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Citation for final published version:

Anno, Rina and Logvinenko, Timothy 2021. Bar category of modules and homotopy adjunction for tensor functors. *International Mathematics Research Notices* 2021 (2) , pp. 1353-1462. 10.1093/imrn/rnaa066

Publishers page: <http://dx.doi.org/10.1093/imrn/rnaa066>

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BAR CATEGORY OF MODULES AND HOMOTOPY ADJUNCTION FOR TENSOR FUNCTORS

RINA ANNO AND TIMOTHY LOGVINENKO

ABSTRACT. Given a DG-category \mathcal{A} we introduce the *bar category of modules* $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$. It is a DG-enhancement of the derived category $D(\mathcal{A})$ of \mathcal{A} which is isomorphic to the category of DG \mathcal{A} -modules with A_∞ -morphisms between them. However, it is defined intrinsically in the language of DG-categories and requires no complex machinery or sign conventions of A_∞ -categories. We define for these bar categories Tensor and Hom bifunctors, dualisation functors, and a convolution of twisted complexes. The intended application is to working with DG-bimodules as enhancements of exact functors between triangulated categories. As a demonstration we develop a homotopy adjunction theory for tensor functors between derived categories of DG-categories. It allows us to show in an enhanced setting that given a functor F with left and right adjoints L and R the functorial complex $FR \xrightarrow{F \text{ act } R} FRFR \xrightarrow{FR \text{ tr} - \text{tr } FR} FR \xrightarrow{\text{tr}} \text{Id}$ lifts to a canonical twisted complex whose convolution is the square of the spherical twist of F . We then write down four induced functorial Postnikov systems computing this convolution.

1. INTRODUCTION

DG-enhancements of triangulated categories were introduced by Bondal and Kapranov in [BK90] to overcome the axiomatic imperfections of the latter [Ver96]. A DG-enhancement of a triangulated category \mathcal{T} is a pretriangulated differentially graded (DG) category \mathcal{A} with $H^0(\mathcal{A}) \simeq \mathcal{T}$. Working formally in \mathcal{A} and truncating down to \mathcal{T} fixes a number of issues. Many constructions of this kind are independent of the choices of lifts to \mathcal{A} and even \mathcal{A} itself. See [Kel94], [Toë11], [AL17, §2] for an introduction to DG-categories, [BK90], [LO10], [AL17, §4] for an introduction to DG-enhancements, and [Toë07] for some key technical results.

Most triangulated categories which arise in algebra and geometry are derived categories. These are $H^0(-)$ truncations of DG-categories of complexes of objects in an abelian category, and thus possess a natural DG-enhancement. Examples include the derived categories of sheaves or constructible sheaves on a topological space or of quasi-coherent sheaves, coherent sheaves, or D -modules on an algebraic variety. Moreover, in a number of these examples (e.g. the derived category of any Grothendieck category) this natural enhancement is unique up to quasi-equivalence [LO10], [CS15]. Thus functorial constructions carried out in a DG-enhancement will produce the same results in the triangulated category regardless of the choice of the enhancement. For technical reasons, it is best to work in a Morita framework: instead of enhancing \mathcal{T} with some DG-category \mathcal{A} we enhance \mathcal{T} with the DG-category $\mathbf{Mod}\text{-}\mathcal{A}$ of modules over \mathcal{A} .

Let \mathcal{T} and \mathcal{U} be triangulated categories. The exact functors $\mathcal{T} \rightarrow \mathcal{U}$ do not form a triangulated category. However, if we Morita enhance \mathcal{T} and \mathcal{U} by DG-categories \mathcal{A} and \mathcal{B} , by a fundamental result of Toën [Toë07] the enhanceable functors form a triangulated category equivalent to the derived category of \mathcal{B} -perfect \mathcal{A} - \mathcal{B} -bimodules. In algebraic geometry, let $\mathcal{T} = D(X)$ and $\mathcal{U} = D(Y)$ be the derived categories of coherent sheaves of separated schemes X and Y of finite type over a field. The DG-category of perfect \mathcal{A} - \mathcal{B} -bimodules DG-enhances $D(X \times Y)$ and the enhanceable functors are the Fourier-Mukai transforms [Toë07], [LS16]. Thus studying \mathcal{A} - \mathcal{B} -bimodules as DG-enhancements of exact functors $D(\mathcal{A}) \rightarrow D(\mathcal{B})$ can be viewed as the universal Fourier-Mukai theory for enhanced triangulated categories.

In their work on spherical functors [AL17] and \mathbb{P}^n -functors [AL19] the authors used the above approach for various technical constructions such as taking cones of natural transformations and, more generally, convolutions of complexes of functors. Though working in quasi-equivalent enhancements produces the same results, we discovered that for carrying out explicit computations choosing a suitable enhancement makes a world of difference. In this paper we define and study the DG-enhancement framework for the derived categories of DG-modules and bimodules which we found most suitable. As a demonstration, we construct an explicit homotopy adjunction theory and thus explicit 2-categorical adjunctions for DG-bimodules.

Let \mathcal{A} be a DG-category and let $\mathbf{Mod}\text{-}\mathcal{A}$ be the DG-category of right \mathcal{A} -modules. Two enhancements commonly used in the literature for $D(\mathcal{A})$ are the subcategory $\mathcal{P}(\mathcal{A})$ of the h -projective modules in $\mathbf{Mod}\text{-}\mathcal{A}$, and the Drinfeld quotient $\mathbf{Mod}\text{-}\mathcal{A}/\mathcal{Ac}(\mathcal{A})$ by the subcategory $\mathcal{Ac}(\mathcal{A})$ of acyclic modules [Dri04]. Neither turned out to be suitable for our purposes. The problem with the Drinfeld quotient is that its morphisms

are inconvenient to work with explicitly. The problem with $\mathcal{P}(\mathcal{A})$ is that when working with bimodules the diagonal bimodule \mathcal{A} , which corresponds to the identity functor $D(\mathcal{A}) \rightarrow D(\mathcal{A})$, is not h -projective. Hence every construction involving the identity functor has to be h -projectively resolved leading to many formulas becoming vastly more complicated than they should be, as seen in [AL17].

This can be fixed by working with DG \mathcal{A} -modules and A_∞ -morphisms between them. In other words, the full subcategory $(\mathbf{Nod}_\infty \mathcal{A})_{dg}$ of the DG-category $\mathbf{Nod}_\infty \mathcal{A}$ of A_∞ \mathcal{A} -modules which consists of DG-modules. In $(\mathbf{Nod}_\infty \mathcal{A})_{dg}$ all quasi-isomorphisms are already homotopy equivalences, there is no need to take resolutions and $D(\mathcal{A}) \simeq H^0((\mathbf{Nod}_\infty \mathcal{A})_{dg})$. However, the machinery of A_∞ -categories lends itself poorly to explicit computations due to e.g. the complicated sign conventions involved. It seems wasteful to use the full generality of the A_∞ -language to only consider A_∞ -morphisms between DG-modules over a DG-category. To this end we introduce the *bar category of modules* over \mathcal{A} , a technical tool which simplifies the A_∞ -machinery involved to the extent actually necessary. It is a category *isomorphic* to $(\mathbf{Nod}_\infty \mathcal{A})_{dg}$, yet it has an intrinsic definition entirely in terms of DG-modules and the bar complex $\bar{\mathcal{A}}$. This builds on the ideas of [Kel94, §6.6]. Keller works with a set of compact generators to obtain a Morita enhancement of $D(\mathcal{A})$. We work with all the \mathcal{A} -modules to obtain a usual enhancement of $D(\mathcal{A})$ and we establish the isomorphism to $(\mathbf{Nod}_\infty \mathcal{A})_{dg}$:

Definition (Definition 3.2). Let \mathcal{A} be a DG-category. Define the *bar category of modules* $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ as follows:

- The object set of $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ is the same as that of $\mathbf{Mod}\text{-}\mathcal{A}$: DG-modules over \mathcal{A} .
- For any $E, F \in \mathbf{Mod}\text{-}\mathcal{A}$ set

$$\mathrm{Hom}_{\overline{\mathbf{Mod}}\text{-}\mathcal{A}}(E, F) = \mathrm{Hom}_{\mathcal{A}}(E \otimes_{\mathcal{A}} \bar{\mathcal{A}}, F)$$

and write $\overline{\mathrm{Hom}}_{\mathcal{A}}(E, F)$ to denote this Hom-complex.

- For any $E \in \mathbf{Mod}\text{-}\mathcal{A}$ set $\mathrm{Id}_E \in \overline{\mathrm{Hom}}_{\mathcal{A}}(E, E)$ to be the element given by

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\mathrm{Id} \otimes \tau} E \otimes_{\mathcal{A}} \mathcal{A} \xrightarrow{\sim} E$$

where $\tau: \bar{\mathcal{A}} \rightarrow \mathcal{A}$ is the canonical projection.

- For any $E, F, G \in \mathbf{Mod}\text{-}\mathcal{A}$ define the composition map

$$\overline{\mathrm{Hom}}_{\mathcal{A}}(F, G) \otimes_k \overline{\mathrm{Hom}}_{\mathcal{A}}(E, F) \longrightarrow \overline{\mathrm{Hom}}_{\mathcal{A}}(E, G)$$

by setting for any $\alpha: E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \rightarrow F$ and $\beta: F \otimes_{\mathcal{A}} \bar{\mathcal{A}} \rightarrow G$ their composition to be

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\mathrm{Id} \otimes \Delta} E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\alpha \otimes \mathrm{Id}} F \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\beta} G$$

where $\Delta: \bar{\mathcal{A}} \rightarrow \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}}$ is the canonical comultiplication.

In Prop. 3.5 we show that $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ is isomorphic to $(\mathbf{Nod}_\infty \mathcal{A})_{dg}$, thus it is also a DG-enhancement of $D(\mathcal{A})$. Let \mathcal{B} be another DG category. We similarly define $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, the bar category of DG-bimodules. We write down bifunctors $\bar{\otimes}$ and $\overline{\mathrm{Hom}}$ for DG-bimodules which correspond to their A_∞ -counterparts. In Prop. 3.14 we prove the Tensor-Hom adjunction for $\bar{\otimes}$ and $\overline{\mathrm{Hom}}$ and give formulas for its adjunction units and counits. Next is the dualisation theory: we define the functors $(-)^{\bar{\mathcal{A}}}$ and $(-)^{\bar{\mathcal{B}}}$ which are equivalences on the subcategories of \mathcal{A} - and \mathcal{B} -perfect bimodules, respectively. We show in Lemma 3.34 that for any $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ the bimodules $M^{\bar{\mathcal{A}}}$ and $M^{\bar{\mathcal{B}}}$ enhance the derived functors $\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M, -)$ and $\mathbf{R}\mathrm{Hom}_{\mathcal{B}}(M, -)$ whenever M is \mathcal{A} - and \mathcal{B} -perfect, respectively. This is crucial for the homotopy adjunction theory we develop later.

The constructions such as cones of natural transformations or convolutions of complexes of functors are usually done via twisted complexes [BK90], [AL17, §3]. Ideally, this needs the DG-enhancement to be strongly pretriangulated, which the bar category $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ is not. However, in Defn. 3.40 we define the convolution functor which is a quasi-equivalence and a homotopy inverse to the inclusion $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \hookrightarrow \mathrm{Pre}\text{-}\mathrm{Tr}(\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B})$. Together with the formulas for Tensor, Hom and dualisation of twisted complexes of bimodules given in Lemmas 3.43 and 3.44 it enables the constructions we need.

Next we examine the biggest drawback of bar categories: the natural map $\mathcal{A} \bar{\otimes}_{\mathcal{A}} M \xrightarrow{\alpha} M$, analogous to the \mathcal{A} -action isomorphism $\mathcal{A} \otimes_{\mathcal{A}} M \simeq M$, is not an isomorphism, but only a homotopy equivalence. We study the higher homotopies involved. It is well known that any homotopy equivalence in a DG-category can be completed to a certain universal system (3.35) of morphisms and relations [Dri04, §3.7][Tab05] [AL17, App]. We do better and write down a homotopy inverse $\beta_0: M \rightarrow \mathcal{A} \bar{\otimes}_{\mathcal{A}} M$ and a degree -1 map θ for which:

Proposition (Prop. 3.22). *Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$. The sub-DG-category of $\mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$ generated by α, β_0 and θ is the free DG-category generated by these modulo the following relations:*

$$\begin{aligned} d\alpha &= d\beta_0 = 0, \\ d\theta &= \text{Id} - \beta_0 \circ \alpha, \\ 0 &= \alpha \circ \beta_0 - \text{Id}, \\ \alpha \circ \theta &= 0. \end{aligned} \quad \begin{array}{c} \begin{array}{ccc} & \xrightarrow{\beta_0} & \\ M & \xrightarrow{\quad} & \mathcal{A} \overline{\otimes}_{\mathcal{A}} M \\ & \xleftarrow{\alpha} & \end{array} \\ \begin{array}{c} \text{---} \circlearrowleft \text{---} \\ \theta \end{array} \end{array} \quad (1.1)$$

The relations in (1.1) can be obtained from those in the universal system (3.35) by setting $\theta_x = 0, \alpha \circ \theta_y = 0$, and $\phi = -\theta_y^2 \circ \beta$ in the notation thereof. In Prop. 3.25 we prove analogous results for the adjoint homotopy equivalence $\gamma: M \rightarrow \overline{\text{Hom}}_{\mathcal{A}}(\mathcal{A}, M)$.

In the latter half of the paper we use the bar categories to give a homotopy adjunction theory for DG-bimodules. Our first main result is the following straightforward, but very useful fact:

Theorem 1.1 (cf. Theorem 4.1). *Let \mathcal{A} and \mathcal{B} be DG-categories and let $f: D(\mathcal{A}) \rightarrow D(\mathcal{B})$ be a tensor functor. Let $M \in \mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$ be any enhancement of f .*

- (1) *The following are equivalent:*
 - (a) *The right adjoint r of f is continuous.*
 - (b) *f restricts to $D_c(\mathcal{A}) \rightarrow D_c(\mathcal{B})$.*
 - (c) *M is \mathcal{B} -perfect.*
 - (d) *$M^{\overline{\mathcal{B}}}$ enhances the right adjoint r of f .*
- (2) *The following are equivalent:*
 - (a) *The left adjoint l of f exists.*
 - (b) *The left adjoint l of f exists and restricts to $D_c(\mathcal{B}) \rightarrow D_c(\mathcal{A})$.*
 - (c) *M is \mathcal{A} -perfect.*
 - (d) *$M^{\overline{\mathcal{A}}}$ enhances the left adjoint l of f .*

In the 2-categorical language of [Bén67], let \mathcal{DGMod} be the bicategory whose objects are DG-categories, whose 1-morphism categories are the derived categories of DG-bimodules, whose composition is given by the derived tensor product, and whose identity object is the diagonal bimodule. Let $M \in \mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$ be \mathcal{A} - and \mathcal{B} -perfect. Write F, L , and R for $M, M^{\overline{\mathcal{A}}}$, and $M^{\overline{\mathcal{B}}}$ considered as 1-morphisms in \mathcal{DGMod} . In Defns. 4.2-4.4 we write down the homotopy adjunction units and counits for (L, F, R)

$$\text{Id}_{\mathcal{A}} \xrightarrow{\text{act}} RF, \quad \text{Id}_{\mathcal{B}} \xrightarrow{\text{act}} FL, \quad FR \xrightarrow{\text{tr}} \text{Id}_{\mathcal{B}}, \quad LF \xrightarrow{\text{tr}} \text{Id}_{\mathcal{A}},$$

and in Proposition 4.6 we show that the following identities hold up to homotopy:

$$F \xrightarrow{F \text{ act}} FRF \xrightarrow{FR \text{ tr}} F = \text{Id}_F \quad \text{and} \quad R \xrightarrow{R \text{ act}} RFR \xrightarrow{RF \text{ tr}} R = \text{Id}_R, \quad (1.2)$$

$$F \xrightarrow{\text{act} F} FLF \xrightarrow{F \text{ tr}} F = \text{Id}_F \quad \text{and} \quad L \xrightarrow{L \text{ act}} LFL \xrightarrow{\text{tr} L} L = \text{Id}_L. \quad (1.3)$$

This implies 2-categorical adjunctions between L, F , and R in \mathcal{DGMod} . Such 2-categorical adjunctions, when they exist, are strongly unique: L and R are determined by F up to the unique isomorphism.

An exact functor $f: D(\mathcal{A}) \rightarrow D(\mathcal{B})$ is a *tensor functor* if it is isomorphic to tensor multiplication by some $M \in \mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$. In Theorem 4.1 we show f has left and right adjoints l and r which are also tensor functors if and only if M is \mathcal{B} - and \mathcal{A} -perfect. Our homotopy adjunction theory then implies that if we fix an enhancement of such f the enhancements of l and r which satisfy (1.2) and (1.3) exist and are unique.

Next we study the homotopies up to which (1.2) and (1.3) hold, and the relations between these homotopies. These can be packaged up as several canonical twisted complexes associated to a homotopy adjunction. We write down explicit degree -1 maps $\xi_{\mathcal{B}}, \xi'_{\mathcal{B}}, \nu_{\mathcal{B}}$, and a degree -2 map $\nu_{\mathcal{B}}$ such that:

Theorem 1.2 (cf. Theorems 4.2 and 4.3). *The following are twisted complexes over $\mathcal{B}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$ and $\mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{A}$:*

$$\begin{array}{ccccc} FR & \xrightarrow{\text{Fact}R} & FRFR & \xrightarrow{FR \text{ tr} - \text{tr} FR} & FR & \xrightarrow{\text{tr}} & \text{Id}_{\mathcal{B}} \end{array} \quad (1.4)$$

$$\begin{array}{ccccc} \text{Id}_{\mathcal{A}} & \xrightarrow{\text{act}} & RF & \xrightarrow{\text{act}RF - R\text{Fact}} & RFRF & \xrightarrow{R\text{tr}F} & RF \end{array} \quad (1.5)$$

Their convolutions are homotopic to T^2 and C^2 , the squares of the spherical twist and co-twist of F .

We also give analogous twisted complexes for FL and LF . The spherical twist and cotwist of a functor are well-studied notions with numerous applications. If F is fully faithful, then $C = 0$ and T is the mutation functor which kills \mathcal{A} and maps ${}^\perp\mathcal{A}$ equivalently to \mathcal{A}^\perp [Bon90]. If F is spherical [AL17], then C and T are autoequivalences. Moreover, as spherical functors are \mathbb{P}^1 -functors [AL19, Prop. 7.1], the theorem above shows that the \mathbb{P} -twist of F is isomorphic to T^2 , generalising similar statements made for split \mathbb{P}^1 -functors in [Add16, Ex. 4.2(3)] and \mathbb{P}^1 -objects in [HT06, Prop. 2.9].

Finally, in §4.4 we interpret the data of (1.4) in terms of the derived category $D(\mathcal{B}\text{-}\mathcal{B})$ of $\mathcal{B}\text{-}\mathcal{B}$ -bimodules. Any twisted complex in the DG-enhancement defines several *Postnikov systems* in the triangulated category which compute its convolution and the convolutions of its subcomplexes, cf. §2.4. The data of (1.4) defines four Postnikov systems in $D(\mathcal{B}\text{-}\mathcal{B})$. These turn out to have an intrinsic description we give in Theorem 4.4 which implies a number of explicit relations between various natural morphisms involving Id , FR , $FRFR$, T , FRT , T^2 and extensions thereof which are highly useful in computations of spherical and \mathbb{P} -twists.

1.1. The layout of the paper. In §2 we give prerequisites on DG-categories and A_∞ -categories, and on their modules and bimodules. In §3 we introduce the bar categories of modules and bimodules. Then in §4 we construct homotopy adjunctions for DG-bimodules and their associated twisted complexes.

1.2. Acknowledgements. The authors would like to thank Alexander Efimov and Sergey Arkhipov for helpful discussions. They would also like to thank Olaf Schnürer for many useful comments which led to significant improvements in the manuscript. The first author would like to thank Kansas State University for a stimulating research environment, as well as for inviting the second author to visit. The second author would like to offer similar thanks to Cardiff University. Both authors would like to thank Banff International Research Station for Mathematical Innovation and Discovery (BIRS) for extending its hospitality to them during the “Homological Mirror Geometry” workshop in March 2016.

2. PRELIMINARIES

Let k be any field. Throughout this paper, we work over k as the base field and all the categories we consider are k -linear. Some of our results, e.g. the definition of the bar-category of modules and its isomorphism to the category of DG-modules with A_∞ -morphisms between them, work just as well when k is only a commutative ring. However, crucially, Lemma 2.16 stops working when k is not a field: DG-tensoring over k with the diagonal bimodule no longer necessarily produces a semi-free module. Consequently, A_∞ -tensoring with the diagonal bimodule is no longer a functorial semi-free resolution as described in §2.10 and A_∞ -quasi-isomorphisms are not necessarily all homotopy equivalences. Thus when k is not a field the bar-category of modules $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ of a DG-category \mathcal{A} , while still being a well-defined pretriangulated DG category, isn’t necessarily a DG-enhancement of the derived category of \mathcal{A} .

We use the following notation for the derived categories we work with. For a DG-category \mathcal{A} we denote by $D(\mathcal{A})$ the derived category of right DG \mathcal{A} -modules, cf. §2.1. For an A_∞ -category \mathcal{A} we denote by $D(\mathcal{A})$ the derived category of right A_∞ \mathcal{A} -modules, cf. §2.8. In case of a DG-category \mathcal{A} we further denote by $D_\infty(\mathcal{A})$ the derived category of right A_∞ \mathcal{A} -modules as per §2.8. Similarly, given two DG or A_∞ -categories \mathcal{A} and \mathcal{B} we denote by $D(\mathcal{A}\text{-}\mathcal{B})$ the derived category of the corresponding $\mathcal{A}\text{-}\mathcal{B}$ -bimodules, etc. For all these triangulated categories, we denote by $D_c(\bullet)$ their full subcategories consisting of compact objects. For a scheme X over k we denote by $D_{qc}(X)$ the derived category of complexes of \mathcal{O}_X -modules with quasi-coherent cohomologies and by $D(X)$ the derived category of complexes with coherent and bounded cohomologies.

We also need to introduce a notation for maps between direct sums of modules. For any two direct sums $E = \bigoplus_{i=1}^n E_i$ and $F = \bigoplus_{j=1}^m F_j$ of objects in an additive category we denote any map

$$\alpha: E \rightarrow F$$

between them by the $m \times n$ -matrix (α_{ij}) where each α_{ij} is the restriction of α to a map $E_j \rightarrow F_i$.

2.1. DG-categories, modules, and bimodules. For a brief introduction to DG-categories, DG-modules, and the technical notions involved we direct the reader to [AL17], §2-4. The present paper was written with that survey in mind. We employ freely any notion or piece of notation introduced in [AL17], §2-4. However, for the convenience of the reader, below we briefly summarise some of the most relevant facts and notation.

Let \mathcal{A} be a DG-category. A (right) \mathcal{A} -module is a DG-functor from \mathcal{A}^{opp} to $\mathbf{Mod}\text{-}k$, the DG-category of DG k -modules. Its underlying graded module is its composition with the forgetful functor to the graded category of graded k -modules. Given an \mathcal{A} -module E for each $a \in \mathcal{A}$ we write E_a for the complex $E(a) \in \mathbf{Mod}\text{-}k$. An \mathcal{A} -module E is *acyclic* if it is acyclic levelwise in $\mathbf{Mod}\text{-}k$, i.e. for any $a \in \mathcal{A}$ the complex E_a is acyclic.

Let E and F be \mathcal{A} -modules. Define $\text{Hom}_{\mathcal{A}}(E, F)$ to be the DG complex of k -modules whose i -th graded part consists of all degree i natural transformations of the graded modules underlying E and F and whose

differential is defined levelwise in $\mathbf{Mod}\text{-}k$. The DG category $\mathbf{Mod}\text{-}\mathcal{A}$ has \mathcal{A} -modules as objects, morphism complexes $\mathrm{Hom}_{\mathcal{A}}(E, F)$, and the composition defined levelwise in $\mathbf{Mod}\text{-}k$. An \mathcal{A} -module E is *h-projective* if the complex $\mathrm{Hom}_{\mathcal{A}}(E, A)$ is acyclic for any acyclic A . Write $\mathcal{Ac}(\mathcal{A})$ and $\mathcal{P}(\mathcal{A})$ for the full subcategories of $\mathbf{Mod}\text{-}\mathcal{A}$ consisting of acyclic and *h*-projective modules, respectively.

A morphism in $\mathbf{Mod}\text{-}\mathcal{A}$ is a *quasi-isomorphism* if it is a quasi-isomorphism levelwise in $\mathbf{Mod}\text{-}k$. The *derived category* $D(\mathcal{A})$ of \mathcal{A} is the localisation of the homotopy category $H^0(\mathbf{Mod}\text{-}\mathcal{A})$ by quasi-isomorphisms. It is constructed as the Verdier quotient of $H^0(\mathbf{Mod}\text{-}\mathcal{A})$ by $H^0(\mathcal{Ac}(\mathcal{A}))$. It comes, in particular, with the canonical projection $H^0(\mathbf{Mod}\text{-}\mathcal{A}) \rightarrow D(\mathcal{A})$ whose composition with the inclusion $H^0(\mathcal{P}(\mathcal{A})) \hookrightarrow H^0(\mathbf{Mod}\text{-}\mathcal{A})$ is an equivalence of categories. An \mathcal{A} -module E is *perfect* if its image in $D(\mathcal{A})$ is a compact object, i.e. the functor $\mathrm{Hom}_{D(\mathcal{A})}(E, -)$ commutes with infinite direct sums.

Let \mathcal{B} be another DG category. An $\mathcal{A}\text{-}\mathcal{B}$ -bimodule is an $\mathcal{A}^{\mathrm{opp}} \otimes_k \mathcal{B}$ -module. We write $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ for the DG category of $\mathcal{A}\text{-}\mathcal{B}$ -bimodules. For any $\mathcal{A}\text{-}\mathcal{B}$ -bimodule M and any $a \in \mathcal{A}$ and $b \in \mathcal{B}$ write ${}_a M_b$, ${}_a M$, and M_b for the complex $M(a, b) \in \mathbf{Mod}\text{-}k$, the \mathcal{B} -module $M(a, -)$, and the $\mathcal{A}^{\mathrm{opp}}$ -module $M(-, b)$, respectively. The bimodule M is \mathcal{A} -perfect if it is perfect levelwise in $\mathbf{Mod}\text{-}\mathcal{A}$, i.e. for any $b \in \mathcal{B}$ the \mathcal{A} -module M_b is perfect. We similarly define \mathcal{B} -perfection and \mathcal{A} - and \mathcal{B} -versions of acyclicity, *h*-projectivity, etc.

The *diagonal bimodule* $\mathcal{A} \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$ is defined by

$${}_a \mathcal{A}_b = \mathrm{Hom}_{\mathcal{A}}(b, a)$$

with the left and right \mathcal{A} -action given by the post- and pre-composition in \mathcal{A} , respectively. The composition in \mathcal{A} defines a canonical map

$$\mathcal{A} \otimes_k \mathcal{A} \mapsto \mathcal{A} \tag{2.1}$$

where \otimes_k on the LHS denotes the tensor product of \mathcal{A} as an $\mathcal{A}\text{-}k$ -bimodule with \mathcal{A} as a $k\text{-}\mathcal{A}$ -bimodule.

For any $\mathcal{A}\text{-}\mathcal{B}$ -bimodule M we have canonical maps

$$\mathcal{A} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{B}}(M, M) \tag{2.2}$$

$$\mathcal{B} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{A}}(M, M) \tag{2.3}$$

in $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$ and $\mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$, respectively. These are called \mathcal{A} - and \mathcal{B} -action maps, respectively, because they represent the action of \mathcal{A} (resp. \mathcal{B}) on M by \mathcal{B} -module (resp. \mathcal{A} -module) morphisms. Note that e.g. the bimodule $\mathrm{Hom}_{\mathcal{B}}(M, M)$ has an \mathcal{A} -algebra structure defined by the composition. It therefore defines a DG-category with the same set of objects as \mathcal{A} . This DG-category is precisely the image of the functor $\mathcal{A} \rightarrow \mathbf{Mod}\text{-}\mathcal{B}$ which corresponds to M , cf. [AL17, §2.1.5].

For any $\mathcal{A}\text{-}\mathcal{B}$ -bimodule M the *shift of M by $n \in \mathbb{Z}$ to the left* is the $\mathcal{A}\text{-}\mathcal{B}$ -bimodule $M[n]$ defined by

$$({}_a (M[n]))_b = ({}_a M_b)_{i+n}$$

where \mathcal{B} acts naturally, \mathcal{A} acts with a sign twist $a \cdot M[n]m = (-1)^{n \deg(a)} a \cdot Mm$, and the differential is $(-1)^n d_M$.

Let \mathcal{C} and \mathcal{D} be DG-categories. For any $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$, $L \in \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$, and $N \in \mathcal{D}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ we have the *composition map* in $\mathcal{D}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$

$$\mathrm{Hom}_{\mathcal{B}}(M, N) \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(L, M) \xrightarrow{\mathrm{cmpr}} \mathrm{Hom}_{\mathcal{B}}(L, N)$$

which is defined levelwise by composition in $\mathbf{Mod}\text{-}\mathcal{B}$.

Let $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$. We have the usual Tensor-Hom adjunction: for any DG-category \mathcal{C}

$$(-) \otimes_{\mathcal{A}} M: \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A} \rightarrow \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$$

is left adjoint to

$$\mathrm{Hom}_{\mathcal{B}}(M, -): \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{B} \rightarrow \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}.$$

The adjunction counit

$$\mathrm{Hom}_{\mathcal{B}}(M, -) \otimes_{\mathcal{A}} M \xrightarrow{\mathrm{ev}} \mathrm{Id} \tag{2.4}$$

is called the *evaluation map*, as it is defined by

$$\alpha \otimes m \mapsto \alpha(m).$$

Similarly, we call the adjunction unit

$$\mathrm{Id} \xrightarrow{\mathrm{mlt}} \mathrm{Hom}_{\mathcal{B}}(M, (-) \otimes_{\mathcal{A}} M) \tag{2.5}$$

the *tensor multiplication map*, as it is defined by

$$s \mapsto s \otimes (-).$$

Analogously,

$$M \otimes_{\mathcal{B}} (-): \mathcal{B}\text{-Mod-}\mathcal{C} \rightarrow \mathcal{A}\text{-Mod-}\mathcal{C}$$

is left adjoint to

$$\mathrm{Hom}_{\mathcal{A}^{\mathrm{opp}}}(M, -): \mathcal{A}\text{-Mod-}\mathcal{C} \rightarrow \mathcal{B}\text{-Mod-}\mathcal{C}$$

with the adjunction counit

$$\begin{aligned} M \otimes_{\mathcal{B}} \mathrm{Hom}_{\mathcal{A}^{\mathrm{opp}}}(M, -) &\xrightarrow{\mathrm{ev}} \mathrm{Id} \\ m \otimes \alpha &\mapsto (-1)^{\deg(m) \deg(\alpha)} \alpha(m) \end{aligned} \quad (2.6)$$

and the adjunction unit

$$\begin{aligned} \mathrm{Id} &\xrightarrow{\mathrm{mlt}} \mathrm{Hom}_{\mathcal{A}^{\mathrm{opp}}}(M, M \otimes_{\mathcal{B}} (-)) \\ s &\mapsto (-1)^{\deg(-) \deg(s)} (-) \otimes s. \end{aligned} \quad (2.7)$$

Let $M \in \mathcal{A}\text{-Mod-}\mathcal{B}$. The action of \mathcal{A} on M defines the canonical isomorphism

$$\mathcal{A} \otimes_{\mathcal{A}} M \xrightarrow{\sim} M \quad (2.8)$$

$$a \otimes m \mapsto a.m. \quad (2.9)$$

The right adjoint of (2.8) with respect to $(-) \otimes_{\mathcal{A}} M$ is the \mathcal{A} -action map $\mathcal{A} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{B}}(M, M)$. The right adjoint of (2.8) with respect to $\mathcal{A} \otimes_{\mathcal{A}} (-)$ is the canonical isomorphism

$$M \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, M) \quad (2.10)$$

$$m \mapsto (-1)^{\deg(-) \deg(m)} (-).m. \quad (2.11)$$

Similarly, we have canonical isomorphisms

$$M \otimes_{\mathcal{B}} \mathcal{B} \xrightarrow{\sim} M, \quad (2.12)$$

$$M \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(\mathcal{B}, M). \quad (2.13)$$

The canonical isomorphisms (2.10) and (2.13) identify evaluation maps with composition maps. E.g.

$$\mathrm{Hom}_{\mathcal{B}}(M, E) \otimes_{\mathcal{A}} M \xrightarrow{\mathrm{Id} \otimes (2.13)} \mathrm{Hom}_{\mathcal{B}}(M, E) \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{B}, M) \xrightarrow{\mathrm{cmps}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{B}, E) \xrightarrow{(2.13)^{-1}} E$$

is the evaluation map (2.4).

Finally, for all $M \in \mathcal{A}\text{-Mod-}\mathcal{B}$, $N \in \mathcal{D}\text{-Mod-}\mathcal{B}$ and $L \in \mathcal{C}\text{-Mod-}\mathcal{A}$ we have a canonical map

$$L \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(N, M) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(N, L \otimes_{\mathcal{A}} M) \quad (2.14)$$

$$l \otimes \alpha \mapsto l \otimes \alpha(-)$$

which is a quasi-isomorphism when N is \mathcal{B} -perfect or L is \mathcal{A} -perfect, cf. [AL17, §2.2]. We can also write (2.14) as the composition

$$L \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(N, M) \xrightarrow{\mathrm{mlt} \otimes \mathrm{Id}} \mathrm{Hom}_{\mathcal{B}}(M, L \otimes_{\mathcal{A}} M) \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(N, M) \xrightarrow{\mathrm{cmps}} \mathrm{Hom}_{\mathcal{B}}(N, L \otimes_{\mathcal{A}} M).$$

2.2. Twisted complexes and their convolutions. Twisted complexes were originally defined in [BK90], and then re-defined in [BLL04]. For the convenience of the reader, we give below a brief summary of the definitions we use in this paper. See [AL17, §3] for the detailed version of the same exposition.

Definition 2.1. Let \mathcal{A} be a DG-category. A *twisted complex* (E_i, α_{ij}) over \mathcal{A} is a collection of

- An object $E_i \in \mathcal{A}$ for each $i \in \mathbb{Z}$.
- An element $\alpha_{ij} \in \mathrm{Hom}_{\mathcal{A}}^{i-j+1}(E_i, E_j)$ for each $i, j \in \mathbb{Z}$.

satisfying:

- $E_i = 0$ for all but a finite number of i ,
- $(-1)^j d\alpha_{ij} + \sum_k \alpha_{kj} \circ \alpha_{ik} = 0$.

A twisted complex is called *one-sided* if $\alpha_{ij} = 0$ for all $i \geq j$.

Definition 2.2. The *DG-category of twisted complexes* $\mathrm{Tw}(\mathcal{A})$ has as objects all twisted complexes over \mathcal{A} . The complex of morphisms

$$\mathrm{Hom}_{\mathrm{Tw}(\mathcal{A})}((E_i, \alpha_{ij}), (F_l, \beta_{lj})),$$

has as its degree p part

$$\bigoplus_{k, l \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}^{p-l+k}(E_k, F_l)$$

and for each $\gamma \in \text{Hom}_{\mathcal{A}}^{p-l+k}(E_k, F_l)$ we have

$$d\gamma = (-1)^l d_{\mathcal{A}}\gamma + \sum_{m \in \mathbb{Z}} (\beta_{lm} \circ \gamma - (-1)^p \gamma \circ \alpha_{mk}),$$

where $d_{\mathcal{A}}$ is the differential on morphisms in \mathcal{A} .

Definition 2.3. Let \mathcal{A} be a DG-category. We define the *convolution functor*

$$\text{Conv}: \text{Tw}(\mathcal{A}) \rightarrow \mathbf{Mod}\text{-}\mathcal{A}$$

as follows. On objects, for any $(E_i, \alpha_{ij}) \in \text{Tw}(\mathcal{A})$ its *convolution* $\text{Conv}(E_i, \alpha_{ij})$ is obtained by taking the \mathcal{A} -module $\bigoplus_i E_i[-i]$, where we use Yoneda embedding to embed E_i into $\mathbf{Mod}\text{-}\mathcal{A}$, and equipping it with the new differential $d_{old} + \sum_{i,j} \alpha_{ij}$, where d_{old} denotes the natural differential on $\bigoplus_i E_i[-i]$. For brevity, we use curly brackets to denote the convolution, e.g. $\{E_i, \alpha_{ij}\}$.

On morphisms, for any $\gamma: (E_i, \alpha_{ij}) \rightarrow (F_i, \beta_{ij})$ its *convolution*

$$\text{Conv}(\gamma): \{E_i, \alpha_{ij}\} \rightarrow \{F_i, \beta_{ij}\}$$

is the \mathcal{A} -module morphism whose morphism of the underlying graded modules

$$\bigoplus_{k \in \mathbb{Z}} E_k[-k] \rightarrow \bigoplus_{l \in \mathbb{Z}} F_l[-l]$$

has as the components $E_k[-k] \rightarrow F_l[-l]$ the corresponding components of $\gamma \in \bigoplus_{k,l \in \mathbb{Z}} \text{Hom}_{\mathcal{A}}^{\bullet}(E_k, F_l)$.

2.3. Pretriangulated categories. Let \mathcal{A}, \mathcal{B} be DG-categories. A functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is a *quasi-equivalence* if it acts by quasi-isomorphisms on morphism complexes and if $H^0(F)$ is an equivalence of categories.

The DG-category \mathcal{A} is *pretriangulated* if $H^0(\mathcal{A}) \subset H^0(\mathbf{Mod}\text{-}\mathcal{A})$ is a triangulated subcategory. The *pretriangulated hull* $\text{Pre-Tr}(\mathcal{A})$ of \mathcal{A} is the full subcategory of $\text{Tw}(\mathcal{A})$ supported at one-sided twisted complexes. The convolution functor $\text{Conv}: \text{Pre-Tr}(\mathcal{A}) \rightarrow \mathbf{Mod}\text{-}\mathcal{A}$ of Defn. 2.3 is fully faithful, and its image descends to the triangulated hull of $H^0(\mathcal{A})$ in $H^0(\mathbf{Mod}\text{-}\mathcal{A})$.

Thus \mathcal{A} is pretriangulated if and only if $\mathcal{A} \hookrightarrow \text{Pre-Tr}(\mathcal{A})$ is a quasi-equivalence. It is *strongly pretriangulated* if $\mathcal{A} \hookrightarrow \text{Pre-Tr}(\mathcal{A})$ is an equivalence. In other words, if it has a quasi-inverse $T: \text{Pre-Tr}(\mathcal{A}) \rightarrow \mathcal{A}$. Then

$$\text{Pre-Tr}(\mathcal{A}) \xrightarrow{T} \mathcal{A} \hookrightarrow \mathbf{Mod}\text{-}\mathcal{A}$$

is isomorphic to the convolution functor. By abuse of notation, we also refer to T as the *convolution functor*.

It is one of the main results of [BK90] that the category $\text{Pre-Tr}(\mathcal{A})$ is strongly pretriangulated. For any strongly pretriangulated \mathcal{C} , the category $\text{DGFun}(\mathcal{A}, \mathcal{C})$ is strongly pretriangulated since convolutions can be defined levelwise in \mathcal{C} . In particular, $\mathbf{Mod}\text{-}\mathcal{A}$ is strongly pretriangulated since $\mathbf{Mod}\text{-}k$ is. Finally, a full subcategory of a strongly pretriangulated DG-category is strongly pretriangulated if it is pretriangulated and closed under homotopy equivalences. Thus $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}^{\text{Perf}}(\mathcal{A})$ are strongly pretriangulated subcategories of $\mathbf{Mod}\text{-}\mathcal{A}$, and $\mathcal{P}^{\mathcal{A}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B})$ and $\mathcal{P}^{\mathcal{B}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B})$ are strongly pretriangulated subcategories of $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$.

2.4. Postnikov systems. A limited version of this notion appears in e.g. [Orl97, §1.3] and [GM03, §IV.2]. Roughly, it is the data of computing a convolution of a complex of objects in a triangulated category. We give a brief summary below.

Let \mathcal{T} be a triangulated category. Let (E_i, f_i) be a complex of objects in \mathcal{T} of length n :

$$E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_3 \rightarrow \cdots \rightarrow E_{n-1} \xrightarrow{f_{n-1}} E_n$$

with all $f_i \circ f_{i-1} = 0$.

Definition 2.4. A *Postnikov system* on the complex (E_i, f_i) is the set of data defined as follows.

For $n = 1$, i.e. the complex consists of a single object E_1 , a Postnikov system on it is an empty set of data.

For any $n > 1$, the data of a Postnikov system on (E_i, f_i) consists of:

- A choice of $i \in \{1, \dots, n-1\}$.
- For the morphism f_i , its completion to an exact triangle in \mathcal{T} :

$$\begin{array}{ccc} E_i & \xrightarrow{f_i} & E_{i+1} \\ & \swarrow h_i \text{ (dashed)} & \searrow g_i \text{ (dotted)} \\ & & E_{i,i+1} \end{array} \quad (2.15)$$

Here, dashed and dotted arrows denote morphisms of degree 1 and -1 respectively.

