of 'mild' revision by A. We then show that these, in turn, under slightly weaker assumptions, jointly amount to the conjunction of a pair of constraints on the extension of (HI) that were recently proposed in the literature. Finally, we consider the consequences of endorsing both suggestions and show that this would yield an identification of rational revision with natural revision. We close the paper by discussing the general prospects for defining iterated revision in terms of iterated contraction.

Keywords: belief revision · iterated belief change · Levi identity.

1 Introduction

The crucial question of iterated belief change–that is, the question of the rationality constraints that govern the beliefs resulting from a sequence of changes in view–remains very much a live one.

In recent work [3], we have studied in some detail the problem of extending, to the iterated case, a principle of single-step change known as the 'Harper Identity' (henceforth '(HI)') [15]. This principle connects single-step contraction and revision, the two main types of change found in the literature, in a manner that allows one to define the former in terms of the latter. We presented a family of extensions of (HI) characterised by the satisfaction of an intuitive pair of principles and showed how these postulates

could be used to translate principles of iterated revision into principles of iterated contraction.

But (HI) also has a well known companion principle which reverses the direction of definition, allowing one to define single-step revision in terms of single-step contraction: the Levi Identity (henceforth '(LI)') [20]. To date, furthermore, the issue of extending (LI) to the iterated case has barely been discussed. Two noteworthy exceptions are the short papers of Nayak et al [21] and of Konieczny & Pino Pérez [17]. The second paper argues that no reasonable extension of (LI) will enable us to reduce iterated revision to iterated contraction. The first paper introduces a non-reductionist extension of (LI) consonant with this claim.

The present contribution aims to provide a more comprehensive discussion of the issue, carried out against the backdrop of the aforementioned recent work on (HI). The plan of the paper is as follows. After a preliminary introduction of the formal framework in Section 2, we provide, in Section 3, a novel result that is of general interest in itself. We collectively characterise the three classic belief revision operators that are the focus of the paper (natural, restrained and lexicographic) under the appellation of 'elementary' operators, showing that they are in fact the *only* operators satisfying a particular set of properties. Section 4 turns to the issue of extending (LI) to the iterated case. We present, in Section 4.1, an extension of (LI) based on the concept of rational closure, which would result in a reduction of two-step revision to two-step contraction. Section 4.2 then discusses the non-reductive proposal of [21]. We first establish that, for elementary revision operators and under mild assumptions regarding contraction, it is in fact equivalent to a new set of postulates formalising the claim that contraction by $\neg A$ should be considered to be a kind of 'mild' revision by A. These, in turn, under slightly weaker assumptions, are proven to jointly amount to the conjunction of the aforementioned constraints on the extension of (HI) that were proposed in [3]. In Section 4.3, we consider the consequences of endorsing both suggestions and show that this would yield an identification of rational revision with natural revision. In Section 5, we briefly discuss the general prospects for defining iterated revision from iterated contraction, critically assessing the central argument of [17]. We conclude, in Section 6, with some remaining open questions.

Due to space limitations, only sketches of the more important proofs have been provided. A version of the paper containing all proofs can be accessed online at https://arxiv.org/abs/1907.01224.

2   Preliminaries

The beliefs of an agent are represented by a *belief state* Ψ. The latter determines a *belief set* $[\Psi]$, a deductively closed set of sentences, drawn from a finitely generated, propositional, truth-functional language L. The set of classical logical consequences of $\Gamma \subseteq L$ will be denoted by $\mathrm{Cn}(\Gamma)$. The set of propositional worlds or valuations will be denoted by W, and the set of models of a given sentence A by $[\![A]\!]$.

We consider the three classic belief change operations mapping a prior state Ψ and input sentence A in L onto a posterior state. The operation of *revision* $*$ returns the posterior state $\Psi * A$ that results from an adjustment of Ψ to accommodate the inclusion

of A, in such a way as to maintain consistency of the resulting belief set when $\neg A \in [\Psi]$. The operation of *expansion* $+$ is similar, save that consistency of the resulting beliefs needn't be ensured. Finally, the operation of *contraction* $\div$ returns the posterior state $\Psi \div A$ that results from an adjustment of Ψ to accommodate the retraction of A.

2.1 Single-step change

In terms of single-step change, revision and contraction are assumed to satisfy the postulates of Alchourrón, Gärdenfors and Makinson [1] (henceforth 'AGM'), while the behaviour of expansion is constrained by $[\Psi + A] = \mathrm{Cn}([\Psi] \cup \{A\})$. AGM ensures a useful order-theoretic representability of the single-shot revision or contraction dispositions of an agent, such that each Ψ is associated with a total preorder (henceforth '*TPO*') $\preccurlyeq_\Psi$ over W, such that $[\![[\Psi * A]]\!] = \min(\preccurlyeq_\Psi, [\![A]\!])$ ([14,16]). In this context, the AGM postulate of *Success* $(A \in [\Psi * A])$ corresponds to the requirement that $\min(\preccurlyeq_{\Psi * A}, W) \subseteq [\![A]\!]$. We denote by $\mathrm{TPO}(W)$ the set of all TPOs over W and shall assume that, for every $\preccurlyeq \in \mathrm{TPO}(W)$, there is a state Ψ such that $\preccurlyeq = \preccurlyeq_\Psi$.

Equivalently, these revision dispositions can be represented by a '*conditional belief set*' $[\Psi]_c$. This set extends the belief set $[\Psi]$ by further including various 'conditional beliefs', expressed by sentences of the form $A \Rightarrow B$, where $\Rightarrow$ is a non-truth-functional conditional connective and $A, B \in L$ (we shall call L_c the language that extends L to include such conditionals). This is achieved by means of the so-called *Ramsey Test*, according to which $A \Rightarrow B \in [\Psi]_c$ iff $B \in [\Psi * A]$. In terms of constraints on $[\Psi]_c$, AGM notably ensures that its conditional subset corresponds to a *rational consequence relation*, in the sense of [19] (we shall say, in this case, that $[\Psi]_c$ is rational).

Following convention, we shall call principles couched in terms of belief sets 'syntactic', and call 'semantic' those principles couched in terms of TPOs, denoting the latter by subscripting the corresponding syntactic principle with '$\preccurlyeq$'.

