ASYMPTOTIC ESTIMATES FOR THE WILLMORE FLOW WITH SMALL ENERGY

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Abstract. Kuwert and Schätzle showed in 2001 that the Willmore flow converges to a standard round sphere, if the initial energy is small. In this situation, we prove stability estimates for the barycenter and the quadratic moment of the surface. Moreover, in codimension one we obtain stability bounds for the enclosed volume and averaged mean curvature. As direct applications, we recover a quasi-rigidity estimate due to De Lellis and Müller (2006) and an estimate for the isoperimetric deficit by Röger and Schätzle (2012), whose original proofs used different methods.

1. introduction

The Willmore flow was introduced in [4, 5] by Schätzle and the first author, and also by Simonett [15]. This paper continues the study of the flow in the class of immersions $f\colon \mathbb{S}^2 \to \mathbb{R}^n$ with small initial energy, that is

$$\mathcal{E}(f) = \int_{\mathbb{S}^2} |A^\circ|^2 \, d\mu_g < \varepsilon_0 \quad \text{for some constant } \varepsilon_0 = \varepsilon_0(n). \tag{1}$$

Here $g = f^*\langle \cdot, \cdot \rangle$ and $A = D^2 f^\perp$ are the first and second fundamental forms; the latter is decomposed as $A = A^\circ + \frac{1}{2}\vec{H} \otimes g$ where A° is tracefree and $\vec{H} = \mathrm{tr}_g A$ is the mean curvature vector. The Willmore energy of any closed surface $f\colon \Sigma \to \mathbb{R}^n$ as introduced in [16] is

$$\mathcal{W}(f) = \frac{1}{4} \int_\Sigma |\vec{H}|^2 \, d\mu_g. \tag{2}$$

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For such f the functionals $\mathcal{E}(f)$ and $\mathcal{W}(f)$ differ only by a topological constant. In fact the Gauß equation and the Gauß-Bonnet theorem yield

$$\frac{1}{4}|\vec{H}|^2 = \frac{1}{2}|A^\circ|^2 + K_g \tag{3}$$

and

$$\mathcal{W}(f) = \frac{1}{2}\,\mathcal{E}(f) + 2\pi\chi(\Sigma). \tag{4}$$

We have $\mathcal{W}(f) \geq 4\pi$ for any closed surface, see [8, 16]. Thus if $\mathcal{E}(f) < 4\pi$ then Σ has automatically the type of $\mathbb{S}^2$, in fact

$$\chi(\Sigma) = \frac{1}{2\pi}\big(\mathcal{W}(f) - \frac{1}{2}\,\mathcal{E}(f)\big) > \frac{1}{2\pi}(4\pi - 2\pi) = 1.$$

This is why we restrict to $\Sigma = \mathbb{S}^2$ from the beginning. We also note that $\mathcal{E}(f) < 8\pi$ implies $\mathcal{W}(f) < 8\pi$, and f is an embedding [8], but this plays no role in the sequel. The first variation formula for $\mathcal{W}(f)$ is derived for instance in [5], it reads

$$\delta\mathcal{W}(f,\phi) = \int_\Sigma \langle \vec{W}(f), \phi\rangle \, d\mu_g \quad \text{where } \vec{W}(f) = \Delta^\perp \vec{H} + Q(A^\circ)\vec{H}. \tag{5}$$

Here $\Delta^\perp$ is the Laplacian of the normal connection, and $Q(A^\circ) = g^{\alpha\beta}g^{\lambda\mu}\langle A^\circ_{\alpha\lambda}, \cdot\rangle A^\circ_{\beta\mu}$. The Willmore flow is then given by the equation

$$\frac{\partial f}{\partial t} = -\vec{W}(f) \quad \text{ on } \Sigma \times (0,T).$$

Schätzle and the first author obtained the following existence and convergence result.

Theorem 1.1 ([4])**.** *There exists a constant $\varepsilon_0 = \varepsilon_0(n) > 0$ such that if $f\colon \mathbb{S}^2 \to \mathbb{R}^n$ is smoothly immersed with*

$$\int_{\mathbb{S}^2} |A^\circ|^2 \, d\mu_g < \varepsilon_0, \tag{6}$$

then the Willmore flow with initial surface f exists for all times and converges to a standard round 2-sphere.

The constant in the theorem stems from L^2 estimates for the curvature, and from applications of the Michael-Simon Sobolev inequality (Theorem 18.6 in [13]). For $n \in \{3,4\}$ the optimal constant in Theorem 1.1 is known to be $\varepsilon_0 = 8\pi$, see [6, 11] and [1, 9].