- For those of the morphisms f_{i-1} and f_{i+1} which exist, their lifts to morphisms $f_{i-1,i+1}$ and $f_{i,i+2}$ to $E_{i,i+1}[-1]$ and from $E_{i,i+1}$, respectively, with $f_{i,i+2} \circ f_{i-1,i+1} = 0$:

$$\begin{array}{ccccccc}
E_{i-1} & \xrightarrow{f_{i-1}} & E_i & \xrightarrow{f_i} & E_{i+1} & \xrightarrow{f_{i+1}} & E_{i+2} \\
& & \swarrow f_{i-1,i+1} & \nwarrow h_i & \swarrow g_i & \searrow f_{i,i+2} & \\
& & & & E_{i,i+1} & &
\end{array} \quad (2.16)$$

- A Postnikov system for the resulting length $n - 1$ complex:

$$E_1[1] \xrightarrow{f_1} E_2[1] \xrightarrow{f_2} E_3[1] \rightarrow \cdots \rightarrow E_{i-1}[1] \xrightarrow{f_{i-1,i+1}} E_{i,i+1} \xrightarrow{f_{i,i+2}} E_{i+2} \rightarrow \cdots \rightarrow E_{n-1} \xrightarrow{f_{n-1}} E_n. \quad (2.17)$$

In other words, the data of a Postnikov system may be viewed as the data of an iterative process. First we choose any two consecutive objects of a complex and replace them by the cone of the differential between them. Then, we lift the neighbouring differentials so that they compose to zero and we thus obtain a complex again. Then we repeat, with the length of the complex decreasing by one at each iteration.

Postnikov systems are not unique and, as it may not always be possible to lift the neighbouring differentials so that they compose to zero, Postnikov systems do not necessarily exist. When they do, they compute convolutions of (E_i, f_i) :

Definition 2.5. The *convolution* of a Postnikov system on the complex (E_i, f_i) as defined in Defn. 2.4 is the E_1 if $n = 1$ and the convolution of the Postnikov system (2.17) if $n > 1$.

That is, it is the single object left in the end of the iterative process described by the Postnikov system.

Example 2.6. Two of the 6 possible types of Postnikov systems on a length 4 complex $E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_4$:

$$\begin{array}{ccc}
\begin{array}{ccccccc}
E_1 & \xrightarrow{f_1} & E_2 & \xrightarrow{f_2} & E_3 & \xrightarrow{f_3} & E_4 \\
& & \swarrow f_{14} & \nwarrow h_{24} & \swarrow f_{24} & \nwarrow h_{34} & \swarrow g_3 \\
& & & & E_{2,3,4} & & \\
& & \swarrow h_{14} & \nwarrow g_{14} & & \swarrow g_{24} & \\
& & & & E_{1,2,3,4} & &
\end{array} &
\begin{array}{ccccccc}
E_1 & \xrightarrow{f_1} & E_2 & \xrightarrow{f_2} & E_3 & \xrightarrow{f_3} & E_4 \\
& & \swarrow h_1 & \nwarrow g_1 & \swarrow f_{24} & \nwarrow h_3 & \swarrow g_3 \\
& & & & E_{1,2} & & \\
& & \swarrow f_{14} & \nwarrow h_{14} & & \swarrow g_{14} & \\
& & & & E_{1,2,3,4} & &
\end{array} &
\end{array} \quad (2.18)$$

Here, dashed and dotted arrows denote morphisms of degree > 0 and < 1 , respectively, the triangles denoted by \star are exact, and the remaining triangles are commutative.

Suppose now we have a strongly pretriangulated DG-enhancement \mathcal{A} of \mathcal{T} , i.e. a strongly pretriangulated \mathcal{A} and a choice of isomorphism $\iota: H^0(\mathcal{A}) \simeq \mathcal{T}$.

Definition 2.7. Let (E_i, f_i) be a length n complex of objects in \mathcal{T} . A twisted complex $(\bar{E}_i, \bar{f}_{ij})$ over \mathcal{A} is a lift of (E_i, f_i) if $E_i = \iota(\bar{E}_{i-n})$ and $f_i = \iota(\bar{f}_{-n,i-n+1})$ for all $i \in \{1, \dots, n-1\}$.

Conversely, for any twisted complex $(\bar{E}_i, \bar{f}_{ij})$ over \mathcal{A} with $\bar{E}_i \neq 0$ only for $i \in \{-n+1, \dots, 0\}$, denote by $\iota(\bar{E}_i, \bar{f}_{ij})$ the complex

$$\iota(\bar{E}_{-n+1}) \xrightarrow{[\bar{f}_{-n+1,-n+2}]} \iota(\bar{E}_{-n+1}) \rightarrow \cdots \rightarrow \iota(\bar{E}_{-1}) \xrightarrow{[\bar{f}_{-1,0}]} \iota(\bar{E}_0),$$

whose lift $(\bar{E}_i, \bar{f}_{ij})$ is.

The index shift in the definition above is introduced to ensure that a lift of a complex $E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_n$ lifts E_n to the degree 0 element of the twisted complex. This is so that notions of convolution for twisted complexes and for ordinary complexes coincide, as per following two results.

Proposition 2.8. Let $\bar{E} = (\bar{E}_i, \bar{f}_{ij})$ be a twisted complex over \mathcal{A} . For any $k \in \mathbb{Z}$ there is a unique twisted complex we denote by $\bar{E}(\hat{k})$ which has the same convolution as \bar{E} and the object set:

$$\bar{E}(\hat{k})_i = \begin{cases} \bar{E}_i, & i > k+1 \\ \left\{ \bar{E}_{i-1} \xrightarrow[\deg.0]{\bar{f}_{i-1,i}} \bar{E}_{i+1} \right\}, & i = k+1, \\ \bar{E}_{i-1}[1], & i < k+1 \end{cases}$$

The degree of this map is $i - j + 1$. Note also that

$$d(\gamma_{ij}) = \begin{array}{ccc} E_i & \xrightarrow{(-1)^i f_{ii}} & F_i \\ d\alpha_{ij} \downarrow & \searrow^{df_{ij} - (-1)^j f_{jj}\alpha_{ij} - (-1)^{i-j+1}(-1)^i \beta_{ij} f_{ii}} & \downarrow d\beta_{ij} \\ E_j & \xrightarrow{(-1)^j f_{jj}} & F_j. \end{array}$$

We claim that $G = (G_i, \gamma_{ij})$ is a twisted complex. For this we need to have for all i and j

$$(-1)^j d\gamma_{ij} + \sum_{k=i+1}^{j-1} \gamma_{kj} \gamma_{ik} = 0.$$

The map on the LHS has three components: $E_i \rightarrow E_j$, $F_i \rightarrow F_j$ and $E_i \rightarrow F_j$. The first two components vanish since E and F are twisted complexes. Computing the $E_i \rightarrow F_j$ component we get

$$\begin{aligned} (-1)^j (df_{ij} - (-1)^j f_{jj}\alpha_{ij} + (-1)^j \beta_{ij} f_{ii}) + \sum_{k=i+1}^{j-1} \beta_{kj} f_{ik} - \sum_{k=i+1}^{j-1} f_{kj} \alpha_{ik} &= \\ = (-1)^j df_{ij} + \beta_{ij} f_{ii} + \sum_{k=i+1}^{j-1} \beta_{kj} f_{ik} - f_{jj}\alpha_{ij} - \sum_{k=i+1}^{j-1} f_{kj} \alpha_{ik} &= (df)_{ij} = 0. \end{aligned}$$

Thus (G_i, γ_{ij}) is a twisted complex. Its total complex and that of $(E \xrightarrow{f} F)$ both have the i -th term

$$E_{i+1} \oplus F_i$$

and the ij -th differential

$$-\alpha_{i+1,j+1} + f_{i+1,j} + \beta_{ij},$$

as desired. \square

Recall [AL17, §3.2] [BK90, §1] that there exists the convolution functor $\text{Pre-Tr Mod-}\mathcal{A} \rightarrow \text{Mod-}\mathcal{A}$ which is a category equivalence and which we denote by $\{-\}$.

Corollary 2.11. *Let \mathcal{A} be a DG-category and let $E = (E_i, \alpha_{ij})$ and $F = (F_i, \beta_{ij})$ be one-sided twisted complexes over $\text{Mod-}\mathcal{A}$ and let $f = (f_{ij})$ be a one-sided closed map $E \rightarrow F$ of degree 0.*

The cone of the induced map

$$\{E\} \xrightarrow{f} \{F\}$$

is isomorphic to the convolution of a twisted complex whose objects are isomorphic to

$$\text{Cone}\left(E_i \xrightarrow{f_{ii}} F_i\right). \quad (2.19)$$

Proof. The convolution of the twisted complex $\text{Tot}(G_i, \gamma_{ij})$ constructed in Lemma 2.10 is isomorphic to the convolution of the twisted complex $(\{G_i\}, \gamma_{ij})$, cf. diagram 3.1 in [BK90, §3]. Each $\{G_i\}$ is isomorphic to $\text{Cone}\left(E_i \xrightarrow{f_{ii}} F_i\right)$. Similarly, the convolution of $\text{Tot}(E \xrightarrow{f} F)$ is isomorphic to $\text{Cone}\left(\{E\} \xrightarrow{f} \{F\}\right)$. The claim now follows from Lemma 2.10. \square

Corollary 2.12. *Let \mathcal{A} be a DG-category and let (E_i, α_{ij}) and (F_i, β_{ij}) be one-sided bounded above twisted complexes over $\text{Mod-}\mathcal{A}$. Let $f = (f_{ij})$ be a one-sided closed map $(E_i, \alpha_{ij}) \rightarrow (F_j, \beta_{ij})$ of degree 0.*

If each component $f_{ii} : E_i \rightarrow F_i$ is a quasi-isomorphism, then so is the induced map

$$\{E_i, \alpha_{ij}\} \xrightarrow{f} \{F_i, \beta_{ij}\}.$$

Proof. By Cor. 2.11 the cone of f is isomorphic in $D(\mathcal{A})$ to the convolution of the bounded above twisted complex whose objects are isomorphic to $\text{Cone}\left(E_i \xrightarrow{f_{ii}} F_i\right)$ in $D(\mathcal{A})$. By assumption f_{ii} are quasi-isomorphisms, thus all the objects of this twisted complex are acyclic. To show that f itself is a quasi-isomorphism, it remains to show that the convolution of a bounded above twisted complex of acyclic modules is itself acyclic. To see this, first note that the cone of any two acyclic modules is acyclic and, by induction, so is the convolution of any finite one-sided twisted complex of acyclic modules. Finally, the convolution of any bounded above twisted complex has an exhaustive filtration by the convolutions of its finite subcomplexes. The induced exhaustive filtration on cohomologies proves the claim. \square

Lemma 2.13 (Extraction Lemma). *Let $E \in \mathbf{Mod}\text{-}\mathcal{A}$ and let $Q \subset E$ be a null-homotopic submodule. Let ν be a contracting homotopy of Q , that is $\nu \in \text{Hom}_{\mathcal{A}}^{-1}(Q, Q)$ such that $d\nu = \text{Id}$. Suppose that on the level of the underlying graded modules E splits as $Q \oplus F$ for some graded \mathcal{A} -submodule F . Let the differential of E with respect to that splitting be*

$$\begin{pmatrix} d_Q & \alpha \\ \beta & \delta \end{pmatrix}. \quad (2.20)$$

Then E is homotopy equivalent to F equipped with the differential $d_F = \delta - \beta \circ \nu \circ \alpha$.

Proof. Since Q is a sub-DG-module of E , we have $d_Q^2 = 0$. Since (2.20) is a differential, it is a derivation and of square zero. The fact (2.20) is a derivation together with the splitting of $E = Q \oplus F$ respecting \mathcal{A} -action implies that δ is also a derivation, while α and β are maps of graded \mathcal{A} -modules. The fact that (2.20) squares to zero implies that

$$\begin{aligned} \alpha \circ \beta &= 0, \\ \delta^2 + \beta \circ \alpha &= 0, \\ \delta \circ \beta + \beta \circ d_Q &= 0, \\ \alpha \circ \delta + d_Q \circ \alpha &= 0. \end{aligned}$$

The map d_F is a derivation as it is the sum of the derivation δ and the graded \mathcal{A} -module map $-\beta \circ \nu \circ \alpha$. Moreover, we have

$$\begin{aligned} d_F^2 &= (\delta - \beta \circ \nu \circ \alpha)^2 = \delta^2 - \delta \circ \beta \circ \nu \circ \alpha - \beta \circ \nu \circ \alpha \circ \delta + \beta \circ \nu \circ \alpha \circ \beta \circ \nu \circ \alpha = \\ &= \delta^2 + \beta \circ d_Q \circ \nu \circ \alpha + \beta \circ \nu \circ d_Q \circ \alpha + 0 = \delta^2 + \beta \circ (d_Q \circ \nu + \nu \circ d_Q) \circ \alpha = \\ &= \delta^2 + \beta \circ d\nu \circ \alpha = \delta^2 + \beta \circ \alpha = 0. \end{aligned}$$

Thus d_F does indeed define a differential and hence a structure of a DG \mathcal{A} -module on F . It can be readily checked that with respect to that structure the maps α and β are both closed.

Consider the following maps of graded \mathcal{A} -modules:

$$Q \oplus F \xrightarrow{(-\beta \circ \nu \text{ Id})} F \quad (2.21)$$

$$F \xrightarrow{\begin{pmatrix} -\nu \circ \alpha \\ \text{Id} \end{pmatrix}} Q \oplus F. \quad (2.22)$$

We claim that they define mutually inverse homotopy equivalences $E \xrightarrow{\sim} F$ and $F \xrightarrow{\sim} E$ in $\mathbf{Mod}\text{-}\mathcal{A}$. Indeed,

$$(2.21) \circ (2.22) = \text{Id} + \beta \circ \nu^2 \circ \alpha = \text{Id} - d(\beta \circ \nu^3 \circ \alpha),$$

while

$$(2.22) \circ (2.21) = \begin{pmatrix} 0 & -\nu \circ \alpha \\ -\beta \circ \nu & \text{Id} \end{pmatrix} = \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix} - \begin{pmatrix} \text{Id} & \nu \circ \alpha \\ \beta \circ \nu & 0 \end{pmatrix} = \text{Id} - d \begin{pmatrix} \nu & 0 \\ 0 & 0 \end{pmatrix}$$

as required. \square

Since the convolution functor $\text{Pre-Tr } \mathbf{Mod}\text{-}\mathcal{A} \rightarrow \mathbf{Mod}\text{-}\mathcal{A}$ is an equivalence, the Extraction Lemma can be applied to any twisted complex over $\mathbf{Mod}\text{-}\mathcal{A}$ with a null-homotopic subcomplex. For example:

Example 2.14. Let \mathcal{A} be a DG-category and let

$$\begin{array}{ccccc} & & \delta_{03} & & \\ & & \text{---} & & \\ & & \begin{pmatrix} \alpha_{02} \\ \delta_{02} \end{pmatrix} & & \\ & & \text{---} & & \\ E_0 & \xrightarrow{\begin{pmatrix} \alpha_{01} \\ \delta_{01} \end{pmatrix}} & X \oplus E_1 & \xrightarrow{\begin{pmatrix} \text{Id} & \alpha_{12} \\ \beta_{12} & \delta_{12} \end{pmatrix}} & X \oplus E_2 & \xrightarrow{(\beta_{23} \ \delta_{23})} & E_3 \\ \text{deg}_0 & & & & & & \\ & & & & \text{---} & & \\ & & & & \begin{pmatrix} \beta_{13} & \delta_{13} \end{pmatrix} & & \end{array} \quad (2.23)$$

be a twisted complex over $\mathbf{Mod}\text{-}\mathcal{A}$. Then the following is a twisted complex homotopy equivalent to (2.23):

$$\begin{array}{ccccc}
 & & \delta_{03} - \beta_{13} \circ \alpha_{02} & & \\
 & \text{---} \text{---} \text{---} & & \text{---} \text{---} \text{---} & \\
 & & \delta_{02} - \beta_{12} \circ \alpha_{02} & & \\
 E_0 & \xrightarrow{\delta_{01}} & E_1 & \xrightarrow{\delta_{12} - \beta_{12} \circ \alpha_{12}} & E_2 & \xrightarrow{\delta_{23}} & E_3 \\
 \text{deg.0} & & & & & & \\
 & & & \text{---} \text{---} \text{---} & & & \\
 & & & \delta_{13} - \beta_{13} \circ \alpha_{12} & & &
 \end{array} \quad (2.24)$$

Proof. In Lemma 2.13 set Q to be the convolution of $\left(\begin{array}{c} X \\ \text{deg.1} \end{array} \xrightarrow{\text{Id}} X \right)$ and set F to be the graded module underlying $\bigoplus E_i[-i]$. Set the contracting homotopy ν to be given by

$$\begin{array}{ccc}
 X & \xrightarrow{\text{Id}} & X \\
 \text{deg.1} & & \\
 & \swarrow \text{Id} & \\
 X & \xrightarrow{\text{Id}} & X \\
 \text{deg.1} & &
 \end{array} \quad (2.25)$$

Then in (2.20) the map δ is $\sum d_{E_i} + \sum \delta_{ij}$, α is $\sum \alpha_{ij}$ and β is $\sum \beta_{ij}$. The map $\beta \circ \nu \circ \alpha$ is therefore

$$\begin{array}{ccccccc}
 E_0 & & E_1 & & E_2 & & E_3 \\
 \text{deg.0} & & & & & & \\
 & \searrow \alpha_{01} & \searrow \alpha_{02} & \searrow \alpha_{12} & & & \\
 & & X & & X & & \\
 & & & & \text{Id} & & \\
 & & X & & X & & \\
 & & & \searrow \beta_{12} & \searrow \beta_{13} & \searrow \beta_{23} & \\
 E_0 & & E_1 & & E_2 & & E_3 \\
 \text{deg.0} & & & & & &
 \end{array}$$

that is $(\beta_{12} + \beta_{13}) \circ (\alpha_{02} + \alpha_{12})$. Thus the differential $\delta - \beta \circ \nu \circ \alpha$ on the graded module $\bigoplus E_i[-i]$ equals

$$\sum d_{E_i} + \sum \delta_{ij} - (\beta_{12} + \beta_{13}) \circ (\alpha_{02} + \alpha_{12}). \quad (2.26)$$

The convolution functor $\text{Pre-Tr } \mathbf{Mod}\text{-}\mathcal{A} \rightarrow \mathbf{Mod}\text{-}\mathcal{A}$ is an equivalence. Every differential on the graded complex $\bigoplus E_i[-i]$ corresponds to a twisted complex whose objects are E_i . By the definition of the convolution functor, the differential (2.26) corresponds to the twisted complex (2.24). \square

2.6. A_∞ -algebras, modules, and bimodules. For an introduction to A_∞ -categories we recommend [Kel06], and for a comprehensive technical text – [LH03]. We refer the reader to the latter for the definitions and the notation we employ. Below, we summarise some of it and prove several minor new results.

An A_∞ -algebra over k is a graded k -bimodule A together with graded maps

$$m_i: A^{\otimes i} \rightarrow A \quad i \geq 1 \quad (2.27)$$

of degree $2 - i$ which lift to a differential which gives the structure of an coaugmented DG coalgebra to the coaugmented tensor coalgebra

$$T^c(A[1]) = \bigoplus_{n \geq 0} A^{\otimes n}[n]$$

generated by $A[1]$. Such differential is necessarily a sum of the zero map on the coaugmented part k and a differential making the reduced tensor coalgebra $\bigoplus_{n \geq 1} A^{\otimes n}[n]$ into a DG coalgebra. The resulting coaugmented DG-coalgebra $\bigoplus_{n \geq 0} A^{\otimes n}[n]$ is the *bar construction* $B_\infty A$ of A , whose counit we denote by $\tau: B_\infty A \rightarrow k$. The resulting DG-coalgebra $\bigoplus_{n \geq 1} A^{\otimes n}[n]$ is the *non-augmented bar construction* $B_\infty^{\text{na}} A$.

Let A and B be A_∞ -algebras. An A_∞ -algebra morphism $f: A \rightarrow B$ is a collection of graded maps

$$f_i: A^{\otimes i} \rightarrow B \quad i \geq 1 \quad (2.28)$$

of degree $1 - i$ which lift to a morphism of augmented coalgebras $B_\infty A \rightarrow B_\infty B$.

Let A be an A_∞ -algebra. A (right) A -module is a graded k -module E together with a collection of graded maps

$$m_i: E \otimes A^{\otimes i-1} \rightarrow E \quad i \geq 1 \quad (2.29)$$

of degree $2 - i$ which lift to a differential on the free graded $B_\infty A$ -comodule $E[1] \otimes_k B_\infty A$ generated by $E[1]$. The resulting DG $B_\infty A$ -comodule is the *bar construction* $B_\infty E$ of E .

Let E and F be two A -modules. An A_∞ -module morphism $f: E \rightarrow F$ is a collection of graded maps

$$f_i: E \otimes_k A^{\otimes i-1} \rightarrow F \quad i \geq 1 \quad (2.30)$$

of degree $1 - i$ which lift to a $B_\infty A$ -comodule morphism $B_\infty E \rightarrow B_\infty F$. An A_∞ -module morphism is *strict* if $f_i = 0$ for $i \geq 2$.

Let A and B be A_∞ -algebras. An A - B -bimodule is a graded k -bimodule M together with a differential $m_{0,0}$ and the action maps

$$m_{i,j}: A^{\otimes i} \otimes_k M \otimes_k B^{\otimes j} \rightarrow M \quad i \geq 0, j \geq 0 \quad (2.31)$$

of degree $1 - i - j$ which together lift to a differential

$$(B_\infty A) \otimes_k M[1] \otimes_k (B_\infty B) \longrightarrow (B_\infty A) \otimes_k M[1] \otimes_k (B_\infty B), \quad (2.32)$$

cf. [LH03, §2.5.1]. The resulting DG $B_\infty A$ - $B_\infty B$ -bicomodule is the *bar construction* $B_\infty M$ of M . Whenever it is necessary to avoid confusion, e.g. in the case of the diagonal bimodule, we will denote the bimodule bar construction as $B_\infty^{\text{bim}} M$.

We also consider the partial bar constructions. The B -bar construction $B_\infty^B M$ is the right DG $(B_\infty B)$ -comodule whose underlying graded $B_\infty B$ -comodule is $M \otimes_k B_\infty B$ and whose differential is constructed from $m_{0,j}$ for $j \geq 0$. It also carries the structure of a left A_∞ A -module, defined by $m_{i,j}$ for $i \geq 1, j \geq 0$. Likewise, the A -bar construction $B_\infty^A M$ is the left DG $(B_\infty A)$ -comodule and right A_∞ B -module defined similarly. Consider now the DG-algebras $\text{End}_{B_\infty B}(B_\infty^B M)$ and $\text{End}_{(B_\infty A)^{\text{opp}}}(B_\infty^A M)$. Specifying the structure of A - B -bimodule on M is equivalent to specifying the natural A -action A_∞ -morphism

$$A \xrightarrow{\text{act}_M} \text{End}_{B_\infty B}(B_\infty^B M). \quad (2.33)$$

Similarly, it is equivalent to specifying the natural B -action A_∞ -morphism

$$B^{\text{opp}} \xrightarrow{\text{act}_M} \text{End}_{(B_\infty A)^{\text{opp}}}(B_\infty^A M), \quad (2.34)$$

cf. the ‘‘lemme clef’’ of [LH03, §5.3]. Explicitly, we define e.g. the A -action morphism by setting for each $a_1 \otimes \cdots \otimes a_n \in A^{\otimes n}$ the endomorphism $\text{act}_M(a_1 \otimes \cdots \otimes a_n) \in \text{End}_{B_\infty B}(B_\infty^B M)$ to be

$$m \otimes b_1 \otimes \cdots \otimes b_m \mapsto \sum_{l=0}^m (-1)^? m_{n,l}(a_1 \otimes \cdots \otimes a_n \otimes m \otimes b_1 \otimes \cdots \otimes b_l) \otimes b_{l+1} \otimes \cdots \otimes b_m \quad (2.35)$$

where the signs are dictated by the definitions in [LH03, §5.3].

The *diagonal bimodule* A is defined by the graded maps

$$m_{i,j}: A^{\otimes i} \otimes_k A \otimes_k A^{\otimes j} \xrightarrow{m_{i+j+1}} A \quad i \geq 0, j \geq 0 \quad (2.36)$$

where m_i are the maps which define the A_∞ -algebra structure on A .

Let M and N be two A - B -bimodules. An A_∞ -bimodule morphism $f: M \rightarrow N$ is a collection of graded maps

$$f_{i,j}: A^{\otimes i} \otimes_k M \otimes_k B^{\otimes j} \rightarrow N \quad i \geq 0, j \geq 0 \quad (2.37)$$

of degree $-i - j$ which lift to a $B_\infty A$ - $B_\infty B$ -bicomodule morphism $B_\infty M \rightarrow B_\infty N$. An A_∞ -module morphism is *strict* if $f_{i,j} = 0$ for $i \geq 1$ or $j \geq 1$.

The DG k -module $\text{Hom}_{B_\infty B}(B_\infty^B M, B_\infty^B N)$ has a natural structure of an A - A -bimodule defined via the A -action maps for $B_\infty^B M$ and $B_\infty^B N$. Similar to (2.33) and (2.34), specifying an A_∞ A - B -bimodule morphism $M \xrightarrow{f} N$ is then equivalent to specifying a DG $B_\infty A$ - $B_\infty A$ -bicomodule morphism

$$B_\infty A \xrightarrow{f^A} B_\infty \left(\text{Hom}_{B_\infty B}(B_\infty^B M, B_\infty^B N) \right). \quad (2.38)$$

Similarly, it is equivalent to specifying a DG $B_\infty B$ - $B_\infty B$ -bicomodule morphism

$$B_\infty B \xrightarrow{f^B} B_\infty \left(\text{Hom}_{B_\infty A}(B_\infty^A M, B_\infty^A N) \right). \quad (2.39)$$

2.7. A_∞ -categories. Let \mathbb{A} be a set. We define $k_{\mathbb{A}}$ to be the category whose set of objects is \mathbb{A} and whose morphisms spaces are

$$\mathrm{Hom}_{k_{\mathbb{A}}}(a, b) = \begin{cases} k & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases} \quad (2.40)$$

For any sets \mathbb{A} , \mathbb{B} and \mathbb{C} , any graded $k_{\mathbb{A}}-k_{\mathbb{B}}$ -bimodule M , and $k_{\mathbb{B}}-k_{\mathbb{C}}$ -bimodule N we denote by \otimes_k their tensor product over $k_{\mathbb{B}}$:

$${}_a(M \otimes_k N)_c = \bigoplus_{b \in \mathbb{B}} {}_a M_b \otimes_k {}_b N_c. \quad (2.41)$$

We give the category of graded $k_{\mathbb{A}}-k_{\mathbb{A}}$ -bimodules a monoidal structure by equipping it with the multiplication given by \otimes_k and the identity element given by the diagonal bimodule $k_{\mathbb{A}}$.

Given a graded $k_{\mathbb{A}}-k_{\mathbb{B}}$ -bimodule M and two maps of sets $\mathbb{A}' \xrightarrow{f} \mathbb{A}$ and $\mathbb{B}' \xrightarrow{g} \mathbb{B}$, we write ${}_f M_g$ for the graded $k_{\mathbb{A}'}-k_{\mathbb{B}'}$ -bimodule obtained by pulling back along f and g , i.e.

$${}_{a'}({}_f M_g)_{b'} = {}_{f(a')} M_{g(b')} \quad \forall a' \in \mathbb{A}', b' \in \mathbb{B}'. \quad (2.42)$$

An A_∞ -category \mathcal{A} is an object set \mathbb{A} and an A_∞ -algebra \mathcal{A} over $k_{\mathbb{A}}$, i.e. in the monoidal category of graded $k_{\mathbb{A}}-k_{\mathbb{A}}$ -bimodules. See [LH03, §1.1-1.2] for an explanation of how the usual notions of ordinary, DG, and A_∞ -algebras over a ring generalise to those over an arbitrary monoidal category. We abuse the notation by also using \mathcal{A} to denote the object set \mathbb{A} where it doesn't cause confusion, e.g. we write $k_{\mathcal{A}}$ for $k_{\mathbb{A}}$.

Given two A_∞ -categories \mathcal{A} and \mathcal{B} an A_∞ -functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is a map $\mathbb{A} \xrightarrow{F} \mathbb{B}$ of their object sets and a morphism $\mathcal{A} \rightarrow {}_F \mathcal{B}_F$ of A_∞ -algebras in the category of graded $k_{\mathcal{A}}-k_{\mathcal{A}}$ -bimodules.

The definitions of modules, bimodules, etc. for A_∞ -algebras in §2.6 generalise similarly to A_∞ -categories by considering the latter as A_∞ -algebras in appropriate categories of graded bimodules, cf. [LH03, §5.1].

Let \mathcal{A} be an A_∞ -category. We denote by $\mathbf{Nod}_\infty \mathcal{A}$ the DG-category of all right A_∞ -modules over \mathcal{A} . Its objects are right \mathcal{A} -modules and for any two objects E and F we have

$$\mathrm{Hom}_{\mathbf{Nod}_\infty \mathcal{A}}(E, F) \simeq \mathrm{Hom}_{B_\infty \mathcal{A}}(B_\infty E, B_\infty F), \quad (2.43)$$

cf. [LH03, §5.2]. It follows that the elements of $\mathrm{Hom}_{\mathbf{Nod}_\infty \mathcal{A}}(E, F)$ can be identified with arbitrary collections of graded $k_{\mathcal{A}}$ -module morphisms $\{E \otimes_k \mathcal{A}^{\otimes i} \rightarrow F\}_{i \geq 0}$. Note that such collection defines a morphism of A_∞ -modules, as per §2.6, if and only if the corresponding element is closed of degree 0.

Let \mathcal{A} and \mathcal{B} be A_∞ -categories. The DG-category $\mathbf{Nod}_\infty \mathcal{A}\text{-}\mathcal{B}$ of A_∞ $\mathcal{A}\text{-}\mathcal{B}$ -bimodules is defined similarly to the above.

Let M be an $\mathcal{A}\text{-}\mathcal{B}$ bimodule. The notions of partial bar constructions and action maps defined in §2.6 extend to A_∞ -functors $\mathcal{A} \rightarrow \mathbf{Nod}_\infty \mathcal{B}$ and $\mathcal{B}^{\mathrm{opp}} \rightarrow \mathbf{Nod}_\infty \mathcal{A}^{\mathrm{opp}}$, cf. [LH03, Cor. 5.3.0.2]. Given $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we write ${}_a M$ and M_b for their images under these functors. When \mathcal{A} and \mathcal{B} are A_∞ -algebras, i.e. A_∞ -categories with a single object \bullet , $\bullet M$ is e.g. the \mathcal{B} -module which corresponds to $B_\infty^{\mathcal{B}} M$, and the functor $\mathcal{A} \rightarrow \mathbf{Nod}_\infty \mathcal{B}$ acts on the morphism spaces by the A_∞ -morphism $\mathcal{A} \rightarrow \mathrm{Hom}_{\mathbf{Nod}_\infty \mathcal{B}}(\bullet M, \bullet M)$ which corresponds to the action map $\mathcal{A} \xrightarrow{\mathrm{act}_M} \mathrm{End}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} M)$.

In case when M is the diagonal bimodule \mathcal{A} , this yields the *Yoneda embedding* $\mathcal{A} \hookrightarrow \mathbf{Nod}_\infty \mathcal{A}$. The modules $\{{}_a \mathcal{A}\}_{a \in \mathcal{A}}$ are the *representable* \mathcal{A} -modules. Explicitly, the graded $k_{\mathcal{A}}$ -module underlying ${}_a \mathcal{A}$ is $\mathrm{Hom}_{\mathcal{A}}(-, a)$ and its A_∞ -module structure is given by the A_∞ -operations m_i of \mathcal{A} .

An \mathcal{A} -module E is *free* if it is isomorphic to a direct sum of shifts of representable modules. An \mathcal{A} -module E is *semi-free* if it admits an ascending filtration whose quotients are free modules.

2.8. The derived category of an A_∞ -category. Let \mathcal{A} be an A_∞ -category. A morphism of A_∞ -modules is a quasi-isomorphism if and only if it is a homotopy equivalence [LH03, Prop. 2.4.1.1]. Thus all quasi-isomorphisms are already invertible in $H^0(\mathbf{Nod}_\infty \mathcal{A})$, and there is no need to formally invert them to construct the derived category of \mathcal{A} . Instead, however, we have to throw away certain \mathcal{A} -modules. For example, given an associative unital algebra, for its derived category as an A_∞ -algebra to coincide with its usual derived category we only want to consider its A_∞ -modules which are quasi-isomorphic to its ordinary, unital modules. It turns out that we need to impose a similar sort of condition even when dealing with an arbitrary A_∞ -category.

We recall various notions of unitality for A_∞ -categories and their modules. Let \mathcal{A} be an A_∞ -category. We say that \mathcal{A} is *strictly unital* if it is equipped with a unit $\eta : k_{\mathcal{A}} \rightarrow \mathcal{A}$ such that $m_2(\eta \otimes \mathrm{Id}_{\mathcal{A}}) = m_2(\mathrm{Id}_{\mathcal{A}} \otimes \eta) = \mathrm{Id}_{\mathcal{A}}$ and $m_i(\mathrm{Id}_{\mathcal{A}} \otimes \cdots \otimes \mathrm{Id}_{\mathcal{A}} \otimes \eta \otimes \mathrm{Id}_{\mathcal{A}} \otimes \cdots \otimes \mathrm{Id}_{\mathcal{A}}) = 0$ for all $i \neq 2$. We say that \mathcal{A} is *homologically unital* if

$H^*(\mathcal{A})$ is a unital graded category. Finally, we say that \mathcal{A} is *H-unital* if its non-augmented bar construction $B_\infty^{\text{na}}\mathcal{A}$ is acyclic. These notions are related as

$$\text{strictly unital} \subset \text{homologically unital} \subset H\text{-unital} . \quad (2.44)$$

The first inclusion is clear, and the second is due to [LH03, Prop. 4.1.2.7].

Let $E \in \mathbf{Nod}_\infty\mathcal{A}$. If \mathcal{A} is strictly unital, we say that E is *strictly unital* if $m_2(\text{Id}_M \otimes \eta) = \text{Id}_M$ and $m_i(\text{Id}_M \otimes \text{Id}_\mathcal{A} \otimes \cdots \otimes \text{Id}_\mathcal{A} \otimes \eta \otimes \text{Id}_\mathcal{A} \otimes \cdots \otimes \text{Id}_\mathcal{A}) = 0$ for all $i \geq 3$. If \mathcal{A} is homologically unital, we say that E is *homologically unital* if $H^*(E)$ is a unital graded module over $H^*(\mathcal{A})$. Finally, for any \mathcal{A} we say that E is *H-unital* if its bar construction $B_\infty E$ is acyclic. Note that the bar construction of \mathcal{A} considered as a right module over itself coincides with its non-augmented bar construction as an A_∞ -category. Thus \mathcal{A} is an *H-unital* A_∞ -category if and only if \mathcal{A} is an *H-unital* \mathcal{A} -module.

Let $(\mathbf{Nod}_\infty\mathcal{A})_{hu}$ be the full subcategory of $\mathbf{Nod}_\infty\mathcal{A}$ consisting of *H-unital* modules. We define the *derived category* $D(\mathcal{A})$ of \mathcal{A} to be $H^0((\mathbf{Nod}_\infty\mathcal{A})_{hu})$. When \mathcal{A} is a DG-category, we denote by $D_\infty(\mathcal{A})$ the derived category of right A_∞ \mathcal{A} -modules as defined above. This is to distinguish it from the derived category $D(\mathcal{A})$ of right DG \mathcal{A} -modules as defined in §2.1.

If \mathcal{A} is *H-unital*, then for any $E \in \mathbf{Nod}_\infty\mathcal{A}$ the following conditions are equivalent:

- (1) E is homotopic to a semi-free module.
- (2) E lies in the smallest full subcategory of $H^0(\mathbf{Nod}_\infty\mathcal{A})$ which is triangulated, cocomplete, closed under isomorphisms, and contains the representable modules.
- (3) E is *H-unital*, that is — its bar-construction $B_\infty E$ is acyclic.

The equivalence of (1) and (2) is straightforward, while that of (2) and (3) is due to [LH03, Prop. 4.1.2.10].

If \mathcal{A} is strictly unital, by [LH03, Prop. 4.1.3.7] the equivalent conditions above are further equivalent to:

- (4) E is *homologically unital*, that is — $H^*(E)$ is a unital graded $H^*(\mathcal{A})$ -module.

Thus, for a strictly unital \mathcal{A} we have a chain of inclusions

$$\mathbf{Mod}_\infty\mathcal{A} \hookrightarrow (\mathbf{Nod}_\infty\mathcal{A})_u \hookrightarrow (\mathbf{Nod}_\infty\mathcal{A})_{hu} \hookrightarrow \mathbf{Nod}_\infty\mathcal{A}$$

where $(\mathbf{Nod}_\infty\mathcal{A})_u$ is the full subcategory consisting of strictly unital modules, and $\mathbf{Mod}_\infty\mathcal{A}$ is its non-full subcategory of strictly unital modules and strictly unital morphisms between them. The first two inclusions are quasi-equivalences, and thus $\mathbf{Mod}_\infty\mathcal{A}$ and $(\mathbf{Nod}_\infty\mathcal{A})_u$ are alternative DG-enhancements of $D(\mathcal{A})$.

The derived categories of A_∞ -bimodules are defined similarly and similar considerations apply.

2.9. Tensor and Hom functors for bimodules. Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be A_∞ -categories. Let $M \in \mathbf{Nod}_\infty\mathcal{A}\text{-}\mathcal{B}$, and $N \in \mathbf{Nod}_\infty\mathcal{B}\text{-}\mathcal{C}$. We define the A_∞ -tensor product $M \overset{\infty}{\otimes}_{\mathcal{B}} N$ to be the A_∞ $\mathcal{A}\text{-}\mathcal{C}$ -bimodule whose bar construction is the (shifted) cotensor product of DG-comodules

$$B_\infty M \otimes_{B_\infty\mathcal{B}} B_\infty N[-1]. \quad (2.45)$$

Explicitly, the underlying graded $k_{\mathcal{A}}\text{-}k_{\mathcal{C}}$ -bimodule is

$$M \otimes_k (B_\infty\mathcal{B}) \otimes_k N \quad (2.46)$$

and its A_∞ $\mathcal{A}\text{-}\mathcal{C}$ -bimodule structure consists of the differential

$$m_{0,0} = -d_{B_\infty M} \otimes \text{Id} - \text{Id} \otimes d_{B_\infty N} + \text{Id} \otimes d_{B_\infty\mathcal{B}} \otimes \text{Id}$$

and of commuting \mathcal{A} and \mathcal{C} actions induced from those on M and N respectively:

$$m_{p,r}((a_1 \otimes \cdots \otimes a_p) \otimes m \otimes (b_1 \otimes \cdots \otimes b_q) \otimes n \otimes (c_1 \otimes \cdots \otimes c_r))$$

equals

$$\begin{cases} 0 & \text{if } p, r \neq 0 \\ \bigoplus_{i=0}^q (-1)^? m_{p,i}^M (a_1 \otimes \cdots \otimes a_p \otimes m \otimes b_1 \otimes \cdots \otimes b_i) \otimes (b_{i+1} \otimes \cdots \otimes b_q) \otimes n & \text{if } p \neq 0, r = 0 \\ \bigoplus_{i=0}^q (-1)^? m \otimes (b_1 \otimes \cdots \otimes b_i) \otimes m_{q-i,r}^N (b_{i+1} \otimes \cdots \otimes b_q \otimes n \otimes c_1 \otimes \cdots \otimes c_r) & \text{if } p = 0, r \neq 0 \end{cases} \quad (2.47)$$

with the signs dictated by (2.45).

Let now $M \xrightarrow{f} M'$ be a morphism in $\mathbf{Nod}_\infty\mathcal{A}\text{-}\mathcal{B}$ and $N \xrightarrow{g} N'$ to be a morphism in $\mathbf{Nod}_\infty\mathcal{B}\text{-}\mathcal{C}$. Define the morphism

$$M \overset{\infty}{\otimes}_{\mathcal{B}} N \xrightarrow{f \otimes g} M' \overset{\infty}{\otimes}_{\mathcal{B}} N'$$

to be the morphism in $\mathbf{Nod}_\infty\mathcal{A}\text{-}\mathcal{C}$ which corresponds to the DG-bicomodule morphism

$$B_\infty M \otimes_{B_\infty\mathcal{B}} B_\infty N \xrightarrow{f \otimes g} B_\infty M' \otimes_{B_\infty\mathcal{B}} B_\infty N'.$$

Explicitly, $f \otimes g$ sends

$$(a_1 \otimes \cdots \otimes a_p) \otimes m \otimes (b_1 \otimes \cdots \otimes b_q) \otimes n \otimes (c_1 \otimes \cdots \otimes c_r)$$

to

$$\bigoplus_{0 \leq i \leq j \leq q} (-1)^? f_{p,i}(a_1 \otimes \cdots \otimes a_p \otimes m \otimes b_1 \otimes \cdots \otimes b_i) \otimes (b_{i+1} \otimes \cdots \otimes b_j) \otimes g_{q-j,r}(b_{j+1} \otimes \cdots \otimes b_q \otimes n \otimes c_1 \otimes \cdots \otimes c_r).$$

We thus obtain a DG-functor:

$$(-) \overset{\infty}{\otimes}_{\mathcal{B}} (-): \mathbf{Nod}_{\infty} \mathcal{A}\text{-}\mathcal{B} \otimes_k \mathbf{Nod}_{\infty} \mathcal{B}\text{-}\mathcal{C} \longrightarrow \mathbf{Nod}_{\infty} \mathcal{A}\text{-}\mathcal{C}. \quad (2.48)$$

Note that $f \otimes \text{Id}$ is \mathcal{C} -strict: $(f \otimes \text{Id})_{i,j} = 0$ if $j > 0$. It follows that for any $M \in \mathbf{Nod}_{\infty} \mathcal{A}\text{-}\mathcal{B}$ the functor

$$(-) \overset{\infty}{\otimes}_{\mathcal{A}} M: \mathbf{Nod}_{\infty} \mathcal{A} \longrightarrow \mathbf{Nod}_{\infty} \mathcal{B}$$

filters through the non-full subcategory $\mathbf{Nod}_{\infty}^{\text{strict}} \mathcal{B} \subset \mathbf{Nod}_{\infty} \mathcal{B}$ consisting of all \mathcal{B} -modules and strict A_{∞} -morphisms between them.