The operations $*$ and $\div$ are assumed to be related in the single-shot case by the Levi and Harper identities, namely

(LI) $[\Psi * A] = \mathrm{Cn}([\Psi \div \neg A] \cup \{A\})$
(HI) $[\Psi \div A] = [\Psi] \cap [\Psi * \neg A]$

with single-shot revision determining single-shot expansion via a third identity:

(TI) $[\Psi + A] = [\Psi * A]$, if $\neg A \notin [\Psi]$
$= L$, otherwise

(LI) can of course alternatively be presented as $[\Psi * A] = [(\Psi \div \neg A) + A]$. Note that, given (HI) and (LI), the constraint $[\![[\Psi * A]]\!] = \min(\preccurlyeq_\Psi, [\![A]\!])$ is equivalent to $[\![[\Psi \div \neg A]]\!] = \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![A]\!])$, so that $\preccurlyeq_\Psi$ equally represents both revision and contraction dispositions.

The motivation for (LI) is the following: The most parsimonious way of modifying $[\Psi]$ so as to include A is to simply add the joint logical consequences of $[\Psi]$ and A. However, $\mathrm{Cn}([\Psi] \cup \{A\})$ needn't be consistent. Hence we first 'make room' for A by considering instead the belief set $[\Psi \div \neg A]$ that results from making the relevant minimal change necessary to achieve consistency.

2.2   Iterated change

In terms of iterated revision, we shall considerably simplify the discussion by restricting our attention to the three principal operators found in the literature. These are natural revision [8]:

$$x \preccurlyeq_{\Psi * _N A} y \text{ iff (1) } x \in \min(\preccurlyeq_\Psi, [\![A]\!]), \text{ or (2) } x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \text{ and } x \preccurlyeq_\Psi y$$

restrained revision [6]:

$$x \preccurlyeq_{\Psi * _R A} y \text{ iff (1) } x \in \min(\preccurlyeq_\Psi, [\![A]\!]), \text{ or (2) } x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \text{ and either (a)}$$
$$x \prec_\Psi y \text{ or (b) } x \sim_\Psi y \text{ and } (x \in [\![A]\!] \text{ or } y \in [\![\neg A]\!])$$

and lexicographic revision [22]:

$$x \preccurlyeq_{\Psi * _L A} y \text{ iff (1) } x \in [\![A]\!] \text{ and } y \in [\![\neg A]\!], \text{ or (2) } (x \in [\![A]\!] \text{ iff } y \in [\![A]\!]) \text{ and}$$
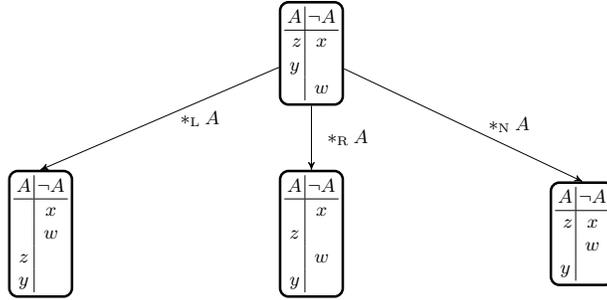$$x \preccurlyeq_\Psi y.^3$$

See Figure 1.

Fig. 1: Elementary revision by A. The boxes represent states and associated TPOs. The lower case letters, which represent worlds, are arranged in such a way that the lower the letter, the lower the corresponding world in the relevant ordering. The columns group worlds according to the sentences that they validate. So, for example, in the initial ordering, we have $w \prec y \prec x \sim z$, with $y, z \in [\![A]\!]$ and $x, w \in [\![\neg A]\!]$ and then, after lexicographic revision by A, $y \prec z \prec w \prec x$.

All three suggestions operate on the assumption that a state Ψ is to be identified with its corresponding TPO $\preccurlyeq_\Psi$ and that belief change functions map pairs of TPOs and sentences onto TPOs, in other words, they entail:

(Red)   If $\preccurlyeq_\Psi = \preccurlyeq_{\Psi'}$, then, for any A, $\preccurlyeq_{\Psi * A} = \preccurlyeq_{\Psi' * A}$

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[3] These are three of the four iterated revision operators mentioned in Rott's influential survey [23]. The remaining operator the irrevocable revision operator of [24], which has the unusual characteristic of ensuring that prior inputs to revision are retained in the belief set after *any* subsequent revision.

The proposals ensure that $*$ satisfies the postulates of Darwiche & Pearl [10]. In their semantic forms, these are:

(C1$^*_\preccurlyeq$)   If $x, y \in [\![A]\!]$ then $x \preccurlyeq_{\Psi*A} y$ iff $x \preccurlyeq_\Psi y$

(C2$^*_\preccurlyeq$)   If $x, y \in [\![\neg A]\!]$ then $x \preccurlyeq_{\Psi*A} y$ iff $x \preccurlyeq_\Psi y$

(C3$^*_\preccurlyeq$)   If $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ and $x \prec_\Psi y$, then $x \prec_{\Psi*A} y$

(C4$^*_\preccurlyeq$)   If $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ and $x \preccurlyeq_\Psi y$, then $x \preccurlyeq_{\Psi*A} y$

Regarding $\div$, we assume that it satisfies the postulates of Chopra et al [9], given semantically by:

(C1$^\div_\preccurlyeq$)   If $x, y \in [\![\neg A]\!]$ then $x \preccurlyeq_{\Psi\div A} y$ iff $x \preccurlyeq_\Psi y$

(C2$^\div_\preccurlyeq$)   If $x, y \in [\![A]\!]$ then $x \preccurlyeq_{\Psi\div A} y$ iff $x \preccurlyeq_\Psi y$

(C3$^\div_\preccurlyeq$)   If $x \in [\![\neg A]\!]$, $y \in [\![A]\!]$ and $x \prec_\Psi y$ then $x \prec_{\Psi\div A} y$

(C4$^\div_\preccurlyeq$)   If $x \in [\![\neg A]\!]$, $y \in [\![A]\!]$ and $x \preccurlyeq_\Psi y$ then $x \preccurlyeq_{\Psi\div A} y$

Concerning the relations between the belief change operators in the iterated case, we will be discussing the extension of (LI), as well as that of (TI), later in the paper. Regarding (HI), a proposal for extending the principle to the two-step case was recently floated in [3]. Semantically speaking, this involved the characterisation of a binary TPO combination operator $\oplus$, such that $\preccurlyeq_{\Psi\div A} = \preccurlyeq_\Psi \oplus \preccurlyeq_{\Psi*\neg A}$. Among the baseline constraints on $\oplus$, were a pair of conditions that were shown to be respectively equivalent, in the presence of (C1$^*_\preccurlyeq$) and (C2$^*_\preccurlyeq$), to the following joint constraints on $\preccurlyeq_{\Psi\div A}$, $\preccurlyeq_\Psi$ and $\preccurlyeq_{\Psi*\neg A}$:

(SPU$_\preccurlyeq$) If $x \prec_\Psi y$ and $x \prec_{\Psi*\neg A} y$ then $x \prec_{\Psi\div A} y$

(WPU$_\preccurlyeq$) If $x \preccurlyeq_\Psi y$ and $x \preccurlyeq_{\Psi*\neg A} y$ then $x \preccurlyeq_{\Psi\div A} y$

We called operators satisfying such postulates, in addition to (HI), 'TeamQueue combinators'.

