NON-SCALE-INVARIANT INVERSE CURVATURE FLOWS IN HYPERBOLIC SPACE

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ABSTRACT. We consider inverse curvature flows in hyperbolic space $\mathbb{H}^{n+1}$ with starshaped initial hypersurface, driven by positive powers of a homogeneous curvature function.
The solutions exist for all time and, after rescaling, converge to a sphere.

CONTENTS

1. INTRODUCTION

During the last decades geometric flows have been studied intensively. Following the ground breaking work of Huisken, [7], who considered the mean curvature flow, several authors started to investigate inverse, or expanding flows, e.g. [2], in which nonconvex hypersurfaces were shown to be driven into spheres. This work, as well as [5], heavily relied on the homogeneity of the curvature function, leading to, at least in Euclidean space, scale invariance of the flow. In both of these settings, the spherical flows exist for all time and thus dictate the behavior of the solution.
In [6] an inverse flow driven by arbitrary positive powers of a homogeneous

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curvature function was considered in $\mathbb{R}^{n+1}$ and for $p > 1$ blow up in finite time was proven.
In the present work we also consider this kind of flow,

$$\dot{x} = F^{-p}\nu, \ 0 < p < \infty,$$

in hyperbolic space $\mathbb{H}^{n+1}$, $n \geq 2$. For $p = 1$ this has been treated in [5], as well as in [1] for mean curvature, however in the latter work the obtained convergence results are of less strength. This flow behaves quite differently compared to the Euclidian case, since the curvature of a geodesic sphere is bounded below by 1, so that the flow exists for all time, regardless of the value of p.
In order to formulate the main result of this work, we first need a definition.

1.1. **Definition.** Let $\Gamma \subset \mathbb{R}^n$ be an open, symmetric and convex cone and $F \in C^\infty(\Gamma)$ a symmetric function. A hypersurface $M_0 \subset \mathbb{H}^{n+1}$ is called F-*admissable*, if at any point $x \in M_0$ the principal curvatures of M_0, $\kappa_1, ..., \kappa_n$, are contained in Γ.

We now state our main result.

1.2. **Theorem.** *Let $\Gamma \subset \mathbb{R}^n$ be a symmetric, convex and open cone, such that*

$$\Gamma_+ = \{(\kappa_i) \in \mathbb{R}^n : \kappa_i > 0 \ \forall 1 \leq i \leq n\} \subset \Gamma \tag{1.1}$$

and $F \in C^\infty(\Gamma) \cap C^0(\bar{\Gamma})$ be a monotone, 1-homogeneous and concave curvature function, such that

$$F_{|\Gamma} > 0, F_{|\partial\Gamma} = 0 \text{ and } F(1, ..., 1) = n. \tag{1.2}$$

Let $p > 0$ and in case $p > 1$ suppose $\Gamma = \Gamma_+$. Let $M \hookrightarrow M_0 \subset \mathbb{H}^{n+1}$ be a smooth and F-admissable embedded closed hypersurface, which can be written as a graph over a geodesic sphere, identified with $\mathbb{S}^n$,

$$M_0 = \operatorname{graph} u(0, \cdot). \tag{1.3}$$

Then
(1) there is a unique smooth curvature flow

$$x : [0, \infty) \times M \to \mathbb{H}^{n+1},$$

which satisfies the flow equation

$$\begin{aligned} \dot{x} &= -\Phi(F)\nu, \\ x(0) &= M_0, \end{aligned} \tag{1.4}$$

where $\nu(t, \xi)$ is the outward normal to $M_t = x(t, M)$ at $x(t, \xi)$, F is evaluated at the principal curvatures of M_t in $x(t, \xi)$,

$$\Phi(r) = -r^{-p} \tag{1.5}$$

and the leaves M_t are graphs over $\mathbb{S}^n$,

$$M_t = \operatorname{graph} u(t, \cdot). \tag{1.6}$$

(2) For all $0 < p \leq 1$ the leaves M_t become more and more umbilic, namely

$$|h^i_j - \delta^i_j| \leq c e^{-\frac{2}{n^p} t}, \ c = c(n, p, M_0). \tag{1.7}$$

In case $p > 1$ there exists $\epsilon = \epsilon(n, p, M_0)$, such that the same conclusion holds, if we impose the C^0-pinching condition

$$\operatorname{osc} u(0, \cdot) < \epsilon. \tag{1.8}$$

(3) Under the appropriate conditions as in (2) we obtain, that the rescaled surfaces

$$\hat{M}_t = \operatorname{graph}\left(u - \frac{t}{n^p}\right) \tag{1.9}$$

converge to a well defined, smooth function in C^∞ and thus the rescaled surfaces

$$\tilde{M}_t = \operatorname{graph} \frac{u}{t} \tag{1.10}$$

converge to a geodesic sphere in C^∞.

2. Setting and general facts

We now state some general facts about hypersurfaces, especially those that can be written as graphs. We basically follow the description of [5], but restrict to Riemannian manifolds. For a detailed discussion we refer to [4].
Let $N = N^{n+1}$ be Riemannian and $M = M^n \hookrightarrow N$ be a hypersurface. The geometric quantities of N will be denoted by $(\bar{g}_{\alpha\beta})$, $(\bar{R}_{\alpha\beta\gamma\delta})$ etc., where greek indices range from 0 to n. Coordinate systems in N will be denoted by (x^α). Quantities for M will be denoted by (g_{ij}), (h_{ij}) etc., where latin indices range from 1 to n and coordinate systems will generally be denoted by (ξ^i), unless stated otherwise.
Covariant differentiation will usually be denoted by indices, e.g. u_{ij} for a function $u\colon M \to \mathbb{R}$, or, if ambiguities are possible, by a semicolon, e.g. $h_{ij;k}$. Usual partial derivatives will be denoted by a comma, e.g. $u_{i,j}$.
Let $x\colon M \hookrightarrow N$ be an embedding and (h_{ij}) be the second fundamental form, then we have the *Gaussian formula*

$$x^\alpha_{ij} = -h_{ij}\nu^\alpha, \tag{2.1}$$

where ν is a differentiable normal, the *Weingarten equation*

$$\nu^\alpha_i = h^k_i x^\alpha_k, \tag{2.2}$$

the *Codazzi equation*

$$h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i x^\gamma_j x^\delta_k \tag{2.3}$$

and the *Gauß equation*

$$R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk}) + \bar{R}_{\alpha\beta\gamma\delta}x^\alpha_i x^\beta_j x^\gamma_k x^\delta_l. \tag{2.4}$$

Since in our case $N = \mathbb{H}^{n+1}$, we have

$$\bar{R}_{\alpha\beta\gamma\delta} = \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma} - \bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} \tag{2.5}$$

and thus the Codazzi equation takes the form

$$h_{ij;k} = h_{ik;j}. \tag{2.6}$$

Now assume that $N = (a, b) \times S_0$, where S_0 is compact Riemannian and that there is a Gaussian coordinate system (x^α) such that

$$d\bar{s}^2 = e^{2\psi}((dx^0)^2 + \sigma_{ij}(x^0, x)dx^i dx^j), \tag{2.7}$$

where σ_{ij} is a Riemannian metric, $x = (x^i)$ are local coordinates for S_0 and $\psi \colon N \to \mathbb{R}$ is a function.
Let $M = \operatorname{graph} u_{|S_0}$ be a hypersurface

$$M = \{(x^0, x) \colon x^0 = u(x), x \in S_0\}, \tag{2.8}$$

then the induced metric has the form

$$g_{ij} = e^{2\psi}(u_i u_j + \sigma_{ij}) \tag{2.9}$$

with inverse

$$g^{ij} = e^{-2\psi}(\sigma^{ij} - v^{-2}u^i u^j), \tag{2.10}$$

where $(\sigma^{ij}) = (\sigma_{ij})^{-1}$, $u^i = \sigma^{ij}u_j$ and

$$v^2 = 1 + \sigma^{ij}u_i u_j \equiv 1 + |Du|^2. \tag{2.11}$$

We use, especially in the Gaussian formula, the normal

$$(\nu^\alpha) = v^{-1}e^{-\psi}(1, -u^i). \tag{2.12}$$

Looking at $\alpha = 0$ in the Gaussian formula, we obtain

$$e^{-\psi}v^{-1}h_{ij} = -u_{ij} - \bar{\Gamma}^0_{00}u_i u_j - \bar{\Gamma}^0_{0i}u_j - \bar{\Gamma}^0_{0j}u_i - \bar{\Gamma}^0_{ij} \tag{2.13}$$

and

$$e^{-\psi}\bar{h}_{ij} = -\bar{\Gamma}^0_{ij}, \tag{2.14}$$

where covariant derivatives are taken with respect to g_{ij}.
Let us state some properties of $\mathbb{H}^{n+1}$. $\mathbb{H}^{n+1}$ is parametrizable over $B_2(0)$ yielding the conformally flat metric

$$d\bar{s}^2 = \frac{1}{(1 - \frac{1}{4}r^2)^2}(dr^2 + r^2\sigma_{ij}dx^i dx^j), \tag{2.15}$$

where (σ_{ij}) is the canonical metric of $\mathbb{S}^n$, cf. [5, p.16]. Also compare [4, Thm. 10.2.1].
Defining τ by

$$\tau = \log(2 + r) - \log(2 - r), \tag{2.16}$$

such that

$$d\tau = \frac{1}{1 - \frac{1}{4}r^2}dr, \tag{2.17}$$

then

$$d\bar{s}^2 = d\tau^2 + \sinh^2\tau\sigma_{ij}dx^i dx^j. \tag{2.18}$$

Thus we have a parametrization of $\mathbb{H}^{n+1}$ over $\mathbb{R}^{n+1}$ and, using [4, Thm. 1.7.5], we see that in geodesic polar coordinates around a given point the metric takes the above representation. In the sequel we will again write r for

τ, for greater clarity.
The geodesic spheres are totally umbilic and, setting

$$\bar{g}_{ij} = \sinh^2 r\sigma_{ij}, \tag{2.19}$$

their second fundamental form is given by

$$\bar{h}_{ij} = \coth r\bar{g}_{ij}. \tag{2.20}$$

Thus $\bar{h}^i_j = \coth r\delta^i_j$ and $\bar{\kappa}_i = \coth r$. The second fundamental form of a graph $M = \operatorname{graph} u$ satisfies

$$h_{ij}v^{-1} = -u_{ij} + \bar{h}_{ij}. \tag{2.21}$$

3. Long time existence

C^0-estimates.

We first construct the spherical barriers of the flow.

3.1. **Proposition.** Consider (1.4) with $x(0) = S_{r_0} = \{x^0 = r_0\}$. Then the corresponding flow $x = x(t,\xi)$ exists for all time. The leaves $M(t) = x(t,M)$ are geodesic spheres with radius

$$x^0(t,M) = \Theta(t,r_0), \tag{3.1}$$

where Θ solves the ODE

$$\begin{aligned} \dot{\Theta} &= F^{-p} = n^{-p}\coth^{-p}\Theta \\ \Theta(0,r_0) &= r_0. \end{aligned} \tag{3.2}$$

Proof. Looking at (2.12), we see that the outer normal of a geodesic sphere is $(1,0,...,0)$ and thus, setting

$$\begin{aligned} x^0(t,\xi) &= \Theta(t,r_0) \\ x^i(t,\xi) &= x^i(0,\xi), \end{aligned} \tag{3.3}$$

where Θ is the unique solution of (3.2), we see that x solves the flow equation, also using that $F(\bar{h}^i_j) = n\coth\Theta$. The solution of the ODE exists for all time, since $0 < \dot{\Theta} \le n^{-p}$. □

We now derive further properties of the spherical flows.