The idea of our paper is to study the stability of certain geometric quantities under the flow. In particular we consider the area, the barycenter and the quadratic moment given by

$$\mathcal{A}(f) = \int_{\mathbb{S}^2} d\mu_g, \tag{7}$$

$$\mathcal{C}(f) = -\!\!\!\!\!\!\int_{\mathbb{S}^2} f\, d\mu_g \quad \text{where } -\!\!\!\!\!\!\int_{\mathbb{S}^2} \ldots d\mu_g = \frac{1}{\mathcal{A}(f)} \int_{\mathbb{S}^2} \ldots d\mu_g, \tag{8}$$

$$\mathcal{Q}(f) = -\!\!\!\!\!\!\int_{\mathbb{S}^2} |f - \mathcal{C}(f)|^2\, d\mu_g. \tag{9}$$

In the case $n = 3$ the surface has a well-defined interior unit normal $\nu \colon \Sigma \to \mathbb{S}^2$, and we can further define the enclosed volume and total mean curvature, putting $\vec{H} = H\nu$,

$$\mathcal{V}(f) = -\frac{1}{3} \int_{\mathbb{S}^2} \langle f, \nu \rangle\, d\mu_g, \tag{10}$$

$$\mathcal{H}(f) = \int_{\mathbb{S}^2} H\, d\mu_g. \tag{11}$$

In the following statement the long-time existence and also the area estimate were already obtained in [4], they are included just for completeness.

Theorem 1.2 (stability)**.** *There exist constants $\varepsilon_0 = \varepsilon_0(n) > 0$, $C = C(n) < \infty$ with the following property. Let $f_0 \colon \mathbb{S}^2 \to \mathbb{R}^n$ be a smoothly immersed surface, normalized* *to area* $\mathcal{A}(f_0) = 4\pi$. *If*

$$\mathcal{E}(f_0) = \int_{\mathbb{S}^2} |A^\circ|^2 \, d\mu_g < \varepsilon_0, \tag{12}$$

then the Willmore flow of f_0 *exists for all times and satisfies*

$$|\mathcal{A}(f) - \mathcal{A}(f_0)| + |\mathcal{C}(f) - \mathcal{C}(f_0)| + |\mathcal{Q}(f) - \mathcal{Q}(f_0)| \leq C\,\mathcal{E}(f_0). \tag{13}$$

For $n = 3$ *one has furthermore the inequalities*

$$\big|\mathcal{V}(f) - \mathcal{V}(f_0)\big| + \big|\mathcal{H}(f) - \mathcal{H}(f_0)\big| \leq C\,\mathcal{E}(f_0). \tag{14}$$

Combining this result with the convergence result in [4], we obtain the following consequence.

Corollary 1 (limit sphere)**.** *For appropriate* $\varepsilon_0 = \varepsilon_0(n) > 0$, *the flow as in Theorem 1.2 converges smoothly to a standard round sphere, having some center* $x \in \mathbb{R}^n$ *and radius* $R > 0$. *Assuming* $\mathcal{A}(f_0) = 4\pi$ *as above, we have the following inequalities:*

$$|R - 1| + |x - \mathcal{C}(f_0)| + |R^2 - \mathcal{Q}(f_0)| \leq C\,\mathcal{E}(f_0), \tag{15}$$

$$\Big|\frac{4\pi}{3}R^3 - \mathcal{V}(f_0)\Big| + \big|8\pi R - \mathcal{H}(f_0)\big| \leq C\,\mathcal{E}(f_0) \quad \text{for } n = 3. \tag{16}$$

Remark 1.1. *For* $n = 3$ *the limit sphere is determined by its center and radius. For* $n \geq 4$ *the sphere lies in some* 3*-dimensional affine subspace passing through the center. It remains open whether that subspace has an estimate similar to the above.*

Remark 1.2. The upper bound for the volume as in (16) follows from the isoperimetric inequality and the radius bound, namely

$$\mathcal{V}(f_0) \leq \frac{1}{\sqrt{36\pi}}\,\mathcal{A}(f_0)^{\frac{3}{2}} = \frac{4\pi}{3} \leq \frac{4}{3}\pi R^3 + C\mathcal{E}(f_0).$$

For the mean curvature integral, the Gauß equation and the radius bound yield

$$\mathcal{H}(f_0) \leq \Big(\int_{\mathbb{S}^2} H^2 \, d\mu_g \Big)^{\frac{1}{2}} \mathcal{A}(f_0)^{\frac{1}{2}} = (16\pi + 2\mathcal{E}(f_0))^{\frac{1}{2}} (4\pi)^{\frac{1}{2}} \leq 8\pi R + C\mathcal{E}(f_0).$$

Therefore we only need to prove the lower bounds in (16).

By Codazzi-Mainardi, a connected immersed surface $f\colon \Sigma \to \mathbb{R}^n$ with $A^\circ \equiv 0$ parametrizes some standard round 2-sphere. In an important paper [2], De Lellis and Müller proved stability for this rigidity type statement in codimension one, assuming that A° is small in the sense of condition (1). In particular they obtained that the curvature is close to a constant in an averaged sense:

$$\int_{\mathbb{S}^2} \big|S - \frac{\bar{H}}{2}\mathrm{Id}\big|^2 \, d\mu_g \leq C \int_{\mathbb{S}^2} |S^\circ|^2 \, d\mu_g \quad \text{where } \bar{H} = \fint_{\mathbb{S}^2} H \, d\mu_g. \tag{17}$$