3 Elementary revision operators

In this section, we demonstrate the relative generality of the results that follow by providing a characterisation result according to which natural, restrained and lexicographic revision operators are the *only* operators satisfying a small set of potentially appealing properties. We shall call operators that satisfy these properties *elementary* revision operators. We define elementary revision operators semantically by:

Definition 1. $*$ *is an* elementary *revision operator iff it satisfies* (C1$^*_\preccurlyeq$)–(C4$^*_\preccurlyeq$), (IIAP$^*_\preccurlyeq$), (IIAI$^*_\preccurlyeq$) *and* (Neut$^*_\preccurlyeq$).

We have already introduced (C1$^*_\preccurlyeq$)-(C4$^*_\preccurlyeq$). The remaining principles are new. We call the first of these 'Independence of Irrelevant Alternatives with respect to the prior TPO', after an analogous precept in social choice. For this, we first define the notion of 'agreement' between TPO's on a pair of worlds:

Definition 2. *Where* $\preccurlyeq_\Psi, \preccurlyeq_{\Psi'} \in \mathrm{TPO}(W)$, $\preccurlyeq_\Psi$ *and* $\preccurlyeq_{\Psi'}$ *agree on* $\{x, y\}$ *iff* $\preccurlyeq_\Psi \cap \{x, y\}^2 = \preccurlyeq_{\Psi'} \cap \{x, y\}^2$.

then offer:

($\mathrm{IIAP}^*_{\preccurlyeq}$) If $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_{\Psi'}, [\![A]\!])$, then, if $\preccurlyeq_\Psi$ and $\preccurlyeq_{\Psi'}$ agree on $\{x, y\}$, so do $\preccurlyeq_{\Psi*A}$ and $\preccurlyeq_{\Psi'*A}$

The second new principle–'Independence of Irrelevant Alternatives with respect to the input'–is formally similar to the first. For this we first introduce some helpful notation:

Definition 3. *(i)* $x \preccurlyeq^A y$ *iff* $x \in [\![A]\!]$ *or* $y \in [\![\neg A]\!]$, *(ii)* $x \sim^A y$ *when* $x \preccurlyeq^A y$ *and* $y \preccurlyeq^A x$, *and (iii)* $x \prec^A y$ *when* $x \preccurlyeq^A y$ *but not* $y \preccurlyeq^A x$.

The principle is then given by:

($\mathrm{IIAI}^*_{\preccurlyeq}$) If $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Psi, [\![B]\!])$, then, if $\preccurlyeq^A$ and $\preccurlyeq^B$ agree on $\{x, y\}$, so do $\preccurlyeq_{\Psi*A}$ and $\preccurlyeq_{\Psi*B}$

The final principle is a principle of 'Neutrality', again named after an analogous condition in social choice. To the best of our knowledge, it appears here for the first time in the context of belief revision. Its presentation makes use of the following concept:

Definition 4. *Where* $A \in L$, π *is an* A-*preserving order isomorphism from* $\langle W, \preccurlyeq_\Psi, \preccurlyeq^A \rangle$ *to* $\langle W, \preccurlyeq_{\Psi'}, \preccurlyeq^A \rangle$ *iff it is a 1:1 mapping from* W *onto itself such that*

(i) $x \preccurlyeq_\Psi y$ *iff* $\pi(x) \preccurlyeq_{\Psi'} \pi(y)$, *and*
(ii) $x \preccurlyeq^A y$ *iff* $\pi(x) \preccurlyeq^A \pi(y)$

and proceeds as follows:

($\mathrm{Neut}^*_{\preccurlyeq}$) $x \preccurlyeq_{\Psi*A} y$ iff $\pi(x) \preccurlyeq_{\Psi'*A} \pi(y)$, for any A-preserving order isomorphism π from $\langle W, \preccurlyeq_\Psi, \preccurlyeq^A \rangle$ to $\langle W, \preccurlyeq_{\Psi'}, \preccurlyeq^A \rangle$

($\mathrm{IIAP}^*_{\preccurlyeq}$) and ($\mathrm{IIAI}^*_{\preccurlyeq}$) say that the relative ordering of x and y after revising by A depends on only (i) their relative order prior to revision (from ($\mathrm{IIAP}^*_{\preccurlyeq}$)) and (ii) their relative positioning with respect to A (i.e., whether or not they satisfy A) *unless* one of x or y is a minimal A-world, in which case this requirement acquiesces to the Success postulate (from ($\mathrm{IIAI}^*_{\preccurlyeq}$)). ($\mathrm{Neut}^*_{\preccurlyeq}$) is a form of language-independence property, stating that the labels (in terms of valuations) of worlds are irrelevant in determining the posterior TPO. The prima facie appeal of these principles is similar to that of their analogues in social choice, substituting a doxastic interpretation of the ordering for a preferential one.

    With this in hand, we can now report that:

Theorem 1. *The only elementary revision operators are lexicographic, restrained and natural revision.*

Proof sketch: The claim that lexicographic, restrained and natural revision operators are elementary operators is straightforward to establish and we shall not provide it here.

Regarding the other direction of the result: $(\text{IIAP}^*_{\preccurlyeq})$ and $(\text{Neut}^*_{\preccurlyeq})$ jointly allows us to represent revision by a given sentence A as a quadruple of functions from prior to posterior relations between two arbitrary worlds x and y, such that $x, y \notin \min(\preccurlyeq_{\Psi}, [\![A]\!])$, one for each of the three following possibilities: (1) $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$, (2) $x, y \in [\![A]\!]$, (3) $x, y \in [\![\neg A]\!]$ (the case in which $x \in [\![\neg A]\!]$, $y \in [\![A]\!]$ is determined by (1), by virtue of $(\text{Neut}^*_{\preccurlyeq})$). These functions can be represented by state diagrams in which the set of states is $\{x \prec y, x \sim y, y \prec x\}$ and the edges represent revisions by A. The postulates $(\text{C1}^*_{\preccurlyeq})$ and $(\text{C2}^*_{\preccurlyeq})$ entail that, whatever degrees of freedom there are, they are associated with (1). Furthermore, $(\text{C3}^*_{\preccurlyeq})$ and $(\text{C4}^*_{\preccurlyeq})$ then leave us with at most *six* possible diagrams for (1) (see Fig 2).

