THE VOLUME AND EHRHART POLYNOMIAL OF THE ALTERNATING SIGN MATRIX POLYTOPE

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Summary

Alternating sign matrices (ASMs), polytopes and partially-ordered sets are fascinating combinatorial objects which form the main themes of this thesis.

In Chapter 1, the origins and various aspects of ASMs are discussed briefly. In particular, bijections between ASMs and other objects, including monotone triangles, corner sum matrices, configurations of the six-vertex model with domain-wall boundary conditions, configurations of simple flow grids and height function matrices, are presented. The ASM lattice and ASM partially ordered set are also introduced.

In Chapter 2, the ASM polytope and related polytopes, including the Birkhoff polytope, Chan-Robbins-Yuen polytope, ASM order polytope and ASM Chan-Robbins-Yuen polytope, are defined and their properties are summarised.

In Chapter 3, new results for the volume and Ehrhart polynomial of the ASM polytope are obtained. In particular, by constructing an explicit bijection between higher spin ASMs and a disjoint union of sets of certain (P, ω)-partitions (where P is a subposet of the ASM poset and ω is a labeling), a formula is derived for the number of higher spin ASMs, or equivalently for the Ehrhart polynomial of the ASM polytope. The relative volume of the ASM polytope is then given by the leading term of its Ehrhart polynomial. Evaluation of the formula involves computing numbers of linear extensions of certain subposets of the ASM poset, and numbers of descents in these linear extensions. Details of this computation are presented for the cases of the ASM polytope of order 4, 5, 6 and 7.

In Chapter 4, some directions for further work are outlined.

A joint paper with Roger Behrend, based on Chapter 3 of the thesis, is currently in preparation for submission.

"It is not our disabilities, it is our abilities that count."

Chris Burke

With my deepest love, gratitude and warmest affection, I dedicate this thesis to my beloved wife

Tahereh Khamseh

who has been a constant source of love, peace, inspiration and unconditional support.

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List of Symbols

$\mathbb{N}$ .... The set of non-negative integers.

$\mathbb{P}$ .... The set of positive integers.

$\mathbb{Z}$ .... The set of all integers.

$\mathbb{R}$ .... The set of all real numbers.

$[m,n]$ .... The set $\{m, m+1, ..., n\}$ for $m, n \in \mathbb{Z}$.

$[n]$ .... The set $\{1, 2, ..., n\}$ for $n \in \mathbb{Z}$.

$[a,b]_{\mathbb{R}}$ .... The set $\{x \in \mathbb{R} \,|\, a \le x \le b\}$ for $a, b \in \mathbb{R}$.

$ASM(n)$ .... The set of all $n \times n$ alternating sign matrices.

$ASM(n,r)$ .... The set of all $n \times n$ higher spin alternating sign matrices with line sum r.

$RASM(n,r)$ .... The set of all $n \times n$ reduced alternating sign matrices.

$MT(n)$ .... The set of all monotone triangles with n rows.

$MT(n,r)$ .... The set of all higher spin monotone triangles with n rows.

$CSM(n,r)$ .... The set of all order n higher spin corner sum matrices.

$SFG(n)$ .... The set of simple flow grids of order n.

$HFM(n)$ .... The set of order n height function matrices.

Br_n .... The strong Bruhat poset of order n.

S_n .... The set of permutations on n elements.

$OIM(n)$ .... The set of $n \times n$ order ideal matrices.

$DOIM(n)$ .... The set of $n \times n$ dual order ideal matrices.

D_8 .... The dihedral group of order 8.

$\mathcal{O}(P)$ .... The order polytope associated with a finite poset P.

$\Omega(P,r)$ .... The order polynomial associated with a finite poset P.

$\mathcal{B}_n$ .... The Birkhoff polytope of order n.

$\mathcal{CRY}_n$ .... The Chan-Robbins-Yuen polytope of order n.

$\mathcal{A}_n$ .... The alternating sign matrix polytope of order n.

$L_{\mathcal{P}}(r)$ .... The Ehrhart polynomial associated with a polytope $\mathcal{P}$.

P_n .... The alternating sign matrix partially ordered set of order n.

$\mathcal{ASMCRY}_n$ .... The alternating sign matrix Chan-Robbins-Yuen polytope of order n.

$SVM(n)$ The set of six vertex model configurations with domain wall boundary conditions.

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CHAPTER 1

THE BIRTH OF ALTERNATING SIGN MATRICES

Introduction

This chapter is organised as follows. We begin with the fascinating story of the birth of alternating sign matrices, or ASMs for short, and their many connections with other mathematical objects. In Section 1.2.3, the notion of monotone triangles is introduced and it is shown that there is a one-to-one correspondence between the set of all $n \times n$ ASMs and the set of all monotone triangles with n rows. In Section 1.2.4, the bijection between ASMs and configurations of the six vertex model is investigated. Similarly to the case of the six vertex model, there is a straightforward one-to-one correspondence between ASMs and other combinatorial objects called simple flow grids which is investigated in Section 1.2.5. The bijection between corner sum matrices and ASMs is one of the key building blocks of this thesis. This is investigated in Section 1.2.6 and later on in Section 1.4 when we provide order relations between ASMs. Utilising the notion of corner sum matrices, we also discuss the bijection between height function matrices and ASMs in Section 1.2.7. The ASM lattice is introduced in Section 1.4 after giving some necessary preliminaries on partially ordered sets in Section 1.3. Then we finish this chapter by investigating the partially ordered set point of view of ASMs by introducing the ASM poset in Section 1.5 and studying it in full detail.

1.1 Literature review

In 1982, Bill Mills, Dave Robbins and Howard Rumsey came up with the idea of **alternating sign matrices**, or ASMs for short, when they were investigating an iterative method to compute the determinant of a given square matrix called the "**Dodgson condensation method**". An ASM of order n is an $n \times n$ matrix with entries from $\{0, \pm 1\}$ such that all row and column sums equal 1 and along each row and column the non-zero entries alternate in sign. The set of all ASMs of order n is denoted by $ASM(n)$. Obviously, each permutation matrix of order n (i.e. each $n \times n$ matrix with entries from $\{0, 1\}$ such that each row and column contains exactly one 1) is an ASM and so the set of all permutation matrices of order n, denoted by $PM(n)$, is a subset of $ASM(n)$.

As an example, the following matrix is a 6×6 ASM.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{1.1}$$

Using the fact that each row and column sum is 1, it follows that each row and column of an ASM contains at least one 1. Using the fact that the non-zero entries alternate in sign along each row and column, it follows that the first and last non-zero entry in each row and column of an ASM is 1. It can also be shown that the first and last row and column of an ASM contain exactly one 1 with all of their other entries being 0's. In particular, consider $A \in ASM(n)$ and assume that $a_{1j} = 1$. Then the next non-zero entry in the first row, if one exists, must be -1. But if $a_{1k} = -1$, then the first non-zero entry of column k is -1, which is not permitted. Therefore, the first row contains exactly one 1, with all other entries being 0. A similar argument holds for the last row, first column and last column.

It can also easily be checked that in each row and column of an ASM, the number of $1's$ is precisely one more than the number of -1's, and that each partial sum of each row and column of an ASM extending from either of its end is non-negative.

Back to the timeline of the discovery of ASMs, the Dodgson condensation method was first introduced by the English writer, mathematician, logician, Anglican deacon and photographer, *Charles Lutwidge Dodgson* who is better known by his pen name "*Lewis Carroll*" [44]. To see the connection between ASMs and the Dodgson condensation method we need to discuss some history behind some well-known determinant formulas. Recall that by the Leibniz formula the determinant of an $n \times n$ matrix $M = (m_{ij})$ is given by:

$$Det(M) = \sum_{\sigma \in S_n} Sgn(\sigma) \prod_{i=1}^{n} m_{i,\sigma(i)} \tag{1.2}$$

where S_n is the set of all permutations of the set $[n] = \{1, 2, ..., n\}$ and $Sgn(\sigma)$ is the sign of permutation σ. Another way to compute (1.2) iteratively is utilizing Dodgson's condensation method. It uses the Desnanot-Jacobi adjoint matrix theorem, which states that

$$Det(M)Det\left(M_{1,n}^{1,n}\right) = Det\left(M_1^1\right)Det\left(M_n^n\right) - Det\left(M_n^1\right)Det\left(M_1^n\right), \quad n \geq 2 \tag{1.3}$$

where the superscripts indicate that the associated rows are deleted from M and the subscripts indicate that the associated columns are deleted from M. Desnanot proved it for $n \leq 6$ in 1819 but Carl Jacobi proved the general case in 1833. For a proof of (1.3) see Chapter 3 of [22].

To see how the Dodgson condensation algorithm works, consider an $n \times n$ matrix $M = (m_{ij})$ and define the determinant of a 1×1 matrix $\begin{pmatrix} m \end{pmatrix}$ to be m and the determinant of a 0×0 matrix to be 1. Then the equation (1.3) enables the determinant of M to be written in terms of the determinant of four $(n-1) \times (n-1)$ matrices and one $(n-2) \times (n-2)$ matrix. Therefore, by iteratively applying (1.3), the determinant of an $n \times n$ matrix is eventually expressed in terms of the determinants of 1×1 and 0×0

matrices. For example, let us apply the above algorithm to the following 4×4 matrix

$$
\begin{vmatrix} 2 & 0 & 1 & 3 \\ -1 & 2 & 1 & -2 \\ 0 & -1 & 1 & 3 \\ 2 & 4 & -3 & 2 \end{vmatrix}
= \frac{\begin{vmatrix} 2 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & -1 & 1 \end{vmatrix}\begin{vmatrix} 2 & 1 & -2 \\ -1 & 1 & 3 \\ 4 & -3 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 3 \\ 2 & 1 & -2 \\ -1 & 1 & 3 \end{vmatrix}\begin{vmatrix} -1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 4 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}}
$$

$$
= \frac{\dfrac{\begin{vmatrix} 2 & 0 \\ -1 & 2 \end{vmatrix}\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix}}{\begin{vmatrix} 2 \end{vmatrix}} \times \dfrac{\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}\begin{vmatrix} 1 & 3 \\ -3 & 2 \end{vmatrix} - \begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix}\begin{vmatrix} -1 & 1 \\ 4 & -3 \end{vmatrix}}{\begin{vmatrix} 1 \end{vmatrix}}}{3}
- \frac{\dfrac{\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}\begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix}\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 \end{vmatrix}} \times \dfrac{\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix}\begin{vmatrix} -1 & 1 \\ 4 & -3 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}\begin{vmatrix} 0 & -1 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} -1 \end{vmatrix}}}{3} \tag{1.4}
$$

$$
= \frac{7 \times 38}{3} - \frac{5 \times 7}{3}
$$

$$
= 77
$$

The main disadvantage of the condensation method is that in some cases it causes division by zeros. This can be avoided by certain reordering of rows and columns, although this introduces additional complications.

A generalization of the determinant was introduced by Robbins and Rumsey [91] by including a new parameter λ in the Desnanot-Jacobi identity (1.3) together with the initial conditions that the $\lambda-$determinant of a 1×1 matrix $\begin{pmatrix} m \end{pmatrix}$ is m and the $\lambda-$determinant of a 0×0 matrix is 1. They called it the $\lambda-$determinant and defined it recursively by

$$
Det_{\lambda}(M) = \frac{Det_{\lambda}(M_1^1)Det_{\lambda}(M_n^n) + \lambda Det_{\lambda}(M_n^1)Det_{\lambda}(M_1^n)}{Det_{\lambda}(M_{1,n}^{1,n})}, \quad n \geq 2 \tag{1.5}
$$

As an illustration, for a given 3×3 matrix

$$
M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{pmatrix}
$$

the $\lambda-$determinant is given by

$$Det_{\lambda}(M) = \frac{\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix}_{\lambda} \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix}_{\lambda} + \lambda \begin{vmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{vmatrix}_{\lambda} \begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix}_{\lambda}}{\begin{vmatrix} m_{22} \end{vmatrix}_{\lambda}}$$

$$= \frac{(m_{11}m_{22} + \lambda m_{12}m_{21})(m_{22}m_{33} + \lambda m_{23}m_{32}) + \lambda(m_{12}m_{23} + \lambda m_{13}m_{22})(m_{21}m_{32} + \lambda m_{22}m_{31})}{m_{22}}$$

$$= m_{11}m_{22}m_{33} + \lambda(m_{11}m_{23}m_{32} + m_{12}m_{21}m_{33}) + \lambda^2(m_{12}m_{23}m_{31} + m_{13}m_{21}m_{32})$$

$$+ \lambda^3 m_{13}m_{22}m_{31} + \lambda^2(1+\lambda^{-1})m_{12}m_{21}m_{23}m_{32}m_{22}^{-1} \tag{1.6}$$

After doing some computations, Robbins and Rumsey noticed that there is a pattern in the λ−determinant. In particular, they observed that the λ−determinant can be expressed as a sum of monomials in the m_{ij}'s and their reciprocals each of which is multiplied by a polynomial in λ. They also noticed that these monomials are of the form

$$\prod_{i,j=1}^{n} m_{ij}^{a_{ij}} \tag{1.7}$$

where $A = (a_{ij})$ is an $n \times n$ ASM. Moreover, they observed that the polynomials in λ can also be expressed in terms of the number of $-1's$ in A and the *inversion number* of A where the inversion number of an ASM will now be defined. For a permutation matrix, the inversion number is defined to be the number of pairs of $1's$ in the matrix for which one of the $1's$ lies to the right and above the other. For an ASM A, the inversion number is defined as

$$\mathcal{I}(A) = \sum_{\substack{1 \le i < k \le n \\ 1 \le l < j \le n}} a_{ij}a_{kl} \tag{1.8}$$

It follows that this definition is consistent with the definition of inversion number of permutation matrices. For example, the inversion number of the ASM in (1.1) is 9 since in the sum (1.8), there are sixteen pairs of entries whose product is +1 and seven pairs of entries whose product is −1, with all other pairs of entries having a product of 0.

Based on the above observations, an analogue of the Leibniz formula (1.2) for the λ−determinant was conjectured by Robbins and later was proved by himself and Rumsey [91], giving the following theorem.

Theorem 1.1. *λ−**Determinant Theorem:** For a given $n \times n$ matrix M, we have*

$$Det(M)_{\lambda} = \sum_{A \in ASM(n)} \lambda^{\mathcal{I}(A)} \left(1+\lambda^{-1}\right)^{\mathcal{N}(A)} \prod_{i,j=1}^{n} m_{ij}^{a_{ij}} \tag{1.9}$$

where $\mathcal{N}(A)$ is the number of -1's in A.

For example, the ASMs corresponding to each of the monomials in (1.6) (which is the $n = 3$ case of (1.9)) are shown in Table 1.1.

Monomial	$ASM's$	$\mathcal{I}(A)$
$m_{11}m_{22}m_{33}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	0
$m_{11}m_{23}m_{32}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	1
$m_{12}m_{21}m_{33}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	1
$m_{12}m_{21}m_{23}m_{32}m_{22}^{-1}$	$\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$	2
$m_{12}m_{23}m_{31}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$	2
$m_{13}m_{21}m_{32}$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	2
$m_{13}m_{22}m_{31}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$	3

Table 1.1: The seven ASMs corresponding to each monomial in the $\lambda-$ determinant formula in (1.6)

It is easy to verify that by substituting $\lambda = -1$ into (1.9), one gets the Leibniz determinant formula again. In their paper [74], Mills, Robbins and Rumsey made some conjectures on the enumeration of ASMs. First of all, they gave the ASM conjecture which states that the number of ASMs of size n is given by

$$A_n = |ASM(n)| = \prod_{i=0}^{n-1} \frac{(3j+1)!}{(n+j)!} \tag{1.10}$$

They also introduced a refinement of the ASM conjecture, according to the position k of the unique 1 in the first row. They conjectured that the number of such ASMs of size n is equal to

$$A_{n,k} = \frac{(n+k-2)!\,(2n+k-1)!}{(k-1)!\,(n-k)!\,(2n-2)!} \prod_{i=0}^{n-2} \frac{(3i+1)!}{(n+i-1)!}, \quad 1 \le k \le n. \tag{1.11}$$

The product formula (1.10) generates the following counting sequence

$$1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, \dots \tag{1.12}$$

Nowadays thanks to technology and the growth of the world wide web, one can easily find adequate information and data regarding many previously known integer sequences in the on-line encyclopedia of integer sequences or OEIS for short. According to the OEIS, the integers in (1.12) are known as *Robbins numbers* or the *Andrews-Mills-Robbins-Rumsey numbers*, see `https://oeis.org/A005130`. On the same page, not only can one find very useful data and references regarding Robbins numbers but also a list of some other objects that are enumerated by the same numbers. But in the 1980s there was no such on-line access to data, so as Robbins describes in [89], he, Mills and Rumsey asked Richard Stanley about their conjecture. In his reply, Stanley indicated that he did not know a proof of the conjecture, but he had seen the sequence before. This sequence was known to enumerate another class of combinatorial objects related to plane partitions which had appeared only a few years earlier in a paper by George Andrews [1].

Basically, a *plane partition*, or PP for short, of a positive integer n is an array of positive integers with non-increasing rows and columns, such that the sum of the entries is equal to n. For instance, the following array is a PP of 56:

$$\begin{array}{cccccc} 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 & \\ 4 & 3 & 2 & 1 & & \\ 3 & 2 & 1 & & & \\ 2 & 1 & & & & \\ 1 & & & & & \end{array} \tag{1.13}$$

In 1979, Andrews tried to enumerate combinatorial objects called *descending plane partitions*, or DPPs for short. He showed that the number of DPPs with no part exceeding n, is equal to

$$\prod_{i=0}^{n-1} \frac{(3j+1)!}{(n+j)!} \tag{1.14}$$

Definition 1.2. A **descending plane partition** is an array (possibly empty), $D = (d_{ij})$ of positive integers defined for $1 \leq i \leq j$ arranged in the form

$$\begin{array}{ccccccc} d_{11} & d_{12} & \cdots & \cdots & \cdots & \cdots & d_{1,\mu_1} \\ & d_{22} & \cdots & \cdots & \cdots & d_{2,\mu_2} & \\ & & \cdots & \cdots & \cdots & & \\ & & d_{rr} & \cdots & d_{r,\mu_r} & & \end{array}$$

(i.e. $\mu_i - i + 1$ is the number of entries in row i) such that

(1) $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r$,

(2) $d_{ij} \geq d_{i,j+1}$ and $d_{ij} > d_{i+1,j}$ whenever both sides are defined, that is, entries along rows are weakly decreasing and entries along columns are strictly decreasing,

(3) $d_{ii} > \mu_i - i + 1$ for $i \leq r$, that is, the first entry in a row is strictly greater than the number of entries in the same row,

(4) $d_{ii} \leq \mu_{i-1} - i + 2$ for $1 < i \leq r$, that is, the first entry in a row is at most the number of entries in the row above it.

Note that condition (3) implies that the first entries of the rows are always greater than one. Since ASMs and DPPs are both enumerated by the same product formula (1.10) and (1.14), the natural question is whether there is a natural bijection between them or not? Mills, Robbins and Rumsey conjecturally found three statistics of DPPs associated with $\mathcal{N}(A)$ the number of -1's in an ASM A, the inversion number $\mathcal{I}(A)$ of A and the position of the unique 1 in the first row of A, see Conjecture 3 [75]. But finding a combinatorial bijection for the general case is still a challenging open problem. Since then some bijections for special cases have been found. For instance, Ayyer and Striker gave such bijections restricted to permutation matrices in [4] and [103]. Later on, Behrend, Di Francesco, and Zinn-Justin gave a computational proof of Conjecture 3 of [75] in [12] and shortly after a proof of a more refined result by introducing one more statistic on ASMs and DPPs in [13].

Back to ASMs, it took more than a decade for the ASM conjecture (1.10) to be proven independently by Zeilberger in 1996 [114], and shortly after by Kuperberg [65]. One of the keys to Zeilberger's proof of the ASM conjecture was a connection between ASMs and certain PPs, see [95]. A PP can be visualized as a stack of unit cubes pushed into a corner. As an illustration, the PP of 56 in (1.13) is shown in Figure 1.1.

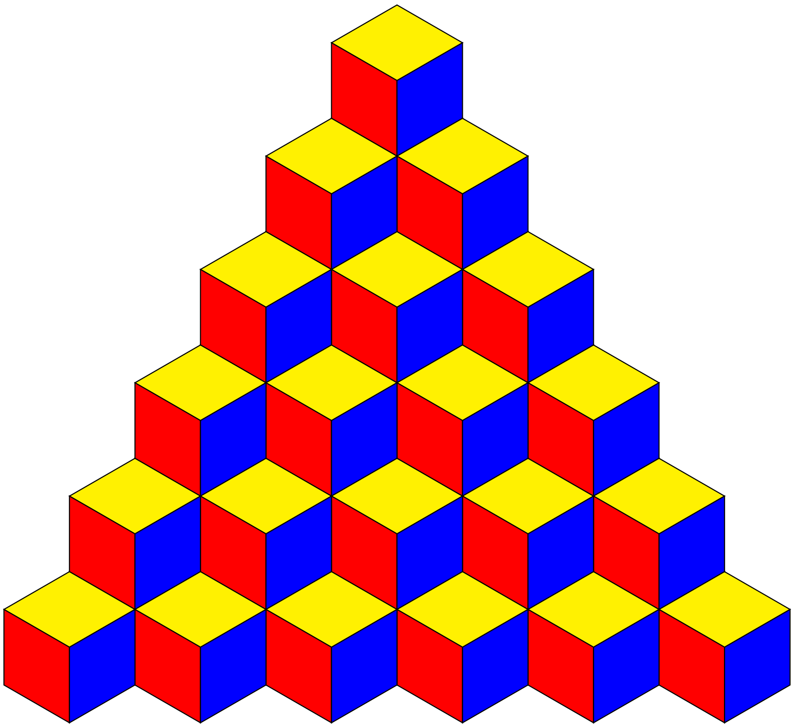


Figure 1.1: The corresponding plane partition of 56 given in (1.13) as a stack of unit cubes.

A *totally symmetric plane partition*, or TSPP for short, is a PP that is invariant under all permutations of its coordinates. In [99], Stembridge proved a conjectured formula for the number of all TSPPs contained in an $n \times n \times n$ box. Another class of PPs is known as *cyclically symmetric plane partition*, or CSPP for short. It is a PP that is invariant under all cyclic permutations of its coordinates. In 1986, Mills, Robbins and Rumsey conjectured that the number of totally symmetric self-complementary plane partitions, or TSSCPPs for short, inside a $2n \times 2n \times 2n$ box (which is defined to be a TSPP which is equal to its complement in the box) is also enumerated by the same product formula (1.10) and (1.14), Conjecture 1, [76]. They also noticed that TSSCPPs are in bijection with certain triangular arrays of integers which later were called $n-$Magog triangles by Zeilberger. Earlier, in [75], Mills, Robbins and Rumsey observed that ASMs are in bijection with certain triangular arrays of integers namely *monotone triangles*, MTs for short, with bottom row $(1, 2, ..., n)$ or $n-$Gog triangles according to Zeilberger. In the early 90s, Andrews proved the TSSCPP conjecture, Theorem 1 in [2]. This reduces the ASM conjecture to showing that the number of $n-$Magog triangles and the number of MTs with bottom row $(1, 2, ..., n)$ are equal. A bijective proof of this is still unknown but these observations led Zeilberger to give a computational proof of the ASM conjecture in an 84-page paper [113], beginning with a long list of people who checked the lemmas within.

Shortly after the proof of the ASM conjecture by Zeilberger, Kuperberg found a bijective correspondence between ASMs and a well-studied model in statistical mechanics known as the *six-vertex model with domain wall boundary conditions* also known as the *square-ice model with domain wall boundary* *conditions*. This bijection enabled him to provide an alternative, shorter proof of the ASM conjecture [65]. A few months later, by utilizing Kuperberg's method, Zeilberger proved the refined ASM conjecture (1.11) in [115]. To see a more detailed and general picture regarding the proof of the ASM conjecture, we encourage the reader to see a well-written book by David Bressoud [22].

For mathematicians proving conjectures is just the beginning of further research and more questioning. So with the two main conjectures related to ASMs being solved by Andrews, Zeilberger and Kuperberg, various questions regarding further aspects of ASMs were raised. One natural aspect was investigating the enumeration of symmetry classes of ASMs and PPs. For instance, how many $n \times n$ ASMs are there which are invariant under the 8 symmetry operations of the square? It turns out that there exist similar product formulae for most of the symmetry classes of ASMs. For more details see [66], [79], [85], [86], [87] or [90]. Another interesting aspect of the enumeration of ASMs is related to the further refinement of them. For instance, if one fixes the position of the unique 1 in the first row, first column, last row and last column simultaneously, then how many such $n \times n$ ASMs exist? This refined enumeration problem was recently solved independently by Behrend [11] and Ayyer and Romik [5] in 2013. The zero-nonzero patterns of ASMs and symmetric ASMs are also investigated by Brualdi, Kiernan, Meyer and Schroeder in [30] in 2013 and by Brualdi and Kim in [31] in 2014, respectively. Another generalization regarding ASMs is given by Fischer in 2006. She found an operator formula [49] for the number of MTs with bottom row $t_1, t_2, ..., t_n$, which in turn enabled her to give an alternative proof of the refined ASM theorem [50] in 2007. Most recently, Behrend, Fischer and Konvalinka proved a product formula for the enumeration of another symmetry class of ASMs called *odd-order diagonally and antidiagonally symmetric alternating sign matrices*, or DASASMs for short [14].

As an application of ASMs and MTs, Elkies, Kuperberg, Larson and Propp introduced another amazing connection between ASMs and particular sorts of tilings in [47] and [48]. They showed how to tile a family of planar regions called *Aztec diamonds*. In fact, in [47] they used two different approaches to show that the Aztec diamond of order n has exactly $2^{n(n+1)/2}$ domino tilings. To do this, firstly they showed a correspondence between a domino tiling of an Aztec diamond and a compatible pair of ASMs. Alternatively, they do the same counting by assigning a suitable weight to MTs with n rows. In [83], Propp also introduced an interesting variant of the square-ice model (and hence ASMs) with a tiling model called "gaskets" and "baskets". For more recent works regarding domino tilings of Aztec diamonds see for example [20], [33], [67], [68] or [78].

Similar to the case of the six vertex configurations, there is another sort of configuration called *fully packed loop configurations*, or FPLs for short, also coming from statistical mechanics that are in bijection with six vertex configurations and hence with ASMs. FPLs are subgraphs of a finite square grid containing n^2 vertices and $4n$ external edges such that each internal vertex has degree 2, and the boundary conditions are alternating. The $2n$ external edges in FPLs are connected by non-crossing paths including internal loops that can be represented on a disk. Such patterns lead to so-called *link patterns*. Having this in mind, then the question is: given a link pattern α, how many FPLs having the pattern α are there? The answer to this question is based on the Razumov-Stroganov conjecture [84] which provides a connection between the enumeration of FPLs with a given link pattern, and the ground-state vector of an $O(1)$ loop model. This conjecture was proved by Cantini and Sportiello in 2010 [36] based on the fact that non-crossing paths in a given link pattern are invariant under rotation or reflection (this was proved by Wieland [111]). For other approaches regarding the enumeration of FPLs see [119] and for a more combinatorial point of view and collection of conjectures related to them see [120].

A partially ordered set, or poset for short, is a set equipped with a binary operation which is reflective, antisymmetric and transitive. Another interesting perspective of ASMs is their poset point of view. In 1996, Lascoux and Schützenberger showed that the MacNeille completion (the unique smallest lattice containing a poset) of the strong Bruhat order on permutations on n elements, S_n, is in fact isomorphic to a poset formed by the set of $n \times n$ ASMs [69], see Section 1.4. Based on this fact, further connections between the Bruhat order and ASMs were investigated by Brualdi and Shroeder [32]. In 2008, Striker in her thesis [100] introduced a poset and a sub-poset whose set of order ideals are in bijection with ASMs. She also investigated the poset perspective of ASMs and other combinatorial objects like TSSCPPs and Catalan objects in [104] and [102]. In Section 1.5 of this thesis a poset is defined and it is shown that the lattice of all order ideals of this poset is in bijection with ASMs. Most recently, Terwilliger also introduced a poset whose maximal chains are in bijection with ASMs, see [107]. Hamaker and Reiner introduced a poset formed by $n \times n$ ASMs which extends the weak Bruhat order for S_n, see [59].

Another interesting aspect of ASMs that is the main theme of this thesis is their polytope point of view. Polytopes play an important role in various mathematical disciplines such as combinatorics, geometry and optimization, with applications in linear programming (a method to find an optimal solution for a linear objective model subject to some linear constraints). In particular, the feasible regions of linear programs are typically convex polytopes (bounded intersections of finitely many halfspaces in Euclidean space), see Chapter 3 of [118]. Among many problems related to convex polytopes, finding the volume of these objects is of high interest since this provides additional algebraic and combinatorial meaning. For instance, it is known that the volume of order polytopes is equal to the number of linear extensions of their associated poset, see Theorem 2.18 in Section 2.2.1. Another example of the combinatorial meaning of the volume of convex bodies is given by the Catalan number (the n^{th} Catalan number is $C_n = \frac{1}{n+1}\binom{2n}{n}$). It turns out that it is the normalized volume of the convex hull of the positive root configuration A_n, see [54]. In addition, Catalan numbers appear as factors in the normalized volume of the Chan-Robbins-Yuen polytope, see Section 2.3.1. Another application of the volumes of polytopes is in the context of algebraic geometry where they are related to the determination of the number of solutions of zero-dimensional systems of polynomial equations and the degrees of algebraic varieties, see [105] or [106]. This motivates researchers to investigate the volumes of polytopes, in particular, those arising as the convex hull of permutation matrices such as Birkhoff polytope and Chan-Robbins-Yuen polytope.

The ASM polytope was introduced independently by Behrend and Knight as a bounded intersection of finitely many halfspaces in Euclidean space (known as the hyperplane description of a polytope) in [15] and Striker in [100], [101] as the convex hull of finitely many points in Euclidean space (known as the vertex description of a polytope). It is known that these two seemingly different descriptions of polytopes are in fact equivalent, see [57] for the sketch of the proof. In [25], Brualdi and Dahl studied the convex cone and polytopes generated by ASMs.

The ASM polytope contains a well-studied polytope known as the Birkhoff polytope, denoted by $\mathcal{B}_n$, with several other names in the literature. The Birkhoff polytope is an example of a very classical $(0,1)$−polytope introduced by Garrett Birkhoff in 1946, see [18]. The Birkhoff polytope is also known as the *assignment polytope*, the *polytope of doubly stochastic matrices*, or the *perfect matching polytope*.

It is the set of $n \times n$ doubly stochastic matrices, that is $n \times n$ matrices with non-negative real entries with each row and column sum equal to 1. As these names suggest, the Birkhoff polytope is a widely studied class of polytope in the literature with many applications in different mathematical disciplines like Representation Theory [21], [58], [72], [80], Statistics [43], [109], Optimization [7], [92], [108], and Enumerative Combinatorics [3], [40], [41], [82], [94]. For more details on the Birkhoff polytope see Chapter 9 of [24] or [112]. The combinatorial types of faces of the Birkhoff polytope were studied thoroughly by Paffenholz in [81]. For many other well-studied analogous polytopes as subsets of the Birkhoff polytope see [24]. In a more general sense, Brualdi and Gibson also investigated the Birkhoff polytope and their connections with $(0,1)-$matrices in a series of papers, [27], [28], [29], [26]. According to the well-known *Birkhoff-von Neumann theorem* (see [18] or [110]), the vertices of $\mathcal{B}_n$ are $n \times n$ permutation matrices. Thus it has $n!$ vertices and it turns out that $\mathcal{B}_n$ is a $(n-1)^2-$dimensional polytope with n^2 facets (maximal faces of $\mathcal{B}_n$). Among many other fascinating related problems to $\mathcal{B}_n$, computing its volume is the most challenging one. Although a closed formula to compute the volume of $\mathcal{B}_n$ was given by De Loera, Liu and Yoshida in 2009 [42], it is not an easy computation at all. The volume of the $\mathcal{B}_n$ was computed up to $n = 10$ by Beck and Pixton in 2003 using the *Ehrhart theory* techniques, see [9]. In [35], Canfield and McKay utilised the methods in [34] on asymptotic enumeration of non-negative integer matrices and presented an asymptotic formula for computing the volume of $\mathcal{B}_n$.

The *Chan-Robbins-Yuen polytope*, denoted by $\mathcal{CRY}(n)$, was first introduced by Chan, Robbins, and Yuen in 1998 as a face of $\mathcal{B}_n$, see [38]. In general, $\mathcal{CRY}(n)$ is defined as the convex hull of all $n \times n$ permutation matrices for which the entries above the superdiagonal are all 0. In their investigation, they constructed a bijection between certain triangular arrays and simplices of $\mathcal{CRY}(n)$. They conjectured that the number of such triangular arrays that encodes the triangulation of $\mathcal{CRY}(n)$ is equal to the product of the first $n-2$ *Catalan numbers*, see Conjecture 1 [39]. This conjecture and two more conjectures in [39], were proved by Zeilberger [116], using the Morris constant term identity [77]. Zeilberger's proof expressed the volume of $\mathcal{CRY}(n)$ as a value of the Kostant partition function and the reformulation of the Morris constant term identity. But the problem of giving a combinatorial proof remains open. The ASM analogue of the Chan-Robbins-Yuen polytope, denoted by $\mathcal{ASMCRY}(n)$ was introduced by Mészáros, Morales and Striker in [73]. It is the convex hull of all $n \times n$ ASMs for which all the entries above the superdiagonal are all 0. Mészáros, Morales and Striker proved that the polytopes in this family are order polytopes and they used Stanley's theory of order polytopes [96] to give a combinatorial proof of formulas for their volumes and Ehrhart polynomials. In [73] it is shown that $\mathcal{ASMCRY}(n)$ is both a flow and order polytope and its Ehrhart polynomial is also given.

A natural generalization of ASMs was first introduced by Behrend and Knight [15], [60] in 2007. A *higher spin ASM* of size n is an $n \times n$ matrix with integer entries for which each complete row and column sum is a non-negative integer r and all partial row and column sums extending from each end of the row or column are non-negative. The set of all such matrices is denoted by $ASM(n,r)$. Similarly to the case of ASMs, higher spin ASMs are also in one-to-one correspondence with configurations of statistical mechanical vertex models with certain boundary conditions related to the spin $r/2$ representation of the Lie algebra $sl(2,\mathbb{C})$ for all non-negative integers r, see [56] for more details on this area. In [37], a determinant formula for the partition functions of these models is given. For more recent works on this see also [53]. Furthermore, it is shown that each member of $ASM(n,r)$ can be written as the sum of r standard ASMs, see [25] or [60]. These matrices not only generalize the standard ASMs but also other well-studied combinatorial objects called *semimagic squares*, or SMSs for short ($n \times n$ non-negative integer-entry matrices in which all complete row and column sums are equal). Higher spin ASMs and SMSs also have a geometrical life. They can be considered as the integer points of the integer dilates of the ASM and the Birkhoff polytopes, see [15] or Chapter 6 of [10]. This is our main motivation to investigate higher spin ASMs and their enumeration.

In Chapter 3 of this thesis, we employ the theory of $P-$partitions (order-reversing maps from a given finite poset P to the non-negative integers) to enumerate the number of higher spin ASMs. From a historical point of view, the idea of $P-$partitions was first discovered by Percy A. MacMahon who worked on plane partition problems in 1911, see [71]. It was also rediscovered independently by Germain Kreweras in 1967 without knowledge of MacMahon's work, see [62], [63] or [64]. It was then generalized by Donald E. Knuth to study three-dimensional partitions also known as solid partitions in 1970, see [61]. But the person who extended the idea of $P-$partitions and considered it in full generality, was Richard Stanley. His Harvard 1971 PhD thesis [93] was on $P-$partitions and plane partitions where he developed a general theory to enumerate the order-reversing maps of P into total orders or chains. For a historical review of the $P-$partition development and applications see [55]. In [96], Stanley introduced two equivalent convex polytopes associated with P, namely, the *order polytope* $\mathcal{O}(P)$ and *chain polytope* $\mathcal{C}(P)$ both of dimension $|P|$. He investigated some interesting connections between P, $\mathcal{O}(P)$ and $\mathcal{C}(P)$, for instance, the connection between the number of $P-$partitions and the number of integer points of the t dilates of $\mathcal{O}(P)$ and $\mathcal{C}(P)$ for a given non-negative integer t, facial structure of $\mathcal{O}(P)$ and a triangulation of $\mathcal{O}(P)$ and $\mathcal{C}(P)$. In this present work, we try to apply the same techniques and methods to the ASM polytope.

1.2 The many faces of ASMs

The title of this section is adapted from a paper by Propp [83] where he briefly describes some bijections between ASMs and other objects. We begin with ASMs and a generalization of them to higher spin ASMs and will continue by providing some bijections related to ASMs.

1.2.1 Alternating sign matrix

In Section 1.1, the definition of ASMs was given and the fundamental properties of ASMs were discussed. The following is an alternative version of the definition of ASMs. The set of all $n \times n$ alternating sign matrices, denoted by $ASM(n)$, is defined such that

$$ASM(n) := \left\{ A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \in \mathbb{Z}^{n\times n} \;\middle|\; \begin{array}{l} \bullet\ \sum_{j'=1}^{n} a_{ij'} = \sum_{i'=1}^{n} a_{i'j} = 1 \ \text{ for all } i,j \in [n]; \\ \bullet\ \sum_{j'=1}^{j} a_{ij'} \geq 0 \ \text{ for all } i \in [n],\ j \in [n-1]; \\ \bullet\ \sum_{j'=j}^{n} a_{ij'} \geq 0 \ \text{ for all } i \in [n],\ j \in [2,n]; \\ \bullet\ \sum_{i'=1}^{i} a_{i'j} \geq 0 \ \text{ for all } i \in [n-1],\ j \in [n]; \\ \bullet\ \sum_{i'=i}^{n} a_{i'j} \geq 0 \ \text{ for all } i \in [2,n],\ j \in [n]. \end{array} \right\}. \tag{1.15}$$

It is clear that each permutation matrix of order n is an ASM. Therefore, the set of all permutation matrices of order n, denoted by $PM(n)$, is a subset of $ASM(n)$.

Example 1.3. As an illustration, all seven order 3 ASMs are given by

$$\text{ASM}(3) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}$$

For the upcoming sections we use the following ASMs as our running example whenever needed.

Example 1.4. Our running example here are the following two ASMs of order 6,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{1.16}$$

We summarize the fundamental properties of ASMs discussed in 1 in the following lemma which will be easier to refer to later on.

Lemma 1.5. *The following statements hold for a given $A \in ASM(n)$.*

(I) The first and last non-zero entry in each row and column of A is 1;

(II) In the first row and column and the last row and column of A there is exactly one 1;

(III) In each row and column of A, the number of 1's is exactly one more than the number of -1's;

(IV) The partial row and column sums extending from each end of A are either 0 or 1, that is, they are non-negative;

1.2.2 Higher spin ASMs

In [15], Behrend and Knight introduced a generalization of the ASMs, namely *higher spin alternating sign matrices*, or HSASMs for short. They are $n \times n$ matrices with integer entries such that each row and column sum is equal to the non-negative integer r, and such that partial line sums extending from either end along each row and column are non-negative. A key motivation for us to study HSASMs is their geometrical interpretation. As already indicated in Section 1.1, they can be considered as the integer points of the integer dilates of the ASM polytope. Hence the enumeration of HSASMs gives the enumeration of integer points in the r^{th} dilate of the ASM polytope. In Chapter 3 we give a formula to enumerate $n \times n$ HSASMs with line sum r, see Theorem 3.24.

Definition 1.6. For any positive integer n and non-negative integer r, the set of *higher spin ASMs* of size n with row and column sum r, denoted by $ASM(n,r)$, is defined by

$$ASM(n,r) := \left\{ A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \in \mathbb{Z}^{n\times n} \;\middle|\; \begin{array}{l} \bullet\ \sum_{j'=1}^{n} A_{ij'} = \sum_{i'=1}^{n} A_{i'j} = r \ \text{ for all } i,j \in [n]; \\ \bullet\ \sum_{j'=1}^{j} a_{ij'} \geq 0 \ \text{ for all } i \in [n],\ j \in [n-1]; \\ \bullet\ \sum_{j'=j}^{n} a_{ij'} \geq 0 \ \text{ for all } i \in [n],\ j \in [2,n]; \\ \bullet\ \sum_{i'=1}^{i} a_{i'j} \geq 0 \ \text{ for all } i \in [n-1],\ j \in [n]; \\ \bullet\ \sum_{i'=i}^{n} a_{i'j} \geq 0 \ \text{ for all } i \in [2,n],\ j \in [n]. \end{array} \right\}. \tag{1.17}$$

Example 1.7. For example, all 26 HSASMs of order 3 with row and column sum 2 are given by

$$
\begin{aligned}
ASM(3,2) = \Bigg\{ & \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 2 & -1 & 1 \\ 0 & 2 & 0 \end{pmatrix}, \\
& \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 1 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 2 & -2 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \\
& \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \\
& \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Bigg\}
\end{aligned}
$$

The following lemma is a straightforward consequence of (1.17) which provides some basic properties of HSASMs.

Lemma 1.8. *The following statements hold for every given $A \in ASM(n,r)$:*

(I) The entries of the first and last row and the first and last column of A are non-negative;

(II) Each entry of A ranges between $-r$ and r.

Remark 1.9. Note that the subset of $ASM(n,r)$ in which all matrix entries are non-negative is equal to the set of well-known objects called *semimagic squares* (an $n \times n$ matrix with non-negative integer entries whose rows and columns sum to the same number) of size n with line sum r, denoted by $SMS(n,r)$ that is the set

$$SMS(n,r) := \{A \in ASM(n,r) \mid a_{ij} \geq 0 \text{ for each } i,j \in [n]\} \tag{1.18}$$

In particular, $SMS(n,1)$ is the set of $n \times n$ permutation matrices and we know that

$$|SMS(n,1)| = n! \tag{1.19}$$

Similar to the case of the enumeration of the ASMs, the enumeration of HSASMs is also challenging. Although there are some trivial cases where there is a simple formula regarding the enumeration of $ASM(n,r)$, there is no general, simple product formula like (1.10). Since $ASM(n,0)$ contains only the $n \times n$ zero matrix, so clearly $|ASM(n,0)| = 1$. Also $|ASM(1,r)| = 1$ since $ASM(1,r) = \left\{\begin{pmatrix} r \end{pmatrix}\right\}$, and $|ASM(2,r)| = r+1$ since we have

$$ASM(2,r) = SMS(2,r) = \left\{ \begin{pmatrix} i & r-i \\ r-i & i \end{pmatrix} : i \in [0,r] \right\} \tag{1.20}$$

Clearly, $ASM(n,1) = ASM(n)$ is the set of *standard ASMs* of size n and hence $|ASM(n,1)| = |ASM(n)|$ with the same enumeration product formula as given by (1.10). The Table 1.2 shows some values for

$|ASM(n,r)|$ where these numbers are computed by Mathematica software. In Chapter 3, we will give a formula for the enumeration of $|ASM(n,r)|$ for any positive integer n and non-negative integer r.

$n \backslash r$	0	1	2	3	4	5	6
1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7
3	1	7	26	70	155	301	532
4	1	42	628	5102	28005	117332	403832
5	1	429	41784	1507128	28226084	335138400	2850715602
6	1	7436	7517457	1749710096	152363972022	6680340618048	177052977183950

Table 1.2: The size of $ASM(n,r)$ for $n,r \in [0,6]$.

In general, we have the following lower and upper bounds for $|ASM(n,r)|$:

$$|SMS(n,r)| \leq |ASM(n,r)| \leq (2r+1)^{n^2}. \tag{1.21}$$

The first inequality holds since $SMS(n,r) \subset ASM(n,r)$, using (1.18). The second inequality holds since $ASM(n,r)$ is contained in the set of all $n \times n$ matrices with entries between $-r$ and r, using part (II) of Lemma 1.8. For the case $r = 1$, by (1.19) and the fact that $ASM(n,1) = ASM(n)$, we can write

$$n! = |SMS(n,1)| \leq |ASM(n)| \leq 3^{n^2} \tag{1.22}$$

where the upper bound in (1.22) comes from the fact that in total there are n^2 positions that can be filled with elements of the set $\{0, \pm 1\}$. The following inequalities can also be easily obtained:

$$\begin{gathered} |ASM(n-1,r)| \leq |ASM(n,r)| \leq |ASM(n+1,r)| \\ \text{and} \\ |ASM(n,r-1)| \leq |ASM(n,r)| \leq |ASM(n,r+1)| \end{gathered} \tag{1.23}$$

The following theorem shows how to decompose $ASM(n,r)$. For the proof see Chapter 2 of [60] or [25]. Note that this decomposition is not unique.

Theorem 1.10. *Every matrix in $ASM(n,r)$ can be written as the sum of r standard ASMs of size n.*

This theorem implies that another upper bound for $|ASM(n,r)|$ is

$$|ASM(n,r)| \leq \binom{|ASM(n)| + r - 1}{r} \tag{1.24}$$

where $|ASM(n)|$ is given explicitly by (1.10). This bound holds since there are $\binom{|ASM(n)|+r-1}{r}$ ways of choosing r elements (with repetition allowed) from the set of standard ASMs of size n. The upper bound in (1.24) is better than that in (1.21). However, due to non-uniqueness of the decomposition in Theorem 1.10, (1.24) is still not a particularly useful bound.

Example 1.11. Our running example for a higher spin ASM is the following matrix of size 6 with line sum 2:

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 2 & -2 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \in ASM(6,2) \tag{1.25}$$

By theorem (1.10) we expect to be able to write the above matrix as the sum of two ASMs. We have

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 2 & -2 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{1.26}$$

It is clear that the converse also holds, that is, every sum of r standard ASMs of size n is a matrix in $ASM(n,r)$.

We will back to the higher spin ASMs in Chapter 3. In particular, the enumeration of HSASMs will be discussed in full details in Chapter 3, particularly Section 3.3 where the reduced form of HSASMs is introduced.

1.2.3 ASMs and monotone triangles

In this section we introduce the notion of monotone triangles, or MTs for short, and their basic properties. Their connection with ASMs was first observed by Mills, Robbins and Rumsey [75] and later was used by Zeilberger who called them n-Gog triangles to prove the ASM Conjecture (1.10) in [113].

Definition 1.12. A *monotone triangle*, or MT for short, of order n is a triangular arrangement of $n(n+1)/2$ integers from the set $[n]$ such that

(1) All rows are strictly increasing, that is $t_{ij} < t_{i,j+1}$, for each $1 \le j < i$,

(2) Any two consecutive rows are weakly interleaved, that is, $t_{ij} \le t_{i-1,j} \le t_{i,j+1}$, for each $1 \le j < i$.

where t_{ij} is the j^{th} entry in the i^{th} row for each $1 \le j \le i$ that is counted from the top.

In other words, a MT with n rows can be described as a triangular array with n numbers along each side such that the bottom row ranges from 1 to n in succession, the numbers in each row are strictly increasing from left to right, and the numbers along diagonals are weakly increasing from left to right. So it is easy to verify that the following pattern holds between any triangle entries in consecutive rows in a given MT

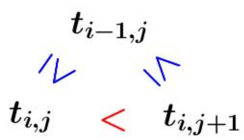


Figure 1.2: Patterns between any triangle of entries in consecutive rows in a given MT.

Let $MT(n)$ denote the set of all MTs of order n. The following lemma provides a bijection between $ASM(n)$ and the set $MT(n)$. For the proof see Proposition 1.2.5 in [88] or [75].

Lemma 1.13. *There is a bijection between the set of MTs of order* n *and the set of* $n \times n$ *ASMs via the following map*

$$\textit{The } i^{th} \textit{ row of MT, } T, \textit{ contains entry } j. \Leftrightarrow \textit{ Top } i \textit{ entries of } j^{th} \textit{ column of ASM, } A, \textit{ sum up to } 1. \tag{1.27}$$

In other words, let $A \in ASM(n)$ be given and denote the corresponding partial row sum matrix of A by R_A. Now whenever we have a 1 in row i and column j of R_A, then the corresponding MT to A, denoted by T_A, contains a j in row i. Conversely, for the inverse mapping, for each $i \in [0, n]$ and $j \in [n]$, $(R_A)_{ij}$ is set to be the number of times that j occurs in row i of $T \in MT(n)$. Finally, A is obtained from R_A by taking differences of successive rows. As an illustration, denote the partial row sum matrix of the ASMs in our current Example 1.4 by R_A and R_B, respectively. Then their corresponding MTs are given by

$$R_A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \Leftrightarrow \quad T_A = \begin{array}{ccccccccccc} & & & & & 2 & & & & & \\ & & & & 1 & & 3 & & & & \\ & & & 1 & & 3 & & 4 & & & \\ & & 1 & & 2 & & 3 & & 6 & & \\ & 1 & & 2 & & 3 & & 4 & & 6 & \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 \end{array}$$

$$R_B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \Leftrightarrow \quad T_B = \begin{array}{ccccccccccc} & & & & & 2 & & & & & \\ & & & & 1 & & 3 & & & & \\ & & & 1 & & 3 & & 4 & & & \\ & & 1 & & 2 & & 3 & & 5 & & \\ & 1 & & 2 & & 3 & & 4 & & 6 & \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 \end{array}$$

The seven MTs of order 3 associated to seven 3×3 ASMs in Example 1.3 are given in Example 1.14.

Example 1.14. Here are all the seven monotone triangles corresponding to the seven ASMs in Example 1.3.

$$MT(3) = \left\{ \begin{array}{ccc} & 1 & \\ 1 & & 2 \\ 1 & 2 & 3 \end{array} , \begin{array}{ccc} & 1 & \\ 1 & & 3 \\ 1 & 2 & 3 \end{array} , \begin{array}{ccc} & 2 & \\ 1 & & 2 \\ 1 & 2 & 3 \end{array} , \begin{array}{ccc} & 2 & \\ 1 & & 3 \\ 1 & 2 & 3 \end{array} , \begin{array}{ccc} & 2 & \\ 2 & & 3 \\ 1 & 2 & 3 \end{array} , \begin{array}{ccc} & 3 & \\ 1 & & 3 \\ 1 & 2 & 3 \end{array} , \begin{array}{ccc} & 3 & \\ 2 & & 3 \\ 1 & 2 & 3 \end{array} \right\} \tag{1.28}$$

Analogously to the case of ASMs, we expect to have a natural generalization of MTs as well. This is done by Behrend and Knight in [15]. First, we need the following definition of the set of *higher spin monotone triangles*.

Definition 1.15. Define $\mathrm{MT}(n,r)$ to be the set of triangular arrays $M = (m_{ij})$ of the form

$$\begin{array}{ccccc} & & m_{1,1} \dots m_{1,r} & & \\ & m_{2,1} & \cdots & m_{2,2r} & \\ & \iddots & \cdots & \ddots & \\ m_{n,1} & & \cdots & & m_{n,nr} \end{array}$$

such that

- Each entry of M is in $[n]$,
- In each row of M, any integer of $[n]$ appears at most r times,
- $m_{ij} \le m_{i,j+1}$ for each $i \in [n]$, $j \in [ir-1]$,
- $m_{i+1,j} \le m_{ij} \le m_{i+1,j+r}$ for each $i \in [n-1]$, $j \in [ir]$.

It follows that the last row of every member of $\mathrm{MT}(n,r)$ consists of each integer of $[n]$ repeated r times. The following lemma provides a bijection between $MT(n,r)$ and $ASM(n,r)$, for the proof of this see [15] or Chapter 2 of [60].

Lemma 1.16. *There is a bijection between $ASM(n,r)$ and $MT(n,r)$.*

In particular, to map $A \in ASM(n,r)$ to $M \in \mathrm{MT}(n,r)$, first find the partial row sum of A, denoted by R_A. Then place the integer j, $(R_A)_{ij}$ times in row i of M for each $i,j \in [n]$.

Example 1.17. For example, the higher spin monotone triangle corresponding to the HSASM in our running Example 1.11 is given by

$$\begin{array}{ccccccccccccccccccccccc}
&&&&&&&&&&2&&4&&&&&&&&&& \\
&&&&&&&&&2&&2&&3&&6&&&&&&& \\
&&&&&&&&1&&1&&3&&4&&5&&6&&&& \\
&&&&&&&1&&1&&2&&4&&4&&5&&5&&6& \\
&&&&&&1&&1&&2&&2&&3&&4&&5&&5&&6&&6 \\
&&&&&1&&1&&2&&2&&3&&3&&4&&4&&5&&5&&6&&6
\end{array}$$

Remark 1.18. Note that the set $MT(n,1)$ is just the MTs of order n corresponding to the standard ASMs. For more details and work regarding $MT(n)$, see [49], [51], [50], [52], [76], [83], [88].

1.2.4 ASMs and configurations of six vertex model with domain wall boundary conditions

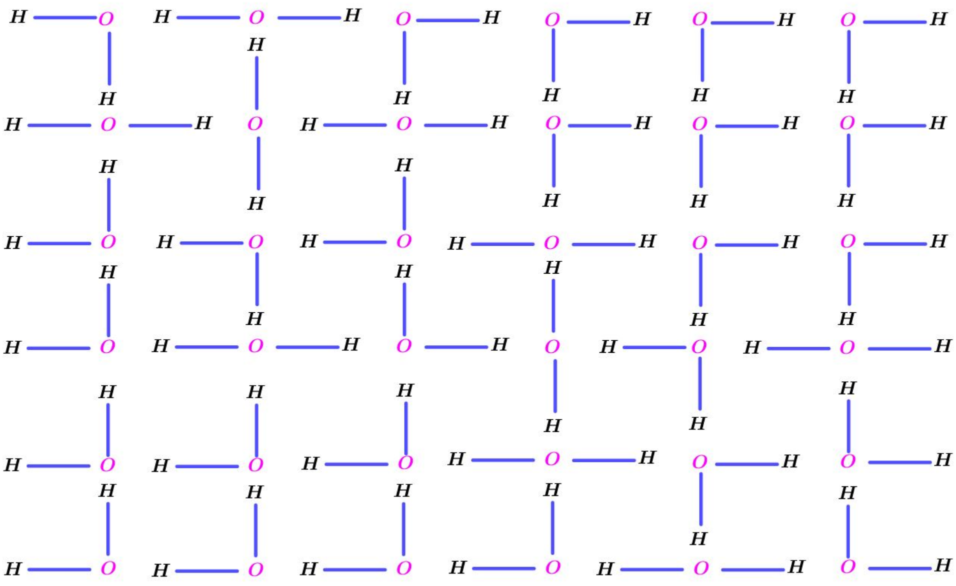
Another amazing appearance of ASMs as interdisciplinary objects is their one-to-one correspondence with configurations of the *six-vertex model*, or SVM for short, also known as configurations of the *square ice model*, with domain-wall boundary conditions, or DWBCs for short. These objects originally come from Statistical Mechanics. As the name "square ice model" suggests, it can be visualized as a system of frozen water molecules in a square lattice, see Figure 1.4a. As illustrated in Figure 1.4a, there is precisely one hydrogen atom, H, between each pair of oxygen atoms, O, with the restriction that exactly two of the four hydrogen atoms surrounding each oxygen atom are attached to it. Now if we consider each oxygen atom as a vertex of the lattice, then there will be precisely six different configurations for each of them depending on which of the four hydrogen atoms are attached to it, see Figure 1.4b. Note that inward arrows in Figure 1.4b means that hydrogen atom, H, is attached to the oxygen atom, O, whereas outwards arrows indicate that they are not attached. This is why it is also called "six vertex model". Figure 1.4c shows the corresponding directed graph of the square ice model of Figure 1.4a. For more details and history of this see Chapter 7 of [22]. They were utilized by Kuperberg to prove the ASM conjecture (1.10), see [65]. To describe the one-to-one correspondence between ASMs and the configurations of six vertex model with DWBCs we need the following definition.