Here S denotes the Weingarten operator of the surface. We show that (17) follows directly from Corollary 1. We further deduce a bound for the isoperimetric deficit due to Röger and Schätzle [12], saying that

$$\frac{\mathcal{A}(f)}{\mathcal{V}(f)^{\frac{2}{3}}} \leq (36\pi)^{\frac{1}{3}} + C\,\mathcal{E}(f) \quad \text{for } \mathcal{E}(f) < \varepsilon_0.$$

Both [2] and [12] employ the estimates by Müller-Šveràk and Hélein [3, 10] as a key tool. In addition to the bound (17), De Lellis and Müller show that a suitable conformal reparametrization $\psi\colon S^2 \to \mathbb{R}^3$ satisfies

$$\big\|\psi - (c + \mathrm{id}_{\mathbb{S}^2})\big\|_{W^{2,2}(\mathbb{S}^2)} \leq C\,\|A^\circ\|_{L^2} \quad \text{for some } c \in \mathbb{R}^3. \tag{18}$$

In higher codimension, the same result (18) is established by Lamm and Schätzle in [7]. These estimates cannot be obtained using the Willmore flow, since it does not give any control on the parametrization. We note that (18) allows for an a priori translation of the surface, therefore our estimate of the center in (1) appears to be an extra information. We should also note that Lamm and Schätzle prove a version of (17) in higher codimension, for which we have no Willmore flow equivalent.

The outline of the paper is as follows. In the next section we recall estimates from [4]. The proof of Theorem 1.2 is given in Section 3. In the final section we deduce the estimates by DeLellis-Müller [2] and Röger-Schätzle [12] from Corollary 1.

2. Known estimates

The proof of the long-term existence under assumption (12) in [4] comes with certain estimates which we now briefly collect. As usual the norms involved are with respect to the metric g and the volume measure μ_g is induced by the time-dependent immersion f. In the present situation, Proposition 3.4 in [4] yields the following.

Theorem 2.1. *([4, Prop. 3.4]) There exist constants $\varepsilon_0(n) > 0$ and $C(n) < \infty$ with the following property. If $f\colon \mathbb{S}^2 \times [0,\infty) \to \mathbb{R}^n$ is a Willmore flow satisfying*

$$\varepsilon := \int_{\mathbb{S}^2} |A^\circ|^2 \, d\mu_g < \varepsilon_0 \qquad \text{at time } t = 0, \tag{19}$$

then the following estimates hold:

$$\int_0^\infty \int_{\mathbb{S}^2} \left(|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^4 |A^\circ|^2\right) d\mu_g \, dt \;\leq\; C\,\varepsilon, \tag{20}$$

$$\int_0^\infty \|A^\circ\|^4_{C^0(\mathbb{S}^2)} \, dt \;\leq\; C\,\varepsilon. \tag{21}$$

A second result estimates the area along the flow.

Theorem 2.2. *([4, Thm. 5.2]) Under the assumptions of Theorem 2.1 one has the further inequalities*

$$|\mathcal{A}(f) - \mathcal{A}(f_0)| \;\leq\; C\,\mathcal{A}(f_0)\,\varepsilon, \tag{22}$$

$$\int_0^\infty \int_{\mathbb{S}^2} \left(|\nabla A|^2 + |A|^2 |A^\circ|^2\right) d\mu_g \, dt \;\leq\; C\,\mathcal{A}(f_0)\,\varepsilon. \tag{23}$$

Finally we will use the following curvature estimate, which implies an energy gap for Willmore surfaces which are not round spheres.

Theorem 2.3. *([4, Thm. 2.9]) There exists an $\varepsilon_0 = \varepsilon_0(n) > 0$, such that for any immersed sphere $f\colon \mathbb{S}^2 \to \mathbb{R}^n$ with $\mathcal{E}(f) < \varepsilon_0$ one has*

$$\|A^\circ\|^2_{L^\infty(\mathbb{S}^2)} \leq C\,\mathcal{A}(f)\,\|\vec{W}(f)\|^2_{L^2(\mathbb{S}^2)}. \tag{24}$$

Proof. This is immediate from Theorem 2.9 in [5], in fact

$$\|A^\circ\|^2_{L^\infty(\mathbb{S}^2)} \leq C\,\|\vec{W}(f)\|_{L^2(\mathbb{S}^2)}\|A^\circ\|_{L^2(\mathbb{S}^2)} \leq C\,\|A^\circ\|_{L^\infty(\mathbb{S}^2)}\mathcal{A}(f)^{\frac{1}{2}}\|\vec{W}(f)\|_{L^2(\mathbb{S}^2)}.$$

The claim follows. ☐

3. Proof of Theorem 1.2

Let $f\colon \Sigma \times [0,T) \to \Omega \subset \mathbb{R}^n$ be the Willmore flow of a compact surface. We start by testing the equation with conformal Killing fields $X\colon \Omega \to \mathbb{R}^n$, that is

$$DX = \frac{1}{n}(\operatorname{div} X)\,\mathrm{Id} \qquad \text{on } \Omega. \tag{25}$$