Diagrams (a), (b) and (c) respectively correspond to $*_{\text{R}} A$, $*_{\text{L}} A$ and $*_{\text{N}} A$. However, (d) and (e) are inconsistent with $(\text{C2}^*_{\preccurlyeq})$, on pains of triviality. Indeed assume that there exist two worlds $y, z \in [\![\neg A]\!]$ and a world $x \in [\![A]\!]$, such that $z \prec_{\Psi} y \prec_{\Psi} x$. Then $z \sim_{\Psi * A} y$, in violation of $(\text{C2}^*_{\preccurlyeq})$. (f) exhibits a similar inconsistency. Consider this time the prior TPO given by $y \prec_{\Psi} \{x, w\}$. We have $z \prec_{\Psi * A} y$. Given $y \prec_{\Psi} z$, this is again inconsistent with $(\text{C2}^*_{\preccurlyeq})$.

So we have established that $(\text{IIAP}^*_{\preccurlyeq})$ and $(\text{Neut}^*_{\preccurlyeq})$ collectively entail that, for any $A, \Psi * A$ is equal to one of either $\Psi *_{\text{R}} A$, $\Psi *_{\text{L}} A$ or $\Psi *_{\text{N}} A$. But it still remains the case that $*$ coincides with one elementary operator for one input but with another elementary operator for another, so that, for example, $\Psi * A = \Psi *_{\text{R}} A$ while $\Psi * A = \Psi *_{\text{L}} A$. This is ruled out by the final condition $(\text{IIAI}^*_{\preccurlyeq})$. $\square$
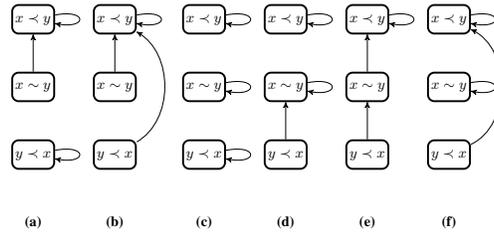

Fig. 2: State diagrams for all $x \in [\![A]\!]$ and all $y \in [\![\neg A]\!]$.

$(\text{IIAP}^*_{\preccurlyeq})$ significantly weakens a principle introduced under the name of '(IIA)' in [12], which simply corresponds to the embedded conditional: If $\preccurlyeq_{\Psi}$ and $\preccurlyeq_{\Psi'}$ agree on $\{x, y\}$, so do $\preccurlyeq_{\Psi * A}$ and $\preccurlyeq_{\Psi' * A}$. $(\text{IIAI}^*_{\preccurlyeq})$ amounts to a similar weakening of a condition found in [7]. An interesting question, therefore, arises as to why the stronger principles do not figure in our characterisation.

The unqualified version of $(\text{IIAP}^*_{\preccurlyeq})$ is only satisfied by $*_{\text{L}}$, assuming $(\text{C1}^*_{\preccurlyeq})$ and $(\text{C2}^*_{\preccurlyeq})$ and that the domain of the revision function is $\text{TPO}(W)$. Indeed, let $x \in [\![A]\!]$ and $y \in [\![\neg A]\!]$. Then, for any $\preccurlyeq_{\Psi}$, there will exist $\preccurlyeq_{\Psi'}$ in $\text{TPO}(W)$ that agrees with $\preccurlyeq_{\Psi}$ on

$\{x, y\}$ and is such that $x \in \min(\preccurlyeq_{\Psi'}, [\![A]\!])$ (and, since $y \in [\![\neg A]\!]$, $y \notin \min(\preccurlyeq_{\Psi'}, [\![A]\!])$). But by AGM, if $x \in \min(\preccurlyeq_{\Psi'}, [\![A]\!])$ but $y \notin \min(\preccurlyeq_{\Psi'}, [\![A]\!])$, then $x \prec_{\Psi'*A} y$. So, by the unqualified version of (IIAP$^*_{\preccurlyeq}$), $x \prec_{\Psi*A} y$. Hence, if $x \in [\![A]\!]$ and $y \in [\![\neg A]\!]$, then $x \prec_{\Psi*A} y$, a condition only satisfied by $*_L$, assuming (C1$^*_{\preccurlyeq}$) and (C2$^*_{\preccurlyeq}$).[4]

Similarly, Booth & Meyer's strong version of (IIAI$^*_{\preccurlyeq}$), in conjunction with (C1$^*_{\preccurlyeq}$)–(C4$^*_{\preccurlyeq}$), can be shown to entail a principle that we have called '($\beta1+^*_{\preccurlyeq}$)' in previous work [5], where we showed (see Corollary 1 there) to characterise lexicographic revision, given AGM and (C1$^*_{\preccurlyeq}$)-(C2$^*_{\preccurlyeq}$).

4   Extending the Levi Identity

4.1   A proposal involving rational closure

The most straightforward syntactic extension of (LI) would involve replacing all belief sets by conditional belief sets, leaving all else unchanged. This would require extending the domain of Cn to subsets of the conditional language L_c, which can be naturally achieved by setting, for $\Delta \subseteq L_c$, $\mathrm{Cn}(\Delta) = \Delta \cup \mathrm{Cn}(\Delta \cap L)$. So we would be considering the claim that $[\Psi * A]_c = \mathrm{Cn}([\Psi \div \neg A]_c \cup \{A\})$. This, however, is a bad idea, since it is easy to show that:

Proposition 1 *If* $[\Psi * A]_c = \mathrm{Cn}([\Psi \div \neg A]_c \cup \{A\})$, *then there are no consistent belief sets, given the two following AGM postulates:*