Definition 1.19. A *six vertex model configuration with domain wall boundary conditions*, or SVM with DWBCs for short, is an orientation of the edges of a square grid with the property that each vertex other than vertices on the boundary has two incoming arrows and two outgoing arrows. The arrows on the external edges on the upper, right, lower and the left boundaries of the grid are all directed upward, leftward, downward and rightward, respectively. These restrictions are called *domain wall boundary conditions*. In this configuration, we have n horizontal lines and n vertical lines meeting in n^2 intersections of degree four with $4n$ vertices of degree one at the boundary. So each vertex or edge is either internal or external. That is, the n^2 vertices of degree four are internal and the remaining $4n$ vertices of degree one are external. Also, the $2n(n-1)$ edges that join two internal vertices are internal, and the remaining $4n$ edges are external. We denote the set of all SVMs with DWBCs of order n by $SVM(n)$.

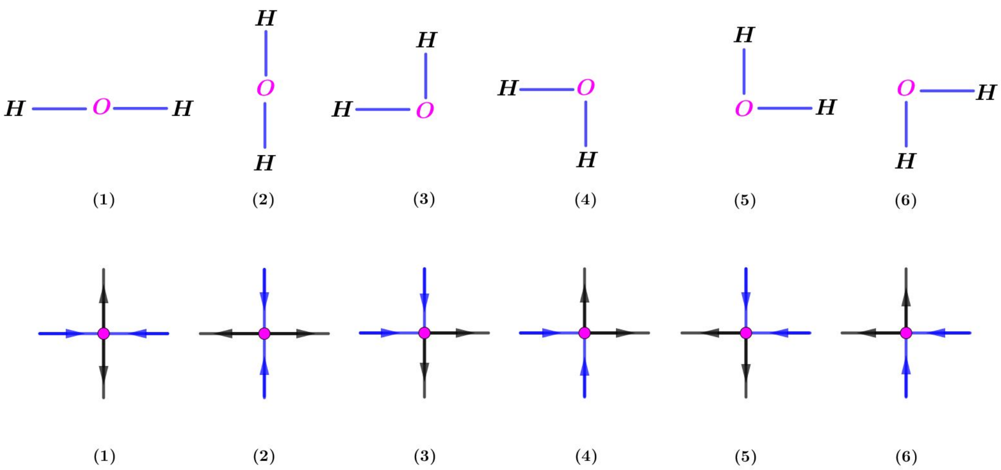
In order to turn a given SVM into an ASM and vice versa, we need to replace the vertices of a given SVM with -1's, 0's or $+1$'s based on the possible configurations in the SVM. As shown in Figure 1.5, the horizontal inwards arrows are corresponding to $+1$'s, the vertical inward arrows are corresponding to -1's and the other four arrangements are associated with the 0's. So we have the following lemma.

Lemma 1.20. *There is a bijection between $ASM(n)$ and $SVM(n)$.*

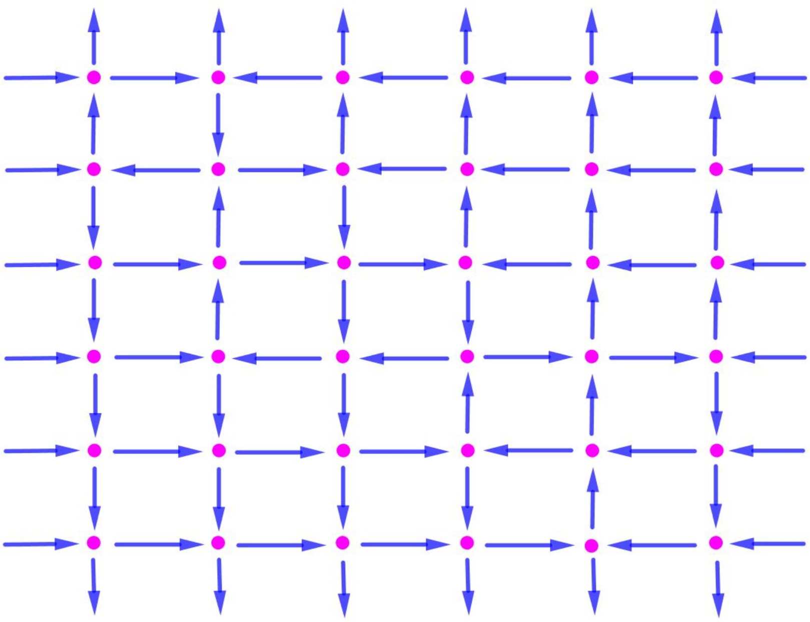
Example 1.21. As an illustration, Figure 1.3a is the SVM with DWBCs given in Figure 1.4c (this is actually the SVM with DWBCs associated with the ASM, A, in our running Example 1.4). In addition, the SVM with DWBCs associated with the ASM, B, in our running Example 1.4 is shown in Figure 1.3b. Moreover, all seven corresponding 3×3 SVMs with DWBCs and ASMs are shown in Figure 1.6.



(a) A configuration of the square ice model.



(b) All six possible configurations for each oxygen atom.



(c) The directed graph corresponding to the configuration of square ice model in 1.4a.

Figure 1.4: The square ice model and its corresponding directed graph.

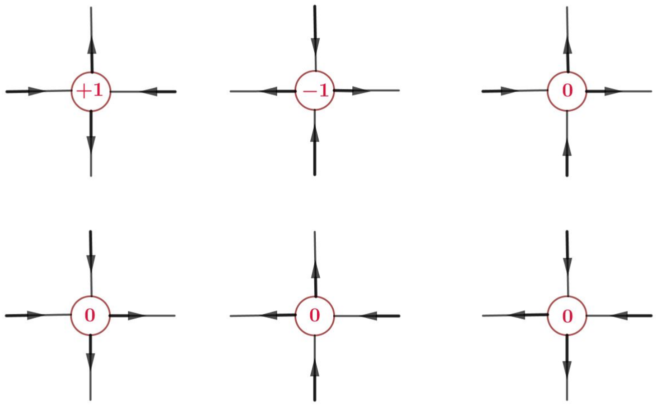
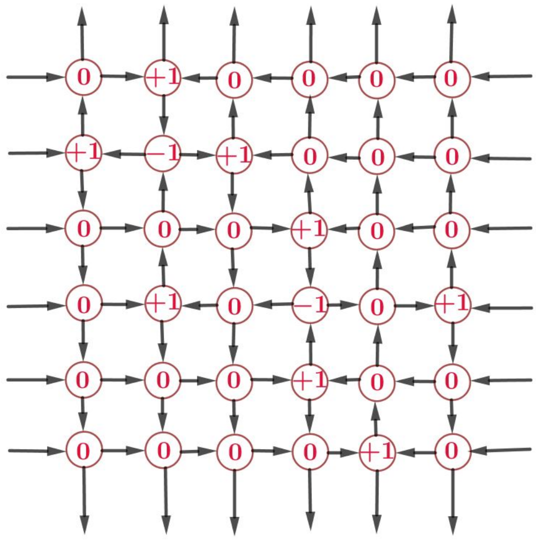
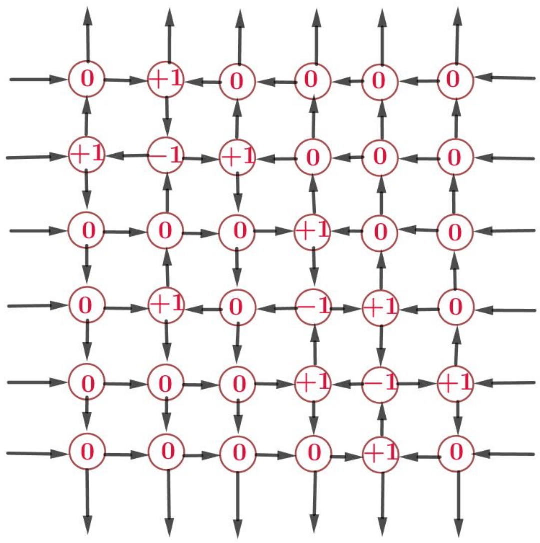


Figure 1.5: The correspondence between six vertex model configurations and ASMs.



(a) The SVM with DWBCs corresponding to the ASM A in our running Example 1.4.



(b) The SVM with DWBCs corresponding to the ASM B in our running Example 1.4.

Figure 1.3: The SVMs with DWBCs corresponding to the ASMs in Example 1.4.

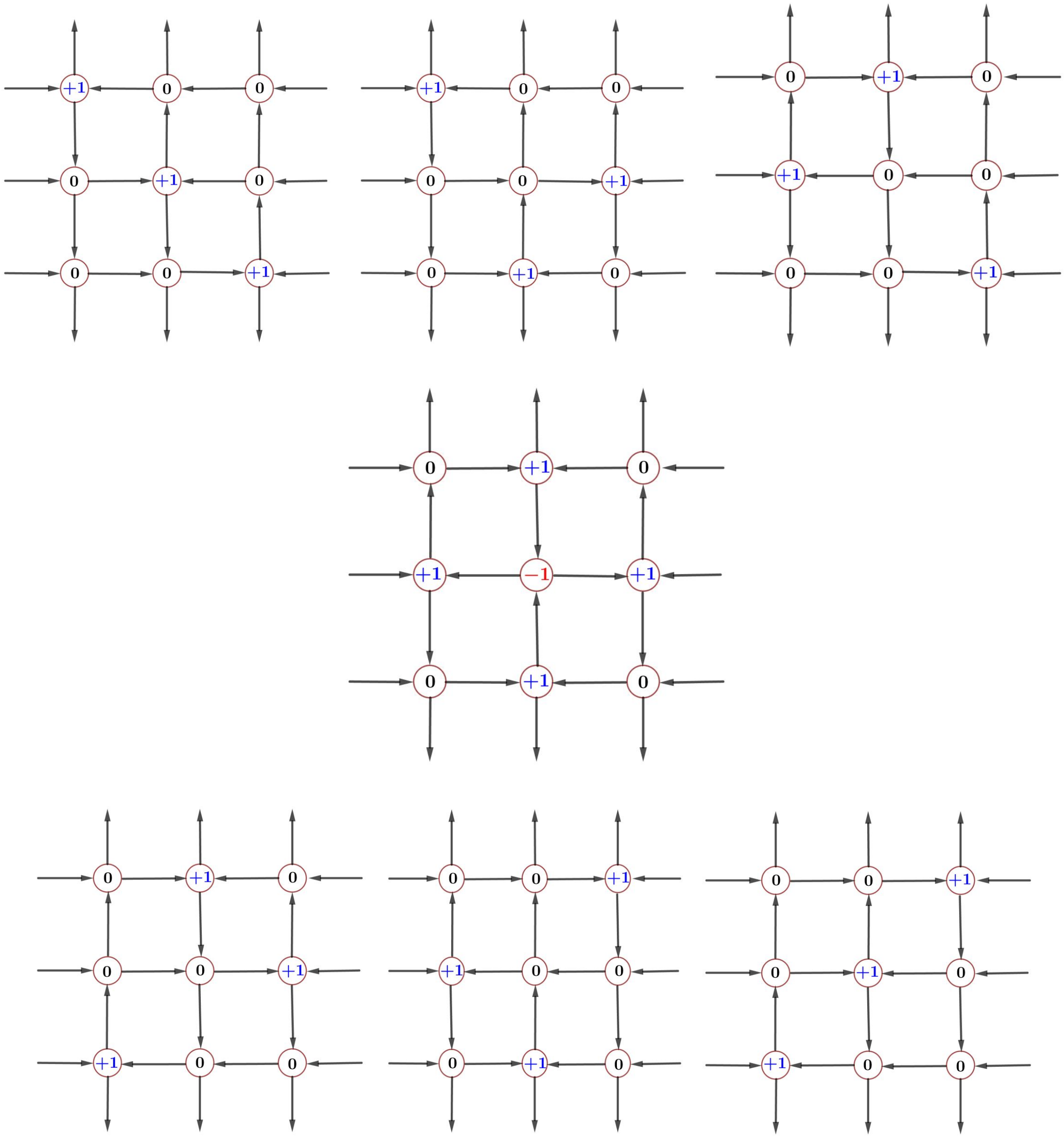


Figure 1.6: All seven configurations of six vertex model corresponding to the seven ASMs of size 3.

1.2.5 ASMs and simple flow grids

In this section, we introduce the notion of *simple flow grids*, or SFGs for short, and their connections with ASMs. The concept of SFGs is needed in Chapter 2 to describe the face structure of the ASM polytope. We need the following definitions first.

Definition 1.22. Consider a directed graph with $n^2 + 4n$ vertices including n^2 internal vertices (i, j) and $4n$ external vertices given by $(i, 0), (0, j), (i, n+1)$, and $(n+1, j)$ where $i, j = 1, ..., n$. One can naturally depict these vertices in a grid in which vertex (i, j) appears in row i and column j (Note that the vertices are identical to those for SVMs). Then we define the *complete flow grid* of order n to be the directed graph on this vertex set with edge set $\{((i,j),(i,j \pm 1)), ((i,j),(i \pm 1, j)) \,|\, i, j = 1, ..., n\}$. Note that it contains directed edges pointing in both directions between neighbouring internal vertices in G

and directed edges from internal vertices to external vertices on the boundary, see Figure 1.10a.

Definition 1.23. A *simple flow grid* of order n is a spanning sub-graph of the complete flow grid of order n in which four edges are incident to each internal vertex, so that either four edges are directed inward, four edges are directed outward, or two horizontal edges point in the same direction and two vertical edges point in the same direction. Denote the set of simple flow grids of order n by $SFG(n)$.

The following proposition provides a bijection between ASMs and simple flow grids.

Lemma 1.24. *There exists an explicit bijection between ASMs of order n, $ASM(n)$, and simple flow grids of order n, $SFG(n)$.*

Proof.
Let $A \in ASM(n)$ be given, then the vertex corresponding to each +1 in A is a *source* and so it has four edges to each of the neighbouring vertices. The vertex corresponding to each -1 in A is a *sink* and thus has four edges from its neighbours to it. To construct the remaining edges just continue the edges from each source until one either hits a sink or a boundary vertex. This determines the simple flow grid corresponding to A.
Conversely, let $\mathcal{S} \in SFG(n)$ be given, then one can easily retrieve its associated ASM simply by replacing all the sources with +1's and all the sinks with -1's. All six possible configurations for each vertex of $\mathcal{S}$ are shown in Figure 1.7. Therefore, ASMs and simple flow grids are in bijection as required. ☐

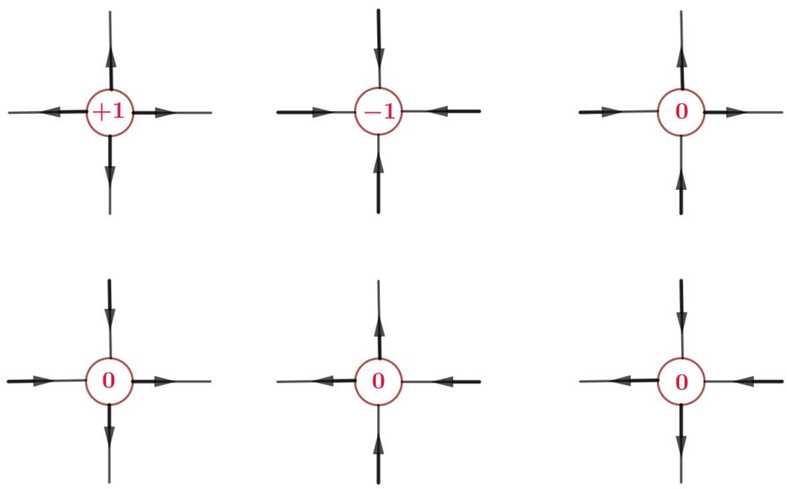
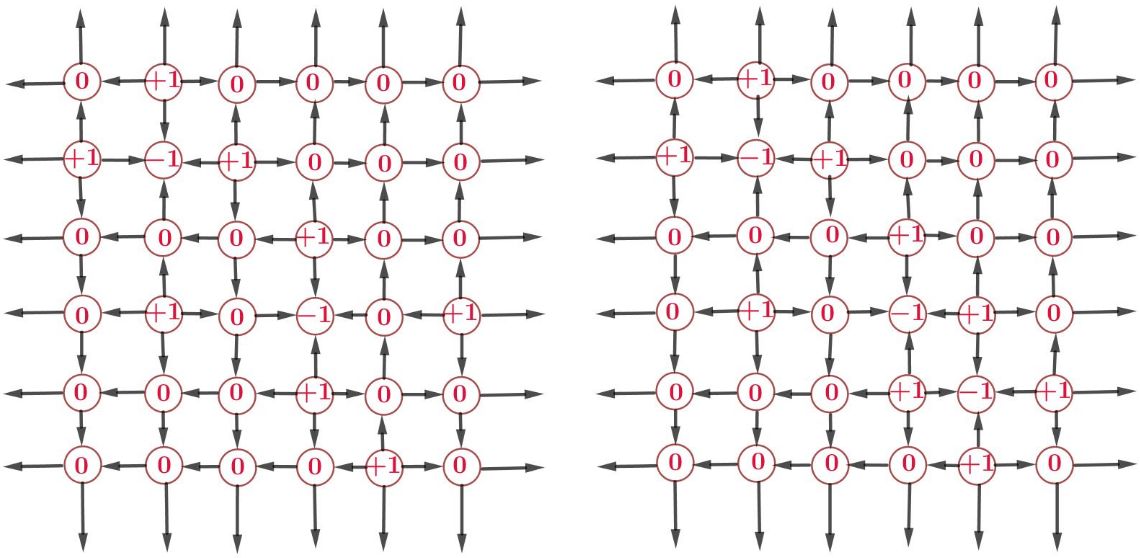


Figure 1.7: All 6 possible vertex configurations of SFG and the corresponding entries of the associated ASM.

Remark 1.25. Note that $SFG(n)$ and $SVM(n)$ have almost the same configurations. The difference is that by reversing the horizontal orientations in a given SVM, we obtain its associated SFG and vice versa.

Example 1.26. For instance, the simple flow grids corresponding to the ASMs A and B in our running Example 1.4 are shown in Figures 1.8a and 1.8b, respectively. Following the observation in Remark 1.25, it is easy to verify that the configurations given in Figures 1.3a and 1.3b are almost the same as the ones in Figures 1.8a and 1.8b except for the horizontal orientations. As further examples, the simple flow grids corresponding to the seven 3×3 ASMs are shown in Figure 1.9.



(a) The SFG configuration corresponding to the ASM A in Example 1.4.

(b) The SFG configuration corresponding to the ASM B in Example 1.4.

Figure 1.8: The simple flow grid configurations associated with the ASMs in our running Example 1.4.

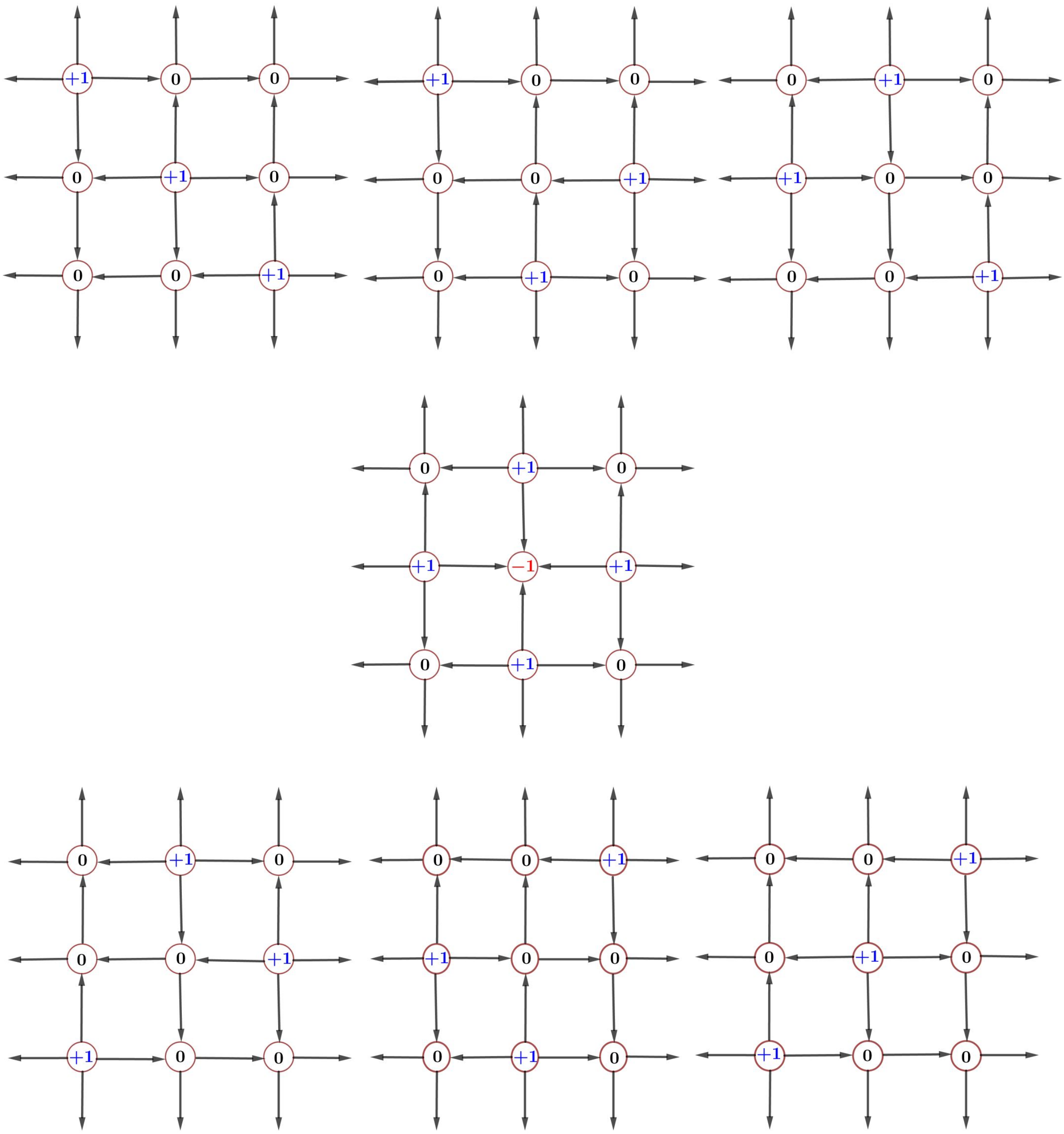
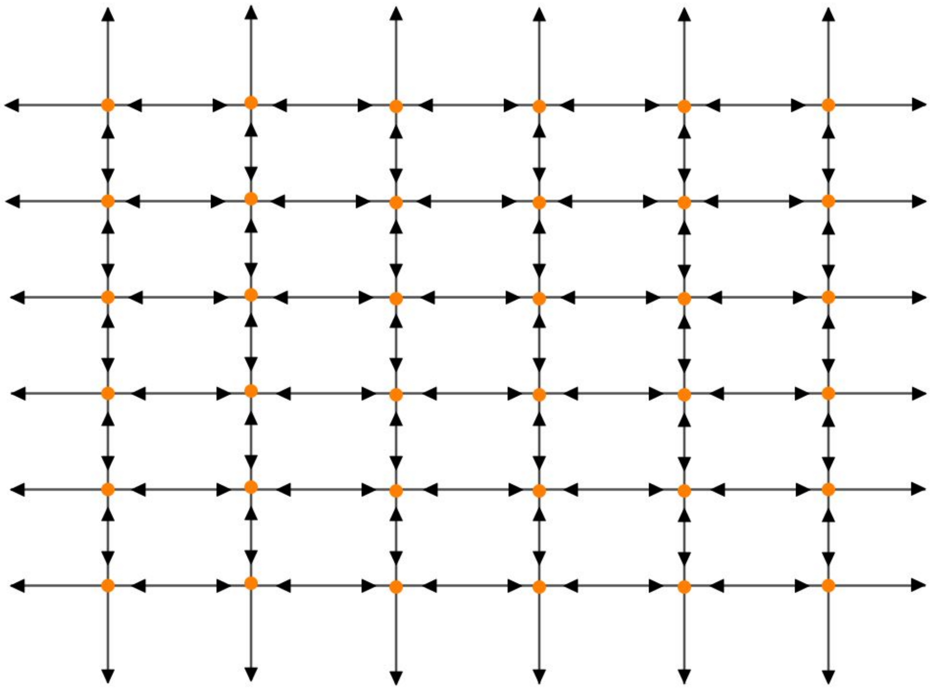


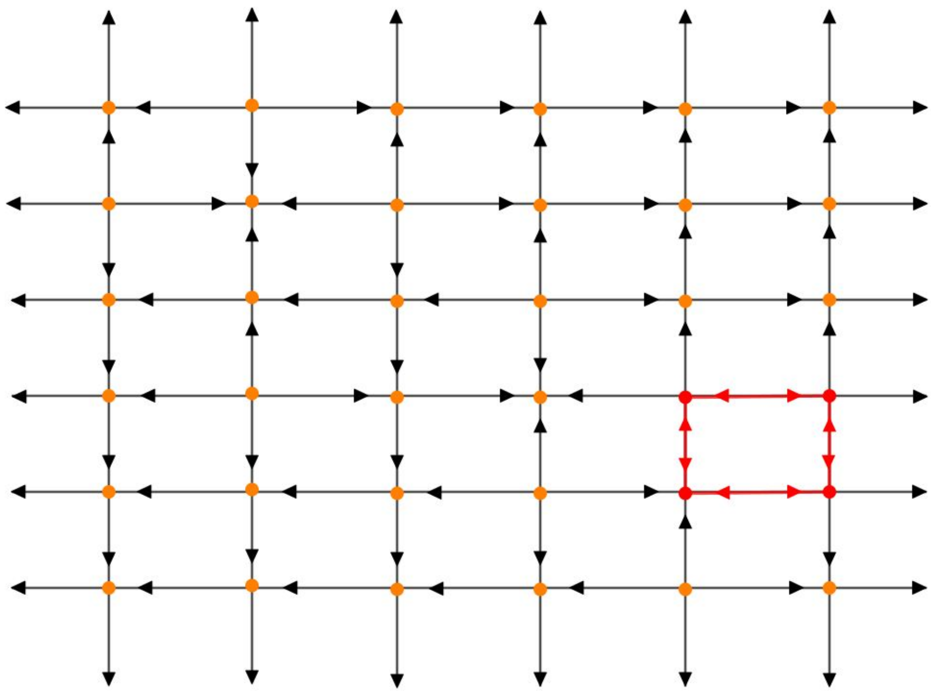
Figure 1.9: All seven simple flow grids corresponding to the seven ASMs of size 3.

The following definition of elementary flow grids is useful for describing the face lattice structure of the ASM polytope, see Section 2.4, Theorem 2.32.

Definition 1.27. An *elementary flow grid* of order n is a spanning subgraph of the complete flow grid of order n such that the edge set is the union of the edge sets of simple flow grids in $SFG(n)$. Given an elementary flow grid, we define a *doubly directed region* as a collection of cells in G completely bounded by doubly directed edges but containing no double directed edges in the interior, see Figure 1.10b.



(a) The complete flow grid of order 6.



(b) An elementary flow grid of order 6 consisting of one doubly directed region.This is the union of SFGs of order 6 given in Figures 1.8a and 1.8b.

Figure 1.10: The complete flow grid and an elementary flow grid of order 6.

1.2.6 ASMs and corner sum matrices

In this section, the notion of corner sum matrices, or CSMs for short, is given. CSMs were first defined by Robbins and Rumsey [91].

Similar to the case for ASMs, one can encode the set of all $n \times n$ CSMs in a compact form as follows.

Definition 1.28. For any positive integer n, the set of all order n CSMs denoted by $CSM(n)$ is given by

$$CSM(n) :=$$
$$\left\{ C = \begin{pmatrix} c_{00} & \dots & c_{1n} \\ \vdots & & \vdots \\ c_{n0} & \dots & c_{nn} \end{pmatrix} \in \mathbb{N}^{(n+1)\times(n+1)} \;\middle|\; \begin{array}{l} \bullet\ c_{0k} = c_{k0} = 0,\ c_{kn} = c_{nk} = k \quad \text{for all } k \in [0,n]; \\ \bullet\ c_{ij} - c_{i,j-1} \quad \text{and} \quad c_{ij} - c_{i-1,j} \quad \text{are each } 0 \quad \text{or } 1 \quad \text{for all } i,j \in [n]. \end{array} \right\}. \tag{1.29}$$

Example 1.29. For instance, $CSM(3)$ is given by

$$\text{CSM(3)=} \left\{ \begin{pmatrix} 0&0&0&0\\0&1&1&1\\0&1&2&2\\0&1&2&3 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&1&1&1\\0&1&1&2\\0&1&2&3 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&1&1\\0&1&2&2\\0&1&2&3 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&1&1\\0&1&1&2\\0&1&2&3 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&1&1\\0&0&1&2\\0&1&2&3 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&1&1&2\\0&1&2&3 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&1&2\\0&1&2&3 \end{pmatrix} \right\} \tag{1.30}$$

Now we present the main theorem of this section which provides a bijection between $ASM(n)$ and $CSM(n)$.

Theorem 1.30. *Let $\phi : ASM(n) \to CSM(n)$ be given by*

$$\phi(A)_{ij} = \sum_{i'=1}^{i} \sum_{j'=1}^{j} a_{i'j'}, \qquad \textit{for} \quad i,j \in [0,n]. \tag{1.31}$$

for each $A \in ASM(n)$, and $\psi : CSM(n) \to ASM(n)$ be given by

$$\psi(C)_{ij} = c_{ij} - c_{i,j-1} - c_{i-1,j} + c_{i-1,j-1}, \qquad \textit{for} \quad i,j \in [n]. \tag{1.32}$$

for each $C \in CSM(n)$. Then

(I) ϕ and ψ are well-defined functions, and

(II) ϕ and ψ are bijective with $\phi^{-1} = \psi$.

In other words, for a given $n \times n$ ASM A, the associated CSM is an $(n+1)\times(n+1)$ matrix $C_A = (c_{ij})$ where $c_{ij} = \phi(A)_{ij}$.

Proof.

(I) Let $A \in ASM(n)$ be given. Then we need to show that $\phi(A)$ is well-defined, that is, $\phi(A) \in CSM(n)$. In other words, $\phi(A)$ satisfies the following conditions, where $c_{ij} = \phi(A)_{ij}$:

(a) $c_{ij} \geq 0$;

(b) $c_{i0} = c_{0i} = 0$ for all $i = 0, ..., n$;

(c) $c_{in} = c_{ni} = i$ for all $i = 0, ..., n$;

(d) $c_{ij} - c_{i-1,j}$ and $c_{ij} - c_{i,j-1}$ are each 0 or 1, for all $i, j = 1, ..., n$.

Part (a) is a straightforward consequence of part (IV) of Lemma 1.5, since partial sums along each row and column of A are non-negative. Part (b) is an immediate consequence of the definition of an empty sum, since $c_{0j} = \sum_{i'=1}^{0}\sum_{j'=1}^{j} a_{i'j'} = 0$ and $c_{i0} = \sum_{i'=1}^{i}\sum_{j'=1}^{0} a_{i'j'} = 0$. For part (c), we can write

$$
\begin{aligned}
c_{in} &= \sum_{i'=1}^{i}\sum_{j'=1}^{n} a_{i'j'} \\
&= \sum_{i'=1}^{i}\left(\sum_{j'=1}^{n} a_{i'j'}\right) \\
&= \sum_{i'=1}^{i} 1 \\
&= i
\end{aligned}
\tag{1.33}
$$

The proof of the case $c_{ni} = i$ for $i = 0, ..., n$ is similar. For part (d), we will only check that $c_{ij} - c_{i-1,j}$ is either 0 or 1 since the argument for the other case is very similar. By definition of ϕ we can write

$$
\begin{aligned}
c_{ij} - c_{i,j-1} &= \sum_{j'=1}^{j}\sum_{i'=1}^{i} a_{i'j'} - \sum_{j'=1}^{j-1}\sum_{i'=1}^{i} a_{i'j'} \\
&= \sum_{i'=1}^{i} a_{i'j}
\end{aligned}
$$

So $c_{ij} - c_{i,j-1}$ is just the partial sum of the j^{th} column in A. Now by part (IV) of Lemma 1.5, we know that the partial sum $\sum_{i'=1}^{i}$ along column j is 0 or 1. This implies that $c_{ij} - c_{i,j-1}$ is either 0 or 1, as required. Thus $\phi(A) \in CSM(n)$ and so it is well defined.

Conversely, we show that ψ is well-defined, that is, $\psi(C) \in ASM(n)$ for all $C \in CSM(n)$. We need to show that for a given $C \in CSM(n)$ satisfying conditions $(a) - (d)$, the row and column sum of $\psi(C)_{ij}$ is 1 and its partial row and column sums extending from each end are non-negative for all $i, j = 1, ..., n$ as in (1.15). Firstly, we show that each complete row sum is 1, that is, for fixed i, we have $\sum_{j'=1}^{n} \psi(C)_{ij'} = 1$. We have

$$
\begin{aligned}
\sum_{j'=1}^{n} \psi(C)_{ij'} &= \sum_{j'=1}^{n}[c_{ij'} - c_{i,j'-1} - c_{i-1,j'} + c_{i-1,j'-1}] \\
&= \left(\sum_{j'=1}^{n} c_{ij'} - \sum_{j'=1}^{n} c_{i,j'-1}\right) - \left(\sum_{j'=1}^{n} c_{i-1,j'} - \sum_{j'=1}^{n} c_{i-1,j'-1}\right) \\
&= (c_{in} - c_{i0}) - (c_{i-1,n} - c_{i-1,0}) \\
&= i - 0 - (i-1) + 0 = 1
\end{aligned}
$$

Similarly for fixed j, we can write

$$
\begin{aligned}
\sum_{i'=1}^{n} \psi(C)_{i'j} &= \sum_{i'=1}^{n} [c_{i'j} - c_{i',j-1} - c_{i'-1,j} + c_{i'-1,j-1}] \\
&= \left(\sum_{i'=1}^{n} c_{i'j} - \sum_{i'=1}^{n} c_{i'-1,j} \right) - \left(\sum_{i'=1}^{n} c_{i',j-1} - \sum_{i'=1}^{n} c_{i'-1,j-1} \right) \\
&= (c_{nj} - c_{0j}) - (c_{n,j-1} - c_{0,j-1}) \\
&= j - 0 - (j-1) + 0 = 1
\end{aligned}
$$

Therefore, each row and column sum of $\psi(C)$ is 1 as required. It remains to verify that the partial row and column sum of $\psi(C)$ extended from each end of each row and each column is non-negative. For fixed i, we can write

$$
\begin{aligned}
\sum_{j'=1}^{j} \psi(C)_{ij'} &= \sum_{j'=1}^{j} [c_{ij'} - c_{i,j'-1} - c_{i-1,j'} + c_{i-1,j'-1}] \\
&= \left(\sum_{j'=1}^{j} c_{ij'} - \sum_{j'=1}^{j} c_{i,j'-1} \right) - \left(\sum_{j'=1}^{j} c_{i-1,j'} - \sum_{j'=1}^{j} c_{i-1,j'-1} \right) \\
&= (c_{ij} - c_{i0}) - (c_{i-1,j} - c_{i-1,0}) \\
&= c_{ij} - 0 - c_{i-1,j} + 0 \\
&= c_{ij} - c_{i-1,j}
\end{aligned}
$$

but by the definition of C, we know that $c_{ij} - c_{i-1,j}$ is either 0 or 1 and so $\sum_{j'=1}^{j} \psi(C)_{ij'}$ and $\sum_{j'=j}^{n} \psi(C)_{ij'}$ are non-negative as required. A similar argument holds for the partial column sums. This completes the proof that ψ is well-defined.

(II) We need to show that $\psi(\phi(A)) = A$ for all $A \in ASM(n)$, and $\phi(\psi(C)) = C$ for all $C \in CSM(n)$. We can write

$$
\begin{aligned}
((\psi \circ \phi)(A))_{ij} &= \psi(\phi(A))_{ij} \\
&= \left(\sum_{i'=1}^{i} \sum_{j'=1}^{j} a_{i'j'} - \sum_{i'=1}^{i} \sum_{j'=1}^{j-1} a_{i'j'} \right) - \left(\sum_{i'=1}^{i-1} \sum_{j'=1}^{j} a_{i'j'} - \sum_{i'=1}^{i-1} \sum_{j'=1}^{j-1} a_{i'j'} \right) \\
&= \sum_{i'=1}^{i} a_{i'j} - \sum_{i'=1}^{i-1} a_{i'j} \\
&= a_{ij}
\end{aligned}
$$

for $i, j \in [n]$. Similarly, for $\phi(\psi(C)) = C$ we can write

$$
\begin{aligned}
((\phi\circ\psi)(C))_{ij} &= \phi(\psi(C))_{ij} \\
&= \sum_{i'=1}^{i}\sum_{j'=1}^{j}\left(c_{i'j'}-c_{i',j'-1}-c_{i'-1,j'}+c_{i'-1,j'-1}\right) \\
&= \left(\sum_{i'=1}^{i}\sum_{j'=1}^{j}c_{i'j'}-\sum_{i'=1}^{i}\sum_{j'=0}^{j-1}c_{i'j'}\right)-\left(\sum_{i'=0}^{i-1}\sum_{j'=1}^{j}c_{i'j'}-\sum_{i'=0}^{i-1}\sum_{j'=0}^{j-1}c_{i'j'}\right) \\
&= \left(\sum_{i'=1}^{i}\sum_{j'=1}^{j}c_{i'j'}-\sum_{i'=1}^{i}\sum_{j'=1}^{j-1}c_{i'j'}\right)-\left(\sum_{i'=1}^{i-1}\sum_{j'=1}^{j}c_{i'j'}-\sum_{i'=1}^{i-1}\sum_{j'=1}^{j-1}c_{i'j'}\right) \\
&= \sum_{i'=1}^{i}c_{i'j}-\sum_{i'=1}^{i-1}c_{i'j} \\
&= c_{ij}
\end{aligned}
$$

for $i,j\in[0,n]$. This argument shows that ϕ and ψ are bijective with $\phi^{-1}=\psi$, as required.

□

Example 1.31. As an illustration, the CSMs corresponding to the ASMs given in our running Example 1.4 are given by

$$
C_A=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&1&1&1&1&1\\0&1&1&2&2&2&2\\0&1&1&2&3&3&3\\0&1&2&3&3&3&4\\0&1&2&3&4&4&5\\0&1&2&3&4&5&6\end{pmatrix}\quad,\quad C_B=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&1&1&1&1&1\\0&1&1&2&2&2&2\\0&1&1&2&3&3&3\\0&1&2&3&3&4&4\\0&1&2&3&4&4&5\\0&1&2&3&4&5&6\end{pmatrix} \tag{1.34}
$$

The following lemma provides a lower and upper bound for each entry of a given CSM.

Lemma 1.32. *For any $C\in CSM(n)$ and $i,j\in\{0,\dots,n\}$, we have*

$$
\max(0,i+j-n)\le c_{ij}\le\min(i,j). \tag{1.35}
$$

As an example, for $n=3$, the first and last matrices on the right hand side of (1.30) correspond to $min(i,j)$ and $max(0,i+j-n)$, respectively.

Proof.
Consider $C\in CSM(n)$ and let $C=\phi(A)$ as in (1.31). Then there are four inequalities in (1.35), which can be verified as follows, using the properties that any partial row or column sum of an ASM is 0 or 1, and any full row or column sum of an ASM is 1.

- $c_{ij}\ge 0$.

  In this case,

$$
c_{ij}=\sum_{i'=1}^{i}\sum_{j'=1}^{j}a_{i'j'}=\sum_{i'=1}^{i}\Big(\sum_{j'=1}^{j}a_{i'j'}\Big)\ge\sum_{i'=1}^{i}0=0.
$$

- $c_{ij} \geq i+j-n$.

  In this case,

$$\begin{aligned}
c_{ij} &= \sum_{i'=1}^{i} \sum_{j'=1}^{j} a_{i'j'} \\
&= \sum_{i'=1}^{i} \Big(\sum_{j'=1}^{n} a_{i'j'} - \sum_{j'=j+1}^{n} a_{i'j'} \Big) \\
&= \sum_{i'=1}^{i} \Big(\sum_{j'=1}^{n} a_{i'j'} \Big) - \sum_{i'=1}^{i} \sum_{j'=j+1}^{n} a_{i'j'} \\
&= \sum_{i'=1}^{i} 1 \ - \sum_{j'=j+1}^{n} \sum_{i'=1}^{i} a_{i'j'} \\
&= i \ - \sum_{j'=j+1}^{n} \Big(\sum_{i'=1}^{i} a_{i'j'} \Big) \\
&\geq i \ - \sum_{j'=j+1}^{n} 1 = i-(n-j) = i+j-n.
\end{aligned}$$

- $c_{ij} \leq i$.

  In this case,

$$\begin{aligned}
c_{ij} &= \sum_{i'=1}^{i} \sum_{j'=1}^{j} a_{i'j'} \\
&= \sum_{i'=1}^{i} \Big(\sum_{j'=1}^{j} a_{i'j'} \Big) \leq \sum_{i'=1}^{i} 1 = i.
\end{aligned}$$

- $c_{ij} \leq j$.

  In this case,

$$\begin{aligned}
c_{ij} &= \sum_{i'=1}^{i} \sum_{j'=1}^{j} a_{i'j'} \\
&= \sum_{j'=1}^{j} \Big(\sum_{i'=1}^{i} a_{i'j'} \Big) \leq \sum_{j'=1}^{j} 1 = j.
\end{aligned}$$

$\square$

In analogy to higher spin ASMs and MTs, Behrend and Knight also introduced a generalization for CSMs in [15]. We have

Definition 1.33. For a positive integer n and a non-negative integer r, the *higher spin corner sum matrices* of order n, or HSCSMs for short, are defined by

$$CSM(n,r) :=$$
$$\left\{ C = \begin{pmatrix} c_{00} & \dots & c_{1n} \\ \vdots & & \vdots \\ c_{n0} & \dots & c_{nn} \end{pmatrix} \in \mathbb{N}^{(n+1)\times(n+1)} \;\middle|\; \begin{array}{l} \bullet\ c_{0k} = c_{k0} = 0,\ c_{kn} = c_{nk} = kr \quad \text{for all } k \in [0,n]; \\ \bullet\ 0 \leq c_{i,j} - c_{i,j-1} \leq r,\ 0 \leq c_{i,j} - c_{i-1,j} \leq r \quad \text{for all } i,j \in [n]. \end{array} \right\}. \tag{1.36}$$

where $CSM(n,r)$ denotes the set of all $n \times n$ HSCSMs.

Clearly the set $CSM(n,1)$ is just the CSMs for standard ASMs, i.e., $CSM(n,1) = CSM(n)$. The following theorem provides a bijection between $ASM(n,r)$ and $CSM(n,r)$. For the proof and more details on HSCSMs see [15] or Chapter 2 of [60].

Theorem 1.34. *For a positive integer n and a non-negative integer r, there is a bijection between $ASM(n,r)$ and $CSM(n,r)$ given by*

$$c_{ij} = \sum_{i'}^{i} \sum_{j'}^{j} a_{i'j'} \quad \text{for each} \quad i,j \in [0,n], \tag{1.37}$$

for a given $A \in ASM(n,r)$ and conversely,

$$a_{ij} = c_{ij} - c_{i,j-1} - c_{i-j,j} + c_{i-1,j-1} \quad \text{for each} \quad i,j \in [n], \tag{1.38}$$

for a given $C \in CSM(n,r)$.

Example 1.35. As an illustration, the HSCSM corresponding to the HSASM in our running Example 1.11 is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 3 & 3 & 3 & 4 \\ 0 & 2 & 2 & 3 & 4 & 5 & 6 \\ 0 & 2 & 3 & 3 & 5 & 7 & 8 \\ 0 & 2 & 4 & 5 & 6 & 8 & 10 \\ 0 & 2 & 4 & 6 & 8 & 10 & 12 \end{pmatrix} \in CSM(6,2) \tag{1.39}$$

1.2.7 ASMs and height function matrices

In this section, we discuss another aspect of ASMs, namely a variant of CSMs called *height function matrices*, or HFMs for short. HFMs were first introduced by Elkies, Kuperberg, Larsen and Propp in [47] when they were investigating the connection between ASMs and domino tilings. HFMs are constructed from CSMs and there exists a simple bijection between these sets of matrices. Hence by Theorem 1.30 they are in bijection with ASMs as well. HFMs will be important in Chapter 3 in the enumeration of higher spin ASMs.

Definition 1.36. Let $C \in CSM(n)$ be given. Then the *height function matrix*, or HFM for short, associated with C is an $(n+1) \times (n+1)$ matrix $H = (h_{i,j})$ given by

$$h_{i,j} = i + j - 2c_{i,j} \quad \text{for} \quad i, j \in [0, n] \tag{1.40}$$

Example 1.37. As an illustration, the HFMs corresponding to CSMs C_A and C_B in Example 1.31 are given by

$$H_A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 1 & 2 & 3 & 4 \\ 3 & 2 & 3 & 2 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 & 2 & 3 & 2 \\ 5 & 4 & 3 & 2 & 1 & 2 & 1 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}, \quad H_B = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 1 & 2 & 3 & 4 \\ 3 & 2 & 3 & 2 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 & 2 & 1 & 2 \\ 5 & 4 & 3 & 2 & 1 & 2 & 1 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix} \tag{1.41}$$

The set of all order n HFMs is denoted by $HFM(n)$. For instance, the set of seven HFMs of order 3 is given by

$$\text{HFM(3)}=\left\{\begin{pmatrix} 0&1&2&3\\1&0&1&2\\2&1&0&1\\3&2&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&2&3\\1&0&1&2\\2&1&2&1\\3&2&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&2&3\\1&2&1&2\\2&1&0&1\\3&2&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&2&3\\1&2&1&2\\2&1&2&1\\3&2&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&2&3\\1&2&1&2\\2&3&2&1\\3&2&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&2&3\\1&2&3&2\\2&1&2&1\\3&2&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&2&3\\1&2&3&2\\2&3&2&1\\3&2&1&0 \end{pmatrix}\right\}$$

Using Definitions 1.28 and 1.36, $HFM(n)$ can also be expressed directly as

$$HFM(n) := \left\{ H = \begin{pmatrix} h_{00} & \dots & h_{1n} \\ \vdots & & \vdots \\ h_{n0} & \dots & h_{nn} \end{pmatrix} \in \mathbb{N}^{(n+1)\times(n+1)} \;\middle|\; \begin{array}{l} \bullet\ h_{0k} = h_{k0} = h_{n,n-k} = h_{n-k,n} = k \ \text{ for all } k \in [0,n]; \\ \bullet\ h_{i,j} - h_{i,j-1} \ \text{ and } h_{i,j} - h_{i-1,j} \ \text{ are each } 1 \ \text{ or } \ -1 \ \text{ for all } i,j \in [n]. \end{array} \right\}. \tag{1.42}$$

Therefore, the characteristics of any matrix in $HFM(n)$ are that its first row and column are $(0, 1, ..., n)$, its last row and column are $(n, n-1, ..., 0)$, and any two adjacent entries differ by 1.

Height function matrices and six vertex model configurations

By Theorem 1.20, there is a bijection between ASMs and SVMs with DWBCs, and by (1.40) a bijection between ASMs and HFMs. Thus we have a bijection between HFMs and SVMs with DWBCs. To describe how this bijection works, consider an $(n+1)\times(n+1)$ grid G and a given order n HFM, H. Then take the midpoints of each face in G and connect them together along each row and column to get an $n \times n$ dual grid, namely G', inside G. Now consider G' with an additional $4n$ external vertices of degree one. We keep the orientations of external edges the same as for SVMs, that is, the arrows on the external edges on the upper, right, lower and left boundaries of the grid are all directed upward, leftward, downward and rightward, respectively. It remains to give proper orientations to the internal edges. To do so, we first place each entry h_{ij} of H in position (i,j) on G for all $i,j \in [0,n]$. Then apply the following simple rule. Assign an arrow to each internal edge of G' such that if one moves in the direction of that arrow, then the smaller and larger entries of H are on one's left and right, respectively. In total, there are six possible configurations for each internal position (i,j) in G', see Figure 1.11.

Example 1.38. As an illustration, the grid of the SVM with DWBCs corresponding to the HFM H_A given in Example 1.37 is shown in Figure 1.12a.

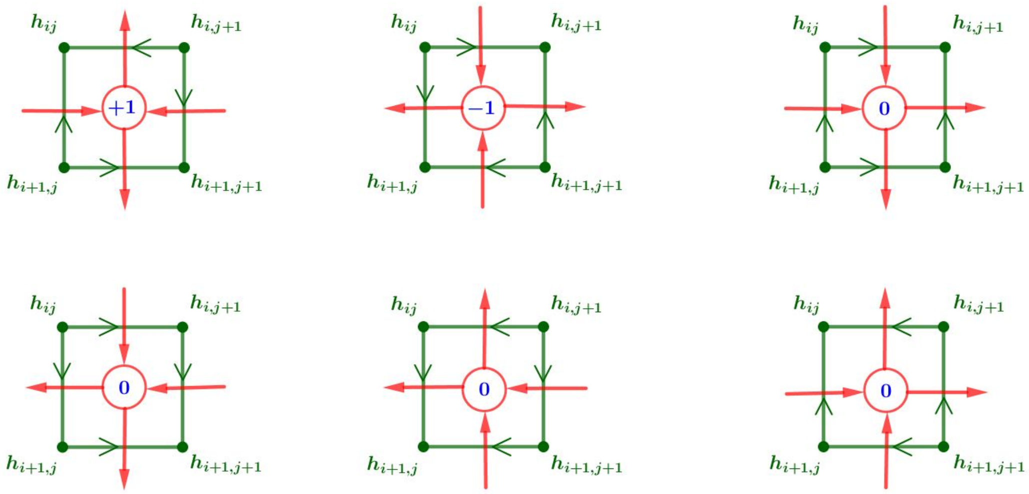


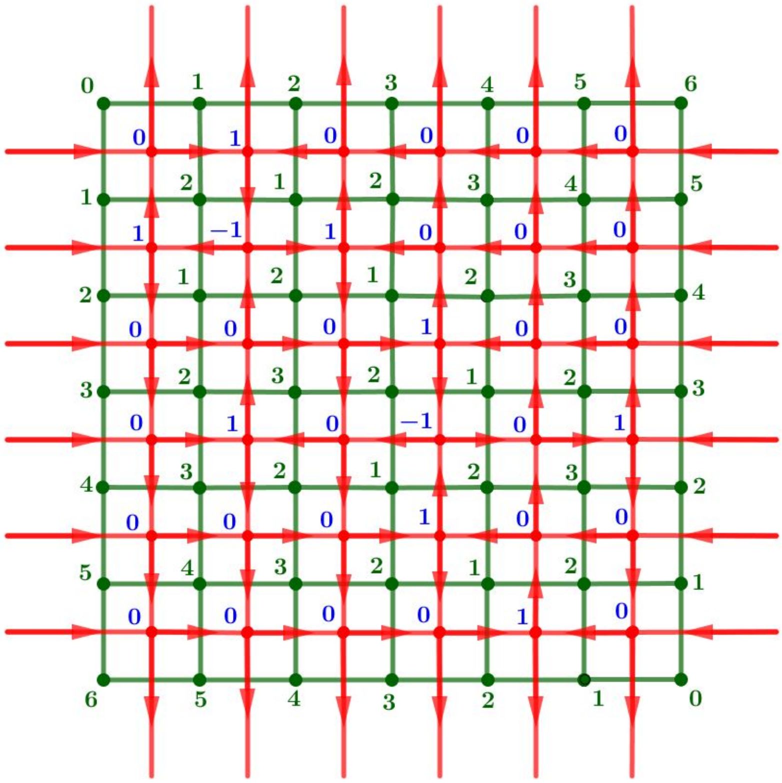
Figure 1.11: All six possible local configurations involving HFMs and SVMs. Note that the green arrows between h_{ij}'s indicate which side is smaller or bigger than the other side. For instance, in the configuration corresponding to the +1, we have $h_{ij} < h_{i,j+1}$, $h_{ij} < h_{i+1,j}$, $h_{i+1,j} > h_{i+1,j+1}$ and $h_{i,j+1} > h_{i+1,j+1}$, respectively.

Height function matrices and simple flow grids

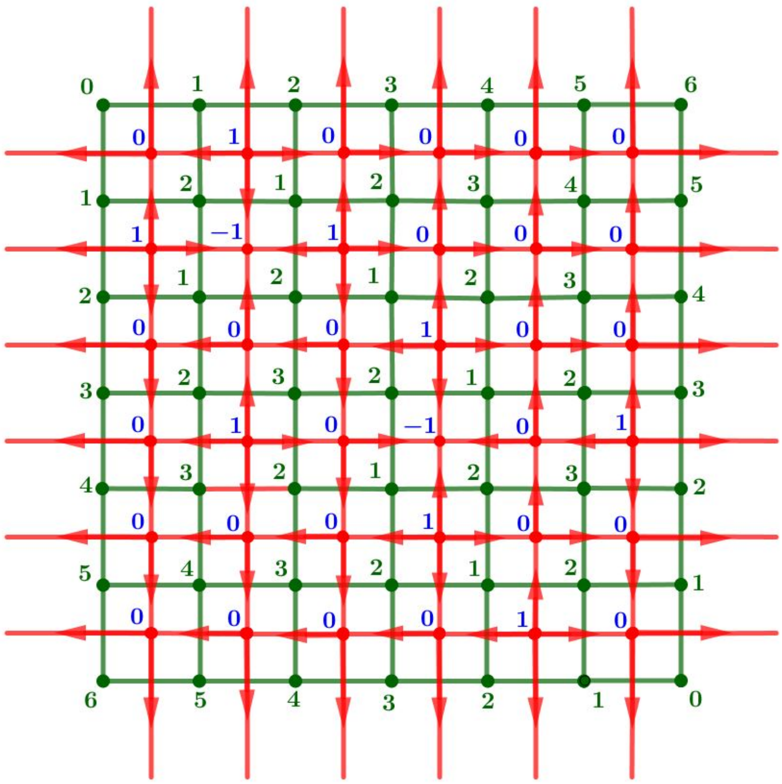
By the same argument as for the case of the bijection between HFMs and SVMs with DWBCs, Theorem 1.24 and Equation (1.40) imply a bijection between HFMs and SFGs. We note that by the argument in Remark 1.25, the steps for constructing a bijection between HFMs and SFGs is exactly the same as the one we described for HFMs and SVMs except for the horizontal orientations. In particular, by reversing the horizontal orientations in the associated SVM of a given HFM, we obtain its associated SFG.

Example 1.39. As an illustration, the grid of the SFG corresponding to the HFM H_A given in

Example 1.37 is shown in Figure 1.12b.



(a) The HFM and SVM configurations of the ASM, A, in our running Example 1.4. Note that the green grid is associated with HFM of A whereas the red one is associated with the corresponding SVM with DWBCs.



(b) The HFM and SFG configurations of the ASM, A, in our running Example 1.4. Note that the only difference between the red grid corresponding to the SFG here and the one in Figure 1.12a is the horizontal orientations along each row.

Figure 1.12: The SVM and SFG configurations associated with the HFM H_A given in Example 1.37.