If the category \mathcal{B} is a DG-category, then the above defined A_{∞} -tensor product over \mathcal{B} is different from the usual DG-tensor product over \mathcal{B} and we need to differentiate the two notions:

Definition 2.15. Let \mathcal{A} and \mathcal{C} be A_{∞} -categories and let \mathcal{B} be a DG-category. Let $M \in \mathbf{Nod}_{\infty} \mathcal{A}\text{-}\mathcal{B}$ and $N \in \mathbf{Nod}_{\infty} \mathcal{B}\text{-}\mathcal{C}$ be such that their partial bar-constructions $B_{\infty}^{\mathcal{A}} M$ and $B_{\infty}^{\mathcal{C}} N$ are DG-modules over \mathcal{B} . In other words, $m_{i,j}^M = 0$ if $j \geq 2$ and $m_{i,j}^N = 0$ if $i \geq 2$.

The *DG-tensor product* $M \otimes_{\mathcal{B}} N$ is the A_{∞} $\mathcal{A}\text{-}\mathcal{C}$ -bimodule which corresponds to the free DG $B_{\infty} \mathcal{A}\text{-}B_{\infty} \mathcal{C}$ bicomodule obtained as the DG-tensor product of the partial bar constructions of M and N :

$$B_{\infty}^{\mathcal{A}} M \otimes_{\mathcal{B}} B_{\infty}^{\mathcal{C}} N. \quad (2.49)$$

Explicitly, the underlying DG $k_{\mathcal{A}}\text{-}k_{\mathcal{C}}$ -bimodule is $M \otimes_{\mathcal{B}} N$ and the commuting \mathcal{A} and \mathcal{C} A_{∞} -actions given by

$$m_{p,r}^{M \otimes_{\mathcal{B}} N}(a_1, \dots, a_p, m \otimes n, c_1, \dots, c_r) = \begin{cases} 0 & p, r \neq 0 \\ (-1)^? m \otimes m_{0,r}^N(n, c_1, \dots, c_r) & p = 0 \\ (-1)^? m_{p,0}^M(a_1, \dots, a_p, m) \otimes n & r = 0 \end{cases}$$

with the signs dictated by (2.49).

In particular, for any A_{∞} -categories \mathcal{A} , \mathcal{B} , and \mathcal{C} and any $M \in \mathbf{Nod}_{\infty} \mathcal{A}\text{-}\mathcal{B}$, and $N \in \mathbf{Nod}_{\infty} \mathcal{B}\text{-}\mathcal{C}$ denote by $M \otimes_k N$ the above construction applied to M and N considered as $\mathcal{A}\text{-}k$ and $k\text{-}\mathcal{C}$ bimodules, respectively. In other words, we simply forget the \mathcal{B} -module structure on M and N , tensor them as DG $k_{\mathcal{B}}$ modules, and then define the commuting \mathcal{A} and \mathcal{C} A_{∞} -actions on the result.

A particularly useful application of this construction is to tensor with the diagonal bimodule. Let \mathcal{A} be an A_{∞} -category and N be a DG $k_{\mathcal{A}}$ -module. The DG-tensor product $N \otimes_k \mathcal{A}$ can be considered as the \mathcal{A} -module generated by N over \mathcal{A} . Explicitly, it has $N \otimes_k \mathcal{A}$ as the underlying DG k -module and for each $p \geq 2$ we have

$$m_p^{N \otimes_k \mathcal{A}}(n \otimes a, a_1, \dots, a_{p-1}) = (-1)^? n \otimes m_p(a, a_1, \dots, a_{p-1}) \quad n \otimes a \in N \otimes_k \mathcal{A}, a_i \in \mathcal{A}$$

with appropriate signs.

Lemma 2.16. *Let \mathcal{A} be an A_{∞} -category and let N be a DG $k_{\mathcal{A}}$ -module. The \mathcal{A} -module $N \otimes_k \mathcal{A}$ admits a filtration of length two whose quotients are free modules. In particular, $N \otimes_k \mathcal{A}$ is semi-free.*

Proof. Suppose first that N is a graded $k_{\mathcal{A}}$ -module considered as a DG-module with zero differential. Then $N \otimes_k \mathcal{A}$ is a free \mathcal{A} -module, as it is isomorphic to

$$\bigoplus_{a \in \mathcal{A}, i \in \mathbb{Z}} (N_a)_i \otimes_k {}_a \mathcal{A}[-i].$$

On the other hand, if N is a DG-module bounded from above, then $N \otimes_k \mathcal{A}$ is semi-free as it admits a filtration whose quotients are $N_i \otimes_k \mathcal{A}$. In particular, if all N_i vanish for $i \notin [a, b]$ for some $a, b \in \mathbb{Z}$, then $N \otimes_k \mathcal{A}$ admits a filtration of length $b - a + 1$ whose factors are free.

Finally, since k is a field, we can (non-canonically) decompose DG $k_{\mathcal{A}}$ -module N as a direct sum of its graded cohomology module $H^*(N)$ and acyclic DG-modules $\text{Im } d_i \rightarrow \text{Im } d_i$ concentrated in degrees i and $i + 1$. Therefore, the \mathcal{A} -module $N \otimes_k \mathcal{A}$ splits into a direct sum of a free module and the modules which each admit a filtration of length 2 whose quotients are free. The desired assertion follows. \square

Now let $L \in \mathbf{Nod}_\infty \mathcal{D}\text{-}\mathcal{B}$, and $M \in \mathbf{Nod}_\infty \mathcal{A}\text{-}\mathcal{B}$. We define the A_∞ -Hom bimodule $\mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(L, M)$ as follows. The underlying graded $k_{\mathcal{A}}\text{-}k_{\mathcal{D}}$ -bimodule is

$$\mathop{\mathrm{Hom}}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} L, B_\infty^{\mathcal{B}} M). \quad (2.50)$$

It has a natural structure of a DG-bimodule over DG-categories $\mathop{\mathrm{End}}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} M)$ and $\mathop{\mathrm{End}}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} L)$. Using the \mathcal{A} - and \mathcal{D} -action functors we restrict this to an A_∞ $\mathcal{A}\text{-}\mathcal{D}$ -bimodule structure, cf. [Kel01, §6.2].

Explicitly, this bimodule structure consists of the standard differential

$$m_{0,0}(\alpha) = d_{B_\infty^{\mathcal{B}} F} \circ \alpha - (-1)^{|\alpha|} \alpha \circ d_{B_\infty^{\mathcal{B}} E} \quad (2.51)$$

and of commuting \mathcal{A} and \mathcal{D} actions: for any $\alpha \in \mathop{\mathrm{Hom}}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} E, B_\infty^{\mathcal{B}} F)$ we have

$$m_{p,r}((a_1 \otimes \cdots \otimes a_p) \otimes \alpha \otimes (d_1 \otimes \cdots \otimes d_r)) = \begin{cases} 0 & \text{if } p, r \neq 0 \\ (-1)^? \mathop{\mathrm{act}}_M(a_1 \otimes \cdots \otimes a_p) \circ \alpha & \text{if } r = 0 \\ (-1)^? \alpha \circ \mathop{\mathrm{act}}_M(d_1 \otimes \cdots \otimes d_r) & \text{if } p = 0. \end{cases}$$

where the signs are dictated by the definition of the restriction functor in [Kel01, §6.2].

Let now further $N \in \mathbf{Nod}_\infty \mathcal{C}\text{-}\mathcal{B}$. It can be readily checked that the *composition map*

$$B_\infty(\mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(M, N)) \otimes_{B_\infty \mathcal{A}} B_\infty(\mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(L, M)) \xrightarrow{\mathop{\mathrm{cmps}}} B_\infty(\mathop{\mathrm{Hom}}_{B_\infty \mathcal{B}}^\infty(L, N)) \quad (2.52)$$

defined by

$$\begin{aligned} & B_\infty \mathcal{C} \otimes_k \mathop{\mathrm{Hom}}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} M, B_\infty^{\mathcal{B}} N) \otimes_k B_\infty \mathcal{A} \otimes_k \mathop{\mathrm{Hom}}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} L, B_\infty^{\mathcal{B}} M) \otimes_k B_\infty \mathcal{D} \xrightarrow{\mathop{\mathrm{Id}}^{\otimes 2} \otimes \tau \otimes \mathop{\mathrm{Id}}^{\otimes 2}} \\ \rightarrow & B_\infty \mathcal{C} \otimes_k \mathop{\mathrm{Hom}}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} M, B_\infty^{\mathcal{B}} N) \otimes_k \mathop{\mathrm{Hom}}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} L, B_\infty^{\mathcal{B}} M) \otimes_k B_\infty \mathcal{D} \xrightarrow{\mathop{\mathrm{Id}} \otimes \mathop{\mathrm{cmps}} \otimes \mathop{\mathrm{Id}}} \\ \rightarrow & B_\infty \mathcal{C} \otimes_k \mathop{\mathrm{Hom}}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} L, B_\infty^{\mathcal{B}} N) \otimes_k B_\infty \mathcal{D} \end{aligned}$$

commutes with the differentials. It defines therefore in $\mathbf{Nod}_\infty \mathcal{C}\text{-}\mathcal{D}$ the *composition map*

$$\mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(M, N) \otimes_{\mathcal{A}}^\infty \mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(L, M) \xrightarrow{\mathop{\mathrm{cmps}}} \mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(L, N). \quad (2.53)$$

Let now $L' \xrightarrow{f} L$ and $M \xrightarrow{g} M'$ be morphisms in $\mathbf{Nod}_\infty \mathcal{D}\text{-}\mathcal{B}$ and $\mathbf{Nod}_\infty \mathcal{A}\text{-}\mathcal{B}$, respectively. Define the morphism

$$\mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(L, M) \xrightarrow{g \circ (-) \circ f} \mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(L', M')$$

in $\mathbf{Nod}_\infty \mathcal{A}\text{-}\mathcal{D}$ by the DG bicomodule morphism

$$\begin{aligned} & B_\infty(\mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(L, M)) \simeq B_\infty \mathcal{A} \otimes_{B_\infty \mathcal{A}} B_\infty(\mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(L, M)) \otimes_{B_\infty \mathcal{D}} B_\infty \mathcal{D} \xrightarrow{g_{\mathcal{A}} \otimes \mathop{\mathrm{Id}} \otimes f_{\mathcal{D}}} \\ \rightarrow & B_\infty(\mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(M, M')) \otimes_{B_\infty \mathcal{A}} B_\infty(\mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(L, M)) \otimes_{B_\infty \mathcal{D}} B_\infty(\mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(L', L)) \xrightarrow{\mathop{\mathrm{cmps}}} B_\infty(\mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(L', M')). \end{aligned}$$

Explicitly, for any $\alpha \in \mathop{\mathrm{Hom}}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} L, B_\infty^{\mathcal{B}} M)$ the map $(f \circ - \circ g)_{p,r}$ sends

$$(a_1 \otimes \cdots \otimes a_p) \otimes \alpha \otimes (d_1 \otimes \cdots \otimes d_r)$$

to the map

$$(-1)^? (g_{p,\bullet} (a_1 \otimes \cdots \otimes a_p \otimes -) \otimes \mathop{\mathrm{Id}}) \circ \Delta \circ \alpha \circ (f_{r,\bullet} (d_1 \otimes \cdots \otimes d_r \otimes -) \otimes \mathop{\mathrm{Id}}) \circ \Delta$$

in $\mathop{\mathrm{Hom}}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} L', B_\infty^{\mathcal{B}} M')$. Here Δ denotes, as usual, the comodule comultiplications.

We thus obtain a DG-functor

$$\mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(-, -): (\mathbf{Nod}_\infty \mathcal{C}\text{-}\mathcal{B})^{\mathop{\mathrm{opp}}} \otimes \mathbf{Nod}_\infty \mathcal{A}\text{-}\mathcal{B} \rightarrow \mathbf{Nod}_\infty \mathcal{A}\text{-}\mathcal{C} \quad (2.54)$$

and, similar to the above, for any $M \in \mathbf{Nod}_\infty \mathcal{A}\text{-}\mathcal{B}$ the functor

$$\mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(M, -): \mathbf{Nod}_\infty \mathcal{B} \rightarrow \mathbf{Nod}_\infty \mathcal{A}$$

filters through $\mathbf{Nod}_\infty^{\mathop{\mathrm{strict}}} \mathcal{A} \subset \mathbf{Nod}_\infty \mathcal{A}$.

We then have the usual Tensor-Hom adjunction: for every $M \in \mathbf{Nod}_\infty \mathcal{A}\text{-}\mathcal{B}$ the functors

$$(-) \otimes_{\mathcal{A}}^\infty M: \mathbf{Nod}_\infty \mathcal{C}\text{-}\mathcal{A} \rightarrow \mathbf{Nod}_\infty \mathcal{C}\text{-}\mathcal{B} \quad (2.55)$$

$$\mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(M, -): \mathbf{Nod}_\infty \mathcal{C}\text{-}\mathcal{B} \rightarrow \mathbf{Nod}_\infty \mathcal{C}\text{-}\mathcal{A} \quad (2.56)$$

are left and right adjoint to each other, respectively. Same holds for the functors $M \otimes_{\mathcal{B}}^\infty (-)$ and $\mathop{\mathrm{Hom}}_{\mathcal{A}^{\mathop{\mathrm{opp}}}}^\infty(M, -)$.

Let $M \in \mathbf{Nod}_\infty \mathcal{A}\text{-}\mathcal{B}$. The DG $k_{\mathcal{A}}\text{-}k_{\mathcal{A}}$ -bimodule underlying $\mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(M, M)$ has an algebra structure given by composition, and thus defines a DG-category with the same object set as \mathcal{A} . By definition, this DG-category can be naturally identified with the DG-category $\mathrm{End}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} M)$. On the other hand, it can be identified with the image of the functor $\mathcal{A} \rightarrow \mathbf{Nod}_\infty \mathcal{B}$ defined by M . Indeed, the assignment $a \mapsto {}_a M$ gives a fully faithful inclusion $\mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(M, M) \hookrightarrow \mathbf{Nod}_\infty \mathcal{B}$, and the functor $\mathcal{A} \rightarrow \mathbf{Nod}_\infty \mathcal{B}$ decomposes as

$$\mathcal{A} \xrightarrow{\mathrm{act}_M} \mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(M, M) \hookrightarrow \mathbf{Nod}_\infty \mathcal{B}. \quad (2.57)$$

Here we write act_M for the composition $\mathcal{A} \xrightarrow{\mathrm{act}_M} \mathrm{End}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} M) \simeq \mathop{\mathrm{Hom}}_{\mathcal{B}}^\infty(M, M)$.

2.10. A functorial semi-free resolution for $(\mathbf{Nod}_\infty \mathcal{A})_{hu}$. To our knowledge, the material presented in this section is original to this paper.

Let \mathcal{A} be an A_∞ -category and let $E \in \mathbf{Nod}_\infty \mathcal{A}$. Consider the A_∞ \mathcal{A} -module $E \otimes_{\mathcal{A}}^\infty \mathcal{A}$. The corresponding DG $B_\infty \mathcal{A}$ -comodule is

$$B_\infty E \otimes_{B_\infty \mathcal{A}} B_\infty^{\mathrm{bim}} \mathcal{A}[-1]. \quad (2.58)$$

As a graded $B_\infty \mathcal{A}$ -comodule (2.58) is isomorphic to

$$E \otimes_k B_\infty \mathcal{A} \otimes_k \mathcal{A}[1] \otimes_k B_\infty \mathcal{A}$$

which decomposes as

$$\bigoplus_{i \geq 0} E \otimes_k (\mathcal{A}[1])^{\otimes i} \otimes_k \mathcal{A}[1] \otimes_k B_\infty \mathcal{A}.$$

Observe that the component of the differential of (2.58) which goes from its i -th summand to its j -th summand is zero if $j > i$. It follows that this differential decomposes into:

- (1) For each $i \geq 0$ a degree 1 square zero $B_\infty \mathcal{A}$ -coderivation

$$E \otimes_k (\mathcal{A}[1])^{\otimes i} \otimes_k \mathcal{A}[1] \otimes_k B_\infty \mathcal{A} \longrightarrow E \otimes_k (\mathcal{A}[1])^{\otimes i} \otimes_k \mathcal{A}[1] \otimes_k B_\infty \mathcal{A} \quad (2.59)$$

- (2) For each $i > j \geq 0$ a degree 1 graded $B_\infty \mathcal{A}$ -comodule morphism

$$E \otimes_k (\mathcal{A}[1])^{\otimes i} \otimes_k \mathcal{A}[1] \otimes_k B_\infty \mathcal{A} \longrightarrow E \otimes_k (\mathcal{A}[1])^{\otimes j} \otimes_k \mathcal{A}[1] \otimes_k B_\infty \mathcal{A}. \quad (2.60)$$

For any $i > 0$ write $E \otimes_k \mathcal{A}^{\otimes i}$ for the A_∞ \mathcal{A} -module

$$(E \otimes_k \mathcal{A}^{\otimes(i-1)}) \otimes_k \mathcal{A}$$

in the sense of Definition 2.15. The corresponding DG $B_\infty \mathcal{A}$ -comodule is the graded $B_\infty \mathcal{A}$ -comodule

$$E \otimes_k \mathcal{A}^{\otimes(i-1)} \otimes_k \mathcal{A}[1] \otimes_k B_\infty \mathcal{A}$$

whose differential is (the shift of) the coderivation (2.59).

Definition 2.17. For any $i > 0$ define a degree $1 - i$ morphism

$$E \otimes_k \mathcal{A}^{\otimes i} \longrightarrow E \quad (2.61)$$

in $\mathbf{Nod}_\infty \mathcal{A}$ by the graded $k_{\mathcal{A}}$ -module maps

$$E \otimes_k \mathcal{A}^{\otimes i} \otimes_k \mathcal{A}^{\otimes n} \xrightarrow{m_{n+i+1}} E.$$

For any i and j with $i > j > 0$ define a degree $i - j + 1$ morphism

$$E \otimes_k \mathcal{A}^{\otimes i} \longrightarrow E \otimes_k \mathcal{A}^{\otimes j} \quad (2.62)$$

in $\mathbf{Nod}_\infty \mathcal{A}$ by the (shift of the) graded $B_\infty \mathcal{A}$ -comodule morphism (2.60).

Explicitly, (2.62) is defined by the maps

$$f_{n+1}: E \otimes_k \mathcal{A}^{\otimes(i-1)} \otimes_k \mathcal{A} \otimes_k \mathcal{A}^{\otimes n} \longrightarrow E \otimes_k \mathcal{A}^{\otimes(j-1)} \otimes_k \mathcal{A}$$

where

$$f_1 = \sum_{\substack{r+1+s=j+1 \\ r \geq 0, s \geq 0}} (-1)^r \mathrm{Id}^{\otimes r} \otimes m_{i+1-r-s} \otimes \mathrm{Id}^{\otimes s}.$$

and for any $n \geq 1$

$$f_{n+1} = (-1)^j \mathrm{Id}^{\otimes j} \otimes m_{i-j+1+n}.$$

Lemma 2.18. *Let \mathcal{A} be an A_∞ -category. The functor*

$$(-)_{\otimes_{\mathcal{A}}^{\infty}} \mathcal{A}: \mathbf{Nod}_{\infty} \mathcal{A} \rightarrow \mathbf{Nod}_{\infty} \mathcal{A}$$

filters through the full subcategory $\mathcal{SF}^{\text{strict}}(\mathcal{A}) \subset \mathbf{Nod}_{\infty} \mathcal{A}$ consisting of semi-free modules and strict A_∞ -morphisms between them.

Proof. As explained in §2.9 for any $M \in \mathbf{Nod}_{\infty} \mathcal{A} \text{-} \mathcal{B}$ the functor

$$(-)_{\otimes_{\mathcal{A}}^{\infty}} M: \mathbf{Nod}_{\infty} \mathcal{A} \longrightarrow \mathbf{Nod}_{\infty} \mathcal{B}$$

filters through $\mathbf{Nod}_{\infty}^{\text{strict}} \mathcal{B}$. It remains to show that for any $E \in \mathbf{Nod}_{\infty} \mathcal{A}$ the module $E_{\otimes_{\mathcal{A}}^{\infty}} \mathcal{A}$ is semi-free.

Recall the decomposition of the differential on the DG comodule corresponding to $E_{\otimes_{\mathcal{A}}^{\infty}} \mathcal{A}$ discussed prior to and employed in Definition 2.17. It follows tautologically that $E_{\otimes_{\mathcal{A}}^{\infty}} \mathcal{A}$ is isomorphic to the convolution of the twisted complex

$$\begin{array}{c} \cdots \longrightarrow E \otimes_k \mathcal{A}^{\otimes 3} \longrightarrow E \otimes_k \mathcal{A}^{\otimes 2} \longrightarrow E \otimes_k \mathcal{A} \\ \text{deg.0} \end{array} \quad (2.63)$$

whose differentials are the A_∞ -morphisms (2.62).

Thus it suffices to show that the convolution of (2.63) is semi-free. As the twisted complex (2.63) is bounded from above and one-sided, its convolution admits an exhaustive filtration whose quotients are (the shifts of) its objects, the modules $E \otimes_k \mathcal{A}^{\otimes i}$. On the other hand, since k is a field, by Lemma 2.16 each of the modules $E \otimes_k \mathcal{A}^{\otimes i}$ in (2.63) admits a filtration of length 2 whose quotients are free modules. We thus obtain an exhaustive filtration on the convolution of $E \otimes_k \mathcal{A}^{\otimes i}$ whose quotients are free modules, as desired. \square

Corollary 2.19. *Let \mathcal{A} be a strictly unital A_∞ -category (resp. a DG-category). The functor*

$$(-)_{\otimes_{\mathcal{A}}^{\infty}} \mathcal{A}: \mathbf{Nod}_{\infty} \mathcal{A} \rightarrow \mathbf{Nod}_{\infty} \mathcal{A}$$

filters through the full subcategory of $\mathbf{Nod}_{\infty}^{\text{strict}} \mathcal{A}$ consisting of strictly unital (resp. DG) modules.

NB: When \mathcal{A} is a DG-category, strict A_∞ -morphisms between DG-modules are simply the DG-morphisms, so the subcategory of $\mathbf{Nod}_{\infty}^{\text{strict}} \mathcal{A}$ consisting of DG-modules is canonically isomorphic to the usual DG-category $\mathbf{Mod}\text{-}\mathcal{A}$ of DG-modules over \mathcal{A} .

For any $E \in \mathbf{Nod}_{\infty} \mathcal{A}$ there is a map of twisted complexes from (2.63) to E concentrated in degree 0 whose individual components are the maps $E \otimes_k \mathcal{A}^{\otimes k} \rightarrow E$ defined in (2.61):

$$\begin{array}{c} \cdots \longrightarrow E \otimes_k \mathcal{A}^{\otimes 3} \longrightarrow E \otimes_k \mathcal{A}^{\otimes 2} \longrightarrow E \otimes_k \mathcal{A} \\ \text{deg.0} \\ \downarrow \\ E \\ \text{deg.0} \end{array} \quad (2.64)$$

It can be readily checked that this map is closed of degree 0. As per the proof of Lemma 2.18, the convolution of the top complex is isomorphic to $E_{\otimes_{\mathcal{A}}^{\infty}} \mathcal{A}$. We can therefore define:

Definition 2.20. Let \mathcal{A} be an A_∞ -category. Define a natural transformation

$$(-)_{\otimes_{\mathcal{A}}^{\infty}} \mathcal{A} \longrightarrow \text{Id}. \quad (2.65)$$

by setting for each $E \in \mathbf{Nod}_{\infty} \mathcal{A}$ the corresponding morphism $E_{\otimes_{\mathcal{A}}^{\infty}} \mathcal{A} \rightarrow E$ to be the convolution of (2.64).

This was defined in different terms in [LH03, Lemme 4.1.1.6] for strictly unital modules.

Proposition 2.21. *For any $E \in \mathbf{Nod}_{\infty} \mathcal{A}$ the morphism $E_{\otimes_{\mathcal{A}}^{\infty}} \mathcal{A} \xrightarrow{(2.65)} E$ is a quasi-isomorphism if and only if $B_\infty E$ is acyclic.*

Proof. The morphism $E_{\otimes_{\mathcal{A}}^{\infty}} \mathcal{A} \xrightarrow{(2.65)} E$ is induced by the twisted complex morphism (2.64). It can be readily checked that the convolution of the total complex of (2.64) is an A_∞ \mathcal{A} -module whose underlying DG $k_{\mathcal{A}}$ -module is the same as that of $B_\infty E$. The claim now follows. \square

Recall, as discussed in §2.8, the full subcategory $(\mathbf{Nod}_\infty \mathcal{A})_{hu} \subset \mathbf{Nod}_\infty \mathcal{A}$ consisting of H -unital modules. These are, equivalently, the modules whose bar construction is acyclic and the modules homotopic to semifree modules. We therefore obtain:

Corollary 2.22. *The natural transformation (2.65) is a functorial semi-free resolution for $(\mathbf{Nod}_\infty \mathcal{A})_{hu}$. If, moreover, \mathcal{A} is strictly unital (resp. DG), then this resolution is also a strictly unital (resp. DG) resolution.*

We note that Prop. 2.21 generalises and simplifies the proofs of several results in [LH03], *Chapitre 4*, e.g. the proof that every module whose bar construction is acyclic is homologically unital.

2.11. The bar complex. In this section, we give an account of the *bar complex*, the notion which lies at the technical heart of this paper. It is obtained from the bar construction on a DG category \mathcal{A} , but the key point is that the resulting object is considered in the monoidal category $(\mathcal{A}\text{-Mod-}\mathcal{A}, \otimes_{\mathcal{A}}, \mathcal{A})$ of \mathcal{A} - \mathcal{A} -bimodules, as opposed to the monoidal category $(k\text{-Mod-}k, \otimes_k, k)$ of DG k - k -bimodules. To this extent, we provide below an alternative construction which works purely in terms of the former monoidal category and is an instance of a more general notion of a *twisted tensor algebra*.

Let \mathcal{A} be a DG category. As per §2.6, the bar construction on \mathcal{A} is the graded k - k -bimodule $\bigoplus_{i \geq 1} \mathcal{A}^i[i]$ with the structure of a (non-unital) coalgebra in the monoidal category $(k\text{-Mod-}k, \otimes_k, k)$ of DG k - k -bimodules. This structure consists of a differential and a comultiplication. The differential, together with the natural left and right actions of \mathcal{A} by composition, makes $\bigoplus_{i \geq 1} \mathcal{A}^i[i]$ into a DG \mathcal{A} - \mathcal{A} -bimodule. The comultiplication map lifts to define on this bimodule a non-unital coalgebra structure in the monoidal category $(\mathcal{A}\text{-Mod-}\mathcal{A}, \otimes_{\mathcal{A}}, \mathcal{A})$.

Our first point of interest is its shift by one to the right. The resulting DG \mathcal{A} - \mathcal{A} -bimodule has the following natural description in the language of the twisted complexes:

Definition 2.23. Let \mathcal{A} be a DG category. Define the *extended bar complex* $\tilde{\mathcal{A}} \in \mathcal{A}\text{-Mod-}\mathcal{A}$ to be the convolution of the following twisted complex of \mathcal{A} - \mathcal{A} bimodules

$$\dots \longrightarrow \mathcal{A} \otimes_k \mathcal{A} \otimes_k \mathcal{A} \xrightarrow{-\text{Id} \otimes (2.1) + (2.1) \otimes \text{Id}} \mathcal{A} \otimes_k \mathcal{A} \xrightarrow{-(2.1)} \mathcal{A}_{\text{deg},0} \quad (2.66)$$

whose differentials $\mathcal{A}^{\otimes(n+1)} \rightarrow \mathcal{A}^{\otimes n}$ are given by

$$\sum_{i=0}^{n-1} (-1)^{i+1} \text{Id}^{\otimes(i)} \otimes (2.1) \otimes \text{Id}^{\otimes(n-i-1)}$$

and all the higher differentials are zero.

The \mathcal{A} - \mathcal{A} -bimodule $\tilde{\mathcal{A}}$ is well-known to be acyclic, since as a k - k DG-bimodule it admits a contracting homotopy of degree -1 whose components are the maps $\mathcal{A}^{\otimes n} \rightarrow \mathcal{A}^{\otimes(n+1)}$ defined by

$$a_1 \otimes \dots \otimes a_n \mapsto 1 \otimes a_1 \otimes \dots \otimes a_n.$$

Thus the twisted complex (2.66) yields a resolution of the diagonal bimodule \mathcal{A} by what is known as the *bar complex*:

Definition 2.24. Let \mathcal{A} be a DG category. The *bar complex* $\bar{\mathcal{A}} \in \mathcal{A}\text{-Mod-}\mathcal{A}$ is the convolution of the twisted complex of free \mathcal{A} - \mathcal{A} bimodules

$$\dots \longrightarrow \mathcal{A} \otimes_k \mathcal{A} \otimes_k \mathcal{A} \xrightarrow{\text{Id} \otimes (2.1) - (2.1) \otimes \text{Id}} \mathcal{A} \otimes_k \mathcal{A}_{\text{deg},0} \quad (2.67)$$

whose differentials $\mathcal{A}^{\otimes(n+1)} \rightarrow \mathcal{A}^{\otimes n}$ are given by

$$\sum_{i=0}^{n-1} (-1)^i \text{Id}^{\otimes(i)} \otimes (2.1) \otimes \text{Id}^{\otimes(n-i-1)} \quad (2.68)$$

and all the higher differentials are zero.

Explicitly, the underlying graded \mathcal{A} - \mathcal{A} -bimodule of $\bar{\mathcal{A}}$ is $\bigoplus_{n \geq 2} \mathcal{A}^{\otimes n}[n-2]$ and its differential $d_{\bar{\mathcal{A}}}$ sends any

$$a_1 \otimes \dots \otimes a_n \in \mathcal{A}^{\otimes n}[n-2]$$

to the sum of

$$\sum_{i=1}^n (-1)^{n+\sum_{j=1}^{i-1} \text{deg}(a_j)} a_1 \otimes \dots \otimes da_i \otimes \dots \otimes a_n, \quad (2.69)$$

which comes from the natural differential on $\mathcal{A}^{\otimes n}$ and

$$\sum_{i=1}^n (-1)^{i-1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, \quad (2.70)$$

which comes from the differential in the twisted complex (2.67).

Since the complex (2.67) is bounded from above and each of its terms is free, $\bar{\mathcal{A}}$ is semi-free. It also comes with a canonical projection to \mathcal{A} :

Definition 2.25. Define the canonical projection

$$\tau: \bar{\mathcal{A}} \rightarrow \mathcal{A} \quad (2.71)$$

to be the convolution of the following map of the twisted complexes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{A}^{\otimes(n+1)} & \longrightarrow & \mathcal{A}^{\otimes n} & \longrightarrow & \cdots & \longrightarrow & \mathcal{A} \otimes_k \mathcal{A} \otimes_k \mathcal{A} & \longrightarrow & \mathcal{A} \otimes_k \mathcal{A} \\ & & & & & & & & & & \text{deg.0} \\ & & & & & & & & & & \downarrow (2.1) \\ & & & & & & & & & & \mathcal{A} \\ & & & & & & & & & & \text{deg.0} \end{array} \quad (2.72)$$

Explicitly, it is the map

$$a_1 \otimes \cdots \otimes a_n \mapsto \begin{cases} a_1 a_2 & n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

By definition of τ , the convolution of

$$\bar{\mathcal{A}} \xrightarrow{\tau} \mathcal{A}_{\text{deg.0}} \quad (2.73)$$

is equal to the convolution of the total complex of (2.72), that is — to $\tilde{\mathcal{A}}$. Since the latter is acyclic, τ is a quasi-isomorphism, and thus $\bar{\mathcal{A}}$ is a canonical semi-free resolution of the diagonal bimodule \mathcal{A} .

The extended bar complex $\tilde{\mathcal{A}}$ admits a structure of an algebra in the monoidal category $(\mathcal{A}\text{-Mod-}\mathcal{A}, \otimes_{\mathcal{A}}, \mathcal{A})$. It comes from a general construction which we now describe. This construction itself is an instance of the cobar construction on a curved A_{∞} -coalgebra, cf. [Pos11, §7.4]. However, it is a degenerate case where the comultiplication and the higher operations are all zero, leaving only the cocurvature and the differential. It is worth it therefore to give a direct definition:

Definition 2.26. Let \mathcal{A} be a DG category. Let $H \in \mathcal{A}\text{-Mod-}\mathcal{A}$ and $\sigma: H \rightarrow \mathcal{A}$ be a closed map of degree 0. The σ -twisted tensor algebra $T_{\sigma}(H)$ of H is the convolution of the twisted complex

$$\cdots \longrightarrow H \otimes_{\mathcal{A}} H \xrightarrow{\text{Id} \otimes \sigma - \sigma \otimes \text{Id}} H \xrightarrow{\sigma} \mathcal{A}_{\text{deg.0}} \quad (2.74)$$

whose differentials $H^{\otimes n} \rightarrow H^{\otimes(n-1)}$ are given by

$$\sum_{i=0}^{n-1} (-1)^i \text{Id}^{\otimes(i)} \otimes \sigma \otimes \text{Id}^{\otimes(n-i-1)}$$

and all the higher differentials are zero.

In other words, as a graded $\mathcal{A}\text{-}\mathcal{A}$ bimodule (T_{σ}, m, e) is just the tensor algebra $\oplus_{i \geq 0} H^{\otimes i}[i]$, but the natural differential on the latter is modified using the map σ , whence the word “twisted” in our choice of the name.

Define further

$$e: \mathcal{A} \rightarrow T_{\sigma}(H)$$

to be the canonical inclusion, and the map

$$m: T_{\sigma}(H) \otimes_{\mathcal{A}} T_{\sigma}(H) \mapsto T_{\sigma}(H)$$

by the natural left and right actions of \mathcal{A} on each $H^{\otimes}[n]$ and by the sign-twisting isomorphisms

$$\begin{aligned} H^{\otimes p}[p] \otimes_{\mathcal{A}} H^{\otimes q}[q] &\rightarrow H^{\otimes(p+q)}[p+q] \\ (h_1 \otimes \cdots \otimes h_p) \otimes (h_{p+1} \otimes \cdots \otimes h_{p+q}) &\mapsto (-1)^q \sum_{i=1}^p \text{deg}(h_i) h_1 \otimes \cdots \otimes h_p \otimes h_{p+1} \otimes \cdots \otimes h_{p+q}. \end{aligned}$$

The latter come from the signless associativity isomorphisms of $\otimes_{\mathcal{A}}$ using the sign-twisting identifications $H^{\otimes p}[p] \simeq (H[1])^{\otimes p}$, cf. [LH03, §1.1.1].

Lemma 2.27. *The triple $(T_\sigma(H), m, e)$ is a unital algebra in the monoidal category $(\mathcal{A}\text{-Mod}-\mathcal{A}, \otimes_{\mathcal{A}}, \mathcal{A})$.*

Proof. It is easy to check that it is precisely the unital algebra obtained via the cobar construction from the curved A_∞ -coalgebra structure given on $H[2]$ by the cocurvature σ , the natural differential, and with the comultiplication and all the higher operations being zero. Indeed, the data of a curved A_∞ -coalgebra is the most general way to define the differential on the tensor algebra of a graded module in order to obtain a DG algebra, see [Pos11, §7.4] for further details. \square

The extended bar complex can be viewed as a twisted tensor algebra in the following way:

Definition 2.28. Define the degree 0 map

$$m: \tilde{\mathcal{A}} \otimes_{\mathcal{A}} \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}} \quad (2.75)$$

by

$$(a_1 \otimes \cdots \otimes a_p) \otimes_{\mathcal{A}} (a_{p+1} \otimes \cdots \otimes a_{p+q}) \mapsto (-1)^{(q-1) \sum_{i=1}^p \deg(a_i)} a_1 \otimes \cdots \otimes a_p a_{p+1} \otimes \cdots \otimes a_{p+q}$$

and let

$$e: \mathcal{A} \hookrightarrow \tilde{\mathcal{A}}$$

be the canonical inclusion.

Corollary 2.29. *The triple $(\tilde{\mathcal{A}}, m, e)$ is a unital algebra in the monoidal category $(\mathcal{A}\text{-Mod}-\mathcal{A}, \otimes_{\mathcal{A}}, \mathcal{A})$.*

Proof. Follows from the Lemma 2.27 by setting $H = \mathcal{A} \otimes_k \mathcal{A}$ and σ to be the map $\mathcal{A} \otimes_k \mathcal{A} \xrightarrow{(2.1)} \mathcal{A}$. \square

We now decompose the multiplication map on the extended bar complex $\tilde{\mathcal{A}}$ into components pertaining to $\bar{\mathcal{A}}$ and to \mathcal{A} . The bimodule $\tilde{\mathcal{A}}$ is not just isomorphic but equal to the convolution of the twisted complex $(\bar{\mathcal{A}} \xrightarrow[\text{deg.0}]{\tau} \mathcal{A})$, these are merely two different descriptions of the same differential on $\bigoplus_{i \geq 1} \mathcal{A}^i[i-1]$. Therefore, by the formula for the tensor product of twisted complexes [AL17, Lemma 3.4], the convolution of the twisted complex

$$\bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\begin{pmatrix} \text{Id} \otimes \tau \\ -\tau \otimes \text{Id} \end{pmatrix}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A} \oplus \mathcal{A} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{(-\tau \ -\tau)} \mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} \quad (2.76)$$

is isomorphic to $\tilde{\mathcal{A}} \otimes_{\mathcal{A}} \tilde{\mathcal{A}}$ via a sign-twisting isomorphism

$$(a_1 \otimes \cdots \otimes a_p) \otimes_{\mathcal{A}} (a_{p+1} \otimes \cdots \otimes a_{p+q}) \mapsto \begin{cases} (a_1 \otimes \cdots \otimes a_p) \otimes_{\mathcal{A}} (a_{p+1}) & q = 1, \\ (-1)^{\deg(a_1)} (a_1) \otimes_{\mathcal{A}} (a_{p+1} \otimes \cdots \otimes a_{p+q}) & p = 1, \\ -1^{p + \sum_{i=1}^p \deg(a_i)} (a_1 \otimes \cdots \otimes a_p) \otimes_{\mathcal{A}} (a_{p+1} \otimes \cdots \otimes a_{p+q}) & p, q > 1. \end{cases}$$

The multiplication map (2.75) is a closed, degree zero map. Composing it with the isomorphism above and applying the natural isomorphisms (2.8) and (2.12), we obtain the closed, degree zero map of twisted complexes

$$\begin{array}{ccc} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} & \xrightarrow{\begin{pmatrix} \text{Id} \otimes \tau \\ -\tau \otimes \text{Id} \end{pmatrix}} \bar{\mathcal{A}} \oplus \bar{\mathcal{A}} & \xrightarrow{(-\tau \ -\tau)} \mathcal{A}_{\text{deg.0}} \\ & \searrow \mu & \downarrow (\text{Id} \ \text{Id}) \\ & & \bar{\mathcal{A}} & \xrightarrow{-\tau} \mathcal{A}_{\text{deg.0}} \\ & & & \downarrow \text{Id} \end{array} \quad (2.77)$$

where the map μ is defined as follows. By [AL17, Lemma 3.4(1)] we can identify $\bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}}$ with the convolution of the twisted complex

$$\cdots \rightarrow (\mathcal{A}^{\otimes 3} \otimes_{\mathcal{A}} \mathcal{A}^{\otimes 2} \oplus \mathcal{A}^{\otimes 2} \otimes_{\mathcal{A}} \mathcal{A}^{\otimes 3}) \rightarrow \mathcal{A}^{\otimes 2} \otimes_{\mathcal{A}} \mathcal{A}^{\otimes 2} \quad (2.78)$$

whose degree zero differentials are defined on each $\mathcal{A}^{\otimes p} \otimes_{\mathcal{A}} \mathcal{A}^{\otimes q}$ by

$$\sum (\text{2.68}) \otimes \text{Id} + (-1)^p \text{Id} \otimes (\text{2.68})$$

and whose higher differentials are all zero.

Definition 2.30. Let \mathcal{A} be a DG-category. Define the degree -1 map

$$\mu: \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \rightarrow \bar{\mathcal{A}} \quad (2.79)$$

in $\mathcal{A}\text{-Mod}-\mathcal{A}$ to be the map induced by the degree -1 map from the twisted complex (2.78) to the twisted complex (2.67) whose only components are the degree zero maps $\bigoplus_{n=p+q} \mathcal{A}^{\otimes p} \otimes_{\mathcal{A}} \mathcal{A}^{\otimes q} \rightarrow \mathcal{A}^{n-1}$ given by

$$(a_1 \otimes \cdots \otimes a_p) \otimes_{\mathcal{A}} (a_{p+1} \otimes \cdots \otimes a_{p+q}) \mapsto (-1)^p a_1 \otimes \cdots \otimes a_p a_{p+1} \otimes \cdots \otimes a_{p+q}. \quad (2.80)$$

We note that the identification of $\bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}}$ with the convolution of (2.78) given in [AL17, Lemma 3.4(1)] involves a sign-twisting isomorphism. Consequently, the explicit formula for μ as a map in $\mathcal{A}\text{-Mod-}\mathcal{A}$ is the formula (2.80) with an extra sign twist $q \sum_{i=1}^p \deg(a_i)$.

We have immediately:

Lemma 2.31. *Let \mathcal{A} be a DG-category. Then*

- (1) $d\mu = \tau \otimes \text{Id} - \text{Id} \otimes \tau$,
- (2) $\tau \circ \mu = 0$.

in $\mathcal{A}\text{-Mod-}\mathcal{A}$.