3.2. **Proposition.** Let $\Theta_i = \Theta(t,r_i)$, $i = 1,2$, be solutions of (3.2), $r_1 < r_2$, then

$$r_i + \frac{t}{n^p\coth^p r_i} \le \Theta_i(t) \le r_i + \frac{t}{n^p} \tag{3.4}$$

and there exists $c = c(r_1,n,p)$, such that

$$0 < \Theta_2(t) - \Theta_1(t) \le c(r_2 - r_1) \quad \forall t \in [0,\infty) \tag{3.5}$$

and such that $p \mapsto c(r_1,n,p)$ is continuous.

Proof. The first inequality follows from

$$\frac{1}{n^p \coth^p r_i} \leq \dot{\Theta} \leq \frac{1}{n^p}, \tag{3.6}$$

since $\dot{\Theta} > 0$ and since coth is decreasing. To prove the second claim, define

$$\rho(t) = \Theta_2(t) - \Theta_1(t). \tag{3.7}$$

ρ is positive, since this is the case at $t = 0$ and different orbits of an ODE flow can not intersect. We have

$$\begin{aligned} \dot{\rho}(t) &= \frac{1}{n^p \coth^p \Theta_2} - \frac{1}{n^p \coth^p \Theta_1} \\ &\leq \frac{1}{n^p}(\coth^p \Theta_1 - \coth^p \Theta_2) \\ &= \frac{1}{n^p}(p \coth^{p-1}(s)(\coth^2(s) - 1))(\Theta_2 - \Theta_1),\ s \in [\Theta_1(t), \Theta_2(t)] \\ &\leq \tilde{c}(n, p, r_1)(\coth^2 \Theta_1 - 1)\rho(t) \\ &= \tilde{c} \sinh^{-2} \Theta_1 \rho(t) \\ &\leq \tilde{c} \sinh^{-2}(r_1 + ct)\rho(t), \end{aligned} \tag{3.8}$$

Thus

$$\begin{aligned} \log \rho(t) &\leq \log \rho(0) + \int_0^t \tilde{c} \sinh^{-2}(cs + r_1) ds \\ &= \log(r_2 - r_1) + \frac{\tilde{c}}{c}[-\coth(cs + r_1)]_0^t \\ &\leq \log(r_2 - r_1) + \frac{\tilde{c}}{c} \coth r_1 \end{aligned} \tag{3.9}$$

and

$$\rho(t) \leq c(n, p, r_1)(r_2 - r_1).$$

□

3.3. **Corollary.** Let $\Theta = \Theta(t, r_0)$ be a solution of (3.2), then there exists $c = c(r_0, n, p)$, such that

$$-c < \Theta - \frac{t}{n^p} < c \quad \forall t \in [0, \infty). \tag{3.10}$$

Proof. The upper estimate follows from Proposition 3.2 immediately. There holds

$$\begin{aligned}\dot{\Theta} - \frac{1}{n^p} &= \frac{1}{n^p \coth^p \Theta} - \frac{1}{n^p} \\ &= \frac{1}{n^p} \frac{1 - \coth^p \Theta}{\coth^p \Theta} \\ &\geq \frac{1}{n^p}(1 - \coth^p \Theta) \\ &\geq \frac{1}{n^p}(1 - \coth^m \Theta), \; p \leq m \in \mathbb{Z} \\ &= \frac{1}{n^p} \sum_{k=0}^{m-1} \coth^k \Theta (1 - \coth \Theta) \\ &\geq c(n, p, r_0)(1 - \coth^2 \Theta) \\ &\geq c(1 - \coth^2(r_0 + \tilde{c}t))\end{aligned} \tag{3.11}$$

and thus

$$\begin{aligned}\Theta(t) - \frac{t}{n^p} &\geq r_0 + c\int_0^t (1 - \coth^2(r_0 + \tilde{c}s))ds \\ &= r_0 + \frac{c}{\tilde{c}}[\coth(r_0 + \tilde{c}s)]_0^t \\ &= r_0 + \frac{c}{\tilde{c}}\coth(r_0 + \tilde{c}t) - \frac{c}{\tilde{c}}\coth r_0 \\ &\geq r_0 - \frac{c}{\tilde{c}}\coth r_0\end{aligned} \tag{3.12}$$

□

3.4. **Remark.** Looking at [4, Thm. 2.5.19] and [4, Thm. 2.6.1], under the assumptions of Theorem 1.2 we obtain short time existence of the flow on a maximal interval $[0, T^*)$, $0 < T^* \leq \infty$, and

$$x \in C^\infty([0, T^*) \times M, \mathbb{H}^{n+1}). \tag{3.13}$$

This includes, that all the leaves $M(t) = x(t, M)$, $0 \leq t < T^*$, are admissable in the sense of Definition 1.1 and can be written as graphs over $\mathbb{S}^n$. Furthermore the flow x exists as long as the scalar flow

$$\dot{u} = \frac{\partial u}{\partial t} = -\Phi v \tag{3.14}$$

does, where

$$u \colon [0, T^*) \times \mathbb{S}^n \to \mathbb{R}, \tag{3.15}$$

also compare [4, Thm. 2.5.17] and [4, p. 98-99]. Thus, for the rest of the next chapters we will most of the time investigate long time existence for (3.14).

3.5. **Lemma.** The solution u of (3.14) satisfies

$$\Theta(t, \inf u(0, \cdot)) \leq u(t, x) \leq \Theta(t, \sup u(0, \cdot)) \;\; \forall t \in [0, T^*) \; \forall x \in \mathbb{S}^n. \tag{3.16}$$

In particular we have

$$\operatorname{osc} u(t,\cdot) = \sup u(t,\cdot) - \inf u(t,\cdot) \le c \operatorname{osc} u(0,\cdot), \tag{3.17}$$

$c = c(n, p, \inf u(0,\cdot))$.

Proof. Let

$$w(t) = \sup u(t,\cdot) = u(t, x_t). \tag{3.18}$$

By [4, Lemma 6.3.2], w is Lipschitz continuous and at a point of differentiability we have

$$\begin{aligned} \dot{w}(t) = \frac{\partial u}{\partial t}(t, x_t) &= \frac{1}{F^p(-g^{ik}u_{kj} + \bar{h}^i_j)} \\ &\le \frac{1}{F^p(\bar{h}^i_j)} \\ &= \frac{1}{n^p \coth^p w} \equiv \mathcal{L}(w) \end{aligned} \tag{3.19}$$

On the other hand

$$\dot{\Theta}(\cdot, \sup u(0,\cdot)) = \mathcal{L}(\Theta(\cdot, \sup u(0,\cdot))) \tag{3.20}$$

as well as

$$w(0) = \Theta(0, \sup u(0,\cdot)), \tag{3.21}$$

from which the upper estimate follows by integration and Gronwall's lemma applied to $w - \Theta$. The estimate from below follows identically. ☐

3.6. **Corollary.** Define

$$\vartheta(r) = \sinh r \tag{3.22}$$

and let u be the solution of (3.14). Then there exists $c = c(n, p, M_0)$, such that

$$0 < c^{-1} \le \vartheta(u) e^{-\frac{t}{n^p}} \le c \quad \forall t \in [0, T^*). \tag{3.23}$$

and

$$\frac{\bar{H}(u)}{n} - 1 = \coth u - 1 \le c e^{-\frac{2}{n^p} t}. \tag{3.24}$$

Proof. We deduce

$$\begin{aligned} \vartheta(u) e^{-\frac{t}{n^p}} &= \frac{1}{2}\left(e^{u - \frac{t}{n^p}} - e^{-(u + \frac{t}{n^p})}\right) \\ &\le \frac{1}{2} e^{\Theta(t, \sup u(0,\cdot)) - \frac{t}{n^p}} \\ &\le c(\sup u(0,\cdot), n, p), \end{aligned} \tag{3.25}$$

by Corollary 3.3, as well as

$$\begin{aligned}\vartheta(u)e^{-\frac{t}{n^p}} &\geq \frac{1}{2}\left(e^{\Theta(t,\inf u(0,\cdot))-\frac{t}{n^p}} - e^{-\Theta(t,\inf u(0,\cdot))-\frac{t}{n^p}}\right)\\ &= \frac{1}{2}\left(e^{\Theta(t,\inf u(0,\cdot))-\frac{t}{n^p}} - e^{\Theta(t,\inf u(0,\cdot))-\frac{t}{n^p}-2\Theta(t,\inf u(0,\cdot))}\right)\\ &\geq \frac{1}{2}e^{-c}\left(1-e^{-2\Theta(t,\inf u(0,\cdot))}\right)\\ &\geq c > 0.\end{aligned} \tag{3.26}$$

Furthermore

$$\begin{aligned}\coth u - 1 &= \frac{\cosh u - \sinh u}{\vartheta(u)} = \frac{e^{-u}}{\vartheta(u)}\\ &= \frac{e^{-(u-\frac{t}{n^p})}e^{-\frac{2}{n^p}t}}{\vartheta(u)e^{-\frac{t}{n^p}}}\\ &\leq \frac{e^{-\Theta(t,\inf u(0,\cdot))+\frac{t}{n^p}}e^{-\frac{2}{n^p}t}}{\vartheta(u)e^{-\frac{t}{n^p}}}\\ &\leq ce^{-\frac{2}{n^p}t}.\end{aligned} \tag{3.27}$$

□

C^1-estimates.

3.7. **Lemma.** Let u be the short time solution of (3.14) in case $p > 1$. Then for the quantity

$$v = \sqrt{1+\bar{g}^{ij}u_iu_j} \equiv \sqrt{1+|Du|^2} \tag{3.28}$$

there exists $c = c(n, p, M_0)$, such that

$$v \leq c \quad \forall t \in [0, T^*). \tag{3.29}$$

Furthermore c depends on p continuously.

Proof. In case $p > 1$ the leaves $M(t)$ are convex. Thus, [4, Thm. 2.7.10], especially estimate (2.7.83)

$$v \leq e^{\bar{\kappa}(\sup u - \inf u)} \tag{3.30}$$

is applicable. Note that in this estimate, an upper bound for the principal curvatures of $\{x^0 = \text{const}\}$ is uniformly given by some $\bar{\kappa} = \bar{\kappa}(\inf u(0,\cdot))$. Thus we obtain the claim in view of Lemma 3.5. □

In case $p \leq 1$ we do not assume convexity. We use the maximum principle to estimate v.

We follow the method in [5].

3.8. **Remark.** Defining

$$\varphi = \int_{r_0}^{u} \vartheta^{-1} \tag{3.31}$$

and having (2.21) in mind, we obtain

$$h^i_j = g^{ik}h_{kj} = v^{-1}\vartheta^{-1}(-(\sigma^{ik} - v^{-2}\varphi^i\varphi^k)\varphi_{jk} + \dot\vartheta\delta^i_j), \tag{3.32}$$

where covariant differentiation and index raising happens with respect to σ_{ij}, cf. [5, (3.26)]. We obtain

$$\dot\varphi = \vartheta^{-1}\dot u = \frac{\vartheta^{p-1}v}{F^p(\vartheta h^i_j)} \equiv \frac{\vartheta^{p-1}v}{F^p(\tilde h^i_j)}. \tag{3.33}$$

There holds

$$g_{ij} = u_iu_j + \vartheta^2\sigma_{ij} = \vartheta^2(\varphi_i\varphi_j + \sigma_{ij}) \equiv \vartheta^2\tilde g_{ij}. \tag{3.34}$$

Defining

$$\tilde h_{ij} = \tilde g_{ik}\tilde h^k_j, \tag{3.35}$$

we see that in (3.33) we are considering the eigenvalues of $\tilde h_{ij}$ with respect to $\tilde g_{ij}$ and thus we define

$$F^{ij} = \frac{\partial F}{\partial \tilde h_{ij}} \text{ and } F^i_j = \frac{\partial F}{\partial \tilde h^j_i}. \tag{3.36}$$

We have

$$\tilde h_{ij} = \tilde g_{ik}\tilde h^k_j = \vartheta^{-2}g_{ik}\vartheta h^k_j = \vartheta^{-1}h_{ij}, \tag{3.37}$$

hence $\tilde h_{ij}$ is symmetric. Furthermore note

$$|Du|^2 = \sigma^{ij}\varphi_i\varphi_j \equiv |D\varphi|^2, \tag{3.38}$$

as well as

$$\tilde h^l_k = -v^{-1}\tilde g^{lj}\varphi_{jk} + v^{-1}\dot\vartheta\delta^l_k. \tag{3.39}$$

3.9. **Lemma.** The various quantities and tensors in (3.33) satisfy

$$(\vartheta^{p-1})_i = (p-1)\vartheta^{p-1}\dot\vartheta\varphi_i, \tag{3.40}$$

$$v_i = v^{-1}\varphi_{ki}\varphi^k, \tag{3.41}$$

$$\tilde g^{lr}{}_{;i} = 2v^{-3}v_i\varphi^l\varphi^r - v^{-2}(\varphi^l_i\varphi^r + \varphi^l\varphi^r_i) \tag{3.42}$$

and

$$\tilde h^l_{k;i} = v^{-2}v_i(\tilde g^{lr}\varphi_{rk} - \dot\vartheta\delta^l_k) - v^{-1}(\tilde g^{lr}{}_{;i}\varphi_{rk} + \tilde g^{lr}\varphi_{rki} - \vartheta^2\varphi_i\delta^l_k), \tag{3.43}$$

where $(\tilde g^{rl}) = (\tilde g_{rl})^{-1}$ and the covariant derivatives as well as index raising are performed with respect to σ_{ij}.