1.3 Partially ordered set preliminaries

In this section we give some preliminaries on partially ordered sets, or posets for short, that are necessary for later sections and chapters. Most of the definitions and lemmas are from Chapter 3 of Stanley's book "Enumerative Combinatorics" [98], unless otherwise stated. For the sake of brevity we skip the proof of the lemmas and theorems but refer the reader to Stanley's book for the proofs. After that an order relationship between ASMs will be given via the CSMs in Section 1.4. This way we will turn ASMs into a poset. Moreover, we will prove that the poset obtained in this way is in fact a distributive lattice equipped with join and meet operations satisfying distributive laws, see Lemma 1.76. In Section 1.5, we will introduce a poset whose lattice of order ideals is anti-isomorphic to the lattice of ASMs.

Definition 1.40. A *partially ordered set* P, or a *poset* for short, is a set equipped with a binary relation denoted by $\leq_P$, or simply by $\leq$ when there is no possibility of ambiguity, satisfying the following axioms:

- *Reflexive property*: For all $x \in P$, $x \leq x$,
- *Antisymmetric property*: If $x \leq y$ and $y \leq x$ then $x = y$,
- *Transitive property*: If $x \leq y$ and $y \leq z$ then $x \leq z$.

We say two elements $x, y \in P$ are *comparable* if either $x \leq y$ or $y \leq x$; otherwise we call them *incomparable*. We use the notation $x < y$ to mean that $x \leq y$ and $x \neq y$.

Definition 1.41. Let P be a poset and $x, y \in P$. Then we say y *covers* x or x is *covered* by y, denoted by $x \lessdot y$ or $y \gtrdot x$, if $x \leq y$ and there does not exist an element $v \in P$ such that $x < v < y$.

As an illustration, for the poset given in Figure 1.13, we have $j \lessdot m$, $k \lessdot m$ and $l \lessdot m$.

Definition 1.42. For a finite poset P, the *Hasse diagram* is defined to be the graph whose vertices are the elements of P and edges are the cover relations in P. Also, if $x \leq y$ then y is drawn above x.

See Figure 1.13 for an example.

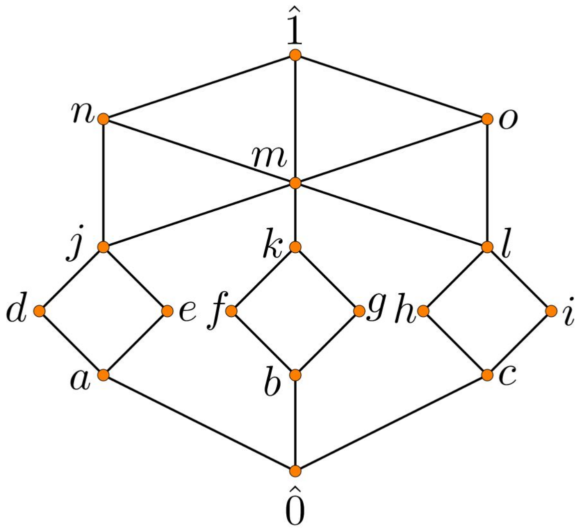


Figure 1.13: A poset P with $\hat{0}$ and $\hat{1}$ elements.

Definition 1.43. A *weak subposet* of a poset P is a poset Q such that $Q \subseteq P$ and if $s \leq_Q t$, then $s \leq_P t$. Moreover, if $Q = P$, then P is called a *refinement* of Q. We say that Q is an *induced subposet* (or simply subposet) of P, if $Q \subseteq P$ and there exists a partial ordering of Q such that for $s,t \in Q$ we have $s \leq_Q t$ if and only if $s \leq_P t$. If this holds we say that the subset Q of P has an *induced order*.

Definition 1.44. An element t in a poset P is called *minimal* if there is no element $s \in P$ such that $s < t$. Similarly, u is said to be a *maximal* element of P if there is no element $v \in P$ such that $u < v$. If there exists an element $\hat{0} \in P$ such that $\hat{0} \leq x$ for any $x \in P$, then we say P has a *minimum* element $\hat{0}$. Similarly, if there exists an element $\hat{1} \in P$ such that $x \leq \hat{1}$ for any $x \in P$, then we say P has a *maximum* element $\hat{1}$.

We note that if s and t are distinct minimal (maximal) elements of P, then s and t are incomparable.

Definition 1.45. A *chain* or a *totally ordered set* in a poset P is a subposet C in which every two elements are comparable. The *length* of a finite chain in P is defined by $l(C) = |C| - 1$. A chain C is called *maximal* if it is not contained in any larger chain of P. A *multichain* of a poset P is a chain with repeated elements, that is, a multiset whose underlying set is a chain of P.

For instance, in the poset shown in Figure 1.13, the chain $C' : \hat{0} \leq b \leq g \leq k \leq m$ is not maximal since it is contained in a maximal chain $C : \hat{0} \leq b \leq g \leq k \leq m \leq \hat{1}$.

Definition 1.46. The dual notion of a chain in a poset P is the notion of an *antichain* (or *Sperner family* or *clutter*). An *antichain* is a subset C of P where elements are not comparable pairwise. The set of all antichains of P is denoted by $Ant(P)$.

For instance, in Figure 1.13, the subset $A = \{d, e, f, g, h, i\}$ is an antichain.

Lemma 1.47. *A maximal chain in a finite non-empty poset P contains a maximal element (and a minimal element) of P.*

Definition 1.48. An *order ideal* of a finite poset P, also known as a *down-set*, *semi-ideal* or *decreasing subset*, is a subset I of P such that if $x \in I$ and $y \leq x$ then $y \in I$. The set of all order ideals of P is denoted by $J(P)$. Similarly, we can define the dual notion of an order ideal known as a *filter*, also known as a *dual order ideal*, *up-set* or *increasing subset*, as a subset F of P such that if $x \in F$ and $y \geq x$ then $y \in F$. The set of all filters of P is denoted by $F(P)$. Note that the sets $J(P)$ and $F(P)$ each form posets, in which the order relation is set inclusion.

Definition 1.49. Let P be a poset and $X \subseteq P$. Then the order ideal generated by X in P is the set

$$< X > := \{s \in P \,|\, s \leq t \text{ for some } t \in X\}$$

In case $X = \{x\}$ for some $x \in P$, then $< X >$ is called the *principal order ideal* generated by x and is denoted by Λ_x.

Definition 1.50. A *rank function* of a finite poset P is a function $\rho : P \to [0, n]$ for some n satisfying the following conditions

- ρ is zero on all minimal elements of P, that is, $\rho(s) = 0$ if s is a minimal element of P;
- ρ preserves covering relations, that is, $\rho(t) = \rho(s) + 1$ whenever s is covered by t in P;

We say an element $s \in P$ has rank i if $\rho(s) = i$.

Note that the rank function is unique, if it exists.

Definition 1.51. A finite poset P is called *graded* of rank k if every maximal chain in P has the same length k. Equivalently, we say P is graded if it admits a certain partition into antichains $\{A_i : i \in \mathbb{N}\}$ such that for each $t \in A_i$, all elements covering t belong to A_{i+1} and all elements covered by t are in A_{i-1}. If P is graded of rank k and has r_i elements of rank i, then *the rank generating function* of P is the polynomial $F_P(x)$ given by

$$F_P(x) := \sum_{i=0}^{k} r_i x^i \tag{1.43}$$

For instance, the poset P shown in Figure 1.13 is graded and its corresponding rank generating function is

$$F_P(x) := 1 + 3x + 6x^2 + 3x^3 + x^4 + 2x^5 + x^6 \tag{1.44}$$

Definition 1.52. Let arbitrary elements s, t in poset P be given. Then we say s and t have an *upper bound* if there exists an element $u \in P$ such that $s \leq u$ and $t \leq u$. We say u is the *least upper bound* or *join* or *supremum* of s and t if for any upper bound v of s and t, we have $u \leq v$. The join of s and t is unique if it exists and is denoted by $s \vee t$. Analogously, we can define the *greatest lower bound* or *meet* or *infimum* of s and t. We say u' is a meet of s and t and denote this as $s \wedge t$ if for any lower bound v' of s and t, we have $u' \geq v'$.

Definition 1.53. An element s of a finite poset is called *join-irreducible* if it covers exactly one element.

Definition 1.54. A *lattice* is a poset L for which every pair of elements has a join and a meet. A lattice L is called *distributive* if it satisfies the distributive laws, that is, for any $s, t, u \in L$ we have

$$\begin{aligned} s \vee (t \wedge u) &= (s \vee t) \wedge (s \vee u) \\ s \wedge (t \vee u) &= (s \wedge t) \vee (s \wedge u) \end{aligned} \tag{1.45}$$

It can be shown that the poset $J(P)$ of order ideals of a given poset P forms a distributive lattice, in which the join and meet are the set union and intersection, respectively.

Definition 1.55. Two posets P and Q are said to be isomorphic, denoted by $P \cong Q$, if there exists a bijection $\phi : P \to Q$ for which $s \leq t$ in P if and only if $\phi(s) \leq \phi(t)$ in Q. Such a bijection ϕ is called an *isomorphism*. Alternatively, if there exists a bijection $\phi : P \to Q$ for which $s \leq t$ in P if and only if $\phi(s) \geq \phi(t)$ in Q, then P and Q are said to be *anti-isomorphic* and ϕ is called *anti-isomorphism*.

The following theorem is known as *the fundamental theorem for finite distributive lattices*. It shows how to obtain a building block poset of a given distributive lattice via the join-irreducible elements.

Theorem 1.56. *Let L be a finite distributive lattice. Then there is a unique (up to isomorphism) finite poset P such that $L \cong J(P)$, where P is the subposet of join irreducibles of L, and $J(P)$ is the lattice of all order ideals of P.*

Definition 1.57. Let P be a poset with p elements. Then a *linear extension* or *topological sorting* of P is an order preserving bijection $\phi : P \to [p]$ where the order relation on $[p] = \{1, 2, ..., p\}$ is the usual numerical order and "'order preserving" means that $s < t$ in P implies $\phi(s) < \phi(t)$. The number of linear extensions of P is denoted by $e(P)$. It can be shown that $e(P)$ is also the number of maximal chains in $J(P)$.

Definition 1.58. Let n be a non-negative integer and let P be a given finite poset. Then the *order polynomial* of P denoted by $\Omega(P, n)$ is defined to be the number of order preserving maps $\phi : P \to [n]$, that is, the number of maps $\phi : P \to [n]$ such that $s < t$ implies that $\phi(s) \leq \phi(t)$.

It turns out that $\Omega(P, n)$ is a polynomial in n of degree $|P|$ and leading coefficient $e(P)/|P|!$.

Remark 1.59. There is a simple connection between *order preserving maps* and *order reversing maps* of a given finite poset P. Let $\sigma : P \to [n]$ be an order preserving map. Then one can easily extend this to an order reversing map $\sigma : P \to [n]$ by replacing $\sigma(t)$ by $n + 1 - \sigma(t)$.

Lemma 1.60. *Let P be a finite poset and $m \in \mathbb{N}$. Then the following quantities are equal:*

(I) The number of order-preserving maps $\phi : P \to [m]$, that is, $\Omega(P, m)$.

(II) The number of multichains $\hat{0} = I_1 \leq I_2 \leq ... \leq I_m = \hat{1}$ of length m in $J(P)$ (where $\hat{0} = \varnothing$ and $\hat{1} = P$ denote the minimal and maximal element of $J(P)$, respectively).

Lemma 1.61. *Let P be a finite poset and $m \in \mathbb{N}$. Then the following quantities are equal:*

(I) The number of surjective order-preserving maps $\sigma : P \to [m]$.

(II) The number of chains $\hat{0} = I_0 < I_1 < ... < I_m = \hat{1}$ of length m in $J(P)$.

Definition 1.62. For a given poset P, an *automorphism* of P is a bijection $\phi : P \to P$ such that $\phi(s) \leq \phi(t)$ if and only if $s \leq t$ in P. Analogously, an *anti-automorphism* is a bijection $\phi : P \to P$ such that $\phi(s) \geq \phi(t)$ if and only if $s \leq t$ in P.

1.4 The alternating sign matrix lattice

In this section, Theorem 1.30 enables us to define an order relation between ASMs by comparing their associated CSMs entry-wise. We will show that the poset defined this way is a graded poset but also is a distributive lattice, see Lemma 1.76.

Definition 1.63. Let $A, B \in ASM(n)$. Then we say that $A \leq B$ if $C_A \geq C_B$ entry-wise, where C_A and C_B are the CSMs corresponding to A and B, respectively. In other words, $A \leq B$ whenever c_{ij} and c'_{ij} in C_A and C_B such that $c_{ij} \geq c'_{ij}$ for all $i, j \in [0, n]$, where $C_A = (c_{ij})$ and $C_B = (c'_{ij})$.

Example 1.64. For instance, for the ASMs given in our running Example 1.4, we have $B \leq A$ since $C_B \geq C_A$. In particular, they are distinguished from each other by red in (1.46).

$$C_A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 2 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 & 3 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \leq \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 2 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 & 3 & 4 & 4 \\ 0 & 1 & 2 & 3 & 4 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = C_B \tag{1.46}$$

In the following lemma we note that $ASM(n)$ with the order relation defined above is in fact a poset. The proof is straightforward.

Lemma 1.65. *$(ASM(n), \leq)$ is a poset where $\leq$ is the order relation of Definition 1.63.*

The following result can also be proved straightforwardly.

Lemma 1.66. *Consider the ASM poset $(ASM(n), \leq)$ and $A, B \in ASM(n)$. Then $A \lessdot B$ if and only if there exist unique $i, j \in [0, n]$ such that $c_{ij} = c'_{ij} + 1$, where (c_{ij}) and (c'_{ij}) are the CSMs corresponding to A and B, respectively.*

Definition 1.67. Let $A \in ASM(n)$ and $C \in CSM(n)$ be its corresponding CSM. Then define $\nu(A)$ to be the sum of entries of C, that is $\nu(A) = \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij}$.

As an example, it can be shown that

$$\nu(I_n) = \frac{n(n+1)(2n+1)}{6} \tag{1.47}$$

and

$$\nu(I'_n) = \binom{n+2}{3} \tag{1.48}$$

where I_n is the $n \times n$ identity matrix and I'_n is the $n \times n$ reflected identity matrix.

Definition 1.68. The *rank* of an $n \times n$ ASM A is defined as

$$rank(A) = \nu(I_n) - \nu(A) \tag{1.49}$$

As an illustration, it is clear that $rank(I_n) = 0$ and $rank(I'_n) = \binom{n+1}{3}$.

Lemma 1.69. *The $n \times n$ identity matrix, I_n, and $n \times n$ reflected identity matrix, I'_n, are the minimum and maximum element within the poset $(ASM(n), \leq)$, respectively.*

Proof.
By (1.31), it can be checked that the CSMs associated with the identity matrix, I_n, and the reflected identity matrix, I'_n, have entries $(C_{I_n})_{ij} = \min(i,j)$ and $(C_{I'_n})_{ij} = \max(0, i+j-n)$, for $i,j \in [0,n]$, respectively. In particular, they have the following forms

$$C_{I_n} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \cdots & 2 & 2 & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & \cdots & n-3 & n-2 & n-2 & n-2 \\ 0 & 1 & 2 & \cdots & n-3 & n-2 & n-1 & n-1 \\ 0 & 1 & 2 & \cdots & n-3 & n-2 & n-1 & n \end{pmatrix}, \quad C_{I'_n} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-5 & n-4 & n-3 & n-2 \\ 0 & 0 & 1 & \cdots & n-4 & n-3 & n-2 & n-1 \\ 0 & 1 & 2 & \cdots & n-3 & n-2 & n-1 & n \end{pmatrix} \tag{1.50}$$

Now it follows from (1.35) that, for any $n \times n$ ASM A, we have $C_{I_n} \geq C_A \geq C_{I'_n}$, entry-wise. Therefore, by Definition 1.63, we have

$$I_n \leq A \leq I'_n \tag{1.51}$$

Thus I_n is the unique minimum element and I'_n is the unique maximum element within the poset $(ASM(n), \leq)$, as required. □

It can now be shown straightforwardly that $(ASM(n), \leq)$ is graded.

Lemma 1.70. *The poset $(ASM(n), \leq)$ is graded of rank $\binom{n+1}{3}$, and the rank of $A \in ASM(n)$ is given by (1.49).*

By considering join and meet operations within $(ASM(n), \leq)$, we will now see that it is in fact a distributive lattice.

Lemma 1.71. *The poset $(ASM(n), \leq)$ is a distributive lattice. The join $A \vee B$ and meet $A \wedge B$ of any $A, B \in ASM(n)$ are given by $(C_{A \vee B})_{ij} = min\left((C_A)_{ij}, (C_B)_{ij}\right)$ and $(C_{A \wedge B})_{ij} = max\left((C_A)_{ij}, (C_B)_{ij}\right)$ for all $i, j \in [0,n]$, where C_A, C_B, $C_{A \vee B}$ and $C_{A \wedge B}$ are the CSMs corresponding to A, B, $A \vee B$ and $A \wedge B$, respectively.*

Proof.
As indicated in Definition 1.54, it needs to be checked that every pair of elements of $ASM(n)$ have a join and a meet and that these satisfy the distributive laws (1.45). This can be done straightforwardly using elementary properties of inequalities. □

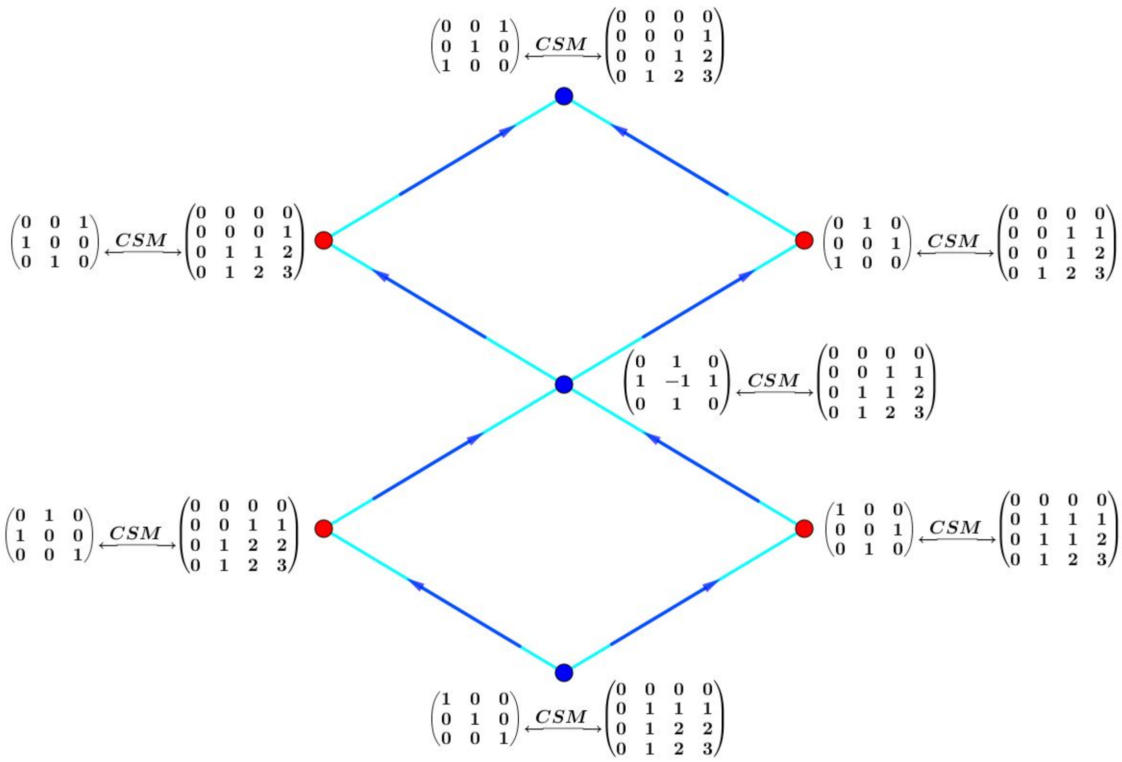


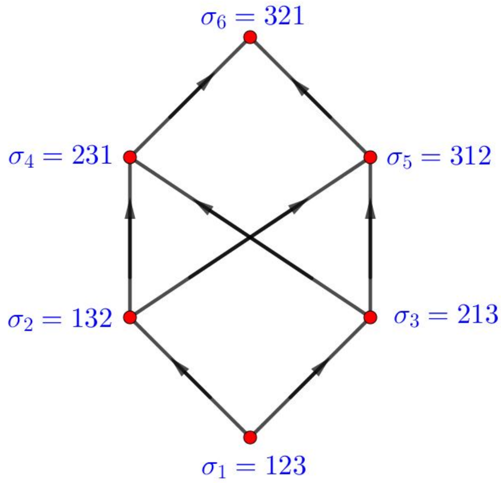
Figure 1.14: The distributive lattice $(ASM(3), \leq)$.

The MacNeille completion of the strong Bruhat order

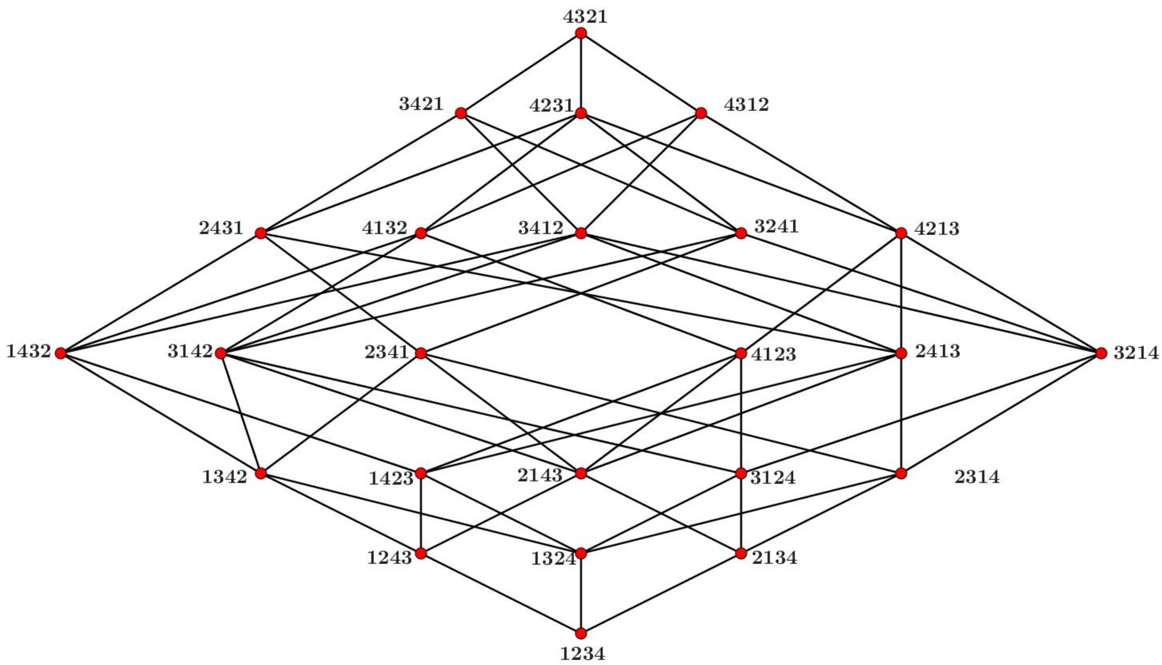
Another way to build up an order relation on the set of $n \times n$ ASMs is via their bijection with MTs with n rows, see Lemma 1.13. There are two ways to do this: The first approach is similar to the one we took to define an order relation between ASMs via their bijection with CSMs. In this approach, we define order relations between ASMs via their bijection with MTs by comparing MTs entry-wise. The second approach is based on an observation made by Lascoux and Schützenberger [69]. They noticed that the MacNeille completion (the unique smallest lattice containing a poset) of the strong Bruhat order on permutations on n elements, denoted by S_n, is isomorphic to the lattice of MTs (and hence the lattice of ASMs). The rest of this section is dedicated to describing this connection. In fact, the *Bruhat order* (also known as *Bruhat-Chevalley order* or *strong Bruhat order*) in a broader sense can be defined on elements of a given *Coxeter group* (where S_n is a Coxeter group with respect to the generating set of adjacent transpositions $s_i = (i, i+1)$). For more details the reader is encouraged to see Chapters 1 and 2 of [19].

Definition 1.72. Strong Bruhat order on permutations. Consider the set of all permutations on n elements, denoted by S_n. Let permutations $\pi_1, \pi_2 \in S_n$ be given. Then the *strong Bruhat order* of S_n, denoted by Br_n, is a partial order on S_n in which $\pi_1 \lessdot \pi_2$ if and only if π_2 can be obtained from π_1 by a single transposition and $\mathcal{I}(\pi_2) = 1 + \mathcal{I}(\pi_1)$, where $\mathcal{I}(\pi_1)$ and $\mathcal{I}(\pi_2)$ are the inversion numbers of π_1 and π_2.

As an illustration, the Hasse diagrams of the strong Bruhat order of sizes 3 and 4 are shown in Figures 1.15a and 1.15b, respectively.



(a) The Strong Bruhat order Br_3 associated with S_3.



(b) The Strong Bruhat order Br_4 associated with S_4.

Figure 1.15: The Hasse diagram of the strong Bruhat order of S_3 and S_4.

The lattice of MTs

Similar to the case for ASMs, we can define the order relation between MTs by comparing them entry-wise. Let $T, T' \in \mathrm{MT}(n)$, then we say $T \leq T'$ if and only if $t_{ij} < t'_{ij}$ for all i, j. We then find that T' covers T if and only if there is a unique position (i, j) such that $t_{ij} < t'_{ij}$ for $1 \leq j \leq i \leq n$. For instance, for MTs T_A and T_B given in Section 1.2.3, we have $T_B \lessdot T_A$ since they only differ in position $(4, 4)$ shown by red in (1.52). Since the set of order n permutation matrices is a subset of ASMs, one can consider their MT representations via the bijection given in Lemma 1.13.

$$
\mathrm{T}_A = \begin{array}{ccccccccccc}
 & & & & & 2 & & & & & \\
 & & & & 1 & & 3 & & & & \\
 & & & 1 & & 3 & & 4 & & & \\
 & & 1 & & 2 & & 3 & & {\color{red}6} & & \\
 & 1 & & 2 & & 3 & & 4 & & 6 & \\
1 & & 2 & & 3 & & 4 & & 5 & & 6
\end{array}
\quad > \quad
\begin{array}{ccccccccccc}
 & & & & & 2 & & & & & \\
 & & & & 1 & & 3 & & & & \\
 & & & 1 & & 3 & & 4 & & & \\
 & & 1 & & 2 & & 3 & & {\color{red}5} & & \\
 & 1 & & 2 & & 3 & & 4 & & 6 & \\
1 & & 2 & & 3 & & 4 & & 5 & & 6
\end{array} = T_B \tag{1.52}
$$

Also note that it is possible to have two MTs incomparable, for example, the MTs given in (1.53) are not comparable since $t_{11} = 2 > 1 = t'_{11}$ but $t_{22} = 2 < 3 = t'_{22}$, shown by red colors in (1.53).

$$
T = \begin{array}{ccccc}
 & & {\color{red}2} & & \\
 & 1 & & {\color{red}2} & \\
1 & & 2 & & 3
\end{array} \quad , \quad
T' = \begin{array}{ccccc}
 & & {\color{red}1} & & \\
 & 1 & & {\color{red}3} & \\
1 & & 2 & & 3
\end{array} \tag{1.53}
$$

Definition 1.73. Let $T \in \mathrm{MT}(n)$ be given. Then define $\nu'(T)$ to be the sum of entries of T, that is $\nu'(T) = \sum_{i=1}^{n} \sum_{j=1}^{i} t_{ij}$.

As an illustration, consider the element T_{min} associated with identity permutation given by

$$
T_{min} = \begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 2 & & & \\
 & & 1 & & 2 & & 3 & & \\
 & \iddots & & \iddots & & \ddots & & \ddots & \\
1 & & 2 & & \cdots & & n-1 & & n
\end{array} \tag{1.54}
$$

Then it can be shown that $\nu'(T_{min})$ is equal to $\binom{n+2}{3}$. Similarly, let T_{max} be the MT associated with the anti-identity permutation given by

$$
T_{max} = \begin{array}{ccccccccc}
 & & & & n & & & & \\
 & & & n-1 & & n & & & \\
 & & n-2 & & n-1 & & n & & \\
 & \iddots & & \iddots & & \ddots & & \ddots & \\
1 & & 2 & & \cdots & & n-1 & & n
\end{array} \tag{1.55}
$$

Then it can also be easily checked that $\nu'(T_{max})$ is equal to $\frac{n(n+1)(2n+1)}{6}$.

Definition 1.74. The *rank* of an order n MT T is defined as

$$
rank(T) = \nu'(T) - \nu'(T_{min}) \tag{1.56}
$$

As an illustration, it is clear that $rank(T_{min}) = 0$ and $rank(T_{max}) = \binom{n+1}{3}$.
Similar to the case for the poset $(ASM(n), \leq)$, it can now be shown straightforwardly that $(\mathrm{MT}(n), \leq)$ is graded.

Lemma 1.75. *The poset* $(\mathrm{MT}(n), \leq)$ *is graded of rank* $\binom{n+1}{3}$*, and the rank of* $T \in \mathrm{MT}(n)$ *is given by* (1.56).

Similar to the case for the poset $(ASM(n), \leq)$, by considering join and meet operations within $(\mathrm{MT}(n), \leq)$, we will now see that it is in fact a distributive lattice.

Lemma 1.76. *The poset* $(\mathrm{MT}(n), \leq)$ *is a distributive lattice. The join* $T \vee T'$ *and meet* $T \wedge T'$ *of any* $T, T' \in \mathrm{MT}(n)$ *are given by* $T''_{T\vee T'} = max\left(t_{ij}, t'_{ij}\right)$ *and* $T''_{T\wedge T'} = min\left(t_{ij}, t'_{ij}\right)$ *for all* i, j*, where* $T''_{T\vee T'}$ *and* $T''_{T\wedge T'}$*, are the MTs corresponding to* $T \vee T'$ *and* $T \wedge T'$*, respectively.*

Proof.
As indicated in Definition 1.54, it needs to be checked that every pair of elements of $\mathrm{MT}(n)$ have a join and a meet and that these satisfy the distributive laws (1.45). This can be done straightforwardly using elementary properties of inequalities. □

As an illustration, the Hasse diagram of the distributive lattice $(\mathrm{MT}(3), \leq)$ is shown in Figure 1.16.

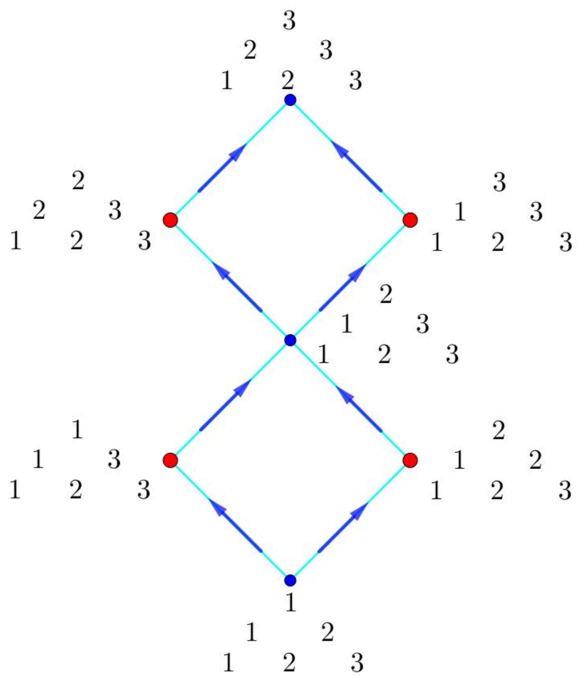


Figure 1.16: The distributive lattice of $(\mathrm{MT}(3), \leq)$.

Now if we restrict the ordering defined for $\mathrm{MT}(n)$ to S_n with each permutation realized with its corresponding MT, then we obtain the strong Bruhat order on S_n. The first person who noticed the connection between S_n and their corresponding MTs through Bruhat ordering under entry-wise comparison was Ehresmann [45]. We also note that the minimum (equivalently maximum) of a pair of elements in S_n is not necessarily a member of S_n, when they are considered as elements of $\mathrm{MT}(n)$. For instance, in Figure 1.16, it is easy to check that the meet of the following MTs in (1.57) is not a member of S_n. Moreover, it is not a MT that is arising from any permutation in S_3 whereas its building blocks are actually permutations 231 and 312, respectively. Also note that as members of the strong Bruhat order, they are not comparable. Therefore, in general, Br_n does not inherit the distributive lattice property of $(\mathrm{MT}(n), \leq)$ as subposet.

$$min\left(\begin{array}{ccccc} & & 2 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 3 \end{array}, \begin{array}{ccccc} & & 3 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array}\right) = \begin{array}{ccccc} & & 2 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array} \tag{1.57}$$

Before stating the connection between between $(\mathrm{MT}(n), \leq)$ and Br_n we briefly discuss how to obtain the MacNeille completion of a given finite poset with a maximum element. For more details and references see [97] and [98].

If every pair of elements of a poset P has a meet, then we call P a *meet-semilattice*. Moreover, it is known that if P is a finite meet-semilattice with maximum element $\hat{1}$, then P is a lattice (For the proof, see Proposition 3.3.1 in [98]).

Now let P be a finite poset with maximum element $\hat{1}$. Then the *MacNeille completion* of P is denoted by $L(P)$ and can be obtained as the meet semilattice of 2^P (boolean algebra of all subsets of P ordered by inclusion) that is generated by the principal order ideals of P.

In particular, to obtain the MacNeille completion of P, we begin with finding all the principal order ideals of P. Then we apply the meet operation (which is set intersection) on all possible pairs of principal order ideals of P (where principal order ideal is defined in Definition 1.49). Finally, the collection of all such sets ordered by inclusion will be our required $L(P)$. As an illustration, consider the poset P shown in Figure 1.17.

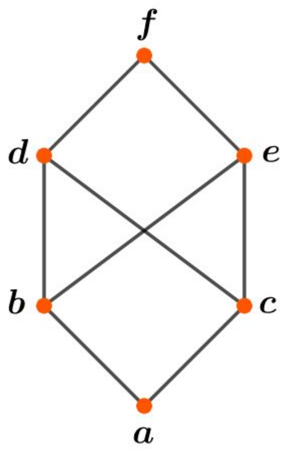


Figure 1.17: The poset P.

The principal order ideals of P are given by

$$\begin{aligned}
\Lambda_a &= \{a\} \\
\Lambda_b &= \{a, b\} \\
\Lambda_c &= \{a, c\} \\
\Lambda_d &= \{a, b, c, d\} \\
\Lambda_e &= \{a, b, c, e\} \\
\Lambda_f &= \{a, b, c, d, e, f\}
\end{aligned} \tag{1.58}$$

Now it is straightforward to check that the meet (which is intersection) of every pair of elements in (1.58) produce existing elements together with the new element

$$\Lambda_d \wedge \Lambda_e = \{a, b, c\}$$

Now as prescribed, the MacNeille completion of P is the set

$$L(P) = \{\{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,c,d\}, \{a,b,c,e\}, \{a,b,c,d,e,f\}\}$$

ordered by inclusion, see Figure 1.18.

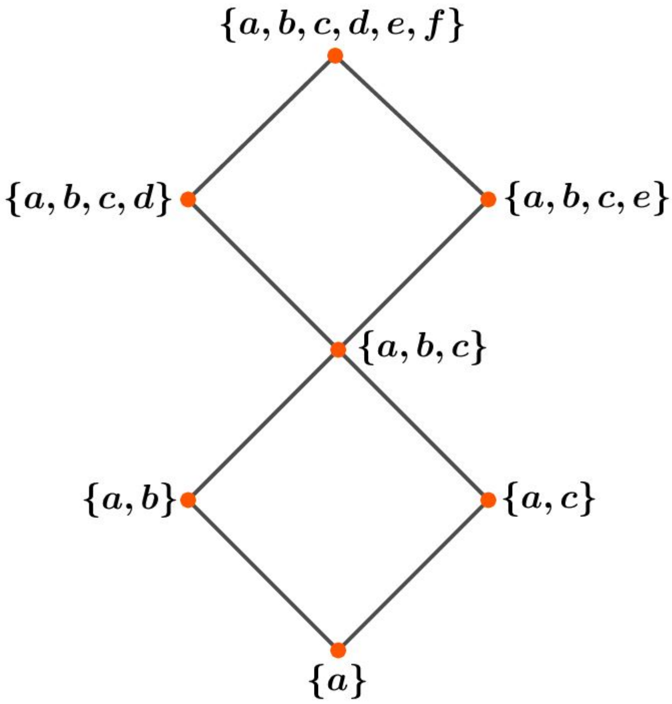


Figure 1.18: The MacNeille completion $L(P)$ associated with the poset P shown in Figure 1.17.

Note that in this example, P is isomorphic to Br_3 and $L(P)$ is isomorphic to $(\mathrm{MT}(3), \leq)$.
Now Theorem 1.77 gives the connection between $(\mathrm{MT}(n), \leq)$ and Br_n which was proved by Lascoux and Schützenberger in 1996 [69].

Theorem 1.77. *The lattice* $(\mathrm{MT}(n), \leq)$ *is isomorphic to the MacNeille completion of the strong Bruhat poset of* S_n.

As an illustration, a poset which is isomorphic to the MacNeille completion of the strong Bruhat order, Br_4, is shown in Figure 1.19. In this Figure, blue vertices depict all 24 permutations in S_4 and the red vertices depict 18 elements that are not permutations but members of $ASM(4)$.
Finally, considering the bijection between ASMs and MTs, Lemma 1.13, and the bijection between ASMs and CSMs, Theorem 1.30, it can be shown that the lattice $(ASM(n), \leq)$ and the lattice $(\mathrm{MT}(n), \leq)$ are isomorphic.

Lemma 1.78. *For a given integer n,* $(ASM(n), \leq)$ *is isomorphic to* $(\mathrm{MT}(n), \leq)$.

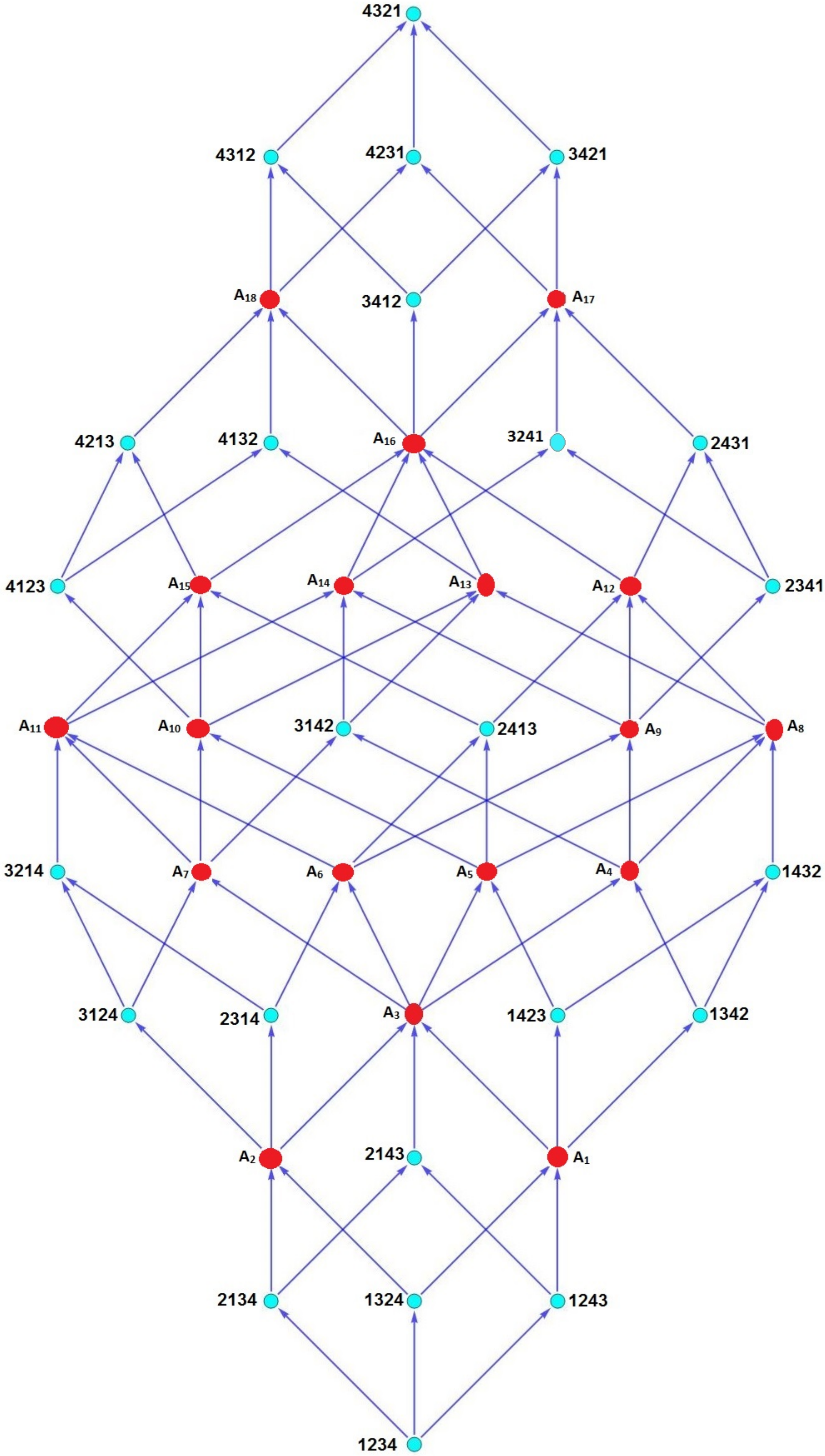


Figure 1.19: A poset which is isomorphic to the MacNeille completion of the strong Bruhat order of S_4. The permutations are shown by blue and the non-permutation MTs are shown in red.

1.5 Alternating sign matrix partially ordered set

In this section, we introduce the *alternating sign matrix partially ordered set*, or *ASM poset* for short, denoted by P_n. The name "ASM poset" comes from the fact that there is an anti-isomorphism between the lattice of order ideals of P_n and the lattice of all ASMs of order n, see Theorem 1.83. The ASM poset P_n is, in fact, the building block of Chapters 2 and 3. In Chapter 2, Section 2.5, we will introduce the ASM order polytope associated with P_n. Furthermore, in Chapter 3, Section 3.2.1, we will introduce some reduction operations and a labelling on P_n which will enable us to enumerate HSASMs.

Definition 1.79. The ASM poset, P_n, for $n \geq 2$ is the set

$$P_n := \{(i,j,k) \mid i,j = 1,\dots,n-1\,;\, k = |n-i-j|, |n-i-j|+2,\dots,n-2-|i-j|\} \tag{1.59}$$

with the order relation given by

$$(i_1,j_1,k_1) \leqslant (i_2,j_2,k_2) \Leftrightarrow k_2-k_1-|i_2-i_1|-|j_2-j_1| \in \{0,2,4,\dots\} \tag{1.60}$$

Furthermore, let $(i_1,j_1,k_1),(i_2,j_2,k_2) \in P_n$ be two arbitrary elements. Then the cover relation in P_n is given by

$$(i_1,j_1,k_1) \lessdot (i_2,j_2,k_2) \Leftrightarrow |i_2-i_1|+|j_2-j_1| = k_2-k_1 = 1 \tag{1.61}$$

Example 1.80. As an illustration, the elements of P_3 and P_4 are given by

$$\begin{aligned} P_3 &= \{(1,1,1),(1,2,0),(2,1,0),(2,2,1)\} \\ P_4 &= \{(1,1,2),(1,2,1),(1,3,0),(2,1,1),(2,2,0),(2,2,2),(2,3,1),(3,1,0),(3,2,1),(3,3,2)\} \end{aligned} \tag{1.62}$$

In general, there are three different ways to represent the Hasse diagram of the ASM poset P_n called the *rectangular representation*, *tetrahedral representation* and the *pyramidal representation*. In what follows, we describe the construction of P_n. As an illustration, all rectangular, tetrahedral and pyramidal representations of P_3, P_4 and P_5 are shown in Figures 1.21, 1.22 and 1.23, respectively.

Definition 1.81. The *central elements* in P_n are the ones with $i,j \in [2,n-2]$. It can be shown that these are precisely the elements in P_n with either highest in-degree or out-degree in the Hasse diagram of P_n. We denote the set of all central elements of P_n by $Cent(P_n)$.

For instance, for the simplest case P_4, $Cent(P_4) = \{(2,2,0),(2,2,2)\}$ and for the ASM poset P_5, we have

$$Cent(P_5) = \{(2,2,1),(2,2,3),(2,3,0),(2,3,2),(3,2,0),(3,2,2),(3,3,1),(3,3,3)\} \tag{1.63}$$

As an illustration, the central elements of the ASM posets P_4 and P_5 are shown in red in Figures 1.22 and 1.23.

The following lemma provides some basic properties of P_n. The proof is straightforward and follows from results in [47].

Lemma 1.82. *Let $n \geq 2$, then the following statements hold for P_n:*

(I) *Let* $Min(P_n)$ *denote the set of minimal elements of* P_n. *Then,* $Min(P_n) = \{(i, n-i, 0) \,|\, i = 1, 2, ..., n-1\}$ *and* $|Min(P_n)| = n - 1$*;*

(II) *Let* $Max(P_n)$ *denote the set of maximal elements of* P_n. *Then,* $Max(P_n) = \{(i, i, n-2) \,|\, i = 1, 2, ..., n-1\}$*, and* $|Max(P_n)| = n - 1$*;*

(III) P_n *is a graded poset of rank* $n - 2$ *with* $rank(i, j, k) = k$ *for any* $(i, j, k) \in P_n$*;*

(IV) *The Hasse diagram of* P_n *has a tetrahedral form;*

(V) $|P_n| = \binom{n+1}{3}$*;*

(VI) *The number of cover relations in* P_n *is equal to* $4\binom{n}{3}$.

1.5.1 The lattice of order ideals of P_n

In this section, we will discuss the set of order ideals of the ASM poset P_n and we will show that they are in bijection with $n \times n$ ASMs.

Let $J(P_n)$ denote the set of all order ideals of P_n. As indicated after Definition 1.54 $(J(P_n), \subseteq)$ forms a distributive lattice. As an example, the set of order ideals of P_3 is given by

$$J(P_3) = \{\{\}, \{(2,1,0)\}, \{(1,2,0)\}, \{(1,2,0),(2,1,0)\}, \{(1,2,0),(2,1,0),(2,2,1)\},$$
$$\{(1,1,1),(1,2,0),(2,1,0)\}, \{(1,1,1),(1,2,0),(2,1,0),(2,2,1)\}$$

and the Hasse diagram of $J(P_3)$ is shown in Figure 1.20.

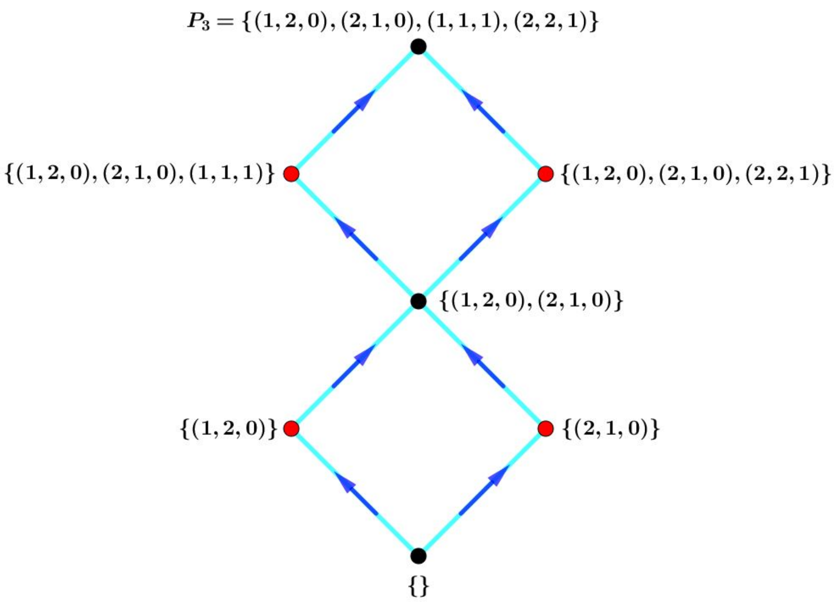
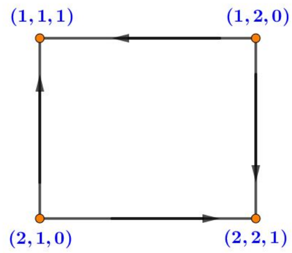
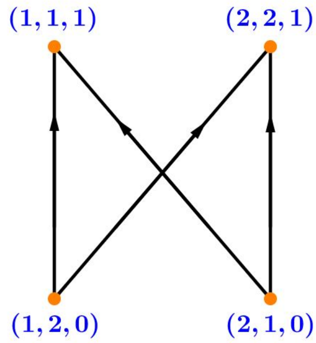


Figure 1.20: The Hasse diagram of $(J(P_3), \subseteq)$.

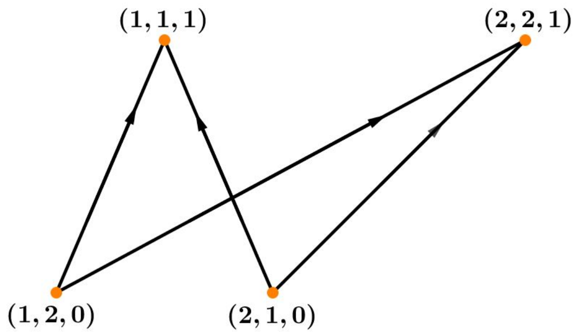
We notice that ASM lattice $(ASM(3), \leq)$ shown in Figure 1.14 and the lattice $(J(P_3), \subseteq)$ shown in Figure 1.20, appear to be similar. In fact, there is a general anti-isomorphism between $(ASM(n), \leq)$ and $(J(P_n), \subseteq)$. Alternatively, it could also be shown that there is a general isomorphism between $(ASM(n), \leq)$ and $(J(P_n), \subseteq)$.



(a) The rectangular representation of P_3.

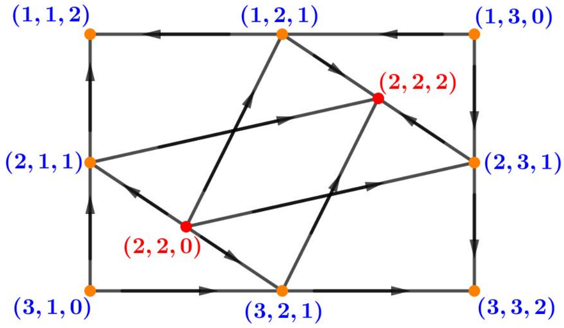


(b) The tetrahedral representation of P_3. In this representation, there are two elements of rank 1 on top of two elements of rank 0 (where the rank of each element in P_n is given in part (III) of Lemma 1.82).

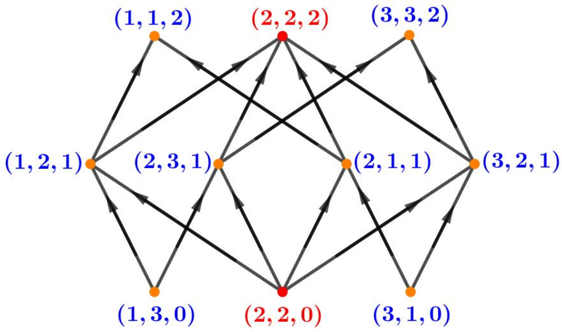


(c) The pyramidal representation of P_3. In this representation, there are two types of pyramidal configurations, namely, the vertical and the diagonal pyramids. In the vertical pyramid, element $(1,1,1)$ of rank 1 with $i+j+k=3$ is on top of two elements $(1,2,0)$ and $(2,1,0)$ of rank 0. The diagonal pyramid consists of one element $(2,2,1)$ of rank 1 with $i+j+k=5$ on top of two elements $(1,2,0)$ and $(2,1,0)$ of rank 0.

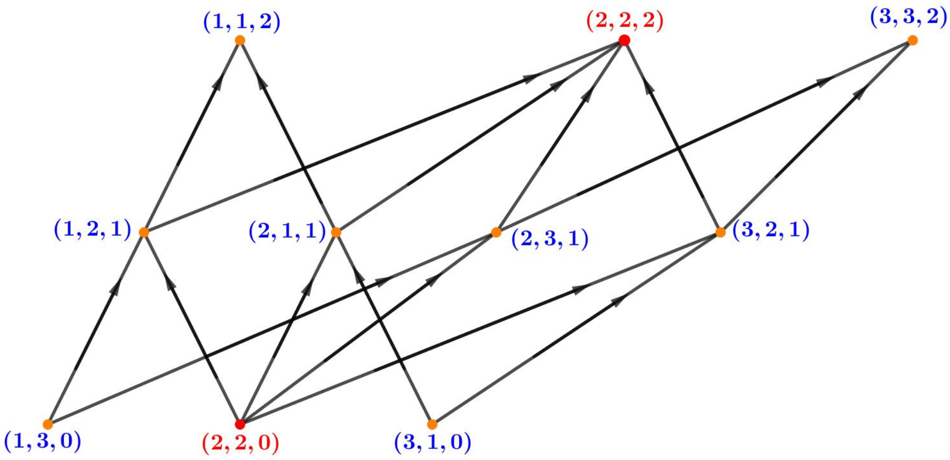
Figure 1.21: Rectangular, tetrahedral and pyramidal representations of the Hasse diagrams of P_3.



(a) The rectangular representation of P_4.

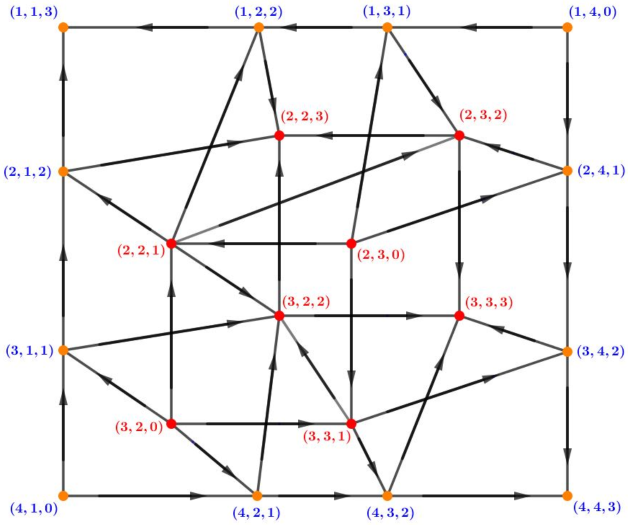


(b) The tetrahedral representation of P_4. In this representation, there are four elements of rank 1 on top of three elements of rank 0, and three elements of rank 2 on top of the four elements of rank 1.