Proof. This follows immediately from (2.77) being a closed, degree zero map of twisted complexes. \square

The bar-complex $\bar{\mathcal{A}}$ has a natural coalgebra structure in $(\mathcal{A}\text{-Mod-}\mathcal{A}, \otimes_{\mathcal{A}}, \mathcal{A})$ which is defined as follows:

Definition 2.32. Define the comultiplication

$$\bar{\mathcal{A}} \xrightarrow{\Delta} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}}, \quad (2.81)$$

to be the map induced by the degree 0 map from the twisted complex (2.67) to the twisted complex (2.78) whose only components are the degree zero maps $\mathcal{A}^{\otimes n} \rightarrow \bigoplus_{n=p+q} \mathcal{A}^{\otimes(p+1)} \otimes_{\mathcal{A}} \mathcal{A}^{\otimes(q+1)}$ given by

$$a_1 \otimes \cdots \otimes a_n \mapsto \sum_{p=1}^{n-1} (a_1 \otimes \cdots \otimes a_p \otimes 1) \otimes_{\mathcal{A}} (1 \otimes a_{p+1} \otimes \cdots \otimes a_{p+q}). \quad (2.82)$$

As explained for the map μ , the explicit formula for Δ as a map in $\mathcal{A}\text{-Mod-}\mathcal{A}$ is the formula (2.82) with the extra sign twist $(-1)^{(n-i+1) \sum_{k=1}^{i-1} \deg(a_k)}$.

Proposition 2.33. *The triple $(\bar{\mathcal{A}}, \Delta, \tau)$ is a unital coalgebra in the monoidal category $(\mathcal{A}\text{-Mod-}\mathcal{A}, \otimes_{\mathcal{A}}, \mathcal{A})$.*

Proof. With the definitions above it is a straightforward verification on the level of twisted complexes over $\mathcal{A}\text{-Mod-}\mathcal{A}$. \square

3. BAR CATEGORY OF MODULES $\overline{\text{Mod-}}\mathcal{A}$

Let X be a scheme of finite type over k . By [BvdB03] the category $D_{qc}(X)$ admits a compact generator. Hence $D_{qc}(X) \simeq D(\mathcal{A})$ for a DG-algebra \mathcal{A} which is the endomorphism algebra of (an h -injective resolution of) such a generator. Similarly, by [Lun10] the category $D(X)$ admits a classical generator and we have $D(X) \simeq D_c(\mathcal{B})$ for the endomorphism DG-algebra \mathcal{B} of such generator. Moreover, the generator can be chosen in such a way that \mathcal{B} is smooth. See [AL17, §4] for a detailed exposition, as well as generalities on DG-enhancements.

This reduces DG-enhancing derived categories of algebraic varieties to DG-enhancing derived categories of DG-modules over DG-algebras or, more generally, DG-categories. Let \mathcal{A} and \mathcal{B} be DG-categories and let

$$D(\mathcal{A}) \xrightarrow{f} D(\mathcal{B})$$

be a DG-enhanceable functor. Recall that f is said to be *continuous* if it commutes with infinite direct sums.

By [Toë07, Theorem 7.2] every DG-enhanceable continuous functor $D(\mathcal{A}) \rightarrow D(\mathcal{B})$ is of the form $(-)\overset{\mathbf{L}}{\otimes} M$ for some bimodule $M \in D(\mathcal{A}\text{-}\mathcal{B})$. It follows that $D(\mathcal{A}\text{-}\mathcal{B})$ can be identified with the triangulated category of DG-enhanceable continuous functors $D(\mathcal{A}) \rightarrow D(\mathcal{B})$. This furthermore identifies the subcategory $D^{\mathcal{B}\text{-Perf}}(\mathcal{A}\text{-}\mathcal{B})$ with the triangulated category of DG-enhanceable functors $D_c(\mathcal{A}) \rightarrow D_c(\mathcal{B})$. This reduces DG-enhancing exact functors between the derived categories of algebraic varieties to DG-enhancing the derived categories of bimodules, cf. [LS16].

Let \mathcal{A} be a DG-category and let $\text{Mod-}\mathcal{A}$ be the DG-category of \mathcal{A} -modules. There are two enhancements commonly used in the literature for $D(\mathcal{A})$: the full subcategory $\mathcal{P}(\mathcal{A})$ of the h -projective modules in $\text{Mod-}\mathcal{A}$, and the Drinfeld quotient $\text{Mod-}\mathcal{A}/\mathcal{Ac}(\mathcal{A})$ where $\mathcal{Ac}(\mathcal{A})$ is the full subcategory of acyclic modules. Neither turned out to be suitable for our purposes. The problem with the Drinfeld quotient is that its morphisms are inconvenient to work with explicitly. The problem with $\mathcal{P}(\mathcal{A})$ manifests itself when working with bimodules. The diagonal bimodule \mathcal{A} , which corresponds to the identity functor $D(\mathcal{A}) \rightarrow D(\mathcal{A})$, is not in general h -projective. Hence every construction involving the identity functor has to be h -projectively resolved by e.g. tensoring with the bar complex. This leads to many formulas becoming vastly more complicated than they should be, cf. [AL17].

We propose a different DG-enhancement framework for the derived categories of DG-categories. We think it is more suitable for identifying the derived categories of DG-bimodules with triangulated categories of

DG-enhanceable functors as described above. Let \mathcal{A} be a DG-category. The proposed enhancement of $D(\mathcal{A})$ admits two different descriptions.

3.1. DG-modules with A_∞ -morphisms between them. The first one is in the language of A_∞ -categories and modules. The enhancement we want is the full subcategory of the DG-category $\mathbf{Nod}_\infty \mathcal{A}$ of A_∞ \mathcal{A} -modules which consists of DG \mathcal{A} -modules. We denote this subcategory by $(\mathbf{Nod}_\infty \mathcal{A})_{dg}$. Note that the subcategory $(\mathbf{Nod}_\infty^{\text{strict}} \mathcal{A})_{dg} \subset \mathbf{Nod}_\infty \mathcal{A}$ which consists of DG \mathcal{A} -modules and strict A_∞ -morphisms between them can be canonically identified with the usual DG-category $\mathbf{Mod}\text{-}\mathcal{A}$ of DG \mathcal{A} -modules. Consider the chain of subcategory inclusions

$$\mathcal{SF}(\mathcal{A}) \hookrightarrow \mathbf{Mod}\text{-}\mathcal{A} \hookrightarrow (\mathbf{Nod}_\infty \mathcal{A})_{dg} \hookrightarrow (\mathbf{Nod}_\infty \mathcal{A})_{hu}. \quad (3.1)$$

In $(\mathbf{Nod}_\infty \mathcal{A})_{hu}$ all quasi-isomorphisms are homotopy equivalences, and thus the functorial resolution $(-)^{\infty}_{\otimes \mathcal{A}} \mathcal{A}$ of $(\mathbf{Nod}_\infty \mathcal{A})_{hu}$ into $\mathcal{SF}(\mathcal{A})$ established in Cor. 2.22 ensures that every full subcategory of $(\mathbf{Nod}_\infty \mathcal{A})_{hu}$ which contains $\mathcal{SF}(\mathcal{A})$ is quasi-equivalent to $\mathcal{SF}(\mathcal{A})$. We thus obtain:

Proposition 3.1. *Let \mathcal{A} be a DG category. The natural inclusions*

$$\mathcal{SF}(\mathcal{A}) \hookrightarrow (\mathbf{Nod}_\infty \mathcal{A})_{dg} \hookrightarrow (\mathbf{Nod}_\infty \mathcal{A})_{hu}$$

are quasi-equivalences. In particular, the induced equivalences

$$D(\mathcal{A}) \simeq H^0((\mathbf{Nod}_\infty \mathcal{A})_{dg}) \simeq D_\infty(\mathcal{A})$$

make $(\mathbf{Nod}_\infty \mathcal{A})_{dg}$ and $(\mathbf{Nod}_\infty \mathcal{A})_{hu}$ into DG-enhancements of $D(\mathcal{A})$.

For any DG-bimodule $M \in \mathcal{A}\text{-Mod}\text{-}\mathcal{B}$ the adjoint functors $(-)^{\infty}_{\otimes \mathcal{A}} M$ and $\mathbf{Hom}_{\mathcal{B}}(M, -)$ restrict from $\mathbf{Nod}_\infty \mathcal{A} \leftrightarrow \mathbf{Nod}_\infty \mathcal{B}$ to $(\mathbf{Nod}_\infty \mathcal{A})_{dg} \leftrightarrow (\mathbf{Nod}_\infty \mathcal{B})_{dg}$. We thus have the usual Tensor-Hom adjunction for the categories $(\mathbf{Nod}_\infty)_{dg}$.

3.2. The category $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$. The second description is a direct one in the language of DG-modules. While less conceptual, it significantly simplifies the computations involved and allows one to avoid having to deal with the sign conventions for A_∞ -categories and modules. It builds on the ideas introduced in [Kel94, §6.6] where it was applied to a set of compact generators of $D(\mathcal{A})$ to obtain a Morita enhancement. We apply it to the whole of $\mathbf{Mod}\text{-}\mathcal{A}$ instead:

Definition 3.2. Let \mathcal{A} be a DG-category. Define the *bar category of modules* $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ as follows:

- The object set of $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ is the same as that of $\mathbf{Mod}\text{-}\mathcal{A}$: DG-modules over \mathcal{A} .
- For any $E, F \in \mathbf{Mod}\text{-}\mathcal{A}$ set

$$\mathbf{Hom}_{\overline{\mathbf{Mod}}\text{-}\mathcal{A}}(E, F) = \mathbf{Hom}_{\mathcal{A}}(E \otimes_{\mathcal{A}} \bar{\mathcal{A}}, F) \quad (3.2)$$

and write $\overline{\mathbf{Hom}}_{\mathcal{A}}(E, F)$ to denote this Hom-complex.

- For any $E \in \mathbf{Mod}\text{-}\mathcal{A}$ set $\text{Id}_E \in \overline{\mathbf{Hom}}_{\mathcal{A}}(E, E)$ to be the element given by

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes \tau} E \otimes_{\mathcal{A}} \mathcal{A} \xrightarrow{\sim} E \quad (3.3)$$

where $\tau: \bar{\mathcal{A}} \rightarrow \mathcal{A}$ is the counit of $\bar{\mathcal{A}}$ as the coalgebra in $\mathcal{A}\text{-Mod}\text{-}\mathcal{A}$, cf. §2.1.

- For any $E, F, G \in \mathbf{Mod}\text{-}\mathcal{A}$ define the composition map

$$\overline{\mathbf{Hom}}_{\mathcal{A}}(F, G) \otimes_k \overline{\mathbf{Hom}}_{\mathcal{A}}(E, F) \longrightarrow \overline{\mathbf{Hom}}_{\mathcal{A}}(E, G) \quad (3.4)$$

by setting for any $E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\alpha} F$ and $F \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\beta} G$ the composition of the corresponding elements to be the element given by

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes \Delta} E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\alpha \otimes \text{Id}} F \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\beta} G. \quad (3.5)$$

where $\Delta: \bar{\mathcal{A}} \rightarrow \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}}$ is the comultiplication of $\bar{\mathcal{A}}$ as the coalgebra in $\mathcal{A}\text{-Mod}\text{-}\mathcal{A}$, cf. §2.1.

Let \mathcal{A} and \mathcal{B} be DG-categories. We define the bimodule category $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ similarly, but with

$$\overline{\mathbf{Hom}}_{\mathcal{A}\text{-}\mathcal{B}}(M, N) = \mathbf{Hom}_{\mathcal{A}\text{-}\mathcal{B}}(\bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}}, N) \quad \forall M, N \in \mathcal{A}\text{-Mod}\text{-}\mathcal{B}, \quad (3.6)$$

with $\text{Id}_M \in \overline{\mathbf{Hom}}_{\mathcal{A}\text{-}\mathcal{B}}(M, M)$ being the element given by

$$\bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\tau \otimes \text{Id} \otimes \tau} M,$$

and with the composition of the $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ morphisms $M \rightarrow N$ and $L \rightarrow M$ corresponding to

$$\beta: \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \rightarrow N \quad \text{and} \quad \alpha: \bar{\mathcal{A}} \otimes_{\mathcal{A}} L \otimes_{\mathcal{B}} \bar{\mathcal{B}} \rightarrow M$$

being the $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ morphism $L \rightarrow N$ corresponding to

$$\bar{\mathcal{A}} \otimes_{\mathcal{A}} L \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\Delta \otimes \text{Id} \otimes \Delta} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} L \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\text{Id} \otimes \alpha \otimes \text{Id}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\beta} N.$$

Proposition 3.3. *Let \mathcal{A} be a DG-category. We have a (non-full) inclusion*

$$\Upsilon: \mathbf{Mod}\text{-}\mathcal{A} \hookrightarrow \overline{\mathbf{Mod}}\text{-}\mathcal{A} \quad (3.7)$$

which is the identity on objects.

Proof. Define Υ to be the identity on objects and for any $E, F \in \mathbf{Mod}\text{-}\mathcal{A}$ define

$$\Upsilon_{E,F}: \text{Hom}_{\mathcal{A}}(E, F) \rightarrow \text{Hom}_{\overline{\mathcal{A}}}(E, F)$$

to be the map which sends any $\alpha \in \text{Hom}_{\mathcal{A}}(E, F)$ to the morphism in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ defined by

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\alpha \otimes \tau} F \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\sim} F.$$

The map $\Upsilon_{E,F}$ is injective as it can be rewritten as the pre-composition of α with the map (3.3) which is surjective. This also shows that it sends Id in $\mathbf{Mod}\text{-}\mathcal{A}$ to Id in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$. It remains to check that $\Upsilon_{E,F}$ is compatible with compositions. Let $G \in \mathbf{Mod}\text{-}\mathcal{A}$ and let $\beta \in \text{Hom}_{\mathcal{A}}(F, G)$. By definition, the composition $\Upsilon_{F,G}(\beta) \circ \Upsilon_{E,F}(\alpha)$ in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ is the element of $\text{Hom}_{\overline{\mathcal{A}}}(E, G)$ defined by

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes \Delta} E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\alpha \otimes \tau \otimes \text{Id}} F \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\beta \otimes \tau} G$$

where we suppress the isomorphisms $(-)\otimes_{\mathcal{A}} \bar{\mathcal{A}} \simeq \text{Id}_{\bar{\mathcal{A}}}$. By functoriality of tensor product, this simplifies to

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{(\beta \circ \alpha) \otimes ((\tau \otimes \tau) \circ \Delta)} G.$$

Since $(\tau \otimes \tau) \circ \Delta = \tau$ the above equals $\Upsilon_{E,G}(\beta \circ \alpha)$, as desired. \square

The inclusion Υ is a special case of a more general identification which relates this section to §3.1:

Definition 3.4. Let \mathcal{A} be a DG-category. Define a DG-functor

$$\Psi: \overline{\mathbf{Mod}}\text{-}\mathcal{A} \rightarrow (\mathbf{Nod}_{\infty} \mathcal{A})_{dg} \quad (3.8)$$

by setting it to be the identity on the objects, and for any $E, F \in \overline{\mathbf{Mod}}\text{-}\mathcal{A}$ setting

$$\Psi(-): \text{Hom}_{\overline{\mathcal{A}}}(E, F) \rightarrow \text{Hom}_{\mathcal{A}}(E, F) \quad (3.9)$$

to be the following map. By definition $\text{Hom}_{\overline{\mathcal{A}}}(E, F) = \text{Hom}_{\mathcal{A}}(E \otimes_{\mathcal{A}} \bar{\mathcal{A}}, F)$, and the module $E \otimes_{\mathcal{A}} \bar{\mathcal{A}}$ is isomorphic to the convolution of the twisted complex

$$\dots \longrightarrow E \otimes_k \mathcal{A}^{\otimes 3} \longrightarrow E \otimes_k \mathcal{A}^{\otimes 2} \longrightarrow E \otimes_k \mathcal{A} \quad (3.10)$$

deg.0

with the degree 0 differentials

$$E \otimes_{\mathcal{A}} \mathcal{A}^{\otimes n+1} \xrightarrow{\sum_{i=0}^n (-1)^i \text{Id}^{\otimes i} \otimes m_2 \otimes \text{Id}^{\otimes (n-i)}} E \otimes_{\mathcal{A}} \mathcal{A}^{\otimes n}$$

where m_2 denotes either the composition map $\mathcal{A} \otimes_k \mathcal{A} \rightarrow \mathcal{A}$ or the action map $E \otimes_k \mathcal{A} \rightarrow E$, as appropriate. Since $\mathbf{Mod}\text{-}\mathcal{A}$ is strongly pre-triangulated, the DG complex $\text{Hom}_{\mathcal{A}}(E \otimes_{\mathcal{A}} \bar{\mathcal{A}}, F)$ is isomorphic to the DG complex of twisted complex morphisms

$$\begin{array}{ccccccc} \dots & \longrightarrow & E \otimes_k \mathcal{A}^{\otimes 3} & \longrightarrow & E \otimes_k \mathcal{A}^{\otimes 2} & \longrightarrow & E \otimes_k \mathcal{A} \\ & & & & & & \text{deg.0} \\ & & & & & & \downarrow \\ & & & & & & \alpha_0 \\ & & & & & & \downarrow \\ & & & & & & F. \end{array} \quad (3.11)$$

$\dots \searrow \alpha_2 \quad \searrow \alpha_1$

The degree i part of this DG complex comprises all $\{\alpha_n\}_{n \geq 0}$ with $\alpha_n \in \text{Hom}_{\mathcal{A}}^{i-n}(E \otimes_k \mathcal{A}^{\otimes (n+1)}, F)$. Since each $E \otimes_k \mathcal{A}^{\otimes (n+1)}$ is a free \mathcal{A} -module generated by the $k_{\mathcal{A}}$ -module $E \otimes_k \mathcal{A}^{\otimes n}$, such collections $\{\alpha_n\}_{n \geq 0}$ are in bijection with $\{\alpha'_n\}_{n \geq 0}$ with $\alpha'_n \in \text{Hom}_{k_{\mathcal{A}}}^{i-n}(E \otimes_k \mathcal{A}^{\otimes n}, F)$, and hence with the elements of $\text{Hom}_{\mathcal{A}}(E, F)$.

Thus, define the action (3.9) of Ψ on morphism complexes to be the composition of the bijective map which sends the elements of $\text{Hom}_{\overline{\mathcal{A}}}(E, F)$ to the twisted complex morphisms (3.11) with the bijective map which sends the latter to the elements of $\text{Hom}_{\mathcal{A}}(E, F)$. The resulting map clearly respects the degrees, and checking that it commutes with the differentials is a straightforward verification, comparing the definition of the differential for morphisms of twisted complexes given in e.g. [AL17, §3.1] or [BK90, §1] with that of the differential for \mathcal{A}_{∞} -module morphisms given in e.g. [LH03, §5.2].

Proposition 3.5. *Let \mathcal{A} be a DG-category. The DG-functor Ψ of Definition 3.4 is an isomorphism*

$$\overline{\mathbf{Mod}}\text{-}\mathcal{A} \simeq (\mathbf{Nod}_\infty \mathcal{A})_{dg}$$

of DG-categories which identifies $\Upsilon: \mathbf{Mod}\text{-}\mathcal{A} \hookrightarrow \overline{\mathbf{Mod}}\text{-}\mathcal{A}$ with $\mathbf{Mod}\text{-}\mathcal{A} = (\mathbf{Nod}_\infty \mathcal{A})_{dg}^{strict} \hookrightarrow (\mathbf{Nod}_\infty \mathcal{A})_{dg}$.

Proof. By construction, Ψ is identity on objects and bijective on morphism complexes. Hence it is an isomorphism of DG categories. For the last assertion, let $\alpha \in \text{Hom}_\mathcal{A}(E, F)$. The corresponding element of $\overline{\text{Hom}}_\mathcal{A}(E, F)$ is the composition of $E \otimes_\mathcal{A} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes \tau} E$ with $E \xrightarrow{\alpha} F$. On the level of twisted complexes, the former map consists of a single component $E \otimes_k \mathcal{A} \xrightarrow{\text{act}} E$. The composition consists therefore of a single component $E \otimes_k \mathcal{A} \xrightarrow{\alpha \circ \text{act}} F$. The corresponding $k_\mathcal{A}$ -module morphism is $E \xrightarrow{\alpha} F$. We conclude that the resulting collection of $k_\mathcal{A}$ -module morphisms $\{E \otimes_k \mathcal{A}^{\otimes n} \rightarrow F\}_{n \geq 0}$ consists of a single non-zero component: $E \xrightarrow{\alpha} F$. This defines the strict A_∞ -morphism $E \rightarrow F$ corresponding to α , as required. \square

Corollary 3.6. *Let \mathcal{A} be a DG category. There is a canonical category isomorphism*

$$\Theta: D(\mathcal{A}) \xrightarrow{\sim} H^0(\overline{\mathbf{Mod}}\text{-}\mathcal{A}) \quad (3.12)$$

giving $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ the structure of a DG-enhancement of $D(\mathcal{A})$.

Proof. In $H^0(\mathbf{Nod}_\infty \mathcal{A})_{dg}$ every acyclic module is isomorphic to zero. By Prop. 3.5 it is also true of $H^0(\overline{\mathbf{Mod}}\text{-}\mathcal{A})$. As $D(\mathcal{A}) = H^0(\mathbf{Mod}\text{-}\mathcal{A})/H^0(\mathcal{A}c\mathcal{A})$ the universal property of Verdier quotient ensures that the inclusion

$$H^0(\Upsilon): H^0(\mathbf{Mod}\text{-}\mathcal{A}) \rightarrow H^0(\overline{\mathbf{Mod}}\text{-}\mathcal{A})$$

factors uniquely into the canonical projection $H^0(\mathbf{Mod}\text{-}\mathcal{A}) \rightarrow D(\mathcal{A})$ and the functor we define to be Θ :

$$H^0(\mathbf{Mod}\text{-}\mathcal{A}) \rightarrow D(\mathcal{A}) \xrightarrow{\Theta} H^0(\overline{\mathbf{Mod}}\text{-}\mathcal{A}). \quad (3.13)$$

To see that Θ is an isomorphism of categories, precompose $H^0(\Upsilon)$ with $H^0(\mathcal{P}(\mathcal{A})) \hookrightarrow H^0(\mathbf{Mod}\text{-}\mathcal{A})$, where $\mathcal{P}(\mathcal{A}) \subset \mathbf{Mod}\text{-}\mathcal{A}$ is the full subcategory of h -projective modules. On the morphism complexes Υ is the precomposition with the quasi-isomorphism $E \otimes_\mathcal{A} \bar{\mathcal{A}} \rightarrow E$ defined in (3.3). In both $H^0(\mathcal{P}(\mathcal{A}))$ and $H^0(\overline{\mathbf{Mod}}\text{-}\mathcal{A})$ quasi-isomorphisms become isomorphisms. Thus the resulting functor $H^0(\mathcal{P}(\mathcal{A})) \rightarrow H^0(\overline{\mathbf{Mod}}\text{-}\mathcal{A})$ is fully faithful. It is furthermore an equivalence, since any module E is isomorphic in $H^0(\overline{\mathbf{Mod}}\text{-}\mathcal{A})$ to the h -projective module $E \otimes_\mathcal{A} \bar{\mathcal{A}}$. Since the composition

$$H^0(\mathcal{P}(\mathcal{A})) \hookrightarrow H^0(\mathbf{Mod}\text{-}\mathcal{A}) \rightarrow D(\mathcal{A})$$

is well-known to be an equivalence, we conclude that Θ is also one. \square

Lemma 3.7. *Let \mathcal{A} be a DG-category and let $E \xrightarrow{\alpha} F$ be the $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ morphism defined by a $\mathbf{Mod}\text{-}\mathcal{A}$ morphism $E \otimes_\mathcal{A} \bar{\mathcal{A}} \xrightarrow{\alpha'} F$. The category isomorphism Θ of Cor. 3.6 identifies α with the $D(\mathcal{A})$ morphism*

$$E \xrightarrow{\text{Id} \otimes \tau^{-1}} E \otimes_\mathcal{A} \bar{\mathcal{A}} \xrightarrow{\alpha'} F. \quad (3.14)$$

Here τ^{-1} is the formal inverse of the quasi-isomorphism $\bar{\mathcal{A}} \xrightarrow{\tau} \mathcal{A}$.

Proof. It suffices to show that $\Theta(\alpha')$ is the composition $\alpha \circ \Theta(\text{Id} \otimes \tau)$ in $H^0(\overline{\mathbf{Mod}}\text{-}\mathcal{A})$. As $\Theta(\alpha') = H^0(\Upsilon)(\alpha')$, we conclude it is the morphism defined by the $\mathbf{Mod}\text{-}\mathcal{A}$ morphism

$$E \otimes_\mathcal{A} \bar{\mathcal{A}} \otimes_\mathcal{A} \bar{\mathcal{A}} \xrightarrow{\alpha' \circ (\text{Id} \otimes \tau)} F.$$

On the other hand, the image of $E \otimes_\mathcal{A} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes \tau} E$ in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ is defined by the $\mathbf{Mod}\text{-}\mathcal{A}$ morphism

$$E \otimes_\mathcal{A} \bar{\mathcal{A}} \otimes_\mathcal{A} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes \tau \otimes \tau} E.$$

Its $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ composition with α is therefore defined by the $\mathbf{Mod}\text{-}\mathcal{A}$ morphism

$$E \otimes_\mathcal{A} \bar{\mathcal{A}} \otimes_\mathcal{A} \bar{\mathcal{A}} \xrightarrow{\alpha' \circ (\text{Id} \otimes \tau \otimes \text{Id})} F.$$

The claim now follows, since the map $\bar{\mathcal{A}} \otimes_\mathcal{A} \bar{\mathcal{A}} \xrightarrow{\tau \otimes \text{Id} - \text{Id} \otimes \tau} \bar{\mathcal{A}}$ is null-homotopic. One choice for the contracting homotopy can be found in Lemma 2.31. \square

Corollary 3.8. *Let \mathcal{A} be a DG-category. A morphism $E \rightarrow F$ in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ is a homotopy equivalence if and only if the corresponding $\mathbf{Mod}\text{-}\mathcal{A}$ morphism $E \otimes_\mathcal{A} \bar{\mathcal{A}} \rightarrow F$ is a quasi-isomorphism.*

We next furnish the categories $\overline{\mathbf{Mod}}$ with adjoint bifunctors which are enhancements of the derived bifunctors $(-)^{\mathbf{L}} \otimes_{\mathcal{B}} (-)$ and $\mathbf{R} \text{Hom}_{\mathcal{B}}(-, -)$.

Definition 3.9. Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be DG-categories. Define the functor

$$\overline{\otimes}_{\mathcal{B}}: \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \otimes_k \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C} \rightarrow \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C} \quad (3.15)$$

by setting

$$M \overline{\otimes}_{\mathcal{B}} N = M \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} N \quad \forall M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}, N \in \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}.$$

Furthermore, for any $\alpha \in \overline{\mathbf{Hom}}_{\mathcal{A}\text{-}\mathcal{B}}(M, M')$ and $\beta \in \overline{\mathbf{Hom}}_{\mathcal{B}\text{-}\mathcal{C}}(N, N')$ define

$$M \overline{\otimes}_{\mathcal{B}} N \xrightarrow{\alpha \overline{\otimes} \beta} M' \overline{\otimes}_{\mathcal{B}} N'$$

to be the morphism corresponding to

$$\overline{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} N \otimes_{\mathcal{C}} \overline{\mathcal{C}} \xrightarrow{\text{Id}^{\otimes 2} \otimes \Delta^2 \otimes \text{Id}^{\otimes 2}} \overline{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} N \otimes_{\mathcal{C}} \overline{\mathcal{C}} \xrightarrow{\alpha \otimes \text{Id} \otimes \beta} M' \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} N'.$$

Explicitly, we have

$$a \otimes m \otimes b \otimes n \otimes c \mapsto \sum (-1)^{\deg(\beta) \deg(a \otimes m \otimes b_{(1)} \otimes b_{(2)})} \alpha(a \otimes m \otimes b_{(1)}) \otimes b_{(2)} \otimes \beta(b_{(3)} \otimes n \otimes c).$$

Here and below we use Sweedler's notation for comultiplications and coactions: the sum above runs over all the summands $b_{(1)} \otimes b_{(2)} \otimes b_{(3)}$ of $\Delta^2(b)$. More generally, given a coalgebra element denoted by e.g. letter b a sum involving expressions $b_{(1)}, \dots, b_{(k)}$ means that the sum is taken over all summands of $\Delta^k(b)$ with each $b_{(i)}$ denoting the i -th factor of each summand.

Definition 3.10. Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be DG-categories. Define the functor

$$\overline{\mathbf{Hom}}_{\mathcal{B}}(-, -): \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \otimes_k (\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B})^{\text{opp}} \rightarrow \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C} \quad (3.16)$$

by setting

$$\overline{\mathbf{Hom}}_{\mathcal{B}}(M, N) = \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \overline{\mathcal{B}}, N) \quad \forall M \in \mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}, N \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}.$$

Furthermore, for any $\alpha \in \overline{\mathbf{Hom}}_{\mathcal{C}\text{-}\mathcal{B}}(M', M)$ and $\beta \in \overline{\mathbf{Hom}}_{\mathcal{A}\text{-}\mathcal{B}}(N, N')$ define

$$\overline{\mathbf{Hom}}_{\mathcal{B}}(M, N) \xrightarrow{\beta \circ (-) \circ \alpha} \overline{\mathbf{Hom}}_{\mathcal{B}}(M', N')$$

by the $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$ map

$$\begin{aligned} & \overline{\mathcal{A}} \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \overline{\mathcal{B}}, N) \otimes_{\mathcal{C}} \overline{\mathcal{C}} \xrightarrow{\beta_{\mathcal{A}} \otimes \text{Id} \otimes \alpha_{\mathcal{C}}} \\ & \rightarrow \text{Hom}_{\mathcal{B}}(N \otimes_{\mathcal{B}} \overline{\mathcal{B}}, N') \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \overline{\mathcal{B}}, N) \otimes_{\mathcal{C}} \text{Hom}_{\mathcal{B}}(M' \otimes_{\mathcal{B}} \overline{\mathcal{B}}, M) \xrightarrow{(3.4)} \\ & \rightarrow \text{Hom}_{\mathcal{B}}(M' \otimes_{\mathcal{B}} \overline{\mathcal{B}}, N'). \end{aligned} \quad (3.17)$$

Here $\overline{\mathcal{A}} \xrightarrow{\beta_{\mathcal{A}}} \text{Hom}(N \otimes_{\mathcal{B}} \overline{\mathcal{B}}, N')$ and $\overline{\mathcal{C}} \xrightarrow{\alpha_{\mathcal{C}}} \text{Hom}(M' \otimes_{\mathcal{B}} \overline{\mathcal{B}}, M)$ are the right adjoints of the $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ and $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ morphisms

$$\begin{aligned} \overline{\mathcal{A}} \otimes_{\mathcal{A}} N \otimes_{\mathcal{B}} \overline{\mathcal{B}} &\rightarrow N', \\ \overline{\mathcal{C}} \otimes_{\mathcal{C}} M' \otimes_{\mathcal{B}} \overline{\mathcal{B}} &\rightarrow M \end{aligned}$$

which correspond to β and α .

Explicitly, the map (3.17) takes any $a \otimes \gamma \otimes c$ to the map

$$m' \otimes b \mapsto \sum (-1)^{\deg(\alpha)(\deg(a) + \deg(\gamma))} \beta(a \otimes \gamma(\alpha(c \otimes m' \otimes b_{(1)} \otimes b_{(2)} \otimes b_{(3)})).$$

We define similarly the functor

$$\overline{\mathbf{Hom}}_{\mathcal{B}^{\text{opp}}}(-, -): \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C} \otimes_k (\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A})^{\text{opp}} \rightarrow \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}. \quad (3.18)$$

Definition 3.11. Let \mathcal{A} , \mathcal{B} , \mathcal{C} , and \mathcal{D} be DG-categories. For any $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, $L \in \mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, and $N \in \mathcal{D}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ define the *composition map* in $\mathcal{D}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$

$$\text{cmps}: \overline{\mathbf{Hom}}_{\mathcal{B}}(M, N) \overline{\otimes}_{\mathcal{A}} \overline{\mathbf{Hom}}_{\mathcal{B}}(L, M) \rightarrow \overline{\mathbf{Hom}}_{\mathcal{B}}(L, N) \quad (3.19)$$

by the corresponding $\mathcal{D}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$ map

$$\begin{aligned} & \overline{\mathcal{D}} \otimes_{\mathcal{D}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \overline{\mathcal{B}}, N) \otimes_{\mathcal{A}} \overline{\mathcal{A}} \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(L \otimes_{\mathcal{B}} \overline{\mathcal{B}}, M) \otimes_{\mathcal{C}} \overline{\mathcal{C}} \xrightarrow{\tau \otimes \text{Id} \otimes \tau \otimes \text{Id} \otimes \tau} \\ & \rightarrow \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \overline{\mathcal{B}}, N) \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(L \otimes_{\mathcal{B}} \overline{\mathcal{B}}, M) \xrightarrow{(3.4)} \text{Hom}_{\mathcal{B}}(L \otimes_{\mathcal{B}} \overline{\mathcal{B}}, N). \end{aligned} \quad (3.20)$$

For any $M \in \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$, $L \in \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$, and $N \in \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{D}$ define similarly $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{D}$ map

$$\text{cmps}: \overline{\mathbf{Hom}}_{\mathcal{B}^{\text{opp}}}(L, M) \overline{\otimes}_{\mathcal{A}} \overline{\mathbf{Hom}}_{\mathcal{B}^{\text{opp}}}(M, N) \xrightarrow{\text{cmps}} \overline{\mathbf{Hom}}_{\mathcal{B}^{\text{opp}}}(L, N). \quad (3.21)$$

When working with bimodules where two different categories act on the left and on the right, there can be no confusion over whether we mean the left or the right action of either category. In such case, we write $\text{Hom}_{\mathcal{A}}$ for $\text{Hom}_{\mathcal{A}^{\text{opp}}}$, for the purposes of brevity.

Proposition 3.12. *Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-Mod-}\mathcal{B}$. The isomorphism Θ of Cor. 3.6 identifies the functors $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M$ and $\mathbf{R}\text{Hom}_{\mathcal{B}}(M, -)$ with the functors $H^0((-)\overline{\otimes}_{\mathcal{A}} M)$ and $H^0(\overline{\text{Hom}}_{\mathcal{B}}(M, -))$. Similarly, Θ identifies $M\overset{\mathbf{L}}{\otimes}_{\mathcal{B}}(-)$ and $\mathbf{R}\text{Hom}_{\mathcal{A}}(M, -)$ with $H^0(M\overline{\otimes}_{\mathcal{B}}(-))$ and $H^0(\overline{\text{Hom}}_{\mathcal{A}}(M, -))$.*

Proof. We only prove the assertion for $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M$ and $(-)\overline{\otimes}_{\mathcal{A}} M$, the others are proved similarly.

For any DG-category \mathcal{C} the following square commutes:

$$\begin{array}{ccc} \mathcal{C}\text{-Mod-}\mathcal{A} & \xrightarrow{(-)\otimes_{\mathcal{A}}\bar{A}\otimes_{\mathcal{A}}M} & \mathcal{C}\text{-Mod-}\mathcal{B} \\ \downarrow \Upsilon & & \downarrow \Upsilon \\ \mathcal{C}\text{-Mod-}\mathcal{A} & \xrightarrow{(-)\overline{\otimes}_{\mathcal{A}}M} & \mathcal{C}\text{-Mod-}\mathcal{B} \end{array}$$

Since $\bar{A}\otimes_{\mathcal{A}} M$ is an \mathcal{A} - h -projective resolution of M , the functor $H^0((-)\otimes_{\mathcal{A}}\bar{A}\otimes_{\mathcal{A}}M)$ descends to the functor $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M: D(\mathcal{C}\text{-}\mathcal{A}) \rightarrow D(\mathcal{C}\text{-}\mathcal{B})$. The factorisation (3.13) then implies that this functor is identified by Θ with the functor $H^0((-)\overline{\otimes}_{\mathcal{A}}M)$. \square

Proposition 3.13. *Let \mathcal{A} and \mathcal{B} be DG-categories. The isomorphism Ψ of Prop. 3.5 identifies the bifunctors $(-)\overline{\otimes}_{\mathcal{B}}(-)$, $\overline{\text{Hom}}_{\mathcal{A}}(-, -)$, and $\overline{\text{Hom}}_{\mathcal{B}}(-, -)$ with the bifunctors $(-)\overset{\infty}{\otimes}_{\mathcal{B}}(-)$, $\overset{\infty}{\text{Hom}}_{\mathcal{B}}(-, -)$, and $\overset{\infty}{\text{Hom}}_{\mathcal{A}}(-, -)$.*

Proof. Straightforward verification. \square

In view of Propositions 3.13 and 3.5 we could deduce the Tensor-Hom adjunction for $\overline{\text{Mod}}$ from the Tensor-Hom adjunction for A_{∞} -modules [LH03, Lemme 4.1.1.4]. However, it is more convenient to prove this adjunction directly in $\overline{\text{Mod}}$ by exhibiting explicit formulas for its unit and counit:

Proposition 3.14. *Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$.*

- (1) *The functors $(-)\overline{\otimes}_{\mathcal{A}}M$ and $\overline{\text{Hom}}_{\mathcal{B}}(M, -)$ are left and right adjoint functors $\mathcal{C}\text{-}\overline{\text{Mod}}\text{-}\mathcal{A} \leftrightarrow \mathcal{C}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$. The unit and the counit of the adjunction are the maps*

$$E \xrightarrow{\text{mlt}} \overline{\text{Hom}}_{\mathcal{B}}(M, E\overline{\otimes}_{\mathcal{A}}M) \quad \forall E \in \mathcal{C}\text{-}\overline{\text{Mod}}\text{-}\mathcal{A}, \quad (3.22)$$

$$\overline{\text{Hom}}_{\mathcal{B}}(M, F)\overline{\otimes}_{\mathcal{A}}M \xrightarrow{\text{ev}} F \quad \forall F \in \mathcal{C}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B} \quad (3.23)$$

in $\mathcal{C}\text{-}\overline{\text{Mod}}\text{-}\mathcal{A}$ and $\mathcal{C}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$ which correspond to the $\mathcal{C}\text{-Mod-}\mathcal{A}$ and $\mathcal{C}\text{-Mod-}\mathcal{B}$ maps

$$\bar{\mathcal{C}} \otimes_{\mathcal{C}} E \otimes_{\mathcal{A}} \bar{A} \xrightarrow{\tau \otimes \text{mlt}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{\mathcal{B}}, E \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}}) \xrightarrow{(\text{Id}^{\otimes 3} \otimes \tau) \circ (-)} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{\mathcal{B}}, E \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M), \quad (3.24)$$

$$\bar{\mathcal{C}} \otimes_{\mathcal{C}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{\mathcal{B}}, F) \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\tau \otimes \text{Id} \otimes \tau \otimes \text{Id}^{\otimes 2}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{\mathcal{B}}, F) \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\text{ev}} F. \quad (3.25)$$

- (2) *The functors $M\overline{\otimes}_{\mathcal{B}}(-)$ and $\overline{\text{Hom}}_{\mathcal{A}}(M, -)$ are left and right adjoint functors $\mathcal{B}\text{-}\overline{\text{Mod}}\text{-}\mathcal{C} \leftrightarrow \mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{C}$. The unit and the counit of the adjunction are given by the maps*

$$E \xrightarrow{\text{mlt}} \overline{\text{Hom}}_{\mathcal{A}}(M, M\overline{\otimes}_{\mathcal{B}}E) \quad \forall E \in \mathcal{B}\text{-}\overline{\text{Mod}}\text{-}\mathcal{C}, \quad (3.26)$$

$$M\overline{\otimes}_{\mathcal{B}}\overline{\text{Hom}}_{\mathcal{A}}(M, F) \xrightarrow{\text{ev}} F \quad \forall F \in \mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{C} \quad (3.27)$$

in $\mathcal{B}\text{-}\overline{\text{Mod}}\text{-}\mathcal{C}$ and $\mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{C}$ which correspond to the in $\mathcal{B}\text{-Mod-}\mathcal{C}$ and $\mathcal{A}\text{-Mod-}\mathcal{C}$ maps

$$\bar{\mathcal{B}} \otimes_{\mathcal{B}} E \otimes_{\mathcal{C}} \bar{\mathcal{C}} \xrightarrow{\text{mlt} \otimes \tau} \text{Hom}_{\mathcal{A}}(\bar{A} \otimes_{\mathcal{A}} M, \bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} E), \xrightarrow{(\tau \otimes \text{Id}^{\otimes 3}) \circ (-)} \text{Hom}_{\mathcal{A}}(\bar{A} \otimes_{\mathcal{A}} M, M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} E), \quad (3.28)$$

$$\bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} \text{Hom}_{\mathcal{A}}(\bar{A} \otimes_{\mathcal{A}} M, F) \otimes_{\mathcal{C}} \bar{\mathcal{C}} \xrightarrow{\text{Id}^{\otimes 2} \otimes \tau \otimes \text{Id} \otimes \tau} \bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \text{Hom}_{\mathcal{A}}(\bar{A} \otimes_{\mathcal{A}} M, F) \xrightarrow{\text{ev}} F. \quad (3.29)$$

Proof. To prove the assertion (1) it suffices to show that for any $E \in \mathcal{C}\text{-}\overline{\text{Mod}}\text{-}\mathcal{A}$ and $F \in \mathcal{C}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$

$$E\overline{\otimes}_{\mathcal{A}}M \xrightarrow{\text{mlt} \otimes \text{Id}} \overline{\text{Hom}}_{\mathcal{B}}(M, E\overline{\otimes}_{\mathcal{A}}M)\overline{\otimes}_{\mathcal{A}}M \xrightarrow{\text{ev}} E\overline{\otimes}_{\mathcal{A}}M \quad (3.30)$$

$$\overline{\text{Hom}}_{\mathcal{B}}(M, F) \xrightarrow{\text{mlt}} \overline{\text{Hom}}_{\mathcal{B}}(M, \overline{\text{Hom}}_{\mathcal{B}}(M, F)\overline{\otimes}_{\mathcal{A}}M) \xrightarrow{\text{ev} \circ (-)} \overline{\text{Hom}}_{\mathcal{B}}(M, F) \quad (3.31)$$

are identity morphisms. We only demonstrate this for (3.30), as (3.31) works out very similarly.