Proof. This is a straightforward computation in any of the cases. Just have in mind that $\vartheta = \vartheta(u)$, such that $\vartheta_i = \dot\vartheta u_i = \dot\vartheta\vartheta\varphi_i$. □

3.10. **Lemma.** Let u be the solution of (3.14) in case $p \le 1$. Then

$$v \le \sup v(0,\cdot). \tag{3.44}$$

Proof. From Remark 3.8 we see, that it suffices to bound $|D\varphi|^2$. Differentiate

$$\dot{\varphi} = -\Phi v\vartheta^{p-1}, \;\; \Phi = \Phi(F(\tilde{h}^k_l)), \tag{3.45}$$

with respect to $\varphi^i D_i$. From Lemma 3.9 we find, setting

$$w = \frac{1}{2}|D\varphi|^2 = \frac{1}{2}\varphi_k\varphi^k, \tag{3.46}$$

$$\dot{w} = \dot{\varphi}_i\varphi^i = -v\vartheta^{p-1}\dot{\Phi}F^k_l\tilde{h}^l_{k;i}\varphi^i - \Phi\vartheta^{p-1}v_i\varphi^i - (p-1)\Phi v\vartheta^{p-1}\dot{\vartheta}|D\varphi|^2. \tag{3.47}$$

Fix $0 < T < T^*$ and suppose

$$\sup_{[0,T]\times\mathbb{S}^n} w = w(t_0, x_0) > 0, \;\; t_0 > 0, \tag{3.48}$$

then at this point we have

$$\begin{aligned}
0 &\le (p-1)F^{-p}v\vartheta^{p-1}\dot{\vartheta}|D\varphi|^2 - v\vartheta^{p-1}\dot{\Phi}F^k_l(-v^{-1}\tilde{g}^{lr}\varphi_{rki}\varphi^i \\
&\quad + v^{-1}\vartheta^2|D\varphi|^2\delta^l_k + v^{-3}\varphi_{rk}\varphi^r\varphi^l_i\varphi^i + v^{-3}\varphi^r_i\varphi^i\varphi^l\varphi_{rk}) \\
&= 2(p-1)F^{-p}v\vartheta^{p-1}\dot{\vartheta}w - 2\dot{\Phi}\vartheta^{p+1}F^{kl}\tilde{g}_{kl}w \\
&\quad + \vartheta^{p-1}\dot{\Phi}F^{kr}\varphi_{rki}\varphi^i \\
&= (2(p-1)F^{-p}v\vartheta^{p-1}\dot{\vartheta} - 2\dot{\Phi}\vartheta^{p+1}F^{kl}\tilde{g}_{kl})w \\
&\quad + \vartheta^{p-1}\dot{\Phi}F^{kr}(\varphi_{irk} + \varphi_k\sigma_{ri} - \varphi_i\sigma_{rk})\varphi^i \\
&= (2(p-1)F^{-p}v\vartheta^{p-1}\dot{\vartheta} - 2\dot{\Phi}\vartheta^{p+1}F^{kl}\tilde{g}_{kl})w \\
&\quad + \vartheta^{p-1}\dot{\Phi}F^{kr}(\varphi_k\varphi_r - |D\varphi|^2\sigma_{kr}) \\
&\quad + \vartheta^{p-1}\dot{\Phi}F^{kr}w_{rk} - \vartheta^{p-1}\dot{\Phi}F^{kr}\varphi_{ir}\varphi^i_k \\
&< 0.
\end{aligned} \tag{3.49}$$

Hence the estimate (3.44) is valid, since T is arbitrary. □

Curvature estimates and long time existence.

3.11. **Proposition.** Let x be a solution of the curvature flow (1.4), $0 < p < \infty$. Then the curvature function is bounded from above and below, i.e. there exists $c = c(n, p, M_0)$, such that

$$0 < c^{-1} \le F(t,\xi) \le c < \infty \;\; \forall (t,\xi) \in [0, T^*) \times M. \tag{3.50}$$

Proof. The proof proceeds similarly to the one in [5, Lemma 4.1].

Define

$$\chi = v\eta(u) \equiv \frac{v}{\sinh u} \tag{3.51}$$

and note

$$\dot{\eta} = -\frac{\bar{H}}{n}\eta, \tag{3.52}$$

where $\eta = \eta(r)$ and $\bar{H}$ is the mean curvature of S_r. Then χ satisfies

$$\dot{\chi} - \dot{\Phi}F^{ij}\chi_{ij} = -\dot{\Phi}F^{ij}h_{ik}h^k_j\chi - 2\chi^{-1}\dot{\Phi}F^{ij}\chi_i\chi_j + (\dot{\Phi}F + \Phi)\frac{\bar{H}}{n}v\chi, \tag{3.53}$$

cf. [3, Lemma 5.8]. Φ, and also $-\Phi$, satisfy

$$\Phi' - \dot{\Phi}F^{ij}\Phi_{ij} = \dot{\Phi}F^{ij}h_{ik}h^k_j\Phi + K_N\dot{\Phi}F^{ij}g_{ij}\Phi, \tag{3.54}$$

where $'$ denotes the time derivative of the evolution and $\dot{\Phi} = \frac{d}{dr}\Phi(r)$, cf. [4, Lemma 2.3.4]. Note that we have $K_N = -1$. The function u satisfies

$$\dot{u} - \dot{\Phi}F^{ij}u_{ij} = (\dot{\Phi}F - \Phi)v^{-1} - \dot{\Phi}F^{ij}\bar{h}_{ij}, \tag{3.55}$$

where $\dot{u}$ is a total derivative, cf. [4, Lemma 3.3.2].

(i) We first prove $F \geq c > 0$. Set

$$\tilde{\chi} = \chi e^{\frac{t}{n^p}}. \tag{3.56}$$

Then there exists $c = c(n, p, M_0)$, such that

$$0 < c^{-1} \leq \tilde{\chi}(t, \xi) \leq c < \infty \quad \forall (t, \xi) \in [0, T^*) \times M, \tag{3.57}$$

where we used Corollary 3.6 and $v \leq c$. Set

$$w = \log(-\Phi) + \log\tilde{\chi}, \tag{3.58}$$

fix $0 < T < T^*$ and suppose

$$\sup_{[0,T]\times M} w = w(t_0, \xi_0) > 0, \; t_0 > 0. \tag{3.59}$$

Then in (t_0, ξ_0) there holds

$$\frac{\Phi_i}{\Phi} = -\frac{\chi_i}{\chi} \tag{3.60}$$

and

$$0 \leq \dot{w} - \dot{\Phi}F^{ij}w_{ij} = -\dot{\Phi}F^{ij}g_{ij} + (\dot{\Phi}F + \Phi)\frac{\bar{H}}{n}v + \frac{1}{n^p}. \tag{3.61}$$

Thus

$$\begin{aligned} 0 &\leq -pF^{ij}g_{ij} + (p-1)F\frac{\bar{H}}{n}v + n^{-p}F^{p+1} \\ &\leq -pn + (p-1)F\frac{\bar{H}}{n}v + n^{-p}F^{p+1}. \end{aligned} \tag{3.62}$$

Moreover

$$\bar{H} = n\coth u \leq n\coth\inf u(0,\cdot), \tag{3.63}$$

so that

$$0 \leq \begin{cases} -pn + n^{-p}F^{p+1}, & 0 < p \leq 1 \\ -pn + (p-1)\coth\inf u(0,\cdot)Fv + n^{-p}F^{p+1}, & p > 1. \end{cases} \tag{3.64}$$

Without loss of generality suppose $w(t_0, \xi_0)$ is so large, that $F(t_0, \xi_0) < 1$. Then

$$F(t_0, \xi_0) \geq \begin{cases} p^{\frac{1}{p+1}}n, & 0 < p \leq 1 \\ \frac{pn - n^{-p}}{(p-1)cv}, & p > 1, \end{cases} \tag{3.65}$$

$c = c(M_0)$. Hence, at a point, where w attains a maximum, we have $F \geq c = c(n, p, M_0)$. Thus

$$w \leq w(t_0, \xi_0) \leq \log\left(\frac{1}{c^p}\right) + c \equiv c(n, p, M_0) \tag{3.66}$$

and

$$\frac{1}{F^p} = e^w \tilde{\chi}^{-1} \leq c(n, p, M_0). \tag{3.67}$$

Thus, F is uniformly bounded below in $[0, T^*)$.

(ii) We prove $F \leq c$.
Define

$$\tilde{u} = u - \frac{t}{n^p}. \tag{3.68}$$

Then, by Corollary 3.3 and Lemma 3.5 we have

$$\tilde{u} > c. \tag{3.69}$$

Set

$$w = -\log(-\Phi) + \tilde{u}. \tag{3.70}$$

Then, in a maximal point $(t_0, \xi_0) \in (0, T] \times M$, $0 < T < T^*$, of w we have

$$\begin{aligned} 0 &\leq \dot{w} - \dot{\Phi} F^{ij} w_{ij} \\ &= -\dot{\Phi} F^{ij} h_{ik} h_j^k + \dot{\Phi} F^{ij} g_{ij} - \dot{\Phi} F^{ij} (\log(-\Phi))_i (\log(-\Phi))_j \\ &\quad + (\dot{\Phi} F - \Phi) v^{-1} - \dot{\Phi} F^{ij} \bar{h}_{ij} - \frac{1}{n^p} \\ &= -\dot{\Phi} F^{ij} h_{ik} h_j^k + \dot{\Phi} F^{ij} (u_i u_j + \bar{g}_{ij} - \coth u \bar{g}_{ij}) \\ &\quad - \dot{\Phi} F^{ij} u_i u_j + (\dot{\Phi} F - \Phi) v^{-1} - \frac{1}{n^p} \\ &\leq (p+1) F^{-p} v^{-1} - \frac{1}{n^p}, \end{aligned} \tag{3.71}$$

where we used $\coth u \geq 1$ and $0 = w_i$ in (t_0, ξ_0). Then

$$F(t_0, \xi_0) \leq c(n, p, M_0), \tag{3.72}$$

leading to

$$w \leq c(n, p, M_0) \tag{3.73}$$

and finally

$$F^p \leq e^w e^{-\tilde{u}} \leq c(n, p, M_0). \tag{3.74}$$

□

3.12. **Proposition.** The leaves $M(t)$ of (1.4) have uniformly bounded principal curvatures, i.e. there exists $c = c(n, p, M_0)$, such that

$$\kappa_i(t, \xi) \leq c \quad \forall (t, \xi) \in [0, T^*) \times M. \tag{3.75}$$

Thus the principal curvatures stay in a compact set $K = K(n, p, M_0) \subset \Gamma$, in view of Proposition 3.11.