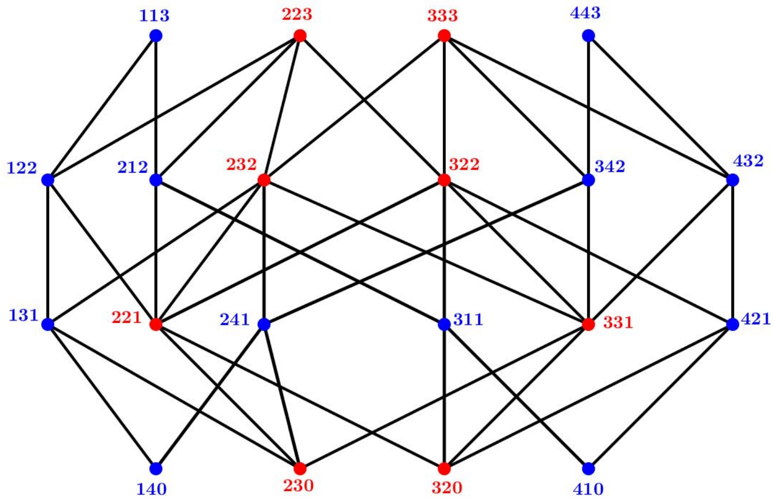


(c) The pyramidal representation of P_4. In this representation, there are two types of pyramidal configurations, namely, the vertical and the diagonal pyramids. In the first vertical pyramid, there are two elements of rank 1 (namely, elements $(1,2,1)$ and $(2,1,1)$) with $i+j+k=4$ on top of three elements of rank 0 (namely elements $(1,3,0)$, $(2,2,0)$ and $(3,1,0)$) with $i+j+k=4$ and one element of rank 2 (namely, element $(1,1,2)$) with $i+j+k=4$ on top of two elements of rank 1. In the second vertical pyramid, there is one element of rank 2 (namely $(2,2,2)$) with $i+j+k=6$ on top of two elements of rank 1 (namely, elements $(2,3,1)$ and $(3,2,1)$). The third vertical pyramid is a single element of rank 2 with $i+j+k=8$ (namely, $(3,3,2)$). In the first diagonal pyramid, there are two elements of rank 1 (namely, elements $(2,3,1)$ and $(3,2,1)$) with $i+j+k=6$ on top of three elements of rank 0 (namely elements $(1,3,0)$, $(2,2,0)$, $(3,1,0)$) and one element of rank 2 with $i+j+k=8$ on top of two elements of rank 1. In the second diagonal pyramid, there is one element of rank 2 (namely $(2,2,2)$) with $i+j+k=6$ on top of two elements of rank 1 (namely, elements $(1,2,1)$, $(2,1,1)$).

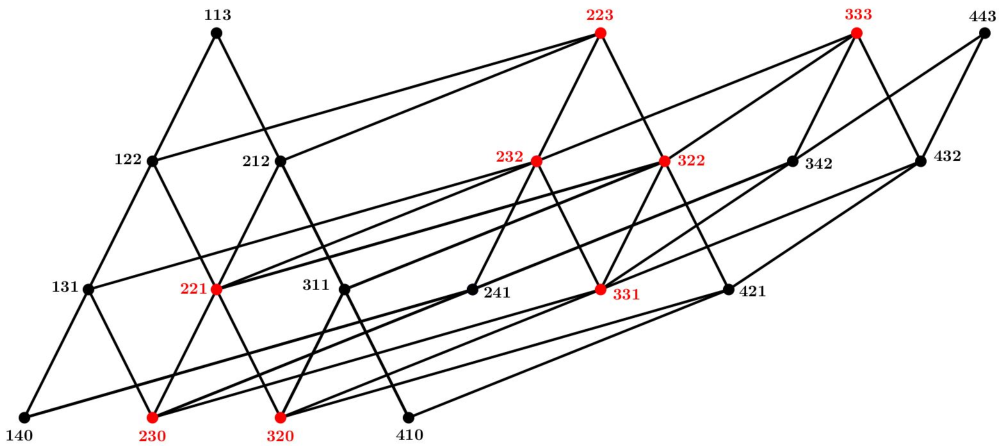
Figure 1.22: Rectangular, tetrahedral and pyramidal representations of the Hasse diagrams of P_3 and P_4.



(a) Rectangular representation of the ASM poset P_5.



(b) Tetrahedral representation of the ASM poset P_5. In this representation, there are six elements of rank 1 on top of four elements of rank 0, six elements of rank 2 on top of six elements of rank 1 and finally four elements of rank 3 on top of six elements of rank 2.



(c) Pyramidal representation of the ASM poset P_5. In this representation, there are four vertical and three diagonal pyramidal configurations. The description of them is very similar to the case for pyramidal representations of P_3 and P_4.

Figure 1.23: Rectangular, tetrahedral and pyramidal representations of the Hasse diagrams of the ASM poset P_5. The central elements of P_5 are shown in red.

Theorem 1.83. *There is an anti-isomorphism between $(ASM(n), \leq)$ and $(J(P_n), \subseteq)$. In particular, for a given $A \in ASM(n)$ its associated order ideal denoted by I_A is given by*

$$I_A = \{(i, j, |n-i-j| + 2l) \mid i, j = 1, 2, ..., n-1 \,;\, l = 0, 1, ..., c_{ij} - max(0, i+j-n) - 1\} \quad (1.64)$$

and c_{ij} is the CSM given in (1.31). Conversely, for a given order ideal $I \in J(P_n)$, the CSM of its associated ASM can be obtained by

$$(|\{k \in \mathbb{N} \mid (i,j,k) \in I\}| + max(0, i+j-n))_{i,j=0}^{n} \quad (1.65)$$

We describe how Theorem 1.83 works by applying it to the ASMs A and B given in our running Example 1.4. Before that we describe how it works in general.

Let $A \in ASM(n)$ be given and let C denotes its associated CSM. Define the matrix $D = (d_{ij})_{i,j=0}^{n}$ where

$$d_{ij} = \begin{cases} 0 & \text{if } i+j \leq n+1 \\ i+j-n & \text{if } i+j > n+1 \end{cases} \quad (1.66)$$

By subtracting D from C and removing the zero entries on the boundaries of $C - D$ (Note that by Lemma 1.32, entries of $C - D$ are non-negative), we obtain an $(n-1) \times (n-1)$ matrix $E = (e_{ij})_{i,j=1}^{n-1}$ where

$$e_{ij} = \begin{cases} c_{ij} & \text{if } i+j \leq n+1 \\ c_{ij} - (i+j-n) & \text{if } i+j > n+1 \end{cases} \quad (1.67)$$

and c_{ij} is the CSM given in 1.29. Now let I_A denote the order ideal associated with the ASM A. Then each element of I_A has the form (i, j, k) where the first and second component i and j correspond to the position (i, j) in matrix E whenever $e_{ij} \neq 0$ and to obtain the third component k, we compare the matrix E with the matrix $K_n = (|n-i-j|)_{i,j=0}^{n}$ entry-wise. Note that the matrix K_n has the form

$$K_n = \begin{pmatrix} n & n-1 & n-2 & \cdots & 2 & 1 & 0 \\ n-1 & n-2 & n-3 & \cdots & 1 & 0 & 1 \\ n-2 & n-3 & n-4 & \cdots & 0 & 1 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 1 & 0 & \cdots & n-4 & n-3 & n-2 \\ 1 & 0 & 1 & \cdots & n-3 & n-2 & n-1 \\ 0 & 1 & 2 & \cdots & n-2 & n-1 & n \end{pmatrix}$$

More precisely, the initial value of the third component k in (i, j, k) is given in the position (i, j) in matrix K. To obtain the other possible values for k, in case $|e_{ij}| > 1$, we note that by the way of definition of I_A, k is growing by adding a multiple of 2 to the initial value.

Example 1.84. Consider the ASMs $A, B \in ASM(6)$ in Example 1.4. Also consider their corresponding CSMs C_A and C_B given in 1.34. By subtracting the matrix $D_6 = (max(0, i+j-n))_{i,j=0}^{6}$ from both C_A and C_B, we obtain

$$C_A - D_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad C_B - D_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then the matrices E_A and E_B corresponding to the ASMs A and B are obtained by removing the entries on the first row and column and last row and column of $C_A - D_6$ and $C_B - D_6$. So we have

$$E_A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \quad ; \quad E_B = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \tag{1.68}$$

Now by considering the matrix K_5 given by

$$K_5 = \begin{pmatrix} 4 & 3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \tag{1.69}$$

and comparing the entries of E_A, E_B and K_5 entry-wise, the sets of order ideals I_A and I_B associated with the ASMs A and B are given by

$$\begin{aligned} I_A = \{&(1,2,3),(1,3,2),(1,4,1),(1,5,0),(2,1,3),(2,2,2),(2,3,1),(2,3,3),(2,4,0),(2,4,2),\\ &(2,5,1),(3,1,2),(3,2,1),(3,3,0),(3,3,2),(3,4,1),(3,4,3),(3,5,2),(4,1,1),(4,2,0),\\ &(4,2,2),(4,3,1),(4,3,3),(4,4,2),(5,1,0),(5,2,1),(5,3,2),(5,4,3)\} \end{aligned}$$

$$\begin{aligned} I_B = \{&(1,2,3),(1,3,2),(1,4,1),(1,5,0),(2,1,3),(2,2,2),(2,3,1),(2,3,3),(2,4,0),(2,4,2),\\ &(2,5,1),(3,1,2),(3,2,1),(3,3,0),(3,3,2),(3,4,1),(3,4,3),(3,5,2),(4,1,1),(4,2,0),\\ &(4,2,2),(4,3,1),(4,3,3),(4,4,2),(4,5,3),(5,1,0),(5,2,1),(5,3,2),(5,4,3)\} \end{aligned}$$

As an example, e_{12} in E_A is 1 and k_{12} in K_5 is 3 thus the element (i,j,k) in I_A is $(1,2,3)$. Note that the first position in E_A where $e_{ij} > 1$ is the position $(2,3)$. In this position, we have $e_{23} = 2$. This means that there are two possible values for the third component k in $(i,j,k) \in I_A$. As indicated above, the initial value for this k is the one in position $(2,3)$ in K_5 which is $k_{23} = 1$. The second value for k is obtained by adding 2 to the initial value (since by (1.64) all k's associated with the position (i,j) are growing by a multiple of 2). Therefore, the two elements of I_A corresponding to the position $(2,3)$ are given by $(2,3,1)$ and $(2,3,3)$. The same approach applies to the other positions within E_A and E_B.

We also note that $I_A \subset I_B$ since $B \lessdot A$ as indicated in (1.46). In Figure 1.24, the ASM poset P_6 together with corresponding order ideals of alternating sign matrices A and B are shown.

Definition 1.85. For a given $A \in ASM(n)$ and all $i,j = 0,1,...,n$, the *ASM order ideal matrix*, or OIM for short, is defined by

$$(O_A)_{ij} = \begin{cases} \sum_{i'=1}^{i} \sum_{j'=1}^{j} a_{i'j'} & \text{if} \quad i+j \leq n; \\ \\ \sum_{i'=i+1}^{n} \sum_{j'=j+1}^{n} a_{i'j'} & \text{if} \quad i+j \geq n. \end{cases} \tag{1.70}$$

The set of all order ideal matrices of order n is denoted by $OIM(n)$.

For example, the seven OIMs of order 3 are given by

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \tag{1.71}$$

Also the corresponding order ideal matrices to the ASMs A and B in our running Example 1.4 are given by

$$O_A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad , \qquad O_B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.72}$$

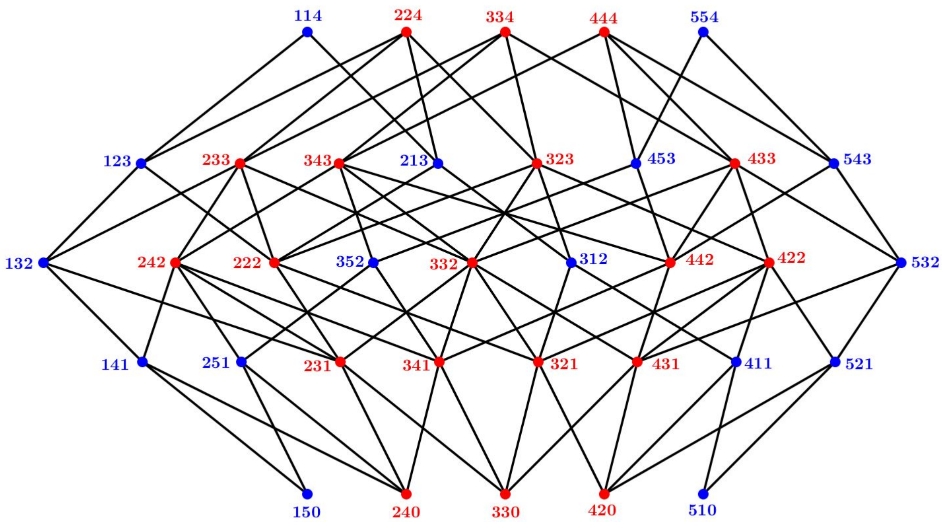
Analogously, the *ASM dual order ideal matrix*, or DOIM for short, of a given $A \in ASM(n)$ is defined by

$$(\overline{O}_A)_{ij} = \begin{cases} \sum_{i'=1}^{i} \sum_{j'=j+1}^{n} a_{i'j'} & \text{if} \quad i \leq j; \\ \\ \sum_{i'=i+1}^{n} \sum_{j'=1}^{j} a_{i'j'} & \text{if} \quad i \geq j. \end{cases} \tag{1.73}$$

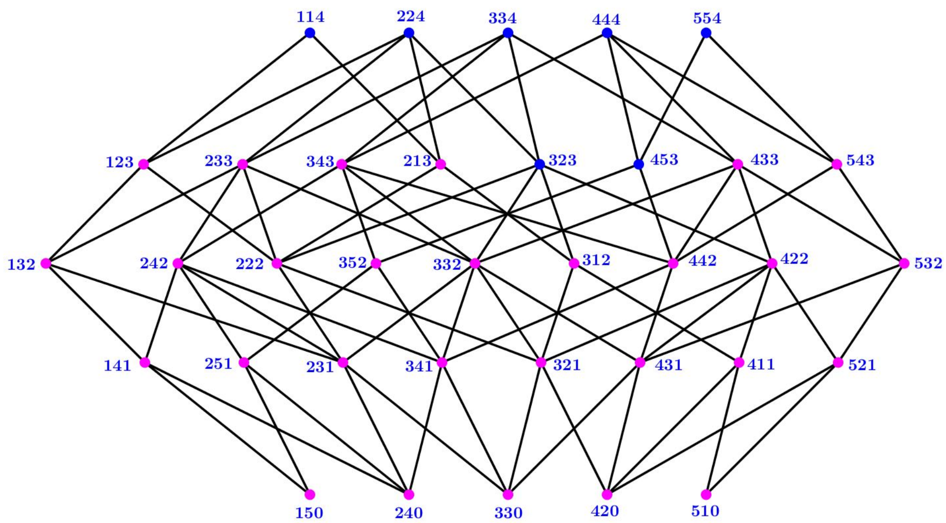
Similarly to the case for OIMs, the set of all DOIMs of order n is denoted by $DOIM(n)$. As an illustration, all seven DOIMs of order 3 are given by

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \tag{1.74}$$

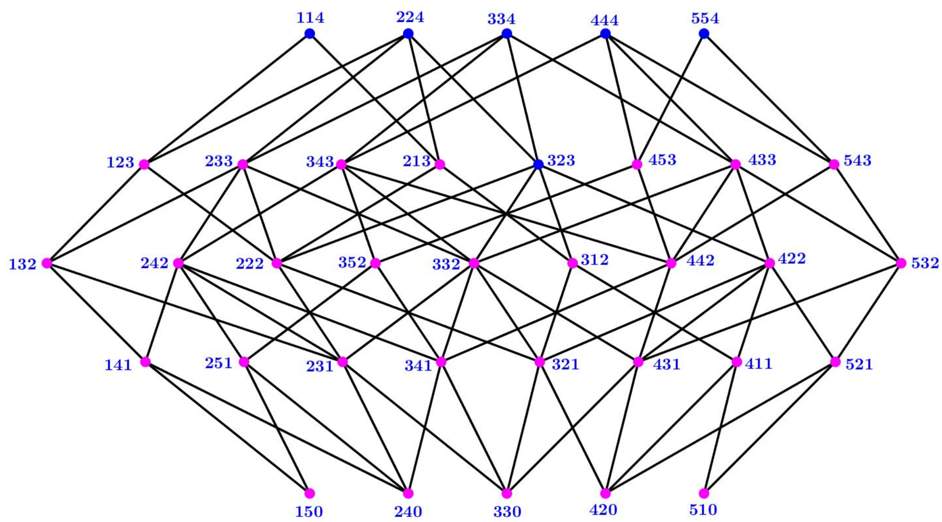
Moreover, the dual order ideal matrices corresponding to the ASMs A and B in our running Example 1.4 are given by



(a) The tetrahedral representation of the Hasse diagram of the ASM poset P_6. The central elements in P_6 are shown in red and for brevity each (i, j, k) in P_6 is encoded by ijk.



(b) The order ideal I_A corresponding to the ASM $A(6)$ given in Example 1.4. All elements of I_A are highlighted with pink and for brevity each (i, j, k) in P_6 is encoded by ijk.



(c) The order ideal I_B corresponding to the ASM $B(6)$ given in Example 1.4. All elements of I_B are highlighted with pink and for brevity each (i, j, k) in P_6 is encoded by ijk.

Figure 1.24: ASM Poset P_6 together with corresponding order ideals of alternating sign matrices A and B.

$$\overline{O}_A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \overline{O}_B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.75}$$

Note that by Theorem 1.30 and definition of the CSMs 1.29, we can write

$$\begin{aligned}
\sum_{i'=i+1}^{n} \sum_{j'=j+1}^{n} a_{i'j'} &= \sum_{i'=i+1}^{n} \left(\sum_{j'=1}^{n} a_{i'j'} - \sum_{j'=1}^{j} a_{i'j'} \right) \\
&= \sum_{i'=i+1}^{n} \sum_{j'=1}^{n} a_{i'j'} - \sum_{i'=i+1}^{n} \sum_{j'=1}^{j} a_{i'j'} \\
&= \sum_{i'=i+1}^{n} 1 - \sum_{j'=1}^{j} \left(\sum_{i'=1}^{n} a_{i'j'} - \sum_{i'=1}^{i} a_{i'j'} \right) \\
&= \sum_{i'=i+1}^{n} 1 - \sum_{j'=1}^{j} \sum_{i'=1}^{n} a_{i'j'} + \sum_{i'=1}^{i} \sum_{j'=1}^{j} a_{i'j'} \\
&= \sum_{i'=i+1}^{n} 1 - \sum_{j'=1}^{j} 1 + \sum_{i'=1}^{i} \sum_{j'=1}^{j} a_{i'j'} \\
&= c_{ij} - (i + j - n)
\end{aligned}$$

Thus we can rewrite $(O_A)_{ij}$ in (1.70) as follows

$$(O_A)_{ij} = \begin{cases} c_{ij} & \text{if } i + j \le n \\ c_{ij} - (i + j - n) & \text{if } i + j \ge n \end{cases} \tag{1.76}$$

Equivalently by the fact that $Min(0, n-i-j) = -Max(0, i+j-n)$, we can also write $(O_A)_{ij}$ as $c_{ij} - Max(0, i+j-n) = Min(0, n-i-j) + c_{ij}$, where $i, j = 0, ..., n$. Similarly, for $i \le j$ we have

$$\begin{aligned}
\sum_{i'=1}^{i} \sum_{j'=j+1}^{n} a_{i'j'} &= \sum_{i'=1}^{i} \left(\sum_{j'=1}^{n} a_{i'j'} - \sum_{j'=1}^{j} a_{i'j'} \right) \\
&= \sum_{i'=1}^{i} \sum_{j'=1}^{n} a_{i'j'} - \sum_{i'=1}^{i} \sum_{j'=1}^{j} a_{i'j'} \\
&= i - \sum_{i'=1}^{i} \sum_{j'=1}^{j} a_{i'j'} \\
&= Min(i, j) - c_{i,j}
\end{aligned}$$

Moreover, if $i \geq j$, then

$$\begin{aligned}
\sum_{i'=i+1}^{n} \sum_{j'=1}^{j} a_{i'j'} &= \sum_{j'=1}^{j} \sum_{i'=i+1}^{n} a_{i'j'} \\
&= \sum_{j'=1}^{j} \left(\sum_{i'=1}^{n} a_{i'j'} - \sum_{i'=1}^{i} a_{i'j'} \right) \\
&= \sum_{j'=1}^{j} \sum_{i'=1}^{n} a_{i'j'} - \sum_{i'=1}^{i} \sum_{j'=1}^{j} a_{i'j'} \\
&= j - \sum_{i'=1}^{i} \sum_{j'=1}^{j} a_{i'j'} \\
&= Min(i,j) - c_{i,j}
\end{aligned}$$

Therefore, for any $i, j = 0, 1, ..., n$ we can write $\left(\overline{O}_A\right)_{ij} = Min(i,j) - c_{ij}$, and Lemma 1.86 immediately follows.

Lemma 1.86. *For a given $A \in ASM(n)$, we have*

(I) $(O_A)_{ij} = Min(0, n-i-j) + c_{ij} = c_{i,j} - Max(0, i+j-1)$*;*

(II) $\left(\overline{O_A}\right)_{ij} = Min(i,j) - c_{ij}$*;*

(III) $(O_A)_{ij} + \left(\overline{O_A}\right)_{ij} = Min(i, j, n-i, n-j)$*.*

The following theorem is the main theorem of this section and it provides a bijection between $ASM(n)$ and $OIM(n)$. The proof is very similar to the proof of Theorem 1.30.

Theorem 1.87. *Let $\phi : ASM(n) \to OIM(n)$ be given by*

$$\phi(A)_{ij} = c_{i,j} - Max(0, i+j-n) \tag{1.77}$$

for all $A \in ASM(n)$ and $i, j = 0, 1, ..., n$. Conversely, let $\psi : OIM(n) \to ASM(n)$ be given by

$$\psi(O)_{ij} = \delta_{i+j,n+1} + o_{ij} - o_{i,j-1} - o_{i-1,j} + o_{i-1,j-1} \tag{1.78}$$

for all $O \in OIM(n)$ where $i, j = 0, 1, ..., n$. Then

(I) ϕ and ψ are well-defined functions, and

(II) ϕ and ψ are bijective with $\phi^{-1} = \psi$.

By definition of DOIMs and Lemma 1.86, the argument for the bijection between $ASM(n)$ and $DOIM(n)$ is very similar to the proof of Theorem 1.87. So we have

Theorem 1.88. *There is a bijection between $ASM(n)$ and $DOIM(n)$.*

Remark 1.89. As a consequence of Theorem 1.87 and (1.40), we expect to have a bijection between HFMs and OIMs as well. For a given order ideal matrix $O \in OIM(n)$, the corresponding *reduced height function matrix* associated to O is an $(n+1) \times (n+1)$ matrix given by $H = (h)_{i,j=0}^{n}$ where

$$h_{ij} = |n-i-j| + 2o_{ij} \tag{1.79}$$

and inversely,

$$o_{ij} = \frac{1}{2}\left(h_{ij} - |n - i - j|\right) \tag{1.80}$$

1.5.2 The action of the dihedral group on P_n

In this section we study the action of the symmetries of the dihedral group D_8 on P_n and on $J(P_n)$. The motivation for this is the fact that ASMs can be acted upon by symmetry operations of the square, see Chapter 1 of [60] and that there is a bijection between the set of order ideals of P_n and the set of $n \times n$ ASMs by Theorem 1.83. Therefore, we are interested in investigating the action of D_8 and its subgroups on P_n and $J(P_n)$. For more details and information on the action of dihedral group on the ASMs see for example [30], [31], [60], [87], or [111].

Definition 1.90. The group of symmetries of the square or the *Dihedral group* denoted by D_8 is the following group

$$D_8 := < \rho , \mu \, : \, \rho^4 = \mu^2 = Id \, , \, \mu\rho = \rho^{-1}\mu > \tag{1.81}$$

where

$Id :=$ The identity element, i.e, leaves the square invariant;

$\rho := R_{\frac{\pi}{2}} =$ Rotation by angle $\frac{\pi}{2}$, counterclockwise;

$\mu := H =$ The reflection of a square through a horizontal line through the centre;

this gives the following extra five elements obtained by iterating the operations:

$D := \rho\mu =$ The diagonal reflection through the centre;

$V := \rho^2\mu =$ The vertical reflection through the centre;

$A := \rho^3\mu =$ The antidiagonal reflection through the centre;

$R_\pi := \rho^2 =$ The half turn rotation of a square or rotation by π;

$R_{\frac{3\pi}{2}} := \rho^3 =$ The rotation of a square by $\frac{3\pi}{2}$.

The group elements of D_8 as operations on the square are shown in Figure 1.25.

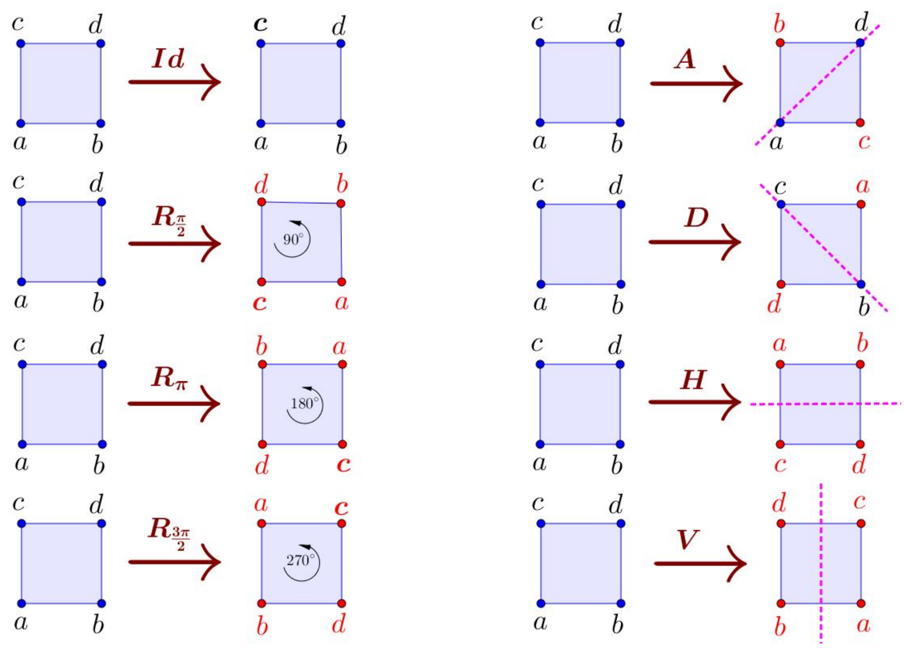


Figure 1.25: Dihedral group elements acting on the square.

Remark 1.91. One useful way to summarize the structure of a finite group is via its so called *Cayley table*. The Cayley table of D_8 is shown in Table 1.3. One can easily derive some useful information about the connections between the elements of D_8, for instance, it can be checked that it is not abelian since it is not symmetrical about the main diagonal. It can also be seen which element is the inverse of which or what is the result of the compositions of elements.

Definition 1.92. Let X be an arbitrary set and G be a given group. Then a *group action* of G on X is a map $\phi : G \times X \to X$ denoted by $\phi(g, x) = g.x$ satisfying the following conditions:

(I) **Identity**: $\phi(e, x) = e.x = x$ for all $x \in X$;

(II) **Compatibility**: $\phi(g_1 g_2, x) = (g_1 g_2).x = g_1(g_2.x)$ for all $g_1, g_2 \in G$ and $x \in X$.

Remark 1.93. We note that the set of subgroups of a finite group G can be turned into a poset via the inclusion ordering. So the Hasse diagram of the poset of subgroups of D_8 ordered by inclusion is shown in Figure 1.26.

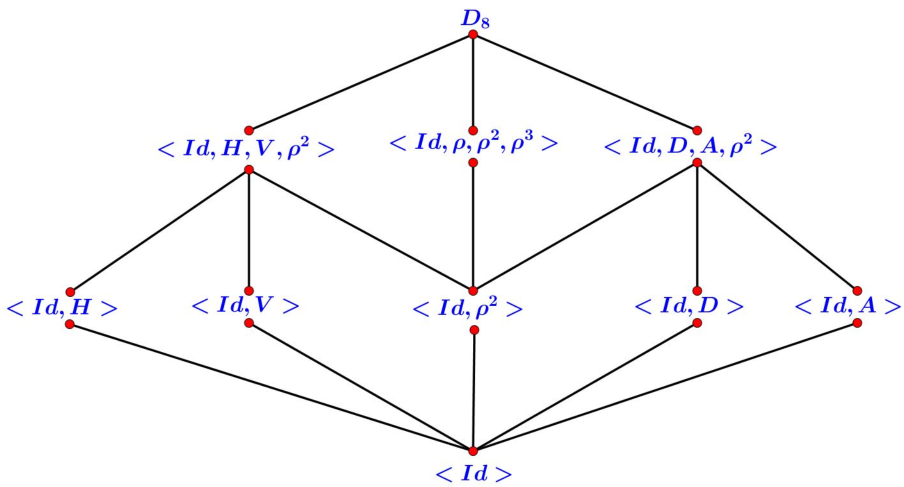


Figure 1.26: Dihedral subgroups ordered by inclusion.

Table 1.4 provides a list of all ten subgroups of D_8 and their generators.

Definition 1.94. Consider the ASM poset P_n as a set. Then the symmetry operations on P_n are given in Table 1.5.

The following lemma determines which of the symmetry operations on P_n is a poset automorphism (order preserving bijection) and which one is a poset anti-automorphism (order reversing bijection).

Lemma 1.95. *The sets* $Aut(P_n) = \{Id, A, D, R_\pi\}$ *and* $\widetilde{Aut}(P_n) = \{H, V, R_{\frac{\pi}{2}}, R_{\frac{3\pi}{2}}\}$ *are sets of automorphisms and anti-automorphisms of* P_n*, respectively.*

Example 1.96. As an illustration, all four automorphisms and anti-automorphisms of P_3 are given pictorially in Table 1.6.

Similarly to the case for the ASM poset P_n, all the symmetries of the square acting on an ASM $A \in ASM(n)$ (or on any square matrix A) are given in Table 1.7.

$*$	Id	$R_{\frac{\pi}{2}}$	R_π	$R_{\frac{3\pi}{2}}$	H	A	V	D
Id	Id	$R_{\frac{\pi}{2}}$	R_π	$R_{\frac{3\pi}{2}}$	H	A	V	D
$R_{\frac{\pi}{2}}$	$R_{\frac{\pi}{2}}$	R_π	$R_{\frac{3\pi}{2}}$	Id	D	H	A	V
R_π	R_π	$R_{\frac{3\pi}{2}}$	Id	$R_{\frac{\pi}{2}}$	V	D	H	A
$R_{\frac{3\pi}{2}}$	$R_{\frac{3\pi}{2}}$	Id	$R_{\frac{\pi}{2}}$	R_π	A	V	D	H
H	H	A	V	D	Id	$R_{\frac{\pi}{2}}$	R_π	$R_{\frac{3\pi}{2}}$
A	A	V	D	H	$R_{\frac{3\pi}{2}}$	Id	$R_{\frac{\pi}{2}}$	R_π
V	V	D	H	A	R_π	$R_{\frac{3\pi}{2}}$	Id	$R_{\frac{\pi}{2}}$
D	D	H	A	V	$R_{\frac{\pi}{2}}$	R_π	$R_{\frac{3\pi}{2}}$	Id

Table 1.3: Cayley table of Dihedral group D_8.

Titles	**Subgroup**	**Generators**
No Symmetry	Id	$< I >$
Horizontal Symmetry	$\{Id, H\}$	$< H >$
Half-turn Symmetry	$\{Id, R_\pi\}$	$< R_\pi >$
Vertical Symmetry	$\{Id, V\}$	$< V >$
Diagonal Symmetry	$\{Id, D\}$	$< D >$
Anti-diagonal Symmetry	$\{Id, A\}$	$< A >$
Both Horizontal and Vertical Symmetry	$\{Id, H, V, R_\pi\}$	$< H, V >$
Quarter-turn Symmetry	$\{Id, \rho, \rho^2, \rho^3\}$	$< \rho >$
Both Diagonal and Anti-diagonal Symmetries	$\{Id, D, A, \rho^2\}$	$< D, A >$
All Symmetries	D_8	$< \rho, H >$

Table 1.4: The subgroups of D_8 and their generators.

Name	Conditions
No symmetry (identity map)	$Id(i,j,k) = (i,j,k)$
Vertical Symmetry (vertical reflection)	$V(i,j,k) = (i, n-j, n-2-k)$
Half-turn symmetry (half-turn rotation)	$R_{\pi}(i,j,k) = (n-i, n-j, k)$
Quarter-turn symmetry (rotation by $\frac{\pi}{2}$)	$R_{\frac{\pi}{2}}(i,j,k) = (j, n-i, n-2-k)$
Diagonal symmetry (diagonal reflection)	$D(i,j,k) = (j,i,k)$
Horizontal Symmetry (horizontal reflection)	$H(i,j,k) = (n-i, j, n-2-k)$
Anti-diagonal symmetry (anti-diagonal reflection)	$A(i,j,k) = (n-j, n-i, k)$
Three quarter-turn symmetry (rotation by $\frac{3\pi}{2}$)	$R_{\frac{3\pi}{2}}(i,j,k) = (n-j, i, n-2-k)$

Table 1.5: All symmetry operations on P_n.

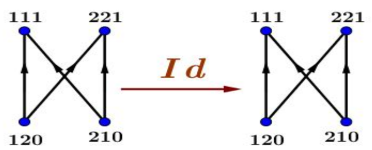
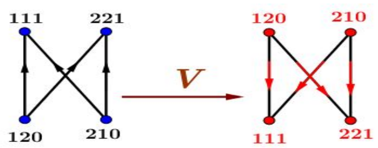
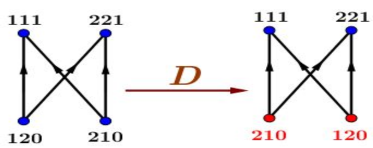
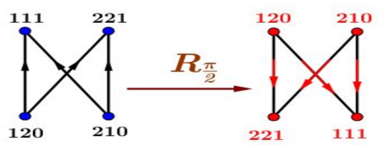
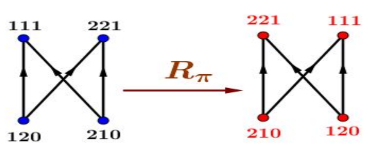
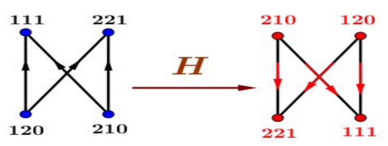
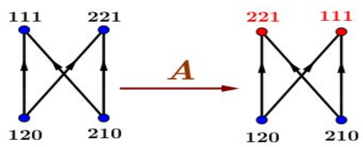
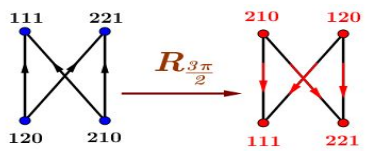
Poset Automorphism	**Poset Anti-automorphism**
	
	
	
	

Table 1.6: ASM poset automorphisms and anti-automorphisms of P_3. The red colored vertices are the ones which are affected by the given symmetry.

Matrix operation	**Conditions**
No Symmetry, I_n	$a_{ij} \to a_{ij}$
Half Turn Rotation, R_π	$a_{ij} \to a_{n-i+1,n-j+1}$
Quarter Turn Rotation, $R_{\frac{\pi}{2}}$	$a_{ij} \to a_{n-j+1,i}$
Three Quarter Turn Rotation, $R_{\frac{3\pi}{2}}$	$a_{ij} \to a_{j,n-i+1}$
Anti-diagonal Reflection, A	$a_{ij} \to a_{n-j+1,n-i+1}$
Diagonal Reflection, D	$a_{ij} \to a_{j,i}$
Vertical Reflection, V	$a_{ij} \to a_{i,n-j+1}$
Horizontal Reflection, H	$a_{ij} \to a_{n-i+1,j}$

Table 1.7: Symmetry operations on square matrices.

Example 1.97. As an illustration, the rotation and reflection of the square acting on a 5×5 matrix are given by

$$\begin{aligned}
&\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} \end{pmatrix} \xrightarrow{R_{\frac{\pi}{2}}} \begin{pmatrix} a_{5,1} & a_{4,1} & a_{3,1} & a_{2,1} & a_{1,1} \\ a_{5,2} & a_{4,2} & a_{3,2} & a_{2,2} & a_{1,2} \\ a_{5,3} & a_{4,3} & a_{3,3} & a_{2,3} & a_{1,3} \\ a_{5,4} & a_{4,4} & a_{3,4} & a_{2,4} & a_{1,4} \\ a_{5,5} & a_{4,5} & a_{3,5} & a_{2,5} & a_{1,5} \end{pmatrix} \\
&\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} \end{pmatrix} \xrightarrow{H} \begin{pmatrix} a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \end{pmatrix}
\end{aligned} \tag{1.82}$$

1.6 Conclusion

The first chapter began with a brief historical review of the birth of ASMs together with several aspects of them. Then the formal definition of standard ASMs and their generalization to so called higher spin alternating sign matrices were discussed. The chapter then considered the many faces of ASMs and their generalizations, including monotone triangles, configurations of six vertex models with domain wall boundary conditions, simple flow grids, corner sum matrices and height function matrices. After some preliminaries on partially ordered sets, we introduced an order relation between ASMs and observed that this poset is in fact a distributive lattice. Moreover, we verified the connection between the ASM lattice and the MacNeille completion of the strong Bruhat order via their connection with monotone triangles. Then in Section 1.5, we discussed the ASM poset P_n and its basic properties. This section will provide the building blocks of the ASM order polytope (Chapter 2, Section 2.5) and of the reduced ASM labelled posets (Chapter 3, Section 3.2.1). We also proved that there is an anti-isomorphism between the ASM lattice and the lattice of order ideals of P_n. We finished the chapter with a brief overview of the action of symmetries of the dihedral group D_8 on P_n.

CHAPTER 2

THE ALTERNATING SIGN MATRIX POLYTOPE

2.1 Introduction

This chapter is organised as follows. We begin this chapter with some basic definitions and concepts on polytopes. In particular, we outline the definitions and basic properties of *order polytopes* and *flow polytopes* in Sections 2.2.1 and 2.2.2, respectively. Then we discuss a classic polytope known as the *Birkhoff polytope* in Section 2.3. This section is followed by a brief review of a well-known face of the Birkhoff polytope called the *Chan-Robbins-Yuen polytope*. Section 2.4 is dedicated to the definition and basic properties of the *alternating sign matrix polytope*, or *ASM polytope* for short. Moreover, we introduce the *ASM order polytope* associated with the ASM poset P_n and provide its fundamental properties. Also we provide a bijection between the ASM polytope and the ASM order polytope for $n = 3$, see Section 2.5.1. Similar to the case for the Birkhoff polytope, in Section 2.5.2 an analogue of the Chan-Robbins-Yuen polytope for the ASM polytope known as the *alternating sign matrix Chan-Robbins-Yuen polytope*, or the *ASMCRY polytope* for short, is given.

2.2 Polytope preliminaries

Definition 2.1. A set $V \subset \mathbb{R}^d$ is called *convex* if for any pair of points $u, v \in V$ the line segment $\overline{uv}$ entirely lies in V. The *convex hull* of $V \subset \mathbb{R}^d$, denoted by $conv(V)$, is the minimal convex set that contains it, i.e., it is the intersection of all convex sets C in $\mathbb{R}^d$ for which $V \subset C$.

It can be shown that for a finite set $\{v_1, ..., v_n\}$ in $\mathbb{R}^d$, the convex hull is given by

$$Conv\{v_1, ..., v_n\} = \{\lambda_1 v_1 + ... + \lambda_n v_n \mid \lambda_1, ..., \lambda_n \in \mathbb{R}\,;\, \lambda_1, ..., \lambda_n \geq 0\,;\, \lambda_1 + ... + \lambda_n = 1\} \quad (2.1)$$

There are two equivalent but different approaches to defining polytopes, namely, the *hyperplane description* (or $\mathcal{H}-description$) and the *vertex description* (or $\mathcal{V}-description$). The $\mathcal{H}-description$ defines the polytope in terms of inequalities whereas the $\mathcal{V}-description$ defines a polytope in terms of a set containing its vertices. In other words, we have

Definition 2.2. ($\mathcal{V}-$*description.*) A *polytope* $\mathcal{P}$ is a convex hull of finitely many points $v_1, v_2, ..., v_k$ in $\mathbb{R}^d$:

$$P = conv(v_1, v_2, ..., v_k)$$

Definition 2.3. ($\mathcal{H}-$*description.*) A *polytope* P is a bounded intersection of finitely many closed halfspaces $\mathbb{R}^d$:

$$\begin{aligned}\mathcal{P} &= \left\{x \in \mathbb{R}^d \mid Ax \leq b\right\}\\ &= \left\{x \in \mathbb{R}^d \mid u_1.x \leq b_1, ..., u_k.x \leq b_k\right\}\end{aligned}$$

where $u_1, ..., u_k \in \mathbb{R}^d$, $b_1, ..., b_k \in \mathbb{R}$ and in the first expression A is a $k \times d$ matrix with rows $u_1, ..., u_k$ and b is the column vector with entries $b_1, ..., b_k$.

The following theorem is known as the *Fundamental Theorem in polytope theory.* For the proof see [57] or Chapter 1 of [118].

Theorem 2.4. *The* $\mathcal{H}-$description *and the* $\mathcal{V}$-description *of polytopes are equivalent. In other words, a subset* $\mathcal{P} \subseteq \mathbb{R}^d$ *is a bounded intersection of finitely many halfspaces if and only if it is a convex hull of a finite set of points of* $\mathcal{P}$.

Remark 2.5. It is very useful to work with both the $\mathcal{V}-$description and $\mathcal{H}-$description of a polytope for both practical and theoretical purposes. For instance, one can show from the $\mathcal{V}-$description that a projection of a polytope is a polytope which is not immediately clear from the $\mathcal{H}-$description. In the same sense, it is not hard to see from the $\mathcal{H}-$description that the intersection of two polytopes is a polytope while it is not immediately clear from the $\mathcal{V}-$description.

Definition 2.6. The *dimension* of a polytope $\mathcal{P}$ is the dimension of the *affine subspace* spanned by $\mathcal{P}$, that is, the dimension of the set

$$aff(\mathcal{P}) := \left\{\sum_{i=1}^{k} \lambda_i v_i \;\middle|\; k \geq 1,\ \sum_{i=1}^{k} \lambda_i = 1\,,\ \lambda_i \in \mathbb{R}\,,\ v_i \in \mathcal{P}\,,\ i = 1, ..., k\right\}$$

If $\mathcal{P}$ has dimension d, then we write $dim(\mathcal{P}) = d$ and say $\mathcal{P}$ is a convex $d-$polytope.

Definition 2.7. Polytopes $\mathcal{P} \subset \mathbb{R}^d$ and $\mathcal{Q} \subset \mathbb{R}^e$ are said to be affinely isomorphic if there is an affine map $\phi : \mathbb{R}^d \to \mathbb{R}^e$ (that is, ϕ is of the form $\phi(x) = Ax + b$ for each $x \in \mathbb{R}^d$ where A is an $e \times d$ matrix and $b \in \mathbb{R}^e$) which is bijective from $\mathcal{P}$ to $\mathcal{Q}$. Such a map ϕ is called an affine isomorphism.

Definition 2.8. For a polytope $\mathcal{P} \subset \mathbb{R}^d$, the hyperplane $\mathcal{H} = \{x \in \mathbb{R}^d \mid u.x = b\}$ is called a *supporting hyperplane* of $\mathcal{P}$ if $\mathcal{P}$ lies entirely on one side of H. That is, we have

$$\mathcal{P} \subset \{x \in \mathbb{R}^d \mid u.x \geq b\} \quad \text{or} \quad \mathcal{P} \subset \{x \in \mathbb{R}^d \mid u.x \leq b\}$$

where $u \in \mathbb{R}^d$ and $b \in \mathbb{R}$.

Definition 2.9. A *vertex* of a polytope $\mathcal{P}$ is a point $v \in \mathcal{P}$ for which there exists a supporting hyperplane H of $\mathcal{P}$ such that $H \cap \mathcal{P} = \{v\}$. The set of vertices of $\mathcal{P}$ is denoted by $V(\mathcal{P})$.

It can be shown that $v \in \mathcal{P}$ is a vertex of $\mathcal{P}$ if and only if it doesn't belong to the interior of a line segment in $\mathcal{P}$ (that is, if and only if there do not exist $u \neq w \in \mathcal{P}$ and $0 < \lambda < 1$ such that $v = \lambda u + (1-\lambda)w$). It can also be shown that a polytope $\mathcal{P}$ is the convex hull of its vertices, that is, $\mathcal{P} = conv(V(\mathcal{P}))$. If all vertices of $\mathcal{P}$ have *integer* coordinates, then $\mathcal{P}$ is called an *integral polytope*.

Definition 2.10. The *interior* of a polytope $\mathcal{P}$ is its topological interior, that is, the set of all points $p \in \mathcal{P}$ such that for some $\epsilon > 0$, the open $\epsilon-$ball around p, denoted by $B_\epsilon(p)$, is contained in $\mathcal{P}$. Analogously, the *relative interior*, is its topological interior as a subset of the affine space $\mathit{aff}(\mathcal{P})$, that is, the set of all points $p \in \mathcal{P}$ such that for some $\epsilon > 0$, the intersection $B_\epsilon(p) \cap \mathit{aff}(\mathcal{P})$ is contained in $\mathcal{P}$. The relative interior of $\mathcal{P}$ is denoted by $\mathcal{P}^\circ$.

Definition 2.11. A **face** of a polytope $\mathcal{P}$ is a set of the form $\mathcal{P} \cap H$ where H is a supporting hyperplane of $\mathcal{P}$.

We note that the empty set is a face of $\mathcal{P}$ corresponding to a hyperplane that does not meet $\mathcal{P}$. Moreover, $\mathcal{P}$ itself is a face of $\mathcal{P}$ corresponding to the degenerate hyperplane (the whole space which contains $\mathcal{P}$). It can be shown that any face of $\mathcal{P}$ is itself a polytope. If $\mathcal{P}$ is a $d-$polytope, then the faces of dimension $0, 1, d-2$ and $d-1$ are called *vertices*, *edges*, *ridges* and *facets* of $\mathcal{P}$, respectively. This definition of vertices matches Definition 2.9.

Example 2.12. As an illustration, we introduce the standard simplex polytope and the permutohedron polytope.

(I) The *standard d-simplex*, denoted by Δ_d, is the convex hull of the set $\{e_1, e_2, ..., e_{d+1}\}$ of standard unit vectors in $\mathbb{R}^{d+1}$. In Figure 2.1a, Δ_2 is shown. In other words, we can write

$$\begin{aligned}\Delta_d &= Conv\{e_1, e_2, ..., e_{d+1}\} \\ &= \left\{(x_1, ..., x_{d+1}) \in \mathbb{R}^{d+1} \mid x_1 + x_2 + ... + x_{d+1} = 1 \text{ and all } x_i \geq 0 \right\}\end{aligned}$$

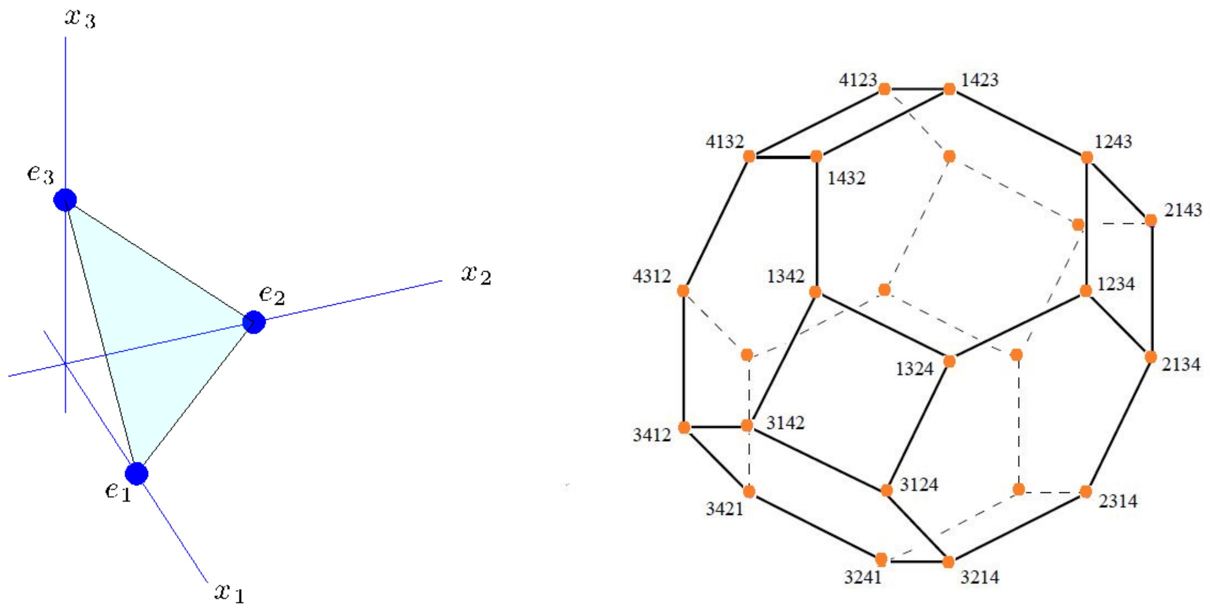
It follows that Δ_d has dimension d.

(II) The *permutohedron* Π_d is the convex hull of all $d-$vectors whose coordinates are $\{1, 2, ..., d\}$, in any order. It is known that Π_d has $d!$ vertices and is a $(d-1)-$dimensional polytope that is contained in the hyperplane $\{X \in \mathbb{R}^d \,|\, \sum_{i=1}^d x_i = \frac{d(d+1)}{2}\}$. For more details on permutohedrons see [118]. The permutohedron Π_4 is shown in Figure 2.1b

Definition 2.13. For $r \in \mathbb{R}$, the r^{th} *dilate* of a polytope $\mathcal{P}$, denoted by $r\mathcal{P}$, is the set

$$r\mathcal{P} = \{rx \mid x \in \mathcal{P}\}$$

Before stating the theory of the Ehrhart polynomial, we will briefly discuss the concepts of the *continuous volume* and the *discrete volume* of a given polytope $\mathcal{P} \subseteq \mathbb{R}^m$. By the *continuous volume* of the polytope $\mathcal{P}$, we simply mean the usual concept of the volume which is given by the Riemannian integral. Now let $\mathbb{Z}^m = \{(x_1, ..., x_m) \,|\, x_i \in \mathbb{Z} \text{ for all } i \in [m]\}$ be a lattice (a discrete collection of equally spaced points in Euclidean space). Then the *discrete volume* of $\mathcal{P} \subseteq \mathbb{R}^m$ is defined to be the number of lattice (or integer) points inside $\mathcal{P}$, i.e., the number of elements in the set $\mathcal{P} \cap \mathbb{Z}^m$. By shrinking the lattice by an integer



(a) The standard simplex Δ_2.

(b) Permutohedron Π_4.

Figure 2.1: The standard simplex Δ_3 and permutohedron Π_4.

factor r, the cardinality $|\mathcal{P} \cap \frac{1}{r}^m|$ of shrunken lattice points inside an integral polytope $\mathcal{P}$ is a polynomial in r known as the *Ehrhart polynomial* of $\mathcal{P}$. If one keep shrinking the lattice by taking the limit, we obtain continuous volume (in usual Riemannian integral sense), that is

$$Vol(\mathcal{P}) = \lim_{r\to\infty} \frac{|\mathcal{P} \cap \frac{1}{r}\mathbb{Z}^m|}{r^m}$$

Now consider the case that $\mathcal{P} \subset \mathbb{R}^m$ is of dimension $d < m$ and let $span(\mathcal{P}) = \{u + \lambda(v - u) \,|\, u, v \in \mathcal{P}, \lambda \in \mathbb{R}\}$ be the affine span of $\mathcal{P}$. Then the *relative volume* of $\mathcal{P}$ is the volume relative to the sublattice $span(\mathcal{P}) \cap \mathbb{Z}^m$. Throughout this thesis, by volume of a polytope we mean the *normalized relative volume* which is defined in what follows unless otherwise stated.

Definition 2.14. For a polytope $\mathcal{P} \subseteq \mathbb{R}^m$ with $\dim \mathcal{P} = d$, the *volume* (or more precisely, the *normalized relative volume*) of $\mathcal{P}$, denoted by $Vol(\mathcal{P})$, can be defined as

$$Vol(\mathcal{P}) = \lim_{r\to\infty} \frac{d!\,|r\mathcal{P} \cap \mathbb{Z}^m|}{r^d} \tag{2.2}$$

It can be shown that this limit exists and is finite for any $\mathcal{P}$. Note that $|r\mathcal{P} \cap \mathbb{Z}^m|$ is the number of points in r^{th} dilate of $\mathcal{P}$ whose coordinates are all integers. For further information on the volume of $\mathcal{P}$, including an alternative definition involving integration see [10].

Definition 2.15. The *face lattice* of the polytope $\mathcal{P}$, denoted by $FL(\mathcal{P})$, is the poset of faces of $\mathcal{P}$ ordered by containment.

It can be shown that it is a lattice equipped with set union and intersection as the *join* and *meet* operations, respectively, and that it is graded with $rank(F) = \dim F + 1$, for any $F \in FL(\mathcal{P})$.
Now we give a brief review of Ehrhart theory for the enumeration of lattice points in integral dilates of integral polytopes via Ehrhart polynomials. These polynomials are named after Eugène Ehrhart who studied them first in the 1960s [46].

The following theorem is very important and is due to results of Ehrhart and Macdonald [70]. For a standard proof the reader is encouraged to see [10] or [98].

Theorem 2.16. *For an integral polytope $\mathcal{P}$ in $\mathbb{R}^m$ with relative interior $\mathcal{P}^\circ$, there exists a unique function $L_P(r)$, where r can be regarded as a complex variable, with the properties that:*

(i) $L_{\mathcal{P}}(r)$ *is a polynomial in r of degree* $\dim \mathcal{P}$.

(ii) $|r\mathcal{P} \cap \mathbb{Z}^m| = L_{\mathcal{P}}(r)$ *for each* $r \in \mathbb{N}$.

(iii) $|r\mathcal{P}^\circ \cap \mathbb{Z}^m| = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}}(-r)$ *for each* $r \in \mathbb{P}$.

(iv) $L_{\mathcal{P}}(r)$ *can be expressed in the form*

$$L_{\mathcal{P}}(r) = \sum_{k=0}^{\dim \mathcal{P}} c_k \binom{r+k}{\dim \mathcal{P}} \tag{2.3}$$

with $c_k \in \mathbb{N}$ *for each* $k \in [0, \dim \mathcal{P}]$.

It follows from Definition 2.14 and from (ii) and (iv) of Theorem 2.16 that for an integral polytope $\mathcal{P}$,

$$Vol\,\mathcal{P} = \sum_{k=0}^{\dim \mathcal{P}} c_k \tag{2.4}$$

where $c_0, ..., c_{\dim \mathcal{P}}$ are the coefficients in (2.3). In particular, once the binomial form with the c_i coefficients is expanded we get the standard polynomial form in the variable r, where the leading coefficient gives the unnormalized relative volume of the polytope and the second leading coefficient gives half the boundary content, see Theorem 5.6 of [10] for further information. This means that each facet of the polytope will have its own Ehrhart polynomial, the leading term of which will give the unnormalized relative volume of the facet in dimension one lower than the polytope.

2.2.1 Order polytope

In this section, we outline the definition of *order polytopes* and their fundamental properties. An example of Theorem 2.18 will be given in Section 2.5 for the order polytope of the ASM poset P_n.

Definition 2.17. The order polytope of a given poset P, denoted by $\mathcal{O}(P)$, is the subset of $\mathbb{R}^P$ (the set of all maps $\theta : P \to [0,1]_{\mathbb{R}}$) whose elements satisfy the following conditions:

(1) $0 \le \theta(t) \le 1$ for all $t \in P$;

(2) $\theta(t) \le \theta(s)$ if $t \le s$ in P.

That is, $\theta(P)$ is the set of all order preserving maps $\theta : P \to \mathbb{R}$. By conditions (1) and (2), since $\mathcal{O}(P)$ is bounded and defined by inequalities, it is a polytope. Theorem 2.18 provides basic properties of the order polytope of a given poset P.