This means that the local flow of X is by conformal transformations. By conformal invariance of the Willmore energy for closed Σ, we then have

$$0 = -\delta\mathcal{W}(f)X\circ f = -\int_\Sigma \langle \vec{W}(f), X\circ f\rangle\,d\mu_g = \int_\Sigma \langle \partial_t f, X\circ f\rangle\,d\mu_g. \tag{26}$$

Lemma 3.1. *Let $X = \operatorname{grad} u$ be a conformal Killing field on Ω. Then*

$$\frac{d}{dt}\int_\Sigma u\circ f\,d\mu_g = \int_\Sigma u\circ f\,\langle \vec{H}, \vec{W}(f)\rangle\,d\mu_g. \tag{27}$$

Proof. We compute by the above and the first variation formula

$$\begin{aligned}\partial_t \int_\Sigma u\circ f\, d\mu_g &= \int_\Sigma (Du)\circ f\cdot \partial_t f\, d\mu_g + \int_\Sigma u\circ f\, \partial_t(d\mu_g)\\ &= \int_\Sigma \langle \partial_t f, X\circ f\rangle\, d\mu_g + \int_\Sigma u\circ f\, \langle \vec{H}, \vec{W}(f)\rangle\, d\mu_g.\end{aligned}$$

The claim follows from (26). ☐

Lemma 3.2. *Let $f\colon \Sigma\to\Omega\subset\mathbb{R}^n$ be a closed immersed surface. Then for any gradient vector field $X = \operatorname{grad} u$ on Ω we have the identity, for $\xi = \langle\,\cdot\,, X\rangle$,*

$$\begin{aligned}\int_\Sigma u\circ f\langle \vec{H}, \Delta^\perp \vec{H}\rangle\, d\mu_g &= -\int_\Sigma u\circ f\, |\nabla\vec{H}|^2\, d\mu_g + 2\int_\Sigma \langle f^*\xi\otimes\nabla\vec{H}, A^\circ\rangle\, d\mu_g\\ &\quad +2\int_\Sigma \langle \nabla(f^*\xi)\otimes\vec{H}, A^\circ\rangle\, d\mu_g.\end{aligned} \tag{28}$$

Proof. By Codazzi we have $\nabla\vec{H} = -2\,\nabla^* A^\circ$, and hence $\Delta^\perp\vec{H} = 2\,\nabla^*\nabla^* A^\circ$. For any function $\varphi\colon\Sigma\to\mathbb{R}$, we compute

$$\begin{aligned}\int_\Sigma \varphi\langle\vec{H},\Delta^\perp\vec{H}\rangle\, d\mu_g &= 2\int_\Sigma \langle \nabla(\varphi\vec{H}), \nabla^* A^\circ\rangle\, d\mu_g\\ &= -\int_\Sigma \varphi\,|\nabla\vec{H}|^2\, d\mu_g + 2\int_\Sigma \langle d\varphi\otimes\vec{H}, \nabla^* A^\circ\rangle\, d\mu_g\\ &= -\int_\Sigma \varphi\,|\nabla\vec{H}|^2\, d\mu_g + 2\int_\Sigma \langle d\varphi\otimes\nabla\vec{H}, A^\circ\rangle\, d\mu_g\\ &\quad +2\int_\Sigma \langle \nabla(d\varphi)\otimes\vec{H}, A^\circ\rangle\, d\mu_g.\end{aligned}$$

For $\varphi = u\circ f = f^*u$ we have $d\varphi = f^*(du) = f^*\xi$, which proves the claim. ☐

In equation (28) the first two integrals on the right are quadratic in A°. Due to a cancellation this is also true for the third integral, in the case when X is a conformal Killing field. This is used for example in our estimate for the barycenter.

Lemma 3.3. *Let $f\colon \Sigma \to \Omega \subset \mathbb{R}^n$ be an immersed surface, and let $\phi\colon \Sigma \to \mathbb{R}^n$ be normal along f. Then for any conformal Killing field $X\colon \Omega \to \mathbb{R}^n$ we have*

$$\langle \nabla(f^*\xi) \otimes \phi, A^\circ \rangle = \langle Q(A^\circ)\phi, X \circ f \rangle \qquad \text{where } \xi = \langle \,\cdot\,, X \rangle. \tag{29}$$

Proof. For $p \in \Sigma$ we chose a local frame e_1, e_2 which is orthonormal for the induced metric g, and such that $\nabla_{e_i} e_j(p) = 0$. At p we then have $\nabla^2 f(e_i, e_j)^\top = Df\,\nabla_{e_i} e_j = 0$ by Levi-Cività, and hence $\nabla^2 f(e_i, e_j) = A_{ij}$. Using this we compute

$$\begin{aligned}
\nabla_{e_i}(f^*\xi)(e_j) &= \partial_{e_i}\big(f^*\xi(e_j)\big) \\
&= \partial_{e_i}\langle Df \cdot e_j, X \circ f \rangle \\
&= \langle \nabla^2 f(e_i, e_j), X \circ f \rangle + \langle Df \cdot e_j, (DX) \circ f Df \cdot e_i \rangle \\
&= \langle A_{ij}, X \circ f \rangle + \frac{1}{n}(\operatorname{div} X) \circ f\, \delta_{ij}.
\end{aligned}$$