Proof. Basically, the proof of the corresponding lemma in [5, Lemma 4.4], applies in our case with slight modifications.

Since $\mathbb{H}^{n+1}$ has constant curvature $K_N = -1$, we have

$$\begin{aligned}\dot{h}^i_j - \dot{\Phi} F^{kl} h^i_{j;kl} = \dot{\Phi} F^{kl} h_{rk} h^r_l h^i_j + (\Phi - \dot{\Phi} F) h^{ki} h_{kj} + \ddot{\Phi} F_j F^i \\ + \dot{\Phi} F^{kl,rs} h_{kl;j} h_{rs;}{}^i - (\Phi + \dot{\Phi} F)\delta^i_j + \dot{\Phi} F^{kl} g_{kl} h^i_j.\end{aligned} \tag{3.76}$$

Let $\tilde{\chi} = \chi e^{\frac{t}{n^p}}$. Setting

$$\hat{\chi} = \tilde{\chi}^{-1}, \tag{3.77}$$

we find a constant $\theta > 0$, such that

$$2\theta \leq \hat{\chi}(t,\xi) \quad \forall (t,\xi) \in [0, T^*) \times M. \tag{3.78}$$

Define the functions

$$\zeta = \sup\{h_{ij}\eta^i\eta^j : \|\eta\|^2 = g_{ij}\eta^i\eta^j = 1\}, \tag{3.79}$$

$$\phi = -\log(\hat{\chi} - \theta) \tag{3.80}$$

and

$$w = \log \zeta + \phi + \lambda \tilde{u}, \tag{3.81}$$

where $\tilde{u} = u - \frac{t}{n^p}$, and λ is to be chosen later. We wish to bound w from above. Thus, suppose w attains a maximal value at $(t_0, \xi_0) \in (0, T] \times M$, $T < T^*$. Choose Riemannian normal coordinates in (t_0, ξ_0), such that in this point we have

$$g_{ij} = \delta_{ij} \ \wedge \ h_{ij} = \kappa_i \delta_{ij} \ \wedge \ \kappa_1 \leq \ldots \leq \kappa_n. \tag{3.82}$$

Since ζ is only continuous in general, we need to find a differentiable version instead. Set

$$\tilde{\zeta} = \frac{h_{ij}\tilde{\eta}^i\tilde{\eta}^j}{g_{ij}\tilde{\eta}^i\tilde{\eta}^j}, \tag{3.83}$$

where $\tilde{\eta} = (\tilde{\eta}^i) = (0, \ldots, 0, 1)$.
At (t_0, ξ_0) we have

$$h_{nn} = h^n_n = \kappa_n = \zeta = \tilde{\zeta} \tag{3.84}$$

and in a neighborhood of (t_0, ξ_0) there holds

$$\tilde{\zeta} \leq \zeta. \tag{3.85}$$

Using $h^n_n = h_{nk} g^{kn}$, we find that at (t_0, ξ_0)

$$\dot{\tilde{\zeta}} = \dot{h}^n_n \tag{3.86}$$

and the spatial derivatives also coincide, cf. [5, p.13]. Replacing w by $\tilde{w} = \log \tilde{\zeta} + \phi + \lambda \tilde{u}$, we see that $\tilde{w}$ attains a maximal value at (t_0, ξ_0), where $\tilde{\zeta}$ satisfies the same differential equation in this point as h^n_n. Thus, without loss of generality, we may pretend h^n_n to be a scalar and w to be given by

$$w = \log h^n_n + \phi + \lambda \tilde{u}. \tag{3.87}$$

Since

$$\dot{\hat{\chi}} - \dot{\Phi}F^{ij}\hat{\chi}_{ij} = -\tilde{\chi}^{-2}(\dot{\tilde{\chi}} - \dot{\Phi}F^{ij}\tilde{\chi}_{ij}) - 2\tilde{\chi}^{-3}\dot{\Phi}F^{ij}\tilde{\chi}_i\tilde{\chi}_j, \tag{3.88}$$

we find

$$\begin{aligned}\dot{\phi} - \dot{\Phi}F^{ij}\phi_{ij} &= (\hat{\chi}-\theta)^{-1}(\tilde{\chi}^{-2}(\dot{\tilde{\chi}} - \dot{\Phi}F^{ij}\tilde{\chi}_{ij}) + 2\tilde{\chi}^{-3}\dot{\Phi}F^{ij}\tilde{\chi}_i\tilde{\chi}_j)\\ &\quad - \dot{\Phi}F^{ij}\frac{(\hat{\chi}-\theta)_i(\hat{\chi}-\theta)_j}{(\hat{\chi}-\theta)^2}\\ &= (\hat{\chi}-\theta)^{-1}(-\dot{\Phi}F^{ij}h_{ik}h^k_j\hat{\chi} + (\dot{\Phi}F+\Phi)\frac{\bar{H}}{n}v\hat{\chi} + \frac{1}{n^p}\hat{\chi})\\ &\quad - \dot{\Phi}F^{ij}(\log(\hat{\chi}-\theta))_i(\log(\hat{\chi}-\theta))_j.\end{aligned} \tag{3.89}$$

Thus, in (t_0, ξ_0) we infer

$$\begin{aligned}0 &\le \dot{w} - \dot{\Phi}F^{ij}w_{ij}\\ &= \dot{\Phi}F^{kl}h_{kr}h^r_l + (\Phi - \dot{\Phi}F)h^n_n + \ddot{\Phi}F_nF^n(h^n_n)^{-1}\\ &\quad + \dot{\Phi}F^{kl,rs}h_{kl;n}h_{rs;}{}^n(h^n_n)^{-1} - (\Phi + \dot{\Phi}F)(h^n_n)^{-1} + \dot{\Phi}F^{kl}g_{kl}\\ &\quad - \dot{\Phi}F^{ij}h_{ik}h^k_j\frac{\hat{\chi}}{\hat{\chi}-\theta} + (\dot{\Phi}F+\Phi)\frac{\bar{H}}{n}v\frac{\hat{\chi}}{\hat{\chi}-\theta} + \frac{1}{n^p}\frac{\hat{\chi}}{\hat{\chi}-\theta}\\ &\quad + \lambda(\dot{\Phi}F - \Phi)v^{-1} - \lambda\dot{\Phi}F^{ij}\bar{h}_{ij} - \frac{\lambda}{n^p}\\ &\quad - \dot{\Phi}F^{ij}(\log(\hat{\chi}-\theta))_i(\log(\hat{\chi}-\theta))_j + \dot{\Phi}F^{ij}(\log h^n_n)_i(\log h^n_n)_j.\end{aligned} \tag{3.90}$$

In the present coordinate system we have

$$F^{kl,rs}\eta_{kl}\eta_{rs} \le \sum_{k\neq l}\frac{F^{kk}-F^{ll}}{\kappa_k - \kappa_l}(\eta_{kl})^2 \le \frac{2}{\kappa_n - \kappa_1}\sum_{k=1}^n(F^{nn}-F^{kk})(\eta_{nk})^2 \tag{3.91}$$

for all symmetric tensors (η_{kl}) and

$$F^{nn} \le \ldots \le F^{11}, \tag{3.92}$$

cf. [5, (4.28), (4.29)] and the references therein. Using those inequalities, $\ddot{\Phi} < 0$ as well as

$$(\log h^n_n)_i = -\phi_i - \lambda\tilde{u}_i \tag{3.93}$$

in (t_0, ξ_0), we obtain from (3.90)

$$\begin{aligned}0 \le &-\dot{\Phi}F^{ij}h_{ik}h^k_j\frac{\theta}{\hat{\chi}-\theta} + (\Phi - \dot{\Phi}F)h^n_n - (\Phi + \dot{\Phi}F)(h^n_n)^{-1} + \dot{\Phi}F^{kl}g_{kl}\\ &+ (\dot{\Phi}F+\Phi)\frac{\bar{H}}{n}v\frac{\hat{\chi}}{\hat{\chi}-\theta} + \frac{1}{n^p}\frac{\hat{\chi}}{\hat{\chi}-\theta}\\ &+ \lambda(\dot{\Phi}F - \Phi)v^{-1} - \lambda\dot{\Phi}F^{ij}\bar{h}_{ij} - \frac{\lambda}{n^p}\\ &+ 2\lambda\dot{\Phi}F^{ij}\phi_i\tilde{u}_j + \lambda^2\dot{\Phi}F^{ij}\tilde{u}_i\tilde{u}_j\\ &+ \frac{2}{\kappa_n - \kappa_1}\dot{\Phi}\sum_{i=1}^n(F^{nn}-F^{ii})(h_{ni;}{}^n)^2(h^n_n)^{-1}.\end{aligned} \tag{3.94}$$

There holds

$$\begin{aligned} F^{ij}\bar{h}_{ij} &= F^{ij}\bar{g}_{ij}\coth u \geq F^{ij}\bar{g}_{ij} = F^{ij}g_{ij} - F^{ij}u_iu_j \\ &\geq F^{ij}g_{ij}(1-\|Du\|^2) = v^{-2}F^{ij}g_{ij} \geq \tilde{c}_0F^{ij}g_{ij}, \end{aligned} \tag{3.95}$$

where $\tilde{c}_0 = c(n,p,M_0)$, and

$$h_{ni;n} = h_{nn;i}, \tag{3.96}$$

in view of the Codazzi equation. We now estimate (3.94).
We distinguish two cases.

Case 1: $\kappa_1 < -\epsilon_1\kappa_n, 0 < \epsilon_1 < 1$.

Then

$$F^{ij}h_{ki}h^k_j \geq \frac{1}{n}F^{ij}g_{ij}\epsilon_1^2\kappa_n^2, \tag{3.97}$$

cf. [5, p.14, (4.47)]. Furthermore, by [3, (5.29)], we have

$$v_i = -v^2h^k_iu_k + v\frac{\bar{H}}{n}u_i = (-v^2\kappa_i + v\frac{\bar{H}}{n})u_i \tag{3.98}$$

and thus

$$\|Dv\| \leq c|\kappa_n|\|Du\| + c\|Du\|, c = c(n,p,M_0) \tag{3.99}$$

so that

$$\|D\phi\| \leq c\|Dv\| + c\|Du\| \leq c|\kappa_n|\|Du\| + c\|Du\|. \tag{3.100}$$

Hence (3.94) can be estimated:

$$\begin{aligned} 0 \leq \dot{\Phi}F^{ij}g_{ij}\Big(&-\frac{1}{n}\epsilon_1^2\kappa_n^2\frac{\theta}{\hat{\chi}-\theta} + 1 - \lambda\tilde{c}_0 + 2\lambda c\|Du\|^2(\kappa_n+1) \\ &+ \lambda^2\|Du\|^2\Big) \\ &- (\Phi + \dot{\Phi}F)\kappa_n^{-1} + (\dot{\Phi}F + \Phi)\frac{\bar{H}}{n}v\frac{\hat{\chi}}{\hat{\chi}-\theta} + \frac{1}{n^p}\frac{\hat{\chi}}{\hat{\chi}-\theta} \\ &+ \lambda(\dot{\Phi}F - \Phi)v^{-1}. \end{aligned} \tag{3.101}$$

The last two lines are uniformly bounded by some $c = c(n,p,M_0)$ and the first line converges to $-\infty$, if $\kappa_n \to \infty$, where we use $\dot{\Phi}F^{ij}g_{ij} \geq c > 0$ and the boundedness of all the other coefficients. We conclude, that in this case any choice of λ yields

$$\kappa_n \leq c(n,p,M_0). \tag{3.102}$$

Case 2: $\kappa_1 \geq -\epsilon_1\kappa_n$.