Theorem 2.18. *Let P be a finite poset with n elements. Then the following statements hold for its associated order polytope $\mathcal{O}(P)$:*

(I) The dimension of $\mathcal{O}(P)$ is $|P|$;

(II) There is a bijection between the vertices of $\mathcal{O}(P)$ and $J(P)$ (the set of all order ideals of P). In particular, the vertex χ_I corresponding to an order ideal $I \in J(P)$ is given by

$$\chi_I(s) = \begin{cases} 1, & s \notin I \\ 0, & s \in I \end{cases} \tag{2.5}$$

(III) The number of facets of $\mathcal{O}(P)$ is equal to $c + p + q$ where c is the number of cover relations in P, p is the number of minimal elements in P and q is the number of maximal elements in P;

(IV) The Ehrhart polynomial of $\mathcal{O}(P)$ is given by

$$L_r(\mathcal{O}(P)) = \Omega(P, r+1) \tag{2.6}$$

where $\Omega(P, r+1)$ is the order polynomial of P;

(V) The volume of $\mathcal{O}(P)$ is $e(P)$, where $e(P)$ is the number of linear extensions of P.

For the proof of Theorem 2.18, see Stanley [96].

2.2.2 Flow polytope

In this section, we will give the definition of flow polytopes and some related topics. Flow polytopes are a family of polytopes with remarkable enumerative and geometric properties. Their combinatorial and geometric study started with work of Baldoni and Vergne [6] and unpublished work of Postnikov and Stanley. For more recent work on flow polytopes, the reader is encouraged to read [16].

Definition 2.19. Let G be a connected graph on the vertex set $[n]$ with edges directed from the smaller vertex to the larger vertex such that each vertex $v \in \{2, 3, ..., n-1\}$ has both incoming and outgoing edges and let $\mathbf{a} = (a_1, a_2, ..., a_n) \in \mathbb{Z}^n$. Then the *flow polytope* of G is given by

$$\mathcal{F}_G(\mathbf{a}) := \{f : E(G) \to \mathbb{R}_{\geq 0} \mid (\text{netflow at vertex } i) = a_i \text{ for each } i \in [n]\} \quad (2.7)$$

where $E(G)$ is the edge set of G, and the netflow at vertex i is given by

$$\sum_{(i,j) \in E(G), i<j} f((i,j)) - \sum_{(j,i) \in E(G), j<i} f((j,i)) \tag{2.8}$$

We call $f \in \mathcal{F}_G(\mathbf{a})$ a flow on G with netflow vector $\mathbf{a}$.

Example 2.20. As an illustration, consider the complete graph K_5. Then its associated flow polytope with netflow vector $(1, 0, 0, 0, -1)$ is given by

$$\mathcal{F}_{K_5}(1,0,0,0,-1) = \left\{ F : E(K_5) \to \mathbb{R}_{\geq 0} \;\middle|\; \begin{array}{l} \bullet\ a + b + c + d = 1; \\[1em] \bullet\ e + f + g - a = 0; \\[1em] \bullet\ h + i - b - c = 0; \\[1em] \bullet\ j - c - f - h = 0. \end{array} \right\} \tag{2.9}$$

where $\{a, b, c, d, e, f, g, h, i, j\}$ is the set of flow variables on the edges of K_5. For example, $F((1,2)) = a$ and $F((1,3)) = b$. Figure 2.2 shows the flow polytope $\mathcal{F}_{K_5}(1,0,0,0,-1)$.

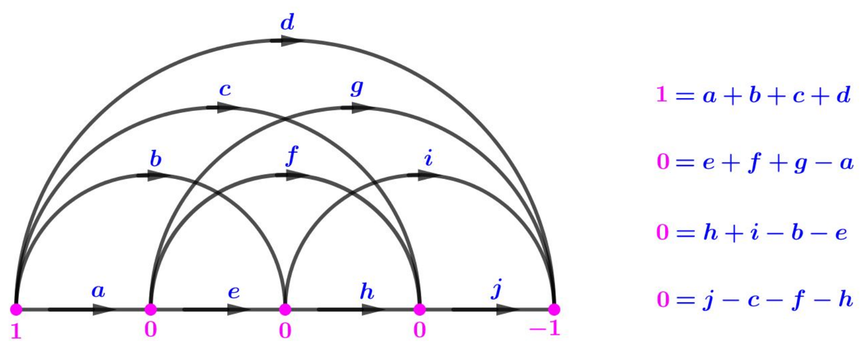


Figure 2.2: The flow polytope associated with complete graph K_5 with edges directed from smaller vertices to the bigger ones. The flow variables on the edges are $a, b, c, d, e, f, g, h, i, j$ with netflow vector $(1, 0, 0, 0, -1)$. The equations defining the flow polytope corresponding to K_5 are also given on the right hand side of the figure.

Theorem 2.22 is due to Postnikov-Stanley (unpublished work) and Baldoni-Vergne [6] and is the main theorem regarding flow polytopes with netflow vector $(1, 0, ..., 0, -1)$. To state it we need the following definition.

Definition 2.21. For a graph G as in Definition 2.19, the *Kostant partition function* $\mathcal{K}_G(V)$ is the number of ways to write a given vector $V \in \mathbb{Z}^n$ as a non-negative integer linear combination of the vectors $e_{ij} = e_i - e_j$, where e_i is the i^{th} standard basis vector in $\mathbb{R}^n$ and $(i,j) \in E(G)$.

As an illustration, for the complete graph of order 3, K_3, and the vector $V = (1,2,-3) \in \mathbb{R}^3$, $\mathcal{K}_{K_3}(V) = 2$ since

$$V = e_{12} + 3e_{23} \qquad \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

or equivalently

$$= e_{13} + 2e_{23} \qquad = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Theorem 2.22. *Let G be a graph as in Definition 2.19 and let $d_i = (\text{indegree of } i) - 1$ where by indegree of i we mean the number of edges that come into the vertex i. Then the volume of the flow polytope $\mathcal{F}_G(1,0,...,0,-1)$ associated with graph G is given by*

$$\mathcal{K}_G\left(0, d_2, ..., d_{n-1}, -\sum_{i=2}^{n-1} d_i\right) \tag{2.10}$$

where $\mathcal{K}_G$ is the Kostant partition function.

2.3 Birkhoff polytope

In this section, we outline the definition of the *Birkhoff polytope* and its fundamental properties. For more details on the Birkhoff polytope as a special case of assignment polytopes see Chapter 9 of [24] or [112].

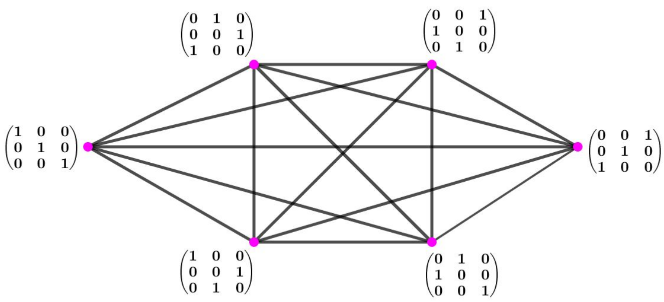
Definition 2.23. The *Birkhoff polytope* of order n is the convex hull of all permutation matrices of size n (that is, $n \times n$ matrices consisting of precisely one 1 in each row and column and zero everywhere else). Alternatively, it can be defined as the convex hull of doubly stochastic matrices of size n (that is, $n \times n$ matrices with non-negative real entries with each complete row and column sum equal to 1). We denote the Birkhoff polytope of order n by $\mathcal{B}_n$. In particular, the hyperplane description of $\mathcal{B}_n$ is given by

$$\mathcal{B}_n = \left\{ B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} \in \mathbb{R}^{n^2} \;\middle|\; \begin{array}{l} \bullet\ b_{ij} \geq 0 \text{ for all } i, j \in [n]; \\ \bullet\ \sum_{j=1}^{n} b_{ij} = 1 \text{ for all } i \in [n]; \\ \bullet\ \sum_{i=1}^{n} b_{ij} = 1 \text{ for all } j \in [n]. \end{array} \right\} \tag{2.11}$$

Example 2.24. As an illustration, a doubly stochastic matrix of size 5 is shown in Figure 2.3b. In addition, for $n = 3$, $\mathcal{B}_3$ is given by

$$\mathcal{B}_3 = conv\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\} \tag{2.12}$$

As an illustration, $\mathcal{B}_3$ is shown in Figure 2.3a.



(a) The Birkhoff Polytope $\mathcal{B}_3$.

$$\begin{pmatrix} \mathbf{0} & \mathbf{0.1} & \mathbf{0.2} & \mathbf{0.3} & \mathbf{0.4} \\ \mathbf{0.4} & \mathbf{0.3} & \mathbf{0.2} & \mathbf{0.1} & \mathbf{0} \\ \mathbf{0.3} & \mathbf{0.2} & \mathbf{0.1} & \mathbf{0} & \mathbf{0.4} \\ \mathbf{0.2} & \mathbf{0.3} & \mathbf{0} & \mathbf{0.4} & \mathbf{0.1} \\ \mathbf{0.1} & \mathbf{0.1} & \mathbf{0.5} & \mathbf{0.2} & \mathbf{0.1} \end{pmatrix}$$

(b) A 5×5 doubly stochastic matrix.

Figure 2.3: The Birkhoff Polytope $\mathcal{B}_3$ and a 5×5 doubly stochastic matrix in $\mathcal{B}_5$.

In Theorem 2.25, some fundamental properties of $\mathcal{B}_n$ are outlined. Part (a) of Theorem 2.25, follows from the Definition 2.6 of the dimension of a polytope by observing that

$$aff(\mathcal{B}_n) = \left\{ B \in \mathbb{R}^{n^2} \;\middle|\; \sum_{i'=1}^{n} b_{i'j} = \sum_{j'=1}^{n} b_{ij'} = 1 \text{ for all } i, j \in [n] \right\}$$

and that out of $2n$ linear equations in n^2 variables within this set, there exist only $2n-1$ independent equations. Part (b) is a consequence of the well-known *Birkhoff-von Neumann Theorem* (see [18] or [110]). Part (c) is a consequence of Definition 2.11 and the fact that each facet of $\mathcal{B}_n$ has dimension $(n-1)^2 - 1$. For the proof of statements $(d)-(e)$, see [17] or [24].

Theorem 2.25. *The following statements hold for $\mathcal{B}_n$.*

(a) The dimension of $\mathcal{B}_n$ is $(n-1)^2$.

(b) The vertices of $\mathcal{B}_n$ are all $n \times n$ permutation matrices. Thus $\mathcal{B}_n$ has $n!$ vertices.

(c) For $n \geq 3$, $\mathcal{B}_n$ has n^2 facets, given by $\{b \in \mathcal{B}_n \,|\, b_{ij} = 0\}$, for all $i, j \in [n]$.

(d) There is a natural isomorphism between the face lattice of $\mathcal{B}_n$ and the lattice of all elementary spanning subgraphs of the complete bipartite graph $K_{n,n}$ ordered by inclusion (A spanning subgraph of $K_{n,n}$ is elementary if its edge set is a union of perfect matchings of $K_{n,n}$). In this isomorphism, the graph corresponding to the face F of $\mathcal{B}_n$ has the edge set

$$\{\{i, j'\} \mid \exists x \in F \text{ with } x_{ij} > 0\}$$

and the face which corresponds to an elementary subgraph G of $K_{n,n}$ is

$$\{x \in \mathcal{B}_n \,|\, x_{ij} = 0 \text{ for all } i, j \text{ such that } \{i, j'\} \notin E(G)\}$$

where $E(G)$ is the edge set of the subgraph G and the vertex and edge sets of $K_{n,n}$ are $\{1, 2, ..., n; 1', 2', ..., n'\}$ and $\{\{i, j'\} \mid i, j \in [n]\}$, respectively.

(e) The dimension of a face of $\mathcal{B}_n$ is given by $e - 2n + c$, where e and c are the numbers of edges and components of its corresponding elementary subgraph of $K_{n,n}$, respectively.

Proceeding to the integer points of the integer dilates of $\mathcal{B}_n$, it can be seen that these are the semimagic squares, introduced in (1.18). More precisely for $r \in \mathbb{N}$,

$$SMS(n,r) = r\mathcal{B}_n \cap \mathbb{Z}^{n^2} = \left\{ \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \in \mathbb{N}^{n^2} \,\middle|\, \textstyle\sum_{j'=1}^{n} a_{ij'} = \sum_{i'=1}^{n} a_{i'j} = r \text{ for all } i, j \in [n] \right\} \tag{2.13}$$

Furthermore, the set of integer points of the integer dilates of the relative interior $\mathcal{B}_n^\circ$ are denoted by $SMS^\circ(n,r)$ and given by

$$SMS^\circ(n,r) = r\mathcal{B}_n^\circ \cap \mathbb{Z}^{n^2} = \left\{ \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \in \mathbb{P}^{n^2} \,\middle|\, \textstyle\sum_{j'=1}^{n} a_{ij'} = \sum_{i'=1}^{n} a_{i'j} = r, \text{ and for all } i, j \in [n] \right\} \tag{2.14}$$

for $r \in \mathbb{P}$. It follows that

$$SMS^\circ(n,r) = \varnothing, \quad \text{for each } n,\, r \in \mathbb{P} \text{ with } r < n. \tag{2.15}$$

Now by Theorem 2.16, since $\mathcal{B}_n$ is an integral polytope, it has an Ehrhart polynomial $L_{\mathcal{B}_n}(r)$, which we denote by $H_n(r)$. The following theorem states the main result for the enumeration of $SMS(n,r)$ and

$SMS^\circ(n,r)$. Note that parts (1) – (3) follows from parts (i) – (iii) of Theorem 2.16. For the complete proof and more details see [10], [15] or [98].

Theorem 2.26. *For fixed $n \in \mathbb{P}$, $H_n(r)$ satisfies the following properties:*

(1) $H_n(r)$ *is a polynomial in* r *of degree* $(n-1)^2$.

(2) $|SMS(n,r)| = H_n(r)$, *for each* $r \in \mathbb{N}$.

(3) $|SMS^\circ(n,r)| = (-1)^{n+1} H_n(r)$, *for each* $r \in \mathbb{P}$.

(4) $H_n(-1) = H_n(-2) = \ldots = H_n(-n+1) = 0$.

(5) $H_n(1) = n!$.

(6) $H_n(r)$ *can be expressed in the form* $H_n(r) = \sum_{k=n-1}^{(n-1)^2} c_k \binom{r+k}{(n-1)^2}$ *with* $c_k \in \mathbb{N}$ *and* $c_k = c_{n(n-1)-k}$ *for each* $k \in [n-1,(n-1)^2]$. *This means that the coefficients of the binomial terms in* $H_n(r)$ *are palindromic (that is, they can be read the same backwards as forwards).*

(7) $H_n(r) = (-1)^{n+1} H_n(-n-r)$ *for all* $n \in \mathbb{P}$ *and* $r \in \mathbb{C}$.

As already indicated in Section 1.1, one of the most challenging open problems related to the Birkhoff polytope is the computation of its volume. The importance of computing the volume of polytopes is also discussed in Section 1.1. As an illustration, the Ehrhart polynomial of $\mathcal{B}_n$ is given in (2.16) for $n \in [5]$.

$$
\begin{aligned}
H_1(r) &= \binom{r}{0} \\
H_2(r) &= \binom{r+1}{1} \\
H_3(r) &= \binom{r+2}{4} + \binom{r+3}{4} + \binom{r+4}{4} \\
H_4(r) &= \binom{r+3}{9} + 14\binom{r+4}{9} + 87\binom{r+5}{9} + 148\binom{r+6}{9} + 87\binom{r+7}{9} + 14\binom{r+8}{9} + \binom{r+9}{9} \qquad (2.16) \\
H_5(r) &= \binom{r+4}{16} + 103\binom{r+5}{16} + 4306\binom{r+6}{16} + 63110\binom{r+7}{16} + 388615\binom{r+8}{16} + 1115068\binom{r+9}{16} \\
&\quad + 1575669\binom{r+10}{16} + 1115068\binom{r+11}{16} + 388615\binom{r+12}{16} + 63110\binom{r+13}{16} + 4306\binom{r+14}{16} \\
&\quad + 103\binom{r+15}{16} + \binom{r+16}{16}
\end{aligned}
$$

2.3.1 Chan-Robbins-Yuen polytope

The *Chan-Robbins-Yuen polytope* was first introduced by Chan, Robbins, and Yuen in 1998 as a face of $\mathcal{B}_n$ [38].

Definition 2.27. The Chan-Robbins-Yuen polytope of order n, denoted by $\mathcal{CRY}(n)$, is given by

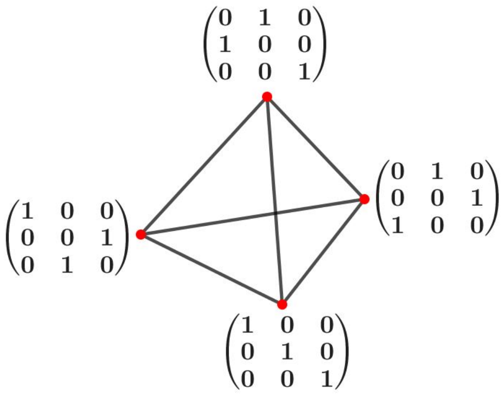
$$\mathcal{CRY}(n) = \left\{B = (b_{ij})_{i,j=1}^{n} \in \mathcal{B}_n \mid b_{ij} = 0 \text{ for } j - i \geq 2\right\} \quad (2.17)$$

Alternatively, the Chan-Robbins-Yuen polytope is the convex hull of all $n \times n$ permutation matrices π such that $\pi_{ij} = 0$ for all $j \geq i + 2$ (that is, permutation matrices with the entries above the main superdiagonal equal to 0).

Example 2.28. As an illustration, for $n = 3$, we have

$$\mathcal{CRY}(3) = conv\left\{\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{pmatrix}\right\}$$

The polytope $\mathcal{CRY}(3)$ and a 5×5 doubly stochastic matrix with all entries above the main superdiagonal equal zero are shown in Figures 2.4a and 2.4b.



(a) Chan-Robbins-Yuen polytope of order 3.

$$\begin{pmatrix}0.6 & 0.4 & 0 & 0 & 0\\0.1 & 0.2 & 0.7 & 0 & 0\\0.1 & 0.3 & 0.1 & 0.5 & 0\\0.1 & 0 & 0.1 & 0.3 & 0.5\\0.1 & 0.1 & 0.1 & 0.2 & 0.5\end{pmatrix}$$

(b) A 5×5 doubly stochastic matrix with all entries above the main superdiagonal equal zero.

Figure 2.4: Chan-Robbins-Yuen polytope of order 3 and a 5×5 doubly stochastic matrix in $\mathcal{CRY}(5)$.

In the following theorem, we summarize previously known key properties of the $\mathcal{CRY}(n)$. Note that the vertices of $\mathcal{CRY}(n)$ are $n \times n$ permutation matrices with all entries above the main superdiagonal equal zero, but $\mathcal{CRY}(n)$ lies in an affine space of dimension $\binom{n}{2}$ within n^2-dimensional ambient space. For instance, $\mathcal{CRY}(3)$ lies in a 3-dimensional affine space of 9-dimensional space. For the proof of the statements $(a) - (c)$ see [39]. The statement (d) was proved by Zeilberger in [117] and for a proof of part (e), see for example [73].

Theorem 2.29. *The following statements hold for* $\mathcal{CRY}(n)$.

(a) $\mathcal{CRY}(n)$ *has dimension* $\binom{n}{2}$.

(b) The vertices of $\mathcal{CRY}(n)$ *are all permutation matrices* $\sigma = (\sigma_{ij})_{i,j=1}^{n}$ *in which* $\sigma_{ij} = 0$ *whenever* $j \geq i + 2$. *In particular,* $\mathcal{CRY}(n)$ *has* 2^{n-1} *vertices.*

(c) *For* $n \geq 3$, $\mathcal{CRY}(n)$ *has* $\frac{n(n+1)}{2} - 2$ *facets.*

(d) *The volume of* $\mathcal{CRY}(n)$ *is given by*

$$Vol(\mathcal{CRY}(n)) = \prod_{i=1}^{n-2} C_i$$

where $C_i = \frac{1}{i+1}\binom{2i}{i}$ *is the* i^{th} *Catalan number. Note that this is the normalized relative volume, as defined in Definition 2.14.*

(e) *There is a natural affine isomorphism between* $\mathcal{CRY}(n)$ *and the flow polytope* $\mathcal{F}_{K_{n+1}}(1, 0, ..., 0, -1)$*, where* K_{n+1} *is the complete graph on vertex set* $[n+1]$*.*

Since $\mathcal{CRY}(n)$ is an integral polytope, by Theorem 2.16, it has an Ehrhart polynomial, which we denote by $H'_n(r)$. We compute $H'_n(r)$ for $n \in [7]$ using some Mathematica code.

$$\begin{aligned}
H'_1(r) =& \binom{r}{0} \\
H'_2(r) =& \binom{r+1}{1} \\
H'_3(r) =& \binom{r+3}{3} \\
H'_4(r) =& \binom{r+5}{6} + \binom{r+6}{6} \\
H'_5(r) =& 4\binom{r+8}{10} + 5\binom{r+9}{10} + \binom{r+10}{10} \\
H'_6(r) =& 9\binom{r+11}{15} + 56\binom{r+12}{15} + 58\binom{r+13}{15} + 16\binom{r+14}{15} + \binom{r+5}{15} \\
H'_7(r) =& 169\binom{r+15}{21} + 1182\binom{r+16}{21} + 2364\binom{r+17}{21} + 1674\binom{r+18}{21} + 448\binom{r+19}{21} \\
&+ 42\binom{r+20}{21} + \binom{r+21}{21}
\end{aligned} \tag{2.18}$$

2.4 Alternating sign matrix polytope

The *alternating sign matrix polytope*, or *ASM polytope* for short, was introduced independently by Behrend and Knight (as a bounded intersection of finitely many halfspaces in Euclidean space known as the hyperplane description of a polytope) in [15] and Striker in [100], [101] (as the convex hull of finitely many points in Euclidean space known as the vertex description of a polytope). We begin with the definition of the ASM polytope and its fundamental properties.

Definition 2.30. The **alternating sign matrix polytope** of order n, denoted by $\mathcal{A}_n$, is the set of $n \times n$ real-entry matrices for which all complete row and column sums are 1, and all partial row and column sums extending from each end of each row or column are non-negative. In other words we have

$$\mathcal{A}_n := \left\{ X = \begin{pmatrix} x_{11} & \dots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2} \;\middle|\; \begin{array}{l} \bullet\ \sum_{j'=1}^{n} x_{ij'} = \sum_{i'=1}^{n} x_{i'j} = 1 \text{ for all } i, j \in [n] \\ \bullet\ \sum_{j'=1}^{j} x_{ij'} \geq 0 \text{ for all } i \in [n],\ j \in [n-1] \\ \bullet\ \sum_{j'=j}^{n} x_{i'j} \geq 0 \text{ for all } i \in [n],\ j \in [2, n] \\ \bullet\ \sum_{i'=1}^{i} x_{i'j} \geq 0 \text{ for all } i \in [n-1],\ j \in [n] \\ \bullet\ \sum_{i'=i}^{n} x_{i'j} \geq 0 \text{ for all } i \in [2, n],\ j \in [n] \end{array} \right\}. \tag{2.19}$$

It is not hard to see that each entry of $X \in \mathcal{A}_n$ is in the real interval $[-1, 1]$ and that entries in the first and last row and column of X belong to the real interval $[0, 1]$. This implies that $\mathcal{A}_n$ is a bounded subset of $\mathbb{R}^{n^2}$. Moreover, since $\mathcal{A}_n$ is an intersection of finitely-many closed halfspaces and hyperplanes in $\mathbb{R}^{n^2}$, it is a polytope. Since doubly stochastic matrices are elements of $\mathcal{A}_n$ whose entries are non-negative, it follows that $\mathcal{B}_n$ is contained in $\mathcal{A}_n$.

Example 2.31. As an illustration, an element of $\mathcal{A}_5$ is given by

$$\begin{pmatrix} 0 & 0.1 & 0.2 & 0.3 & 0.4 \\ 0.4 & 0.3 & 0.3 & -0.3 & 0.3 \\ 0.3 & -0.3 & 0.4 & 0.4 & 0.2 \\ 0.2 & 0.8 & -0.4 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.5 & 0.3 & 0 \end{pmatrix} \in \mathcal{A}_5$$

Similar to the case for the Birkhoff polytope, we outline the basic properties of $\mathcal{A}_n$ in the following single theorem. For the proof, see [15] or [100], [101].

Theorem 2.32. *The following statements hold for the ASM polytope $\mathcal{A}_n$.*

(a) The dimension of $\mathcal{A}_n$ is $(n-1)^2$.

(b) The vertices of $\mathcal{A}_n$ are all ASMs of size n. Thus the number of vertices of $\mathcal{A}_n$ is given by

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

(c) For $n \geq 3$, $\mathcal{A}_n$ has $4(1 + (n-2)^2)$ facets, given by

$$\{A \in \mathcal{A}_n \,|\, a_{11} = 0\} \quad , \quad \{A \in \mathcal{A}_n \,|\, a_{1n} = 0\}$$
$$\{A \in \mathcal{A}_n \,|\, a_{n1} = 0\} \quad , \quad \{A \in \mathcal{A}_n \,|\, a_{nn} = 0\}$$

and

$$\left\{A \in \mathcal{A}_n \,\middle|\, \sum_{j'=1}^{j-1} a_{ij'} = 0\right\} \quad , \quad \left\{A \in \mathcal{A}_n \,\middle|\, \sum_{i'=1}^{i-1} a_{i'j} = 0\right\}.$$
$$\left\{A \in \mathcal{A}_n \,\middle|\, \sum_{j'=j+1}^{n} a_{ij'} = 0\right\} \quad , \quad \left\{A \in \mathcal{A}_n \,\middle|\, \sum_{i'=i+1}^{n} a_{i'j} = 0\right\}$$

for each $i, j = 2, \ldots, n-1$.

(d) *There is a natural isomorphism between the face lattice of* $\mathcal{A}_n$ *and a lattice of all elementary flow grids of order n (as defined in Definition 1.27) ordered by inclusion.*

(e) *The dimension of a face* F *of* $\mathcal{A}_n$ *is the number of doubly directed regions (as defined in Definition 1.27) in the elementary flow grid associated with* F *using the isomorphism of* (d).

We note that the higher spin ASMs of size n with line sum r as introduced in Definition 1.6 are the integer points of the r^{th} dilates of $\mathcal{A}_n$. That is, we can write

$$ASM(n,r) = r\mathcal{A}_n \cap \mathbb{Z}^{n^2} \tag{2.20}$$

Furthermore, the sets of integer dilates of the relative interior $\mathcal{A}_n^\circ$ (which is obtained by simply replacing each weak inequality in (2.19) by a strict inequality) are denoted by $\mathrm{ASM}^\circ(n,r)$ and given by

$$\mathrm{ASM}^\circ(n,r) := r\mathcal{A}_n^\circ \cap \mathbb{Z}^{n^2} = \left\{ \begin{pmatrix} A_{11} & \ldots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \ldots & A_{nn} \end{pmatrix} \in \mathbb{Z}^{n\times n} \,\middle|\, \begin{array}{l} \bullet\ \sum_{j'=1}^{n} A_{ij'} = \sum_{i'=1}^{n} A_{i'j} = r \ \text{ for all } i,j \in [n] \\ \bullet\ \sum_{j'=1}^{j} A_{ij'} \geq 1 \ \text{ for all } i \in [n],\ j \in [n-1] \\ \bullet\ \sum_{j'=j}^{n} A_{ij'} \geq 1 \ \text{ for all } i \in [n],\ j \in [2,n] \\ \bullet\ \sum_{i'=1}^{i} A_{i'j} \geq 1 \ \text{ for all } i \in [n1],\ j \in [n] \\ \bullet\ \sum_{i'=i}^{n} A_{i'j} \geq 1 \ \text{ for all } i \in [2,n],\ j \in [n] \end{array} \right\} \tag{2.21}$$

Thus, $\mathrm{ASM}^\circ(n,r)$ is the set of $n \times n$ integer-entry matrices for which all complete row and column sums are r, and all partial row and column sums extending from each end of the row or column are positive. Note that

$$\mathrm{ASM}^\circ(n,r) = \varnothing, \quad \text{for each } n, r \in \mathbb{P} \text{ with } r < n. \tag{2.22}$$

Similar to the case for $\mathcal{B}_n$, by Therem 2.16, $\mathcal{A}_n$ has an Ehrhart polynomial which satisfy the following properties. See [15] for the proof.

Theorem 2.33. *For fixed* $n \in \mathbb{P}$, *there exists a function* $A_n(r)$, *the Ehrhart polynomial of the alternating sign matrix polytope* $\mathcal{A}_n$, *which satisfies:*

(1) $A_n(r)$ *is a polynomial in* r *of degree* $(n-1)^2$.

(2) $|ASM(n,r)| = A_n(r)$, *for each* $r \in \mathbb{N}$.

(3) $|\mathrm{ASM}^\circ(n,r)| = (-1)^{n+1} A_n(-r)$, *for each* $r \in \mathbb{P}$.

(4) $A_n(-1) = A_n(-2) = \ldots = A_n(-n+1) = 0$.

(5) $A_n(1) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$.

(6) $A_n(r)$ *can be expressed in the form* $A_n(r) = \sum_{k=n-1}^{(n-1)^2} c_k \binom{r+k}{(n-1)^2}$ *with* $c_k \in \mathbb{N}$ *for each* $k \in [n-1,(n-1)^2]$.

It follows that the explicit polynomial $A_n(r)$ for a particular $n \in \mathbb{P}$ can be found by interpolation using the $n+1$ values provided by $A_n(0) = 1$ and properties (4) and (5), together with $n^2 - 3n + 1$ further values obtained by the direct enumeration of cases of $ASM(n,r)$ or $\mathrm{ASM}^\circ(n,r)$ and the application of properties (2) and (3). This was done for trivial cases $n = 1$ and 2 and $n = 3$, 4 and 5 in [15]. The results, expressed in the form of (6) of Theorem 2.33 are given by

$$
\begin{aligned}
A_1(r) &= \binom{r}{0} \\
A_2(r) &= \binom{r+1}{1} \\
A_3(r) &= \binom{r+2}{4} + 2\binom{r+3}{4} + \binom{r+4}{4} \\
A_4(r) &= 3\binom{r+3}{9} + 80\binom{r+4}{9} + 415\binom{r+5}{9} + 592\binom{r+6}{9} \\
&\quad + 253\binom{r+7}{9} + 32\binom{r+8}{9} + \binom{r+9}{9}
\end{aligned}
\tag{2.23}
$$

and

$$
\begin{aligned}
A_5(r) &= 70\binom{r+4}{16} + 14468\binom{r+5}{16} + 521651\binom{r+6}{16} + 6002192\binom{r+7}{16} \\
&\quad + 28233565\binom{r+8}{16} + 61083124\binom{r+9}{16} + 64066830\binom{r+10}{16} \\
&\quad + 32866092\binom{r+11}{16} + 7998192\binom{r+12}{16} + 854464\binom{r+13}{16} \\
&\quad + 34627\binom{r+14}{16} + 412\binom{r+15}{16} + \binom{r+16}{16}
\end{aligned}
\tag{2.24}
$$

Note that using (2.4), the sums of the coefficients of the binomial terms in (2.23)- (2.24) give the volume of $\mathcal{A}_n$. Table 2.1 shows the volume of $\mathcal{A}_n$ for $n \in [7]$. The computation of the volumes of $\mathcal{A}_6$ and $\mathcal{A}_7$ depends on new results which will be presented in Chapter 3. These results will also be used to compute the Ehrhart polynomials $A_4(r)$, $A_5(r)$ and $A_6(r)$.

2.5 Alternating sign matrix order polytope

In this section, we introduce the *alternating sign matrix order polytope*, or ASM order polytope for short. This polytope is associated with the ASM poset P_n that was introduced in Section 1.5. In addition, we outline its fundamental properties in Theorem 2.36 as an application of Theorem 2.18.

Definition 2.34. The *ASM order polytope* associated with the ASM poset P_n, denoted by $\mathcal{O}(P_n)$, is defined by

$$\mathcal{O}(P_n) := \left\{ \theta : P_n \to [0,1]_{\mathbb{R}} \;\middle|\; \begin{array}{l} \bullet\ 0 \le \theta(i,j,k) \le 1 \text{ for any } (i,j,k) \in P_n; \\ \bullet\ \theta(i',j',k') \le \theta(i,j,k) \text{ if } (i',j',k') \le (i,j,k) \text{ in } P_n. \end{array} \right\}. \tag{2.25}$$

In other words, $\mathcal{O}(P_n)$ is the set of all order preserving maps $\theta : P_n \to [0,1]_{\mathbb{R}}$ that satisfy conditions given in (2.25).

Example 2.35. As an illustration, the ASM order polytope $\mathcal{O}(P_3)$ is given by

$$\mathcal{O}(P_3) = \left\{ \theta : P_3 \to [0,1]_{\mathbb{R}} \;\middle|\; \begin{array}{l} \bullet\ \theta(1,2,0) \le \theta(1,1,1); \\ \bullet\ \theta(1,2,0) \le \theta(2,2,1); \\ \bullet\ \theta(2,1,0) \le \theta(1,1,1); \\ \bullet\ \theta(2,1,0) \le \theta(2,2,1). \end{array} \right\}. \tag{2.26}$$

See Figure 1.21 for three alternative representations of the ASM poset P_3.

Consider the basic properties of the ASM poset given in Lemma 1.82. Then an application of Theorem 2.18 for the ASM order polytope $\mathcal{O}(P_n)$ is given in the following theorem.

Theorem 2.36. *The following statements hold for the ASM order polytope $\mathcal{O}(P_n)$:*

(I) The dimension of $\mathcal{O}(P_n)$ is $\binom{n+1}{3}$;

(II) There is a bijection (given by (2.5)) between the vertices of $\mathcal{O}(P_n)$ and the set of all order ideals $J(P_n)$ of P_n. Since there is also a bijection between $J(P_n)$ and $ASM(n)$, the number of vertices of $\mathcal{O}(P_n)$ is the number of $n \times n$ ASMs, i.e.,

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

(III) $\mathcal{O}(P_n)$ has $\frac{2}{3}(n-1)\left(n^2-2n+3\right)$ facets.

(IV) The volume of $\mathcal{O}(P_n)$ is $e(P_n)$, where $e(P_n)$ is the number of linear extensions of P_n;

(V) The Ehrhart polynomial of $\mathcal{O}(P_n)$ is given by

$$L_r(\mathcal{O}(P_n)) = \Omega(P_n, r+1) \tag{2.27}$$

Example 2.37. As an illustration, the Ehrhart polynomial of $\mathcal{O}(P_n)$ (or equivalently the order polynomial of P_n), for $n = 3, 4, 5$ are given by

$$L_r(\mathcal{O}(P_3)) = \binom{r+2}{4} + 2\binom{r+3}{4} + \binom{r+4}{4}$$

$$\begin{aligned} L_r(\mathcal{O}(P_4)) = &\binom{r+3}{10} + 31\binom{r+4}{10} + 237\binom{r+5}{10} + 627\binom{r+6}{10} \\ &+ 627\binom{r+7}{10} + 237\binom{r+8}{10} + 31\binom{r+9}{10} + \binom{r+10}{10} \end{aligned}$$

$$\begin{aligned} L_r(\mathcal{O}(P_5)) = &\binom{r+4}{20} + 408\binom{r+5}{20} + 37153\binom{r+6}{20} + 1194730\binom{r+7}{20} \\ &+ 1714020\binom{r+8}{20} + 124268754\binom{r+9}{20} + 490129375\binom{r+10}{20} + 1099041516\binom{r+11}{20} \\ &+ 1434897770\binom{r+12}{20} + 1099041516\binom{r+13}{20} + 490129375\binom{r+14}{20} + 124268754\binom{r+15}{20} \\ &+ 17140202\binom{r+16}{20} + 1194730\binom{r+17}{20} + 37153\binom{r+18}{20} + 408\binom{r+19}{20} + \binom{r+20}{20} \end{aligned} \tag{2.28}$$

In Table 2.1, the dimension and the volume of the ASM polytope $\mathcal{A}_n$ and the ASM order polytope $\mathcal{O}(P_n)$ are given for $n \in [7]$.

n	$dim(\mathcal{O}(P_n)) = \binom{n+1}{3}$	$dim(\mathcal{A}_n) = (n-1)^2$	$Vol(\mathcal{O}(P_n))$	$Vol(\mathcal{A}_n)$
3	4	4	4	4
4	10	9	1792	1376
5	20	16	4898522048	201675688
6	35	25	7076584870863697681920	37350087969236232
7	56	36	6878929475060252893322454706925941246664064	20423967603561169141089171040

Table 2.1: The volumes of the order polytope $\mathcal{O}(P_n)$ and ASM polytope $\mathcal{A}_n$.

Remark 2.38. Considering the coefficients of the polynomials in 2.28, it is clear that the coefficients of the binomial terms in the Ehrhart polynomial of $\mathcal{O}(P_n)$ are palindromic (that is, they can be read the same backwards as forwards). For more details and historic review on polytopes with palindromic Ehrhart polynomials the reader is encouraged to see [8]. It can be shown that the Ehrhart polynomial of the order polytope of any graded poset has this property.

Remark 2.39. Table 2.1 together with the Ehrhart polynomial of $\mathcal{A}_n$ (2.23) and $\mathcal{O}(P_n)$ (2.28), suggest that $\mathcal{O}(P_3)$ and $\mathcal{A}_3$ are affinely isomorphic since they have the same dimension and Ehrhart polynomial. This will be seen explicitly in the next section.

2.5.1 Affine isomorphism between $\mathcal{A}_3$ and $\mathcal{O}(P_3)$

Here we show that $\mathcal{A}_3$ and $\mathcal{O}(P_3)$ are affinely isomorphic. Consider the mappings $\phi : \mathcal{A}_3 \to \mathcal{O}(P_3)$ and $\psi : \mathcal{O}(P_3) \to \mathcal{A}_3$ as follows.

For $X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \in \mathcal{A}_3$, let $\phi(X)$, which we denote by ϕ_X, be given by

$$\begin{aligned} \phi_X(1,2,0) &= x_{11} \\ \phi_X(2,1,0) &= x_{33} \\ \phi_X(1,1,1) &= 1 - x_{13} \\ \phi_X(2,2,1) &= 1 - x_{31} \end{aligned} \tag{2.29}$$

For $\theta \in \mathcal{O}(P_3)$, let $\psi(\theta)$ be given by

$$\psi(\theta) = \begin{pmatrix} \theta(1,2,0) & \theta(1,1,1) - \theta(1,2,0) & 1 - \theta(1,1,1) \\ \theta(2,2,1) - \theta(1,2,0) & 1 + \theta(1,2,0) + \theta(2,1,0) - \theta(1,1,1) - \theta(2,2,1) & \theta(1,1,1) - \theta(2,1,0) \\ 1 - \theta(2,2,1) & \theta(2,2,1) - \theta(2,1,0) & \theta(2,1,0) \end{pmatrix} \tag{2.30}$$

It can be checked straightforwardly that ψ and ϕ are well-defined and inverse of each other. For example, the well-definedness of ψ can be observed using the conditions for $\mathcal{O}(P_3)$ given in Example 2.26. Furthermore, ϕ and ψ can be regarded as restrictions of affine maps between $\mathbb{R}^{3\times 3}$ and $\mathbb{R}^{P_3}$, so that $\mathcal{A}_3$ and $\mathcal{O}(P_3)$ are affinely isomorphic. The way that the mapping ϕ works is shown in Figure 2.5.

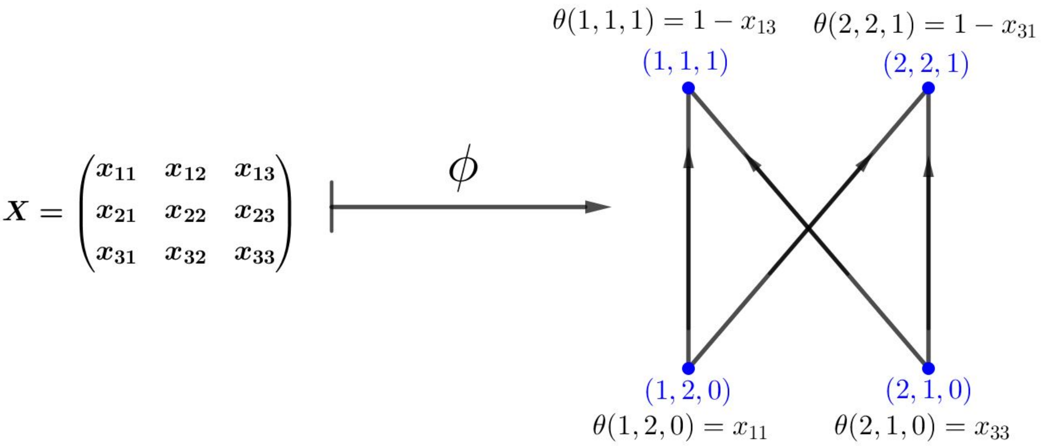


Figure 2.5: The affine isomorphism between $\mathcal{A}_3$ and $\mathcal{O}(P_3)$.

2.5.2 Alternating sign matrix Chan-Robbins-Yuen polytope

In this section we introduce an analogous ASM polytope version of the Chan-Robbins-Yuen polytope known as the *alternating sign matrix Chan-Robbins-Yuen polytope*, or *ASMCRY polytope* for short. The ASMCRY polytope was introduced by Mészáros, Morales and Striker in [73].

Definition 2.40. The ASM Chan-Robbins-Yuen polytope of order n, denoted by $\mathcal{ASMCRY}(n)$ is given by

$$\mathcal{ASMCRY}(n) \coloneqq \{A \in \mathcal{A}_n \mid a_{ij} = 0 \ \text{ if } j - i \geq 2 \ \text{ for all } i, j \in [n]\} \tag{2.31}$$

Alternatively, it can be shown that $\mathcal{ASMCRY}(n)$ is the convex hull of all $n \times n$ ASMs A such that $a_{ij} = 0$ for all $j - i \geq 2$.

Example 2.41. As an illustration, for $n = 3$, $\mathcal{ASMCRY}(3)$ is given by

$$\mathcal{ASMCRY}(3) = conv \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\}$$

The polytope $\mathcal{ASMCRY}(3)$ and a matrix in $\mathcal{ASMCRY}(5)$ are shown in Figures 2.6a and 2.6b, respectively.

Similar to the case for $\mathcal{CRY}(n)$, we outline the main properties of the $\mathcal{ASMCRY}(n)$ in the following single theorem. For the proof see [73].

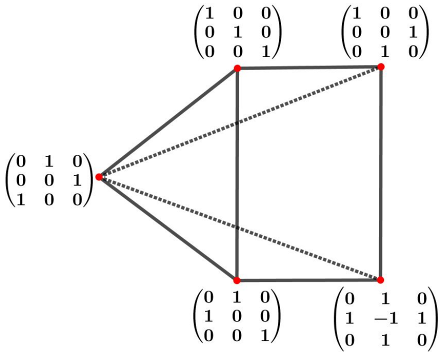
Theorem 2.42. *The following statements hold for $\mathcal{ASMCRY}(n)$:*

1. *$\mathcal{ASMCRY}(n)$ has dimension $\binom{n}{2}$.*
2. *The vertices of $\mathcal{ASMCRY}(n)$ are all ASMs $A = (a_{ij})_{i,j=1}^{n}$ in which $a_{ij} = 0$ whenever $j \geq i + 2$. It can be shown that $\mathcal{ASMCRY}(n)$ has $C_n = \frac{1}{n+1}\binom{2n}{n}$ vertices, where C_n is the n^{th} Catalan number.*
3. *For $n \geq 3$, $\mathcal{ASMCRY}(n)$ has $(n-1)^2 + 1$ facets.*
4. *The volume of the $\mathcal{ASMCRY}(n)$ is given by*

$$Vol(\mathcal{ASMCRY}(n)) = \frac{\binom{n}{2}!}{1^{n-1}3^{n-2}...(2n-3)^1}$$

5. *There is an affine isomorphism between $\mathcal{ASMCRY}(n)$ and the flow polytope $\mathcal{F}_{G_n}(1, 0, ..., 0, -1)$, of a certain graph G_n (See [73] for the exact definition of G_n).*
6. *There is an affine isomorphism between $\mathcal{ASMCRY}(n)$ and the order polytope of a certain poset δ_n^* (See [73] for the exact definition of δ_n^*).*
7. *The Ehrhart polynomial of $\mathcal{ASMCRY}(n)$ is given by*

$$L_{\mathcal{ASMCRY}_n}(r) = \Omega_{\delta_n^*}(r+1) = \prod_{1 \leq i < j \leq n} \frac{2r + i + j - 1}{i + j - 1}$$



(a) The polytope $\mathcal{ASMCRY}(3)$.

$$\begin{pmatrix} 0.7 & 0.3 & 0 & 0 & 0 \\ 0.1 & -0.1 & 1 & 0 & 0 \\ 0.1 & 0.1 & -0.2 & 1 & 0 \\ 0 & 0.7 & 0.2 & -0.4 & 0.5 \\ 0.1 & 0 & 0 & 0.4 & 0.5 \end{pmatrix}$$

(b) A matrix in $\mathcal{ASMCRY}(5)$ with all entries above the main superdiagonal equal to zero.

Figure 2.6: The polytope $\mathcal{ASMCRY}(3)$ and a matrix in $\mathcal{ASMCRY}(5)$.

2.6 Conclusion

This chapter began with some basic definitions and properties of polytopes. Then we outlined fundamental properties of some polytopes related to the ASM polytope, such as the Birkhoff polytope, Chan-Robbins-Yuen polytope, alternating sign matrix order polytope and alternating sign matrix Chan-Robbins-Yuen polytope.

CHAPTER 3

THE ENUMERATION OF HIGHER SPIN ALTERNATING SIGN MATRICES

3.1 Introduction

This chapter is organised as follows. First we provide a brief review of the theory of (P, ω)−partitions. Then we introduce a reduction operation on the ASM poset P_n (1.5) and define a labelling ω_n (in Definition 3.12) for this newly obtained reduced ASM poset in Section 3.2.1. By applying the theory of (P, ω)−partitions to this labelled reduced ASM poset, we prove the main theorem of this chapter, Theorem 3.23. This theorem enables us to enumerate $n \times n$ HSASMs for an arbitrary non-negative integer r. In particular, it enables us to give the Ehrhart polynomial of the ASM polytope $\mathcal{A}_n$ as the sum of order polynomials associated with the reduced ASM order polytopes, see Theorem 3.24.

3.2 (P, ω)-Partitions

We begin this section with an introduction to the theory of (P, ω)-partitions and some related properties and theorems. For more details and proof of the lemmas and theorems see Chapter 3 of Stanley's book [98], in particular, Section 3.15.

Definition 3.1. Let P be a finite poset of size p and let $\omega : P \to [p]$ be a bijection called a *labelling* of P. We then say that the pair (P, ω) is a *labelled poset*. We say ω is a *natural labelling* of P if $a < b$ in P implies that $\omega(a) < \omega(b)$. Hence a natural labelling is a linear extension of P. Analogously, we say ω is a *dual natural labelling* if $a < b$ in P implies that $\omega(a) > \omega(b)$.

Definition 3.2. Let (P, ω) be a finite labelled poset. Then a (P, ω)−partition is a map $\sigma : P \to \mathbb{N}$ satisfying the following conditions:

(a) σ is an *order-reversing map*, i.e., if $a \leq b$ in P then $\sigma(a) \geq \sigma(b)$;

(b) If $a < b$ in P and $\omega(a) > \omega(b)$, then $\sigma(a) > \sigma(b)$.

If the labelling ω is natural, then a (P, ω)−partition is just an order reversing map and σ is called a P−partition. Analogously, if ω is a dual natural labelling, then (P, ω)−partition is a *strict order-reversing map* and σ is called a *strict P-partition.*

The following fundamental statistics associated with a permutation $\pi \in S_n$ play a key role in the theory of P-partitions.

Definition 3.3. Let $\pi = \pi_1\pi_2...\pi_n \in S_n$ and $1 \leq i \leq n-1$. Then the position i is a *descent* of π if $\pi_i > \pi_{i+1}$. The set of descent positions of π is denoted by $Des(\pi)$ and we define $des(\pi) = |Des(\pi)|$ to be the number of descents in π. In other words, $des(\pi) := |\{i \in [n-1] \,|\, \pi_i > \pi_{i+1}\}|$.

Remark 3.4. Recall from Definition 1.57 that a linear extension of a finite poset P of size p is an order preserving bijection $\phi : P \to [p]$.

Definition 3.5. Let P be a finite poset of size p and ω be a labelling of P. Then the set of all $\omega-$labelled *linear extensions* (or *Jordan-Hölder set*) of P is denoted by $\mathcal{L}(P,\omega)$ and is given by

$$\mathcal{L}(P,\omega) := \left\{\left(\omega(\phi^{-1}\ (1)),...,\omega(\phi^{-1}\ (p))\right) \,|\, \phi \text{ is a linear extension of } P\right\} (3.1)$$

We think of the elements of $\mathcal{L}(P,\omega)$ as permutations of the labels $\omega(t)$, so we have $\mathcal{L}(P,\omega) \subset S_p$.

Of particular interest here are $(P,\omega)-$partitions into the set $[r]$, i.e., maps $\sigma : P \to [r]$ which satisfy conditions (a) and (b) of Definition 3.2, where (P,ω) is a labelled poset and r is a positive integer.

Definition 3.6. For a labelled poset (P,ω), the *order polynomial* associated with (P,ω) is denoted by $\Omega_{(P,\omega)}(r)$ and is defined to be the number of $(P,\omega)-$partitions into $[r]$.

Then the fundamental theorem regarding such order polynomials is the following.

Theorem 3.7. *Let (P,ω) be a labelled poset and p be the size of P. Then $\Omega_{(P,\omega)}(r)$ is a polynomial in r and is given by*

$$\Omega_{(P,\omega)}(r) = \sum_{\pi \in \mathcal{L}(P,\omega)} \binom{r+p-des(\pi)-1}{p} \tag{3.2}$$

For a proof of this theorem, see Theorem 3.15.8 of Stanley [98].

Example 3.8. As an illustration, consider the ASM poset P_3 with natural labelling ω that is shown in Figure 3.1.

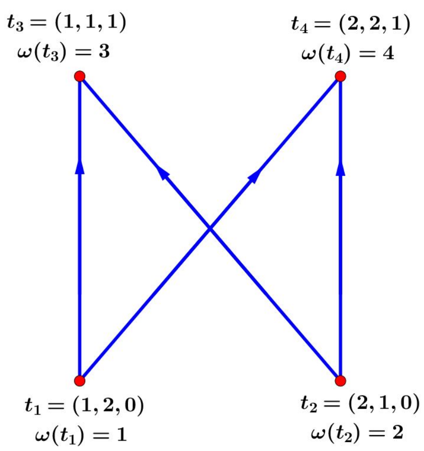


Figure 3.1: The ASM poset P_3 with natural labelling ω.

The set of $\omega-$labelled linear extensions of P_3 is given by

$$\mathcal{L}(P_3,\omega) = \{1234, 2134, 1243, 2143\} \tag{3.3}$$

Diagrams corresponding to each linear extension ϕ of P_3 are shown in Figure 3.2.

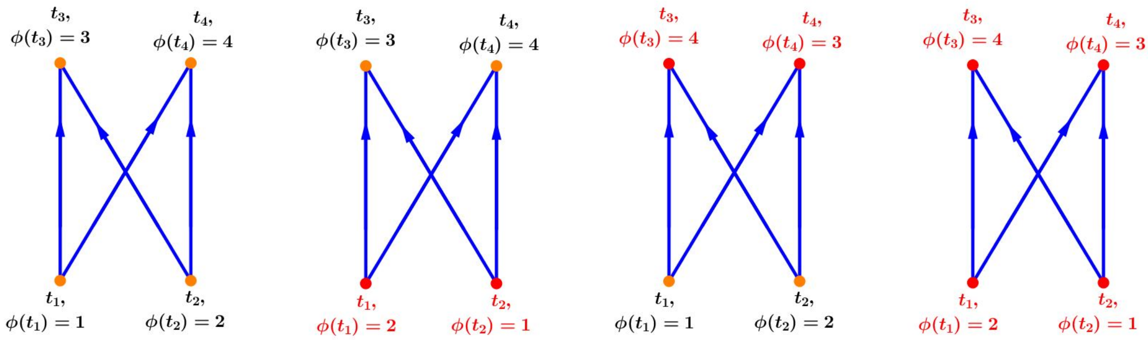


Figure 3.2: The four linear extensions associated with the ASM poset P_3.

Now let $\mathcal{A}(P_3,\omega,r)$ be the set of all (P_3,ω)−partitions into $[r]$, that is, the set

$$\mathcal{A}(P_3,\omega,r) = \left\{ \sigma : P_3 \to [r] \;\middle|\; \begin{array}{l} \bullet\ \sigma(t_1) \geq \sigma(t_3); \\ \bullet\ \sigma(t_1) \geq \sigma(t_4); \\ \bullet\ \sigma(t_2) \geq \sigma(t_3); \\ \bullet\ \sigma(t_2) \geq \sigma(t_4); \end{array} \right\} \tag{3.4}$$

The number of descents associated with each linear extension in (3.3) are given in Table 3.1. So using (3.2) we can write

$$|\mathcal{A}(P_3,\omega,r)| = \Omega_{(P_3,\omega)}(r) = \binom{r+3}{4} + 2\binom{r+2}{4} + \binom{r+1}{4} \tag{3.5}$$

π_i	$des(\pi_i)$
$\pi_1 = 1234$	0
$\pi_2 = 2134$	1
$\pi_3 = 1243$	1
$\pi_4 = 2143$	2

Table 3.1: The number of descents associated with each ω−labelled linear extension given in (3.3).

3.2.1 Reduced ASM labelled poset

In this section, we introduce a reduction operation on the ASM poset P_n for a given $I \in J(P_{n-2})$. Then by applying this reduction operation on P_n for each $I \in J(P_{n-2})$, we obtain a subdivision of P_n into $|ASM(n-2)|$ subposets each of which is equipped with the labelling ω_n defined in Definition 3.12. Then the basic properties of the reduced ASM labelled posets are discussed.

Definition 3.9. Let $n \geq 4$ and for a given order ideal $I \in J(P_{n-2})$ consider the following sets

$$\begin{aligned} \Gamma_1 &= \{(i+1, j+1, k) \mid (i,j,k) \in I\} \\ \Gamma_2 &= \{(i+1, j+1, k+2) \mid (i,j,k) \in P_{n-2} \smallsetminus I\} \end{aligned} \tag{3.6}$$

Then, the *reduced ASM poset* associated with the order ideal $I \in J(P_{n-2})$ is the induced subposet given by

$$P_n(I) \coloneqq P_n \smallsetminus (\Gamma_1 \cup \Gamma_2) \tag{3.7}$$

Remark 3.10. Note that by applying Γ_1 and Γ_2 to a given order ideal $I \in J(P_{n-2})$, we obtain central elements of P_n, that is, $\Gamma_1 \cup \Gamma_2 \subseteq Cent(P_n)$. Therefore, each $P_n(I)$ is the ASM poset P_n from which some central elements (that are obtained from the reduction operations (3.6)) have been removed. By Theorem 1.83, $J(P_{n-2})$ is anti-isomorphic to the lattice $ASM(n-2)$. This implies that $|J(P_{n-2})| = |ASM(n-2)|$, so in total, there are $|ASM(n-2)|$ reduced ASM posets induced by each $I \in J(P_{n-2})$. It follows from (3.7) that $|P_n(I)| = (n-1)^2$ for each $I \in J(P_{n-2})$.