(K2*) $A \in [\Psi * A]$

(K2$^\div$) $[\Psi \div A] \subseteq [\Psi]$

Proof: Assume $[\Psi * A]_c = \mathrm{Cn}([\Psi \div \neg A]_c \cup \{A\})$. By Success, $A \in [\Psi * A]$. By the Ramsey Test, $\top \Rightarrow A \in [\Psi * A]_c$ and so $\top \Rightarrow A \in \mathrm{Cn}([\Psi \div \neg A]_c \cup \{A\})$. But then, since $\top \Rightarrow A \notin L$ and, as we have stipulated, for $\Delta \subseteq L_c$, $\mathrm{Cn}(\Delta) = \Delta \cup \mathrm{Cn}(\Delta \cap L)$, it must be the case that $\top \Rightarrow A \in [\Psi \div \neg A]_c$. Hence, by the Ramsey Test again, it follows that $A \in [\Psi \div \neg A]$. From (K2$^\div$), we then have $A \in [\Psi]$. By a similar chain of reasoning, we can establish that $\neg A \in [\Psi]$.   □

The core issue highlighted by this result is that the right hand side of the equality won't generally correspond to a rational consequence relation, due to the fact that Cn simply yields too small a set of consequences. So a natural suggestion here would be to make use of the rational closure operator $\mathrm{C}_{\mathrm{rat}}$ of [19] instead of Cn. Indeed, $\mathrm{C}_{\mathrm{rat}}$ has been touted as offering the appropriately conservative way of extending a set of conditionals to something that corresponds to a rational consequence relation (see [19]). This gives us the 'iterated Levi Identity using Rational Closure' (or '(iLIRC)' for short):

(iLIRC) $[\Psi * A]_c = \mathrm{C}_{\mathrm{rat}}([\Psi \div \neg A]_c \cup \{A\})$[5]

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[4] We note that [12] offers a rather different characterisation of lexicographic revision that also involves the unqualified version of (IIAP$^*_{\preccurlyeq}$).

[5] Strictly speaking, $\mathrm{C}_{\mathrm{rat}}$ is an operation on *purely* conditional belief sets. However, it can be obviously generalised to the case in which the set includes non-conditionals, since for any $A \in L$, $A \in [\Psi]_c$ iff $\top \Rightarrow A \in [\Psi]_c$.

4.2   Nayak *et al*'s 'New Levi Identity'

An alternative extension of (LI) can be obtained by using an iterable expansion operator +. This is the 'New Levi Identity' of Nayak *et al*, which is briefly presented in [21]. Semantically, it is given by:

$$(\text{NLI}_{\preccurlyeq}) \quad \preccurlyeq_{\Psi * A} = \preccurlyeq_{(\Psi \div \neg A) + A}$$

Syntactically, in terms of conditional belief sets, we then would have: $[\Psi * A]_c = [(\Psi \div \neg A) + A]_c$.

It is easily verified that (LI) follows from $(\text{NLI}_{\preccurlyeq})$, if one assumes, for instance, that $\div$ satisfies $(\text{C1}_{\preccurlyeq}^{\div})$. Indeed, (LI) amounts to $\min(\preccurlyeq_\Psi, \llbracket A \rrbracket) = \min(\preccurlyeq_{\Psi \div \neg A}, W) \cap \llbracket A \rrbracket = \min(\preccurlyeq_{\Psi \div \neg A}, \llbracket A \rrbracket)$, which immediately follows from $(\text{C1}_{\preccurlyeq}^{\div})$. $(\text{NLI}_{\preccurlyeq})$ also has some other interesting general properties. For example, one can show, rather trivially, that:

Proposition 2 *If $*$ and $\div$ satisfy* $(\text{NLI}_{\preccurlyeq})$*, then, for $i \in \{1, 2, 3, 4\}$, $(\text{C}i_{\preccurlyeq}^{\div})$ entails $(\text{C}i_{\preccurlyeq}^{*})$, if $+$ also satisfies $(\text{C}i_{\preccurlyeq}^{*})$.*

This result mirrors a result in [3], in which it was shown that TeamQueue combination allows one to move from each $(\text{C}i_{\preccurlyeq}^{*})$ to the corresponding $(\text{C}i_{\preccurlyeq}^{\div})$.

Assuming, as Nayak *et al* do, the following natural semantic iterated version of (TI):

$$(\text{iTI}_{\preccurlyeq}) \quad \preccurlyeq_{\Psi + A} = \preccurlyeq_{\Psi * A}, \text{ if } \min(\preccurlyeq, W) \nsubseteq \llbracket \neg A \rrbracket$$
$$= \preccurlyeq_{\Psi_\perp}, \text{ otherwise}$$

where $\Psi_\perp$ is an 'absurd' epistemic state such that $[\Psi_\perp] = L$,[6] $(\text{NLI}_{\preccurlyeq})$ is equivalent to:

$$(\text{iLI}*_{\preccurlyeq}) \quad \preccurlyeq_{\Psi * A} = \preccurlyeq_{(\Psi \div \neg A) * A}$$

In what follows, then, we shall use $(\text{NLI}_{\preccurlyeq})$ and $(\text{iLI}*_{\preccurlyeq})$ interchangeably. Importantly, while the proposal considered in the previous section was reductive, in the sense that the operator $*$ on the left-hand side of the identity did not appear on the right, $(\text{iLI}*_{\preccurlyeq})$ features $*$ on both sides.

To date, however, the implications of this principle have not been studied in any kind of detail. In what follows, we offer some new results of interest. We first note:

Theorem 2. *If $*$ is an elementary revision operator and $\div$ satisfies* $(\text{C1}_{\preccurlyeq}^{\div})$-$(\text{C4}_{\preccurlyeq}^{\div})$*, then $*$ and $\div$ satisfy* $(\text{NLI}_{\preccurlyeq})$ *iff they satisfy the following:*

$(\text{C1}_{\preccurlyeq}^{\div/*})$ *If $x, y \in \llbracket A \rrbracket$, then $x \preccurlyeq_{\Psi \div \neg A} y$ iff $x \preccurlyeq_{\Psi * A} y$*

$(\text{C2}_{\preccurlyeq}^{\div/*})$ *If $x, y \in \llbracket \neg A \rrbracket$, then $x \preccurlyeq_{\Psi \div \neg A} y$ iff $x \preccurlyeq_{\Psi * A} y$*

$(\text{C3}_{\preccurlyeq}^{\div/*})$ *If $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$ and $x \prec_{\Psi \div \neg A} y$, then $x \prec_{\Psi * A} y$.*

$(\text{C4}_{\preccurlyeq}^{\div/*})$ *If $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$ and $x \preccurlyeq_{\Psi \div \neg A} y$, then $x \preccurlyeq_{\Psi * A} y$.*

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[6] Nayak *et al* have little to say about $\Psi_\perp$, aside from its being the case that $\preccurlyeq_{\Psi_\perp \div A}$ is such that $x \sim_{\Psi_\perp \div A} y$ for all $x, y \in W$. More recently, [11] have suggested that the state resulting from expansion into inconsistency be defined in a more fine-grained manner, in a proposal that involves introducing an 'impossible' world such that $w_\perp \models A$ for all $A \in L$. We refer the reader to their paper for further details, since nothing here hinges on the distinction between these views.