Then

$$\begin{aligned} &\frac{2}{\kappa_n-\kappa_1}\dot{\Phi}\sum_{i=1}^n(F^{nn}-F^{ii})(h_{ni;}{}^n)^2(h^n_n)^{-1} \\ &\leq \frac{2}{1+\epsilon_1}\dot{\Phi}\sum_{i=1}^n(F^{nn}-F^{ii})(\log h^n_n)_i^2, \end{aligned} \tag{3.103}$$

so that

$$\begin{aligned}&\dot\Phi F^{ij}(\log h^n_n)_i(\log h^n_n)_j+\frac{2}{\kappa_n-\kappa_1}\dot\Phi\sum_{i=1}^n(F^{nn}-F^{ii})(h_{ni;}{}^n)^2(h^n_n)^{-1}\\&\le\frac{2}{1+\epsilon_1}\dot\Phi\sum_{i=1}^nF^{nn}(\log h^n_n)^2_i-\frac{1-\epsilon_1}{1+\epsilon_1}\dot\Phi\sum_{i=1}^nF^{ii}(\log h^n_n)^2_i\\&\le\frac{2}{1+\epsilon_1}\dot\Phi\sum_{i=1}^nF^{nn}(\log h^n_n)^2_i-\frac{1-\epsilon_1}{1+\epsilon_1}\dot\Phi F^{nn}\sum_{i=1}^n(\log h^n_n)^2_i\\&=\dot\Phi F^{nn}\|D\phi+\lambda Du\|^2\\&=\dot\Phi F^{nn}(\|D\phi\|^2+\lambda^2\|Du\|^2+2\lambda\langle D\phi,D\tilde u\rangle),\end{aligned}\tag{3.104}$$

where we used $g_{ij}=\delta_{ij}$. We now choose $\lambda=\lambda(n,p,M_0)$, such that

$$\lambda>\tilde c_0^{-1}.\tag{3.105}$$

Estimating (3.90) again yields

$$\begin{aligned}0\le&-\dot\Phi F^{nn}\kappa_n^2\frac{\theta}{\hat\chi-\theta}-(\Phi+\dot\Phi F)\kappa_n^{-1}+\dot\Phi F^{kl}g_{kl}(1-\lambda\tilde c_0)\\&+(\Phi-\dot\Phi F)\kappa_n+(\dot\Phi F+\Phi)\frac{\bar H}{n}v\frac{\hat\chi}{\hat\chi-\theta}+\frac{1}{n^p}\frac{\hat\chi}{\hat\chi-\theta}\\&+\lambda(\dot\Phi F-\Phi)v^{-1}-\frac{\lambda}{n^p}\\&+\dot\Phi F^{nn}(\lambda^2\|Du\|^2+2\lambda\|D\phi\|\|Du\|),\end{aligned}\tag{3.106}$$

implying

$$\kappa_n(t_0,\xi_0)\le c(n,p,M_0).\tag{3.107}$$

Thus, w and ζ as well, are bounded from above, implying the claim. □

3.13. **Theorem.** Under the hypothesis of Theorem 1.2 we have

$$T^*=\infty.\tag{3.108}$$

Proof. Following [4, 2.6.2], all we have to show is that we have a uniform $C^2(\mathbb{S}^n)$ estimate on finite intervals, since we have already shown the uniform ellipticity on such intervals. There holds

$$h^i_j=-v^{-1}\vartheta^{-1}\tilde g^{ik}\varphi_{kj}+v^{-1}\frac{\dot\vartheta}{\vartheta}\delta^i_j,\tag{3.109}$$

where $\tilde g^{ik}=\sigma^{ik}-v^{-2}\varphi^i\varphi^k$. We have

$$\varphi_j=\vartheta^{-1}u_j\tag{3.110}$$

and

$$\varphi_{jk}=-\vartheta^{-2}\dot\vartheta u_ju_k+\vartheta^{-1}u_{jk},\tag{3.111}$$

where covariant derivatives are taken with respect to σ_{ij}. Thus

$$h^i_j = \frac{\dot\vartheta}{v\vartheta}\delta^i_j + v^{-1}\vartheta^{-3}\dot\vartheta\tilde g^{ik}u_ju_k - v^{-1}\vartheta^{-2}\tilde g^{ik}u_{jk} = \frac{\dot\vartheta}{v\vartheta}\delta^i_j + \frac{\dot\vartheta}{v^3\vartheta^3}u^iu_j - \frac{\tilde g^{ik}}{v\vartheta^2}u_{kj}, \tag{3.112}$$

where $u^i = \sigma^{ik}u_k$. Since $v \le c$, σ_{ik} and $\tilde g_{ik}$ generate equivalent norms. All the other tensors are bounded in finite time and thus

$$|u|_{2,\mathbb{S}^n} \le c = c(n,p,M_0,T^*). \tag{3.113}$$

Then, using Krylov-Safonov, [9], [4, Thm. 2.5.9] and Remark 3.4 we conclude the result. □

4. Decay estimates in C^1 and C^2

Decay of the C^1-norm.

4.1. **Theorem.** Under the hypotheses of Theorem 1.2, for all $0 < p \le 1$ there exist constants $0 < \lambda$ and $0 < c$ depending on n, p and M_0, such that

$$v - 1 \le ce^{-\lambda t} \quad \forall t \in [0,\infty). \tag{4.1}$$

In case $p > 1$ there exist constants $0 < \epsilon, \lambda, c$, depending on n, p and M_0, such that

$$\operatorname{osc} u(0,\cdot) < \epsilon \Rightarrow \ v - 1 \le ce^{-\lambda t} \quad \forall t \in [0,\infty). \tag{4.2}$$

Proof. Considering the equation for v, cf. [3, (5.28)] and, using

$$\frac{\dot{\bar H}}{n} = 1 - \frac{\bar H^2}{n^2}, \tag{4.3}$$

we obtain

$$\begin{aligned} \dot v - \dot\Phi F^{ij}v_{ij} &= -\dot\Phi F^{ij}h_{ik}h^k_jv - 2v^{-1}\dot\Phi F^{ij}v_iv_j + 2\dot\Phi F^{ij}v_iu_j\frac{\bar H}{n} \\ &\quad - \dot\Phi F^{ij}g_{ij}\frac{\bar H^2}{n^2}v - \dot\Phi F^{ij}u_iu_jv + \dot\Phi F^{ij}u_iu_j\frac{\bar H^2}{n^2}v \\ &\quad + \frac{\bar H}{n}(v^2-1)(\Phi - \dot\Phi F) + 2\dot\Phi F\frac{\bar H}{n}v^2 \\ &= -\dot\Phi F^{ij}(h_{ik}h^k_j - 2h_{ij} + g_{ij})v - 2v^{-1}\dot\Phi F^{ij}v_iv_j \\ &\quad + 2\dot\Phi F^{ij}v_iu_j\frac{\bar H}{n} - \dot\Phi F^{ij}g_{ij}\left(\frac{\bar H^2}{n^2} - 1\right)v \\ &\quad + \dot\Phi F^{ij}u_iu_j\left(\frac{\bar H^2}{n^2} - 1\right)v + \frac{\bar H}{n}(v^2-1)\Phi \\ &\quad + \dot\Phi F\frac{\bar H}{n} - 2\dot\Phi Fv + \dot\Phi F\frac{\bar H}{n}v^2 \end{aligned} \tag{4.4}$$

Let $\lambda > 0$ and set

$$w = (v-1)e^{\lambda t}. \tag{4.5}$$

Fix $T > 0$ and suppose

$$\sup_{[0,T]\times M} w = w(t_0, \xi_0) > 1. \tag{4.6}$$

Then at this point

$$\begin{aligned}
0 &\le \dot\Phi F^{ij} u_i u_j \left(\frac{\bar H^2}{n^2} - 1\right) v e^{\lambda t} + \frac{\bar H}{n}(v^2-1)\Phi e^{\lambda t} + \dot\Phi F \frac{\bar H}{n}(v-1)^2 e^{\lambda t} \\
&\quad + 2\dot\Phi F \left(\frac{\bar H}{n} - 1\right) v e^{\lambda t} + \lambda w \\
&= \dot\Phi F^{ij} u_i u_j \left(\frac{\bar H^2}{n^2} - 1\right) v e^{\lambda t} + 2\dot\Phi F \left(\frac{\bar H}{n} - 1\right) v e^{\lambda t} \\
&\quad + \left(\frac{\bar H}{n} F^{-p}(p(v-1) - (v+1)) + \lambda\right) w. \\
&\le c e^{(\lambda - \frac{2}{n^p})t} + \left(\frac{\bar H}{n} F^{-p}(p(v-1) - (v+1)) + \lambda\right) w,
\end{aligned} \tag{4.7}$$

where the last estimate follows from the estimates of the curvature function, the principal curvatures and Corollary 3.6. The constant in this inequality depends on n, p and M_0.

Consider $p > 1$. In view of (3.30) we deduce

$$v \le e^{\bar\kappa \operatorname{osc} u}, \tag{4.8}$$

where $\bar\kappa$ is an upper bound for the curvatures of the slices, which in our case converge to 1, as $t \to \infty$. Choosing $\beta > 0$, such that

$$\beta < \frac{1}{\bar\kappa} \log \frac{p+1}{p-1} \quad \forall t \in [0, \infty), \tag{4.9}$$

there exists $\epsilon > 0$, such that

$$\operatorname{osc} u(0,\cdot) < \epsilon \Rightarrow \sup_{t\in[0,\infty)} \operatorname{osc} u(t,\cdot) < \beta, \tag{4.10}$$

due to the estimates (3.5) and (3.16) and we conclude further

$$v \le e^{\bar\kappa\beta} < \frac{p+1}{p-1} \quad \forall t \in [0,\infty). \tag{4.11}$$

Using

$$0 < c^{-1} \le F \le c, \ \ c = c(n, p, M_0) \tag{4.12}$$

and

$$\frac{\bar H}{n} \ge 1, \tag{4.13}$$

we obtain from (4.7)

$$0 \le c e^{(\lambda - \frac{2}{n^p} t)} + c((p-1)v - (p+1) + \lambda) w. \tag{4.14}$$

In this inequality the coefficient of the linear term is strictly negative in view of the previous considerations, if $\lambda(n, p, M_0)$ is small, while the first term converges to 0, which leads to a contradiction, if t_0 is sufficiently large. Thus w is bounded, completing the proof.

□

Curvature asymptotics.

4.2. **Lemma.** Let $f \in C^{0,1}(\mathbb{R}_+)$ and let D be the set of points of differentiability of f. Suppose that for all $\epsilon > 0$ there exist $T_\epsilon > 0$ and $\delta_\epsilon > 0$, such that

$$\{t \in D \cap [T_\epsilon, \infty) \colon f(t) \geq \epsilon\} \subset \{t \in D \cap [T_\epsilon, \infty) \colon f'(t) < -\delta_\epsilon\}. \tag{4.15}$$

Then there holds

$$\limsup_{t \to \infty} f(t) \leq 0. \tag{4.16}$$

Proof. Suppose first, that

$$\liminf_{t \to \infty} f(t) \geq 2\epsilon > 0. \tag{4.17}$$

Then there exists $\tilde{T} > 0$, such that

$$f(t) > \epsilon \quad \forall t \geq \tilde{T} \tag{4.18}$$

and hence, there exists $T_\epsilon \geq \tilde{T}$ and $\delta_\epsilon > 0$, such that

$$f'(t) < -\delta_\epsilon \quad \forall t \in D \cap [T_\epsilon, \infty) \tag{4.19}$$

and we infer for all $t \geq T_\epsilon$

$$f(t) \leq f(T_\epsilon) + \int_{T_\epsilon}^{t} (-\delta_\epsilon) = f(T_\epsilon) - \delta_\epsilon(t - T_\epsilon) \to -\infty, \tag{4.20}$$

as $t \to \infty$, which is a contradiction.