Example 3.11. For $n = 4$, we have $J(P_2) = \{\varnothing, \{(1,1,0)\}\}$. Let $I_1 = \varnothing$ and $I_2 = \{(1,1,0)\}$. Then

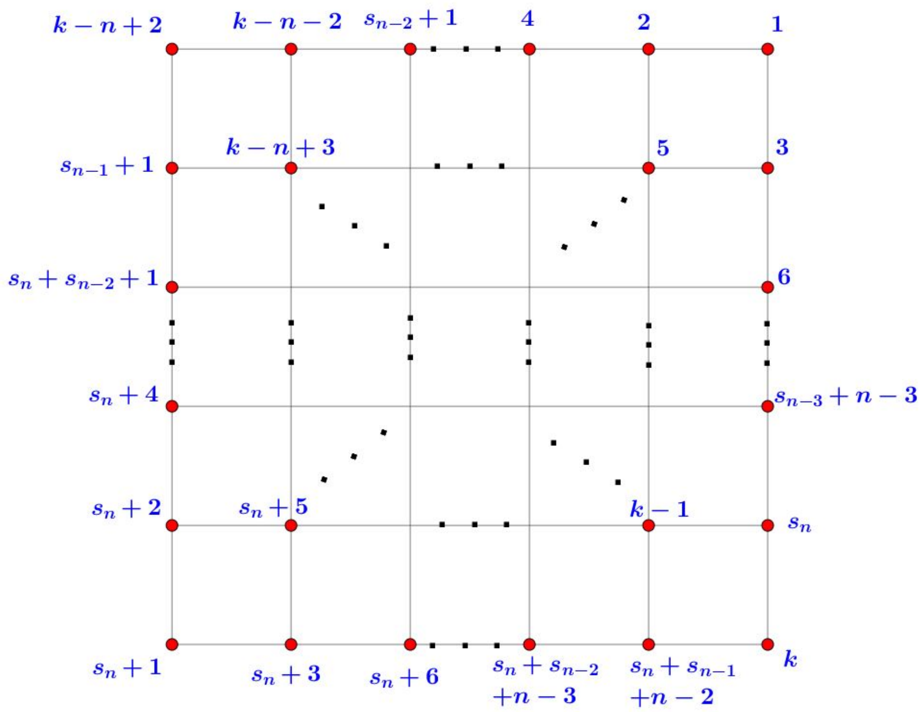
$$\begin{aligned} P_4(I_1) &= P_4 \smallsetminus (\varnothing \cup \{(2,2,2)\}) \\ &= \{(1,1,2),(1,2,1),(1,3,0),(2,1,1),(2,2,0),(2,3,1),(3,1,0),(3,2,1),(3,3,2)\} \\ P_4(I_2) &= P_4 \smallsetminus (\{(2,2,0)\} \cup \varnothing) \\ &= \{(1,1,2),(1,2,1),(1,3,0),(2,1,1),(2,2,2),(2,3,1),(3,1,0),(3,2,1),(3,3,2)\} \end{aligned}$$

Definition 3.12. For a given reduced ASM poset $P_n(I)$, the *labelling* $\omega_n : P_n(I) \to [(n-1)^2]$ is defined by

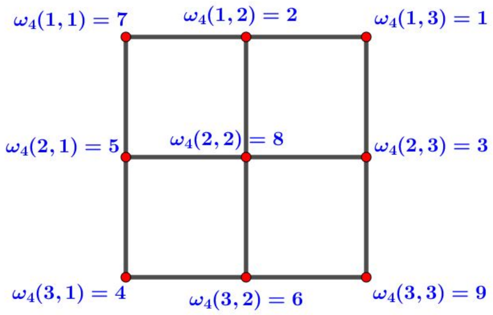
$$\omega_n(i,j,k) = \begin{cases} \frac{(n+i-j-1)(n+i-j-2)}{2} + i & \text{if } i < j \\ (n-1)^2 - n + i + 1 & \text{if } i = j \\ \frac{(n+j-i-1)(n+j-i-2)+(n-1)(n-2)}{2} + j & \text{if } i > j \end{cases} \tag{3.8}$$

for all $i, j = 1, 2, ..., n-1$. Since $\omega_n(i,j,k)$ is independent of k, it will often be abbreviated as $\omega_n(i,j)$. We can depict ω_n on a $(n-1) \times (n-1)$ grid of the form that is shown in Figure 3.3a.

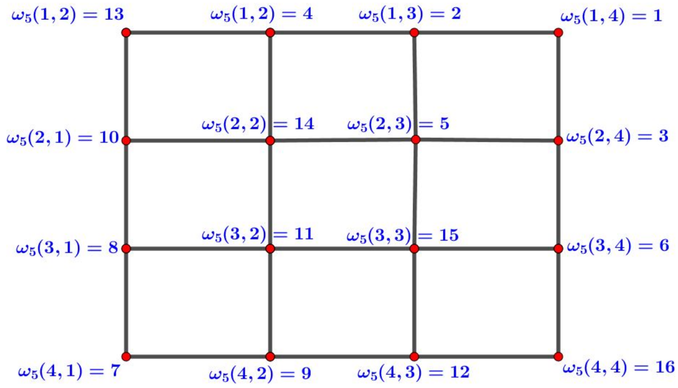
Consider the entries of ω_n on an $(n-1) \times (n-1)$ grid (as shown in Figure 3.3a). We can describe its construction as follows: the entries of ω_n are strictly increasing from the top right corner (that is the position $(1, n-1)$) to the main superdiagonal (that is positions $(i, i+1)$ for $i = 1, ..., n-1$) position-wise in a zigzag manner. Similarly, the entries are strictly increasing from bottom left corner (position $(n-1, 1)$) to the main subdiagonal (positions $(i+1, i)$ for $i = 1, ..., n-1$) position-wise in a zigzag manner. Finally,



(a) The labelling ω_n where $s_n = (n-1)(n-2)/2$ and $k = (n-1)^2$.



(b) The labelling ω_4.



(c) The labelling ω_5.

Figure 3.3: The labelling ω_n on an $(n-1)\times(n-1)$ grid and the labelling ω_4 and ω_5, respectively.

the entries on the main diagonal are also strictly increasing from the top left corner (position $(1,1)$) all the way down to the bottom right corner (position $(n-1,n-1)$) along the main diagonal. As an illustration, the cases ω_4 and ω_5 are shown in Figures 3.3b and 3.3c, respectively.

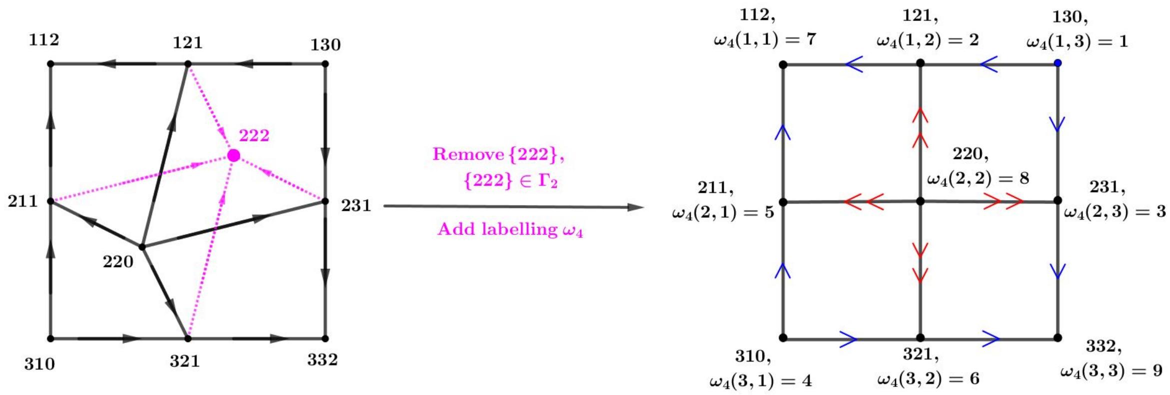
Definition 3.13. Let $n \geq 4$ and $I \in J(P_{n-2})$. Then the *labelled reduced ASM poset* is given by $(P_n(I), \omega_n)$.

As an illustration, the two labelled reduced ASM posets $P_4(I_1)$ and $P_4(I_2)$ are shown in Figures 3.4a and 3.4b, respectively.

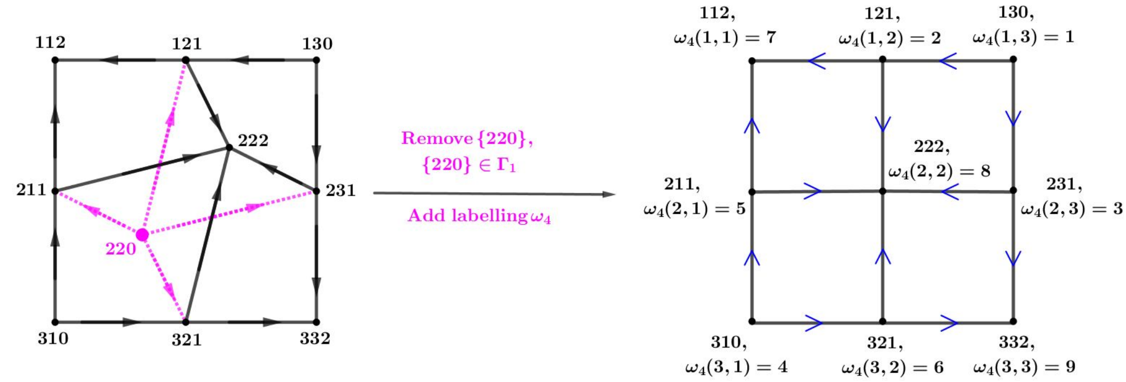
Lemma 3.14. *The following statements are valid:*

(I) $P_n = \bigcup_{I\in J(P_{n-2})} P_n(I)$;

(II) $\bigcap_{I\in J(P_{n-2})} P_n(I) = P_n \setminus Cent(P_n)$;



(a) The poset P_4 and the labelled reduced ASM poset $(P_4(I_1), \omega_4)$ corresponding to order ideal $I_1 = \varnothing \in J(P_2)$. Note that cover relations for which ω_4 gives a strict labelling are distinguished by double red arrows.



(b) The poset P_4 and labelled reduced ASM poset $(P_4(I_2), \omega_4)$ corresponding to order ideal $I_2 = \{(1,1,0)\} \in J(P_2)$. Note that, in this case, ω_4 is a natural labelling.

Figure 3.4: The two labelled reduced ASM posets $(P_4(I_1), \omega_4)$ and $(P_4(I_2), \omega_4)$.

(III) The Hasse diagram of each $P_n(I)$ can be depicted as an $(n-1) \times (n-1)$ grid, with the element in position (i,j) given by (i,j,k) for a unique k.

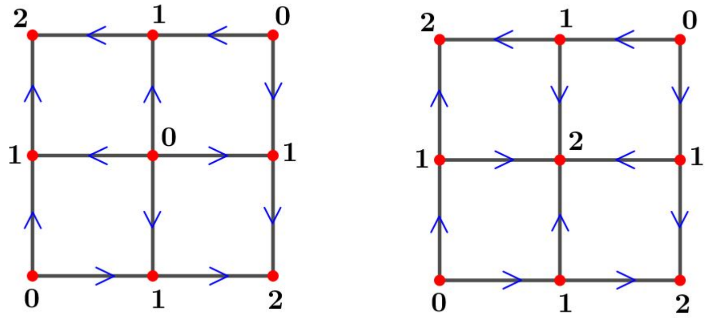
(IV) The number of cover relations in each $P_n(I)$ is $2(n-1)(n-2)$.

Remark 3.15. There is another way to think of each $P_n(I)$ where $I \in J(P_{n-2})$. If in the grid depiction of the Hasse diagram of $P_n(I)$, we consider the component k of the unique element (i,j,k) in position (i,j) we get a matrix $(n-2-h_{ij})_{i,j=0}^{n-2}$ where $(h_{ij})_{i,j=0}^{n-2}$ is the height function matrix associated with I. Note that the entries of the matrix give the direction of the edges in the Hasse diagram. In particular, each edge is directed from the smaller to a larger matrix entry. We can therefore regard this poset as a *reduced height function matrix poset*, or RHFM poset for short. As an illustration, the Hasse diagrams of the RHFM posets associated with $P_4(I_1)$ and $P_4(I_2)$ are shown in Figures 3.5a and 3.5b, respectively.

Definition 3.16. We now define the set of $(P_n(I), \omega_n)$–partitions from $P_n(I)$ to $[0,r]$ for each reduced ASM labelled poset $P_n(I)$ as

$$\mathcal{X}(P_n(I), \omega_n, r) := \left\{ \theta : P_n(I) \to [0,r] \;\middle|\; \begin{array}{l} \bullet\ 0 \le \theta(i,j,k) \le r \ \text{ for all } (i,j,k) \in P_n(I); \\ \bullet\ \theta(i,j,k) \ge \theta(i',j',k') \ \text{ if } (i,j,k) < (i',j',k') \ \text{ and } \omega_n(i,j) < \omega_n(i',j') \ \text{ in } P_n(I); \\ \bullet\ \theta(i,j,k) > \theta(i',j',k') \ \text{ if } (i,j,k) < (i',j',k') \ \text{ and } \omega_n(i,j) > \omega_n(i',j') \ \text{ in } P_n(I). \end{array} \right\} \tag{3.9}$$

We denote the union of these sets by $\Lambda(n,r)$. In other words, we define



(a) The reduced ASM height function poset associated with $P_4(I_1)$.

(b) The reduced ASM height function poset associated with $P_4(I_2)$.

Figure 3.5: The reduced ASM height function matrix posets associated with $P_4(I_1)$ and $P_4(I_2)$.

$$\Lambda(n,r) = \bigcup_{I \in J(P_{n-2})} \mathcal{X}\left(P_n(I), \omega_n, r\right) \tag{3.10}$$

Example 3.17. As an illustration, consider the labelled reduced ASM posets $P_4(I_1,\omega_4)$ and $P_4(I_2,\omega_4)$ given in Example 3.11. Then their $(P_n(I),\omega_n)$–partitions corresponding to the two order ideals $I_1 = \varnothing$ and $I_2 = \{(1,1,0)\}$ of P_2 are given as follows.

(I) For the labelled reduced ASM poset $(P_4(I_1),\omega_4)$ shown in Figure 3.4a, we have

$$\mathcal{X}\left(P_4(I_1),\omega_4,r\right) := \left\{ \theta : P_4(I_1) \to [0,r] \;\middle|\; \begin{array}{l} \bullet\ 0 \le \theta(i,j,k) \le r \ \text{ for all } (i,j,k) \in P_4(I_1); \\ \bullet\ \theta(i,j,k) \ge \theta(i',j',k') \ \text{ if } (i,j,k) < (i',j',k') \ \text{ and } \ \omega_4(i,j) < \omega_4(i',j') \ \text{ in } P_4(I_1); \\ \bullet\ \theta(i,j,k) > \theta(i',j',k') \ \text{ if } (i,j,k) < (i',j',k') \ \text{ and } \ \omega_4(i,j) > \omega_4(i',j') \ \text{ in } P_4(I_1). \end{array} \right\} \tag{3.11}$$

or equivalently

$$\mathcal{X}\left(P_4(I_1),\omega_4,r\right) = \left\{ \theta : P_4(I_1) \to [0,r] \;\middle|\; \begin{array}{ll} \bullet\ \theta(1,3,0) \ge \theta(1,2,1) & \bullet\ \theta(3,1,0) \ge \theta(2,1,1) \\ \bullet\ \theta(1,3,0) \ge \theta(2,3,1) & \bullet\ \theta(3,1,0) \ge \theta(3,2,1) \\ \bullet\ \theta(2,2,0) >> \theta(1,2,1) & \bullet\ \theta(1,2,1) \ge \theta(1,1,2) \\ \bullet\ \theta(2,2,0) >> \theta(2,1,1) & \bullet\ \theta(2,1,1) \ge \theta(1,1,2) \\ \bullet\ \theta(2,2,0) >> \theta(2,3,1) & \bullet\ \theta(2,3,1) \ge \theta(3,3,2) \\ \bullet\ \theta(2,2,0) >> \theta(3,2,1) & \bullet\ \theta(3,2,1) \ge \theta(3,3,2) \end{array} \right\} \tag{3.12}$$

Note that strict inequalities are distinguished by the red color $>>$ from the weak inequalities. Also note that the strict inequality occurs due to the third condition in the definitition (3.9).

(II) For the labelled reduced ASM poset $P_4(I_2)$ shown in Figure 3.4b, we have

$$\mathcal{X}\left(P_4(I_2,\omega_4,r\right) :=$$

$$\left\{ \theta : P_4(I_2) \to [0,r] \;\middle|\; \begin{array}{l} \bullet\ 0 \le \theta(i,j,k) \le r \ \text{ for all } (i,j,k) \in P_4(I_2); \\ \bullet\ \theta(i,j,k) \ge \theta(i',j',k') \ \text{ if } (i,j,k) < (i',j',k') \quad \text{and } \omega_4(i,j) < \omega_4(i',j') \quad \text{in } P_4(I_2); \\ \bullet\ \theta(i,j,k) > \theta(i',j',k') \ \text{ if } (i,j,k) < (i',j',k') \quad \text{and } \omega_4(i,j) > \omega_4(i',j') \quad \text{in } P_4(I_2). \end{array} \right\} \tag{3.13}$$

or equivalently

$$\mathcal{X}\left(P_4(I_2),\omega_4,r\right) =$$

$$\left\{ \theta : P_4(I_2) \to [0,r] \;\middle|\; \begin{array}{ll} \bullet\ \theta(1,3,0) \ge \theta(1,2,1) & \bullet\ \theta(3,1,0) \ge \theta(2,1,1) \\ \bullet\ \theta(1,3,0) \ge \theta(2,3,1) & \bullet\ \theta(3,1,0) \ge \theta(3,2,1) \\ \bullet\ \theta(1,2,1) \ge \theta(2,2,2) & \bullet\ \theta(2,3,1) \ge \theta(3,3,2) \\ \bullet\ \theta(2,1,1) \ge \theta(2,2,2) & \bullet\ \theta(3,2,1) \ge \theta(3,3,2) \\ \bullet\ \theta(2,3,1) \ge \theta(2,2,2) & \bullet\ \theta(1,2,1) \ge \theta(1,1,2) \\ \bullet\ \theta(3,2,1) \ge \theta(2,2,2) & \bullet\ \theta(2,1,1) \ge \theta(1,1,2) \end{array} \right\} \tag{3.14}$$

3.3 The enumeration of higher spin ASMs

In this section we give the main theorem of this thesis. First we introduce the notion of reduced higher spin ASMs together with a partition of them into $|ASM(n-2)|$ subsets.

Definition 3.18. Let $X \in ASM(n,r)$ be given, and let C_X be its corresponding CSM (1.37). Define the matrix $E = (e_{ij})_{i,j=1}^{n-1}$ with

$$e_{ij} = \begin{cases} c_{ij} & \text{if } i+j \le n \\ c_{ij} - r(i+j-n) & \text{if } i+j > n \end{cases} \tag{3.15}$$

We call E a *reduced ASM*, or RASM for short. We denote the set of all $n \times n$ RASMs by $RASM(n,r)$. Since all the steps to obtain a RASM are reversible, the following lemma is straightforward.

Lemma 3.19. *There is a bijection between $ASM(n,r)$ and $RASM(n,r)$.*

Definition 3.20. One can classify $RASM(n,r)$ into $|ASM(n-2)|$ subsets according to the growth of entries in each $E \in RASM(n,r)$ from top left corner to the bottom right corner. Consider $E \in RASM(n,r)$ and $i,j \in [n-1]$. Then $Min(i,j,n-i,n-j)$ determines certain rows and columns of E called the *shell* of E. In particular, for a given $t \in \left[\lfloor \frac{n}{2} \rfloor\right]$, $Min(i,j,n-i,n-j) = t$ determines entries on the i^{th} row, j^{th} column, $(n-i)^{th}$ row and $(n-j)^{th}$ column. These entries satisfy $0 \le e_{ij} \le rt$. For instance, $Min(i,j,n-i,n-j) = 1$ determines the entries on the first shell in E, that is, entries on the first and last row and column of E. Also $Min(i,j,n-i,n-j) = 2$ determines the entries on the second shell in E. This continues all the way down to the innermost shell which is determined by $Min(i,j,n-i,n-j) = k$ where $k = \lfloor \frac{n}{2} \rfloor$. If n is even, then the innermost shell is a single entry matrix $(e_{k,k})$. Otherwise, it is a 2×2 matrix in the following form

$$\begin{pmatrix} e_{k,k} & e_{k,k+1} \\ e_{k+1,k} & e_{k+1,k+1} \end{pmatrix} \tag{3.16}$$

As an illustration, the shelling structure of each $E \in RASM(n,r)$ is shown in Figure 3.6 when n is even. Therefore, one way to subdivide RASMs into disjoint union of $|ASM(n-2)|$ subsets is to partition each interval $[0,rt]$ into disjoint subintervals and then compare the values of the entries within the internal shells. In what follows, we will describe the process of subdividing RASMs.

Remark 3.21. For a given $E \in RASM(n,r)$, consider the shell $Min(i,j,n-i,n-j) = t$ where $t \in \left[\lfloor \frac{n}{2} \rfloor\right]$ (note that for each entry in this shell, we have $0 \le e_{ij} \le rt$). Then each interval $[0,rt]$ can be partitioned into at most t subintervals given by

$$[0,rt] = \bigcup_{s=0}^{t-1} \left([rs-(t-s-1), r(s+1)-(t-s-1)] \bigcap \mathbb{N} \right) \tag{3.17}$$

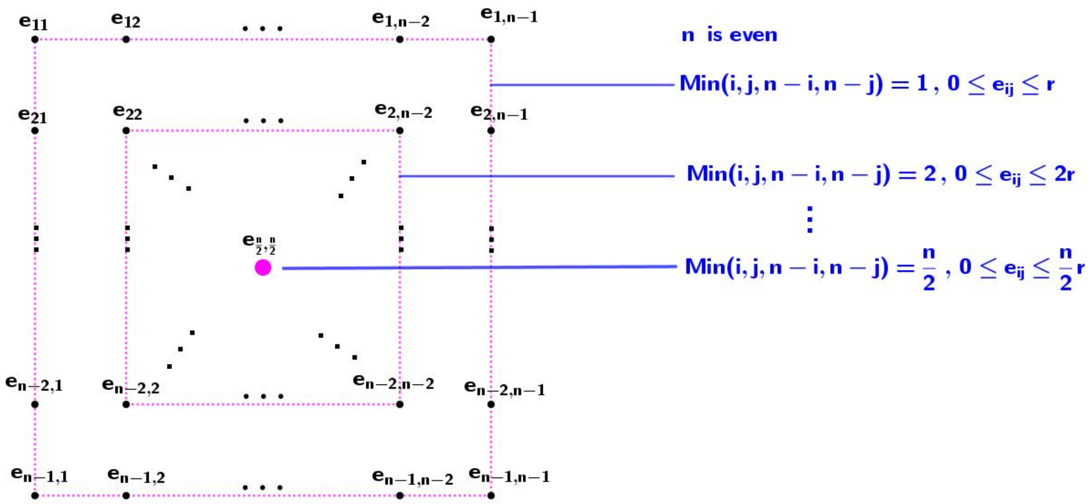


Figure 3.6: The shelling structure in a given $E \in RASM(n,r)$ when n is even.

Definition 3.22. For a given n, the set of *central positions* in an $(n-1) \times (n-1)$ matrix E is defined by

$$CPO(n) = \{ (i,j) \mid 2 \le i,j \le n-2\} \tag{3.18}$$

Note that $CPO(n)$ is simply $Cent(P_n)$, as defined in Definition 1.81.

Now depending on the value of e_{ij}'s in each internal shell, one can write $RASM(n,r)$ as a disjoint union of $|ASM(n-2)|$ subsets, namely

$$RASM(n,r) = \bigcup_{i=1}^{|ASM(n-2)|} R_i(r) \tag{3.19}$$

where each $R_i(r)$ is a subset of $RASM(n,r)$ with certain conditions on matrices. For example, for $RASM(4,r)$, there are two shells, namely, $S_1 = \{(i,j) \mid Min(i,j,4-i,4-j) = 1\}$ and $S_2 = \{(i,j) \mid Min(i,j,4-i,4-j) = 2\}$. Moreover, $CPO(4) = \{(2,2)\}$ and $0 \le e_{22} \le 2r$, see Figure 3.7. By definition of $RASM(4,r)$, since $0 \le e_{22} \le 2r$, so we can write $RASM(4,r) = R_1(r) \cup R_2(r)$ where

$$R_1(r) = \{E \in RASM(4,r) \mid 0 \le e_{22} \le r-1\} \quad , \quad R_2(r) = \{E \in RASM(4,r) \mid r \le e_{22} \le 2r\} \tag{3.20}$$

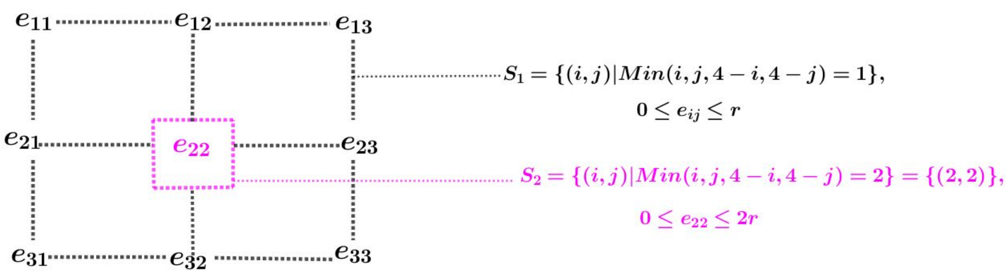


Figure 3.7: The $RASM(4,r)$ and its two shells.

Now we are ready to state and prove the main theorem of this thesis. To do this, we construct a bijection between $RASM(n,r)$ and $\Lambda(n,r)$ when n is even, since the same map works for the case where n is odd. Note that the only difference between even and odd cases is the innermost shell, since for n is even it includes a single element whereas for the odd case, it consists of a 2×2 matrix.

Theorem 3.23. *Let $Min(i,j,n-i,n-j) = t$ where $t \in [\lfloor \frac{n}{2} \rfloor]$ and $i,j \in [n-1]$. Consider the map $\Phi : RASM(n,r) \to \Lambda(n,r)$ given by $\Phi(E) = \theta_E$ where*

$$\theta_E(i,j,k) = e_{ij} - rs + t - s - 1 \quad \text{if} \quad e_{ij} \in [rs-(t-s-1), r(s+1)-(t-s-1)] \tag{3.21}$$

for all $E \in RASM(n,r)$, $(i,j,k) \in P_n$. Then Φ is a bijection.

Proof.
First we show that Φ is well-defined, that is, $\Phi(E) \in \Lambda(n,r)$ for a given $E \in RASM(n,r)$. In other words, we need to verify the following conditions for $(\Phi(E))_{ij} = \theta_E(i,j,k)$ and $(i,j,k) \in P_n$:

$$\begin{aligned}
&(1)\ 0 \le \theta_E(i,j,k) \le r \quad \text{for all } (i,j,k) \in P_n(I);\\
&(2)\ \theta_E(i,j,k) \ge \theta_E(i',j',k') \quad \text{if } (i,j,k) \le (i',j',k') \quad \text{in } P_n(I);\\
&(3)\ \theta_E(i,j,k) > \theta_E(i',j',k') \quad \text{if } (i,j,k) \le (i',j',k') \quad \text{and} \quad \omega_n(i,j) > \omega_n(i',j') \quad \text{in } P_n(I),
\end{aligned} \tag{3.22}$$

for some $I \in J(P_{n-2})$. Without loss of generality, consider the position (i,j) such that $Min(i,j,n-i,n-j) = t$ for all $t \in [\lfloor \frac{n}{2} \rfloor]$ and that $e_{ij} \in [rs-(t-s-1)\,,\, r(s+1)-(t-s-1)]$. Then we can write

$$\begin{aligned}
rs-(t-s-1) \le e_{ij} \le r(s+1)-(t-s-1) &\Leftrightarrow 0 \le e_{ij} - rs + t - s - 1 \le r\\
&\Leftrightarrow 0 \le \theta_E(i,j,k) \le r.
\end{aligned}$$

as required. To investigate the other two conditions (under the above assumptions) we consider the following regions in E:

$$\text{Reg}(1):\ i+j<n;$$

$$\text{Reg}(2):\ i+j=n;$$

$$\text{Reg}(3):\ i+j>n.$$

Note that by symmetries described in Table 1.7, a given configuration in $Reg(2)$ or $Reg(3)$ is just the anti-main diagonal reflection of the same configuration in $Reg(1)$ and vice versa. Therefore, without loss of generality, it is sufficient to verify conditions (2) and (3) for all possible configurations for a given position (i,j) and its four adjacent neighbours in $Reg(1)$. Also assume that $(i,j,k) \in P_n$ is the associated position with (i,j) in $P_n(I) \in \Lambda(n,r)$ in $Reg(1)$. Then depending on the value of (i,j,k), the value of $\omega_n(i,j)$ and the values of the entries adjacent to it, we can classify all the possible configurations into five disjoint categories each of which contains four (weak or strong) inequalities that need to be verified. As an illustration, the entries on the first shell of a given $P_n(I)$ together with the regions $Reg(1)-Reg(3)$ are shown in Figure 3.8. We begin with the configurations on the first shell that is, when $t=1$. We investigate the configurations on the first shell separately since each entry in this shell belongs to the interval $[0,r]$. In the rest of this section, we will show the strict order relations between adjacent positions in each shell with "$>>$" or "$<<$" and each weak inequality with "$>$" or "$<$", respectively.

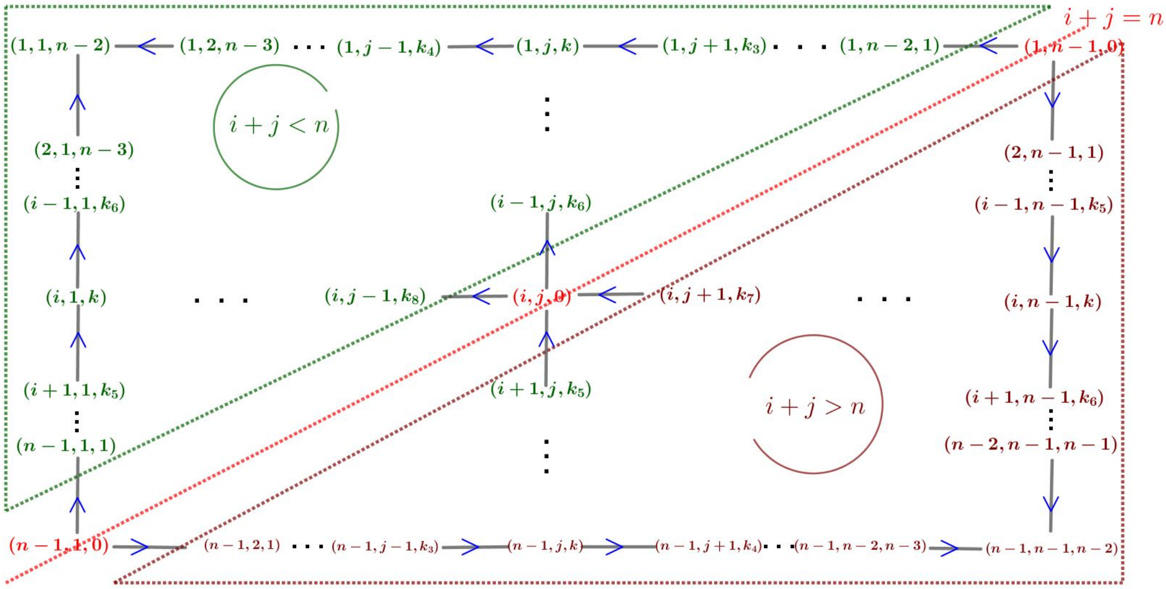


Figure 3.8: The possible configurations on the first shell of $P_n(I)$.

(a) Horizontal configurations: There are three possible configurations in this case shown in Figure 3.9.

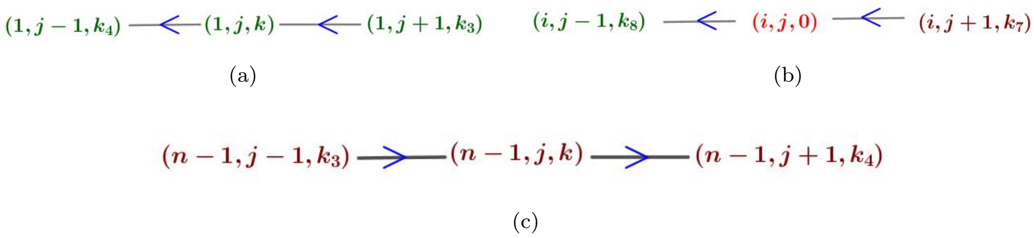


Figure 3.9: Horizontal configurations.

Without loss of generality, consider the configuration shown in Figure 3.9a. For this configuration, we need to verify the following two sub-cases:

(1) We need to verify that:

$$\theta_E(1,j,k) \geq \theta_E(1,j-1,k_4) \quad \Leftrightarrow \quad e_{1j} \geq e_{1,j-1}$$

The last inequality $e_{1j} \geq e_{1,j-1}$ is valid since by the definition of E, $e_{1j} - e_{1,j-1} \geq 0$ for $j \in [n]$.

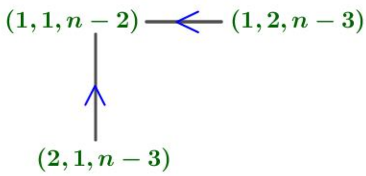
(2) Secondly, we need to verify that

$$\theta_E(1,j+1,k_3) \geq \theta_E(1,j,k) \quad \Leftrightarrow \quad e_{1,j+1} \geq e_{1,j}$$

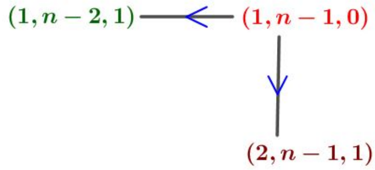
Again the last inequality $e_{1,j+1} \geq e_{1,j}$ is valid since by the definition of E, $e_{1,j+1} - e_{1j} \geq 0$ for $j \in [n]$.

The verification of the inequalities associated with the configurations shown in 3.9b and 3.11c is very similar to the argument for the case (1) and (2).

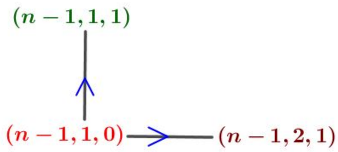
(b) Right angled corner configurations: In this configuration, there are four possible scenarios that are shown in Figure 3.10.



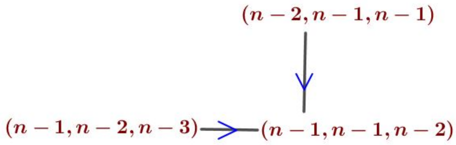
(a) The top left corner configuration.



(b) The top right corner configuration.



(c) The bottom left corner configuration.



(d) The bottom right corner configuration.

Figure 3.10: Four possible right angled corner configurations for the first shell in $P_n(I)$.

Without loss of generality, consider the configuration in Figure 3.10a. Then we need to consider the following two subcases:

(1) We need to verify that

$$\theta_E(1,2,n-3) \geq \theta_E(1,1,n-2) \quad \Leftrightarrow \quad e_{12} \geq e_{11}$$

The last inequality $e_{12} \geq e_{11}$ clearly holds since by the definition of E, $e_{12} - e_{11} \geq 0$.

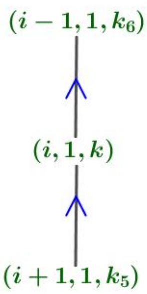
(2) We need to verify that

$$\theta_E(2,1,n-3) \geq \theta_E(1,1,n-2) \quad \Leftrightarrow \quad e_{21} \geq e_{11}$$

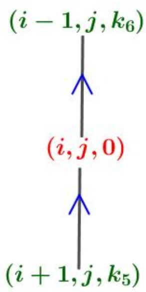
By the same argument $e_{21} \geq e_{11}$ holds since by definition of E, $e_{21} - e_{11} \geq 0$.

Similar to the case for the horizontal case, the argument for the other three right angled corner configurations shown in Figures 3.10b, 3.11c and 3.10d is very similar.

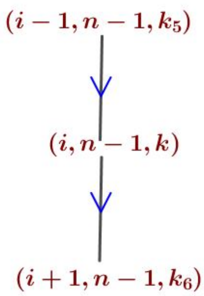
(c) Vertical configurations: There are three vertical configurations that are shown in Figure 3.11.



(a) The vertical configuration on $Reg(1)$.



(b) The vertical configuration on $Reg(3)$.



(c) The vertical configuration on $Reg(2)$.

Figure 3.11: All three vertical configurations in the first shell of $P_n(I)$.

Without loss of generality, consider the configuration in Figure 3.11a. Then we have the following two subcases:

(1) We need to verify that

$$\theta_E(i,1,k) \geq \theta_E(i-1,1,k_6) \iff e_{i,1} \geq e_{i-1,1}$$

Clearly the inequality $e_{i,1} \geq e_{i-1,1}$ holds since by the definition of E, $e_{i,1} - e_{i-1,1} \geq 0$ in $Reg(1)$.

(2) We need to verify that

$$\theta_E(i+1,1,k_5) \geq \theta_E(i,1,k) \iff e_{i+1,1} \geq e_{i,1}$$

Clearly the inequality $e_{i+1,1} \geq e_{i,1}$ holds since by the definition of E, $e_{i+1,1} - e_{i,1} \geq 0$ in $Reg(1)$. The verification for the other configurations in Figures 3.11b and 3.11c is very similar.

Now we investigate the five categories of configurations for positions in internal shells in $Reg(1)$. For brevity, we pick one arbitrary configuration from each category to verify the validity of the associated inequalities with it since the argument for the rest is very similar by symmetry.

cat(1): The first category contains 16 configurations (in all regions) each of which consists of four strict inequalities (either >> or << regarding the position (i,j,k) and its four adjacent positions $(i,j-1,k_1)$, $(i-1,j,k_2)$, $(i,j+1,k_3)$ and $(i+1,j,k_4)$. Without loss of generality, consider the configuration shown in Figure (3.12) in $Reg(1)$.

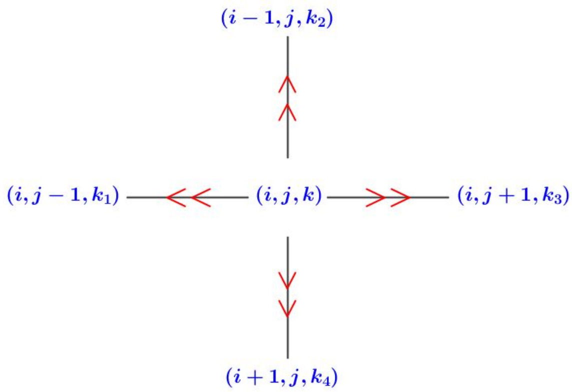


Figure 3.12: The first category containing one configuration with four strict relations.

In this configuration, we need to verify the following inequalities:

$$\theta_E(i,j,k) \quad >> \quad \theta_E(i,j-1,k_1) \tag{3.23}$$

$$\theta_E(i,j,k) \quad >> \quad \theta_E(i-1,j,k_2) \tag{3.24}$$

$$\theta_E(i,j,k) \quad >> \quad \theta_E(i,j+1,k_3) \tag{3.25}$$

$$\theta_E(i,j,k) \quad >> \quad \theta_E(i+1,j,k_4) \tag{3.26}$$

In order to verify the validity of inequalities (3.23)- (3.26), assume that

$$
\begin{aligned}
Min(i,j-1) = t_{ij}-1 \quad &, \quad e_{i,j-1} \in [rs_{i,j-1}-(t_{ij}-s_{i,j-1}-2)\,,\,r(s_{i,j-1}+1)-(t_{ij}-s_{i,j-1}-2)]\,;\\
Min(i-1,j) = t_{ij}-1 \quad &, \quad e_{i-1,j} \in [rs_{i-1,j}-(t_{ij}-s_{i-1,j}-2)\,,\,r(s_{i-1,j}+1)-(t_{ij}-s_{i-1,j}-2)]\,;\\
Min(i,j+1) = t_{ij} \quad &, \quad e_{i,j+1} \in [rs_{i,j+1}-(t_{ij}-s_{i,j+1}-1)\,,\,r(s_{i,j+1}+1)-(t_{ij}-s_{i,j+1}-1)]\,;\\
Min(i+1,j) = t_{ij} \quad &, \quad e_{i+1,j} \in [rs_{i+1,j}-(t_{ij}-s_{i+1,j}-1)\,,\,r(s_{i+1,j}+1)-(t_{ij}-s_{i+1,j}-1)]\,.
\end{aligned} \tag{3.27}
$$

Now by definition of Φ, for (3.23) we need to verify that

$$
\begin{aligned}
\theta_E(i,j,k) >> \theta_E(i,j-1,k_1) &\Leftrightarrow e_{ij}-rs_{ij}+t_{ij}-s_{ij}-1 >> e_{i,j-1}-rs_{i,j-1}+t_{i,j-1}-s_{i,j-1}-2\\
&\Leftrightarrow e_{ij}-(r+1)(s_{ij}-s_{i,j-1})+(t_{ij}-t_{i,j-1})+1 >> e_{i,j-1}
\end{aligned} \tag{3.28}
$$

To verify the validity of the last inequality in (3.28), we note that by the definition of E (3.15), $e_{ij}-e_{i,j-1} \geq 0$ (or equivalently $e_{ij} \geq e_{i,j-1})$). Moreover, by our assumptions $t_{ij} = 1+t_{i,j-1}$ implies that $s_{ij} \leq s_{i,j-1}$ (and thus $-(r+1)(s_{ij}-s_{i,j-1}) \geq 0$). Therefore, we can write $e_{ij}-(r+1)(s_{ij}-s_{i,j-1}) \geq e_{i,j-1}$ and subsequently $e_{ij}-(r+1)(s_{ij}-s_{i,j-1})+1 >> e_{i,j-1}$ as required.

The verification for inequality (3.24) is very similar since it is the anti-main diagonal reflection of the given configuration with respect to the position (i,j,k).

For inequality (3.25), we need to verify that

$$
\begin{aligned}
\theta_E(i,j,k) >> \theta_E(i,j+1,k_3) &\Leftrightarrow e_{ij}-rs_{ij}+t_{ij}-s_{ij}-1 >> e_{i,j+1}-rs_{i,j+1}+t_{i,j+1}-s_{i,j+1}-1\\
&\Leftrightarrow e_{ij}-(r+1)(s_{ij}-s_{i,j+1}) >> e_{i-1,j}
\end{aligned} \tag{3.29}
$$

To verify the last inequality in (3.29), first we note that $t_{ij} = t_{i,j+1}$ implies that $s_{i,j+1} \geq 1+s_{ij}$ (and thus $-(r+1)(s_{ij}+1) \geq -(r+1)(s_{i,j+1})$). Also by the definition of E (3.15), $e_{i,j+1}-e_{ij} \leq r$ (or equivalently $e_{ij} \geq e_{i,j+1}-r$). Then we can write

$$
\begin{aligned}
e_{ij}-(r+1)s_{ij}-(r+1) \geq e_{i,j+1}-(r+1)s_{i,j+1}-r \quad &\Leftrightarrow \quad e_{ij}-(r+1)\,(s_{ij}-s_{i,j+1})-1 \geq e_{i,j+1}\\
&\Leftrightarrow \quad e_{ij}-(r+1)\,(s_{ij}-s_{i,j+1}) >> e_{i,j+1}
\end{aligned}
$$

The verification for the inequality in (3.26) is very similar to the case (3.25) since it is the anti-main diagonal reflection of the given configuration with respect to position (i,j,k).

cat(2): This category consists of 56 configurations (in all regions) each of which contains three strict inequalities (either $>>$ or $<<$) and one weak inequality (either $>$ or $<$) regarding the position (i,j,k) and its four adjacent positions $(i,j-1,k_1)$, $(i-1,j,k_2)$, $(i,j+1,k_3)$ and $(i+1,j,k_4)$. Without loss of generality, consider the configuration in Figure 3.13 in $Reg(1)$.

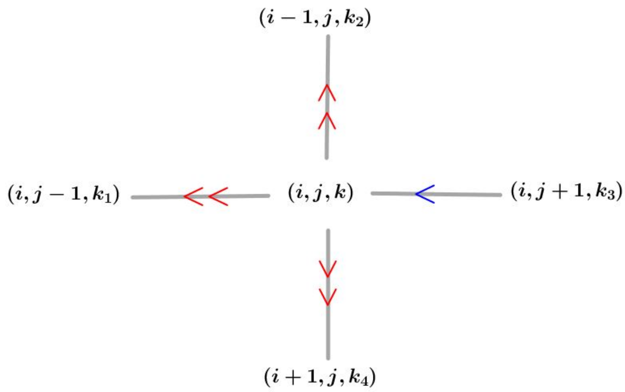


Figure 3.13: An arbitrary configuration from second category that consists of three strict inequalities and one weak inequality, respectively.

For this configuration, we need to verify the following inequalities:

$$\theta_E(i,j,k) \quad >> \quad \theta_E(i,j-1,k_1) \tag{3.30}$$

$$\theta_E(i,j,k) \quad >> \quad \theta_E(i-1,j,k_2) \tag{3.31}$$

$$\theta_E(i,j,k) \quad < \quad \theta_E(i,j+1,k_3) \tag{3.32}$$

$$\theta_E(i,j,k) \quad >> \quad \theta_E(i+1,j,k_4) \tag{3.33}$$

To verify them, let us assume the same assumptions described in (3.27) regarding the shells that these positions belong to. Then the argument for (3.30), (3.31) and (3.33) is exactly the same as the ones we had for (3.23), (3.24) and (3.26) in the first category, respectively. So we only need to verify the inequality (3.32). We have

$$\begin{aligned}\theta_E(i,j+1,k_4) \geq \theta_E(i,j,k) &\Leftrightarrow e_{i,j+1} - rs_{i,j+1} + t_{i,j+1} - s_{i,j+1} - 1 \geq e_{ij} - rs_{ij} + t_{ij} - s_{ij} - 1 \\ &\Leftrightarrow e_{i,j+1} - (r+1)(s_{i,j+1} - s_{ij}) > e_{ij}\end{aligned} \tag{3.34}$$

The last inequality in (3.34) is valid since $s_{i,j+1} \leq s_{ij}$ (and hence $-(r+1)s_{i,j+1} \geq -(r+1)s_{ij}$) and by the definition of E, $e_{i,j+1} \geq e_{ij}$.

cat(3): This category contains 64 configurations (in all regions) each of which contains two strict inequalities (either $>>$ or $<<$) and two weak inequalities (either $>$ or $<$) regarding the position (i,j,k) and its four adjacent positions $(i,j-1,k_1)$, $(i-1,j,k_2)$, $(i,j+1,k_3)$ and $(i+1,j,k_4)$. Without loss of generality, consider the configuration shown in Figure 3.14 in $Reg(1)$.

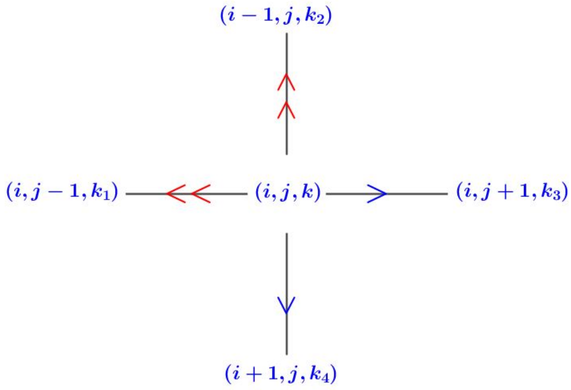


Figure 3.14: An arbitrary configuration from the third category consists of two strict inequalities $>>$ and two weak inequalities $>$, respectively.

We need to verify the following inequalities:

$$\theta_E(i,j,k) \quad >> \quad \theta_E(i,j-1,k_1) \tag{3.35}$$

$$\theta_E(i,j,k) \quad >> \quad \theta_E(i-1,j,k_2) \tag{3.36}$$

$$\theta_E(i,j,k) \quad > \quad \theta_E(i,j+1,k_3) \tag{3.37}$$

$$\theta_E(i,j,k) \quad > \quad \theta_E(i+1,j,k_4) \tag{3.38}$$

Consider the same assumptions given in (3.27) regarding the value of the shells that these positions belong to. Then the argument for inequalities given in (3.35) and (3.36) are exactly the same as the ones we had for (3.23) and (3.24), respectively. In addition, the argument for (3.37) and (3.38) is precisely the same as the argument we had for (3.32) in $cat(2)$. (We note that (3.38) is in fact the anti-main diagonal reflection of the given configuration with respect to position (i,j,k)).

cat(4): This category contains 64 configurations (in all regions) each of which contains one strict inequality (either $>>$ or $<<$) and three weak inequalities (either $>$ or $<$) regarding the position (i,j,k) and its four adjacent positions $(i,j-1,k_1)$, $(i-1,j,k_2)$, $(i,j+1,k_3)$ and $(i+1,j,k_4)$. Without loss of generality, consider the configuration shown in Figure 3.15 in $Reg(1)$.

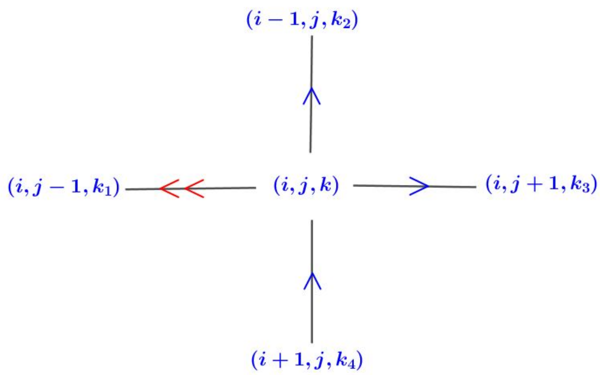


Figure 3.15: An arbitrary configuration from the fourth category consists of one strict inequality $<<$ and three weak inequalities (either $>$ or $<$), respectively.

We need to verify the following inequalities for this configuration:

$$\begin{aligned}
\theta_E(i,j,k) &\gg \theta_E(i,j-1,k_1) && (3.39)\\
\theta_E(i,j,k) &> \theta_E(i-1,j,k_2) && (3.40)\\
\theta_E(i,j,k) &> \theta_E(i,j+1,k_3) && (3.41)\\
\theta_E(i,j,k) &< \theta_E(i+1,j,k_4) && (3.42)
\end{aligned}$$

To do this, we consider the following scenarios for the value of the shells where these positions belong:

$$\begin{aligned}
Min(i,j-1) = t_{ij}-1 \quad &, \quad e_{i,j-1} \in [rs_{i,j-1} - (t_{ij} - s_{i,j-1} - 2)\,,\, r(s_{i,j-1}+1) - (t_{ij} - s_{i,j-1} - 2)]\,;\\
Min(i-1,j) = t_{ij} \quad &, \quad e_{i-1,j} \in [rs_{i-1,j} - (t_{ij} - s_{i-1,j} - 1)\,,\, r(s_{i-1,j}+1) - (t_{ij} - s_{i-1,j} - 1)]\,;\\
Min(i,j+1) = t_{ij}+1 \quad &, \quad e_{i,j+1} \in [rs_{i,j+1} - (t_{ij} - s_{i,j+1})\,,\, r(s_{i,j+1}+1) - (t_{ij} - s_{i,j+1})]\,;\\
Min(i+1,j) = t_{ij} \quad &, \quad e_{i+1,j} \in [rs_{i+1,j} - (t_{ij} - s_{i+1,j} - 1)\,,\, r(s_{i+1,j}+1) - (t_{ij} - s_{i+1,j} - 1)]\,.
\end{aligned} \tag{3.43}$$

The argument for the inequality (3.39) is exactly the same as the one for (3.28). For (3.40), we need to verify that

$$\begin{aligned}
\theta_E(i,j,k) > \theta_E(i-1,j,k_2) &\Leftrightarrow e_{ij} - rs_{ij} + t_{ij} - s_{ij} - 1 > e_{i-1,j} - rs_{i-1,j} + t_{ij} - s_{i-1,j} - 1\\
&\Leftrightarrow e_{ij} - (r+1)(s_{ij} - s_{i-1,j}) > e_{i-1,j}
\end{aligned} \tag{3.44}$$

The last inequality in (3.44) holds since $s_{ij} \le s_{i-1,j}$ (and hence $-(r+1)s_{ij} \ge -(r+1)s_{i-1,j}$) and by the construction of E, $e_{ij} \ge e_{i-1,j}$ in $Reg(1)$.

For inequality (3.41), we need to verify that

$$\begin{aligned}
\theta_E(i,j,k) > \theta_E(i,j+1,k_3) &\Leftrightarrow e_{ij} - rs_{ij} + t_{ij} - s_{ij} - 1 > e_{i,j+1} - rs_{i,j+1} + t_{ij} - s_{i,j+1}\\
&\Leftrightarrow e_{ij} - (r+1)(s_{ij} - s_{i,j+1}) - 1 > e_{i,j+1}
\end{aligned} \tag{3.45}$$

To check the last inequality in (3.45) we note that $s_{ij} \le s_{i,j+1} - 1$ (and hence $-(r+1)s_{ij} \ge -(r+1)s_{i-1,j} - (r+1)$) and by the construction of E, $e_{i,j+1} - e_{ij} \le r$ (or equivalently $e_{ij} \ge e_{i,j+1} - r$). Thus we can write

$$e_{ij} - (r+1)s_{ij} \ge e_{i,j+1} - (r+1)s_{i,j+1} + (r+1) - r \;\Leftrightarrow\; e_{ij} - (r+1)\left(s_{ij} - s_{i-1,j}\right) - 1 > e_{i,j+1}$$

as required.

Finally, for the inequality (3.42), we need to verify that

$$\begin{aligned}
\theta_E(i+1,j,k_4) \ge \theta_E(i,j,k) &\Leftrightarrow e_{i+1,j} - s_{i+1,j}r + t_{i+1,j} - s_{i+1,j} - 1 \ge e_{ij} - s_{ij}r + t_{ij} - s_{ij} - 1\\
&\Leftrightarrow e_{i+1,j} - (s_{ij} - s_{i+1,j})(r+1) > e_{ij}
\end{aligned} \tag{3.46}$$

The last inequality in (3.46) is valid since $s_{i+1,j} \le s_{ij}$ (which implies that $-(r+1)s_{i+1,j} \ge -(r+1)s_{ij}$) and by the construction of E, $e_{i+1,j} \ge e_{ij}$.

cat(5): The fifth and last category contains 64 possible configurations (in all regions) each of which has four weak inequalities (either $>$ or $<$) regarding the position (i, j, k) and its four adjacent positions $(i, j-1, k_1)$, $(i-1, j, k_2)$, $(i, j+1, k_3)$ and $(i+1, j, k_4)$. Without loss of generality, consider the configuration shown in Figure 3.16.

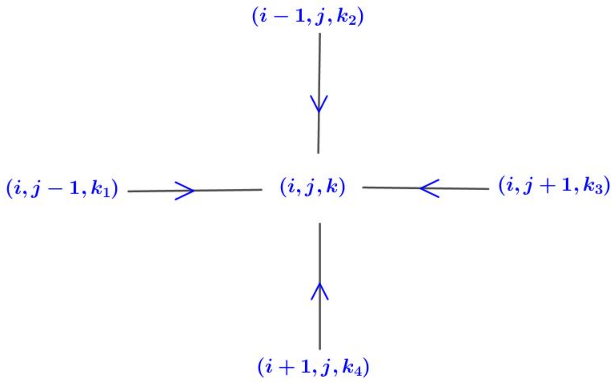


Figure 3.16: The fifth category containing four weak inequality $<$.