Proof sketch: Each direction is proven separately. Regarding the right-to-left direction:

Lemma 1 *If $*$ is an elementary revision operator, $\div$ satisfies $(C1^{\div}_{\preccurlyeq})$-$(C4^{\div}_{\preccurlyeq})$, and $\div$ and $*$ satisfy $(C1^{\div/*}_{\preccurlyeq})$-$(C4^{\div/*}_{\preccurlyeq})$, then $*$ and $\div$ satisfy $(NLI_{\preccurlyeq})$.*

Regarding the left-to-right direction, we actually prove the following stronger claim:

Lemma 2 *If $*$ satisfies $(C1^{*}_{\preccurlyeq})$–$(C4^{*}_{\preccurlyeq})$, then there exists $\div$ such that $*$ and $\div$ satisfy $(NLI_{\preccurlyeq})$ only if $*$ and $\div$ satisfy $(C1^{\div/*}_{\preccurlyeq})$–$(C4^{\div/*}_{\preccurlyeq})$.*   □

The principles $(C1^{\div/*}_{\preccurlyeq})$-$(C4^{\div/*}_{\preccurlyeq})$ are new to the literature and bear an obvious formal resemblance to the postulates of Darwiche & Pearl and of Chopra *et al*. Taken together, they require contraction by $\neg A$ to be a kind of '*mild revision*' by A, since they tell us that the position of any A-world with respect to any $\neg A$-world is at least as good after revision by A as it is after contraction by $\neg A$.

Somewhat surprisingly (to us), it turns out that these principles are *also* closely connected to the semantic 'TeamQueue combinator' approach to extending the Harper Identity to the iterated case that was proposed in [3]. Indeed, one can show that:

Theorem 3. *If $*$ satisfies $(C1^{*}_{\preccurlyeq})$-$(C4^{*}_{\preccurlyeq})$ and $\div$ satisfies $(C1^{\div}_{\preccurlyeq})$-$(C4^{\div}_{\preccurlyeq})$, then $*$ and $\div$ satisfy $(C1^{\div/*}_{\preccurlyeq})$-$(C4^{\div/*}_{\preccurlyeq})$ iff they satisfy $(SPU_{\preccurlyeq})$ and $(WPU_{\preccurlyeq})$.*

Proof: We prove the result in two parts. Firstly we establish the following strengthening of the right-to-left direction of the claim:

Lemma 3 *Given $(SPU_{\preccurlyeq})$ and $(WPU_{\preccurlyeq})$, for all $1 \leq i \leq 4$, $(Ci^{*}_{\preccurlyeq})$ entails $(Ci^{\div/*}_{\preccurlyeq})$.*

Regarding $i = 1, 2$: We provide the proof for $i = 1$, since the case in which $i = 2$ is handled analogously. Assume $x, y \in [\![A]\!]$. From left to right: Assume $x \preccurlyeq_{\Psi \div \neg A} y$. By the contrapositve of $(SPU_{\preccurlyeq})$, either $x \preccurlyeq_{\Psi} y$ or $x \preccurlyeq_{\Psi * A} y$. If the latter holds, we are done. So assume that $x \preccurlyeq_{\Psi} y$. Then the required result follows by $(C1^{*}_{\preccurlyeq})$. From right to left: Assume $x \preccurlyeq_{\Psi * A} y$. By $(C1^{*}_{\preccurlyeq})$, $x \preccurlyeq_{\Psi} y$. By $(WPU_{\preccurlyeq})$, $x \preccurlyeq_{\Psi \div \neg A} y$, as required.

Regarding $i = 3, 4$: We provide the proof for $i = 3$, since the case in which $i = 4$ is handled analogously (using $(SPU_{\preccurlyeq})$ rather than $(WPU_{\preccurlyeq})$). We derive the contrapositive. Assume $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ and $y \preccurlyeq_{\Psi * A} x$. If $y \preccurlyeq_{\Psi} x$, then, from $y \preccurlyeq_{\Psi * A} x$, we have $y \preccurlyeq_{\Psi \div \neg A} x$, by $(WPU_{\preccurlyeq})$, as required. So assume $x \prec_{\Psi} y$. By $(C3^{*}_{\preccurlyeq})$, $x \prec_{\Psi * A} y$. Contradiction. This completes the proof of Lemma 3.

Concerning the left-to-right direction of our principal claim, we show:

Lemma 4 *$(C3^{\div}_{\preccurlyeq})$, $(C4^{\div}_{\preccurlyeq})$, and $(C1^{\div/*}_{\preccurlyeq})$–$(C4^{\div/*}_{\preccurlyeq})$ entail $(SPU_{\preccurlyeq})$ and $(WPU_{\preccurlyeq})$.*

We just prove this in relation to $(WPU_{\preccurlyeq})$, using $(C1^{\div/*}_{\preccurlyeq})$, $(C2^{\div/*}_{\preccurlyeq})$, $(C3^{\div/*}_{\preccurlyeq})$ and $(C4^{\div}_{\preccurlyeq})$. The proof in relation to $(SPU_{\preccurlyeq})$ is analogous but uses $(C1^{\div/*}_{\preccurlyeq})$, $(C2^{\div/*}_{\preccurlyeq})$, $(C4^{\div/*}_{\preccurlyeq})$ and $(C3^{\div}_{\preccurlyeq})$ instead. Assume that $x \preccurlyeq_{\Psi} y$ and $x \preccurlyeq_{\Psi * A} y$. We want to show $x \preccurlyeq_{\Psi \div \neg A} y$. If (a) $x, y \in [\![A]\!]$, (b) $x, y \in [\![\neg A]\!]$, or (c) $x \in [\![\neg A]\!]$ and $y \in [\![A]\!]$, this follows from $x \preccurlyeq_{\Psi * A} y$, by $(C1^{\div/*}_{\preccurlyeq})$, $(C2^{\div/*}_{\preccurlyeq})$ or $(C3^{\div/*}_{\preccurlyeq})$, respectively. If (d) $x \in [\![A]\!]$ and $y \in [\![\neg A]\!]$, then it follows from $x \preccurlyeq_{\Psi} y$, by $(C4^{\div}_{\preccurlyeq})$.   □