We conclude, observing $A_{ij} = A^\circ_{ij} + \frac{1}{2}\vec{H}\delta_{ij}$ where $\delta_{ij} A^\circ_{ij} = \operatorname{tr}_g A^\circ = 0$,

$$\begin{aligned}
\langle \nabla(f^*\xi) \otimes \phi, A^\circ \rangle &= \langle \nabla_{e_i}(f^*\xi)(e_j)\phi, A^\circ(e_i, e_j) \rangle \\
&= \langle A_{ij}, X \circ f \rangle \langle \phi, A^\circ_{ij} \rangle + \frac{1}{n}(\operatorname{div} X) \circ f\, \delta_{ij} \langle \phi, A^\circ_{ij} \rangle \\
&= \langle Q(A^\circ)\phi, X \circ f \rangle.
\end{aligned}$$

Here we recall from [4] that $Q(A^\circ)\phi = A^\circ_{ij}\langle A^\circ_{ij}, \phi \rangle$. $\square$

Combining Lemma 3.1, Lemma 3.2 and Lemma 3.3, we arrive at the following.

Lemma 3.4. *Let $f\colon \Sigma \times (0,T) \to \Omega \subset \mathbb{R}^n$ be the Willmore flow of a closed surface. Then for any conformal Killing field $X = \operatorname{grad} u$ on Ω we have, putting $\xi = \langle \cdot, X\rangle$,*

$$\begin{aligned}\frac{d}{dt}\int_\Sigma u\circ f\,d\mu_g &= 2\int_\Sigma \langle Q(A^\circ)\vec{H}, X\circ f\rangle + 2\int_\Sigma \langle f^*\xi\otimes\nabla\vec{H}, A^\circ\rangle\,d\mu_g \\ &\quad -\int_\Sigma u\circ f\,|\nabla\vec{H}|^2\,d\mu_g + \int_\Sigma u\circ f\,\langle Q(A^\circ)\vec{H},\vec{H}\rangle\,d\mu_g.\end{aligned} \tag{30}$$

In particular we have the estimate

$$\begin{aligned}\Big|\frac{d}{dt}\int_\Sigma u\circ f\,d\mu_g\Big| &\le C\int_\Sigma \big(|A^\circ|^2|\vec{H}| + |\nabla\vec{H}|\,|A^\circ|\big)|X\circ f|\,d\mu_g \\ &\quad + C\int_\Sigma \big(|\nabla\vec{H}|^2 + |A^\circ|^2|\vec{H}|^2\big)|u\circ f|\,d\mu_g.\end{aligned} \tag{31}$$

We now turn to the proof of Theorem 1.2, in particular we assume from now on that Σ is just the standard sphere $\mathbb{S}^2$.

Area estimate: We refer to Theorem 5.2 in [4].

Barycenter estimate: Put $\mathcal{C}(t) = \mathcal{C}(f(\cdot,t))$ and assume without loss of generality that $\mathcal{C}(0) = 0$. By Simon's diameter bound, see Lemma 1.2 in [14], we know that

$$|f(p,t) - \mathcal{C}(t)| \le C \quad \text{for all } (p,t)\in\mathbb{S}^2\times[0,\infty). \tag{32}$$

As $\mathcal{A}(f_0) = 4\pi$ by assumption, the area is bounded from above and below. Taking $u(x) = x^i$, hence $X(x) \equiv e_i$, we now obtain in vector notation

$$\begin{aligned}|\mathcal{C}(t)| \le \Big|\int_{\mathbb{S}^2} f\,d\mu_g\Big| &= \Big|\int_0^t \frac{d}{ds}\int_{\mathbb{S}^2} f\,d\mu_g\,ds\Big| \\ &\le C\int_0^t\int_{\mathbb{S}^2}\big(|A^\circ|^2|\vec{H}| + |\nabla\vec{H}|\,|A^\circ|\big)\,d\mu_g \\ &\quad + C\int_0^t\int_{\mathbb{S}^2}\big(|\nabla\vec{H}|^2 + |A^\circ|^2|\vec{H}|^2\big)\,|f|\,d\mu_g\,ds \\ &\le C\Big(b(t) + \int_0^t \alpha(s)|\mathcal{C}(s)|\,ds\Big).\end{aligned}$$

Here we used that $|f(p,t)| \le |\mathcal{C}(t)| + C$ by (32), and $\alpha(t)$, $b(t)$ are defined by

$$\begin{aligned}
\alpha(t) &= \int_{\mathbb{S}^2} \big(|\nabla \vec{H}|^2 + |A^\circ|^2 |\vec{H}|^2\big)\, d\mu_g, \\
b(t) &= \int_0^t \int_{\mathbb{S}^2} \big(|\nabla \vec{H}|^2 + |A^\circ|^2 |\vec{H}|^2\big)\, d\mu_g\, dt + \int_0^t \int_{\mathbb{S}^2} \big(|A^\circ|^2 |\vec{H}| + |\nabla \vec{H}||A^\circ|\big)\, d\mu_g\, dt.
\end{aligned}$$