We need to verify the following inequalities for this configuration:

$$\theta_E(i,j,k) < \theta_E(i,j-1,k_1) \tag{3.47}$$

$$\theta_E(i,j,k) < \theta_E(i-1,j,k_2) \tag{3.48}$$

$$\theta_E(i,j,k) < \theta_E(i,j+1,k_3) \tag{3.49}$$

$$\theta_E(i,j,k) < \theta_E(i+1,j,k_4) \tag{3.50}$$

Consider the same assumptions as in (3.27) regarding the shells that each position in this configuration belongs to. Then for the inequality (3.47), we need to verify that

$$\begin{aligned} \theta_E(i,j-1,k_1) > \theta_E(i,j,k) &\Leftrightarrow e_{i,j-1} - r s_{i,j-1} + t_{ij} - s_{i,j-1} - 2 > e_{ij} - s_{ij} r + t_{ij} - s_{ij} - 1 \\ &\Leftrightarrow e_{i,j-1} - (r+1)(s_{i,j-1} - s_{ij})(r+1) - 1 > e_{ij} \end{aligned} \tag{3.51}$$

To check the last inequality in (3.51), we note that by our assumption (3.27), $1 + s_{i,j-1} \le s_{ij}$ (and hence $-(r+1) - (r+1)s_{i,j-1} \ge -(r+1)s_{ij}$) and by the construction of E, $e_{ij} - e_{i,j-1} \le r$ (or equivalently $e_{i,j-1} + r \ge e_{ij}$). So we can write

$$e_{i,j-1} - (r+1)s_{i,j-1} - r - 1 + r \ge e_{ij} - (r+1)s_{ij} \;\Leftrightarrow\; e_{i,j-1} - (r+1)\left(s_{i,j-1} - s_{ij}\right) - 1 > e_{ij}$$

as required.

Now the argument for inequality (3.48), is exactly the same as the one for (3.47). (In fact, it is the main-diagonal reflection with respect to the position (i, j, k)). Similarly, inequalities (3.49) and (3.50) are vertical and horizontal reflections with respect to the position (i, j, k), respectively. Thus the same argument as the one for (3.47) holds for them as well.

So far, we have shown that Φ is well-defined. Now we show that Φ is injective, that is, if $E \ne E'$ then $\Phi(E) \ne \Phi(E')$ for all $E, E' \in RASM(n, r)$. Let $E, E' \in RASM(n, r)$ and $E \ne E'$. Then there exist $e_{ij} \in E$ and $e_{i'j'} \in E'$ such that $e_{ij} \ne e_{i'j'}$ for some $i, i', j, j' = 1, ..., n$. Let us assume that $Min(i, j, n-i, n-j) = t_{ij}$ and $Min(i', j', n-i', n-j') = t_{i'j'}$ where $t_{ij}, t_{i'j'} \in \left[\lfloor \frac{n}{2} \rfloor\right]$. Then by the construction of E and E', we know that $0 \le e_{ij} \le rt_{ij}$ and $0 \le e_{i'j'} \le rt_{i'j'}$, respectively. So without loss of generality, by (3.17), assume that

$$\begin{array}{rl} rs_{ij} - (t_{ij} - s_{ij} - 1) \le e_{ij} \le r(s_{ij}+1) - (t_{ij} - s_{ij} - 1) & , \quad s_{ij} = 0, 1, \dots, t_{ij} - 1 \\ rs_{i'j'} - (t_{i'j'} - s_{i'j'} - 1) \le e_{i'j'} \le r(s_{i'j'}+1) - (t_{i'j'} - s_{i'j'} - 1) & , \quad s_{i'j'} = 0, 1, \dots, t_{i'j'} - 1. \end{array} \tag{3.52}$$

If $(i,j) = (i',j')$, $t_{ij} = t_{i'j'}$ and so $s_{ij} = s_{i'j'}$, and there is nothing to prove. So without loss of generality, assume that $(i,j) \ne (i',j')$ and $t_{ij} \ne t_{i'j'}$. Then by definition of Φ, we have

$$\begin{aligned} &\Phi(E)_{ij} = \theta_E(i,j,k) = e_{ij} - s_{ij}r + (t_{ij} - s_{ij} - 1) \text{ for all } i,j = 1,2,\dots,n; \\ &\Phi(E')_{i'j'} = \theta_{E'}(i',j',k') = e_{i'j'} - s_{i'j'}r + (t_{i'j'} - s_{i'j'} - 1) \text{ for all } i',j' = 1,2,\dots,n. \end{aligned} \tag{3.53}$$

To see that $\Phi(E) \ne \Phi(E')$ for all $E, E' \in RASM(n,r)$, we need to consider the following cases:

$$t_{ij} < t_{i'j'} \text{ and } s_{ij} = s_{i'j'}; \tag{3.54a}$$

$$t_{ij} < t_{i'j'} \text{ and } s_{ij} < s_{i'j'}; \tag{3.54b}$$

$$t_{ij} < t_{i'j'} \text{ and } s_{ij} > s_{i'j'}; \tag{3.54c}$$

$$t_{ij} > t_{i'j'} \text{ and } s_{ij} = s_{i'j'}; \tag{3.54d}$$

$$t_{ij} > t_{i'j'} \text{ and } s_{ij} < s_{i'j'}; \tag{3.54e}$$

$$t_{ij} > t_{i'j'} \text{ and } s_{ij} > s_{i'j'}. \tag{3.54f}$$

Without loss of generality, we consider (3.54b). Then

$$\Phi(E)_{ij} = \theta_E(i,j,k) = e_{ij} + t_{ij} \ne e_{i'j'} + t_{i'j'} = \theta_B(i',j',k') = \Phi(E')_{i'j'} \text{ since } t_{ij} < t_{i'j'}.$$

The argument for cases (3.54c)- (3.54f) is very similar and we skip the verifications for brevity. Therefore, Φ is injective as required.

It remains to show that Φ is surjective, that is, for all $\theta \in \Lambda(n,r)$ there exists $E \in RASM(n,r)$ such that $\Phi(E) = \theta$. In what follows, for a fixed n and r, we construct the map $\Pi : \Lambda(n,r) \to RASM(n,r)$ which maps any given $\theta \in \Lambda(n,r)$ to a member of $RASM(n,r)$, namely $\Pi(\theta)$ with $\Phi(\Pi(\theta)) = \theta$. Let $\theta \in \Lambda(n,r)$ be given. Since $I \in J(P_{n-2})$ is then given, by comparing the central entries of θ and entries of I position-wise, it is easy to find the shell and the subinterval which each $\theta(i,j)$ belongs to. Note that $\theta(i,j,k)$ is abbreviated here to $\theta(i,j)$. Let $Min(i,j,n-i,n-j) = t_{ij}$ and its corresponding value in the position (i,j) lie within the subinterval $[s_{ij}r - (t_{ij} - s_{ij} - 1), (s_{ij}+1)r - (t_{ij} - s_{ij} - 1)]$. Then $\Pi(\theta)$ is given by

$$\Pi(\theta)_{ij} := \theta(i,j) + s_{ij}r - (t_{ij} - s_{ij} - 1) \text{ for all } i,j = 1,\dots,n. \tag{3.55}$$

We are considering i and j with $i + j \le n$ (that is, (i,j) is in $Reg(1)$). Then we need to verify that

$$0 \le \Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} \le r. \tag{3.56}$$

Let (i,j) be associated with s_{ij} and t_{ij}, and let $(i-1,j)$ be associated with $s_{i-1,j}$ and $t_{i-1,j}$. We have

$$t_{ij} = \min(i,j) \qquad \text{and} \qquad t_{i-1,j} = \min(i-1,j),$$

and therefore

$$t_{ij} - t_{i-1,j} = \min(i,j) - \min(i-1,j). \tag{3.57}$$

We also have height matrix entries

$$h_{ij} = 2s_{ij} + n - i - j \qquad \text{and} \qquad h_{i-1,j} = 2s_{i-1,j} + n - i - j + 1,$$

and therefore

$$h_{ij} - h_{i-1,j} = 2(s_{ij} - s_{i-1,j}) - 1. \tag{3.58}$$

Due to the conditions of (P,ω)-partitions, we have:

- If $h_{ij} - h_{i-1,j} = -1$, then $\theta(i,j) \geq \theta(i-1,j)$. (3.59a)
- If $h_{ij} - h_{i-1,j} = -1$ and $\omega_n(i,j) > \omega_n(i-1,j)$, then $\theta(i,j) > \theta(i-1,j)$. (3.59b)
- If $h_{ij} - h_{i-1,j} = 1$, then $\theta(i,j) \leq \theta(i-1,j)$. (3.59c)
- If $h_{ij} - h_{i-1,j} = 1$ and $\omega_n(i,j) < \omega_n(i-1,j)$, then $\theta(i,j) < \theta(i-1,j)$. (3.59d)

We have defined

$$\Pi(\theta)_{ij} = \theta(i,j) + (r+1)s_{ij} - t_{ij} + 1 \qquad \text{and} \qquad \Pi(\theta)_{i-1,j} = \theta(i-1,j) + (r+1)s_{i-1,j} - t_{i-1,j} + 1,$$

which gives

$$\Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} = \theta(i,j) - \theta(i-1,j) + (r+1)(s_{ij} - s_{i-1,j}) - t_{ij} + t_{i-1,j}. \tag{3.60}$$

We also have $0 \leq \theta(i,j) \leq r$ and $0 \leq \theta(i-1,j) \leq r$, which gives

$$-r \leq \theta(i,j) - \theta(i-1,j) \leq r. \tag{3.61}$$

By the construction of $P_n(I)$ and definition of θ, we only need to verify four possibilities since $s_{ij} - s_{i-1,j} \in \{0,1\}$ and $t_{ij} - t_{i-1,j} \in \{0,1\}$.

(a) <u>The case $s_{ij} = s_{i-1,j}$ and $t_{ij} = t_{i-1,j}$.</u>

In this case, (3.60) gives

$$\Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} = \theta(i,j) - \theta(i-1,j). \tag{3.62}$$

The second inequality of (3.61), together with (3.62), now gives $\Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} \leq r$. Also, (3.58) gives $h_{ij} - h_{i-1,j} = -1$, which (using (3.59a)) implies that $\theta(i,j) - \theta(i-1,j) \geq 0$, and so (3.62) gives $\Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} \geq 0$. Therefore, we have $0 \leq \Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} \leq r$, as required.

(b) <u>The case $s_{ij} - s_{i-1,j} = 1$ and $t_{ij} - t_{i-1,j} = 1$.</u>

In this case, (3.60) gives

$$\Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} = \theta(i,j) - \theta(i-1,j) + r. \tag{3.63}$$

The first inequality of (3.61), together with (3.63), now gives $\Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} \geq 0$. Also, (3.58) gives $h_{ij} - h_{i-1,j} = 1$, which (using (3.59c)) implies that $\theta(i,j) - \theta(i-1,j) \leq 0$, and so (3.63) gives $\Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} \leq r$. Therefore, we have $0 \leq \Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} \leq r$, as required.

(c) The case $s_{ij} = s_{i-1,j}$ and $t_{ij} - t_{i-1,j} = 1$.

In this case, (3.60) gives

$$\Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} = \theta(i,j) - \theta(i-1,j) - 1. \tag{3.64}$$

The second inequality of (3.61), together with (3.64), gives $\Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} \leq r - 1$. Now note that (3.57) gives $\min(i,j) - \min(i-1,j) = 1$, which implies that $i \leq j$ (since if we had $i > j$, then we would have $\min(i,j) - \min(i-1,j) = j - j = 0$). Therefore, we have $\omega_n(i,j) > \omega_n(i-1,j)$. Also, (3.58) gives $h_{ij} - h_{i-1,j} = -1$, which (using (3.59b)) implies that $\theta(i,j) - \theta(i-1,j) > 0$, i.e., $\theta(i,j) - \theta(i-1,j) \geq 1$. Hence, (3.64) gives $\Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} \geq 0$. Therefore, $0 \leq \Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} \leq r - 1$, and so we have $0 \leq \Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} \leq r$, as required.

(d) The case $s_{ij} - s_{i-1,j} = 1$ and $t_{ij} = t_{i-1,j}$.

In this case, (3.60) gives

$$\Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} = \theta(i,j) - \theta(i-1,j) + r + 1. \tag{3.65}$$

The first inequality of (3.61), together with (3.65), gives $\Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} \geq 1$. Now note that (3.57) gives $\min(i,j) - \min(i-1,j) = 0$, which implies that $i \geq j$ (since if we had $i < j$, then we would have $\min(i,j) - \min(i-1,j) = i - (i-1) = 1$). Therefore, we have $\omega_n(i,j) < \omega_n(i-1,j)$. Also, (3.58) gives $h_{ij} - h_{i-1,j} = 1$, which (using (3.59d)) implies that $\theta(i,j) - \theta(i-1,j) < 0$, i.e., $\theta(i,j) - \theta(i-1,j) \leq -1$. Hence, (3.65) gives $\Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} \leq r$. Therefore, $1 \leq \Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} \leq r$, and so we have $0 \leq \Pi(\theta)_{ij} - \Pi(\theta)_{i-1,j} \leq r$, as required.

The other three conditions that need to be verified are

(i) $i + j \leq n$ and $0 \leq \Pi(\theta)_{ij} - \Pi(\theta)_{i,j-1} \leq r$;

(ii) $i + j \geq n$ and $0 \leq \Pi(\theta)_{ij} - \Pi(\theta)_{i+1,j} \leq r$;

(iii) $i + j \geq n$ and $0 \leq \Pi(\theta)_{ij} - \Pi(\theta)_{i,j+1} \leq r$.

We note that by symmetry, (i) is just the diagonal reflection of the case we have verified and conditions (ii) and (iii) are just the anti-main diagonal reflection of what we have already discussed. So the argument for these three cases are very similar. Therefore, Φ is surjective. This completes the proof. □

Now by the discussion in Section 3.2 (in particular Theorem 3.7), we are ready to provide a formula to enumerate the higher spin ASMs as a corollary to Theorem 3.23. We have

Theorem 3.24. *The Ehrhart polynomial of the ASM polytope $\mathcal{A}_n$, or equivalently the size of $ASM(n,r)$, is given by*

$$\begin{aligned} |ASM(n,r)| &= \sum_{I \in J(P_{n-2})} \Omega_{(P_n(I),\omega_n)}(r+1) \\ &= \sum_{I \in J(P_{n-2})} \sum_{\pi \in \mathcal{L}(P_n(I),\omega_n)} \binom{r+(n-1)^2 - des(\pi)}{(n-1)^2} \end{aligned} \tag{3.66}$$

where $\Omega_{(P_n(I),\omega_n)}(r+1)$ is the order polynomial of the labelled reduced ASM poset $(P_n(I),\omega_n)$ and $\mathcal{L}(P_n(I),\omega_n)$ is the set of all ω_n−labelled linear extensions of $P_n(I)$.

The derivation of (3.66) in Theorem 3.24 provides an example of the bijective method in enumerative combinatorics. It is a useful result since it reduces the enumeration of the HSASMs to the enumeration of descents within the ω_n−labelled linear extensions of the reduced ASM posets $P_n(I)$ for each $I \in J(P_{n-2})$. The enumeration of these linear extensions and their descents can be done recursively. However, in general, the enumeration of linear extensions of a finite poset is $\#P$−complete, see [23]. To reduce the complexity of these computations, it is expected that the symmetry operations given in Table 1.5 can be applied to $P_n(I)$ for each $I \in J(P_{n-2})$. See also the discussion in Section 4.3.

As an illustration, in what follows we apply Theorems 3.23 and 3.24 to $ASM(4,r)$, $ASM(5,r)$, $ASM(6,r)$ and $ASM(7,r)$, respectively.

3.3.1 The enumeration of $ASM(4,r)$

Now we apply Theorems 3.23 and 3.24 to $ASM(4,r)$ for a given positive integer r. Consider the set of all reduced HSASMs of order 4, $RASM(4,r)$. Note that since $n=4$ is even, we expect to have a single element in the innermost shell (that is, the second shell in the central position $(i,j)=(2,2)$). Then depending on the value of the entry e_{22} in this position, $RASM(r,4)$ can be written as disjoint union of the subsets $R_1(r)$ and $R_2(r)$ given in (3.20). Table 3.2 shows the cardinality of $RASM(4,r)$ and its subsets $R_1(r)$ and $R_2(r)$, respectively, for $r\in[10]$. As an illustration, for $r=1$, the set $R_1(1)$ consisting of 4 RASMs is given by

$$R_1(1)=\left\{\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\right\}$$

and the set $R_2(1)$ consisting of 38 RASMs is given by

$$R_2(1)=\left\{\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&0\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&0\\1&1&0\end{pmatrix},\begin{pmatrix}0&0&0\\1&1&0\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\1&1&0\\1&1&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&1&0\\0&0&0\end{pmatrix},\right.$$
$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&1&0\\1&1&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&1&1\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&1&1\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&1&1\\1&1&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&1&1\\1&1&1\end{pmatrix},$$
$$\begin{pmatrix}0&0&1\\1&1&0\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\1&1&0\\1&1&0\end{pmatrix},\begin{pmatrix}0&0&1\\1&1&1\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\1&1&1\\1&1&0\end{pmatrix},\begin{pmatrix}0&0&1\\1&1&1\\1&1&1\end{pmatrix},\begin{pmatrix}0&1&1\\0&1&0\\0&0&0\end{pmatrix},$$
$$\begin{pmatrix}0&1&1\\0&1&0\\1&0&0\end{pmatrix},\begin{pmatrix}0&1&1\\0&1&0\\1&1&0\end{pmatrix},\begin{pmatrix}0&1&1\\0&1&1\\0&0&0\end{pmatrix},\begin{pmatrix}0&1&1\\0&1&1\\1&0&0\end{pmatrix},\begin{pmatrix}0&1&1\\0&1&1\\1&1&0\end{pmatrix},\begin{pmatrix}0&1&1\\0&1&1\\1&1&1\end{pmatrix},$$
$$\begin{pmatrix}0&1&1\\1&1&0\\1&0&0\end{pmatrix},\begin{pmatrix}0&1&1\\1&1&0\\1&1&0\end{pmatrix},\begin{pmatrix}0&1&1\\1&1&1\\1&0&0\end{pmatrix},\begin{pmatrix}0&1&1\\1&1&1\\1&1&0\end{pmatrix},\begin{pmatrix}0&1&1\\1&1&1\\1&1&1\end{pmatrix},\begin{pmatrix}0&1&1\\1&2&1\\1&1&0\end{pmatrix},$$
$$\begin{pmatrix}0&1&1\\1&2&1\\1&1&1\end{pmatrix},\begin{pmatrix}1&1&1\\1&1&0\\1&0&0\end{pmatrix},\begin{pmatrix}1&1&1\\1&1&0\\1&1&0\end{pmatrix},\begin{pmatrix}1&1&1\\1&1&1\\1&0&0\end{pmatrix},\begin{pmatrix}1&1&1\\1&1&1\\1&1&0\end{pmatrix},\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix},$$
$$\left.\begin{pmatrix}1&1&1\\1&2&1\\1&1&0\end{pmatrix},\begin{pmatrix}1&1&1\\1&2&1\\1&1&1\end{pmatrix}\right\}$$

Now by Theorem 3.23, the map $\Phi : RASM(4,r) \to \Lambda(4,r)$ is given by $\Phi(E)_{ij} = \theta_E(i,j,k)$ where

$$\theta_E(i,j,k) = \begin{cases} e_{ij} & \text{if} \quad (i,j) \neq (2,2) \\ e_{22}+1 & \text{if} \quad (i,j) = (2,2) \quad \text{and} \quad 0 \leq e_{22} \leq r-1 \\ e_{22}-r & \text{if} \quad (i,j) = (2,2) \quad \text{and} \quad r \leq e_{22} \leq 2r \end{cases}$$

for each $E \in RASM(4,r)$ and $(i,j,k) \in P_4$.

r	$\lvert RASM(4,r)\rvert$	$\lvert R_1(r)\rvert$	$\lvert R_2(r)\rvert$
1	42	4	38
2	628	126	502
3	5102	1376	3726
4	28005	8818	19187
5	117332	40592	76740
6	403832	148780	255052
7	1197116	461272	735844
8	3156329	1257652	1898677
9	7572510	3096676	4475834
10	16810508	7018154	9792354

Table 3.2: The size of $RASM(4,r)$ and the size of its disjoint subsets for $r \in [10]$.

Now by Theorem 3.24, the Ehrhart polynomial of $ASM(4,r)$ is given by

$$\lvert ASM(4,r)\rvert = \sum_{I \in J(P_2)} \sum_{\pi \in \mathcal{L}(P_4(I),\omega_4)} \binom{r+9-des(\pi)}{9}$$

$$= \Omega_{(P_4(I_1),\omega_4)}(r+1) + \Omega_{(P_4(I_2),\omega_4)}(r+1)$$

$$= 3\binom{r+3}{9} + 80\binom{r+4}{9} + 415\binom{r+5}{9} + 592\binom{r+6}{9} + 253\binom{r+7}{9} + 32\binom{r+8}{9} + \binom{r+9}{9}$$

where I_1 and I_2 are as given in Example 3.11,

$$\Omega_{(P_4(I_1),\omega_4)}(r+1) = 2\binom{r+3}{9} + 52\binom{r+4}{9} + 248\binom{r+5}{9} + 296\binom{r+6}{9} + 86\binom{r+7}{9} + 4\binom{r+8}{9} \tag{3.67}$$

and

$$\Omega_{(P_4(I_2),\omega_4)}(r+1) = \binom{r+3}{9} + 28\binom{r+4}{9} + 167\binom{r+5}{9} + 296\binom{r+6}{9} + 167\binom{r+7}{9} + 28\binom{r+8}{9} + \binom{r+9}{9} \tag{3.68}$$

Note that the ω_n–labelled linear extensions and their numbers of descents were computed straightforwardly on Mathematica. Note also that this result matches (2.23).

3.3.2 The enumeration of $ASM(5,r)$

Consider $RASM(5,r)$. Then by the definition of central positions, Definition 3.22, we have

$$CPO(5) = \{(i,j) \mid i,j = 2,3\} = \{(2,2),(2,3),(3,2),(3,3)\}$$

Since $n = 5$ is odd, the innermost shell is a 2×2 matrix, namely $\begin{pmatrix} e_{22} & e_{23} \\ e_{32} & e_{33} \end{pmatrix}$, such that $0 \le e_{ij} \le 2r$ for $i,j = 2,3$. Now according to the value of these central entries, we can partition $RASM(5,r)$ into seven disjoint subsets $R_j(r)$ each of which satisfies certain boundary conditions. For brevity, we summarize the list of all seven R_j's and the certain boundary conditions associated with each central entry in Table 3.3. Thus we need to consider the following partition for the interval $[0,2r]$ as prescribed in (3.17):

$$[0,2r] = [0,r-1] \cup [r,2r]$$

Table 3.4 shows the size of $RASM(5,r)$ and the size of each subset $R_j(r)$ for each $j \in [7]$ and $r \in [10]$.

Now we turn our attention to the right hand side of the equation (3.21) in Theorem 3.23. We begin with the ASM poset P_5. Both rectangular and tetrahedral representations of P_5 are shown in Figure 3.17. We apply the reduction operations (3.6) on seven order ideals of P_3 given by

$$\begin{aligned} I_1 &= \{\}, \\ I_2 &= \{(2,1,0)\}, \\ I_3 &= \{(1,2,0)\}, \\ I_4 &= \{(1,2,0),(2,1,0)\}, \\ I_5 &= \{(1,2,0),(2,1,0),(2,2,1)\}, \\ I_6 &= \{(1,1,1),(1,2,0),(2,1,0)\}, \\ I_7 &= \{(1,1,1),(1,2,0),(2,1,0),(2,2,1)\}. \end{aligned} \tag{3.69}$$

We obtain seven reduced ASM posets $P_5(I)$ associated with seven order ideals in (3.69) each of which is equipped with the labelling ω_5 that is shown in Figure 3.3c. Then we consider seven $(P_5(I),\omega_5)$−partitions associated with each $P_5(I)$. Let $\Lambda(5,r) = \bigcup_{m=1}^{7} \mathcal{X}(P_5(I_m),\omega_5,r)$. Now by Theorem 3.23, the map $\Phi : RASM(5,r) \to \Lambda(5,r)$ is given by $\Phi(E) = \theta_E$ where

$$\theta_E(i,j,k) = \begin{cases} e_{ij} & \text{if} \quad Min(i,j,5-i,5-j) = 1 \\ e_{ij}+1 & \text{if} \quad Min(i,j,5-i,5-j) = 2 \quad \text{and} \quad 0 \le e_{i,j} \le r-1 \\ e_{ij}-r & \text{if} \quad Min(i,j,5-i,5-j) = 2 \quad \text{and} \quad r \le e_{i,j} \le 2r \end{cases} \tag{3.70}$$

Conditions / RHSASMs	e_{22}	e_{23}	e_{32}	e_{33}
$R_1(r)$	$< r$	$< r$	$< r$	$< r$
$R_2(r)$	$< r$	$< r$	$\geq r$	$< r$
$R_3(r)$	$< r$	$\geq r$	$< r$	$< r$
$R_4(r)$	$< r$	$\geq r$	$\geq r$	$< r$
$R_5(r)$	$< r$	$\geq r$	$\geq r$	$\geq r$
$R_6(r)$	$\geq r$	$\geq r$	$\geq r$	$< r$
$R_7(r)$	$\geq r$	$\geq r$	$\geq r$	$\geq r$

Table 3.3: The partition of $RASM(5,r)$ into 7 disjoint subsets according to the value of the central elements in the central positions in each $E \in RASM(5,r)$.

r	$\lvert RASM(5,r)\rvert$	$\lvert R_1(r)\rvert$	$\lvert R_2(r)\rvert$	$\lvert R_3(r)\rvert$	$\lvert R_4(r)\rvert$	$\lvert R_5(r)\rvert$	$\lvert R_6(r)\rvert$	$\lvert R_7(r)\rvert$
1	429	4	10	10	25	53	53	274
2	41784	951	2145	2145	5475	7407	7407	16254
3	1507128	47675	105204	105204	278608	278099	278099	414239
4	28226084	1057462	2297998	2297998	6267320	5108076	5108076	6089154
5	335138400	13922006	29858518	29858518	83220422	58809234	58809234	60660468
6	2850715602	126991780	881099884	269276252	762799010	485123780	485123780	452124748
7	18804467974	881099884	1850098898	1850098898	5306617888	3112618348	3112618348	2691315710
8	101540207493	4943470322	10293046034	10293046034	29814488556	16403646960	16403646960	13388862627
9	466452104685	23401887758	48373206304	48373206304	141227732427	73774360119	73774360119	57527351654
10	1875368841270	96394044547	197998681519	197998681519	581829274865	291177735389	291177735389	218792688042

Table 3.4: The size of $RASM(5,r)$ and the size of its disjoint subsets for $r \in [10]$.

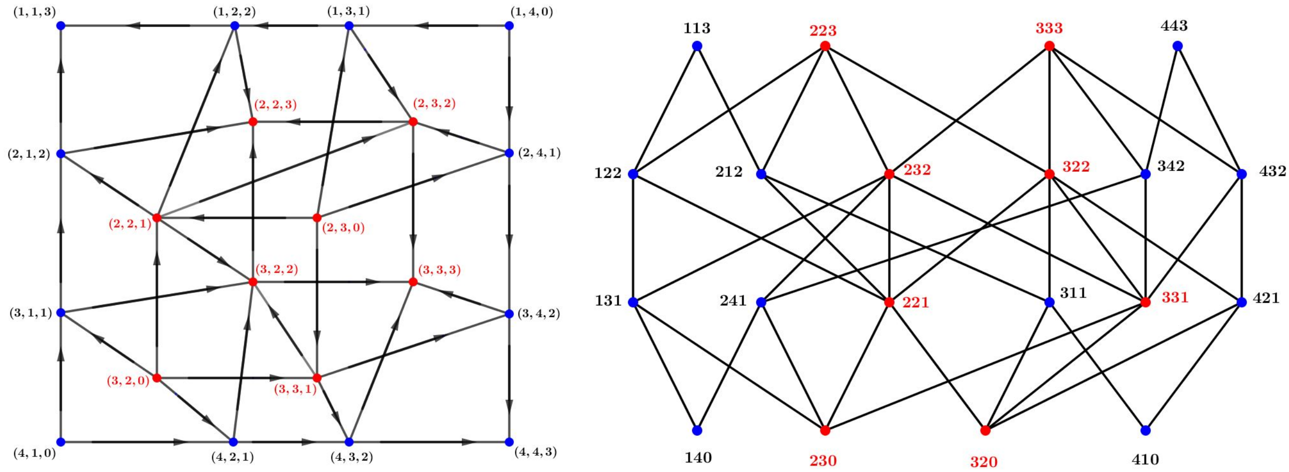
for all $E \in RASM(5,r)$ and $(i,j,k) \in P_5(I)$. Finally, by Theorem 3.24, we can write

$$\begin{aligned}
|ASM(5,r)| = &\sum_{I \in J(P_3)} \Omega_{(P_5(I),\omega_5)}(r+1) \\
=&\Omega_{(P_5(I_1),\omega_5)}(r+1) + \Omega_{(P_5(I_2),\omega_5)}(r+1) + \Omega_{(P_5(I_3),\omega_5)}(r+1) + \Omega_{(P_5(I_4),\omega_5)}(r+1) \\
&+ \Omega_{(P_5(I_5),\omega_5)}(r+1) + \Omega_{(P_5(I_6),\omega_5)}(r+1) + \Omega_{(P_5(I_7),\omega_5)}(r+1) \\
=&70\binom{r+4}{16} + 14468\binom{r+5}{16} + 521651\binom{r+6}{16} + 6002192\binom{r+7}{16} \qquad (3.71) \\
&+ 28233565\binom{r+8}{16} + 61083124\binom{r+9}{16} + 64066830\binom{r+10}{16} \\
&+ 32866092\binom{r+11}{16} + 7998192\binom{r+12}{16} + 854464\binom{r+13}{16} \\
&+ 34627\binom{r+14}{16} + 412\binom{r+15}{16} + \binom{r+16}{16}
\end{aligned}$$

where again the ω_n−labelled linear extensions and their number of descents were computed using Mathematica. Note that this result matches (2.24).

In what follows, we outline all seven corresponding reduced ASM labelled posets $(P_5(I_m), \omega_5)$ together with their associated $(P_5(I_m), \omega_5)$−partitions and their corresponding order polynomials $\Omega_{(P_5(I_m),\omega_5)}(r+$

1) for each $m \in [7]$.



(a) Rectangular representation of ASM poset P_5.

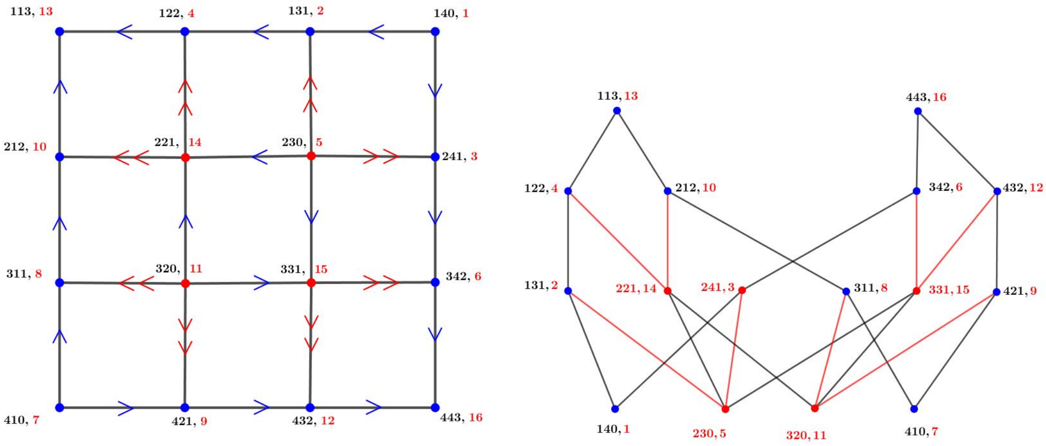
(b) Tetrahedral representation of ASM poset P_5.

Figure 3.17: Two equivalent representations of the ASM poset P_5. The red color vertices are the central elements of P_5.

(I) The labelled reduced ASM poset $(P_5(I_1), \omega_5)$ corresponding to the order ideal $I_1 = \{\}$ is given by (3.72) and its two equivalent representations are shown in Figure 3.18.

$$
\begin{aligned}
P_5(I_1) &= P_5 \setminus \{(2,2,3),(2,3,2),(3,2,2),(3,3,3)\} \\
&= \{(1,1,3),(1,2,2),(1,3,1),(1,4,0),(2,1,2),(2,2,1),(2,3,0),(2,4,1), \\
&\quad (3,1,1),(3,2,0),(3,3,1),(3,4,2),(4,1,0),(4,2,1),(4,3,2),(4,4,3)\}
\end{aligned}
\tag{3.72}
$$

Moreover, its associated $(P_5(I_1), \omega_5)$–partition is given by



(a) Rectangular representation of $P_5(I_1)$.

(b) Tetrahedral representation of $P_5(I_1)$.

Figure 3.18: The rectangular and tetrahedral representations of the labelled reduced ASM poset $P_5(I_1)$. The double arrows and red colored edges in these representations denote the strict inequality between the corresponding elements.

$$\mathcal{X}(P_5(I_1),\omega_5,r) = \left\{ \theta : P_5(I_1) \to [0,r] \;\middle|\; \begin{array}{ll} \bullet\ \theta(1,4,0) \geq \theta(1,3,1) & \bullet\ \theta(1,3,1) \geq \theta(1,2,2) \\ \bullet\ \theta(1,4,0) \geq \theta(2,4,1) & \bullet\ \theta(2,2,1) \gg \theta(1,2,2) \\ \bullet\ \theta(2,3,0) \gg \theta(1,3,1) & \bullet\ \theta(2,2,1) \gg \theta(2,1,2) \\ \bullet\ \theta(2,3,0) \gg \theta(2,4,1) & \bullet\ \theta(2,4,1) \geq \theta(3,4,2) \\ \bullet\ \theta(2,3,0) \geq \theta(2,2,1) & \bullet\ \theta(3,1,1) \geq \theta(2,1,2) \\ \bullet\ \theta(2,3,0) \geq \theta(3,3,1) & \bullet\ \theta(3,3,1) \gg \theta(3,4,2) \\ \bullet\ \theta(3,2,0) \gg \theta(3,1,1) & \bullet\ \theta(3,3,1) \gg \theta(4,3,2) \\ \bullet\ \theta(3,2,0) \gg \theta(4,2,1) & \bullet\ \theta(4,2,1) \geq \theta(4,3,2) \\ \bullet\ \theta(3,2,0) \geq \theta(2,2,1) & \bullet\ \theta(1,2,2) \geq \theta(1,1,3) \\ \bullet\ \theta(3,2,0) \geq \theta(3,3,1) & \bullet\ \theta(2,1,2) \geq \theta(1,1,3) \\ \bullet\ \theta(4,1,0) \geq \theta(3,1,1) & \bullet\ \theta(3,4,2) \geq \theta(4,4,3) \\ \bullet\ \theta(4,1,0) \geq \theta(4,2,1) & \bullet\ \theta(4,3,2) \geq \theta(4,4,3) \end{array} \right\} \tag{3.73}$$

Now if $E \in R_1$, then by Theorem 3.23, we define the map $\theta : R_1(r) \to \mathcal{X}(P_5(I_1),\omega_5,r)$ by

$$\theta_E(i,j,k) = \begin{cases} e_{ij} & \text{if} \quad Min(i,j,5-i,5-j) = 1 \\ e_{ij}+1 & \text{if} \quad Min(i,j,5-i,5-j) = 2 \end{cases} \tag{3.74}$$

and its associated order polynomial is given by

$$\Omega_{(P_5(I_1),\omega_5)}(r+1) = 7\binom{r+4}{16} + 1388\binom{r+5}{16} + 46487\binom{r+6}{16} + 493524\binom{r+7}{16} + 2128738\binom{r+8}{16}$$
$$+ 4189496\binom{r+9}{16} + 3952138\binom{r+10}{16} + 1791792\binom{r+11}{16} + 373603\binom{r+12}{16} \quad (3.75)$$
$$+ 32052\binom{r+13}{16} + 883\binom{r+14}{16} + 4\binom{r+15}{16}$$

(II) By applying the operation of reduction to $I_2 = \{(2,1,0)\}$ we obtain

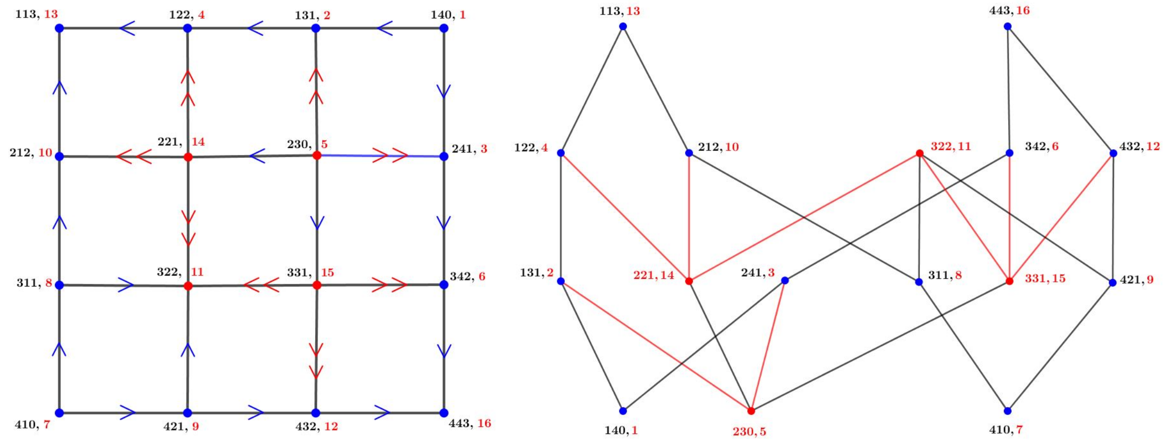
$$P_5(I_2) = P_5 \smallsetminus \{(2,2,3),(2,3,2),(3,2,0),(3,3,3)\}$$
$$= \{(1,1,3),(1,2,2),(1,3,1),(1,4,0),(2,1,2),(2,2,1),(2,3,0),(2,4,1), \quad (3.76)$$
$$(3,1,1),(3,2,2),(3,3,1),(3,4,2),(4,1,0),(4,2,1),(4,3,2),(4,4,3)\}$$

Both representations of $P_5(I_2)$ are shown in Figure 3.19.

$$\mathcal{X}(P_5(I_2),\omega_5,r) = \left\{ \theta : P_5(I_2) \to [0,r] \;\middle|\; \begin{array}{ll} \bullet\ \theta(1,4,0) \geq \theta(1,3,1) & \bullet\ \theta(1,2,2) \geq \theta(1,1,3) \\ \bullet\ \theta(1,4,0) \geq \theta(2,4,1) & \bullet\ \theta(2,4,1) \geq \theta(3,4,2) \\ \bullet\ \theta(2,3,0) >> \theta(1,3,1) & \bullet\ \theta(3,4,2) \geq \theta(4,4,3) \\ \bullet\ \theta(2,3,0) >> \theta(2,4,1) & \bullet\ \theta(3,1,1) \geq \theta(2,1,2) \\ \bullet\ \theta(2,3,0) \geq \theta(2,2,1) & \bullet\ \theta(3,1,1) \geq \theta(3,2,2) \\ \bullet\ \theta(2,3,0) \geq \theta(3,3,1) & \bullet\ \theta(3,3,1) >> \theta(3,3,2) \\ \bullet\ \theta(1,3,1) \geq \theta(1,2,2) & \bullet\ \theta(3,3,1) >> \theta(3,4,2) \\ \bullet\ \theta(2,2,1) >> \theta(1,2,2) & \bullet\ \theta(3,3,1) > \theta(4,3,2) \\ \bullet\ \theta(2,2,1) >> \theta(2,1,2) & \bullet\ \theta(4,2,1) \geq \theta(3,2,2) \\ \bullet\ \theta(2,2,1) >> \theta(3,2,2) & \bullet\ \theta(4,2,1) \geq \theta(4,3,2) \\ \bullet\ \theta(4,1,0) \geq \theta(3,1,1) & \bullet\ \theta(2,1,2) \geq \theta(1,1,3) \\ \bullet\ \theta(4,1,0) \geq \theta(4,2,1) & \bullet\ \theta(4,3,2) \geq \theta(4,4,3) \end{array} \right\} \quad (3.77)$$

If $E \in R_2$ then by Theorem 3.23, we define the map $\theta : R_2(r) \to \mathcal{X}(P_5(I_2),\omega_5,r)$ by

$$\theta_E(i,j,k) = \begin{cases} e_{ij} & \text{if} \quad Min(i,j,5-i,5-j) = 1 \\ e_{ij}+1 & \text{if} \quad Min(i,j,5-i,5-j) = 2 \quad \text{and} \quad (i,j) \neq (3,2) \\ e_{32}-r & \text{if} \quad (i,j) = (3,2) \end{cases} \quad (3.78)$$



(a) Rectangular representation of $P_5(I_2)$.

(b) Tetrahedral representation of $P_5(I_2)$.

Figure 3.19: Two equivalent representations of labelled reduced ASM poset $P_5(I_2)$. The double arrows and red colored edges in these representations mean strict inequality between the elements.

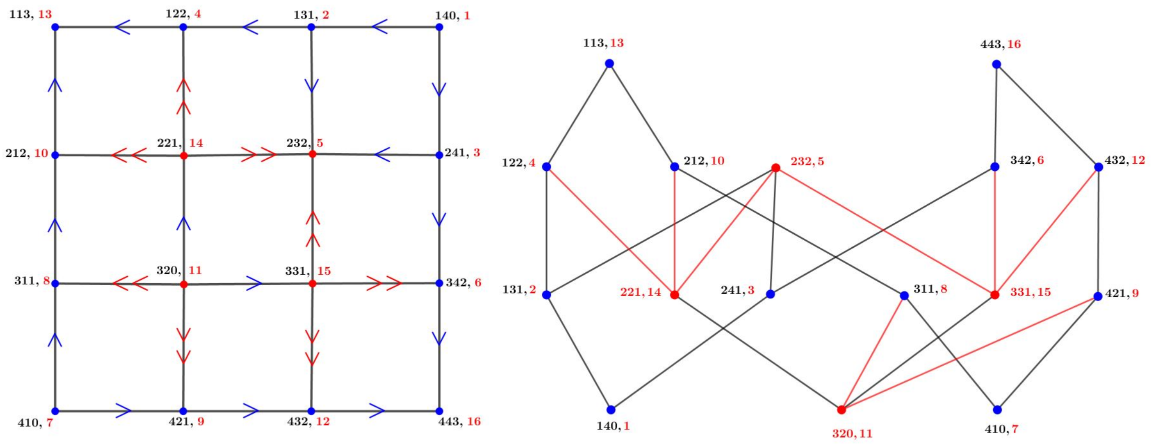
and its corresponding order polynomial is given by

$$\begin{aligned}\Omega_{(P_5(I_2),\omega_5)}(r+1) =& 10\binom{r+4}{16} + 2041\binom{r+5}{16} + 72417\binom{r+6}{16} + 815554\binom{r+7}{16} + 3727728\binom{r+8}{16} \\ &+ 7758442\binom{r+9}{16} + 7713674\binom{r+10}{16} + 3665496\binom{r+11}{16} + 794450\binom{r+12}{16} \\ &+ 70099\binom{r+13}{16} + 1975\binom{r+14}{16} + 10\binom{r+15}{16}\end{aligned} \tag{3.79}$$

(III) By applying the operation of reduction to $I_3 = \{(1,2,0)\}$ we obtain

$$\begin{aligned}P_5(I_3) &= P_5 \setminus \{(2,3,0),(2,2,3),(3,2,2),(3,3,3)\} \\ &= \{(1,1,3),(1,2,2),(1,3,1),(1,4,0),(2,1,2),(2,2,1),(2,3,2),(2,4,1), \\ &\quad (3,1,1),(3,2,0),(3,3,1),(3,4,2),(4,1,0),(4,2,1),(4,3,2),(4,4,3)\}\end{aligned} \tag{3.80}$$

The labelled reduced ASM poset $(P_5(I_3),\omega_5)$ is shown in Figure 3.21. Moreover, the corresponding set of $(P_5(I_3),\omega_5)$–partitions is given by (3.81).



(a) Rectangular representation of $P_5(I_3)$.

(b) Tetrahedral representation of $P_5(I_2)$.

Figure 3.20: Two equivalent representations of labelled reduced ASM poset $P_5(I_3)$. The double arrows and red colored edges in these representations mean strict inequality between the elements.

$$\mathcal{X}\left(P_5(I_3),\omega_5,r\right) = \left\{ \theta : P_5(I_3) \to [0,r] \;\middle|\; \begin{array}{ll} \bullet\ \theta(1,4,0) \ge \theta(1,3,1) & \bullet\ \theta(1,3,1) \ge \theta(2,3,2) \\ \bullet\ \theta(1,4,0) \ge \theta(2,4,1) & \bullet\ \theta(2,4,1) \ge \theta(2,3,2) \\ \bullet\ \theta(3,2,0) \gg \theta(2,2,1) & \bullet\ \theta(2,4,1) \ge \theta(3,4,2) \\ \bullet\ \theta(3,2,0) \gg \theta(3,1,1) & \bullet\ \theta(3,1,1) \ge \theta(2,1,2) \\ \bullet\ \theta(3,2,0) \ge \theta(3,3,1) & \bullet\ \theta(3,3,1) \gg \theta(2,3,2) \\ \bullet\ \theta(3,2,0) \gg \theta(4,2,1) & \bullet\ \theta(3,3,1) \gg \theta(3,4,2) \\ \bullet\ \theta(2,2,1) \gg \theta(1,2,2) & \bullet\ \theta(3,3,1) \gg \theta(4,3,2) \\ \bullet\ \theta(2,2,1) \gg \theta(2,1,2) & \bullet\ \theta(4,2,1) \ge \theta(4,3,2) \\ \bullet\ \theta(2,2,1) \gg \theta(2,3,2) & \bullet\ \theta(1,2,2) \ge \theta(1,1,3) \\ \bullet\ \theta(1,3,1) \ge \theta(1,2,2) & \bullet\ \theta(2,1,2) \ge \theta(1,1,3) \\ \bullet\ \theta(4,1,0) \ge \theta(3,1,1) & \bullet\ \theta(3,4,2) \ge \theta(4,4,3) \\ \bullet\ \theta(4,1,0) \ge \theta(4,2,1) & \bullet\ \theta(4,3,2) \ge \theta(4,4,3) \end{array} \right\} \tag{3.81}$$

If $E \in R_3$, then by Theorem 3.23, we define the map $\theta : R_3(r) \to \mathcal{X}\left(P_5(I_3),\omega_5,r\right)$ by

$$\theta_E(i,j,k) = \begin{cases} e_{ij} & \text{if} \quad Min(i,j,5-i,5-j) = 1 \\ e_{ij}+1 & \text{if} \quad Min(i,j,5-i,5-j) = 2 \quad \text{and} \quad (i,j) \ne (2,3) \\ e_{23}-r & \text{if} \qquad (i,j) = (2,3) \end{cases} \tag{3.82}$$

and its corresponding order polynomial is given by

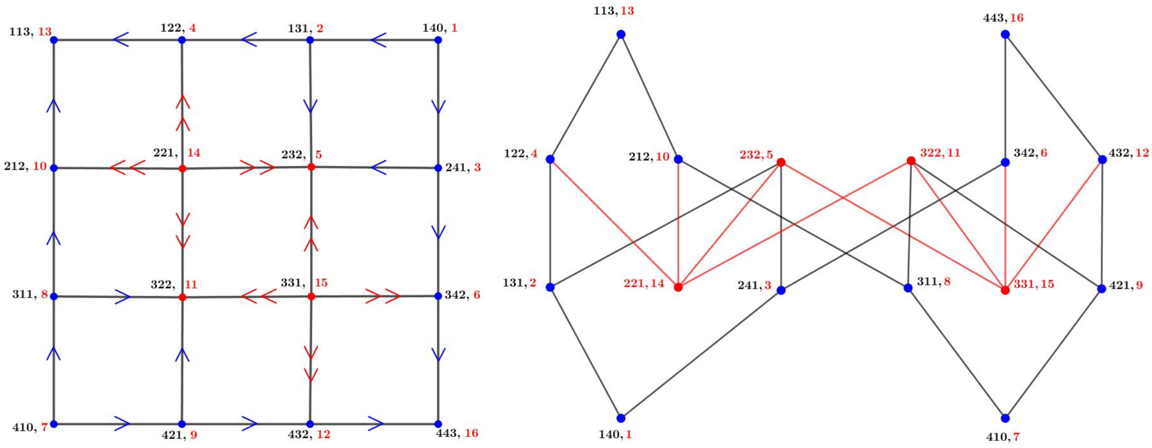
$$
\begin{aligned}
\Omega_{(P_5(I_3),\omega_5)}(r+1) = & 10\binom{r+4}{16} + 2041\binom{r+5}{16} + 72417\binom{r+6}{16} + 815554\binom{r+7}{16} + 3727728\binom{r+8}{16} \\
& + 7758442\binom{r+9}{16} + 7713674\binom{r+10}{16} + 3665496\binom{r+11}{16} + 794450\binom{r+12}{16} \\
& + 70099\binom{r+13}{16} + 1975\binom{r+14}{16} + 10\binom{r+15}{16}
\end{aligned} \tag{3.83}
$$

(IV) By applying the operation of reduction to the order ideal $I_4 = \{(1,2,0),(2,1,0)\}$, we obtain

$$
\begin{aligned}
P_5(I_4) &= P_5 \setminus \{(2,3,0),(2,2,3),(3,2,0),(3,3,3)\} \\
&= \{(1,1,3),(1,2,2),(1,3,1),(1,4,0),(2,1,2),(2,2,1),(2,3,2),(2,4,1), \\
&\quad (3,1,1),(3,2,2),(3,3,1),(3,4,2),(4,1,0),(4,2,1),(4,3,2),(4,4,3)\}
\end{aligned} \tag{3.84}
$$

The labelled reduced ASM poset $(P_5(I_4),\omega_5)$ is shown in Figure 3.21. Also its associated set of $(P_5(I_4),\omega_5)$–partitions is given by (3.85).

$$
\mathcal{X}\left(P_5(I_4),\omega_5,r\right) = \left\{ \theta : P_5(I_4) \to [0,r] \;\middle|\; \begin{array}{ll}
\bullet\ \theta(1,4,0) \geq \theta(1,3,1) & \bullet\ \theta(1,3,1) \geq \theta(1,2,2) \\
\bullet\ \theta(1,4,0) \geq \theta(2,4,1) & \bullet\ \theta(1,3,1) \geq \theta(2,3,2) \\
\bullet\ \theta(2,2,1) >> \theta(1,2,2) & \bullet\ \theta(2,4,1) \geq \theta(2,3,2) \\
\bullet\ \theta(2,2,1) >> \theta(2,1,2) & \bullet\ \theta(2,4,1) \geq \theta(3,4,2) \\
\bullet\ \theta(2,2,1) >> \theta(2,3,2) & \bullet\ \theta(3,1,1) \geq \theta(2,1,2) \\
\bullet\ \theta(2,2,1) >> \theta(3,2,2) & \bullet\ \theta(3,1,1) \geq \theta(3,2,2) \\
\bullet\ \theta(3,3,1) >> \theta(2,3,2) & \bullet\ \theta(4,2,1) \geq \theta(3,2,2) \\
\bullet\ \theta(3,3,1) >> \theta(3,2,2) & \bullet\ \theta(4,2,1) \geq \theta(4,3,2) \\
\bullet\ \theta(3,3,1) >> \theta(3,4,2) & \bullet\ \theta(1,2,2) \geq \theta(1,1,3) \\
\bullet\ \theta(3,3,1) >> \theta(4,3,2) & \bullet\ \theta(2,1,2) \geq \theta(1,1,3) \\
\bullet\ \theta(4,1,0) \geq \theta(3,1,1) & \bullet\ \theta(3,4,2) \geq \theta(4,4,3) \\
\bullet\ \theta(4,1,0) \geq \theta(4,2,1) & \bullet\ \theta(4,3,2) \geq \theta(4,4,3)
\end{array} \right\} \tag{3.85}
$$



(a) Rectangular representation of $P_5(I_4)$. (b) Tetrahedral representation of $P_5(I_4)$.

Figure 3.21: Two equivalent representations of labelled reduced ASM poset $P_5(I_4)$. The double arrows and red color edges in these representations mean strict inequality between the elements.

If $E \in R_4$, then by Theorem 3.23, we define the map $\theta : R_4(r) \to \mathcal{X}\,(P_5(I_4), \omega_5, r)$ by

$$\theta_E(i,j,k) = \begin{cases} e_{ij} & \text{if} \quad Min(i,j,5-i,5-j) = 1 \\ e_{ij}+1 & \text{if} \quad (i,j) \in \{(2,2),(3,3)\} \\ e_{ij}-r & \text{if} \quad (i,j) \in \{(2,3),(3,2)\} \end{cases} \tag{3.86}$$

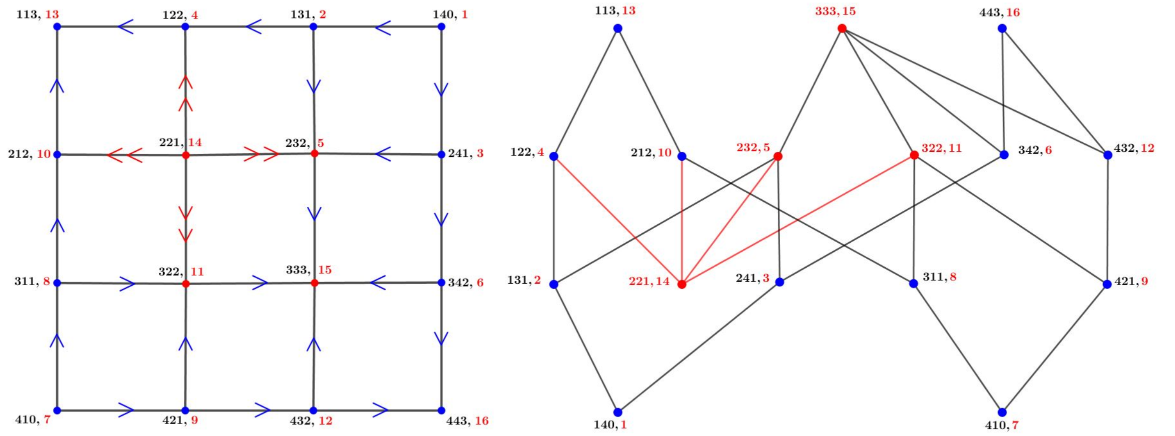
and its corresponding order polynomial is given by

$$\begin{aligned}\Omega_{(P_5(I_4),\omega_5)}(r+1) =& 36\binom{r+4}{16} + 7177\binom{r+5}{16} + 249282\binom{r+6}{16} + 2743701\binom{r+7}{16} + 12230556\binom{r+8}{16} \\ &+ 24751650\binom{r+9}{16} + 23829716\binom{r+10}{16} + 10903170\binom{r+11}{16} + 2258584\binom{r+12}{16} \\ &+ 188933\binom{r+13}{16} + 5050\binom{r+14}{16} + 25\binom{r+15}{16}\end{aligned} \tag{3.87}$$

(V) By applying the operation of reduction to the order ideal $I_5 = \{(1,2,0),(2,1,0),(2,2,1)\}$, we obtain

$$\begin{aligned} P_5(I_5) &= P_5 \setminus \{(2,3,0),(2,2,3),(3,2,0),(3,3,1)\} \\ &= \{(1,1,3),(1,2,2),(1,3,1),(1,4,0),(2,1,2),(2,2,1),(2,3,2),(2,4,1), \\ &\quad (3,1,1),(3,2,2),(3,3,3),(3,4,2),(4,1,0),(4,2,1),(4,3,2),(4,4,3)\} \end{aligned} \tag{3.88}$$

Both rectangular and tetrahedral representations of the labelled reduced ASM poset $(P_5(I_5), \omega_5)$ are shown in Figure 3.22. In addition, its associated set of $(P_5(I_5), \omega_5)$–partitions is given by (3.89).