Now suppose that

$$\liminf_{t \to \infty} f(t) \leq 0 \ \wedge \ \limsup_{t \to \infty} f(t) \geq 2\epsilon > 0. \tag{4.21}$$

Then there exist $(t_k)_{k \in \mathbb{N}}$ and $(s_k)_{k \in \mathbb{N}}$, such that

$$\begin{aligned} &t_k < s_k, \\ &t_k \to \infty, k \to \infty, \\ &\frac{\epsilon}{2} \leq f(t_k) \leq \epsilon, \\ &f(s_k) > \frac{3}{2}\epsilon, \\ &f_{|[t_k, s_k)} \geq \frac{\epsilon}{2}. \end{aligned} \tag{4.22}$$

Since $D \subset \mathbb{R}_+$ is dense and f continuous, we may suppose that $t_k, s_k \in D$. Choose $T_{\frac{\epsilon}{2}} > 0$ and $\delta_{\frac{\epsilon}{2}} > 0$, such that

$$f(t) \geq \frac{\epsilon}{2} \Rightarrow f'(t) < -\delta_{\frac{\epsilon}{2}} \quad \forall t \in D \cap [T_{\frac{\epsilon}{2}}, \infty). \tag{4.23}$$

We conclude

$$f(s_k) - f(t_k) \le -\int_{t_k}^{s_k} \delta_{\frac{\epsilon}{2}} = -\delta_{\frac{\epsilon}{2}}(s_k - t_k) \quad \forall t_k, s_k \ge T_{\frac{\epsilon}{2}}, \tag{4.24}$$

hence

$$f(s_k) < f(t_k), \tag{4.25}$$

which is a contradiction. □

4.3. **Lemma.** Under the hypotheses of Theorem 1.2 the principal curvatures of the flow hypersurfaces converge to 1,

$$\sup_M |\kappa_i(t,\cdot) - 1| \to 0, \; t \to \infty, \quad \forall 1 \le i \le n. \tag{4.26}$$

Proof. (i) As in the proof of Proposition 3.12 we consider the function

$$\zeta = \sup\{h_{ij}\eta^i\eta^j : \|\eta\|^2 = g_{ij}\eta^i\eta^j = 1\}. \tag{4.27}$$

Set

$$w = (\log\zeta + \log\tilde\chi + \tilde u - \log 2)t, \tag{4.28}$$

where

$$\tilde\chi = \chi e^{\frac{t}{n^p}} = \frac{v}{\sinh u} e^{\frac{t}{n^p}}, \; \tilde u = u - \frac{t}{n^p}. \tag{4.29}$$

Fix $0 < T < \infty$ and suppose

$$\sup_{[0,T]\times M} w = w(t_0, \xi_0), \; t_0 > 0. \tag{4.30}$$

As in the proof of Proposition 3.12, we choose coordinates such that in (t_0, ξ_0) there holds $g_{ij} = \delta_{ij}$, $h_{ij} = \kappa_i\delta_{ij}$ and

$$w = (\log h^n_n + \log\tilde\chi + \tilde u - \log 2)t. \tag{4.31}$$

First note, that

$$\begin{aligned} (\log\tilde\chi + \tilde u - \log 2)t &= \left(\log v - \log(\sinh u) + \frac{t}{n^p} + u - \frac{t}{n^p} - \log 2\right)t \\ &= \left(\log v - \log\frac{1}{2}(e^u - e^{-u}) + u - \log 2\right)t \\ &= (\log v - \log(e^u - e^{-u}) + u)t \end{aligned} \tag{4.32}$$

is bounded. To prove this claim, note that

$$t\log v = \log(1 + v - 1)^t \le \log(1 + ce^{-\lambda t})^t, \tag{4.33}$$

which follows from Theorem 4.1. Furthermore

$$e^{t(u - \log(e^u - e^{-u}))} = \left(\frac{e^u}{e^u - e^{-u}}\right)^t = (1 - e^{-2u})^{-t} \le (1 - e^{c - \frac{t}{n^p}})^{-t}, \tag{4.34}$$

following from Corollary 3.3 and Lemma 3.5. But for large t we have

$$e^{-\lambda t} \le \frac{c}{t} \tag{4.35}$$

and thus

$$(1+ce^{-\lambda t})^t \le (1+\frac{c}{t})^t \le \text{const}. \tag{4.36}$$

The term

$$t(u-\log(e^u-e^{-u})) \tag{4.37}$$

is bounded for the same reason. Using the equations for h^n_n, $\tilde{\chi}$ and $\tilde{u}$, cf. Proposition 3.12, we obtain

$$\begin{aligned}\dot{w}-\dot{\Phi}F^{ij}w_{ij} = &\Big((\Phi-\dot{\Phi}F)h^{kn}h_{kn}(h^n_n)^{-1}+\ddot{\Phi}F_nF^n(h^n_n)^{-1}\\ &+\dot{\Phi}F^{kl,rs}h_{kl;n}h_{rs;}{}^n(h^n_n)^{-1}-(\Phi+\dot{\Phi}F)(h^n_n)^{-1}\\ &+\dot{\Phi}F^{kl}g_{kl}+\dot{\Phi}F^{kl}(\log h^n_n)_k(\log h^n_n)_l\\ &-\dot{\Phi}F^{kl}(\log\tilde{\chi})_k(\log\tilde{\chi})_l+(\dot{\Phi}F+\Phi)\frac{\bar{H}}{n}v\\ &+(\dot{\Phi}F-\Phi)v^{-1}-\dot{\Phi}F^{ij}\bar{h}_{ij}\Big)t_0\\ &+(\log h^n_n+\log\tilde{\chi}+\tilde{u}-\log 2)\\ \le\ &\Phi\left(h^n_n-(h^n_n)^{-1}-v^{-1}+\frac{\bar{H}}{n}v\right)t_0\\ &+\dot{\Phi}F\left(\frac{\bar{H}}{n}v+v^{-1}-(h^n_n+(h^n_n)^{-1})\right)t_0\\ &+\dot{\Phi}F^{kl}\bar{g}_{kl}(1-\coth u)t_0+\dot{\Phi}F^{kl}u_ku_lt_0\\ &+\dot{\Phi}F^{kl}((\log h^n_n)_k(\log h^n_n)_l-(\log\tilde{\chi})_k(\log\tilde{\chi})_l)t_0\\ &+\log h^n_n+\log\tilde{\chi}+\tilde{u}-\log 2.\end{aligned} \tag{4.38}$$

At (t_0,ξ_0) we have

$$(\log h^n_n)_k = -(\log\tilde{\chi})_k-u_k \tag{4.39}$$

and thus

$$\begin{aligned}0\le\ &\Phi\left(h^n_n-(h^n_n)^{-1}-v^{-1}+\frac{\bar{H}}{n}v\right)t_0\\ &+\dot{\Phi}F\left(\frac{\bar{H}}{n}v+v^{-1}-(h^n_n+(h^n_n)^{-1})\right)t_0\\ &+\log h^n_n+\log\tilde{\chi}+\tilde{u}-\log 2+2\dot{\Phi}F^{kl}u_ku_lt_0+2\dot{\Phi}F^{kl}(\log\tilde{\chi})_ku_lt_0\end{aligned} \tag{4.40}$$

We have

$$(\log\tilde{\chi})_k=\frac{\chi_k}{\chi}=\frac{\sinh u}{v}\frac{v_k\sinh u-vu_k\cosh u}{\sinh^2 u}\to 0, \tag{4.41}$$

since

$$v_k=-v^2h^i_ku_i+v\frac{\bar{H}}{n}u_k, \tag{4.42}$$

the principal curvatures are bounded by Proposition 3.12 and $|Du|^2\to 0$ by Theorem 4.1. In view of

$$x+x^{-1}\ge 2 \quad \forall x>0 \tag{4.43}$$

and by Theorem 4.1 we have in (t_0, ξ_0) :

$$0 \le \Phi(h^n_n - (h^n_n)^{-1})t_0 + c \tag{4.44}$$

for some $c = c(n, p, M_0)$, which implies

$$h^n_n - (h^n_n)^{-1} \le \frac{cF^p}{t_0}. \tag{4.45}$$

Thus we find

$$\begin{aligned} w &\le t_0 \log\left(1 + \frac{cF^p}{t_0}\right) + t_0(\log\tilde{\chi} + \tilde{u} - \log 2) \\ &\le c = c(n, p, M_0). \end{aligned} \tag{4.46}$$

Hence w is a priori bounded and thus

$$\limsup_{t\to\infty} \sup_M \kappa_i(t, \cdot) \le 1 \quad \forall 1 \le i \le n. \tag{4.47}$$

(ii) Now we investigate the function

$$z = \log(-\Phi) + \log\tilde{\chi} + \tilde{u} - \log 2 - \log\frac{1}{n^p} \tag{4.48}$$

and show that

$$\limsup_{t\to\infty} \sup_M z(t, \cdot) \le 0. \tag{4.49}$$

The Lipschitz function

$$\tilde{z} = \sup_{\xi\in M} z(\cdot, \xi) \tag{4.50}$$

satisfies for almost every $t \ge 0$

$$\begin{aligned} \dot{\tilde{z}} &\le \dot{\Phi}F^{kl}((\log(-\Phi))_k(\log(-\Phi))_l - (\log\tilde{\chi})_k(\log\tilde{\chi})_l) \\ &\quad + \Phi\left(\frac{\bar{H}}{n}v - v^{-1}\right) + \dot{\Phi}\left(F\frac{\bar{H}}{n}v - F^{kl}\bar{h}_{kl}\right) \\ &\quad + \dot{\Phi}(Fv^{-1} - F^{kl}g_{kl}) \\ &\le o(1) + \dot{\Phi}\left(F\frac{\bar{H}}{n}v + Fv^{-1} - 2F^{kl}g_{kl}\right). \end{aligned} \tag{4.51}$$

Claim: $\forall\epsilon > 0\ \exists T > 0\ \exists\delta > 0$:

$$A_\epsilon = \{t \in [T, \infty) \cap D : \tilde{z}(t) > \epsilon\} \subset \{t \in [T, \infty) \cap D : \dot{\tilde{z}}(t) \le -\delta\},$$

where D is the set of points of differentiability of $\tilde{z}$.

To prove this claim, let $\epsilon > 0$ and choose $T > 0$ such that

$$\log\tilde{\chi} + \tilde{u} - \log 2 < \frac{\epsilon}{2} \quad \forall(t, \xi) \in [T, \infty) \times M. \tag{4.52}$$

Then for $t \in A_\epsilon$ we have

$$\left(\log(-\Phi) - \log\frac{1}{n^p}\right)(t, \xi_t) > \frac{\epsilon}{2}, \tag{4.53}$$

where $\tilde{z}(t) = z(t, \xi_t)$. Thus there exists $0 < \gamma = \gamma(\epsilon)$, such that

$$F(t, \xi_t) < n - \gamma, \tag{4.54}$$

implying

$$F\frac{\bar{H}}{n}v + Fv^{-1} - 2F^{kl}g_{kl} \leq \bar{H}v - n - \frac{\bar{H}}{n}v\gamma. \tag{4.55}$$

One may enlarge T, such that

$$|o(1) + \dot{\Phi}(\bar{H}v - n)| \leq \frac{(\inf \dot{\Phi})\gamma}{2} \quad \forall (t,\xi) \in [T,\infty) \times M. \tag{4.56}$$

Thus

$$\dot{\tilde{z}}(t) \leq -(\inf \dot{\Phi})\frac{\gamma}{2} =: -\delta. \tag{4.57}$$

Now it follows from Lemma 4.2, that

$$\limsup_{t\to\infty} \tilde{z}(t) \leq 0. \tag{4.58}$$

Thus

$$\begin{aligned} \limsup_{t\to\infty} \sup_M \log(-\Phi) - \log\frac{1}{n^p} &= \limsup_{t\to\infty} \sup_M (z - \log\tilde{\chi} - \tilde{u} + \log 2) \\ &\leq \limsup_{t\to\infty} \tilde{z} + \limsup_{t\to\infty} \sup_M (-\log\tilde{\chi} - \tilde{u} + \log 2) \\ &\leq 0, \end{aligned} \tag{4.59}$$

implying

$$\begin{aligned} \frac{1}{n^p} &\geq \limsup_{t\to\infty} \sup_M \frac{1}{F^p} = \limsup_{t\to\infty} (\inf_M F^p)^{-1} \\ &= (\liminf_{t\to\infty} \inf_M F^p)^{-1}. \end{aligned} \tag{4.60}$$