The Gronwall inequality yields

$$|\mathcal{C}(t)| \le C\, e^{Ca(t)} b(t) \quad \text{ where } a(t) = \int_0^t \alpha(s)\, ds. \tag{33}$$

From (23) we know that $a(t) \le C\mathcal{E}(f_0)$ for all $t \in [0,\infty)$. Furthermore, by applying Cauchy-Schwarz twice we can estimate

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{S}^2} |A^\circ|^2 |\vec{H}|\, d\mu_g\, dt &\le \int_0^\infty \Big(\int_{\mathbb{S}^2} |A^\circ|^2 |\vec{H}|^2\, d\mu_g\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{S}^2} |A^\circ|^2\, d\mu_g\Big)^{\frac{1}{2}} dt \\
&\le \Big(\int_0^\infty \int_{\mathbb{S}^2} |A^\circ|^2 |\vec{H}|^2\, d\mu_g\, dt\Big)^{\frac{1}{2}} \Big(\int_0^\infty \int_{\mathbb{S}^2} |A^\circ|^2\, d\mu_g\, dt\Big)^{\frac{1}{2}} \\
&\le C\, \mathcal{E}(f_0).
\end{aligned}$$

In the last step we used (23) for the first integral, the second integral is estimated by combining (24), the area bound and the energy identity. The remaining integral in $b(t)$ is estimated similarly by

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{S}^2} |\nabla \vec{H}|\, |A^\circ|\, d\mu_g\, dt &\le \int_0^\infty \Big(\int_{\mathbb{S}^2} |\nabla \vec{H}|^2\, d\mu_g\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{S}^2} |A^\circ|^2\, d\mu_g\Big)^{\frac{1}{2}} dt \\
&\le \Big(\int_0^\infty \int_{\mathbb{S}^2} |\nabla \vec{H}|^2\, d\mu_g\, dt\Big)^{\frac{1}{2}} \Big(\int_0^\infty \int_{\mathbb{S}^2} |A^\circ|^2\, d\mu_g\, dt\Big)^{\frac{1}{2}} \\
&\le C\, \mathcal{E}(f_0).
\end{aligned}$$

The estimate for the barycenter now follows from (33).

The quadratic moment estimate: We continue assuming $\mathcal{C}(f_0) = 0$, in particular the barycenter estimate and (32) imply

$$|f(p,t)| \le |f(p,t) - \mathcal{C}(t)| + |\mathcal{C}(t)| \le C.$$

We now apply (31) for $u(x) = \frac{1}{2}|x|^2$, whose gradient $X(x) = x$ generates the dilations. Combining with our previous estimates we see easily

$$\begin{aligned}
\Big[\int_{\mathbb{S}^2} |f|^2\, d\mu_g\Big]_{s=0}^{s=t} &= \int_0^t \frac{d}{ds}\int_{\mathbb{S}^2} |f|^2\, d\mu_g\, ds \\
&\le C\int_0^t \int_{\mathbb{S}^2} \big(|A^\circ|^2|\vec{H}| + |\nabla \vec{H}|\,|A^\circ|\big)|f|\, d\mu_g\, ds \\
&\quad + C\int_0^t \int_{\mathbb{S}^2} \big(|\nabla \vec{H}|^2 + |A^\circ|^2|\vec{H}|^2\big)|f|^2\, d\mu_g\, ds \\
&\le C\,\mathcal{E}(f_0).
\end{aligned}$$

We conclude, putting $\mathcal{I}(t) = \int_{\mathbb{S}^2} |f|^2\, d\mu_g$,

$$\begin{aligned}
|\mathcal{Q}(t) - \mathcal{Q}(0)| &= \Big|\Big(\frac{\mathcal{I}(t)}{\mathcal{A}(t)} - |\mathcal{C}(t)|^2\Big) - \Big(\frac{\mathcal{I}(0)}{\mathcal{A}(0)} - |\mathcal{C}(0)|^2\Big)\Big| \\
&\le \frac{|\mathcal{I}(t) - \mathcal{I}(0)|}{\mathcal{A}(t)} + \frac{|\mathcal{A}(0) - \mathcal{A}(t)|}{\mathcal{A}(t)\mathcal{A}(0)}\mathcal{I}(0) + \big(|\mathcal{C}(t)| + |\mathcal{C}(0)|\big)\,|\mathcal{C}(t) - \mathcal{C}(0)| \\
&\le C\,\mathcal{E}(f_0).
\end{aligned}$$

From now on we assume $n = 3$, in other words codimension one.