In conjunction with Theorem 2, Theorem 3 entails:

Corollary 1 *If $*$ is an elementary revision operator and $\div$ satisfies* $(\mathrm{C1}_{\preccurlyeq}^{\div})$-$(\mathrm{C4}_{\preccurlyeq}^{\div})$*, then $*$ and $\div$ satisfy* $(\mathrm{NLI}_{\preccurlyeq})$ *iff they satisfy* $(\mathrm{SPU}_{\preccurlyeq})$ *and* $(\mathrm{WPU}_{\preccurlyeq})$*.*

In this particular context, then, $(\mathrm{NLI}_{\preccurlyeq})$ simply amounts to the conjunction of a pair of constraints proposed in the context of extending (HI) to the iterated case.

4.3   Rational closure and the New Levi Identity

At this stage, we have considered both a potentially promising reductive proposal and a promising non-reductive one. A natural question, then, is: How would these two suggestions fare in conjunction with one another? To answer this question, we provide the semantic counterpart for our first principle, which was formulated only syntactically:

Theorem 4. *Given AGM,* (iLIRC) *is equivalent to :*

$(\mathrm{iLIRC}_{\preccurlyeq})$ $\preccurlyeq_{\Psi * A} = \preccurlyeq_{(\Psi \div \neg A) *_{\mathrm{N}} A}$[7]

Proof sketch: We prove the claim by establishing that (iLIRC) ensures that $\preccurlyeq_{\Psi * A}$ is the 'flattest' TPO such that the following *lower bound principle* is satisfied: $[\Psi \div \neg A]_{\mathrm{c}} \cup \{A\} \subseteq [\Psi * A]_{\mathrm{c}}$. In view of Definitions 20 and 21 of [19], the upshot of this is then that $\preccurlyeq_{(\Psi \div \neg A) *_{\mathrm{N}} A}$ corresponds to the rational closure of $[\Psi \div \neg A]_{\mathrm{c}} \cup \{A\}$.

   We first note that, given AGM, the lower bound principle can be semantically expressed as follows: (a) If $x \prec_{\Psi \div \neg A} y$, then $x \prec_{\Psi * A} y$ and (b) $\min(\preccurlyeq_{\Psi * A}, W) \subseteq [\![A]\!]$. With this in hand, we prove two lemmas, which are individually easy to establish. First:

Lemma 5 *If $*$ and $\div$ satisfy* $(\mathrm{iLIRC}_{\preccurlyeq})$*, then they satisfy the lower bound principle.*

For the second lemma, we make use of the convenient representation of TPOs by their corresponding *ordered partitions* of W. The ordered partition $\langle S_1, S_2, \ldots S_m \rangle$ of W corresponding to a TPO $\preccurlyeq$ is such that $x \preccurlyeq y$ iff $r(x, \preccurlyeq) \leq r(y, \preccurlyeq)$, where $r(x, \preccurlyeq)$ denotes the 'rank' of x with respect to $\preccurlyeq$ and is defined by taking $S_{r(x, \preccurlyeq)}$ to be the cell in the partition that contains x. The lemma is given as follows:

Lemma 6 $\preccurlyeq_{(\Psi \div \neg A) *_{\mathrm{N}} A}$ $\sqsupseteq$ $\preccurlyeq$*, for any TPO $\preccurlyeq$ satisfying the lower bound principle.*

where:

Definition 5. $\sqsupseteq$ *is a binary relation on the set of TPOs over W such such that, for any TPOs $\preccurlyeq_1$ and $\preccurlyeq_2$, whose corresponding ordered partitions are given by $\langle S_1, S_2, \ldots, S_m \rangle$ and $\langle T_1, T_2, \ldots, T_m \rangle$ respectively, $\preccurlyeq_1$ $\sqsupseteq$ $\preccurlyeq_2$ iff either (i) $S_i = T_i$ for all $i = 1, \ldots, m$, or (ii) $S_i \supset T_i$ for the first i such that $S_i \neq T_i$.*

$\sqsupseteq$ partially orders $\mathrm{TPO}(W)$ according to comparative 'flatness', with the flatter TPOs appearing 'greater' in the ordering, so that $\preccurlyeq_1$ $\sqsupseteq$ $\preccurlyeq_2$ iff $\preccurlyeq_1$ is at least as as flat as $\preccurlyeq_2$.

   Let $\langle T_1, \ldots, T_m \rangle$ be the ordered partition corresponding to the TPO $\preccurlyeq_{(\Psi \div \neg A) *_{\mathrm{N}} A}$, which we will denote by $\preccurlyeq_{\mathbb{N}}$. Let $\preccurlyeq$ be any TPO satisfying the lower bound condition: (a) If $x \prec_{\Psi \div \neg A} y$, then $x \prec y$ and (b) $\min(\preccurlyeq, W) \subseteq [\![A]\!]$. Let $\langle S_1, \ldots, S_n \rangle$ be its

corresponding ordered partition. The proof of the lemma then amounts to showing that $\preccurlyeq_{\mathbb{N}} \sqsupseteq \preccurlyeq$.   □

With this in hand, the consequences of endorsing (iLIRC) on the heels of $(\mathrm{NLI}_\preccurlyeq)$ should be obvious: rational iterated revision would have to coincide with natural revision.

This raises an interesting question: For each remaining elementary operator $*$, does there exist a suitable alternative closure operator C, such that $[\Psi * A]_{\mathrm{c}} = \mathrm{C}([\Psi \div \neg A]_{\mathrm{c}} \cup \{A\})$ iff $\preccurlyeq_{\Psi * A} = \preccurlyeq_{(\Psi \div \neg A) * A}$?[8] Indeed, although rational closure is by far the most popular closure operator in the literature, alternative closure operators have been proposed, including, for instance the lexicographic closure operator of [18] or again the maximum entropy closure operator of [13]. Furthermore, there has been some limited work on potential connections between closure operators and revision operators (namely [2]). However, this work has only focussed on the relation between lexicographic closure and lexicographic revision and its pertinence to the current problem remains unclear.