This leads to

$$\liminf_{t\to\infty} \inf_M F^p \geq n^p \tag{4.61}$$

Together with (i) we obtain

$$\sup_M |F - n| \to 0. \tag{4.62}$$

Now suppose there was a sequence (t_k, ξ_k) such that for the smallest eigenvalue we had

$$\kappa_1(t_k, \xi_k) \to \delta < 1. \tag{4.63}$$

Then

$$\begin{aligned} \limsup_{k\to\infty} F(\kappa_1, \ldots, \kappa_n) - n &= \limsup_{k\to\infty} \sum_{i=1}^n \frac{\partial F}{\partial \kappa_i}(\tilde{\kappa}_k)(\kappa^i - 1) \\ &\leq \limsup_{k\to\infty} F^1(\delta - 1) < 0, \end{aligned} \tag{4.64}$$

which is a contradiction. □

Optimal rates of convergence. We now derive the optimal speed of convergence of the second fundamental form to δ^i_j, which, of course, can not be better than what we expect from the spherical flow, i.e.

$$\begin{aligned}|\bar{h}_{ij}-\bar{g}_{ij}| &= |\coth\Theta - 1||\delta^i_j| \\ &\le c\left|\frac{\cosh\Theta-\sinh\Theta}{\sinh\Theta}\right| \\ &\le c\frac{e^{-\Theta}}{e^{\Theta}-e^{-\Theta}} \\ &\le ce^{-2\Theta}\le ce^{-\frac{2}{np}t}.\end{aligned} \tag{4.65}$$

4.4. **Theorem.** The principal curvatures of the flow hypersurfaces of (1.4) converge to 1 exponentially fast. There exists $c = c(n, p, M_0)$, such that

$$|h^i_j-\delta^i_j|\le ce^{-\frac{2}{np}t}\quad \forall (t,\xi)\in[0,\infty)\times M. \tag{4.66}$$

Proof. Also compare [1, Thm. 5.1], where the author uses the same function G.

(i) Define

$$G=\frac{1}{2}|h^i_j-\delta^i_j|^2e^{\lambda t}=\frac{1}{2}(h^i_j-\delta^i_j)(h^j_i-\delta^j_i)e^{\lambda t},\lambda>0. \tag{4.67}$$

Then

$$\begin{aligned}\dot G-\dot\Phi F^{kl}G_{kl} &= \Big((\dot h^i_j-\dot\Phi F^{kl}h^i_{j;kl})(h^j_i-\delta^j_i)-\dot\Phi F^{kl}h^i_{j;k}h^j_{i;l}\Big)e^{\lambda t}+\lambda G \\ &= \Big(\dot\Phi F^{kl}h_{kr}h^r_l h^i_j(h^j_i-\delta^j_i)+(\Phi-\dot\Phi F)h^{ki}h_{kj}(h^j_i-\delta^j_i) \\ &\quad+\ddot\Phi F_jF^i(h^j_i-\delta^j_i)+\dot\Phi F^{kl,rs}h_{kl;j}h_{rs;}{}^i(h^j_i-\delta^j_i) \\ &\quad-(\Phi+\dot\Phi F)\delta^i_j(h^j_i-\delta^j_i)+\dot\Phi F^{kl}g_{kl}h^i_j(h^j_i-\delta^j_i) \\ &\quad-\dot\Phi F^{kl}h^i_{j;k}h^j_{i;l}\Big)e^{\lambda t}+\lambda G.\end{aligned} \tag{4.68}$$

Fix $0 < T < \infty$ and suppose

$$\sup_{[0,T]\times M} G = G(t_0,\xi_0)>0,\ t_0>0. \tag{4.69}$$

Since $|h^i_j-\delta^i_j|\to 0$ using Lemma 4.3, we may suppose that t_0 is so large, that bad terms involving derivatives of the second fundamental form can be absorbed by the term $-\dot\Phi F^{kl}h^i_{j;k}h^j_{i;l}$. There holds

$$h^i_kh^k_j=(h^i_k-\delta^i_k)(h^k_j-\delta^k_j)+2(h^i_j-\delta^i_j)+\delta^i_j. \tag{4.70}$$

Thus there exists $T_0 = T_0(n, p, M_0)$, such that for $t_0 > T_0$ we have

$$\begin{aligned} 0 \le &\Big(\dot\Phi F^{kl} h_{kr} h^r_l h^i_j (h^j_i - \delta^j_i) + (\Phi - \dot\Phi F)(h^i_k - \delta^i_k)(h^k_j - \delta^k_j)(h^j_i - \delta^j_i) \\ &+ 2(\Phi - \dot\Phi F)(h^i_j - \delta^i_j)(h^j_i - \delta^j_i) + (\Phi - \dot\Phi F)\delta^i_j (h^j_i - \delta^j_i) \\ &- (\Phi + \dot\Phi F)\delta^i_j (h^j_i - \delta^j_i) + \dot\Phi F^{kl} g_{kl} h^i_j (h^j_i - \delta^j_i)\Big) e^{\lambda t_0} + \lambda G \\ = &\Big(\dot\Phi F^{kl}(h_{kr}h^r_l - 2h_{kl} + g_{kl}) h^i_j (h^j_i - \delta^j_i) \\ &+ (\Phi - \dot\Phi F)(h^i_k - \delta^i_k)(h^k_j - \delta^k_j)(h^j_i - \delta^j_i) \\ &+ 2\Phi (h^i_j - \delta^i_j)(h^j_i - \delta^j_i)\Big) e^{\lambda t_0} + \lambda G. \end{aligned} \tag{4.71}$$

In (t_0, ξ_0) choose coordinates, such that

$$g_{ij} = \delta_{ij},\ h_{ij} = \kappa_i \delta_{ij},\ \kappa_1 \le \ldots \le \kappa_n. \tag{4.72}$$

Then

$$\begin{aligned} 0 \le &\Big(\dot\Phi F^{ii}(\kappa_i - 1)^2 \sum_{j=1}^n \kappa_j(\kappa_j - 1) \\ &+ (\Phi - \dot\Phi F) \sum_{i=1}^n (\kappa_i - 1)^3\Big) e^{\lambda t_0} + (4\Phi + \lambda) G \\ \le &\Big(-4F^{-p} + \lambda + 2\dot\Phi \sum_{j=1}^n |\kappa_j||\kappa_j - 1| \sum_{m=1}^n F^{mm} \\ &+ 2|\Phi - \dot\Phi F| \max_{1 \le j \le n} |\kappa_j - 1|\Big) G. \end{aligned} \tag{4.73}$$

Enlarging T_0, we obtain a contradiction, if $\lambda > 0$ is small.

(ii) By Proposition 3.12 we know that $\Phi = \Phi(F(\kappa_i))$ is uniformly Lipschitz continuous with respect to (κ_i) during the flow. Thus

$$|-4F^{-p} + \frac{4}{n^p}| \le c \max_i |\kappa_i - 1| \le c e^{-\frac{\lambda}{2} t}. \tag{4.74}$$

Now define

$$\tilde G = \sup_M \frac{1}{2} |h^i_j - \delta^i_j|^2 e^{\frac{4}{n^p} t}. \tag{4.75}$$

Then for $t \ge T_0$, cf. (i), we obtain from (4.73)

$$\begin{aligned} \dot{\tilde G} \le &\Big(-4F^{-p} + \frac{4}{n^p} + 2\dot\Phi \sum_{j=1}^n |\kappa_j||\kappa_j - 1| \sum_{m=1}^n F^{mm} \\ &+ 2|\Phi - \dot\Phi F| \max_{1 \le j \le n} |\kappa_j - 1|\Big) \tilde G \\ \le &\, c e^{-\frac{\lambda}{2} t} \tilde G,\ c = c(n, p, M_0),\ \lambda = \lambda(n, p, M_0). \end{aligned} \tag{4.76}$$

Thus

$$\tilde G \le c(n, p, M_0), \tag{4.77}$$

which implies the claim. ☐

4.5. **Theorem.** In both cases of Theorem 4.1 the conclusions hold with $\lambda = \frac{2}{n^p}$.

Proof. We come back to the proof of Lemma 3.10 and define

$$\tilde{w} = \sup_{x \in \mathbb{S}^n} \frac{1}{2} |D\varphi(\cdot, x)|^2 = w(t, x_t). \tag{4.78}$$

Using the same calculation as in (3.49), we obtain

$$\begin{aligned} \dot{\tilde{w}} &\le \big(2(p-1)F^{-p} v \vartheta^{p-1} \dot{\vartheta} - 2pF^{-(p+1)} \vartheta^{p+1} F^{kl} \tilde{g}_{kl}\big) \tilde{w} \\ &\le \big(2p(F^{-p} v \vartheta^{p-1} \dot{\vartheta} - F^{-(p+1)} \vartheta^{p+1} F^{kl} \tilde{g}_{kl}) - 2F^{-p} v \vartheta^p\big) \tilde{w}, \end{aligned} \tag{4.79}$$

where $F = F(\tilde{h}^i_j) = F(\vartheta h^i_j)$.

We have

$$|v\dot{\vartheta} - \vartheta| \le |v\dot{\vartheta} - \dot{\vartheta}| + |\dot{\vartheta} - \vartheta| \le c\dot{\vartheta} e^{-\lambda t} + e^{-\frac{t}{n^p}}, \tag{4.80}$$

and thus

$$\dot{\tilde{w}} \le (2pF^{-p}(v\dot{\vartheta} - \vartheta)\vartheta^{-1} + 2p(F^{-p} - F^{-(p+1)} n) - 2F^{-p})\tilde{w}, \tag{4.81}$$

where now $F = F(h^i_j)$. Since

$$|F - n| \le ce^{-\frac{\lambda}{2} t}, \tag{4.82}$$

we obtain

$$\dot{\tilde{w}} \le \left(ce^{-\frac{\lambda}{2} t} - \frac{2}{n^p} \right) \tilde{w}, \tag{4.83}$$

which implies

$$\tilde{w} \le ce^{-\frac{2}{n^p} t}. \tag{4.84}$$

☐

4.6. **Theorem.** For the function φ in Remark 3.8 there exists $c = c(n, p, M_0)$, such that

$$|D^2 \varphi| \le ce^{-\frac{t}{n^p}}, \tag{4.85}$$

where the derivatives as well as the norm are taken with respect to σ_{ij}.

Proof. (3.32) implies

$$\varphi^i_j = \sigma^{ik} \varphi_{kj} = v^{-2} \varphi^i \varphi^k \varphi_{kj} + \dot{\vartheta} \delta^i_j - v\vartheta h^i_j. \tag{4.86}$$

In view of Theorem 4.4 and Theorem 4.5 we deduce

$$|\dot{\vartheta} - \vartheta| = e^{-u} \le ce^{-\frac{t}{n^p}} \tag{4.87}$$

and, using (3.23), we obtain

$$\begin{aligned}|D^2\varphi| &\le |v^{-2}\varphi^i\varphi^k\varphi_{kj}| + |\dot\vartheta\delta^i_j - v\vartheta h^i_j|\\ &\le c|D\varphi|^2|D^2\varphi| + |\dot\vartheta\delta^i_j - \vartheta\delta^i_j| + |\vartheta\delta^i_j - v\vartheta\delta^i_j| + |v\vartheta\delta^i_j - v\vartheta h^i_j|\\ &\le c|D\varphi|^2|D^2\varphi| + ce^{-\frac{t}{n^p}}\\ &\le \tilde c e^{-\frac{2}{n^p}t}|D^2\varphi| + ce^{-\frac{t}{n^p}},\end{aligned} \tag{4.88}$$

where $c, \tilde c$ depend on n, p and M_0. Choosing $T = T(n, p, M_0)$ such that

$$\tilde c e^{-\frac{2}{n^p}t} < \frac{1}{2} \quad \forall t \ge T. \tag{4.89}$$

we obtain the claim. □

5. Decay estimates of higher order

We first need a definition to simplify the notation, compare [5, Def. 6.6], and the remark afterwards.