Volume estimate: Let $f\colon \Sigma \to \mathbb{R}^3$ be a closed surface with orientation induced by the interior unit normal ν. According to our definition of the volume,

$$\mathcal{V}(f) = -\int_\Sigma f^*\omega \quad \text{ where } \omega = \frac{1}{3}(x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2).$$

For any flow we have, using $d\omega = dx^1 \wedge dx^2 \wedge dx^3$,

$$\frac{d}{dt}\int_\Sigma f^*\omega = \int_\Sigma \sum_{i=1}^3 \partial_t f^i\, f^*(e_i \llcorner d\omega) = \int_\Sigma \langle \partial_t f, \nu\rangle\, d\mu_g.$$

Now the Willmore flow is given by $\partial_t f = -\vec{W}(f) = -(\Delta H + |A^\circ|^2 H)\nu$. Putting $\mathcal{V}(t) = \mathcal{V}(f(\cdot,t))$ we see

$$\mathcal{V}'(t) = \int_\Sigma (\Delta H + |A^\circ|^2 H)\, d\mu_g = \int_\Sigma |A^\circ|^2 H\, d\mu_g. \tag{34}$$

We note that the term ΔH cancels. Under the assumption of the theorem, we get

$$|\mathcal{V}(t) - \mathcal{V}(0)| \leq \int_0^t \int_{\mathbb{S}^2} |A^\circ|^2\, |H|\, d\mu_g \leq C\,\mathcal{E}(f_0).$$

Integral mean curvature estimate: Writing $\vec{W} = W\nu$ we have

$$\partial_t(d\mu_g) \;=\; HW\, d\mu_g, \tag{35}$$

$$\partial_t H \;=\; -(\Delta W + |A|^2 W). \tag{36}$$

Using $W = \Delta H + |A^\circ|^2 H$ we compute, again with a cancellation,

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}^2} H\, d\mu_g &= -\int_{\mathbb{S}^2} (\Delta W + |A|^2 W)\, d\mu_g + \int_{\mathbb{S}^2} H^2 W\, d\mu_g \\
&= -\int_{\mathbb{S}^2} (|A^\circ|^2 - \frac{1}{2}H^2)(\Delta H + |A^\circ|^2 H)\, d\mu_g.
\end{aligned}$$

Under the assumptions of Theorem 1.2, the space-time integrals of the right hand side are estimated as follows, using Theorem 2.1 and Theorem 2.2.

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{S}^2} |A^\circ|^2 |\Delta H|\, d\mu_g\, dt &\leq \int_0^\infty \Big(\int_{\mathbb{S}^2} |A^\circ|^4\, d\mu_g\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{S}^2} |\Delta H|^2\, d\mu_g\Big)^{\frac{1}{2}} dt \\
&\leq \Big(\int_0^\infty \int_{\mathbb{S}^2} |A^\circ|^4\, d\mu_g\, dt\Big)^{\frac{1}{2}} \Big(\int_0^\infty \int_{\mathbb{S}^2} |\Delta H|^2\, d\mu_g\, dt\Big)^{\frac{1}{2}} \\
&\leq C\,\mathcal{E}(f_0).
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
\int_0^\infty \Big| \int_{\mathbb{S}^2} H^2 \Delta H \, d\mu_g \Big| \, dt &\leq C \int_0^\infty \int_{\mathbb{S}^2} |H| |\nabla H|^2 \, d\mu_g \, dt \\
&\leq \int_0^\infty \Big(\int_{\mathbb{S}^2} |H|^2 |\nabla H|^2 \, d\mu_g \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{S}^2} |\nabla H|^2 \, d\mu_g \Big)^{\frac{1}{2}} \, dt \\
&\leq \Big(\int_0^\infty \int_{\mathbb{S}^2} H^2 |\nabla H|^2 \, d\mu_g \, dt \Big)^{\frac{1}{2}} \Big(\int_0^\infty \int_{\mathbb{S}^2} |\nabla H|^2 \, d\mu_g \, dt \Big)^{\frac{1}{2}} \\
&\leq C \, \mathcal{E}(f_0).
\end{aligned}$$

Finally we have

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{S}^2} |A^\circ|^2 |A|^3 \, d\mu_g \, dt &\leq \int_0^\infty \Big(\int_{\mathbb{S}^2} |A^\circ|^2 |A|^2 \, d\mu_g \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{S}^2} |A^\circ|^2 |A|^4 \, d\mu_g \Big)^{\frac{1}{2}} \, dt \\
&\leq \Big(\int_0^\infty \int_{\mathbb{S}^2} |A^\circ|^2 |A|^2 \, d\mu_g \, dt \Big)^{\frac{1}{2}} \Big(\int_0^\infty \int_{\mathbb{S}^2} |A^\circ|^2 |A|^4 \, d\mu_g \, dt \Big)^{\frac{1}{2}} \\
&\leq C \, \mathcal{E}(f_0).
\end{aligned}$$

This gives the bound for $\mathcal{H}(f)$, which completes the proof of Theorem 1.2.

4. Applications

For nearly umbilical immersions $f \colon \mathbb{S}^2 \to \mathbb{R}^3$, in the sense of small energy $\mathcal{E}(f)$, we recover a well-known rigidity estimate due to S. Müller and C. DeLellis [2]. We also show an estimate for the isoperimetric deficit due to M. Röger and R. Schätzle [12]. For both results, the original proofs were based on nontrivial estimates for conformal parametrisations from [3, 10]. Our proof relies instead on the geometric estimates for the Willmore flow.