By definition of the composition in $\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, (3.30) corresponds to the $\mathbf{Mod}\text{-}\mathcal{B}$ map

$$\begin{aligned} & \bar{\mathcal{C}} \otimes_{\mathcal{C}} E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\Delta \otimes \text{Id} \otimes \Delta \otimes \text{Id}^{\otimes 2}} \bar{\mathcal{C}} \otimes_{\mathcal{C}} \bar{\mathcal{C}} \otimes_{\mathcal{C}} E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\text{Id} \otimes \tau \otimes \text{mlt} \otimes \text{Id}^{\otimes 3}} \\ & \rightarrow \bar{\mathcal{C}} \otimes_{\mathcal{C}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{\mathcal{B}}, E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}}) \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\text{Id} \otimes ((\tau \otimes \text{Id}^{\otimes 3}) \circ (-)) \otimes \text{Id}^{\otimes 3}} \\ & \rightarrow \bar{\mathcal{C}} \otimes_{\mathcal{C}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{\mathcal{B}}, E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\tau \otimes \text{Id} \otimes \tau \otimes \text{Id}^{\otimes 2}} \\ & \rightarrow \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{\mathcal{B}}, E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\text{ev}} E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M. \end{aligned}$$

By functoriality of the tensor product the above composition equals

$$\begin{aligned} & \bar{\mathcal{C}} \otimes_{\mathcal{C}} E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\Delta \otimes \text{Id} \otimes \Delta \otimes \text{Id}^{\otimes 2}} \bar{\mathcal{C}} \otimes_{\mathcal{C}} \bar{\mathcal{C}} \otimes_{\mathcal{C}} E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\tau \otimes \text{Id}^{\otimes 3} \otimes \tau \otimes \text{Id}^{\otimes 2}} \\ & \rightarrow \bar{\mathcal{C}} \otimes_{\mathcal{C}} E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\tau \otimes \text{mlt} \otimes \text{Id}^{\otimes 2}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{\mathcal{B}}, E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}}) \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{((\text{Id}^{\otimes 3} \otimes \tau) \circ (-)) \otimes \text{Id}^{\otimes 2}} \\ & \rightarrow \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{\mathcal{B}}, E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\text{ev}} E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M. \end{aligned}$$

By Prop. 2.33 the maps $\tau \otimes \text{Id}$ and $\text{Id} \otimes \tau$ are left inverses to Δ , thus the first two maps compose to Id . On the other hand, the last two maps compose to $(\text{Id}^{\otimes 3} \otimes \tau) \circ \text{ev}$. By the Tensor-Hom adjunction for $M \otimes_{\mathcal{B}} \bar{\mathcal{B}}$ the total composition is therefore $\tau \otimes \text{Id}^{\otimes 3} \otimes \tau$. The corresponding map in $\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ is Id , as desired.

The assertion (2) is settled similarly. \square

It is worth writing out the maps defining the units and the counits of these two adjunctions explicitly. The compositions (3.24), (3.25), (3.28), and (3.29) are the maps

$$\begin{aligned} c \otimes e \otimes a & \mapsto (\tau(c)e \otimes a \otimes -) \circ (\text{Id} \otimes \tau) \\ c \otimes \alpha \otimes a \otimes m \otimes b & \mapsto \tau(c)\alpha(\tau(a).m \otimes b) \\ b \otimes e \otimes c & \mapsto \left((-1)^{\deg(-)(\deg(b)+\deg(e))} (-) \otimes b \otimes e\tau(c) \right) \circ (\tau \otimes \text{Id}) \\ a \otimes m \otimes b \otimes \alpha \otimes c & \mapsto (-1)^{\deg(\alpha)(\deg(a)+\deg(m)+\deg(b))} \alpha(a \otimes m.\tau(b))\tau(c). \end{aligned}$$

To sum up, we have a DG-enhancement framework which to every DG-category \mathcal{A} associates an enhancement $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ of its derived category $D(\mathcal{A})$. These enhancements $\overline{\mathbf{Mod}}$ admit genuinely adjoint (in each argument) bifunctors $(-) \otimes_{\bullet} (-)$ and $\text{Hom}_{\bullet}(-, -)$.

3.3. On non-invertibility of the semi-free resolution $\mathcal{A} \overline{\otimes} M \rightarrow M$. Recall the semi-free resolution

$$\mathcal{A} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{(2.65)} M$$

discussed in §2.10. Consider moreover its right adjoint with respect to $\mathcal{A} \overline{\otimes}_{\mathcal{A}} (-)$:

$$M \longrightarrow \text{Hom}_{\mathcal{A}}(\mathcal{A}, M).$$

The category isomorphism Ψ of Prop. 3.5 identifies these with the maps

$$\mathcal{A} \overline{\otimes}_{\mathcal{A}} M \longrightarrow M \tag{3.32}$$

$$M \longrightarrow \text{Hom}_{\mathcal{A}}(\mathcal{A}, M) \tag{3.33}$$

defined by the $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ maps

$$\begin{aligned} & \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\tau \otimes \tau \otimes \text{Id} \otimes \tau} M \\ & \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\tau \otimes \text{Id} \otimes \tau} M \xrightarrow{(2.10)} \text{Hom}_{\mathcal{A}}(\mathcal{A}, M) \xrightarrow{(-) \circ \tau} \text{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, M). \end{aligned}$$

We therefore see that the $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ maps (3.32) and (3.33) are the analogues of the canonical $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ isomorphisms $\mathcal{A} \otimes_{\mathcal{A}} M \xrightarrow{(2.8)} M$ and $M \xrightarrow{(2.10)} \text{Hom}_{\mathcal{A}}(\mathcal{A}, M)$. Indeed, they induce the same isomorphisms $\mathcal{A} \overline{\otimes}_{\mathcal{A}} M \simeq M$ and $M \simeq \mathbf{R}\text{Hom}_{\mathcal{A}}(\mathcal{A}, M)$ in the derived category $D(\mathcal{A}\text{-}\mathcal{B})$ as (2.8) and (2.10).

The biggest drawback of the categories $\overline{\mathbf{Mod}}$ is that the maps (3.32) and (3.33) are not themselves isomorphisms, like (2.8) and (2.10), but merely homotopy equivalences.

In this section, we show that this can be controlled. The maps (3.32) and (3.33) have natural semi-inverses. These are genuine inverses on one side, but only homotopy inverses on the other. However, the arising higher homotopies are induced by endomorphisms of the bar complex and thus independent of M .

To put this into context, recall [Dri04, §3.7], [Tab05], [AL17, Appendix A] that for any DG-category \mathcal{A} and any objects $x, y \in \mathcal{A}$ we can (non-canonically) complete any homotopy equivalence

$$x \xrightarrow{\beta} y \quad (3.34)$$

to the following system of morphisms and relations between them. The dotted arrows denote the morphisms of degree -1 and the dashed arrow the morphism of degree -2 :

$$\begin{aligned} d\theta_x &= \alpha \circ \beta - \text{Id}_x, \\ d\theta_y &= \text{Id}_y - \beta \circ \alpha, \\ d\alpha &= d\beta = 0, \\ d\phi &= -\beta \circ \theta_x - \theta_y \circ \beta. \end{aligned} \quad \begin{array}{c} \phi \\ \text{---} \\ \beta \\ \text{---} \\ \alpha \end{array} \quad \begin{array}{c} \theta_x \\ \text{---} \\ x \\ \text{---} \\ y \\ \text{---} \\ \theta_y \end{array} \quad (3.35)$$

In other words, we can find:

- a homotopy inverse α of β ,
- a degree -1 homotopy θ_x from $\alpha \circ \beta$ to Id_x
- a degree -1 homotopy θ_y from $\beta \circ \alpha$ to Id_y
- a degree -2 homotopy ϕ from $\beta \circ \theta_x$ to $\theta_y \circ \beta$.

The key assertion here is that we can choose θ_x and θ_y so that ϕ exists.

It turns out that in the case of homotopy equivalences (3.32) and (3.33) we can do quite a bit better than (3.35). Firstly, they admit natural one-sided inverses:

Definition 3.15. Let $M \in \mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$. Define the maps

$$M \longrightarrow \mathcal{A} \overline{\otimes}_{\mathcal{A}} M \quad (3.36)$$

$$\overline{\text{Hom}}_{\mathcal{A}}(\mathcal{A}, M) \longrightarrow M \quad (3.37)$$

in $\mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$ by the $\mathcal{A}\text{-Mod-}\mathcal{B}$ maps

$$\begin{aligned} \overline{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \overline{\mathcal{B}} &\xrightarrow{\text{Id} \otimes \tau} \overline{\mathcal{A}} \otimes_{\mathcal{A}} M \\ \overline{\mathcal{A}} \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{A}}(\overline{\mathcal{A}}, M) \otimes_{\mathcal{B}} \overline{\mathcal{B}} &\xrightarrow{\text{Id} \otimes \tau} \overline{\mathcal{A}} \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{A}}(\overline{\mathcal{A}}, M) \xrightarrow{\text{ev}} M. \end{aligned}$$

It can be readily checked that (3.36) is a right inverse to (3.32), while (3.37) is a left inverse to (3.33). We can apply these to give a more natural description of Tensor-Hom adjunction counits, and to show action maps to be instances of Tensor-Hom adjunction units:

Lemma 3.16. Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$.

The composition

$$\overline{\text{Hom}}_{\mathcal{B}}(M, -) \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\text{Id} \overline{\otimes} (3.33)} \overline{\text{Hom}}_{\mathcal{B}}(M, -) \overline{\otimes}_{\mathcal{A}} \overline{\text{Hom}}_{\mathcal{B}}(\mathcal{A}, M) \xrightarrow{\text{cmpr}} \overline{\text{Hom}}_{\mathcal{B}}(\mathcal{A}, -) \xrightarrow{(3.37)} \text{Id}_{\mathcal{C}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}} \quad (3.38)$$

is the counit of the $((-) \overline{\otimes}_{\mathcal{A}} M, \overline{\text{Hom}}_{\mathcal{B}}(M, -))$ adjunction. The counit of the $(M \otimes_{\mathcal{B}} (-), \overline{\text{Hom}}_{\mathcal{A}}(M, -))$ adjunction admits an analogous description.

Proof. Direct verification. □

Lemma 3.17. Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$. The compositions

$$\mathcal{A} \xrightarrow{\text{mlt}} \overline{\text{Hom}}_{\mathcal{B}}(M, \mathcal{A} \overline{\otimes}_{\mathcal{A}} M) \xrightarrow{(3.32)} \overline{\text{Hom}}_{\mathcal{B}}(M, M)$$

$$\mathcal{B} \xrightarrow{\text{mlt}} \overline{\text{Hom}}_{\mathcal{A}}(M, M \overline{\otimes}_{\mathcal{B}} \mathcal{B}) \xrightarrow{(3.32)} \overline{\text{Hom}}_{\mathcal{A}}(M, M)$$

are the maps

$$\mathcal{A} \xrightarrow{\text{act}} \overline{\text{Hom}}_{\mathcal{B}}(M, M) \quad (3.39)$$

$$\mathcal{B} \xrightarrow{\text{act}} \overline{\text{Hom}}_{\mathcal{A}}(M, M) \quad (3.40)$$

in $\mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{A}$ and $\mathcal{B}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$ induced by the corresponding action maps.

Proof. Direct verification. □

In order to define degree -1 and -2 homotopies as per (3.35) we need to introduce certain natural endomorphisms of the bar complex. In Prop. 2.33 we have shown that $\tau \otimes \text{Id}$ and $\text{Id} \otimes \tau$ are left inverses to Δ . Since τ is a quasi-isomorphism, the maps $\tau \otimes \text{Id}$ and $\text{Id} \otimes \tau$ are also quasi-isomorphisms and thus homotopy equivalences. Hence so is Δ , and the following morphism, which composes to zero with Δ , is a boundary:

$$\bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\tau \otimes \text{Id} - \text{Id} \otimes \tau} \bar{\mathcal{A}}. \quad (3.41)$$

In §2.11 we have produced, out of a natural algebra structure on the extended bar complex, a degree -1 map

$$\mu: \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \mapsto \bar{\mathcal{A}}$$

which lifts this boundary, i.e. $d\mu = \tau \otimes \text{Id} - \text{Id} \otimes \tau$, and which satisfies $\tau \circ \mu = 0$. We next look at the compositions of this lift μ with the comultiplication Δ . For this, we need the following definition:

Definition 3.18. Let \mathcal{A} be a DG-category and let $k \in \mathbb{Z}_{\geq 0}$. Define the *insertion of k 1s* map

$$\lambda_k: \bar{\mathcal{A}} \rightarrow \bar{\mathcal{A}} \quad (3.42)$$

by the degree $-k$ map from the twisted complex (2.67) to itself which sends any $a_1 \otimes \cdots \otimes a_n \in \mathcal{A}^{\otimes n}$ to

$$\sum_{i_1+i_2+\cdots+i_{k+1}=n} (-1)^{ki_1+(k-1)i_2+\cdots+1i_k+k} a_1 \otimes \cdots \otimes a_{i_1} \otimes 1 \otimes a_{i_1+1} \otimes \cdots \otimes a_{i_1+i_2} \otimes 1 \otimes \cdots \otimes 1 \otimes a_{i_1+\cdots+i_k+1} \otimes \cdots \otimes a_n.$$

We have established in §2.11 that $\bar{\mathcal{A}}$ has a natural structure of coalgebra. In particular, its comultiplication map Δ is coassociative. We therefore write Δ^k for the unique map $\bar{\mathcal{A}} \rightarrow \bar{\mathcal{A}}^{\otimes(k+1)}$ which is a composition of k applications of Δ .

Proposition 3.19. (1) *The composition $\bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\mu} \bar{\mathcal{A}} \xrightarrow{\Delta} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}}$ equals the sum*

$$(\text{Id} \otimes \mu) \circ (\Delta \otimes \text{Id}) + (\mu \otimes \text{Id}) \circ (\text{Id} \otimes \Delta).$$

(2) *The composition $\bar{\mathcal{A}} \xrightarrow{\Delta} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\mu} \bar{\mathcal{A}}$ is the map λ_1 .*

(3) *For any $k \geq 0$ the map λ_k equals the composition*

$$\bar{\mathcal{A}} \xrightarrow{\Delta^k} \bar{\mathcal{A}}^{\otimes(k+1)} \xrightarrow{\text{Id}^{\otimes(k-1)} \otimes \mu} \bar{\mathcal{A}}^{\otimes k} \xrightarrow{\text{Id}^{\otimes(k-2)} \otimes \mu} \bar{\mathcal{A}}^{\otimes(k-1)} \rightarrow \cdots \rightarrow \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\mu} \bar{\mathcal{A}}. \quad (3.43)$$

(4) *For any $k \geq 0$ the map $(-1)^{\frac{k(k-1)}{2}} \lambda_k$ equals the composition*

$$\bar{\mathcal{A}} \xrightarrow{\Delta^k} \bar{\mathcal{A}}^{\otimes(k+1)} \xrightarrow{\mu \otimes \text{Id}^{\otimes(k-1)}} \bar{\mathcal{A}}^{\otimes k} \xrightarrow{\mu \otimes \text{Id}^{\otimes(k-2)}} \bar{\mathcal{A}}^{\otimes(k-1)} \rightarrow \cdots \rightarrow \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\mu} \bar{\mathcal{A}}. \quad (3.44)$$

(5) *For any $k \geq 1$ the map λ_k equals the compositions $\lambda_k = \mu \circ (\text{Id} \otimes \lambda_{k-1}) \circ \Delta = (-1)^{k+1} \mu \circ (\lambda_{k-1} \otimes \text{Id}) \circ \Delta$.*

(6) *For any $k \geq 1$ the map $d\lambda_k$ equals λ_{k-1} if k is even, and 0 if k is odd.*

Proof. As explained in §2.11 the bimodules $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}}$ can be identified with the convolutions of the twisted complexes (2.67) and (2.78). The maps τ , μ and δ were then defined on the level of these twisted complexes in $\text{Pre-Tr}(\mathcal{A}\text{-Mod-}\mathcal{A})$. We therefore perform all the computations in this proof with these maps in $\text{Pre-Tr}(\mathcal{A}\text{-Mod-}\mathcal{A})$. The reason for doing this, as explained in §2.11, is that the signs in the formulas become significantly simpler on the level of twisted complexes.

(1): For any $(a_1 \otimes \cdots \otimes a_n) \otimes_{\mathcal{A}} (b_1 \otimes \cdots \otimes b_m)$ in the twisted complex (2.78) its image under $\Delta \circ \mu$ is:

$$\begin{aligned} & (-1)^n \sum_{i=1}^{n-1} (a_1 \otimes \cdots \otimes a_i \otimes 1) \otimes (1 \otimes \cdots \otimes a_n b_1 \otimes \cdots \otimes b_m) + \\ & + (-1)^n \sum_{i=1}^{m-1} (a_1 \otimes \cdots \otimes a_n b_1 \otimes \cdots \otimes b_i \otimes 1) \otimes (1 \otimes b_{i+1} \otimes \cdots \otimes b_m). \end{aligned}$$

Its image under $(\text{Id} \otimes \mu) \circ (\Delta \otimes \text{Id})$ is

$$\sum_{i=1}^{n-1} (-1)^{i+1} (-1)^{n-i+1} (a_1 \otimes \cdots \otimes a_i \otimes 1) \otimes (1 \otimes \cdots \otimes a_n b_1 \otimes \cdots \otimes b_m)$$

where the sign $(-1)^{i+1}$ comes from the definition of a tensor product of two maps applied to $\text{Id} \otimes \mu$, while the sign $(-1)^{n-i+1}$ comes from the definition of μ . Finally, its image under $(\mu \otimes \text{Id}) \circ (\text{Id} \otimes \Delta)$ is

$$\sum_{i=1}^{m-1} (-1)^n (a_1 \otimes \cdots \otimes a_n b_1 \otimes \cdots \otimes b_i \otimes 1) \otimes (1 \otimes b_{i+1} \otimes \cdots \otimes b_m).$$

where the sign $(-1)^n$ comes from the definition of μ . The desired result now follows.

(2): For any $a_1 \otimes \cdots \otimes a_n \in \bar{\mathcal{A}}$ we have

$$\begin{aligned} & \mu(\Delta(a_1 \otimes \cdots \otimes a_n)) = \\ &= \mu\left(\sum_{i=1}^{n-1} (a_1 \otimes \cdots \otimes a_i \otimes 1) \otimes_{\mathcal{A}} (1 \otimes a_{i+1} \otimes \cdots \otimes a_n)\right) = \\ &= \sum_{i=1}^{n-1} (-1)^{i+1} a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n = \lambda_1(a_1 \otimes \cdots \otimes a_n). \end{aligned}$$

(3), (4): This follows by a direct computation analogous to the one for (2).

(5): By the assertion (3) we have

$$\begin{aligned} \lambda_k &= \mu \circ (\text{Id} \otimes \mu) \circ \cdots \circ (\text{Id}^{\otimes(k-1)} \otimes \mu) \circ \Delta^k = \\ &= \mu \circ (\text{Id} \otimes \mu) \circ \cdots \circ (\text{Id}^{\otimes(k-1)} \otimes \mu) \circ (\text{Id}^{\otimes(k-1)} \otimes \Delta) \circ \cdots \circ (\text{Id} \otimes \Delta) \circ \Delta = \\ &= \mu \circ \left(\text{Id} \otimes \left(\mu \circ \cdots \circ (\text{Id}^{\otimes(k-2)} \otimes \mu) \circ (\text{Id}^{\otimes(k-2)} \otimes \Delta) \circ \cdots \circ \Delta \right) \right) \circ \Delta = \\ &= \mu \circ (\text{Id} \otimes \lambda_{k-1}) \circ \Delta. \end{aligned}$$

The second part is proven similarly using the assertion (4).

(6): By the assertion (2) we have

$$d\lambda_1 = d(\mu \circ \Delta) = (d\mu) \circ \Delta = (\tau \otimes \text{Id} - \text{Id} \otimes \tau) \circ \Delta = \text{Id} - \text{Id} = 0.$$

Suppose now we have proved our claim for $k = n > 1$. Then by the assertion (5) we have

$$\begin{aligned} d\lambda_{n+1} &= d(\mu \circ (\text{Id} \otimes \lambda_n) \circ \Delta) = d\mu \circ (\text{Id} \otimes \lambda_n) \circ \Delta - \mu \circ (\text{Id} \otimes d\lambda_n) \circ \Delta = \\ &= (\tau \otimes \text{Id} - \text{Id} \otimes \tau) \circ (\text{Id} \otimes (\mu \circ (\text{Id} \otimes \lambda_{n-1}) \circ \Delta)) \circ \Delta - \mu \circ (\text{Id} \otimes d\lambda_n) \circ \Delta = \\ &= (\mu \circ (\text{Id} \otimes \lambda_{n-1}) \circ \Delta) \circ (\tau \otimes \text{Id}) \circ \Delta - (\text{Id} \otimes (\tau \circ \mu \circ (\text{Id} \otimes \lambda_{n-1}) \circ \Delta)) \circ \Delta - \mu \circ (\text{Id} \otimes d\lambda_n) \circ \Delta. \end{aligned}$$

Since $\tau \circ \mu = 0$ by the Lemma 2.31(2) and since $(\tau \otimes \text{Id}) \circ \Delta = \text{Id}$ the above further equals

$$\mu \circ (\text{Id} \otimes \lambda_{n-1}) \circ \Delta - \mu \circ (\text{Id} \otimes d\lambda_n) \circ \Delta.$$

Since this is zero when $d\lambda_n = \lambda_{n-1}$ and λ_n when $d\lambda_n = 0$, the desired assertion for $k = n + 1$ follows. \square

Having established these properties of the maps μ and λ_k , we can now proceed to our main objective:

Definition 3.20. Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. Define the maps

$$M \xrightarrow{\beta_k} \mathcal{A} \overline{\otimes}_{\mathcal{A}} M \tag{3.45}$$

$$\mathcal{A} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\theta} \mathcal{A} \overline{\otimes}_{\mathcal{A}} M \tag{3.46}$$

$$\mathcal{A} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\alpha} M \tag{3.47}$$

of degree $-k$, -1 , and 0 , respectively, in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ by the corresponding $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ maps

$$\begin{aligned} \bar{\mathcal{A}} \otimes_{\mathcal{A}} (M) \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\xrightarrow{\lambda_k \otimes \text{Id} \otimes \tau} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \\ \bar{\mathcal{A}} \otimes_{\mathcal{A}} (\bar{\mathcal{A}} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\xrightarrow{\mu \otimes \text{Id} \otimes \tau} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \\ \bar{\mathcal{A}} \otimes_{\mathcal{A}} (\bar{\mathcal{A}} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\xrightarrow{\tau \otimes \tau \otimes \text{Id} \otimes \tau} M. \end{aligned}$$

NB: The map α is the canonical map (3.32), while β_0 is its left inverse (3.36).

Proposition 3.21. Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. We have:

- (1) $d\theta = \text{Id} - \beta_0 \circ \alpha$.
- (2) $0 = \alpha \circ \beta_0 - \text{Id}$.
- (3) $\beta_k = \theta^k \circ \beta_0$.
- (4) For any $k \geq 1$ we have $\alpha \circ \beta_k = 0$.
- (5) For any $k \geq 1$ the map $d\beta_k$ equals 0 when k is odd, and β_{k-1} when k is even.

Proof. These follow immediately from the properties of the maps μ and λ_k established in Lemma 2.31 and Prop. 3.19. For example, by the definition of the map θ and by Lemma 2.31 the map $d\theta$ corresponds to

$$d(\mu \otimes \text{Id} \otimes \tau) = (d\mu) \otimes \text{Id} \otimes \tau = \tau \otimes \text{Id} \otimes \text{Id} \otimes \tau - \text{Id} \otimes \tau \otimes \text{Id} \otimes \tau.$$

The map

$$\bar{\mathcal{A}} \otimes_{\mathcal{A}} (\bar{\mathcal{A}} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\tau \otimes \text{Id} \otimes \text{Id} \otimes \tau} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M$$

is the map in $\mathcal{A}\text{-Mod-}\mathcal{B}$ which corresponds to the $\mathcal{A}\text{-Mod-}\mathcal{B}$ identity map $\mathcal{A} \bar{\otimes}_{\mathcal{A}} M \xrightarrow{\text{Id}} \mathcal{A} \bar{\otimes}_{\mathcal{A}} M$. On the other hand, the map $\text{Id} \otimes \tau \otimes \text{Id} \otimes \tau$ is readily checked to be the map in $\mathcal{A}\text{-Mod-}\mathcal{B}$ which corresponds to $\beta_0 \circ \alpha$ in $\mathcal{A}\text{-Mod-}\mathcal{B}$. The assertion (1) follows. \square

Proposition 3.22. *Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-Mod-}\mathcal{B}$. Let α , β_0 , and θ be the maps, respectively, introduced in Definition 3.20:*

$$\begin{array}{ccc} & \xrightarrow{\beta_0} & \\ M & \xrightarrow{\quad} & \mathcal{A} \bar{\otimes}_{\mathcal{A}} M \\ & \xleftarrow{\alpha} & \\ & & \xleftarrow{\theta} \end{array} \quad (3.48)$$

The sub-DG-category of $\mathcal{A}\text{-Mod-}\mathcal{B}$ generated by α, β_0 and θ is the free DG-category generated by those elements modulo the following relations:

- (1) $d\alpha = d\beta_0 = 0$,
- (2) $d\theta = \text{Id} - \beta_0 \circ \alpha$,
- (3) $0 = \alpha \circ \beta_0 - \text{Id}$,
- (4) $\alpha \circ \theta = 0$.

NB: The relations in Prop. 3.22 can be obtained by taking those in (3.35) and demanding further that $\theta_x = 0$, $\alpha \circ \theta_y = 0$, and $\phi = -\theta_y^2 \circ \beta$.

Proof. It follows from Prop. 3.21 that the relations (1)-(4) do hold. It remains to show that no other relations are necessary. This is equivalent to showing that:

- (1) $\theta^k \beta_0$ for $k \geq 0$ are linearly independent elements of $\text{Hom}_{\mathcal{A}\text{-}\mathcal{B}}(M, \mathcal{A} \bar{\otimes}_{\mathcal{A}} M)$.
- (2) Id , θ^k , $\theta^k \circ \beta_0 \circ \alpha$ for all $k \geq 0$ are linearly independent elements of $\text{Hom}_{\mathcal{A}\text{-}\mathcal{B}}(\mathcal{A} \bar{\otimes}_{\mathcal{A}} M, \mathcal{A} \bar{\otimes}_{\mathcal{A}} M)$.

For (1) first note that each $\theta^k \circ \beta_0$ is of degree $-k$. For degree reasons, it is enough therefore to show that each is non-zero. For this, we describe the maps $\theta^k \circ \beta_0$ explicitly. By Prop. 3.21 we have $\theta^k \circ \beta_0 = \beta_k$. Thus it is induced by the map $\bar{\mathcal{A}} \xrightarrow{\lambda_k} \bar{\mathcal{A}}$ of Definition 3.18 which inserts k 1s into an element of $\bar{\mathcal{A}}$.

For (2), we similarly note that for each $k \geq 1$ the maps θ^k and $\theta^k \circ \beta_0 \circ \alpha$ have degree $-k$. For degree reasons, it is enough to show for each $k \geq 0$ that the maps θ^k and $\theta^k \circ \beta_0 \circ \alpha$ are linearly independent. This is clear since they are induced by the maps

$$\begin{aligned} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} &\xrightarrow{(-1)^{k-1} \mu \circ (\lambda_{k-1} \otimes \text{Id})} \bar{\mathcal{A}}, \\ \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} &\xrightarrow{\lambda_k \otimes \tau} \bar{\mathcal{A}}, \end{aligned}$$

respectively. \square

Similarly, we have:

Definition 3.23. Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-Mod-}\mathcal{B}$. Define the maps

$$\text{Hom}_{\mathcal{A}}(\mathcal{A}, M) \xrightarrow{\delta_k} M \quad (3.49)$$

$$\text{Hom}_{\mathcal{A}}(\mathcal{A}, M) \xrightarrow{\kappa} \text{Hom}_{\mathcal{A}}(\mathcal{A}, M) \quad (3.50)$$

$$M \xrightarrow{\gamma} \text{Hom}_{\mathcal{A}}(\mathcal{A}, M) \quad (3.51)$$

of degree $-k$, -1 , and 0 , respectively, in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ by the corresponding $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ maps

$$\begin{aligned} & \bar{\mathcal{A}} \otimes_{\mathcal{A}} (\mathrm{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, M)) \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\mathrm{ev} \circ (\lambda_k \otimes \mathrm{Id} \otimes \tau)} M \\ \bar{\mathcal{A}} \otimes_{\mathcal{A}} (\mathrm{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, M)) \otimes_{\mathcal{B}} \bar{\mathcal{B}} & \xrightarrow{\mathrm{cmps} \circ ((\mu \circ (-)) \circ \mathrm{mlt}) \otimes \mathrm{Id} \otimes \tau} \mathrm{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, M) \\ \bar{\mathcal{A}} \otimes_{\mathcal{A}} (M) \otimes_{\mathcal{B}} \bar{\mathcal{B}} & \xrightarrow{((-) \circ \tau) \circ (\tau \otimes (2.10) \otimes \tau)} \mathrm{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, M) \end{aligned}$$

where $(\mu \circ (-)) \circ \mathrm{mlt}$ denotes the composition $\bar{\mathcal{A}} \xrightarrow{\mathrm{mlt}} \mathrm{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}}) \xrightarrow{\mu \circ (-)} \mathrm{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, \bar{\mathcal{A}})$.

NB: The map γ is the canonical map (3.33), while δ_0 is its left inverse (3.37).

Proposition 3.24. *Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. We have:*

- (1) $d\kappa = \gamma \circ \delta_0 - \mathrm{Id}$.
- (2) $0 = \mathrm{Id} - \delta_0 \circ \gamma$.
- (3) $\delta_k = \delta_0 \circ \kappa^k$.
- (4) For any $k \geq 1$ we have $\delta_k \circ \gamma = 0$.
- (5) For any $k \geq 1$ the map $d\delta_k$ equals 0 when k is odd, and δ_{k-1} when k is even.

Proof. Analogous to the proof of Prop. 3.21. □

Proposition 3.25. *Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. Let γ , δ_0 , and κ be the maps, respectively, introduced in Definition 3.23:*

$$\begin{array}{ccc} & & \delta_0 \\ & \searrow & \nearrow \\ \kappa \circlearrowleft & \mathrm{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, M) & M \\ & \swarrow & \searrow \\ & & \gamma \end{array} \quad (3.52)$$

The sub-DG-category of $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ generated by γ , δ_0 , and κ is the free DG-category generated by those elements modulo the following relations:

- (1) $d\gamma = d\delta_0 = 0$,
- (2) $d\kappa = \gamma \circ \delta_0 - \mathrm{Id}$,
- (3) $0 = \mathrm{Id} - \delta_0 \circ \gamma$,
- (4) $\kappa \circ \gamma = 0$.

Proof. Analogous to the proof of Prop. 3.22. □

Finally, the results in this section have been related so far to the left action of \mathcal{A} on M . For the right action of \mathcal{B} on M we need to define the maps as follows: the maps

$$M \xrightarrow{\beta_k} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \quad (3.53)$$

$$M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\theta} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \quad (3.54)$$

$$M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\alpha} M \quad (3.55)$$

of degree $-k$, -1 , and 0 , are defined, respectively, in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ by the corresponding $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ maps

$$\begin{aligned} & \bar{\mathcal{A}} \otimes_{\mathcal{A}} (M) \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{(-1)^{\frac{k(k+1)}{2}} \tau \otimes \mathrm{Id} \otimes \lambda_k} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \\ \bar{\mathcal{A}} \otimes_{\mathcal{A}} (M \otimes_{\mathcal{B}} \bar{\mathcal{B}}) \otimes_{\mathcal{B}} \bar{\mathcal{B}} & \xrightarrow{-\tau \otimes \mathrm{Id} \otimes \mu} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \\ \bar{\mathcal{A}} \otimes_{\mathcal{A}} (M \otimes_{\mathcal{B}} \bar{\mathcal{B}}) \otimes_{\mathcal{B}} \bar{\mathcal{B}} & \xrightarrow{\tau \otimes \mathrm{Id} \otimes \tau \otimes \tau} M, \end{aligned}$$

and the maps

$$\mathrm{Hom}_{\mathcal{B}}(\bar{\mathcal{B}}, M) \xrightarrow{\delta_k} M \quad (3.56)$$

$$\mathrm{Hom}_{\mathcal{B}}(\bar{\mathcal{B}}, M) \xrightarrow{\kappa} \mathrm{Hom}_{\mathcal{B}}(\bar{\mathcal{B}}, M) \quad (3.57)$$

$$M \xrightarrow{\gamma} \mathrm{Hom}_{\mathcal{B}}(\bar{\mathcal{B}}, M) \quad (3.58)$$

of degree $-k$, -1 , and 0 , are defined, respectively, in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ by the corresponding $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ maps

$$\begin{aligned} \bar{\mathcal{A}} \otimes_{\mathcal{A}} (\mathrm{Hom}_{\mathcal{B}}(\bar{\mathcal{B}}, M)) \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\xrightarrow{(-1)^{\frac{k(k+1)}{2}} \mathrm{ev} \circ (\tau \otimes \mathrm{Id} \otimes \lambda_k)} M \\ \bar{\mathcal{A}} \otimes_{\mathcal{A}} (\mathrm{Hom}_{\mathcal{B}}(\bar{\mathcal{B}}, M)) \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\xrightarrow{-\mathrm{cmps} \circ (\tau \otimes \mathrm{Id} \otimes ((\mu \circ (-)) \circ \mathrm{mlt}))} \mathrm{Hom}_{\mathcal{B}}(\bar{\mathcal{B}}, M) \\ \bar{\mathcal{A}} \otimes_{\mathcal{A}} (M) \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\xrightarrow{((-) \circ \tau) \circ (\tau \otimes (2.10) \otimes \tau)} \mathrm{Hom}_{\mathcal{B}}(\bar{\mathcal{B}}, M). \end{aligned}$$

Then Propositions 3.21 and 3.22 hold for these versions of the maps α , β_k , and θ , and Propositions 3.24 and 3.25 hold for these versions of the maps γ , δ_k , and κ , as well as for their left counterparts.

Setting M to be \mathcal{B} in

$$\begin{aligned} \mathcal{B} \otimes_{\mathcal{B}} M &\xrightarrow{\alpha_M} M \\ M \otimes_{\mathcal{B}} \mathcal{B} &\xrightarrow{\alpha_M} M \end{aligned}$$

we obtain two maps $\mathcal{B} \otimes_{\mathcal{B}} \mathcal{B} \rightarrow \mathcal{B}$. They are, in fact, equal since they both correspond to the $\mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ map

$$\bar{\mathcal{B}} \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\tau^{\otimes 3}} \bar{\mathcal{B}}. \quad (3.59)$$

We generally denote this map by $\alpha_{\mathcal{B}}$, but occasionally use $\alpha_{\mathcal{B}}^l$ and $\alpha_{\mathcal{B}}^r$ to stress it to be the instance of the former or of the latter map, respectively. Similarly for the analogous map $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} \rightarrow \mathcal{A}$ which we denote by $\alpha_{\mathcal{A}}$, or occasionally $\alpha_{\mathcal{A}}^l$ or $\alpha_{\mathcal{A}}^r$.

On the other hand, setting M to be \mathcal{B} in

$$\begin{aligned} M &\xrightarrow{\beta_M} \mathcal{B} \otimes_{\mathcal{B}} M \\ M &\xrightarrow{\beta_M} M \otimes_{\mathcal{B}} \mathcal{B} \end{aligned}$$

we obtain two maps $\mathcal{B} \rightarrow \mathcal{B} \otimes_{\mathcal{B}} \mathcal{B}$ which are not the same and which we denote by $\beta_{\mathcal{B}}^l$ and $\beta_{\mathcal{B}}^r$, respectively. They are, however, homotopic:

Definition 3.26. Let \mathcal{B} be a DG-category. Define the degree -1 map

$$\pi_{\mathcal{B}}: \mathcal{B} \rightarrow \mathcal{B} \otimes_{\mathcal{B}} \mathcal{B} \quad (3.60)$$

in $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ to be the map corresponding to the $\mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ map $\bar{\mathcal{B}} \otimes \bar{\mathcal{B}} \xrightarrow{\mu} \bar{\mathcal{B}}$.

Lemma 3.27. Let \mathcal{B} be a DG-category. We have

$$d\pi_{\mathcal{B}} = \beta_{\mathcal{B}}^r - \beta_{\mathcal{B}}^l.$$

Proof. The maps $\beta_{\mathcal{B}}^l$ and $\beta_{\mathcal{B}}^r$ correspond to $\mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ maps $\bar{\mathcal{B}} \otimes_{\mathcal{B}} \bar{\mathcal{B}} \rightarrow \bar{\mathcal{B}}$ given by $\mathrm{Id} \otimes \tau$ and $\tau \otimes \mathrm{Id}$. The desired assertion now follows from Lemma 2.31(1) where the map $\tau \otimes \mathrm{Id} - \mathrm{Id} \otimes \tau$ was established to be the differential of the degree -1 map μ . \square

3.4. Functoriality of α and β . We next consider the functorial properties of the maps α and β . The latter give rise to genuine natural transformations $\mathrm{Id} \rightarrow \mathcal{A} \otimes_{\mathcal{A}} (-)$ and $\mathrm{Id} \rightarrow (-) \otimes_{\mathcal{B}} \mathcal{B}$, while the former — only to homotopy ones. We also compute the commutators of the corresponding squares for the non-closed maps θ :

Proposition 3.28. Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \xrightarrow{f} N$ be a morphism in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$.

(1) The compositions

$$M \xrightarrow{\beta_M} \mathcal{A} \otimes_{\mathcal{A}} M \xrightarrow{\mathrm{Id} \otimes f} \mathcal{A} \otimes_{\mathcal{A}} N \xrightarrow{\alpha_N} N \quad (3.61)$$

$$M \xrightarrow{\beta_M} M \otimes_{\mathcal{B}} \mathcal{B} \xrightarrow{f \otimes \mathrm{Id}} N \otimes_{\mathcal{B}} \mathcal{B} \xrightarrow{\alpha_N} N \quad (3.62)$$

both equal $M \xrightarrow{f} N$.

(2) The squares

$$\begin{array}{ccc} \mathcal{A} \otimes_{\mathcal{A}} M & \xrightarrow{\mathrm{Id} \otimes f} & \mathcal{A} \otimes_{\mathcal{A}} N \\ \downarrow \alpha_M & & \downarrow \alpha_N \\ M & \xrightarrow{f} & N \end{array} \quad \text{and} \quad \begin{array}{ccc} M \otimes_{\mathcal{B}} \mathcal{B} & \xrightarrow{f \otimes \mathrm{Id}} & N \otimes_{\mathcal{B}} \mathcal{B} \\ \downarrow \alpha_M & & \downarrow \alpha_N \\ M & \xrightarrow{f} & N \end{array} \quad (3.63)$$

have the commutators

$$\alpha_N \circ (\mathrm{Id} \otimes f) - f \circ \alpha_M = (-1)^{\deg(f)} (d(\alpha_N \circ (\mathrm{Id} \otimes f) \circ \theta_M) - \alpha_N \circ (\mathrm{Id} \otimes df) \circ \theta_M), \quad (3.64)$$

$$\alpha_N \circ (f \otimes \mathrm{Id}) - f \circ \alpha_M = (-1)^{\deg(f)} (d(\alpha_N \circ (f \otimes \mathrm{Id}) \circ \theta_M) - \alpha_N \circ (df \otimes \mathrm{Id}) \circ \theta_M). \quad (3.65)$$

(3) The following squares commute:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow \beta_M & & \downarrow \beta_N \\ \mathcal{A} \overline{\otimes}_{\mathcal{A}} M & \xrightarrow{\text{Id} \overline{\otimes} f} & \mathcal{A} \overline{\otimes}_{\mathcal{A}} N \end{array} \quad \text{and} \quad \begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow \beta_M & & \downarrow \beta_N \\ M \overline{\otimes}_{\mathcal{B}} \mathcal{B} & \xrightarrow{f \overline{\otimes} \text{Id}} & N \overline{\otimes}_{\mathcal{B}} \mathcal{B}. \end{array} \quad (3.66)$$

(4) The squares

$$\begin{array}{ccc} \mathcal{A} \overline{\otimes}_{\mathcal{A}} M & \xrightarrow{\text{Id} \overline{\otimes} f} & \mathcal{A} \overline{\otimes}_{\mathcal{A}} N \\ \downarrow \theta_M & & \downarrow \theta_N \\ \mathcal{A} \overline{\otimes}_{\mathcal{A}} M & \xrightarrow{\text{Id} \overline{\otimes} f} & \mathcal{A} \overline{\otimes}_{\mathcal{A}} N \end{array} \quad \text{and} \quad \begin{array}{ccc} M \overline{\otimes}_{\mathcal{B}} \mathcal{B} & \xrightarrow{f \overline{\otimes} \text{Id}} & N \overline{\otimes}_{\mathcal{B}} \mathcal{B} \\ \downarrow \theta_M & & \downarrow \theta_N \\ M \overline{\otimes}_{\mathcal{B}} \mathcal{B} & \xrightarrow{f \overline{\otimes} \text{Id}} & N \overline{\otimes}_{\mathcal{B}} \mathcal{B} \end{array} \quad (3.67)$$

have the commutators

$$\theta_N \circ (\text{Id} \overline{\otimes} f) - (\text{Id} \overline{\otimes} f) \circ \theta_M = -\beta_N \circ \alpha_N \circ (\text{Id} \overline{\otimes} f) \circ \theta_M, \quad (3.68)$$

$$\theta_N \circ (f \overline{\otimes} \text{Id}) - (f \overline{\otimes} \text{Id}) \circ \theta_M = -\beta_N \circ \alpha_N \circ (f \overline{\otimes} \text{Id}) \circ \theta_M. \quad (3.69)$$

Proof. (1): We only prove the first equality, the second one is proved similarly. By the definitions of the morphisms involved the composition

$$M \xrightarrow{\beta_M} \mathcal{A} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\text{Id} \overline{\otimes} f} \mathcal{A} \overline{\otimes}_{\mathcal{A}} N \xrightarrow{\alpha_N} N$$

in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ corresponds to the composition

$$\begin{aligned} \bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}} &\xrightarrow{\Delta^2 \otimes \text{Id} \otimes \Delta^2} \bar{\mathcal{A}} \otimes \bar{\mathcal{A}} \otimes (\bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}}) \otimes \bar{\mathcal{B}} \otimes \bar{\mathcal{B}} \xrightarrow{\text{Id}^{\otimes 5} \otimes \tau \otimes \text{Id}^{\otimes 2}} \bar{\mathcal{A}} \otimes (\bar{\mathcal{A}} \otimes \bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}}) \otimes \bar{\mathcal{B}} \xrightarrow{\text{Id}^{\otimes 2} \otimes \Delta \otimes \text{Id}^{\otimes 3}} \\ &\rightarrow \bar{\mathcal{A}} \otimes (\bar{\mathcal{A}} \otimes \bar{\mathcal{A}} \otimes \bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}}) \otimes \bar{\mathcal{B}} \xrightarrow{\text{Id} \otimes \tau \otimes f \otimes \text{Id}} \bar{\mathcal{A}} \otimes \bar{\mathcal{A}} \otimes N \otimes \bar{\mathcal{B}} \xrightarrow{\tau^{\otimes 2} \otimes \text{Id} \otimes \tau} N \end{aligned}$$

in $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$. The latter simplifies via the identity $\tau \circ \Delta = \text{Id}$ and the functoriality of \otimes to

$$\bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}} \xrightarrow{f} N$$

as required.