Although we do not currently have an answer to our question, we can report that the existence of suitable relevant closure operators will very much depend on the manner in which one extends (HI) to the iterated case. To illustrate, in a previous discussion of the issue [3], we considered a particular TeamQueue combinator, $\oplus_{\mathrm{STQ}}$. We showed, in Section 6 of that paper, that for $* = *_{\mathrm{L}}$ or $* = *_{\mathrm{R}}$, the equality $\preccurlyeq_{\Psi \div A} = \preccurlyeq_\Psi \oplus_{\mathrm{STQ}} \preccurlyeq_{\Psi * \neg A}$ entails that $\div = \div_{\mathrm{STQL}}$, where $\div_{\mathrm{STQL}}$ is an iterated contraction operator that we call 'STQ-Lex'. We can, however, show the following:

Proposition 3 *If* $* = *_{\mathrm{L}}$ *or* $* = *_{\mathrm{R}}$ *and* $\div = \div_{\mathrm{STQL}}$, *then there exists no closure operator* C, *satisfying the property of Rational Identity:*

$(\mathrm{RID^c})$ *If* Δ *is rational, then* $\mathrm{C}(\Delta) = \Delta$.

such that both $(\mathrm{NLI}_\preccurlyeq)$ *and* $[\Psi * A]_{\mathrm{c}} = \mathrm{C}([\Psi \div \neg A]_{\mathrm{c}} \cup \{A\})$ *are true.*

$(\mathrm{RID^c})$ seems a desirable property of closure operators, which aim to extend a set of conditionals Δ to that rational set of conditionals whose endorsement is mandated by that of Δ. The standard postulate of Inclusion $(\Delta \subseteq \mathrm{C}(\Delta))$ tell us that C must extend Δ to a rational superset of Δ. $(\mathrm{RID^c})$ adds to this the notion that if Δ 'ain't broke', it needn't be 'fixed'.

Interestingly, the proof of this impossibility result fails to go through when $* = *_{\mathrm{L}}$ and $\div = \div_{\mathrm{P}}$, where $\div_{\mathrm{P}}$ is the priority contraction operator of [21]. In [3] we note that priority contraction can be recovered from lexicographic revision via a particular TeamQueue combinator. Furthermore, the same combinator can be used to define a contraction operator from restrained revision (call it $\div_{\mathrm{R}}$). Again, the proof of the above result breaks down when $* = *_{\mathrm{R}}$ and $\div = \div_{\mathrm{R}}$.

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[7] Note that [17] explicitly mention (iLIRC$_\preccurlyeq$) and flag it out as a potentially desirable principle.

[8] Note the importance of (Red) in making this kind of correspondence even *prima facie* possible. Indeed, if (Red) fails, then $\preccurlyeq_{\Psi \div \neg A}$ and A will fail to jointly determine $\preccurlyeq_{\Psi * A}$. In syntactic terms, $[\Psi \div \neg A]_{\mathrm{c}}$ and A will fail to jointly determine $[\Psi * A]_{\mathrm{c}}$.

5   Is iterated revision reducible to iterated contraction?

Konieczny and Pino Pérez [17, Theorem 5] plausibly claim that, for a finitely generated language, the cardinality of (i) the set of revision operators that satisfy both the AGM postulates for revision and $(\text{C}1^*_{\preccurlyeq})$–$(\text{C}4^*_{\preccurlyeq})$ is strictly greater than the cardinality of (ii) the set of contraction operators that satisfy both the AGM postulates for contraction and $(\text{C}1^{\div}_{\preccurlyeq})$–$(\text{C}4^{\div}_{\preccurlyeq})$. From this, they conclude that there is no bijection between rational iterated revision and contraction operators and hence no reduction of iterated revision to iterated contraction.

But this conclusion is not warranted without a further argument to the effect that every member of (i) is rational. In other words, it could be the case that $(\text{C}1^*_{\preccurlyeq})$–$(\text{C}4^*_{\preccurlyeq})$ need supplementing. This has certainly been the belief of the proponents of the various elementary revision operators that we have discussed in the present paper. And indeed, the proponent of $*_{\text{N}}$ could claim, endorsing our $\oplus_{\text{STQ}}$-based extension of (HI), that rational contraction goes by natural contraction. By the same principle, proponents of $*_{\text{R}}$ or $*_{\text{L}}$ could respectively claim that rational contraction goes by natural contraction or STQ-Lex contraction, respectively (see [3, Section 6]). Those are three candidate bijections that are all consistent, furthermore, with $(\text{NLI}_{\preccurlyeq})$.

One could nevertheless run an arguably plausible argument to Konieczny and Pino Pérez's desired conclusion based on the observation that natural and restrained revision are both mapped onto natural contraction by the $\oplus_{\text{STQ}}$ method. Even if one thinks that it is implausible to claim that iterated change *must* comply with one of either restrained or natural revision, it is not implausible to claim that it sometimes *may* comply with either. In other words: There plausibly exists at least one prior TPO that is rationally consistent with two distinct potential posterior TPOs, respectively obtained via natural and restrained revision by a given sentence A. Given the $\oplus_{\text{STQ}}$-based extension of (HI), only one posterior TPO can be obtained by contraction by $\neg A$, namely the one obtained by natural contraction by $\neg A$. But if this is true, iterated revision dispositions cannot be recovered from iterated contraction dispositions.

6   Conclusions and further work

We have considered two possible extensions of (LI) to the iterated case: a reductive proposal (iLIRC) based on the rational closure operator, and a non-reductive proposal $(\text{NLI}_{\preccurlyeq})$ that involves a contraction step, followed by an expansion. We have shown that, when restricted to a popular class of 'elementary' revision operators, $(\text{NLI}_{\preccurlyeq})$ is in fact equivalent, under weak assumptions, to both (i) a new set of postulates $(\text{C}1^{\div/*}_{\preccurlyeq})$–$(\text{C}4^{\div/*}_{\preccurlyeq})$ and (ii) a pair of principles recently defended in the literature on (HI).

However, it has also been noted that (iLIRC) has strong consequences when conjoined with $(\text{NLI}_{\preccurlyeq})$. This suggests the need for (1) a future consideration of various alternatives to the former that make use of surrogate closure operators.

Furthermore, the revision operators of the class that we have focussed on have been criticised for their equation of belief states with TPOs (the principle (Red); see [4]). One obvious extension of our work would be (2) an exploration of the extent to which the results reported in Section 4 carry over to operators that avoid this identification, such as the POI operators of [5].

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