Theorem 4.1 (DeLellis & Müller [2])**.** *There is a universal constant $C < \infty$, such that for any immersed sphere $f \colon \mathbb{S}^2 \to \mathbb{R}^3$ with Weingarten operator S we have*

$$\int_{\mathbb{S}^2} \big| S - \frac{\bar{H}}{2} \mathrm{Id} \big|^2 \, d\mu_g \leq C \int_{\mathbb{S}^2} |S^\circ|^2 \, d\mu_g \quad \textit{where } \bar{H} = \fint_{\mathbb{S}^2} H \, d\mu_g.$$

Proof. We assume by scaling that $\mathcal{A}(f) = 4\pi$. Using orthogonality we see that

$$\int_{\mathbb{S}^2} \left|S - \frac{\bar{H}}{2}\mathrm{Id}\right|^2 d\mu_g \leq \int_{\mathbb{S}^2} |S - \lambda\,\mathrm{Id}|^2\, d\mu_g \quad \text{ for all } \lambda \in \mathbb{R}.$$

Let $0 < \varepsilon_0 < 4\pi$ be the constant of Theorem 1.2. If $\int_{\mathbb{S}^2} |A^\circ|^2\, d\mu_g \geq \varepsilon_0$, then we obtain trivially from the Gauß equations and Gauß-Bonnet, see (3),

$$\int_{\mathbb{S}^2} |S|^2\, d\mu_g = 2\int_{\mathbb{S}^2} (|S^\circ|^2 + K_g)\, d\mu_g \leq \left(2 + \frac{8\pi}{\varepsilon_0}\right) \int_{\mathbb{S}^2} |S^\circ|^2\, d\mu_g.$$

For $\int_{\mathbb{S}^2} |A^\circ|^2\, d\mu_g < \varepsilon_0$ we expand

$$\int_{\mathbb{S}^2} \left|S - \frac{\bar{H}}{2}\mathrm{Id}\right|^2 d\mu_g = \int_{\mathbb{S}^2} |S|^2\, d\mu_g - 2\pi\bar{H}^2.$$

By the Gauß equations and Gauß-Bonnet, see above, we have

$$\int_{\mathbb{S}^2} |S|^2\, d\mu_g = 2\int_{\mathbb{S}^2} |S^\circ|^2\, d\mu_g + 8\pi.$$

Now the Willmore flow with initial surface f converges to a round sphere with radius $R > 0$, where by Corollary 1

$$\left|\bar{H} - 2R\right| \leq C\mathcal{E}(f) \quad \text{ and } \quad |R - 1| \leq C\mathcal{E}(f).$$

We conclude

$$2\pi\bar{H}^2 \geq 2\pi\left(2R - C\,\mathcal{E}(f)\right)^2 \geq 8\pi R^2 - C\,\mathcal{E}(f) \geq 8\pi - C\,\mathcal{E}(f).$$

The desired inequality follows. $\square$

Remark 4.1. *Theorem 4.1 holds also for closed surfaces Σ of type other than the sphere, with a simple proof. Namely we have by (3) and the Willmore inequality*

$$\int_\Sigma |S^\circ|^2\, d\mu_g = \frac{1}{2}\int_\Sigma H^2\, d\mu_g - 2\int_\Sigma K_g\, d\mu_g \geq 8\pi - 4\pi\chi(\Sigma) \geq 4\pi,$$

since $\chi(\Sigma) \leq 1$. *Therefore*

$$\int_\Sigma |S|^2 \, d\mu_g = 2 \int_\Sigma |S^\circ|^2 \, d\mu_g + 4\pi\chi(\Sigma) \leq 3 \int_\Sigma |S^\circ|^2 \, d\mu_g.$$

We finally come to the bound for the isoperimetric deficit, recalling again that an immersed closed surface with $\mathcal{E}(f) < 4\pi$ is embedded and has the type of the sphere. Our definition (10) of the volume implies that $\mathcal{V}(f) > 0$ for f embedded.

Theorem 4.2 (Röger & Schätzle [12])**.** *There exist universal constants* $\varepsilon_0 > 0$ *and* $C < \infty$, *such that for any immersed surface* $f : \mathbb{S}^2 \to \mathbb{R}^3$ *with* $\mathcal{E}(f) < \varepsilon_0$, *one has*

$$\frac{\mathcal{A}(f)}{\mathcal{V}(f)^{\frac{2}{3}}} \leq (36\pi)^{\frac{1}{3}} + C\,\mathcal{E}(f). \tag{37}$$

Proof. By scaling we can assume that $\mathcal{A}(f) = 4\pi$. Then Corollary 1 implies

$$\mathcal{V}(f) \geq \frac{4\pi}{3} R^3 - C\,\mathcal{E}(f) \geq \frac{4\pi}{3} - C\,\mathcal{E}(f).$$

The desired estimate follows. □

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