(2): We only treat the left square in (3.67), the right one is treated similarly. By (1) we have

$$\alpha_N \circ (\text{Id} \overline{\otimes} f) - f \circ \alpha_M = \alpha_N \circ (\text{Id} \overline{\otimes} f) - \alpha_N \circ (\text{Id} \overline{\otimes} f) \circ \beta_M \circ \alpha_M.$$

By Prop.3.21 we have $d\theta_M = \text{Id} - \beta_M \circ \alpha_M$ and thus

$$\alpha_N \circ (\text{Id} \overline{\otimes} f) - \alpha_N \circ (\text{Id} \overline{\otimes} f) \circ \beta_M \circ \alpha_M = \alpha_N \circ (\text{Id} \overline{\otimes} f) \circ (\text{Id} - \beta_M \circ \alpha_M) = \alpha_N \circ (\text{Id} \overline{\otimes} f) \circ d\theta_M$$

whence the assertion (3.64) readily follows.

(3): We only prove that the right square in (3.66) commutes, as the proof for the left square is similar. By the definition of the morphisms involved the compositions

$$\begin{array}{ccc} M & \xrightarrow{f} & N \xrightarrow{\beta_N} N \overline{\otimes}_{\mathcal{B}} \mathcal{B} \\ M \xrightarrow{\beta_M} M \overline{\otimes}_{\mathcal{B}} \mathcal{B} & \xrightarrow{f \overline{\otimes} \text{Id}} & N \overline{\otimes}_{\mathcal{B}} \mathcal{B} \end{array}$$

in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ correspond to the compositions

$$\begin{aligned} \bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}} &\xrightarrow{\Delta \otimes \text{Id} \otimes \Delta} \bar{\mathcal{A}} \otimes (\bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}}) \otimes \bar{\mathcal{B}} \xrightarrow{\text{Id} \otimes f \otimes \text{Id}} \bar{\mathcal{A}} \otimes N \otimes \bar{\mathcal{B}} \xrightarrow{\tau \otimes \text{Id} \otimes \text{Id}} N \otimes \bar{\mathcal{B}} \\ \bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}} &\xrightarrow{\Delta \otimes \text{Id} \otimes \Delta} \bar{\mathcal{A}} \otimes (\bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}}) \otimes \bar{\mathcal{B}} \xrightarrow{\tau \otimes \text{Id} \otimes \text{Id}} \bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}} \otimes \bar{\mathcal{B}} \xrightarrow{\text{Id}^{\otimes 2} \otimes \Delta \otimes \tau} \bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}} \otimes \bar{\mathcal{B}} \xrightarrow{f \otimes \text{Id}} N \otimes \bar{\mathcal{B}} \end{aligned}$$

in $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$. Both these $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ compositions simplify to

$$\bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}} \xrightarrow{\text{Id} \otimes \text{Id} \otimes \Delta} \bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}} \otimes \bar{\mathcal{B}} \xrightarrow{f \otimes \text{Id}} N \otimes \bar{\mathcal{B}},$$

as required.

(4): We only treat the left square in (3.67), the right one is treated similarly. Consider the maps

$$\mathcal{A} \overline{\otimes} M \longrightarrow \mathcal{A} \overline{\otimes} N$$

given by the compositions $(\text{Id} \overline{\otimes} f) \circ \theta_M$, $\theta_N \circ (\text{Id} \overline{\otimes} f)$, and $\beta_N \circ \alpha_N \circ (\text{Id} \overline{\otimes} f) \circ \theta_M$ in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. After simplification, the corresponding $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ maps

$$\bar{\mathcal{A}} \otimes \bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}} \longrightarrow \bar{\mathcal{A}} \otimes N$$

are given by the compositions

$$\bar{\mathcal{A}} \otimes \bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}} \xrightarrow{(\dots) \otimes \text{Id} \otimes \tau} \bar{\mathcal{A}} \otimes \bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}} \xrightarrow{\text{Id} \otimes f} \bar{\mathcal{A}} \otimes N$$

where (\dots) denotes the maps $\Delta \circ \mu$, $(\mu \otimes \text{Id}) \otimes (\text{Id} \otimes \Delta)$, $(\text{Id} \otimes \mu) \otimes (\Delta \otimes \text{Id})$, respectively. The desired result now follows from Prop. 3.19(1). \square

Definition 3.29. Let \mathcal{A} and \mathcal{B} be DG-categories and let $f: M \rightarrow N$ be a morphism in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. We say that f is a *fiberwise $\mathbf{Mod}\text{-}\mathcal{A}^{\text{opp}}$ morphism* if its fiber $M_b \rightarrow N_b$ over each $b \in \mathcal{B}$ lies in $\Upsilon(\mathbf{Mod}\text{-}\mathcal{A}^{\text{opp}})$, where Υ is the inclusion of Prop. 3.3. Similarly, f is a *fiberwise $\mathbf{Mod}\text{-}\mathcal{B}$ morphism* if its fiber ${}_a M \rightarrow {}_a N$ over each $a \in \mathcal{A}$ lies in $\Upsilon(\mathbf{Mod}\text{-}\mathcal{B})$.

Corollary 3.30. Let \mathcal{A} and \mathcal{B} be DG-categories and let $f: M \rightarrow N$ be a morphism in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. If f is a fiberwise $\mathbf{Mod}\text{-}\mathcal{A}^{\text{opp}}$ (resp. fiberwise $\mathbf{Mod}\text{-}\mathcal{B}$) morphism, the left (resp. right) squares (3.63) and (3.67) commute.

Proof. We only treat the left squares in (3.63) and (3.67), the right squares are treated similarly.

Writing out and simplifying the $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ morphism corresponding to

$$\mathcal{A} \otimes M \xrightarrow{\theta_M} \mathcal{A} \otimes M \xrightarrow{\text{Id} \otimes f} \mathcal{A} \otimes N \xrightarrow{\alpha_N} N$$

in a fashion similar to the one employed in the proof of Prop. 3.28(2) we obtain

$$\bar{\mathcal{A}} \otimes \bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}} \xrightarrow{\mu \otimes \text{Id} \otimes \text{Id}} \bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}} \xrightarrow{f} N. \quad (3.70)$$

Let now f be a fiberwise $\mathbf{Mod}\text{-}\mathcal{A}$ morphism. Then $\bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}} \xrightarrow{f} N$ factors through

$$\bar{\mathcal{A}} \otimes M \otimes \bar{\mathcal{B}} \xrightarrow{\tau \otimes \text{Id} \otimes \text{Id}} M \otimes \bar{\mathcal{B}},$$

and therefore the composition (3.70) factors through $(\tau \circ \mu) \otimes \text{Id} \otimes \text{Id}$. Since $\tau \circ \mu = 0$ we conclude that $\alpha_N \circ (\text{Id} \otimes f) \circ \theta_M = 0$. Furthermore, if f is a fiberwise $\mathbf{Mod}\text{-}\mathcal{A}$ morphism, then so is df , and therefore $\alpha_N \circ (\text{Id} \otimes df) \circ \theta_M = 0$. It now follows by Prop. 3.28 that the left squares in (3.63) and (3.67) commute. \square

3.5. Dualisation. In this section we look at the dualising functors for bar categories of bimodules:

Definition 3.31. Let \mathcal{A} and \mathcal{B} be DG-categories. Define the *dualising functors* $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \rightarrow (\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A})^{\text{opp}}$

$$\begin{aligned} (-)^{\bar{\mathcal{A}}} &\stackrel{\text{def}}{=} \overline{\text{Hom}}_{\mathcal{A}}(-, \mathcal{A}) \\ (-)^{\bar{\mathcal{B}}} &\stackrel{\text{def}}{=} \overline{\text{Hom}}_{\mathcal{B}}(-, \mathcal{B}). \end{aligned}$$

By Tensor-Hom adjunction we have for any $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ and $N \in \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$

$$\overline{\text{Hom}}_{(\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A})^{\text{opp}}}(M^{\bar{\mathcal{A}}}, N) \simeq \overline{\text{Hom}}_{\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}}(M \otimes_{\mathcal{B}} N, \mathcal{A}) \simeq \overline{\text{Hom}}_{\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}}(M, N^{\bar{\mathcal{A}}}) \quad (3.71)$$

It follows that the functor

$$(-)^{\bar{\mathcal{A}}}: (\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A})^{\text{opp}} \rightarrow \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$$

is left adjoint to the functor

$$(-)^{\bar{\mathcal{A}}}: \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \rightarrow (\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A})^{\text{opp}}.$$

The adjunction unit is the natural transformation

$$\text{Id}_{\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}} \longrightarrow (-)^{\bar{\mathcal{A}}\bar{\mathcal{A}}} \quad (3.72)$$

of endofunctors of $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ defined on every $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ by the right adjoint of the evaluation map

$$M \otimes_{\mathcal{B}} \overline{\text{Hom}}_{\mathcal{A}}(M, \mathcal{A}) \xrightarrow{\text{ev}} \mathcal{A}$$

with respect to the functor $(-)^{\bar{\mathcal{A}}}$. The adjunction counit is the natural transformation

$$(-)^{\bar{\mathcal{A}}\bar{\mathcal{A}}} \rightarrow \text{Id}_{(\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A})^{\text{opp}}}$$

which corresponds to the natural transformation $\text{Id}_{\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}} \rightarrow (-)^{\bar{\mathcal{A}}\bar{\mathcal{A}}}$ defined in the same way as (3.72).

We define similarly natural transformations

$$\text{Id}_{\bullet} \longrightarrow (-)^{\bar{\mathcal{B}}\bar{\mathcal{B}}} \quad (3.73)$$

which give the unit and the counit of the analogous adjunction of $(-)^{\bar{\mathcal{B}}}$ with itself.

Lemma 3.32. Let \mathcal{A} and \mathcal{B} be DG-categories.

- (1) The natural transformations (3.72) and (3.73) are homotopy equivalences on \mathcal{A} - and \mathcal{B} -perfect bimodules, respectively.
(2) The functors $(-)^{\bar{\mathcal{A}}}$ and $(-)^{\bar{\mathcal{B}}}$ restrict to quasi-equivalences

$$((\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B})^{\mathcal{A}\text{-}\mathcal{P}erf})^{opp} \xrightarrow{(-)^{\bar{\mathcal{A}}}} (\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A})^{\mathcal{A}\text{-}\mathcal{P}erf}, \quad (3.74)$$

$$((\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B})^{\mathcal{B}\text{-}\mathcal{P}erf})^{opp} \xrightarrow{(-)^{\bar{\mathcal{B}}}} (\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A})^{\mathcal{B}\text{-}\mathcal{P}erf}, \quad (3.75)$$

Proof. (1): We proceed by reduction to a similar result for ordinary categories of DG-bimodules proved in [AL17, §2]. Let $M \in \overline{\mathbf{Mod}}\text{-}\mathcal{B}$ and let $a \in \mathcal{A}$. Since $(M^{\bar{\mathcal{B}}})_a = ({}_aM)^{\bar{\mathcal{B}}}$, it is clear that the fiber over a of

$$M \xrightarrow{(3.73)} M^{\bar{\mathcal{B}}\bar{\mathcal{B}}} \quad (3.76)$$

is the analogous natural transformation (3.73) of endofunctors of $\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ applied to ${}_aM$.

It follows from the description of the adjunction unit (3.26) that the $\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ map

$${}_aM \xrightarrow{(3.73)} ({}_aM)^{\bar{\mathcal{B}}\bar{\mathcal{B}}}$$

is defined by the $\mathbf{Mod}\text{-}\mathcal{B}$ map

$$\begin{aligned} & {}_aM \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\text{mlt}} \text{Hom}_{\mathcal{B}} \left(({}_aM \otimes_{\mathcal{B}} \bar{\mathcal{B}})^{\mathcal{B}}, {}_aM \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} ({}_aM \otimes_{\mathcal{B}} \bar{\mathcal{B}})^{\mathcal{B}} \right) \xrightarrow{\text{ev} \circ (-)} \\ & \rightarrow \text{Hom}_{\mathcal{B}} \left(({}_aM \otimes_{\mathcal{B}} \bar{\mathcal{B}})^{\mathcal{B}}, \mathcal{B} \right) \xrightarrow{(-) \circ (\text{Id} \otimes \tau)} \text{Hom}_{\mathcal{B}} \left(({}_aM \otimes_{\mathcal{B}} \bar{\mathcal{B}})^{\mathcal{B}} \otimes_{\mathcal{B}} \bar{\mathcal{B}}, \mathcal{B} \right). \end{aligned} \quad (3.77)$$

If M is \mathcal{B} -perfect, ${}_aM$ is perfect. On the other hand, $\bar{\mathcal{B}}$ is h -projective and both left and right perfect. By [AL17, Prop. 2.5 and 2.14] when tensoring an h -projective (resp. perfect) bimodule with a bimodule which is h -projective (resp. perfect) on the side not involved in the tensor product the result is h -projective (resp. perfect). It follows that ${}_aM \otimes_{\mathcal{B}} \bar{\mathcal{B}}$ is h -projective and perfect, and hence so is $({}_aM \otimes_{\mathcal{B}} \bar{\mathcal{B}})^{\mathcal{B}}$. Thus the last map in the composition above is a quasi-isomorphism. On the other hand, the first two maps define a natural transformation $\text{Id} \rightarrow (-)^{\mathcal{B}\bar{\mathcal{B}}}$ of endofunctors of \mathcal{B} . It is an isomorphism on representables, and hence a quasi-isomorphism on all h -projective and perfect modules, cf. [AL17, §2.2]. In particular, it is a quasi-isomorphism on ${}_aM \otimes_{\mathcal{B}} \bar{\mathcal{B}}$. Thus (3.77) is a quasi-isomorphism.

We conclude that for a \mathcal{B} -perfect M the $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ map which defines (3.76) is a quasi-isomorphism, since its every fiber over \mathcal{A} is. It follows from Cor. 3.8 that (3.76) itself is a homotopy equivalence, as desired.

A similar argument shows that (3.72) is a homotopy equivalence on \mathcal{A} -perfect bimodules.

(2): This follows from (1) since both the units and the counits of the adjunctions of $(-)^{\bar{\mathcal{A}}}$ and $(-)^{\bar{\mathcal{B}}}$ with themselves were shown to be homotopy equivalences on the corresponding subcategories. \square

The following is the $\overline{\mathbf{Mod}}$ analogue of the map (2.14):

Definition 3.33. Let \mathcal{A} , \mathcal{B} , \mathcal{C} , and \mathcal{D} be DG-categories. Let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, $N \in \mathcal{D}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, and $L \in \mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$.

Define the $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{D}$ map

$$L \otimes_{\mathcal{A}} \overline{\mathbf{Hom}}_{\mathcal{B}}(N, M) \longrightarrow \overline{\mathbf{Hom}}_{\mathcal{B}}(N, L \otimes_{\mathcal{A}} M) \quad (3.78)$$

as the composition

$$L \otimes_{\mathcal{A}} \overline{\mathbf{Hom}}_{\mathcal{B}}(N, M) \xrightarrow{\text{mlt} \otimes \overline{\text{Id}}} \overline{\mathbf{Hom}}_{\mathcal{B}}(M, L \otimes_{\mathcal{A}} M) \otimes_{\mathcal{A}} \overline{\mathbf{Hom}}_{\mathcal{B}}(N, M) \xrightarrow{\text{cmps}} \overline{\mathbf{Hom}}_{\mathcal{B}}(N, L \otimes_{\mathcal{A}} M).$$

Lemma 3.34. Let \mathcal{A} , \mathcal{B} , \mathcal{C} , and \mathcal{D} be DG-categories and let $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$, $N \in \mathcal{D}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$, and $L \in \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$. The $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{D}$ map

$$L \otimes_{\mathcal{A}} \overline{\mathbf{Hom}}_{\mathcal{B}}(N, M) \xrightarrow{(3.78)} \overline{\mathbf{Hom}}_{\mathcal{B}}(N, L \otimes_{\mathcal{A}} M)$$

is a homotopy equivalence when N is \mathcal{B} -perfect or L is \mathcal{A} -perfect.

Proof. It follows from the definitions of the adjunction unit (3.22) and the composition map (3.19) that the $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{D}$ map defining (3.78) is

$$\begin{aligned} & L \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(N \otimes_{\mathcal{B}} \bar{\mathcal{B}}, M) \xrightarrow{\text{mlt} \otimes \text{Id}} \text{Hom}_{\mathcal{B}}(M, L \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(N \otimes_{\mathcal{B}} \bar{\mathcal{B}}, M) \xrightarrow{\text{cmps}} \\ & \rightarrow \text{Hom}_{\mathcal{B}}(N \otimes_{\mathcal{B}} \bar{\mathcal{B}}, L \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M). \end{aligned}$$

Thus it is an instance of the map (2.14) and is therefore a quasi-isomorphism when $N \otimes_{\mathcal{B}} \bar{\mathcal{B}} \in \mathcal{P}^{\mathcal{B}\text{-}\mathcal{P}erf}(\mathcal{A}\text{-}\mathcal{B})$ or $L \otimes_{\mathcal{A}} \bar{\mathcal{A}} \in \mathcal{P}^{\mathcal{A}\text{-}\mathcal{P}erf}(\mathcal{A}\text{-}\mathcal{B})$. Hence when N is \mathcal{B} -perfect or L is \mathcal{A} -perfect the $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{D}$ map defining (3.78) is a quasi-isomorphism, and by Cor. 3.8 it follows that (3.78) is a homotopy equivalence, as desired. \square

Definition 3.35. Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$.

Define the natural transformations

$$(-) \otimes_{\mathcal{B}} M^{\overline{\mathcal{B}}} \xrightarrow{\eta_{\mathcal{B}}} \overline{\mathbf{Hom}}_{\mathcal{B}}(M, -), \quad (3.79)$$

$$M^{\overline{\mathcal{A}}} \otimes_{\mathcal{A}} (-) \xrightarrow{\eta_{\mathcal{A}}} \overline{\mathbf{Hom}}_{\mathcal{A}}(M, -) \quad (3.80)$$

of functors $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \rightarrow \mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ and $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C} \rightarrow \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$, respectively, as the compositions

$$\begin{aligned} (-) \otimes_{\mathcal{B}} \overline{\mathbf{Hom}}_{\mathcal{B}}(M, \mathcal{B}) &\xrightarrow{(3.33) \otimes \text{Id}} \overline{\mathbf{Hom}}_{\mathcal{B}}(\mathcal{B}, -) \otimes_{\mathcal{B}} \overline{\mathbf{Hom}}_{\mathcal{B}}(M, \mathcal{B}) \xrightarrow{\text{cmpr}} \overline{\mathbf{Hom}}_{\mathcal{B}}(M, -), \\ \overline{\mathbf{Hom}}_{\mathcal{A}}(M, \mathcal{A}) \otimes_{\mathcal{A}} (-) &\xrightarrow{\text{Id} \otimes (3.33)} \overline{\mathbf{Hom}}_{\mathcal{A}}(M, \mathcal{A}) \otimes_{\mathcal{A}} \overline{\mathbf{Hom}}_{\mathcal{A}}(\mathcal{A}, -) \xrightarrow{\text{cmpr}} \overline{\mathbf{Hom}}_{\mathcal{A}}(M, -). \end{aligned}$$

Lemma 3.36. Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$.

(1) M is \mathcal{B} -perfect if and only if the map

$$(-) \otimes_{\mathcal{B}} M^{\overline{\mathcal{B}}} \xrightarrow{\eta_{\mathcal{B}}} \overline{\mathbf{Hom}}_{\mathcal{B}}(M, -)$$

is a homotopy equivalence of functors $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \rightarrow \mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ for any DG-category \mathcal{C} .

(2) M is \mathcal{A} -perfect if and only if the map

$$M^{\overline{\mathcal{A}}} \otimes_{\mathcal{A}} (-) \xrightarrow{\eta_{\mathcal{A}}} \overline{\mathbf{Hom}}_{\mathcal{A}}(M, -)$$

is a homotopy equivalence of functors $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C} \rightarrow \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$ for any DG-category \mathcal{C} .

Proof. We only give the proof for the assertion (1), as the proof for (2) is identical.

Assume that M is \mathcal{B} -perfect. Since $(-)\xrightarrow{(3.33)}\overline{\mathbf{Hom}}_{\mathcal{B}}(\mathcal{B}, -)$ is the right adjoint of (3.32), it equals the composition

$$(-) \xrightarrow{\text{mlt}} \overline{\mathbf{Hom}}_{\mathcal{B}}(\mathcal{B}, (-) \otimes_{\mathcal{B}} \mathcal{B}) \xrightarrow{(3.32) \circ (-)} \overline{\mathbf{Hom}}_{\mathcal{B}}(\mathcal{B}, -).$$

It follows that the map

$$(-) \otimes_{\mathcal{B}} M^{\overline{\mathcal{B}}} \xrightarrow{\eta_{\mathcal{B}}} \overline{\mathbf{Hom}}_{\mathcal{B}}(M, -)$$

equals the composition

$$(-) \otimes_{\mathcal{B}} \overline{\mathbf{Hom}}_{\mathcal{B}}(M, \mathcal{B}) \xrightarrow{\text{mlt} \otimes \text{Id}} \overline{\mathbf{Hom}}_{\mathcal{B}}(\mathcal{B}, (-) \otimes_{\mathcal{B}} \mathcal{B}) \otimes_{\mathcal{B}} \overline{\mathbf{Hom}}_{\mathcal{B}}(M, \mathcal{B}) \xrightarrow{(3.32) \circ (-)} \overline{\mathbf{Hom}}_{\mathcal{B}}(\mathcal{B}, -) \otimes_{\mathcal{B}} \overline{\mathbf{Hom}}_{\mathcal{B}}(M, \mathcal{B}) \xrightarrow{\text{cmpr}} \overline{\mathbf{Hom}}_{\mathcal{B}}(M, -)$$

and hence, by the definition of the map (3.78), to

$$(-) \otimes_{\mathcal{B}} M^{\overline{\mathcal{B}}} \xrightarrow{(3.78)} \overline{\mathbf{Hom}}_{\mathcal{B}}(M, (-) \otimes_{\mathcal{B}} \mathcal{B}) \xrightarrow{(3.32) \circ (-)} \overline{\mathbf{Hom}}_{\mathcal{B}}(M, -).$$

The first map in this composition is a homotopy equivalence by Lemma 3.34, and the second one is a homotopy equivalence since (3.32) is. We conclude that $\eta_{\mathcal{B}}$ is also a homotopy equivalence, as desired.

Conversely, assume that

$$(-) \otimes_{\mathcal{B}} M^{\overline{\mathcal{B}}} \xrightarrow{\eta_{\mathcal{B}}} \overline{\mathbf{Hom}}_{\mathcal{B}}(M, -)$$

is a homotopy equivalence on all of $\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. By Prop. 3.12 it follows that

$$(-) \otimes_{\mathcal{B}}^{\mathbf{L}} M^{\overline{\mathcal{B}}} \simeq \mathbf{R}\overline{\mathbf{Hom}}_{\mathcal{B}}(M, -).$$

Thus $\mathbf{R}\overline{\mathbf{Hom}}_{\mathcal{B}}(M, -)$ commutes with infinite direct sums, i.e. M is \mathcal{B} -perfect. \square

Lemma 3.37. Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be DG-categories. Let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ and $N \in \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$.

If M is \mathcal{B} -perfect, there is a $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ homotopy equivalence

$$N^{\overline{\mathcal{C}}} \otimes_{\mathcal{B}} M^{\overline{\mathcal{B}}} \longrightarrow (M \otimes_{\mathcal{B}} N)^{\overline{\mathcal{C}}}. \quad (3.81)$$

If N is \mathcal{B} -perfect, there is a $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ homotopy equivalence

$$N^{\overline{\mathcal{B}}} \otimes_{\mathcal{B}} M^{\overline{\mathcal{A}}} \longrightarrow (M \otimes_{\mathcal{B}} N)^{\overline{\mathcal{A}}}. \quad (3.82)$$

Proof. Similarly to the proof of Lemma 2.12 in [AL17], define (3.81) to be the composition

$$N^{\overline{\mathcal{C}}} \otimes_{\mathcal{B}} M^{\overline{\mathcal{B}}} \xrightarrow{\eta_{\mathcal{B}}} \overline{\mathbf{Hom}}_{\mathcal{B}}(M, N^{\overline{\mathcal{C}}}) \xrightarrow{\text{adjunction}} \overline{\mathbf{Hom}}_{\mathcal{C}}(M \otimes_{\mathcal{B}} N, \mathcal{C}). \quad (3.83)$$

The first component is a homotopy equivalence by Lemma 3.36 and the second component is the Tensor-Hom adjunction isomorphism. Thus (3.81) is itself a homotopy equivalence.

We define (3.82) similarly. \square

3.6. Convolution functor for $\text{Pre-Tr}(\overline{\mathbf{Mod}})$. Let \mathcal{A} be a DG-category. It is well known that $\mathbf{Mod}\text{-}\mathcal{A}$ is a strongly pretriangulated category, cf. [BK90], [AL17, §3.2]. That is, the natural inclusion

$$\mathbf{Mod}\text{-}\mathcal{A} \hookrightarrow \text{Pre-Tr } \mathbf{Mod}\text{-}\mathcal{A}$$

is an equivalence of categories. Its quasi-inverse

$$\text{Pre-Tr } \mathbf{Mod}\text{-}\mathcal{A} \xrightarrow{T} \mathbf{Mod}\text{-}\mathcal{A}$$

is the *convolution functor*. Similarly, $\mathbf{Nod}_\infty\mathcal{A}$ is strongly pretriangulated and admits a convolution functor. This is because the DG category of all DG $B_\infty\mathcal{A}$ -comodules is strongly pre-triangulated, and $\mathbf{Nod}_\infty\mathcal{A}$ is equivalent to its full subcategory consisting of the DG $B_\infty\mathcal{A}$ -comodules which are free as graded comodules. On the level of graded DG-comodules the convolution functor is a direct sum with shifts. Thus if every object in the twisted complex is free as a graded comodule, so is its convolution. We thus obtain the convolution functor for $\mathbf{Nod}_\infty\mathcal{A}$, as required.

The category $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ is not strongly pretriangulated. It is however *pretriangulated*. In other words,

$$\overline{\mathbf{Mod}}\text{-}\mathcal{A} \hookrightarrow \text{Pre-Tr } \overline{\mathbf{Mod}}\text{-}\mathcal{A}$$

is only a quasi-equivalence. This is readily seen via the isomorphism $\overline{\mathbf{Mod}}\text{-}\mathcal{A} \simeq (\mathbf{Nod}_\infty\mathcal{A})_{dg}$ of Prop. 3.5. While $\mathbf{Nod}_\infty\mathcal{A}$ is strongly pretriangulated, its full subcategory $(\mathbf{Nod}_\infty\mathcal{A})_{dg}$ is not. This is because when we restrict the convolution functor

$$\text{Pre-Tr } \mathbf{Nod}_\infty\mathcal{A} \xrightarrow{T_\infty} \mathbf{Nod}_\infty\mathcal{A}$$

to $\text{Pre-Tr}(\mathbf{Nod}_\infty\mathcal{A})_{dg}$ its image doesn't restrict to $(\mathbf{Nod}_\infty\mathcal{A})_{dg}$. Indeed, let (E_i, α_{ij}) be a twisted complex over $(\mathbf{Nod}_\infty\mathcal{A})_{dg}$. That is, E_i are DG \mathcal{A} -modules and α_{ij} are A_∞ -morphisms between them. Then, $T_\infty(E_i, \alpha_{ij})$ is the A_∞ -module whose underlying DG-module is $T(E_i, \alpha_{ij})$ and whose higher A_∞ -module structure is defined by the higher operations of the differentials α_{ij} . This structure will not in general be trivial unless the higher operations are all zero, i.e. unless α_{ij} are regular DG morphisms.

The fact that $T_\infty(\text{Pre-Tr}(\mathbf{Nod}_\infty\mathcal{A})_{dg})$ doesn't land in $(\mathbf{Nod}_\infty\mathcal{A})_{dg}$ can be readily fixed by applying the semi-free resolution $(-)\overset{\infty}{\otimes}_{\mathcal{A}}\mathcal{A}$ of §2.10. We then obtain a functor

$$\text{Pre-Tr}(\mathbf{Nod}_\infty\mathcal{A})_{dg} \xrightarrow{T_\infty} (\mathbf{Nod}_\infty\mathcal{A})_{hu} \xrightarrow{(-)\overset{\infty}{\otimes}_{\mathcal{A}}\mathcal{A}} (\mathbf{Nod}_\infty\mathcal{A})_{dg} \quad (3.84)$$

which is a quasi-equivalence because both of its components are. It is also, by construction, a quasi-inverse of the natural inclusion $(\mathbf{Nod}_\infty\mathcal{A})_{dg} \rightarrow \text{Pre-Tr}(\mathbf{Nod}_\infty\mathcal{A})_{dg}$ and thus an analogue of a convolution functor. In this section, we translate these considerations to the case of $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$.

Throughout this section, we adopt the following notation. Given a morphism $\alpha: E \rightarrow F$ in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ we denote by $\hat{\alpha}$ the underlying $\mathbf{Mod}\text{-}\mathcal{A}$ morphism $E \otimes_{\mathcal{A}} \overline{\mathcal{A}} \rightarrow F$.

Recall the natural inclusion

$$\mathbf{Mod}\text{-}\mathcal{A} \xrightarrow{(3.7)} \overline{\mathbf{Mod}}\text{-}\mathcal{A}$$

of Prop. 3.3. It gets identified under the isomorphism $\overline{\mathbf{Mod}}\text{-}\mathcal{A} \simeq (\mathbf{Nod}_\infty\mathcal{A})_{dg}$ of Prop. 3.5 with the inclusion

$$\mathbf{Mod}\text{-}\mathcal{A} = (\mathbf{Nod}_\infty\mathcal{A})_{dg}^{\text{strict}} \hookrightarrow (\mathbf{Nod}_\infty\mathcal{A})_{dg}.$$

On the other hand, by Cor. 2.19 the semi-free resolution

$$(\mathbf{Nod}_\infty\mathcal{A})_{hu} \xrightarrow{(-)\overset{\infty}{\otimes}_{\mathcal{A}}\mathcal{A}} (\mathbf{Nod}_\infty\mathcal{A})_{dg}$$

factors through $\mathbf{Mod}\text{-}\mathcal{A} = (\mathbf{Nod}_\infty\mathcal{A})_{dg}^{\text{strict}}$. As $(-)\overset{\infty}{\otimes}_{\mathcal{A}}\mathcal{A}$ is identified with $(-)\overline{\otimes}_{\mathcal{A}}\mathcal{A}$ by the isomorphism $\overline{\mathbf{Mod}}\text{-}\mathcal{A} \simeq (\mathbf{Nod}_\infty\mathcal{A})_{dg}$, it follows that

$$\overline{\mathbf{Mod}}\text{-}\mathcal{A} \xrightarrow{(-)\overline{\otimes}_{\mathcal{A}}\mathcal{A}} \overline{\mathbf{Mod}}\text{-}\mathcal{A}$$

factors as

$$\overline{\mathbf{Mod}}\text{-}\mathcal{A} \rightarrow \mathbf{Mod}\text{-}\mathcal{A} \xrightarrow{(3.7)} \overline{\mathbf{Mod}}\text{-}\mathcal{A}. \quad (3.85)$$

Definition 3.38. Define the functor

$$\overline{\mathbf{Mod}}\text{-}\mathcal{A} \xrightarrow{\widetilde{(-)}} \mathbf{Mod}\text{-}\mathcal{A} \quad (3.86)$$

to be the first half of the factorisation of $(-)\overline{\otimes}_{\mathcal{A}}\mathcal{A}$ given in (3.85).

Lemma 3.39. *Let $E \in \overline{\mathbf{Mod}}\text{-}\mathcal{A}$, then*

$$\tilde{E} = E \otimes_{\mathcal{A}} \bar{\mathcal{A}}. \quad (3.87)$$

Let $\alpha \in \text{Hom}_{\overline{\mathbf{Mod}}\text{-}\mathcal{A}}(E, F)$, then

$$\tilde{\alpha} = (\hat{\alpha} \otimes \text{Id}) \circ (\text{Id} \otimes \Delta). \quad (3.88)$$

Proof. This follows directly from the definition of the $\bar{\otimes}$ bifunctor given in Definition 3.9. Note that it needs to be adjusted in an obvious way for having right modules, and not bimodules, in the left argument.

The first assertion is immediate. For the second one, applying the definition to the morphisms $\alpha: E \rightarrow F$ and $\text{Id}_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{A}$ we see that the underlying $\mathbf{Mod}\text{-}\mathcal{A}$ morphism of $\alpha \bar{\otimes} \text{Id}_{\mathcal{A}}$ is the composition

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes \Delta^2 \otimes \text{Id}^{\otimes 2}} E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\hat{\alpha} \otimes \text{Id} \otimes \hat{\text{Id}}_{\mathcal{A}}} F \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A}.$$

Since $\hat{\text{Id}}_{\mathcal{A}} = \tau \otimes \text{Id} \otimes \tau$, this can be rewritten as

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\text{Id}^{\otimes 2} \otimes \tau} E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes \Delta} E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\hat{\alpha} \otimes \text{Id}} F \otimes_{\mathcal{A}} \bar{\mathcal{A}}$$

which is the image of (3.88) under the category inclusion (3.7), as desired. \square

Definition 3.40. Define the *convolution functor*

$$\text{Pre-Tr } \overline{\mathbf{Mod}}\text{-}\mathcal{A} \xrightarrow{\bar{T}} \overline{\mathbf{Mod}}\text{-}\mathcal{A} \quad (3.89)$$

as the composition

$$\text{Pre-Tr } \overline{\mathbf{Mod}}\text{-}\mathcal{A} \xrightarrow{\widetilde{(-)}} \text{Pre-Tr } \mathbf{Mod}\text{-}\mathcal{A} \xrightarrow{T} \mathbf{Mod}\text{-}\mathcal{A} \xrightarrow{(3.7)} \overline{\mathbf{Mod}}\text{-}\mathcal{A}. \quad (3.90)$$

Proposition 3.41. *The functor \bar{T} gets identified by the isomorphism $\overline{\mathbf{Mod}}\text{-}\mathcal{A} \simeq (\mathbf{Nod}_{\infty}\mathcal{A})_{dg}$ with the functor*

$$\text{Pre-Tr } (\mathbf{Nod}_{\infty}\mathcal{A})_{dg} \xrightarrow{(3.84)} (\mathbf{Nod}_{\infty}\mathcal{A})_{dg}.$$

Proof. By its definition, $\overline{\mathbf{Mod}}\text{-}\mathcal{A} \xrightarrow{\widetilde{(-)}} \mathbf{Mod}\text{-}\mathcal{A}$ gets identified by the isomorphism $\overline{\mathbf{Mod}}\text{-}\mathcal{A} \simeq (\mathbf{Nod}_{\infty}\mathcal{A})_{dg}$ with the functor $(\mathbf{Nod}_{\infty}\mathcal{A})_{dg} \xrightarrow{(-)^{\infty} \otimes_{\mathcal{A}} \mathcal{A}} \mathbf{Mod}\text{-}\mathcal{A}$. Therefore \bar{T} gets identified with the path in the diagram below which travels along the upper-right half of the lower rectangle:

$$\begin{array}{ccccc} (\mathbf{Nod}_{\infty}\mathcal{A})_{hu} & \xrightarrow{(-)^{\infty} \otimes_{\mathcal{A}} \mathcal{A}} & \mathbf{Mod}\text{-}\mathcal{A} & & \\ \downarrow & & \downarrow & & \\ \text{Pre-Tr } (\mathbf{Nod}_{\infty}\mathcal{A})_{dg} & \xrightarrow{\text{Pre-Tr } (-)^{\infty} \otimes_{\mathcal{A}} \mathcal{A}} & \text{Pre-Tr } \mathbf{Mod}\text{-}\mathcal{A} & & \\ \downarrow T_{\infty} & & \downarrow T & & \\ (\mathbf{Nod}_{\infty}\mathcal{A})_{hu} & \xrightarrow{(-)^{\infty} \otimes_{\mathcal{A}} \mathcal{A}} & \mathbf{Mod}\text{-}\mathcal{A} & \xrightarrow{\quad} & (\mathbf{Nod}_{\infty}\mathcal{A})_{dg}. \end{array}$$

On the other hand, the path which travels along the lower-left half of the lower rectangle is precisely (3.84). The upper rectangle and the outer perimeter of the two rectangles clearly commute. Since all the vertical arrows are category equivalences, the lower rectangle commutes as well. The desired assertion follows. \square

Corollary 3.42. *The convolution functor*

$$\text{Pre-Tr } \overline{\mathbf{Mod}}\text{-}\mathcal{A} \xrightarrow{\bar{T}} \overline{\mathbf{Mod}}\text{-}\mathcal{A}$$

is a quasi-equivalence and a homotopy inverse of the natural inclusion $\overline{\mathbf{Mod}}\text{-}\mathcal{A} \hookrightarrow \text{Pre-Tr } \overline{\mathbf{Mod}}\text{-}\mathcal{A}$.

Let \mathcal{A} and \mathcal{B} be DG-categories. We define the convolution functor

$$\text{Pre-Tr } \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \xrightarrow{\bar{T}} \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$$

similarly.

Lemma 3.43. *Let (E_i, α_{ij}) be a one-sided twisted complex in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, and let E be its convolution.*

(1) *Let (F_i, β_{ij}) be a one-sided twisted complex in $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$, and let F be its convolution. Then there is an $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$ homotopy equivalence*

$$\left\{ \bigoplus_{k+l=i} E_k \bar{\otimes}_{\mathcal{B}} F_l, \sum_{l+m=j} (-1)^{l(k-m+1)} \alpha_{km} \bar{\otimes} \text{Id}_l + \sum_{k+n=j} (-1)^k \text{Id}_k \bar{\otimes} \beta_{ln} \right\} \rightarrow E \bar{\otimes}_{\mathcal{B}} F. \quad (3.91)$$

(2) Let (F_i, β_{ij}) be a one-sided twisted complex in $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, and let F be its convolution. Then there is a $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ homotopy equivalence

$$\left\{ \bigoplus_{l-k=i} \overline{\mathbf{Hom}}_{\mathcal{B}}(E_k, F_l), \sum_{l-m=j} (-1)^{m(m-k)+l+1} (-) \circ \alpha_{mk} + \sum_{n-k=j} (-1)^{(l-n+1)k} \beta_{ln} \circ (-) \right\} \rightarrow \overline{\mathbf{Hom}}_{\mathcal{B}}(E, F). \quad (3.92)$$

(3) Let (F_i, β_{ij}) be a one-sided twisted complex in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$, and let F be its convolution. Then there is a $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$ homotopy equivalence

$$\left\{ \bigoplus_{l-k=i} \overline{\mathbf{Hom}}_{\mathcal{A}}(E_k, F_l), \sum_{l-m=j} (-1)^{m(m-k)+l+1} (-) \circ \alpha_{mk} + \sum_{n-k=j} (-1)^{(l-n+1)k} \beta_{ln} \circ (-) \right\} \rightarrow \overline{\mathbf{Hom}}_{\mathcal{A}}(E, F). \quad (3.93)$$

Proof. (1): By the definition of the convolution functor as the composition (3.90) it suffices to show that $T \circ (-)$ applied to the twisted complex in the LHS of (3.91) is quasi-isomorphic to $E \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} F$ in $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$. Since the natural inclusion $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C} \xrightarrow{(3.7)} \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$ maps quasi-isomorphisms to homotopy equivalences, the desired assertion then follows.

By [AL17, Lemma 3.4] we have the following isomorphism in $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$:

$$\left\{ \bigoplus_{k+l=i} \tilde{E}_k \otimes_{\mathcal{B}} \tilde{F}_l, \sum_{l+m=j} (-1)^{l(k-m+1)} \tilde{\alpha}_{km} \otimes \text{Id}_l + \sum_{k+n=j} (-1)^k \text{Id}_k \otimes \tilde{\beta}_{ln} \right\} \simeq E \otimes_{\mathcal{B}} F. \quad (3.94)$$

Let (G_i, γ_{ij}) be the image of the twisted complex in the LHS of (3.91) under $\widetilde{(-)}$. We therefore have

$$G_i = \bigoplus_{k+l=i} \bar{\mathcal{A}} \otimes_{\mathcal{A}} E_k \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} F_l \otimes_{\mathcal{C}} \bar{\mathcal{C}}$$

and thus there is a map from (G_i, γ_{ij}) to the twisted complex in the LHS of (3.94) whose only non-zero components are degree 0 homotopy equivalences

$$\bar{\mathcal{A}} \otimes_{\mathcal{A}} E_k \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} F_l \otimes_{\mathcal{C}} \bar{\mathcal{C}} \xrightarrow{\text{Id}^{\otimes 2} \otimes \Delta \otimes \text{Id}^{\otimes 2}} (\bar{\mathcal{A}} \otimes_{\mathcal{A}} E_k \otimes_{\mathcal{B}} \bar{\mathcal{B}}) \otimes_{\mathcal{B}} (\bar{\mathcal{B}} \otimes_{\mathcal{B}} F_l \otimes_{\mathcal{C}} \bar{\mathcal{C}}).$$

One readily checks that these commute with the differentials of the twisted complexes, thus the resulting map from $\{G_i, \gamma_{ij}\}$ to the LHS of (3.94) is closed. Therefore by Cor. 2.12 it is a quasi-isomorphism. Thus $\{G_i, \gamma_{ij}\}$ is quasi-isomorphic to $E \otimes_{\mathcal{B}} F$, and thus to $E \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} F$, as desired.