(a) Rectangular representation of $P_5(I_5)$. (b) Tetrahedral representation of $P_5(I_5)$.

Figure 3.22: Two equivalent representations of labelled reduced ASM poset $P_5(I_5)$. The double arrows and red colored edges in these representations mean strict inequality between the elements.

$$\mathcal{X}\left(P_5(I_5),\omega_5,r\right) = \left\{ \theta : P_5(I_5) \to [0,r] \;\middle|\; \begin{array}{ll} \bullet\ \theta(1,4,0) \geq \theta(1,3,1) & \bullet\ \theta(1,3,1) \geq \theta(1,2,2) \\ \bullet\ \theta(1,4,0) \geq \theta(2,4,1) & \bullet\ \theta(1,3,1) \geq \theta(2,3,2) \\ \bullet\ \theta(2,2,1) >> \theta(1,2,2) & \bullet\ \theta(2,1,2) \geq \theta(1,1,3) \\ \bullet\ \theta(2,2,1) >> \theta(2,1,2) & \bullet\ \theta(2,4,1) \geq \theta(2,3,2) \\ \bullet\ \theta(2,2,1) >> \theta(2,3,2) & \bullet\ \theta(2,4,1) \geq \theta(3,4,2) \\ \bullet\ \theta(2,2,1) >> \theta(3,2,2) & \bullet\ \theta(3,1,1) \geq \theta(2,1,2) \\ \bullet\ \theta(3,2,2) \geq \theta(3,3,3) & \bullet\ \theta(3,1,1) \geq \theta(3,2,2) \\ \bullet\ \theta(3,4,2) \geq \theta(3,3,3) & \bullet\ \theta(4,2,1) \geq \theta(4,3,2) \\ \bullet\ \theta(4,3,2) \geq \theta(3,3,3) & \bullet\ \theta(4,2,1) \geq \theta(3,2,2) \\ \bullet\ \theta(2,3,2) \geq \theta(3,3,3) & \bullet\ \theta(2,1,2) \geq \theta(1,1,3) \\ \bullet\ \theta(4,1,0) \geq \theta(3,1,1) & \bullet\ \theta(3,4,2) \geq \theta(4,4,3) \\ \bullet\ \theta(4,1,0) \geq \theta(4,2,1) & \bullet\ \theta(4,3,2) \geq \theta(4,4,3) \end{array} \right\} \tag{3.89}$$

Now if $E \in R_5$, then by Theorem 3.23, we define the map $\theta : R_5(r) \to \mathcal{X}\left(P_5(I_5),\omega_5,r\right)$ by

$$\theta_E(i,j,k) = \begin{cases} e_{ij} & \text{if} \quad Min(i,j,5-i,5-j) = 1 \\ e_{22}+1 & \text{if} \quad (i,j) = (2,2) \\ e_{ij}-r & \text{if} \quad (i,j) \in \{(2,3),(3,2),(3,3)\} \end{cases} \tag{3.90}$$

and its corresponding order polynomial is given by

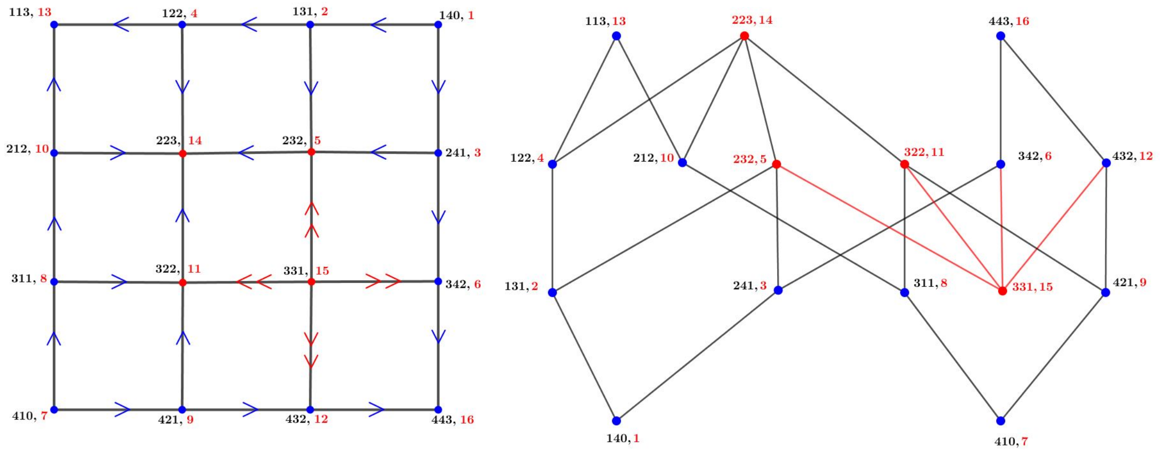
$$
\begin{aligned}
\Omega_{(P_5(I_5),\omega_5)}(r+1) =& 3\binom{r+4}{16} + 782\binom{r+5}{16} + 34658\binom{r+6}{16} + 479677\binom{r+7}{16} + 2672560\binom{r+8}{16} \\
&+ 6775264\binom{r+9}{16} + 8258514\binom{r+10}{16} + 4882786\binom{r+11}{16} + 1351705\binom{r+12}{16} \\
&+ 159388\binom{r+13}{16} + 6506\binom{r+14}{16} + 53\binom{r+15}{16}
\end{aligned} \tag{3.91}
$$

(VI) By applying the operation of reduction to the order ideal $I_6 = \{(1,1,1),(1,2,0),(2,1,0)\}$ we obtain

$$
\begin{aligned}
P_5(I_6) &= P_5 \smallsetminus \{(2,3,0),(2,2,1),(3,2,0),(3,3,3)\} \\
&= \{(1,1,3),(1,2,2),(1,3,1),(1,4,0),(2,1,2),(2,2,3),(2,3,2),(2,4,1), \\
&\quad (3,1,1),(3,2,2),(3,3,1),(3,4,2),(4,1,0),(4,2,1),(4,3,2),(4,4,3)\}
\end{aligned} \tag{3.92}
$$

The labelled reduced ASM poset $(P_5(I_6),\omega_5)$ is shown in Figure 3.23. Also its associated set of $(P_5(I_6),\omega_5)$–partitions is given by (3.93).

$$
\mathcal{X}\left(P_5(I_6),\omega_5,r\right) =
\left\{ \theta : P_5(I_6) \to [0,r] \;\middle|\;
\begin{array}{ll}
\bullet\ \theta(1,4,0) \geq \theta(1,3,1) & \bullet\ \theta(3,1,1) \geq \theta(2,1,2) \\
\bullet\ \theta(1,4,0) \geq \theta(2,4,1) & \bullet\ \theta(3,1,1) \geq \theta(3,2,2) \\
\bullet\ \theta(3,3,1) >> \theta(2,3,2) & \bullet\ \theta(4,2,1) \geq \theta(3,2,2) \\
\bullet\ \theta(3,3,1) >> \theta(3,2,2) & \bullet\ \theta(4,2,1) \geq \theta(4,3,2) \\
\bullet\ \theta(3,3,1) >> \theta(3,4,2) & \bullet\ \theta(1,2,2) \geq \theta(1,1,3) \\
\bullet\ \theta(3,3,1) >> \theta(4,3,2) & \bullet\ \theta(1,2,2) \geq \theta(2,2,3) \\
\bullet\ \theta(2,4,1) \geq \theta(3,4,2) & \bullet\ \theta(2,1,2) \geq \theta(1,1,3) \\
\bullet\ \theta(1,3,1) \geq \theta(1,2,2) & \bullet\ \theta(2,1,2) \geq \theta(2,2,3) \\
\bullet\ \theta(1,3,1) \geq \theta(2,3,2) & \bullet\ \theta(2,3,2) \geq \theta(2,2,3) \\
\bullet\ \theta(2,4,1) \geq \theta(2,3,2) & \bullet\ \theta(3,2,2) \geq \theta(2,2,3) \\
\bullet\ \theta(4,1,0) \geq \theta(3,1,1) & \bullet\ \theta(3,4,2) \geq \theta(4,4,3) \\
\bullet\ \theta(4,1,0) \geq \theta(4,2,1) & \bullet\ \theta(4,3,2) \geq \theta(4,4,3)
\end{array}
\right\} \tag{3.93}
$$



(a) Rectangular representation of $P_5(I_6)$.

(b) Tetrahedral representation of $P_5(I_6)$.

Figure 3.23: Two equivalent representations of labelled reduced ASM poset $P_5(I_6)$. The double arrows and red color edges in these representations mean strict inequality between the elements.

Now if $E \in R_6$, then by Theorem 3.23, we define the map $\theta : R_6(r) \to \mathcal{X}(P_5(I_6), \omega_5, r)$ by

$$\theta_E(i,j,k) = \begin{cases} e_{ij} & \text{if} \quad Min(i,j,5-i,5-j) = 1 \\ e_{ij} - r & \text{if} \quad (i,j) \in \{(2,2),(2,3),(3,2)\} \\ e_{33} + 1 & \text{if} \quad (i,j) = (3,3) \end{cases} \tag{3.94}$$

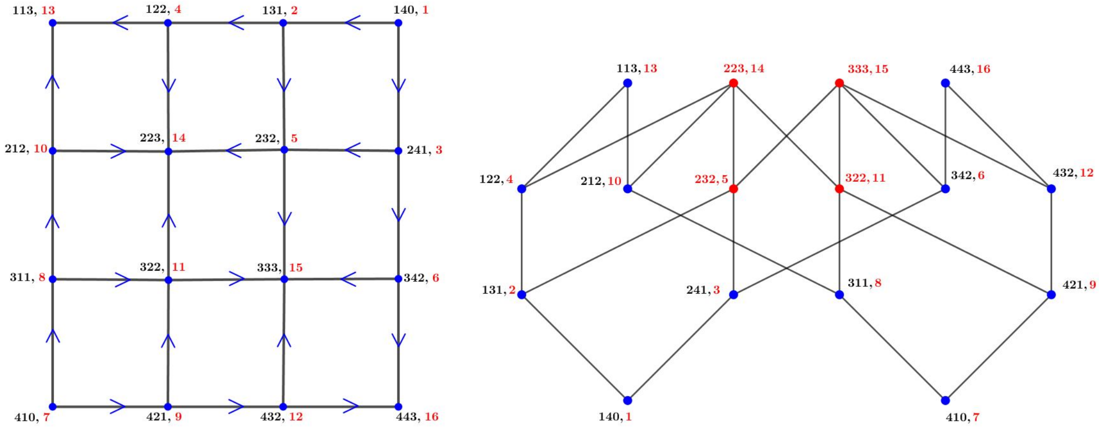
and its associated order polynomial is given by

$$\begin{aligned} \Omega_{(P_5(I_6),\omega_5)}(r+1) =& 3\binom{r+4}{16} + 782\binom{r+5}{16} + 34658\binom{r+6}{16} + 479677\binom{r+7}{16} + 2672560\binom{r+8}{16} \\ &+ 6775264\binom{r+9}{16} + 8258514\binom{r+10}{16} + 4882786\binom{r+11}{16} + 1351705\binom{r+12}{16} \\ &+ 159388\binom{r+13}{16} + 6506\binom{r+14}{16} + 53\binom{r+15}{16} \end{aligned} \tag{3.95}$$

(VII) Finally, by applying the operation of reduction to the order ideal $I_7 = \{(1,1,1),(1,2,0),(2,1,0),(2,2,1)\}$, we obtain

$$\begin{aligned} P_5(I_7) &= P_5 \setminus \{(2,3,0),(2,2,1),(3,2,0),(3,3,1)\} \\ &= \{(1,1,3),(1,2,2),(1,3,1),(1,4,0),(2,1,2),(2,2,3),(2,3,2),(2,4,1), \\ &\quad (3,1,1),(3,2,2),(3,3,3),(3,4,2),(4,1,0),(4,2,1),(4,3,2),(4,4,3)\} \end{aligned} \tag{3.96}$$

The labelled reduced ASM poset $(P_5(I_7),\omega_5)$ is shown in Figure 3.24. Moreover, its associated $(P_5(I_7),\omega_5)$–partition is given by (3.97). Note that in this case, since ω_5 is natural so all the inequalities in (3.97) are weak.



(a) Rectangular representation of $P_5(I_7)$.

(b) Tetrahedral representation of $P_5(I_7)$.

Figure 3.24: Two equivalent representations of labelled reduced ASM poset $P_5(I_7)$. The double arrows and red colored edges in these representations mean strict inequality between the elements.

$$\mathcal{X}\left(P_5(I_7), \omega_5, r\right) = \left\{ \theta : P_5(I_7) \to [0, r] \;\middle|\; \begin{array}{ll} \bullet\ \theta(1,4,0) \geq \theta(1,3,1) & \bullet\ \theta(1,2,2) \geq \theta(1,1,3) \\ \bullet\ \theta(1,4,0) \geq \theta(2,4,1) & \bullet\ \theta(1,2,2) \geq \theta(2,2,3) \\ \bullet\ \theta(1,3,1) \geq \theta(1,2,2) & \bullet\ \theta(2,1,2) \geq \theta(1,1,3) \\ \bullet\ \theta(1,3,1) \geq \theta(2,3,2) & \bullet\ \theta(2,1,2) \geq \theta(2,2,3) \\ \bullet\ \theta(2,4,1) \geq \theta(2,3,2) & \bullet\ \theta(2,3,2) \geq \theta(2,2,3) \\ \bullet\ \theta(2,4,1) \geq \theta(3,4,2) & \bullet\ \theta(2,3,2) \geq \theta(3,3,3) \\ \bullet\ \theta(3,1,1) \geq \theta(2,1,2) & \bullet\ \theta(3,2,2) \geq \theta(2,2,3) \\ \bullet\ \theta(3,1,1) \geq \theta(3,2,2) & \bullet\ \theta(3,2,2) \geq \theta(3,3,3) \\ \bullet\ \theta(4,2,1) \geq \theta(3,2,2) & \bullet\ \theta(3,4,2) \geq \theta(3,3,3) \\ \bullet\ \theta(4,2,1) \geq \theta(4,3,2) & \bullet\ \theta(4,3,2) \geq \theta(3,3,3) \\ \bullet\ \theta(4,1,0) \geq \theta(3,1,1) & \bullet\ \theta(3,4,2) \geq \theta(4,4,3) \\ \bullet\ \theta(4,1,0) \geq \theta(4,2,1) & \bullet\ \theta(4,3,2) \geq \theta(4,4,3) \end{array} \right\} \tag{3.97}$$

If $E \in R_7$, then by Theorem 3.23, we define the map $\theta : R_7(r) \to \mathcal{X}\left(P_5(I_7), \omega_5, r\right)$ by

$$\theta_E(i,j,k) = \begin{cases} e_{ij} & \text{if} \quad Min(i,j,5-i,5-j) = 1 \\ e_{ij} - r & \text{if} \quad Min(i,j,5-i,5-j) = 2 \end{cases} \tag{3.98}$$

and its associated order polynomial is given by

$$\Omega_{(P_5(I_7),\omega_5)}(r+1) = \binom{r+4}{16} + 257\binom{r+5}{16} + 11732\binom{r+6}{16} + 174505\binom{r+7}{16} + 1073695\binom{r+8}{16}$$

$$+ 3074566\binom{r+9}{16} + 4340600\binom{r+10}{16} + 3074566\binom{r+11}{16} + 1073695\binom{r+12}{16} \quad (3.99)$$

$$+ 174505\binom{r+13}{16} + 11732\binom{r+14}{16} + 257\binom{r+15}{16} + \binom{r+16}{16}$$

Now by adding the order polynomials in the above seven cases $(I) - (VII)$, it is easy to check that the identity (3.71) holds.

3.3.3 The enumeration of $ASM(6,r)$

We begin with the left hand side of the function Φ in Theorem 3.23. Let $E \in RASM(6,5)$ be given. Then by the definition of RASMs, each E consists of 3 shells. The set of central positions in E is given by the union of

$$A = \{(2,2),(2,3),(2,4),(3,2),(3,4),(4,2),(4,3),(4,4)\} \quad \text{and} \quad B = \{(3,3)\} \tag{3.100}$$

where A contains the central positions in the second shell and B contains the central position in the innermost shell. Also note that by the definition of reduced ASMs, Definition 3.18, for $(i,j) \in A$ we have $0 \le e_{ij} \le 2r$ and $0 \le e_{33} \le 3r$. Therefore, by the discussion in Remark 3.21, we consider the following partitions for the intervals $[0,2r]$ and $[0,3r]$:

$$\begin{aligned} [0,2r] &= [0,r-1] \cup [r,2r] \\ [0,3r] &= [0,r-2] \cup [r-1,2r-1] \cup [2r,3r] \end{aligned} \tag{3.101}$$

Similar to the case for $RASM(5,r)$, according to the value of the entries in the central positions in each $E \in RASM(6,r)$, we can classify $RASM(6,r)$ into 42 subsets. In particular, $RASM(6,r) = \bigcup_{i=1}^{42} R_i(r)$ where each $R_i(r)$ is the set of all order 6 RASMs with line sum r such that central entries of its elements satisfy certain conditions. Table 3.6 shows the cardinality of $RASM(6,r)$ and its 42 subsets for $r \in [2]$.

Now we turn our attention to the right hand side of the function Φ in Theorem 3.23 and describe briefly how it is constructed. We begin with the ASM poset P_6. The rectangular and the tetrahedral representations of the Hasse diagram of P_6 are shown in Figures 3.25a and 3.26b, respectively. Now by applying the reduction operation on each order ideal $I \in J(P_4)$, we obtain 42 reduced ASM posets $P_6(I)$ each of which is equipped with the labelling ω_6 (Note that by Theorem 1.87, $|J(P_4)| = |ASM(4)| = 42$). As an illustration, the reduced ASM labelled posets $(P_6(I_1),\omega_6)$ and $(P_6(I_{42}),\omega_6)$ associated with order ideals $I_1 = \varnothing$ and $I_{42} = P_4$ are shown in Figure 3.25a and Figure 3.26b, respectively. Therefore, there are 42 labelled reduced ASM posets $(P_6(I),\omega_6)$. So we can consider 42 sets of $(P_6(I),\omega_6)$–partitions associated with 42 ASM order ideals $I \in J(P_4)$. In particular, we can write $\Lambda(6,r) = \bigcup_{I \in J(P_4)} \mathcal{X}(P_6(I),\omega_6,r)$.

Now by Theorem 3.23, the map $\Phi : RASM(6,r) \to \Lambda(6,r)$ is given by $\Phi(E) = \theta_E$ where

$$\theta_E(i,j,k) = \begin{cases} e_{ij} & \text{if} \quad Min(i,j,6-i,6-j) = 1 \\ e_{ij}+1 & \text{if} \quad (i,j) \in A \quad \text{and} \quad 0 \le e_{i,j} \le r-1 \\ e_{ij}-r & \text{if} \quad (i,j) \in A \quad \text{and} \quad r \le e_{i,j} \le 2r \\ e_{33}+2 & \text{if} \quad (i,j) = (3,3) \quad \text{and} \quad 0 \le e_{33} \le r-2 \\ e_{33}-r+1 & \text{if} \quad (i,j) = (3,3) \quad \text{and} \quad r-1 \le e_{33} \le 2r-1 \\ e_{33}-2r & \text{if} \quad (i,j) = (3,3) \quad \text{and} \quad 2r \le e_{33} \le 3r \end{cases} \tag{3.102}$$

RHSASMs \ **Conditions**	e_{22}	e_{23}	e_{24}	e_{32}	e_{33}	e_{34}	e_{42}	e_{43}	e_{44}
$R_1(r)$	$< r$	$< r$	$< r$	$< r$	$\leq r-2$	$< r$	$< r$	$< r$	$< r$
$R_2(r)$	$< r$	$< r$	$< r$	$< r$	$\leq r-2$	$< r$	$\geq r$	$< r$	$< r$
$R_3(r)$	$< r$	$< r$	$< r$	$< r$	$r-1 \leq e_{33} \leq 2r-1$	$< r$	$< r$	$< r$	$< r$
$R_4(r)$	$< r$	$< r$	$< r$	$< r$	$r-1 \leq e_{33} \leq 2r-1$	$< r$	$\geq r$	$< r$	$< r$
$R_5(r)$	$< r$	$< r$	$< r$	$< r$	$r-1 \leq e_{33} \leq 2r-1$	$< r$	$\geq r$	$\geq r$	$< r$
$R_6(r)$	$< r$	$< r$	$< r$	$\geq r$	$r-1 \leq e_{33} \leq 2r-1$	$< r$	$\geq r$	$< r$	$< r$
$R_7(r)$	$< r$	$< r$	$< r$	$\geq r$	$r-1 \leq e_{33} \leq 2r-1$	$< r$	$\geq r$	$\geq r$	$< r$
$R_8(r)$	$< r$	$< r$	$\geq r$	$< r$	$\leq r-2$	$< r$	$< r$	$< r$	$< r$
$R_9(r)$	$< r$	$< r$	$\geq r$	$< r$	$\leq r-2$	$< r$	$\geq r$	$< r$	$< r$
$R_{10}(r)$	$< r$	$< r$	$\geq r$	$< r$	$r-1 \leq e_{33} \leq 2r-1$	$< r$	$< r$	$< r$	$< r$
$R_{11}(r)$	$< r$	$< r$	$\geq r$	$< r$	$r-1 \leq e_{33} \leq 2r-1$	$< r$	$\geq r$	$< r$	$< r$
$R_{12}(r)$	$< r$	$< r$	$\geq r$	$< r$	$r-1 \leq e_{33} \leq 2r-1$	$< r$	$\geq r$	$\geq r$	$< r$
$R_{13}(r)$	$< r$	$< r$	$\geq r$	$< r$	$r-1 \leq e_{33} \leq 2r-1$	$\geq r$	$< r$	$< r$	$< r$
$R_{14}(r)$	$< r$	$< r$	$\geq r$	$< r$	$r-1 \leq e_{33} \leq 2r-1$	$\geq r$	$\geq r$	$< r$	$< r$
$R_{15}(r)$	$< r$	$< r$	$\geq r$	$< r$	$r-1 \leq e_{33} \leq 2r-1$	$\geq r$	$\geq r$	$\geq r$	$< r$
$R_{16}(r)$	$< r$	$< r$	$\geq r$	$< r$	$r-1 \leq e_{33} \leq 2r-1$	$\geq r$	$\geq r$	$\geq r$	$\geq r$
$R_{17}(r)$	$< r$	$< r$	$\geq r$	$\geq r$	$r-1 \leq e_{33} \leq 2r-1$	$< r$	$\geq r$	$< r$	$< r$
$R_{18}(r)$	$< r$	$< r$	$\geq r$	$\geq r$	$r-1 \leq e_{33} \leq 2r-1$	$< r$	$\geq r$	$\geq r$	$< r$
$R_{19}(r)$	$< r$	$< r$	$\geq r$	$\geq r$	$r-1 \leq e_{33} \leq 2r-1$	$\geq r$	$\geq r$	$< r$	$< r$
$R_{20}(r)$	$< r$	$< r$	$\geq r$	$\geq r$	$r-1 \leq e_{33} \leq 2r-1$	$\geq r$	$\geq r$	$\geq r$	$< r$
$R_{21}(r)$	$< r$	$< r$	$\geq r$	$\geq r$	$r-1 \leq e_{33} \leq 2r-1$	$\geq r$	$\geq r$	$\geq r$	$\geq r$
$R_{22}(r)$	$< r$	$\geq r$	$\geq r$	$< r$	$r-1 \leq e_{33} \leq 2r-1$	$< r$	$< r$	$< r$	$< r$
$R_{23}(r)$	$< r$	$\geq r$	$\geq r$	$< r$	$r-1 \leq e_{33} \leq 2r-1$	$< r$	$\geq r$	$< r$	$< r$
$R_{24}(r)$	$< r$	$\geq r$	$\geq r$	$< r$	$r-1 \leq e_{33} \leq 2r-1$	$< r$	$\geq r$	$\geq r$	$< r$
$R_{25}(r)$	$< r$	$\geq r$	$\geq r$	$< r$	$r-1 \leq e_{33} \leq 2r-1$	$\geq r$	$< r$	$< r$	$< r$
$R_{26}(r)$	$< r$	$\geq r$	$\geq r$	$< r$	$r-1 \leq e_{33} \leq 2r-1$	$\geq r$	$\geq r$	$< r$	$< r$
$R_{27}(r)$	$< r$	$\geq r$	$\geq r$	$< r$	$r-1 \leq e_{33} \leq 2r-1$	$\geq r$	$\geq r$	$\geq r$	$< r$
$R_{28}(r)$	$< r$	$\geq r$	$\geq r$	$< r$	$r-1 \leq e_{33} \leq 2r-1$	$\geq r$	$\geq r$	$\geq r$	$\geq r$
$R_{29}(r)$	$< r$	$\geq r$	$\geq r$	$\geq r$	$r-1 \leq e_{33} \leq 2r-1$	$< r$	$\geq r$	$< r$	$< r$
$R_{30}(r)$	$< r$	$\geq r$	$\geq r$	$\geq r$	$r-1 \leq e_{33} \leq 2r-1$	$< r$	$\geq r$	$\geq r$	$< r$
$R_{31}(r)$	$< r$	$\geq r$	$\geq r$	$\geq r$	$r-1 \leq e_{33} \leq 2r-1$	$\geq r$	$\geq r$	$< r$	$< r$
$R_{32}(r)$	$< r$	$\geq r$	$\geq r$	$\geq r$	$r-1 \leq e_{33} \leq 2r-1$	$\geq r$	$\geq r$	$\geq r$	$< r$
$R_{33}(r)$	$< r$	$\geq r$	$\geq r$	$\geq r$	$r-1 \leq e_{33} \leq 2r-1$	$\geq r$	$\geq r$	$\geq r$	$\geq r$
$R_{34}(r)$	$< r$	$\geq r$	$\geq r$	$\geq r$	$\geq 2r$	$\geq r$	$\geq r$	$\geq r$	$< r$
$R_{35}(r)$	$< r$	$\geq r$	$\geq r$	$\geq r$	$\geq 2r$	$\geq r$	$\geq r$	$\geq r$	$\geq r$
$R_{36}(r)$	$\geq r$	$\geq r$	$\geq r$	$\geq r$	$r-1 \leq e_{33} \leq 2r-1$	$< r$	$\geq r$	$< r$	$< r$
$R_{37}(r)$	$\geq r$	$\geq r$	$\geq r$	$\geq r$	$r-1 \leq e_{33} \leq 2r-1$	$< r$	$\geq r$	$\geq r$	$< r$
$R_{38}(r)$	$\geq r$	$\geq r$	$\geq r$	$\geq r$	$r-1 \leq e_{33} \leq 2r-1$	$\geq r$	$\geq r$	$< r$	$< r$
$R_{39}(r)$	$\geq r$	$\geq r$	$\geq r$	$\geq r$	$r-1 \leq e_{33} \leq 2r-1$	$\geq r$	$\geq r$	$\geq r$	$< r$
$R_{40}(r)$	$\geq r$	$\geq r$	$\geq r$	$\geq r$	$r-1 \leq e_{33} \leq 2r-1$	$\geq r$	$\geq r$	$\geq r$	$\geq r$
$R_{41}(r)$	$\geq r$	$\geq r$	$\geq r$	$\geq r$	$\geq 2r$	$\geq r$	$\geq r$	$\geq r$	$< r$
$R_{42}(r)$	$\geq r$	$\geq r$	$\geq r$	$\geq r$	$\geq 2r$	$\geq r$	$\geq r$	$\geq r$	$\geq r$

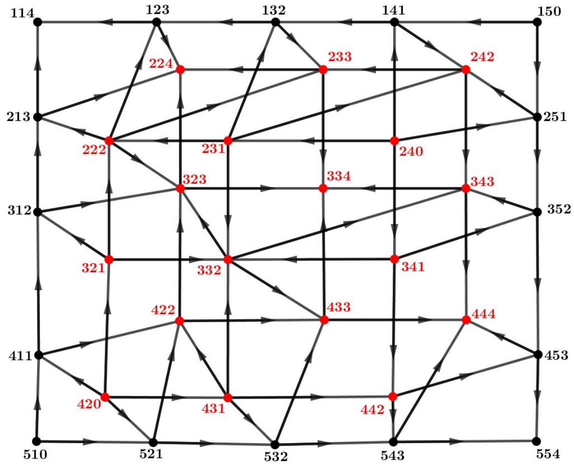
Table 3.5: The partition of $RASM(6, r)$ into 42 subsets according to the value of the central elements.

r	1	2
$\lvert R_1(r)\rvert$	0	144
$\lvert R_2(r)\rvert$	0	168
$\lvert R_3(r)\rvert$	8	19300
$\lvert R_4(r)\rvert$	20	50120
$\lvert R_5(r)\rvert$	14	34338
$\lvert R_6(r)\rvert$	14	34338
$\lvert R_7(r)\rvert$	28	51032
$\lvert R_8(r)\rvert$	0	168
$\lvert R_9(r)\rvert$	0	196
$\lvert R_{10}(r)\rvert$	20	50120
$\lvert R_{11}(r)\rvert$	50	136086
$\lvert R_{12}(r)\rvert$	35	100603
$\lvert R_{13}(r)\rvert$	14	34338
$\lvert R_{14}(r)\rvert$	35	100603
$\lvert R_{15}(r)\rvert$	49	129560
$\lvert R_{16}(r)\rvert$	85	128610
$\lvert R_{17}(r)\rvert$	35	100603
$\lvert R_{18}(r)\rvert$	70	156884
$\lvert R_{19}(r)\rvert$	49	126004
$\lvert R_{20}(r)\rvert$	98	212509
$\lvert R_{21}(r)\rvert$	181	228737
$\lvert R_{22}(r)\rvert$	14	34338
$\lvert R_{23}(r)\rvert$	35	100603
$\lvert R_{24}(r)\rvert$	49	126004
$\lvert R_{25}(r)\rvert$	28	51032
$\lvert R_{26}(r)\rvert$	70	156884
$\lvert R_{27}(r)\rvert$	98	212509
$\lvert R_{28}(r)\rvert$	181	228737
$\lvert R_{29}(r)\rvert$	49	129560
$\lvert R_{30}(r)\rvert$	98	212509
$\lvert R_{31}(r)\rvert$	98	212509
$\lvert R_{32}(r)\rvert$	196	384720
$\lvert R_{33}(r)\rvert$	386	453520
$\lvert R_{34}(r)\rvert$	196	297684
$\lvert R_{35}(r)\rvert$	586	438596
$\lvert R_{36}(r)\rvert$	85	128610
$\lvert R_{37}(r)\rvert$	181	228737
$\lvert R_{38}(r)\rvert$	181	228737
$\lvert R_{39}(r)\rvert$	386	453520
$\lvert R_{40}(r)\rvert$	778	558134
$\lvert R_{41}(r)\rvert$	586	438596
$\lvert R_{42}(r)\rvert$	2350	747457
$\lvert RASM(6,r)\rvert$	7436	7517457

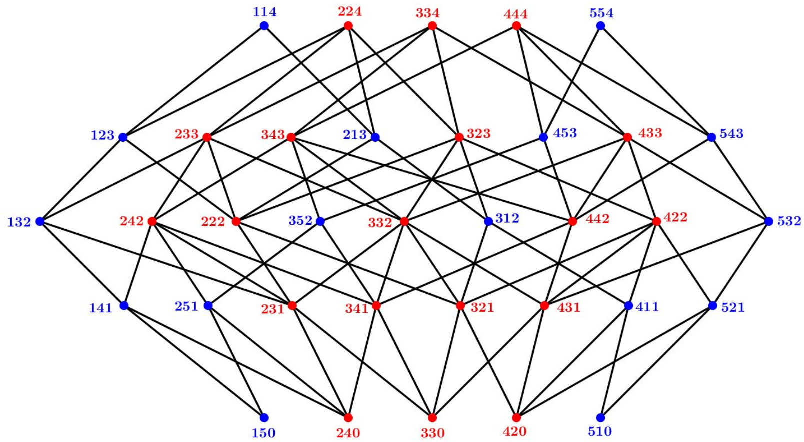
Table 3.6: The cardinality of $RASM(6,r)$ and its 42 subsets for $r \in [2]$.

Now by Theorem 3.24, the cardinality of $ASM(6,r)$ is given by

$$
\begin{aligned}
|ASM(6,r)| = &\sum_{I\in J(P_4)} \Omega_{(P_6(I),\omega_5)}(r+1) \\
=&28005\binom{r+5}{25} + 37487402\binom{r+6}{25} \\
&+ 6969176678\binom{r+7}{25} + 397997364306\binom{r+8}{25} \\
&+ 9598325929253\binom{r+9}{25} + 115380639727880\binom{r+10}{25} \\
&+ 762307595324496\binom{r+11}{25} + 2942082216947224\binom{r+12}{25} \\
&+ 6891053963453162\binom{r+13}{25} + 10016535526744820\binom{r+14}{25} \\
&+ 9125692783636604\binom{r+15}{25} + 5205953846638436\binom{r+16}{25} \\
&+ 1837564143208010\binom{r+17}{25} + 391689902517936\binom{r+18}{25} \\
&+ 48445062844552\binom{r+19}{25} + 3268098840896\binom{r+20}{25} \\
&+ 109295364401\binom{r+21}{25} + 1556670314\binom{r+22}{25} \\
&+ 7324446\binom{r+23}{25} + 7410\binom{r+24}{25} + \binom{r+25}{25}
\end{aligned}
\tag{3.103}
$$

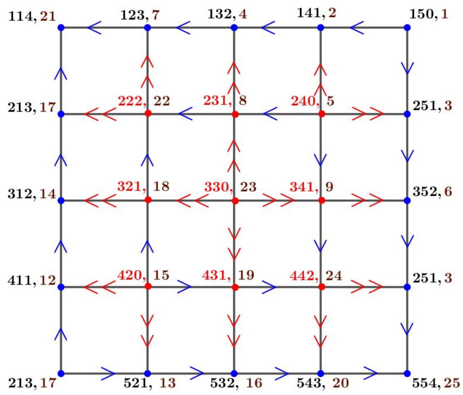


(a) Rectangular representation of P_6.

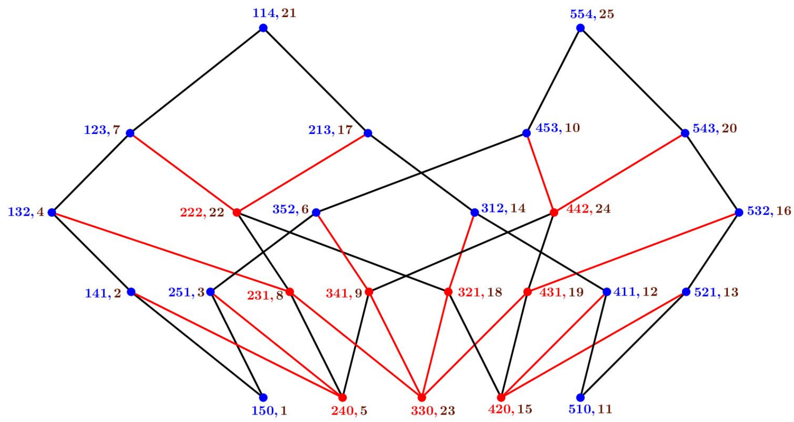


(b) Tetrahedral representation of P_6.

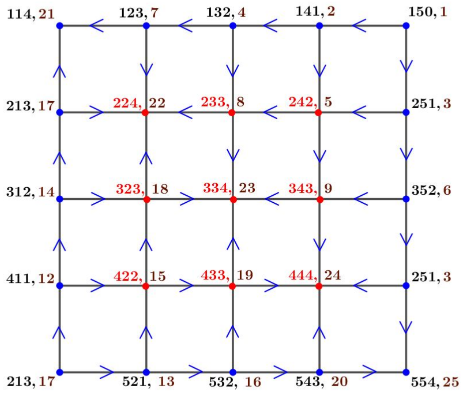
Figure 3.25: Two equivalent representations of the reduced ASM poset P_6. Note that the central elements of P_6 are distinguished by red color.



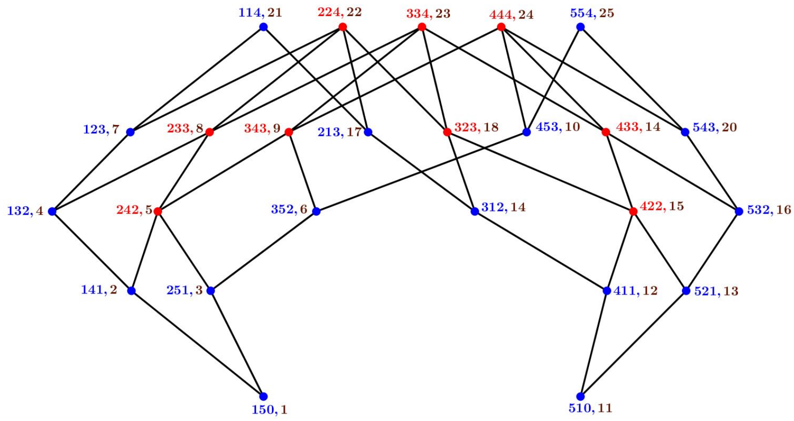
(a) Rectangular representation of $(P_6(I_1), \omega_6)$ where $I_1 = \varnothing$.



(b) Tetrahedral representation of $(P_6(I_1), \omega_6)$ where $I_1 = \varnothing$.



(c) Rectangular representation of $(P_6(I_{42}), \omega_6)$ where $I_{42} = P_4$.



(d) Tetrahedral representation of $(P_6(I_{42}), \omega_6)$ where $I_{42} = P_4$.

Figure 3.26: The rectangular and the tetrahedral representations of the labelled reduced ASM posets $(P_6(I_1), \omega_6)$ and $(P_6(I_{42}), \omega_6)$ associated with order ideals $I_1 = \varnothing$ and $I_{42} = P_4$. The red color vertices in each representation indicate the central elements and the red color edges and double red arrows indicate the strict order relations between the corresponding elements. Moreover, as it can be seen, similarly to the case for P_5, the labelled reduced ASM poset $(P_6(I_{42}), \omega_6)$ has a natural labelling and thus has no strict order relations between its elements.

3.3.4 The enumeration of $ASM(7,r)$

Similar to the case for $n=6$, on the one hand, we have the set $RASM(7,r)$ where each $E \in RASM(7,r)$ consists of 3 shells with a 2×2 matrix of the form $\begin{pmatrix} e_{33} & e_{34} \\ e_{43} & e_{44} \end{pmatrix}$ as its innermost shell. In particular, the set of central positions in E is given by the union of

$$C = \{(2,2),(2,3),(2,4),(2,5),(3,2),(3,5),(4,2),(4,5),(5,2),(5,3),(5,4),(5,5)\} \tag{3.104}$$

and

$$D = \{(3,3),(3,4),(4,3),(4,4)\} \tag{3.105}$$

where C contains the central positions in the second shell and D contains the central positions in the innermost shell. Moreover, if $(i,j) \in C$ then $0 \le e_{ij} \le 2r$ and if $(i,j) \in D$ then $0 \le e_{ij} \le 3r$. Therefore, we can consider the same partitions for the intervals $[0,2r]$ and $[0,3r]$ as the ones given in (3.101). Also according to the value of the entries in the central positions in each $E \in RASM(7,r)$ we can write

$$RASM(7,r) = \bigcup_{i=1}^{429} R_i(r)$$

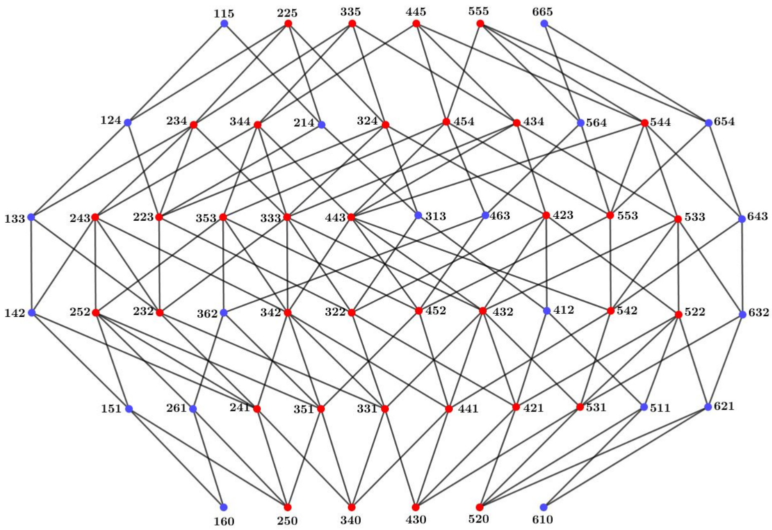
where each $R_i(r)$ is the set of all order 7 RASMs with line sum r such that central entries of its elements satisfy certain conditions.

On the other hand, we have the ASM poset P_7. The tetrahedral representation of the Hasse diagram of P_7 is shown in Figure 3.27a. Now by applying the reduction operation on each order ideal $I \in J(P_5)$, we obtain 429 reduced ASM posets $P_7(I)$ each of which is equipped with the labelling ω_7 (Note that by Theorem 1.87, $|J(P_5)| = |ASM(5)| = 429$). As an illustration, the reduced ASM labelled posets $(P_7(I_1),\omega_7)$ and $(P_7(I_{429}),\omega_6)$ associated with order ideals $I_1 = \varnothing$ and $I_{429} = P_5$ are shown in Figures 3.27b and 3.27c, respectively. Therefore, in total, there are 429 labelled reduced ASM posets $(P_7(I),\omega_7)$ for $I \in J(P_5)$. Thus we need to consider 429 sets of $(P_7(I),\omega_7)$–partitions associated with 429 ASM order ideals $I \in J(P_5)$. In particular, we can write

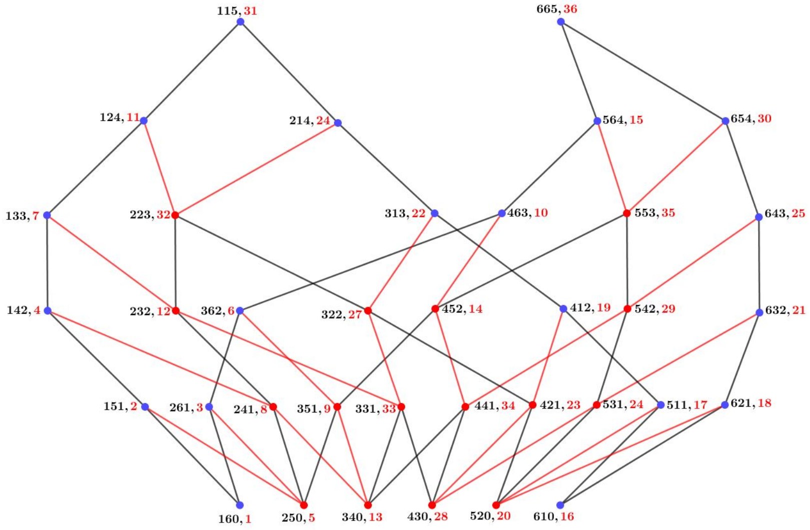
$$\Lambda(7,r) = \bigcup_{I\in J(P_5)} \mathcal{X}(P_7(I),\omega_7,r)$$

Now by Theorem 3.23, the map $\Phi : RASM(7,r) \to \Lambda(7,r)$ is given by $\Phi(E) = \theta_E$ where

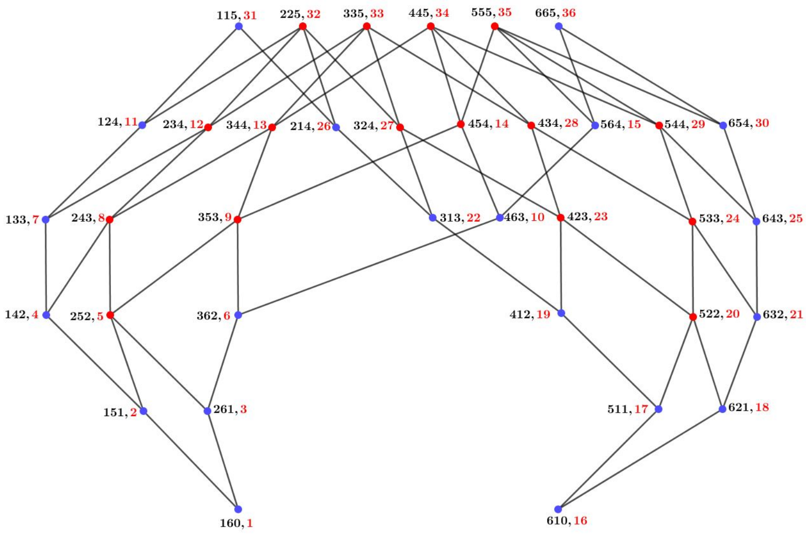
$$\theta_E(i,j,k) = \begin{cases} e_{ij} & \text{if} \quad Min(i,j,7-i,7-j) = 1 & & \\ e_{ij}+1 & \text{if} \quad (i,j) \in C & \text{and} & 0 \le e_{i,j} \le r-1 \\ e_{ij}-r & \text{if} \quad (i,j) \in C & \text{and} & r \le e_{i,j} \le 2r \\ e_{33}+2 & \text{if} \quad (i,j) \in D & \text{and} & 0 \le e_{ij} \le r-2 \\ e_{33}-r+1 & \text{if} \quad (i,j) \in D & \text{and} & r-1 \le e_{ij} \le 2r-1 \\ e_{33}-2r & \text{if} \quad (i,j) \in D & \text{and} & 2r \le e_{ij} \le 3r \end{cases} \tag{3.106}$$



(a) Tetrahedral representation of the ASM poset P_7 .



(b) Tetrahedral representation of $(P_7(I_1), \omega_7)$ where $I_1 = \varnothing$.



(c) Tetrahedral representation of $(P_7(I_{429}), \omega_7)$ where $I_{429} = P_5$.

Figure 3.27: The tetrahedral representations of the ASM poset (P_7 and the labelled reduced ASM posets $(P_7(I_1), \omega_7)$ and $(P_7(I_{429}), \omega_7)$ associated with order ideals $I_1 = \varnothing$ and $I_{429} = P_5$. The red color vertices in each representation indicate the central elements and the red color edges indicate the strict order relations between the corresponding elements. Moreover, as can be seen, similarly to the case for P_6, the labelled reduced ASM poset $(P_7(I_{429}), \omega_7)$ has a natural labelling and thus has no strict order relations between its elements.

where the sets C and D are given in (3.104) and (3.105). Now by Theorem 3.24, we have

$$\begin{aligned} |ASM(7,r)| &= \sum_{I \in J(P_5)} \Omega_{(P_7(I),\omega_7)}(r+1) \\ &= \sum_{I \in J(P_5)} \sum_{\pi \in \mathcal{L}(P_7(I),\omega_7)} \binom{r+36-des(\pi)}{36} \end{aligned} \tag{3.107}$$

For brevity we just give the general formula for the cardinality of $ASM(7,r)$. The sum of the coefficients of the binomial terms in (3.107) is the volume of the ASM polytope $\mathcal{A}_7$ which we computed by Mathematica and is given by

$$vol(\mathcal{A}_7) = 2042396760356116914108917 1040$$

which was given earlier in Table 2.1.

CHAPTER 4

CONCLUSION

4.1 Conclusions

This section is a summary of the work presented within this thesis.

Chapter 1 began with a historical review of the birth and development of ASMs. It continued with an overview of the bijections between ASMs and other objects including monotone triangles, configurations of six vertex models with domain wall boundary conditions, configurations of simple flow grids, corner sum matrices and height function matrices. The last sections of this chapter were dedicated to the partially ordered set point of view of ASMs.

In Chapter 2, after some preliminaries on polytopes, we provided an overview of results on the Birkhoff polytope, Chan-Robbins-Yuen polytope, ASM polytope, ASM order polytope and ASM Chan-Robbins-Yuen polytope.

Chapter 3 was dedicated to the enumeration of higher spin ASMs by relating these numbers to numbers of certain (P, ω)−partitions. To do this, we introduced a reduced ASM poset by applying a reduction operation to the ASM poset P_n, for a given order ideal $I \in J(P_{n-2})$. This enabled us to subdivide P_n into $|ASM(n-2)|$ different subposets each of which was equipped with a labelling ω_n. Then we considered the (P_n, ω_n)−partitions for each of these reduced ASM labelled posets. This, together with some established theorems from the theory of (P, ω)−partitions, enabled us to give a formula, in Theorem 3.24, for the number of higher spin ASMs, or equivalently for the Ehrhart polynomial of the ASM polytope. This chapter ended with explicitly investigating the enumerations of $ASM(4, r)$, $ASM(5, r)$ and $ASM(6, r)$.

4.2 The investigation of the ASM polytope: The continuous case

In Section 2.4 we saw that

$$ASM(n,r) = r\mathcal{A}_n \cap \mathbb{Z}^{n^2} \tag{4.1}$$

and in Theorem 3.24 we gave a formula to enumerate the cardinality of $ASM(n,r)$. We can think of this as the discrete enumeration of the r^{th} dilate of the ASM polytope $\mathcal{A}_n$. The question is: Can one apply the same technique to compute the volume of $\mathcal{A}_n$ directly (the continuous case)? In what follows, we highlight some differences between the computations in $ASM(n,r)$ and $\mathcal{A}_n$.

For the continuous case, on the one hand, we can define reduced forms of elements of $\mathcal{A}_n$, analogously to the definition of reduced forms of elements of $ASM(n,r)$.

On the other hand, we have $|ASM(n-2)|$ reduced ASM order polytopes $\mathcal{O}(P_n(I))$ associated with the reduced ASM poset $P_n(I)$ for each order ideal $I \in J(P_{n-2})$. Now consider the set $\Gamma(n)$ defined by

$$\Gamma(n) := \bigcup_{I \in J(P_{n-2})} \mathcal{O}(P_n(I)) \tag{4.2}$$

Then, similarly to the discrete case, the aim is to define a bijective map $\Phi : \mathcal{A}_n \to \Gamma(n)$. Although we have worked on it and have made some observations, this still needs further investigation.

4.3 The symmetry of the square and the enumeration of the linear extensions of P_n

By Theorem 2.18, part (V), the volume of the order polytope of a finite poset P is $e(P)$, where $e(P)$ is the number of linear extensions of P. There are several ways to compute $e(P)$. In particular, it can be shown that $e(P)$ is the number of maximal chains in the lattice of order ideals $J(P)$. Now consider the ASM poset P_n and its associated lattice of order ideals $J(P_n)$. In general, obtaining $e(P)$ can be a difficult computation. So another interesting open problem is to use the symmetry classes of the dihedral group D_8 (see Section 1.5) on $J(P_n)$ to reduce the complexity of the computation of the maximal chains in $J(P_n)$. In particular, it is expected that many of the maximal chains are related by symmetries. As an illustration, a highlighted path in red color in the Hasse diagram of $J(P_4)$ as shown in Figure 4.1 is a maximal chain. For $n = 4$, there are 1792 such maximal chains, which is hence the volume of $\mathcal{O}(P_4)$. Therefore, reducing the complexity of computing $e(P_n)$ would lead to reducing the complexity of computing the volume of the ASM order polytope $\mathcal{O}(P_n)$ associated to the ASM poset P_n.

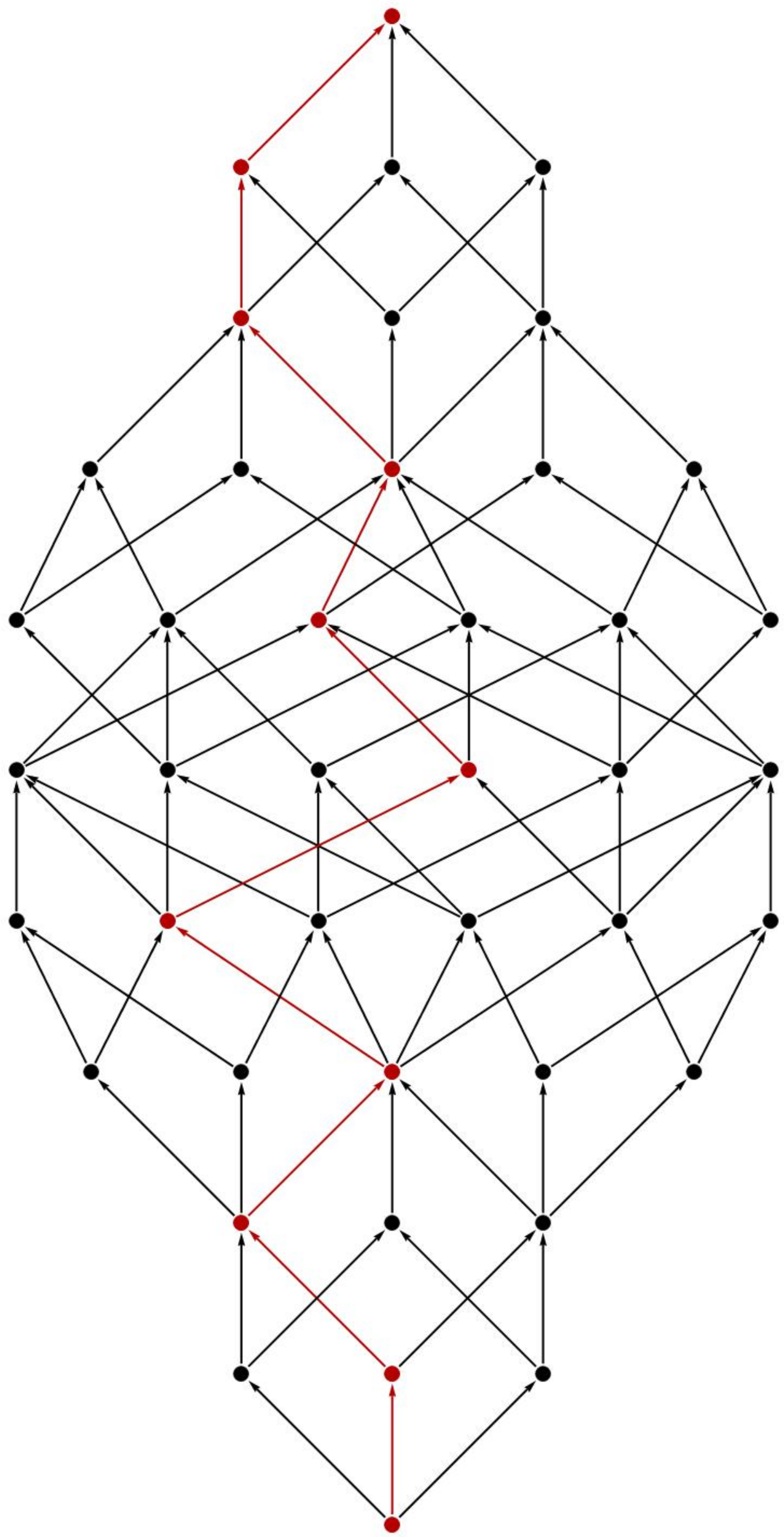


Figure 4.1: A maximal chain in the Hasse diagram of the lattice $J(P_4)$.

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