(2): Similar to the proof of (1) let (G_i, γ_{ij}) be the image of the twisted complex in the LHS of (3.92) under $\widetilde{(-)}$. It suffices to show that $\{G_i, \gamma_{ij}\}$ is quasi-isomorphic to $\text{Hom}_{\mathcal{B}}(E \otimes \bar{\mathcal{B}}, F)$ in $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$.

By [AL17, Lemma 3.4], $\text{Hom}_{\mathcal{B}}(E \otimes \bar{\mathcal{B}}, F)$ is isomorphic in $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$ to the convolution of

$$\left(\bigoplus_{l-k=i} \text{Hom}_{\mathcal{B}}(\tilde{E}_k \otimes \bar{\mathcal{B}}, \tilde{F}_l), \sum_{l-m=j} (-1)^{m(m-k)+l+1} (-) \circ (\tilde{\alpha}_{mk} \otimes \text{Id}) + \sum_{n-k=j} (-1)^{(l-n+1)k} \tilde{\beta}_{ln} \circ (-) \right). \quad (3.95)$$

We have

$$G_i = \bigoplus_{l-k=i} \bar{\mathcal{C}} \otimes_{\mathcal{C}} \text{Hom}_{\mathcal{B}}(E_k \otimes_{\mathcal{B}} \bar{\mathcal{B}}, F_l) \otimes_{\mathcal{A}} \bar{\mathcal{A}}.$$

Consider therefore the following twisted complex over $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$:

$$\left(\bigoplus_{l-k=i} \text{Hom}_{\mathcal{B}}(\tilde{E}_k \otimes \bar{\mathcal{B}}, F_l), \sum_{l-m=j} (-1)^{m(m-k)+l+1} (-) \circ (\tilde{\alpha}_{mk} \otimes \text{Id}) + \sum_{n-k=j} (-1)^{(l-n+1)k} \beta_{ln} \circ (-) \right). \quad (3.96)$$

Consider the map from (3.95) to (3.96) whose components

$$\text{Hom}_{\mathcal{B}}(\bar{\mathcal{A}} \otimes_{\mathcal{A}} E_k \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} \bar{\mathcal{B}}, \bar{\mathcal{C}} \otimes_{\mathcal{C}} F_l \otimes_{\mathcal{B}} \bar{\mathcal{B}}) \xrightarrow{(\tau \text{Id} \otimes \tau) \circ (-)} \text{Hom}_{\mathcal{B}}(\bar{\mathcal{A}} \otimes_{\mathcal{A}} E_k \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} \bar{\mathcal{B}}, F_l)$$

These are homotopy equivalences in $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$, and thus by Cor. 2.12 the induced map between the convolutions of (3.95) and (3.96) is a quasi-isomorphism.

On the other hand, consider the map of twisted complexes from (G_i, γ_{ij}) to (3.96) whose components are the maps

$$\bar{\mathcal{C}} \otimes_{\mathcal{C}} \text{Hom}_{\mathcal{B}}(E_k \otimes_{\mathcal{B}} \bar{\mathcal{B}}, F_l) \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\tau \circ ((-)\circ(\text{Id} \otimes \text{Id} \otimes \tau)) \otimes \tau} \text{Hom}_{\mathcal{B}}(E_k \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} \bar{\mathcal{B}}, F_l).$$

Likewise, these are homotopy equivalences in $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$ and therefore $\{G_i, \gamma_{ij}\}$ is quasi-isomorphic to the convolution of (3.96). It is therefore quasi-isomorphic to (3.95). We conclude that $\{G_i, \gamma_{ij}\}$ is quasi-isomorphic to $\text{Hom}_{\mathcal{B}}(E \otimes \bar{\mathcal{B}}, F)$, as desired.

(3): This is proved similarly to the assertion (2). \square

Lemma 3.44. *Let (E_i, α_{ij}) be a twisted complex over $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. Then there are homotopy equivalences:*

$$\{E_i, \alpha_{ij}\}^{\mathcal{B}} \rightarrow \{E_{-i}^{\mathcal{B}}, (-1)^{j^2+ij+1} \alpha_{(-i)(-j)}^{\mathcal{B}}\}; \quad (3.97)$$

$$\{E_i, \alpha_{ij}\}^{\mathcal{A}} \rightarrow \{E_{-i}^{\mathcal{A}}, (-1)^{j^2+ij+1} \alpha_{(-i)(-j)}^{\mathcal{A}}\}. \quad (3.98)$$

Proof. This is proved similarly to Lemma 3.43. Use [AL17, Lemma 3.5] to write the LHS of (3.98) and (3.97) as twisted complexes over $\mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$. There are obvious maps from these to the images under $\widetilde{(-)}$ of the RHS twisted complexes in (3.98) and (3.97) whose components are all homotopy equivalences. The claim of the Lemma then follows by Cor. 2.12. \square

4. HOMOTOPY ADJUNCTION FOR TENSOR FUNCTORS

Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be DG-categories. In this paper, we frequently consider $\mathcal{A}\text{-}\mathcal{B}$ -, $\mathcal{B}\text{-}\mathcal{C}$ -, etc. bimodules as DG-enhancements of continuous exact functors $D(\mathcal{A}) \rightarrow D(\mathcal{B})$, $D(\mathcal{B}) \rightarrow D(\mathcal{C})$, etc. Accordingly, whenever it is convenient we adopt the following ‘‘functorial’’ notation: given $F \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ and $G \in \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$ we write

$$GF \quad \text{for} \quad F \overline{\otimes}_{\mathcal{B}} G \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}.$$

This is because we work with categories of right modules, and for any $E \in \overline{\mathbf{Mod}}\text{-}\mathcal{A}$ we have

$$E \overline{\otimes}_{\mathcal{A}} (F \overline{\otimes}_{\mathcal{B}} G) = (E \overline{\otimes}_{\mathcal{A}} F) \overline{\otimes}_{\mathcal{B}} G.$$

Thus $F \overline{\otimes}_{\mathcal{B}} G$ enhances the functor which is the composition of first the functor enhanced by F , and then the functor enhanced by G , whence our shorthand GF .

4.1. Tensor functors. Let \mathcal{A} and \mathcal{B} be DG categories and let $f : D(\mathcal{A}) \rightarrow D(\mathcal{B})$ be a continuous exact functor. As the following proposition demonstrates, this is equivalent to f being a *tensor functor*, that is — a functor given by tensoring by an $\mathcal{A}\text{-}\mathcal{B}$ -bimodule:

Proposition 4.1. *The following are equivalent:*

- (1) f has a right adjoint $r : D(\mathcal{B}) \rightarrow D(\mathcal{A})$.
- (2) f is continuous.
- (3) f is isomorphic to $H^0((-) \overline{\otimes}_{\mathcal{A}} M)$ for some $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$.

Proof. The implication (1) \Rightarrow (2) is well-known and straightforward, the implication (2) \Rightarrow (3) follows from [Kel94, §6.4], and the implication (3) \Rightarrow (1) follows since by the Tensor-Hom adjunction the functor $(-) \overline{\otimes}_{\mathcal{A}} M$ has the right adjoint $\overline{\text{Hom}}_{\mathcal{B}}(M, -)$. \square

Let f satisfy these equivalent conditions. Fix $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ such that $(-) \overline{\otimes}_{\mathcal{A}} M$ enhances f as above. By Prop. 3.14 the functor $\overline{\text{Hom}}_{\mathcal{B}}(M, -)$ is genuinely adjoint to $(-) \overline{\otimes}_{\mathcal{A}} M$ and thus enhances r . We conclude that any adjoint pair (f, r) of functors $D(\mathcal{B}) \leftarrow D(\mathcal{A})$ with f enhanceable can be enhanced by a pair of genuinely adjoint DG-functors.

We are however, more interested in the case where f has left and right adjoints l and r which are also tensor functors. Note, that r always exists, but might not be a tensor functor, while l may not exist, but when it does — it is automatically a tensor functor. The conditions of the existence of l and of r being a tensor functor are easily stated in terms of the properties of the DG-bimodule enhancing f :

Theorem 4.1. *Let \mathcal{A} and \mathcal{B} be DG-categories and let $f : D(\mathcal{A}) \rightarrow D(\mathcal{B})$ be a tensor functor. Let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ be any enhancement of f .*

- (1) *The following are equivalent:*
 - (a) *The right adjoint r of f is continuous.*
 - (b) *f restricts to $D_c(\mathcal{A}) \rightarrow D_c(\mathcal{B})$.*
 - (c) *M is \mathcal{B} -perfect.*
 - (d) *$H^0((-) \overline{\otimes}_{\mathcal{B}} M^{\mathcal{B}})$ is the right adjoint of f (see Definition 3.31).*
- (2) *The following are equivalent:*
 - (a) *The left adjoint l of f exists.*
 - (b) *The left adjoint l of f exists and restricts to $D_c(\mathcal{B}) \rightarrow D_c(\mathcal{A})$.*
 - (c) *M is \mathcal{A} -perfect.*
 - (d) *$H^0((-) \overline{\otimes}_{\mathcal{B}} M^{\mathcal{A}})$ is the left adjoint of f (see Definition 3.31).*

Proof. (1): By the definition of adjunction we have $\mathrm{Hom}_{D(\mathcal{B})}(f(-), \bigoplus_{\infty}(-)) \simeq \mathrm{Hom}_{D(\mathcal{A})}(-, r(\bigoplus_{\infty}(-)))$. Thus if r is continuous then f preserves compact objects, i.e. (1a) \Rightarrow (1b). For any $a \in \mathcal{A}$ we have $f({}_a\mathcal{A}) \simeq {}_aM$ in $D(\mathcal{B})$ which shows (1b) \Rightarrow (1c). When M is \mathcal{B} -perfect $(-)\overline{\otimes}_{\mathcal{B}}M^{\overline{\mathcal{B}}}$ is homotopy right adjoint to $(-)\overline{\otimes}_{\mathcal{A}}M$ by Prop. 4.6, whence (1c) \Rightarrow (1d). Finally, the implication (1d) \Rightarrow (1a) is trivial since tensor functors are continuous.

(2): If l exists, then, by above, it has to preserve compact objects since f is continuous, so (2a) \Rightarrow (2b). The implications (2b) \Rightarrow (2c) and (2c) \Rightarrow (2d) are proved analogously to (1b) \Rightarrow (1c) and (1c) \Rightarrow (1d) and again the implication (2d) \Rightarrow (2a) is trivial. \square

4.2. Homotopy adjunction for tensor functors. Let $f: D(\mathcal{A}) \rightarrow D(\mathcal{B})$ be a continuous functor which has left and right adjoints l and r which are also continuous. By Prps. 4.1 we can enhance f , l , and r by DG-bimodules. It is not, to our knowledge, always possible to lift the adjunctions of f , l , and r to genuine adjunctions between the corresponding DG tensor functors. In this section we demonstrate that it is always possible to lift them to homotopy adjunctions in an economical and mutually compatible way.

First, we demonstrate that when M is \mathcal{B} -perfect the functors $((-)\overline{\otimes}_{\mathcal{A}}M, (-)\overline{\otimes}_{\mathcal{B}}M^{\overline{\mathcal{B}}})$ form a homotopy adjoint pair. That is, there exist maps of bimodules in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ and $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ such that the corresponding natural transformations of tensor functors define the unit and the counit of the adjunction in homotopy categories. Similarly, when M is \mathcal{A} -perfect $((-)\overline{\otimes}_{\mathcal{B}}M^{\overline{\mathcal{A}}}, (-)\overline{\otimes}_{\mathcal{A}}M)$ form a homotopy adjoint pair.

It follows immediately from Lemma 3.36 and the Tensor-Hom adjunction that when M is \mathcal{B} -perfect $((-)\overline{\otimes}_{\mathcal{A}}M, (-)\overline{\otimes}_{\mathcal{B}}M^{\overline{\mathcal{B}}})$ are homotopy adjoint. Similarly, when M is \mathcal{A} -perfect $((-)\overline{\otimes}_{\mathcal{B}}M^{\overline{\mathcal{A}}}, (-)\overline{\otimes}_{\mathcal{A}}M^{\overline{\mathcal{A}}})$ are homotopy adjoint. When M is \mathcal{A} -perfect the natural map $M \rightarrow M^{\overline{\mathcal{A}}}$ is an isomorphism in $D(\mathcal{A}\text{-}\mathcal{B})$, thus $((-)\overline{\otimes}_{\mathcal{B}}M^{\overline{\mathcal{A}}}, (-)\overline{\otimes}_{\mathcal{A}}M)$ are also homotopy adjoint.

However, the abstract fact of these functors being homotopy adjoint is not enough. We next write down certain natural lifts of adjunctions units and counits involved to the maps in $\overline{\mathbf{Mod}}$ between the DG-bimodules involved. We then compute the homotopies which arise when writing down relations between these maps. Our choice of natural lifts significantly reduces the number of choices involved and thus the number of higher differentials in the explicit computations:

Definition 4.2. Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. Define the *homotopy trace maps*

$$M \overline{\otimes}_{\mathcal{B}}M^{\overline{\mathcal{A}}} \xrightarrow{\mathrm{tr}} \mathcal{A} \quad \text{and} \quad M^{\overline{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}}M \xrightarrow{\mathrm{tr}} \mathcal{B} \quad (4.1)$$

to be the Tensor-Hom adjunction counits applied to the diagonal bimodules \mathcal{A} and \mathcal{B} , respectively.

To define homotopy action maps and to work with the resulting homotopy adjunctions we need to choose and fix the following homotopy inverses and higher homotopies as per (3.35):

Definition 4.3. Let \mathcal{A} and \mathcal{B} be DG-categories.

- (1) For every \mathcal{B} -perfect $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ fix once and for all a homotopy inverse

$$\mathrm{Hom}_{\mathcal{B}}(M, M) \xrightarrow{\zeta_{\mathcal{B}}} M \overline{\otimes}_{\mathcal{B}}M^{\overline{\mathcal{B}}} \quad (4.2)$$

of the map $\eta_{\mathcal{B}}$ defined in Defn. 3.35. Furthermore, choose and fix

$$\omega_{\mathcal{B}} \in \mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}^{-1}(\mathrm{Hom}_{\mathcal{B}}(M, M), \mathrm{Hom}_{\mathcal{B}}(M, M)) \quad \text{such that } d\omega_{\mathcal{B}} = \eta_{\mathcal{B}} \circ \zeta_{\mathcal{B}} - \mathrm{Id}, \quad (4.3)$$

$$\omega'_{\mathcal{B}} \in \mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}^{-1}\left(M \overline{\otimes}_{\mathcal{B}}M^{\overline{\mathcal{B}}}, M \overline{\otimes}_{\mathcal{B}}M^{\overline{\mathcal{B}}}\right) \quad \text{such that } d\omega'_{\mathcal{B}} = \mathrm{Id} - \zeta_{\mathcal{B}} \circ \eta_{\mathcal{B}}, \quad (4.4)$$

$$\phi_{\mathcal{B}} \in \mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}^{-2}\left(M \overline{\otimes}_{\mathcal{B}}M^{\overline{\mathcal{B}}}, \mathrm{Hom}_{\mathcal{B}}(M, M)\right) \quad \text{such that } d\phi_{\mathcal{B}} = \omega_{\mathcal{B}} \circ \eta + \eta \circ \omega'_{\mathcal{B}} \quad (4.5)$$

- (2) For every \mathcal{A} -perfect $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ fix once and for all a homotopy inverse

$$\mathrm{Hom}_{\mathcal{A}}(M, M) \xrightarrow{\zeta_{\mathcal{A}}} M^{\overline{\mathcal{A}}} \overline{\otimes}_{\mathcal{A}}M \quad (4.6)$$

of the map $\eta_{\mathcal{A}}$ defined in Defn. 3.35. Furthermore, choose and fix $\omega_{\mathcal{A}}$, $\omega'_{\mathcal{A}}$ and $\phi_{\mathcal{A}}$ analogously.

We now define the homotopy action maps:

Definition 4.4. Let \mathcal{A} and \mathcal{B} be DG-categories. For all \mathcal{B} -perfect (resp. \mathcal{A} -perfect) $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ define the *homotopy \mathcal{A} -action (resp. \mathcal{B} -action) map*

$$\mathcal{A} \xrightarrow{\mathrm{act}} M \overline{\otimes}_{\mathcal{B}}M^{\overline{\mathcal{B}}}, \quad (4.7)$$

$$\text{resp. } \mathcal{B} \xrightarrow{\mathrm{act}} M^{\overline{\mathcal{A}}} \overline{\otimes}_{\mathcal{A}}M \quad (4.8)$$

to be the composition

$$\mathcal{A} \xrightarrow{\text{act}} \overline{\text{Hom}}_{\mathcal{B}}(M, M) \xrightarrow{\zeta_{\mathcal{B}}} M \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{B}}}, \quad (4.9)$$

$$\text{resp. } \mathcal{B} \xrightarrow{\text{act}} \overline{\text{Hom}}_{\mathcal{A}}(M, M) \xrightarrow{\zeta_{\mathcal{A}}} M^{\overline{\mathcal{A}}} \overline{\otimes}_{\mathcal{A}} M. \quad (4.10)$$

Finally, we define the maps whose boundaries we prove below to be the difference between our homotopy adjunctions and genuine ones:

Definition 4.5. (1) Define $\chi_{\mathcal{B}}, \chi_{\mathcal{A}} \in \overline{\text{Hom}}_{\mathcal{A}\text{-}\mathcal{B}}^{-1}(M, M)$ to be the compositions

$$M \xrightarrow{\text{act} \overline{\otimes} \text{Id}} \overline{\text{Hom}}_{\mathcal{B}}(M, M) \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\omega_{\mathcal{B}} \overline{\otimes} \text{Id}} \overline{\text{Hom}}_{\mathcal{B}}(M, M) \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\text{ev}} M, \quad (4.11)$$

$$M \xrightarrow{\text{act} \overline{\otimes} \text{Id}} M \overline{\otimes}_{\mathcal{B}} \overline{\text{Hom}}_{\mathcal{A}}(M, M) \xrightarrow{\text{Id} \overline{\otimes} \omega_{\mathcal{A}}} M \overline{\otimes}_{\mathcal{B}} \overline{\text{Hom}}_{\mathcal{A}}(M, M) \xrightarrow{\text{ev}} M. \quad (4.12)$$

(2) Define $\chi'_{\mathcal{B}} \in \overline{\text{Hom}}_{\mathcal{B}\text{-}\mathcal{A}}^{-1}(M^{\overline{\mathcal{B}}}, M^{\overline{\mathcal{B}}})$ and $\chi'_{\mathcal{A}} \in \overline{\text{Hom}}_{\mathcal{B}\text{-}\mathcal{A}}^{-1}(M^{\overline{\mathcal{A}}}, M^{\overline{\mathcal{A}}})$ to be the $\mathcal{B}\text{-Mod-}\mathcal{A}$ maps

$$M^{\overline{\mathcal{B}}} \xrightarrow{\text{Id} \overline{\otimes} \text{act}} M^{\overline{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} \overline{\text{Hom}}_{\mathcal{B}}(M, M) \xrightarrow{\text{Id} \overline{\otimes} \omega_{\mathcal{B}}} M^{\overline{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} \overline{\text{Hom}}_{\mathcal{B}}(M, M) \xrightarrow{\text{cmps}} M^{\overline{\mathcal{B}}} \quad (4.13)$$

$$M^{\overline{\mathcal{A}}} \xrightarrow{\text{act} \overline{\otimes} \text{Id}} \overline{\text{Hom}}_{\mathcal{A}}(M, M) \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{A}}} \xrightarrow{\omega_{\mathcal{A}} \overline{\otimes} \text{Id}} \overline{\text{Hom}}_{\mathcal{A}}(M, M) \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{A}}} \xrightarrow{\text{cmps}} M^{\overline{\mathcal{A}}} \quad (4.14)$$

(3) Define $\xi'_{\mathcal{B}} \in \overline{\text{Hom}}_{\mathcal{B}\text{-}\mathcal{B}}^{-1}(M^{\overline{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} M, M^{\overline{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} M)$ and $\xi'_{\mathcal{A}} \in \overline{\text{Hom}}_{\mathcal{A}\text{-}\mathcal{A}}^{-1}(M \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{A}}}, M \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{A}}})$ to be the maps

$$M^{\overline{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\omega_{\mathcal{B}}(\text{Id}_M)^{\overline{\mathcal{B}}} \overline{\otimes} \text{Id} - \text{Id} \overline{\otimes} \omega_{\mathcal{B}}(\text{Id}_M)} M^{\overline{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} M, \quad (4.15)$$

$$M \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{A}}} \xrightarrow{\text{Id} \overline{\otimes} \omega_{\mathcal{A}}(\text{Id}_M)^{\overline{\mathcal{A}}} - \omega_{\mathcal{A}}(\text{Id}_M) \overline{\otimes} \text{Id}} M \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{A}}}, \quad (4.16)$$

where e.g. $\omega_{\mathcal{B}}(\text{Id}_M) \in \overline{\text{Hom}}_{\mathcal{B}}^{-1}(M, M)$ is the image of Id_M under the map $\omega_{\mathcal{B}}$ defined in (4.3).

It is now convenient to adopt the functorial notation explained in the beginning of this section. Let F denote the bimodule $M \in \mathcal{A}\text{-Mod-}\mathcal{B}$ and R and L denote the bimodules $M^{\overline{\mathcal{B}}}$ and $M^{\overline{\mathcal{A}}}$ in $\mathcal{B}\text{-Mod-}\mathcal{A}$.

We introduce further conventions regarding the diagonal bimodules \mathcal{A} and \mathcal{B} :

- When they occur on their own, they are denoted as $\text{Id}_{\mathcal{A}}$ or $\text{Id}_{\mathcal{B}}$, respectively.
- When working with tensor products of several bimodules, we suppress all appearance of diagonal bimodules in them by implicit use of the homotopy equivalences α and β defined in §3.3. This is analogous to implicit use of the equalities $\Phi \circ \text{Id} = \text{Id} \circ \Phi = \Phi$ which exist for any functor Φ .
- More specifically, given a map whose source is a diagonal bimodule we write it as applied “in between” two factors of a tensor product of bimodules. This means first applying β to either of the two factors. It doesn’t matter which — the resulting map is the same. The corresponding map in the ordinary module category applies Δ to the bar complex in the middle.

For example, we write $FR \xrightarrow{F \text{ act } R} FRFR$ to denote the composition

$$M^{\overline{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\beta \overline{\otimes} \text{Id}} M^{\overline{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} \mathcal{A} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\text{Id} \overline{\otimes} \text{act} \overline{\otimes} \text{Id}} M^{\overline{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} M \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} M.$$

We could have used $\text{Id} \overline{\otimes} \beta$ instead of $\beta \overline{\otimes} \text{Id}$ as they both correspond to the $\mathcal{B}\text{-Mod-}\mathcal{B}$ map

$$\overline{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} \text{Hom}_{\mathcal{B}}(M \otimes \overline{\mathcal{B}}, \mathcal{B}) \otimes_{\mathcal{A}} \overline{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \overline{\mathcal{B}} \xrightarrow{\tau \otimes \text{Id} \otimes \Delta \otimes \text{Id} \otimes \tau} \text{Hom}_{\mathcal{B}}(M \otimes \overline{\mathcal{B}}, \mathcal{B}) \otimes_{\mathcal{A}} \overline{\mathcal{A}} \otimes_{\mathcal{A}} \overline{\mathcal{A}} \otimes_{\mathcal{A}} M.$$

- Similarly, when we apply a map whose target is a diagonal bimodule to a part of a tensor product of bimodules, we suppress this diagonal bimodule in the resulting product. This stands for using α on the product of this diagonal bimodule with one of its two possible neighbouring factors. When both are present we choose the left one. This choice matters: in the underlying ordinary module category it chooses which of the two bar complexes in the middle to contract with the map τ .

For example, we write $FLFR \xrightarrow{F \text{ tr } L} FR$ to denote the composition

$$M^{\overline{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} M \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{A}}} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\text{Id} \overline{\otimes} \text{tr} \overline{\otimes} \text{Id}} M^{\overline{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} \mathcal{A} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\alpha \overline{\otimes} \text{Id}} M^{\overline{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} M.$$

The map $\alpha \overline{\otimes} \text{Id}$ corresponds to the $\mathcal{B}\text{-Mod-}\mathcal{B}$ map

$$\overline{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} \text{Hom}_{\mathcal{B}}(M \otimes \overline{\mathcal{B}}, \mathcal{B}) \otimes_{\mathcal{A}} \overline{\mathcal{A}} \otimes_{\mathcal{A}} \overline{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \overline{\mathcal{B}} \xrightarrow{\tau \otimes \text{Id} \otimes \tau \otimes \text{Id} \otimes \text{Id} \otimes \tau} \text{Hom}_{\mathcal{B}}(M \otimes \overline{\mathcal{B}}, \mathcal{B}) \otimes_{\mathcal{A}} \overline{\mathcal{A}} \otimes_{\mathcal{A}} M,$$

while $\text{Id} \overline{\otimes} \alpha$ corresponds to the map $\tau \otimes \text{Id} \otimes \text{Id} \otimes \tau \otimes \text{Id} \otimes \tau$.

- Finally, we have a special convention regarding the maps $RFR \xrightarrow{R \text{tr}} R$ and $LFL \xrightarrow{\text{tr} L} L$. By the general rules laid out above $RFR \xrightarrow{R \text{tr}} R$ should denote the map

$$M^{\mathcal{B}} \overline{\otimes}_{\mathcal{A}} M \overline{\otimes}_{\mathcal{B}} M^{\mathcal{B}} \xrightarrow{\text{tr} \overline{\otimes} \text{Id}} \mathcal{B} \overline{\otimes}_{\mathcal{B}} M^{\mathcal{B}} \xrightarrow{\alpha} M^{\mathcal{B}}. \quad (4.17)$$

Instead, we denote by $R \text{tr}$ the map

$$M^{\mathcal{B}} \overline{\otimes}_{\mathcal{A}} M \overline{\otimes}_{\mathcal{B}} M^{\mathcal{B}} \xrightarrow{\text{Id} \overline{\otimes} \gamma \overline{\otimes} \text{Id}} M^{\mathcal{B}} \overline{\otimes}_{\mathcal{A}} \overline{\text{Hom}}_{\mathcal{B}}(\mathcal{B}, M) \overline{\otimes}_{\mathcal{B}} M^{\mathcal{B}} \xrightarrow{\text{cmps}} M^{\mathcal{B}}. \quad (4.18)$$

Note that it follows from Lemma 3.16 that the maps (4.17) and (4.18) are homotopic in $\mathcal{B}\text{-}\overline{\text{Mod}}\text{-}\mathcal{A}$ and thus are isomorphic in $D(\mathcal{B}\text{-}\mathcal{A})$.

Similarly, we denote by $\text{tr} L$ the map

$$M^{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \overline{\otimes}_{\mathcal{B}} M^{\mathcal{A}} \xrightarrow{\text{Id} \overline{\otimes} \gamma \overline{\otimes} \text{Id}} M^{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} \overline{\text{Hom}}_{\mathcal{A}}(\mathcal{A}, M) \overline{\otimes}_{\mathcal{B}} M^{\mathcal{A}} \xrightarrow{\text{cmps}} M^{\mathcal{A}}. \quad (4.19)$$

In a tensor product of several maps we always evaluate all instances of the maps $R \text{tr}$ and $\text{tr} L$ first. For example, $FRFR \xrightarrow{\text{tr} \text{tr}} \text{Id}$ denotes the map

$$M^{\mathcal{B}} \overline{\otimes}_{\mathcal{A}} M \overline{\otimes}_{\mathcal{B}} M^{\mathcal{B}} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\text{Id} \overline{\otimes} \gamma \overline{\otimes} \text{Id} \overline{\otimes} \text{Id}} M^{\mathcal{B}} \overline{\otimes}_{\mathcal{A}} \overline{\text{Hom}}_{\mathcal{B}}(\mathcal{B}, M) \overline{\otimes}_{\mathcal{B}} M^{\mathcal{B}} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\text{cmps} \overline{\otimes} \text{Id}} M^{\mathcal{B}} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\text{tr}} \mathcal{B}$$

and not the map

$$M^{\mathcal{B}} \overline{\otimes}_{\mathcal{A}} M \overline{\otimes}_{\mathcal{B}} M^{\mathcal{B}} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\text{Id} \overline{\otimes} \text{Id} \overline{\otimes} \text{tr}} M^{\mathcal{B}} \overline{\otimes}_{\mathcal{A}} \overline{\text{Hom}}_{\mathcal{B}}(\mathcal{B}, M) \overline{\otimes}_{\mathcal{B}} \mathcal{B} \xrightarrow{\text{Id} \overline{\otimes} \alpha} M^{\mathcal{B}} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\text{tr}} \mathcal{B}.$$

Proposition 4.6. *Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$.*

- (1) *If M is \mathcal{A} -perfect, we have in $\mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$ and $\mathcal{B}\text{-}\overline{\text{Mod}}\text{-}\mathcal{A}$, respectively:*

$$F \xrightarrow{\text{act} F} FLF \xrightarrow{F \text{tr}} F = \text{Id} + d\chi_{\mathcal{A}}, \quad (4.20)$$

$$L \xrightarrow{L \text{act}} LFL \xrightarrow{\text{tr} L} L = \text{Id} + d\chi'_{\mathcal{A}}. \quad (4.21)$$

- (2) *If M is \mathcal{B} -perfect, we have in $\mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$ and $\mathcal{B}\text{-}\overline{\text{Mod}}\text{-}\mathcal{A}$, respectively:*

$$F \xrightarrow{F \text{act}} FRF \xrightarrow{\text{tr} F} F = \text{Id} + d\chi_{\mathcal{B}}, \quad (4.22)$$

$$R \xrightarrow{\text{act} R} RFR \xrightarrow{R \text{tr}} R = \text{Id} + d\chi'_{\mathcal{B}}. \quad (4.23)$$

Proof. We only prove the assertion (1), the assertion (2) is proved similarly.

Consider the following diagram of morphisms in $\mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$:

$$\begin{array}{ccccc} M & \xrightarrow{\text{Id} \overline{\otimes} \text{act}} & M \overline{\otimes}_{\mathcal{B}} M^{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M & \xrightarrow{\text{tr} \overline{\otimes} \text{Id}} & \mathcal{A} \overline{\otimes}_{\mathcal{A}} M \\ & \searrow^{\text{Id} \overline{\otimes} \text{act}} & \downarrow^{\text{Id} \overline{\otimes} \eta_{\mathcal{A}}} & \swarrow^{\text{Id} \overline{\otimes} \gamma} & \downarrow^{\alpha} \\ & & M \overline{\otimes}_{\mathcal{B}} \overline{\text{Hom}}_{\mathcal{A}}(M, M) & \xrightarrow{\text{ev}} & M \\ & & & \swarrow^{\text{ev}} & \end{array}$$

(A)

The triangle (A) commutes up to $d((\text{Id} \overline{\otimes} \omega_{\mathcal{A}}) \circ (\text{Id} \overline{\otimes} \text{act}))$. The rest of the diagram commutes: the pentagon — by definition of the evaluation map, and the triangle — by direct verification. Therefore the perimeter of the diagram commutes up to $d\chi_{\mathcal{A}}$.

The composition of the lower-left half of the perimeter is readily verified to be the identity morphism. On the other hand, the composition of the upper-right half of the perimeter is the LHS of (4.20). Since the perimeter of the diagram commutes up to $d\chi_{\mathcal{A}}$, the equality in (4.20) follows.

Next, consider the following diagram of morphisms in $\mathcal{B}\text{-}\overline{\text{Mod}}\text{-}\mathcal{A}$:

$$\begin{array}{ccccc} M^{\mathcal{A}} & \xrightarrow{\text{act} \overline{\otimes} \text{Id}} & M^{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \overline{\otimes}_{\mathcal{B}} M^{\mathcal{A}} & \xrightarrow{\text{Id} \overline{\otimes} \gamma \overline{\otimes} \text{Id}} & M^{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} \overline{\text{Hom}}_{\mathcal{A}}(\mathcal{A}, M) \overline{\otimes}_{\mathcal{B}} M^{\mathcal{A}} \\ & \searrow^{\text{act} \overline{\otimes} \text{Id}} & \downarrow^{\eta_{\mathcal{A}} \overline{\otimes} \text{Id}} & \swarrow^{\text{cmps}} & \downarrow^{\text{cmps}} \\ & & \overline{\text{Hom}}_{\mathcal{A}}(M, M) \overline{\otimes}_{\mathcal{B}} M^{\mathcal{A}} & \xrightarrow{\text{cmps}} & M^{\mathcal{A}}. \\ & & & \swarrow^{\text{cmps}} & \end{array}$$

(1)

Similar to the above, the lower-left half of the perimeter composes to Id , while the upper-right — to the LHS of (4.21). The triangle (A) commutes up to $d((\omega_{\mathcal{A}} \overline{\otimes} \text{Id}) \circ (\text{act} \overline{\otimes} \text{Id}))$ and the quadrilateral commutes

by the associativity of the composition in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$. It follows that the perimeter commutes up to $d\chi'_A$, whence (4.21). \square

Corollary 4.7. *Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$.*

(1) *If M is \mathcal{A} -perfect, we have in $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, respectively:*

$$LF \xrightarrow{L \text{ act } F} LFLF \xrightarrow{\text{tr } LF - LF \text{ tr}} LF = d\xi'_A, \quad (4.24)$$

(2) *If M is \mathcal{B} -perfect, we have in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$, respectively:*

$$FR \xrightarrow{F \text{ act } R} FRFR \xrightarrow{FR \text{ tr} - \text{tr } FR} FR = d\xi'_B, \quad (4.25)$$

Proof. By Prop. 4.6 the LHS of (4.24) equals $d(\chi'_A F - L\chi_A)$, and $\chi'_A F - L\chi_A$ is the composition

$$M \otimes_{\mathcal{B}} M^{\bar{A}} \xrightarrow{\text{Id} \otimes (\omega_{\mathcal{A} \text{ oact}}) \otimes \text{Id}} M \otimes_{\mathcal{B}} \overline{\mathbf{Hom}}_{\mathcal{A}}(M, M) \otimes_{\mathcal{B}} M^{\bar{A}} \xrightarrow{\text{Id} \otimes \text{cmps} - \text{ev} \otimes \text{Id}} M \otimes_{\mathcal{B}} M^{\bar{A}}$$

which is precisely ξ'_A . The other assertion works out similarly. \square

Next we treat the two compositions dual to (4.24) and (4.25):

$$FL \xrightarrow{FL \text{ act} - \text{act } FL} FLFL \xrightarrow{F \text{ tr } L} FL, \quad (4.26)$$

$$RF \xrightarrow{\text{act } RF - RF \text{ act}} RFRF \xrightarrow{R \text{ tr } F} RF. \quad (4.27)$$

Here things do not work out so well, because, for instance, in (4.26) after contracting $M^{\bar{A}} \otimes M \otimes M^{\bar{A}} \otimes M$ to $M^{\bar{A}} \otimes \overline{\mathbf{Hom}}_{\mathcal{A}}(\mathcal{A}, \mathcal{A}) \otimes M$ the map $F \text{ tr } L$ composes $\overline{\mathbf{Hom}}_{\mathcal{A}}(\mathcal{A}, \mathcal{A})$ with $M^{\bar{A}}$ on the left. This works well when composed with $FL \text{ act}$, but not with $\text{act } FL$. Thus we do not get something as simple as $d(F\chi'_A - \chi_A L)$.

We can fix this by composing (4.26) with the homotopy equivalence $M^{\bar{A}} \otimes M \xrightarrow{\eta} \overline{\mathbf{Hom}}_{\mathcal{A}}(M, M)$. Then it doesn't matter whether in $M^{\bar{A}} \otimes \overline{\mathbf{Hom}}_{\mathcal{A}}(\mathcal{A}, \mathcal{A}) \otimes M$ we contract $\overline{\mathbf{Hom}}_{\mathcal{A}}(\mathcal{A}, \mathcal{A})$ to the left or to the right. And any boundary which lifts $\eta \circ (4.26)$ induces a lift of (4.26) itself. More generally:

Lemma 4.8. *Let $E, F, F' \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ and let $F \xrightarrow{s} F'$ and $F' \xrightarrow{s'} F$ be closed maps of degree 0 such that there exists $F \xrightarrow{t} F$ with $dt = \text{Id} - s' \circ s$.*

(1) *Let $E \xrightarrow{f} F$ be a closed map and let $E \xrightarrow{g} F'$ be such that $dg = s \circ f$. Then*

$$d(t \circ f + s' \circ g) = f.$$

We say that $t \circ f + s' \circ g$ is the lift of f induced by the lift g of $s \circ f$.

(2) *Let f and g be as above and let h be another lift of f . Let $E \xrightarrow{j} F'$ be such that $dj = g - s \circ h$. Then*

$$d(s' \circ j - t \circ h) = t \circ f + s' \circ g - h.$$

We say that $s' \circ j - t \circ h$ is the lift of $t \circ f + s' \circ g - h$ induced by the lift j of $g - s \circ h$.

Proof. Direct computation. \square

Proposition 4.9. *Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$.*

(1) *If M is \mathcal{A} -perfect, then in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ the map $\eta \circ (4.26)$ is the boundary of*

$$\text{cmps} \circ \left((\omega \otimes \eta) \circ (\text{act} \otimes \text{Id}) - (\eta \otimes \omega) \circ (\text{Id} \otimes \text{act}) \right). \quad (4.28)$$

Define ξ_A to be the induced lift of (4.26) as per Lemma 4.8(1) with $s = \eta_A$, $s' = \zeta_A$, $t = \omega'_A$, $f = (4.26)$, and $g = (4.28)$.

(2) *If M is \mathcal{B} -perfect, then in $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ the map $\eta \circ (4.27)$ is the boundary of*

$$\text{cmps} \circ \left((\eta \otimes \omega) \circ (\text{Id} \otimes \text{act}) - (\omega \otimes \eta) \circ (\text{act} \otimes \text{Id}) \right). \quad (4.29)$$

Define ξ_B to be the induced lift of (4.27) as per Lemma 4.8(1) with $s = \eta_B$, $s' = \zeta_B$, $t = \omega'_B$, $f = (4.27)$, and $g = (4.29)$.

Proof. We only prove the assertion (1), the assertion (2) is proved analogously. Consider the diagram:

$$\begin{array}{ccccc} M^{\bar{A}} \otimes_{\mathcal{A}} M & \xrightarrow{\text{act} \otimes \text{Id} - \text{Id} \otimes \text{act}} & M^{\bar{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} M^{\bar{A}} \otimes_{\mathcal{A}} M & \xrightarrow{\text{cmps} \otimes \text{Id}} & M^{\bar{A}} \otimes_{\mathcal{A}} M \\ \downarrow \eta & & \downarrow \eta \otimes \eta & & \downarrow \eta \\ \overline{\mathbf{Hom}}_{\mathcal{A}}(M, M) & \xrightarrow{\text{act} \otimes \text{Id} - \text{Id} \otimes \text{act}} & \overline{\mathbf{Hom}}_{\mathcal{A}}(M, M) \otimes_{\mathcal{B}} \overline{\mathbf{Hom}}_{\mathcal{A}}(M, M) & \xrightarrow{\text{cmps}} & \overline{\mathbf{Hom}}_{\mathcal{A}}(M, M) \end{array}$$

The left rectangle commutes up to

$$d\left((\omega \overline{\otimes} \eta) \circ (\text{act} \overline{\otimes} \text{Id}) - (\eta \overline{\otimes} \omega) \circ (\text{Id} \overline{\otimes} \text{act})\right),$$

while the right rectangle genuinely commutes. The composition in the bottom row is zero. As the composition in the top row is (4.26), the assertion (1) follows. \square

Corollary 4.10. *Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$.*

(1) *If M is \mathcal{A} -perfect, we have in $\mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{A}$*

$$FL \xrightarrow{FL \text{ act} - \text{act} FL} FLFL \xrightarrow{F \text{ tr} L} FL = d\xi_{\mathcal{A}}. \quad (4.30)$$

(2) *If M is \mathcal{B} -perfect, we have in $\mathcal{B}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$, respectively:*

$$RF \xrightarrow{\text{act} RF - RF \text{ act}} RFRF \xrightarrow{R \text{ tr} F} RF = d\xi_{\mathcal{B}}. \quad (4.31)$$

4.3. Canonical twisted complexes associated to a homotopy adjunction. In Corollaries 4.7 and 4.10 we've shown that each of the four compositions in (4.24)-(4.27) is a boundary. We thus obtain four natural three-term twisted complexes e.g.

$$\begin{array}{ccc} & \xrightarrow{\quad -\xi_{\mathcal{A}} \quad} & \\ & \text{---} & \\ FL & \xrightarrow{FL \text{ act} - \text{act} FL} & FLFL \xrightarrow{F \text{ tr} L} FL \end{array} \quad (4.32)$$

In this section, we show that these extend to natural four-term twisted complexes whose convolutions are the squares of the twist T , the dual twist T' , the co-twist C and the dual co-twist C' of F .

Definition 4.11. Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$.

(1) If M is \mathcal{B} -perfect, define the degree -1 morphism

$$\text{Id}_{\mathcal{A}} \xrightarrow{v_{\mathcal{B}}} RFRF$$

to be the composition

$$\mathcal{A} \xrightarrow{\pi_{\mathcal{A}}} \mathcal{A} \overline{\otimes}_{\mathcal{A}} \mathcal{A} \xrightarrow{\text{act} \overline{\otimes} \text{act}} M \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} M \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{B}}} \quad (4.33)$$

(2) If M is \mathcal{A} -perfect, define the degree -1 morphism

$$\text{Id}_{\mathcal{B}} \xrightarrow{v_{\mathcal{A}}} FLFL$$

to be the composition

$$\mathcal{B} \xrightarrow{-\pi_{\mathcal{B}}} \mathcal{B} \overline{\otimes}_{\mathcal{B}} \mathcal{B} \xrightarrow{\text{act} \overline{\otimes} \text{act}} M^{\overline{\mathcal{A}}} \overline{\otimes}_{\mathcal{A}} M \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{A}}} \overline{\otimes}_{\mathcal{A}} M. \quad (4.34)$$

Here $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ are the maps defined in (3.26).

Proposition 4.12. *Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$.*

(1) *If M is \mathcal{A} -perfect, then in $\mathcal{B}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$ the map*

$$(4.28) \circ \text{act} - \eta_{\mathcal{A}} \circ F \text{ tr} L \circ v_{\mathcal{A}} \quad (4.35)$$

is the boundary of

$$\text{cmpr} \circ (\omega \overline{\otimes} (\eta \circ \zeta) + \text{Id} \overline{\otimes} \omega) \circ (\text{act} \overline{\otimes} \text{act}) \circ \pi - \text{cmpr} \circ (\omega \overline{\otimes} \omega) \circ (\text{act} \overline{\otimes} \text{act}) \circ \beta^r. \quad (4.36)$$

Define $v_{\mathcal{A}}$ to be the induced lift of $\xi_{\mathcal{A}} \circ \text{act} - F \text{ tr} L \circ v_{\mathcal{A}}$ as per Lemma 4.8(2) with $s = \eta_{\mathcal{A}}$, $s' = \zeta_{\mathcal{A}}$, $t = \omega'_{\mathcal{A}}$, $f = (4.26) \circ \text{act}$, $g = (4.28) \circ \text{act}$, $h = F \text{ tr} L \circ v_{\mathcal{A}}$, and $j = (4.36)$.

(2) *If M is \mathcal{B} -perfect, then in $\mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{A}$ the map*

$$(4.29) \circ \text{act} - \eta_{\mathcal{B}} \circ F \text{ tr} R \circ v_{\mathcal{B}}$$

is the boundary of

$$\text{cmpr} \circ (\omega \overline{\otimes} \omega) \circ (\text{act} \overline{\otimes} \text{act}) \circ \beta^l - \text{cmpr} \circ ((\eta \circ \zeta) \overline{\otimes} \omega + \omega \overline{\otimes} \text{Id}) \circ (\text{act} \overline{\otimes} \text{act}) \circ \pi \quad (4.37)$$

Denote by $v_{\mathcal{B}}$ the induced lift of $\xi_{\mathcal{B}} \circ \text{act} - F \text{ tr} R \circ v_{\mathcal{B}}$ as per Lemma 4.8(2) with $s = \eta_{\mathcal{B}}$, $s' = \zeta_{\mathcal{B}}$, $t = \omega'_{\mathcal{B}}$, $f = (4.27) \circ \text{act}$, $g = (4.29) \circ \text{act}$, $h = F \text{ tr} R \circ v_{\mathcal{B}}$, and $j = (4.37)$.

Proof. We only prove the assertion (1) as the proof of (2) is similar. Applying the differential (4.36) we obtain

$$\begin{aligned} & \text{cmps} \circ ((\eta \circ \zeta - \text{Id}) \otimes (\eta \circ \zeta) + \text{Id} \otimes (\eta \circ \zeta - \text{Id})) \circ (\text{act} \otimes \text{act}) \circ \pi \quad - \\ & - \text{cmps} \circ (\omega \otimes (\eta \circ \zeta) + \text{Id} \otimes \omega) \circ (\text{act} \otimes \text{act}) \circ (\beta^r - \beta^l) \quad - \\ & - \text{cmps} \circ ((\eta \circ \zeta - \text{Id}) \otimes \omega - \omega \otimes (\eta \circ \zeta - \text{Id})) \circ (\text{act} \otimes \text{act}) \circ \beta^r \end{aligned}$$

which simplifies to

$$\begin{aligned} & \text{cmps} \circ ((\eta \circ \zeta) \otimes (\eta \circ \zeta) - \text{Id} \otimes \text{Id}) \circ (\text{act} \otimes \text{act}) \circ \pi \quad + \\ & + \text{cmps} \circ (\omega \otimes (\eta \circ \zeta) + \text{Id} \otimes \omega) \circ (\text{act} \otimes \text{act}) \circ \beta^l \quad - \\ & - \text{cmps} \circ ((\eta \circ \zeta) \otimes \omega + \omega \otimes \text{Id}) \circ (\text{act} \otimes \text{act}) \circ \beta^r. \end{aligned}$$

By Prop. 3.28(3) the maps $\text{cmps} \circ (\text{Id} \otimes \omega) \circ (\text{act} \otimes \text{act}) \circ \beta^l$ and $\text{cmps} \circ (\omega \otimes \text{Id}) \circ (\text{act} \otimes \text{act}) \circ \beta^r$ are readily seen to both be equal to the map $\omega \circ \text{act}$. On the other hand, the map $\text{cmps} \circ (\text{act} \otimes \text{act}) \circ \pi$ corresponds in $\mathcal{B}\text{-Mod-}\mathcal{B}$ to the map

$$\bar{\mathcal{B}} \otimes \bar{\mathcal{B}} \xrightarrow{\mu} \bar{\mathcal{B}} \xrightarrow{\tau} \mathcal{B} \xrightarrow{\text{act}} \text{Hom}_{\mathcal{A}}(M, M) \xrightarrow{(-) \circ (\tau \otimes \text{Id})} \text{Hom}_{\mathcal{A}}(\bar{\mathcal{A}} \otimes M, M)$$

which vanishes as $\tau \circ \mu = 0$.

Hence the expression above for the boundary of (4.36) simplifies further to

$$\begin{aligned} & \text{cmps} \circ ((\eta \circ \zeta) \otimes (\eta \circ \zeta)) \circ (\text{act} \otimes \text{act}) \circ \pi \quad + \\ & + \text{cmps} \circ (\omega \otimes (\eta \circ \zeta)) \circ (\text{act} \otimes \text{act}) \circ \beta^l \quad - \\ & - \text{cmps} \circ ((\eta \circ \zeta) \otimes \omega) \circ (\text{act} \otimes \text{act}) \circ \beta^r. \end{aligned}$$

which is precisely the map (4.35) as desired. \square

Theorem 4.2. *Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-Mod-}\mathcal{B}$.*

(1) *If M is \mathcal{B} -perfect, the following is a twisted complex over $\mathcal{B}\text{-Mod-}\mathcal{B}$*

$$\begin{array}{ccccccc} & & \xi'_B & & & & \\ & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \\ FR & \xrightarrow{F \text{act} R} & FRFR & \xrightarrow{FR \text{tr} - \text{tr} FR} & FR & \xrightarrow{\text{tr}} & \text{Id}_{\mathcal{B}}, \\ & & & & & & \text{deg.0} \end{array} \quad (4.38)$$

and the following is a twisted complex over $\mathcal{A}\text{-Mod-}\mathcal{A}$

$$\begin{array}{ccccccc} & & \nu_B & & & & \\ & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \\ \text{Id}_{\mathcal{A}} & \xrightarrow{\text{act}} & RF & \xrightarrow{\text{act} RF - RF \text{act}} & RFRF & \xrightarrow{F \text{tr} R} & RF. \\ & & & & & & \text{deg.0} \end{array} \quad (4.39)$$

(2) *If M is \mathcal{A} -perfect, the following is a twisted complex over $\mathcal{A}\text{-Mod-}\mathcal{A}$*

$$\begin{array}{ccccccc} & & \xi'_A & & & & \\ & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \\ LF & \xrightarrow{L \text{act} F} & LFLF & \xrightarrow{\text{tr} LF - LF \text{tr}} & LF & \xrightarrow{\text{tr}} & \text{Id}_{\mathcal{A}}, \\ & & & & & & \text{deg.0} \end{array} \quad (4.40)$$

and the following is a twisted complex over $\mathcal{B}\text{-Mod-}\mathcal{B}$

$$\begin{array}{ccccccc} & & \nu_A & & & & \\ & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \\ \text{Id}_{\mathcal{B}} & \xrightarrow{\text{act}} & FL & \xrightarrow{FL \text{act} - \text{act} FL} & FLFL & \xrightarrow{F \text{tr} L} & FL. \\ & & & & & & \text{deg.0} \end{array} \quad (4.41)$$

Proof. We only prove that (4.38) and (4.39) are twisted complexes, the proofs for (4.40) and (4.41) are similar.

The definition of a twisted complex [AL17, §3.1] implies that (4.38) is a twisted complex if and only if:

- (1) $(FR \operatorname{tr} - \operatorname{tr} FR) \circ \operatorname{Fact} R = d\xi'_B$,
- (2) $\operatorname{tr} \circ (FR \operatorname{tr} - \operatorname{tr} FR) = 0$,
- (3) $\operatorname{tr} \circ \xi'_B = 0$,

The identity (1) is the content of Corollary 4.7. For the identity (2), observe that following our conventions $FRFR \xrightarrow{\operatorname{tr} \circ FR \operatorname{tr}} \operatorname{Id}$ denotes the composition

$$M^{\bar{\mathcal{B}}} \bar{\otimes} M \bar{\otimes} M^{\bar{\mathcal{B}}} \bar{\otimes} M \xrightarrow{\operatorname{Id} \bar{\otimes} \gamma \bar{\otimes} \operatorname{Id}^2} M^{\bar{\mathcal{B}}} \bar{\otimes} \operatorname{Hom}_{\mathcal{B}}(\mathcal{B}, M) \bar{\otimes} M^{\bar{\mathcal{B}}} \bar{\otimes} M \xrightarrow{\operatorname{cmps} \bar{\otimes} \operatorname{Id}} M^{\bar{\mathcal{B}}} \bar{\otimes} M \xrightarrow{\operatorname{ev}} \mathcal{B} \quad (4.42)$$

On the other hand, $FRFR \xrightarrow{\operatorname{tr} \circ \operatorname{tr} FR} \operatorname{Id}$ denotes the composition

$$M^{\bar{\mathcal{B}}} \bar{\otimes} M \bar{\otimes} M^{\bar{\mathcal{B}}} \bar{\otimes} M \xrightarrow{\operatorname{Id}^2 \bar{\otimes} \operatorname{ev}} M^{\bar{\mathcal{B}}} \bar{\otimes} M \bar{\otimes} \mathcal{B} \xrightarrow{\operatorname{Id} \bar{\otimes} \alpha} M^{\bar{\mathcal{B}}} \bar{\otimes} M \xrightarrow{\operatorname{ev}} \mathcal{B} \quad (4.43)$$

Finally, recall that the map $M \bar{\otimes} \mathcal{B} \xrightarrow{\alpha} M$ is the same as the composition

$$M \bar{\otimes} \mathcal{B} \xrightarrow{\gamma \bar{\otimes} \operatorname{Id}} \operatorname{Hom}_{\mathcal{B}}(\mathcal{B}, M) \bar{\otimes} \mathcal{B} \xrightarrow{\operatorname{ev}} M.$$

It follows that (4.42) and (4.43) are the same map, and thus (2) holds as required.

For the identity (3), we observe that, by definition, $\operatorname{tr} \circ \xi'_B$ is the map

$$\operatorname{Hom}_{\mathcal{B}}(M, \mathcal{B}) \bar{\otimes}_{\mathcal{A}} M \xrightarrow{(-\circ \omega_{\mathcal{B}}(\operatorname{Id}_M)) \bar{\otimes} \operatorname{Id} - \operatorname{Id} \bar{\otimes} \omega_{\mathcal{B}}(\operatorname{Id}_M)} \operatorname{Hom}_{\mathcal{B}}(M, \mathcal{B}) \bar{\otimes}_{\mathcal{A}} M \xrightarrow{\operatorname{ev}} \mathcal{B}$$

which is zero by the associativity of the composition of morphisms in the category $\overline{\mathbf{Mod}}\text{-}\mathcal{B}$.

Similarly, (4.39) is a twisted complex if and only if:

- (1) $F \operatorname{tr} R \circ (\operatorname{act} RF - R \operatorname{Fact}) = d\xi_B$,
- (2) $(\operatorname{act} RF - R \operatorname{Fact}) \circ \operatorname{act} = d\nu_B$,
- (3) $\xi_B \circ \operatorname{act} - F \operatorname{tr} L \circ \nu_B = d\nu_B$.

The identities (1) and (3) follow from Cor. 4.10 and Prop. 4.12, respectively. It remains to establish (2).

Following our conventions, the maps $\operatorname{Id} \xrightarrow{\operatorname{act} RF \circ \operatorname{act}} RFRF$ and $\operatorname{Id} \xrightarrow{R \operatorname{Fact} \circ \operatorname{act}} RFRF$ denote the compositions

$$\begin{aligned} \mathcal{A} &\xrightarrow{\operatorname{act}} M \bar{\otimes} M^{\bar{\mathcal{B}}} \xrightarrow{\operatorname{Id} \bar{\otimes} \beta} M \bar{\otimes} M^{\bar{\mathcal{B}}} \bar{\otimes} \mathcal{A} \xrightarrow{\operatorname{Id}^2 \bar{\otimes} \operatorname{act}} M \bar{\otimes} M^{\bar{\mathcal{B}}} \bar{\otimes} M \bar{\otimes} M^{\bar{\mathcal{B}}} \\ \mathcal{A} &\xrightarrow{\operatorname{act}} M \bar{\otimes} M^{\bar{\mathcal{B}}} \xrightarrow{\operatorname{act} \bar{\otimes} \operatorname{Id}} \mathcal{A} \bar{\otimes} M \bar{\otimes} M^{\bar{\mathcal{B}}} \xrightarrow{\operatorname{act} \bar{\otimes} \operatorname{Id}^2} M \bar{\otimes} M^{\bar{\mathcal{B}}} \bar{\otimes} M \bar{\otimes} M^{\bar{\mathcal{B}}} \end{aligned}$$

which by Prop. 3.28(3) are equal to

$$\begin{aligned} \mathcal{A} &\xrightarrow{\beta^r} \mathcal{A} \bar{\otimes} \mathcal{A} \xrightarrow{\operatorname{act} \bar{\otimes} \operatorname{act}} M \bar{\otimes} M^{\bar{\mathcal{B}}} \bar{\otimes} M \bar{\otimes} M^{\bar{\mathcal{B}}} \\ \mathcal{A} &\xrightarrow{\beta^l} \mathcal{A} \bar{\otimes} \mathcal{A} \xrightarrow{\operatorname{act} \bar{\otimes} \operatorname{act}} M \bar{\otimes} M^{\bar{\mathcal{B}}} \bar{\otimes} M \bar{\otimes} M^{\bar{\mathcal{B}}}. \end{aligned}$$

Thus the map $(\operatorname{act} RF - R \operatorname{Fact}) \circ \operatorname{act}$ equals

$$\mathcal{A} \xrightarrow{\beta^r - \beta^l} \mathcal{A} \bar{\otimes} \mathcal{A} \xrightarrow{\operatorname{act} \bar{\otimes} \operatorname{act}} M \bar{\otimes} M^{\bar{\mathcal{B}}} \bar{\otimes} M \bar{\otimes} M^{\bar{\mathcal{B}}}.$$

The desired assertion follows since $d\pi = \beta^r - \beta^l$. \square

We next prove that the convolutions of (4.38)-(4.41) are isomorphic in $D(\mathcal{B}\text{-}\mathcal{B})$ and $D(\mathcal{A}\text{-}\mathcal{A})$ to the squares of twists and co-twists of F . It follows, in particular, that the isomorphism classes in $D(\mathcal{B}\text{-}\mathcal{B})$ and $D(\mathcal{A}\text{-}\mathcal{A})$ of the convolutions of (4.38)-(4.41) depend only on the isomorphism class in $D(\mathcal{A}\text{-}\mathcal{B})$ of F . We can therefore think of them as canonical twisted complexes associated to the homotopy adjunctions (F, R) and (L, F) .

Theorem 4.3. *Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$.*

- (1) *If M is \mathcal{B} -perfect, then we have in $D(\mathcal{B}\text{-}\mathcal{B})$*

$$\left\{ \begin{array}{c} \begin{array}{ccccccc} FR & \xrightarrow{\operatorname{Fact} R} & FRFR & \xrightarrow{FR \operatorname{tr} - \operatorname{tr} FR} & FR & \xrightarrow{\operatorname{tr}} & \operatorname{Id}_{\mathcal{B}} \\ & & & & & & \operatorname{deg.0} \end{array} \\ \begin{array}{c} \xrightarrow{\xi'_B} \\ \xrightarrow{\xi'_B} \end{array} \end{array} \right\} \simeq T^2 \quad (4.44)$$

where $T = \left\{ FR \xrightarrow[\text{deg.0}]{\text{tr}} \text{Id}_{\mathcal{B}} \right\}$ is the twist of F . We also have in $D(\mathcal{A}\text{-}\mathcal{A})$

$$\left\{ \begin{array}{c} \text{Id}_{\mathcal{A}} \\ \text{deg.0} \end{array} \xrightarrow{\text{act}} RF \xrightarrow{\text{act } RF - RF \text{ act}} RFRF \xrightarrow{F \text{ tr } R} RF \right\} \simeq C^2 \quad (4.45)$$

$\nu_{\mathcal{B}}$ (top dashed arrow), $-v_{\mathcal{B}}$ (middle dashed arrow), $\xi_{\mathcal{B}}$ (bottom dashed arrow)

where $C = \left\{ \text{Id}_{\mathcal{A}} \xrightarrow[\text{deg.0}]{\text{act}} RF \right\}$ is the co-twist of F .

(2) If M is \mathcal{A} -perfect, then we have in $D(\mathcal{A}\text{-}\mathcal{A})$

$$\left\{ LF \xrightarrow{L \text{ act } F} LFLF \xrightarrow{LF \text{ tr} - \text{tr } LF} LF \xrightarrow{\text{tr}} \text{Id}_{\mathcal{A}} \right\} \simeq C'^2 \quad (4.46)$$

$\xi'_{\mathcal{A}}$ (top dashed arrow)

where $C' = \left\{ LF \xrightarrow[\text{deg.0}]{\text{tr}} \text{Id}_{\mathcal{A}} \right\}$ is the dual co-twist of F . We also have in $D(\mathcal{B}\text{-}\mathcal{B})$

$$\left\{ \text{Id}_{\mathcal{B}} \xrightarrow{\text{act}} FL \xrightarrow{FL \text{ act} - \text{act } FL} FLFL \xrightarrow{F \text{ tr } L} FL \right\} \simeq T'^2 \quad (4.47)$$

$\nu_{\mathcal{A}}$ (top dashed arrow), $-v_{\mathcal{A}}$ (middle dashed arrow), $\xi_{\mathcal{A}}$ (bottom dashed arrow)

where $T' = \left\{ \text{Id}_{\mathcal{B}} \xrightarrow[\text{deg.0}]{\text{act}} FL \right\}$ is the dual twist of F .

Proof. We only construct the isomorphism (4.44). The isomorphisms (4.45), (4.46), and (4.47) are constructed similarly. By Lemma 3.43 (1) the object T^2 is isomorphic in $D(\mathcal{B}\text{-}\mathcal{B})$ to the convolution of

$$M^{\bar{\mathcal{B}}} \otimes M \otimes M^{\bar{\mathcal{B}}} \otimes M \xrightarrow{\begin{pmatrix} -\text{Id}^2 \otimes \text{tr} \\ \text{tr} \otimes \text{Id}^2 \end{pmatrix}} (M^{\bar{\mathcal{B}}} \otimes M \otimes \mathcal{B}) \oplus (\mathcal{B} \otimes M^{\bar{\mathcal{B}}} \otimes M) \xrightarrow{(\text{tr} \otimes \text{Id} \quad \text{Id} \otimes \text{tr})} \mathcal{B} \otimes \mathcal{B} \text{ deg.0}$$

The twisted complex above is readily seen to be homotopy equivalent to the twisted complex

$$FRFR \xrightarrow{\begin{pmatrix} -\text{tr } FR \\ FR \text{ tr} \end{pmatrix}} FR \oplus FR \xrightarrow{(\text{tr} \text{ tr})} \text{Id}_{\mathcal{B}} \text{ deg.0} \quad (4.48)$$

Indeed, by the Rectangle Lemma and Prop. 3.28 the following is a homotopy equivalence between the two:

$$\begin{array}{ccccc}
M^{\mathcal{B}} \overline{\otimes} M \overline{\otimes} M^{\mathcal{B}} \overline{\otimes} M & \xrightarrow{\begin{pmatrix} -\text{Id}^2 \overline{\otimes} \text{tr} \\ \text{tr} \overline{\otimes} \text{Id}^2 \end{pmatrix}} & (M^{\mathcal{B}} \overline{\otimes} M \overline{\otimes} \mathcal{B}) \oplus (\mathcal{B} \overline{\otimes} M^{\mathcal{B}} \overline{\otimes} M) & \xrightarrow{(\text{tr} \overline{\otimes} \text{Id} \quad \text{Id} \overline{\otimes} \text{tr})} & \mathcal{B} \overline{\otimes} \mathcal{B} \\
\parallel & \dashrightarrow & \downarrow \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} & \dashrightarrow & \downarrow \alpha \\
M^{\mathcal{B}} \overline{\otimes} M \overline{\otimes} M^{\mathcal{B}} \overline{\otimes} M & \xrightarrow{\begin{pmatrix} -\alpha \circ (\text{Id}^2 \overline{\otimes} \text{tr}) \\ \alpha \circ (\text{tr} \overline{\otimes} \text{Id}^2) \end{pmatrix}} & (M^{\mathcal{B}} \overline{\otimes} M) \oplus (M^{\mathcal{B}} \overline{\otimes} M) & \xrightarrow{(\text{tr} \quad \text{tr})} & \mathcal{B} \\
& & & & \downarrow \\
& & & & \mathcal{B} \\
& & & & \text{deg.0}
\end{array}$$

Consider now the following closed degree zero map of twisted complexes

$$\begin{array}{ccccc}
FRFR & \xrightarrow{\begin{pmatrix} -\text{tr} FR \\ FR \text{tr} \end{pmatrix}} & FR \oplus FR & \xrightarrow{(\text{tr} \quad \text{tr})} & \text{Id}_{\mathcal{B}} \\
\parallel & & \downarrow (\text{Id} \quad \text{Id}) & & \parallel \\
FR & \xrightarrow{\text{Fact}R} & FRFR & \xrightarrow{FR \text{tr} - \text{tr} FR} & FR \\
& & \dashrightarrow \xi'_{\mathcal{B}} & & \downarrow \\
& & & & \text{Id}_{\mathcal{B}} \\
& & & & \text{deg.0}
\end{array} \tag{4.49}$$

By the Rectangle Lemma (Lemma 2.10) the total complex of (4.49) is isomorphic to the total complex of

$$\begin{array}{ccccccc}
FR & \xrightarrow{0 \rightarrow 0: \text{Fact}R} & (FRFR \xrightarrow{\text{Id}} FRFR)_{\text{deg.0}} & \xrightarrow[0 \rightarrow 0: FR \text{tr} - \text{tr} FR]{-1 \rightarrow -1: -\begin{pmatrix} -\text{tr} FR \\ FR \text{tr} \end{pmatrix}} & (FR \oplus FR \xrightarrow{-(\text{Id} \quad \text{Id})} FR)_{\text{deg.0}} & \xrightarrow[0 \rightarrow 0: \text{tr}]{-1 \rightarrow -1: -(\text{tr} \quad \text{tr})} & (\text{Id}_{\mathcal{B}} \xrightarrow{\text{Id}} \text{Id}_{\mathcal{B}})_{\text{deg.0}} \\
& & \dashrightarrow \xi'_{\mathcal{B}} & & & &
\end{array}$$

Here and below, we use the following convention for labelling the arrows which correspond to maps of twisted complexes. To specify such map we list all its non-zero components in the format $i \rightarrow j: \alpha$. This means that the component of the map which goes from the degree i element of the source complex to the degree j element of the target complex is α .

By the Extraction Lemma (Lemma 2.13) the complex above is homotopy equivalent to

$$\begin{array}{ccc}
FR & \xrightarrow{\quad} & 0 \xrightarrow{\quad} \{FR \oplus FR \xrightarrow{-(\text{Id} \quad \text{Id})} FR\}_{\text{deg.0}} \\
& \dashrightarrow & \downarrow \\
& & \{FR \oplus FR \xrightarrow{-(\text{Id} \quad \text{Id})} FR\}_{\text{deg.-1}} \\
& & \dashrightarrow \xi'_{\mathcal{B}}
\end{array}$$

and that can be rearranged as

$$\{FR \xrightarrow{-\text{tr} FR \circ \text{Fact}R} FR\}_{\text{deg.0}} \xrightarrow[0 \rightarrow 0: \text{Id}, -1 \rightarrow 0: \xi'_{\mathcal{B}}]{-1 \rightarrow -1: FR \text{tr} \circ \text{Fact}R} \{FR \xrightarrow{-\text{Id}} FR\}_{\text{deg.0}}$$

which is null-homotopic by the Extraction Lemma. Here and below, for any DG-category \mathcal{C} , we say that two \mathbb{Z} -graded collections of elements of $\text{Pre-Tr } \mathcal{C}$ and maps between them can be *rearranged* one into another if they have the same totalisation, considered as a \mathbb{Z} -graded collection of elements of \mathcal{C} and maps between them. Note that a \mathbb{Z} -graded collection of elements of $\text{Pre-Tr } \mathcal{C}$ and maps between them is a twisted complex if and only if its totalisation is a twisted complex. Thus if we start with an element of $\text{Pre-Tr Pre-Tr } \mathcal{C}$, anything we can rearrange it into is also an element of $\text{Pre-Tr Pre-Tr } \mathcal{C}$. Note further that this implies that a map of two elements of $\text{Pre-Tr } \mathcal{C}$ is closed of degree 0 if and only if its totalisation is a twisted complex. \square

4.4. Derived category perspective. As described in §2.4 any twisted complex over a DG-category defines several Postnikov systems in the homotopy category which compute its convolution. In this section, we study in detail four Postnikov systems in $D(\mathcal{B}\text{-}\mathcal{B})$ obtained from the twisted complex (4.38) of Theorem 4.2 by repeated application of Corollary 2.9. These turn out to depend only on the isomorphism class of M in $D(\mathcal{A}\text{-}\mathcal{B})$ and thus induce canonical functorial Postnikov systems which exist for any adjoint pair (f, r) of enhanced derived functors. Similarly, each of the twisted complexes (4.39), (4.40) and (4.41) of Theorem 4.2 defines four canonical Postnikov systems in the derived category. An interested reader would have no trouble working these out following our treatment of the complex (4.38) in this section.

Let

$$FR \xrightarrow{\text{tr}} \text{Id} \xrightarrow{p} T \xrightarrow{q} FR[1] \quad (4.50)$$

$$C \xrightarrow{r} \text{Id} \xrightarrow{\text{act}} RF \xrightarrow{s} C[1]. \quad (4.51)$$

be any exact triangles in $D(\mathcal{A}\text{-}\mathcal{B})$ which complete $FR \xrightarrow{\text{tr}} \text{Id}$ and $\text{Id} \xrightarrow{\text{act}} RF$. The objects T and C are called the *twist* and the *co-twist* of F , respectively.

Since F and R are genuinely adjoint in $D(\mathcal{A}\text{-}\mathcal{B})$

$$R \xrightarrow{\text{act } R} RFR \xrightarrow{R \text{tr}} R$$

$$F \xrightarrow{F \text{act}} FRF \xrightarrow{\text{tr } F} F$$

are retracts. In a triangulated category all retracts are split, and thus we have isomorphisms

$$FR \oplus FRT[-1] \xrightarrow{(F \text{act } R \quad FRq)} FRFR \quad (4.52)$$

$$FR \oplus TFR[-1] \xrightarrow{(F \text{act } R \quad qFR)} FRFR \quad (4.53)$$

$$FRFR \xrightarrow{\begin{pmatrix} FR \text{tr} \\ F s R \end{pmatrix}} FR \oplus FCR[1] \quad (4.54)$$

$$FRFR \xrightarrow{\begin{pmatrix} \text{tr } FR \\ F s R \end{pmatrix}} FR \oplus FCR[1]. \quad (4.55)$$

Below, we adopt the following convention. Given objects A , B and C in a triangulated category we say that a triangle

$$A \xrightarrow{\text{deg } i} B \xrightarrow{\text{deg } j} C \xrightarrow{\text{deg } k} A$$

with $i + j + k = 1$ is exact if the following induced triangle is exact:

$$A \xrightarrow{\text{deg } 0} B[i] \xrightarrow{\text{deg } 0} C[i + j] \xrightarrow{\text{deg } 0} A[1].$$

Theorem 4.4. *Let $M \in \mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$ be \mathcal{B} -perfect and let (F, R) be the corresponding homotopy adjoint pair. For any exact triangle (4.50) completing $FR \xrightarrow{\text{tr}} \text{Id}$:*

(1) *The following is a Postnikov system in $D(\mathcal{A}\text{-}\mathcal{B})$:*

$$\begin{array}{ccccccc}
 FR & \xrightarrow{\begin{matrix} F \text{act } R \\ \simeq \begin{pmatrix} \text{Id} \\ 0 \end{pmatrix} \end{matrix}} & FRFR & \xrightarrow{\begin{matrix} FR \text{tr} - \text{tr } FR \\ \simeq \begin{pmatrix} 0 & -q \circ \text{tr } T \end{pmatrix} \end{matrix}} & FR & \xrightarrow{\text{tr}} & \text{Id}_{\mathcal{B}} \\
 & & & & \swarrow q & \searrow p & \\
 & & & & & & T \\
 & & & & \swarrow (0 \quad -\text{tr } T) & & \\
 & & & & & & \\
 & & & & \swarrow \begin{pmatrix} \text{Id} & 0 \\ 0 & qT \end{pmatrix} & & \\
 & & & & & & \\
 & & & & & & FR[2] \oplus T^2 \\
 & & & & \swarrow \begin{pmatrix} 0 \\ pT \end{pmatrix} & & \\
 & & & & & & \\
 & & & & \swarrow (0 \quad \text{Id}) & & \\
 & & & & & & T^2
 \end{array}
 \quad (4.56)$$

Here the triangles denoted by \star are exact, the remaining triangles are commutative, the morphisms of $\text{deg} > 0$ are drawn with dashed arrows, and the morphisms of $\text{deg} < 0$ are drawn with dotted arrows.

(2) *For any exact triangles*

$$FRT[-1] \xrightarrow{-q \circ \text{tr } T} FR \xrightarrow{t} X \xrightarrow{u} FRT \quad (4.57)$$

$$Y \xrightarrow{v} \text{Id} \xrightarrow{p^2} T^2 \xrightarrow{w} Y[1] \quad (4.58)$$

in $D(\mathcal{A}\text{-}\mathcal{B})$ completing $-q \circ \text{tr} T$ and p^2 , there exists an isomorphism $X \simeq Y$ in $D(\mathcal{A}\text{-}\mathcal{B})$ such that the following are Postnikov systems:

$$\begin{array}{ccccccc}
 FR & \xrightarrow[\simeq(\text{Id})]{\text{Fact}R} & FRFR & \xrightarrow[\simeq(0 \ -q\text{otr}T)]{FR \text{tr} - \text{tr} FR} & FR & \xrightarrow{\text{tr}} & \text{Id}_{\mathcal{B}} \\
 & \searrow^{0} & \downarrow^{(0 \ \text{Id})} & \nearrow^{-q\text{otr}T} & \nearrow^t & \nearrow^v & \nearrow^{p^2} \\
 & & FRT[-1] & & & & \\
 & & \downarrow^u & & & & \\
 & & X \simeq Y & & & & \\
 & & \downarrow^w & & & & \\
 & & T^2 & & & &
 \end{array}
 \tag{4.59}$$

$$\begin{array}{ccccccc}
 FR & \xrightarrow[\simeq(\text{Id})]{\text{Fact}R} & FRFR & \xrightarrow[\simeq(0 \ -q\text{otr}T)]{FR \text{tr} - \text{tr} FR} & FR & \xrightarrow{\text{tr}} & \text{Id}_{\mathcal{B}} \\
 & \searrow^{0} & \downarrow^{(\text{Id} \ 0)} & \nearrow^{(\text{Id} \ 0 \ v)} & \nearrow^{(0 \ t)} & \nearrow^v & \nearrow^{p^2} \\
 & & & FR[1] \oplus X \simeq Y & & & \\
 & & \downarrow^{(0 \ \text{Id})} & & & & \\
 & & X \simeq Y & & & & \\
 & & \downarrow^w & & & & \\
 & & T^2 & & & &
 \end{array}
 \tag{4.60}$$

$$\begin{array}{ccccccc}
 FR & \xrightarrow[\simeq(\text{Id})]{\text{Fact}R} & FRFR & \xrightarrow[\simeq(0 \ -q\text{otr}T)]{FR \text{tr} - \text{tr} FR} & FR & \xrightarrow{\text{tr}} & \text{Id}_{\mathcal{B}} \\
 & \searrow^{0} & \downarrow^{(\text{Id} \ 0)} & \nearrow^{(\text{Id} \ 0 \ v)} & \nearrow^{(0 \ t)} & \nearrow^{(0 \ u)} & \nearrow^{(0 \ p^2)} \\
 & & & FR[1] \oplus X \simeq Y & & & \\
 & & \downarrow^{(\text{Id} \ 0 \ w)} & & & & \\
 & & & FR[2] \oplus T^2 & & & \\
 & & \downarrow^{(0 \ \text{Id})} & & & & \\
 & & T^2 & & & &
 \end{array}
 \tag{4.61}$$

Here the triangles denoted by \star are exact, the remaining triangles are commutative, the morphisms of $\text{deg} > 0$ are drawn with dashed arrows, and the morphisms of $\text{deg} < 0$ are drawn with dotted arrows.

NB: There are also versions of this theorem for each of the three other splittings (4.53)-(4.55) of $FRFR$. We leave them as an exercise to the reader.

- The homotopy equivalence

$$\begin{array}{c} FRFR \xrightarrow[\text{deg.0}]{\begin{pmatrix} 0 \\ FR \text{tr} \end{pmatrix}} FR \oplus FR \\ \downarrow \text{Id} \\ FRFR. \\ \text{deg.0} \end{array}$$

It descends in $D(\mathcal{A}\text{-}\mathcal{B})$ to the splitting isomorphism $FR \oplus FRT[-1] \xrightarrow{(4.52)} FRFR.$

- The homotopy equivalence (4.49) which was demonstrated in the proof of Theorem 4.3 to descend in $D(\mathcal{A}\text{-}\mathcal{B})$ to an isomorphism

$$T^2 \simeq \left\{ FR \longrightarrow FRFR \longrightarrow FR \longrightarrow \text{Id}_{\mathcal{B}} \right\}_{\text{deg.0}}.$$

- The homotopy equivalence

$$\begin{array}{ccccc} FR \oplus FRFR & \xrightarrow{\begin{pmatrix} 0 & -\text{tr} FR \\ 0 & FR \text{tr} \end{pmatrix}} & FR \oplus FR & \xrightarrow{(\text{tr} \quad \text{tr})} & \text{Id}_{\mathcal{B}} \\ \downarrow (F \text{ act } R \quad \text{Id}) & \searrow (\xi'_{\mathcal{B}} \quad 0) & \downarrow (\text{Id} \quad \text{Id}) & & \parallel \\ FRFR & \xrightarrow{FR \text{tr} - \text{tr} FR} & FR & \xrightarrow{\text{tr}} & \text{Id}_{\mathcal{B}} \\ & & & & \text{deg.0} \end{array} \quad (4.67)$$

obtained by rearranging the terms of (4.49). It descends in $D(\mathcal{A}\text{-}\mathcal{B})$ to an isomorphism

$$FR[2] \oplus T^2 \simeq \left\{ FRFR \longrightarrow FR \longrightarrow \text{Id}_{\mathcal{B}} \right\}_{\text{deg.0}}.$$

It can be readily checked on the level of twisted complexes that these equivalences identify up to homotopy the natural lift of (4.56) and (4.66). Hence the corresponding isomorphisms identify (4.56) and the image of (4.66) in $D(\mathcal{A}\text{-}\mathcal{B})$. Since the latter is a Postnikov system, so must be the former.

(2):

As in the proof of (1), it is enough to prove the desired assertion when

$$T = \left\{ FR \xrightarrow{\text{tr}} \text{Id} \right\}_{\text{deg.0}}$$

and (4.50) is the canonical exact triangle (4.62). For similar reasons, it is enough to assume that the exact triangles (4.57) and (4.58) are

$$\left\{ FRFR \xrightarrow[\text{deg.0}]{-FR \text{tr}} FR \right\} \xrightarrow{0,0: -\text{tr} FR} FR \xrightarrow{0,0: \begin{pmatrix} 0 \\ \text{Id} \end{pmatrix}} \left\{ FRFR \xrightarrow[\text{deg.0}]{\begin{pmatrix} -\text{tr} FR \\ FR \text{tr} \end{pmatrix}} FR \oplus FR \right\} \xrightarrow[-1,0: \text{Id}]{0,1: \begin{pmatrix} 0 & \text{Id} \\ -1 & 0 \end{pmatrix}} \quad (4.68)$$

$$\left\{ FRFR \xrightarrow[\text{deg.0}]{\begin{pmatrix} -\text{tr} FR \\ FR \text{tr} \end{pmatrix}} FR \oplus FR \right\} \xrightarrow{0,0: (\text{tr} \quad \text{tr})} \text{Id} \xrightarrow{0,0: \text{Id}} \left\{ FRFR \xrightarrow[\text{deg.0}]{\begin{pmatrix} -\text{tr} FR \\ FR \text{tr} \end{pmatrix}} FR \oplus FR \xrightarrow{(\text{tr} \quad \text{tr})} \text{Id} \right\}_{\text{deg.0}} \xrightarrow[-2,-1: \text{Id}]{-1,0: \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix}} \quad (4.69)$$

In particular,

$$X = Y = \left\{ FRFR \xrightarrow[\text{deg.0}]{\begin{pmatrix} -\text{tr} FR \\ FR \text{tr} \end{pmatrix}} FR \oplus FR \right\}$$

and we can take the requisite isomorphism $X \simeq Y$ to be the identity map.

The rest of the proof proceeds analogously to that of (1). We lift the objects of (4.59)-(4.61) to $\text{Pre-Tr}(\mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B})$ as in the proof of (1), plus:

- We lift $X = Y$ to the twisted complex

$$FRFR \xrightarrow[\text{deg.0}]{\begin{pmatrix} -\text{tr} FR \\ FR \text{tr} \end{pmatrix}} FR \oplus FR. \quad (4.70)$$

- We lift $FR[1] \oplus X = Y$ to the twisted complex

$$FR \oplus FRFR \xrightarrow[\text{deg.0}]{\begin{pmatrix} 0 & -\text{tr} FR \\ 0 & FR \text{tr} \end{pmatrix}} FR \oplus FR. \quad (4.71)$$

Then the morphisms in (4.59)-(4.61) all have natural lifts to $\text{Pre-Tr}(\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B})$, since they are all written in terms of the trace map and the maps in the exact triangles (4.62), (4.68), and (4.69). And these were all defined by maps in $\text{Pre-Tr}(\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B})$.

Thus we have natural lifts of (4.59)-(4.61) to $\text{Pre-Tr}(\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B})$. It is then straightforward to verify on the level of twisted complexes that (4.59)-(4.61) are Postnikov systems. Alternatively, these lifts can be identified up to homotopy with the three remaining diagrams induced, in addition to (4.66), by the canonical twisted complex (4.65). The identifying homotopy equivalences are those used in the proof of (1) plus:

- The following restriction of (4.49) to corresponding subcomplexes:

$$\begin{array}{ccc}
 FRFR & \xrightarrow{\begin{pmatrix} -\text{tr } FR \\ FR \text{ tr} \end{pmatrix}} & FR \oplus FR \\
 \downarrow -\text{Id} & & \downarrow (\text{Id} \ \text{Id}) \\
 FR & \xrightarrow{-F \text{ act } R} FRFR & \xrightarrow{-(FR \text{ tr} - \text{tr } FR)} FR \\
 & \searrow -\xi'_B & \nearrow \\
 & & FR_{\text{deg.0}}
 \end{array} \tag{4.72}$$

It descends in $D(\mathcal{A}\text{-}\mathcal{B})$ to an isomorphism

$$(X = Y) \xrightarrow{\sim} \left\{ FR \xrightarrow{-F \text{ act } R} FRFR \xrightarrow{-(FR \text{ tr} - \text{tr } FR)} FR_{\text{deg.0}} \right\}.$$

- The following restriction of (4.67) to corresponding subcomplexes:

$$\begin{array}{ccc}
 FR \oplus FRFR & \xrightarrow{\begin{pmatrix} 0 & -\text{tr } FR \\ 0 & FR \text{ tr} \end{pmatrix}} & FR \oplus FR \\
 \downarrow \begin{pmatrix} -F \text{ act } R & -\text{Id} \end{pmatrix} & \searrow \begin{pmatrix} -\xi'_B \\ 0 \end{pmatrix} & \downarrow (\text{Id} \ \text{Id}) \\
 FRFR & \xrightarrow{-(FR \text{ tr} - \text{tr } FR)} & FR \\
 & & \nearrow \\
 & & FR_{\text{deg.0}}
 \end{array} \tag{4.73}$$

obtained by rearranging the terms of (4.72). It descends in $D(\mathcal{A}\text{-}\mathcal{B})$ to an isomorphism

$$FR[1] \oplus X \simeq Y \xrightarrow{\sim} \left\{ FRFR \xrightarrow{FR \text{ tr} - \text{tr } FR} FR_{\text{deg.0}} \right\}.$$

It follows that (4.59)-(4.61) are Postnikov systems as desired. \square

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