5.1. **Definition.** (1) For tensors S and T, the symbol $S \star T$ denotes an arbitrary linear combination of contractions of $S \otimes T$. We do not distinguish between $S \star T$ and $cS \star T$, $c = c(n, p, M_0)$.

(2) For $\epsilon \in \mathbb{R}$, the symbol $\mathcal{O}^\epsilon$ denotes an arbitrary tensor, which can be estimated like

$$|\mathcal{O}^\epsilon| \le c_\epsilon e^{\frac{\epsilon t}{n^p}}, c_\epsilon = c(n, p, M_0, \epsilon), \tag{5.1}$$

where the norm is taken with respect to the spherical metric.

(3) For a tensor T, the symbol D^kT denotes an arbitrary covariant derivative of order k with respect to the spherical metric.

(4) If a derivative of order m is expressed as an algebraic combination of terms involving $\mathcal{O}^\epsilon$, then the corresponding constants may additionally depend on m.

Until now we have shown, that the function

$$\varphi = \int_{r_0}^{u} \vartheta^{-1} \tag{5.2}$$

satisfies the scalar parabolic equation

$$\frac{\partial\varphi}{\partial t} \equiv \dot\varphi = -\vartheta^{p-1} v\Phi \text{ on } [0, \infty) \times \mathbb{S}^n, \tag{5.3}$$

where $F = F(\tilde h^i_j) = F(\vartheta h^i_j)$. Furthermore, we have proven the estimates

$$D\varphi = \mathcal{O}^{-1} \tag{5.4}$$

and

$$D^2\varphi = \mathcal{O}^{-1}. \tag{5.5}$$

On the following pages we prove analogous estimates for higher derivatives of φ by differentiating (3.33). We prepare the final result by examining all of the terms separately first. In the sequel, we suppose $m \geq 3$.

5.2. **Lemma.** For functions $g, f^i \colon M \to \mathbb{R}$ on a manifold the following generalizations of the product- and chain rule hold for higher derivatives.

$$D^m\left(\prod_{i=1}^k f^i\right) = \sum_{j_1+\ldots+j_k=m} c_{m,k} \prod_{i=1}^k D^{j_i} f^i, \tag{5.6}$$

$$D^m(f \circ g) = \sum_{k_1+\ldots+mk_m=m} \frac{m!}{k_1! \ldots k_m!} D^{\sum_{i=1}^m k_i} f(g) \prod_{i=1}^m \left(\frac{D^i g}{i!}\right)^{k_i}. \tag{5.7}$$

Proof. For $m = 1$ this is the ordinary product rule. If the claim holds for $m \geq 1$, we find

$$\begin{aligned} D^{m+1}\left(\prod_{i=1}^k f^i\right) &= D^m\left(\sum_{j_1+\ldots+j_k=1} \prod_{i=1}^k D^{j_i} f^i\right) \\ &= \sum_{j_1+\ldots+j_k=1} \sum_{l_1+\ldots+l_k=m} c_{m,k} \prod_{i=1}^k D^{l_i+j_i} f^i \\ &= \sum_{(j_1+l_1)+\ldots(j_k+l_k)=m+1} \tilde{c}_{m,k} \prod_{i=1}^k D^{j_i+l_i} f^i, \end{aligned} \tag{5.8}$$

as desired.
The generalized chain rule is known as formula of Faá di Bruno, cf. [8, p. 17, Thm. 1.3.2]. □

Let us remark, that the cited version of the generalized chain rule is the one, which holds for functions depending on one variable. Although our functions depend on n variables, all that matters is the order of the multiindex in most of the cases, so that such a formal version is all we need.

5.3. **Lemma.** Let φ be the solution of (3.33) and suppose, that there exists $0 < \gamma \leq 1$, such that

$$D^k \varphi = \mathcal{O}^{-\gamma} \quad \forall 1 \leq k \leq m-1, \tag{5.9}$$

then

$$D^k v = \mathcal{O}^{-2\gamma} \quad \forall 1 \leq k \leq m-2, \tag{5.10}$$

$$v_{j_1 \ldots j_{m-1}} = \mathcal{O}^{-2\gamma} + v^{-1} \varphi_{k j_1 \ldots j_{m-1}} \varphi^k, \tag{5.11}$$

and

$$v_{i_1 \ldots i_m} = \mathcal{O}^{-2\gamma} + \mathcal{O}^{-\gamma} \star D^m \varphi + v^{-1} \varphi_{k i_1 \ldots i_m} \varphi^k. \tag{5.12}$$

Proof. For $k=1$ we have

$$v_{i_1} = v^{-1}\varphi_{ai_1}\varphi^a = \mathcal{O}^{-2\gamma}. \tag{5.13}$$

Suppose the first claim to hold for $1 \leq j \leq l \leq m-3$. Then

$$D^{l+1}v = \sum_{s+r=l} D^s(v^{-1}) \star D^r(\varphi_{ai_1}\varphi^a) = \mathcal{O}^{-2\gamma}, \tag{5.14}$$

since

$$D^s(v^{-1}) = \mathcal{O}^{-2\gamma} \ \forall 1 \leq s \leq l \tag{5.15}$$

and

$$D^r(\varphi_{ai_1}\varphi^a) = \mathcal{O}^{-2\gamma} \ \forall 0 \leq r \leq l, \tag{5.16}$$

by Lemma 5.2. To prove (5.11) we infer from (5.13)

$$\begin{aligned} v_{j_1\dots j_{m-1}} &= (\varphi_{aj_1}\varphi^a v^{-1})_{;j_2\dots j_{m-1}} \\ &= \varphi_{aj_1\dots j_{m-1}}\varphi^a v^{-1} + \mathcal{O}^{-2\gamma}, \end{aligned} \tag{5.17}$$

where we used (5.10) to estimate $D^{m-2}(v^{-1})$. Finally, we deduce

$$\begin{aligned} v_{i_1\dots i_m} &= (\varphi_{ai_1}\varphi^a v^{-1})_{;i_2\dots i_m} \\ &= \varphi_{ai_1\dots i_m}\varphi^a v^{-1} + D^m\varphi \star D(\varphi^a v^{-1}) + D^m\varphi \star \mathcal{O}^{-\gamma} + \mathcal{O}^{-2\gamma} \\ &= \varphi_{ai_1\dots i_m}\varphi^a v^{-1} + D^m\varphi \star \mathcal{O}^{-\gamma} + \mathcal{O}^{-2\gamma}, \end{aligned} \tag{5.18}$$

where we used (5.10) and (5.11). □

5.4. **Lemma.** Let φ be the solution of (3.33) and suppose, that there exists $0 < \gamma \leq 1$, such that

$$D^k\varphi = \mathcal{O}^{-\gamma} \quad \forall 1 \leq k \leq m-1, \tag{5.19}$$

then

$$D^k u = \mathcal{O}^{k(1-\gamma)} \quad \forall 1 \leq k \leq m-1. \tag{5.20}$$

Proof. Note, that

$$D\varphi = \vartheta^{-1}Du \Rightarrow Du = \vartheta D\varphi, \tag{5.21}$$

where $\vartheta = \vartheta(u)$. Thus the claim holds for $k=1$ in view of Corollary 3.6. Suppose the claim to hold for $1 \leq l \leq m-2$. Then

$$D^{l+1}u = D^l(\vartheta D\varphi) = \sum_{r+s=l} D^s(\vartheta) \star D^r(D\varphi) = \mathcal{O}^{(l+1)(1-\gamma)}, \tag{5.22}$$

since $D^r(D\varphi) = \mathcal{O}^{-\gamma}$ $\forall 0 \leq r \leq l$ and

$$D^s\vartheta = \sum_{k_1+\ldots+sk_s=s} c_s \vartheta^{(\sum_{i=1}^s k_i)}(u) \prod_{i=1}^s \left(\frac{D^i u}{i!}\right)^{k_i} = \mathcal{O}^{s(1-\gamma)+1}, \tag{5.23}$$

since $D^i u = \mathcal{O}^{i(1-\gamma)}$ $\forall 1 \leq i \leq s$ and

$$\vartheta^{(a)} = \begin{cases} \vartheta, & a \text{ even} \\ \dot{\vartheta}, & a \text{ odd} \end{cases} = \mathcal{O}^1. \tag{5.24}$$

□

5.5. **Lemma.** Let φ be the solution of (3.33) and suppose, that there exists $0 < \gamma \leq 1$, such that

$$D^k \varphi = \mathcal{O}^{-\gamma} \quad \forall 1 \leq k \leq m-1, \tag{5.25}$$

then

$$(\vartheta^{p-1})_{i_1 \dots i_m} = (p-1)\vartheta^{p-1}\dot{\vartheta}\varphi_{i_1 \dots i_m} + \mathcal{O}^{p-1+m(1-\gamma)} \tag{5.26}$$

and

$$D^k(\vartheta^{p-1}) = \mathcal{O}^{p-1+k(1-\gamma)} \quad \forall 0 \leq k \leq m-1. \tag{5.27}$$

Proof. For the real function $f(x) = x^{p-1}$ and $g = f \circ \vartheta$ there holds

$$g^{(s)} = \sum_{k_1 + \dots + s k_s = s} c_s f^{(\sum_{i=1}^s k_i)}(\vartheta) \prod_{i=1}^{s} \left(\frac{\vartheta^{(i)}}{i!} \right)^{k_i} = \mathcal{O}^{p-1}, \tag{5.28}$$

since

$$f^{(a)}(\vartheta) = \prod_{i=1}^{a} (p-i) \vartheta^{p-1-a} = \mathcal{O}^{p-1-a}, \tag{5.29}$$

$a = \sum_{i=1}^s k_i$, and

$$\prod_{i=1}^{s} \left(\frac{\vartheta^{(i)}}{i!} \right)^{k_i} = \mathcal{O}^{a}. \tag{5.30}$$

Thus, (5.27) follows from the the di Bruno formula again, (5.7), applied to $g \circ u$, and by Lemma 5.4. Note that (5.27) also holds for $\dot{\vartheta}$ instead of ϑ, because they share the same growth behavior and there holds $\ddot{\vartheta} = \vartheta$.

In order to prove (5.26), observe that

$$(\vartheta^{p-1})_{i_1} = (p-1)\dot{\vartheta}\vartheta^{p-1}\varphi_{i_1} \tag{5.31}$$

and

$$\begin{aligned} (\vartheta^{p-1})_{i_1 \dots i_m} &= (p-1)\vartheta^{p-1}\dot{\vartheta}\varphi_{i_1 \dots i_m} \\ &\quad + \sum_{\substack{s+r=m-1 \\ s \geq 1}} D^s((p-1)\dot{\vartheta}\vartheta^{p-1}) \star D^r(\varphi_{i_1}) \\ &= (p-1)\vartheta^{p-1}\dot{\vartheta}\varphi_{i_1 \dots i_m} + \mathcal{O}^{p-1+m(1-\gamma)}, \end{aligned} \tag{5.32}$$

where we used $D^k \varphi = \mathcal{O}^{-\gamma}$ and (5.26) applied to ϑ and $\dot{\vartheta}$ as well, also using

$$D^s(\dot{\vartheta}\vartheta) = \sum_{s_1 + s_2 = s} D^{s_1}\dot{\vartheta} \star D^{s_2}\vartheta = \mathcal{O}^{1+s_1(1-\gamma)} \star \mathcal{O}^{p-1+s_2(1-\gamma)}. \tag{5.33}$$

□

5.6. **Lemma.** Let φ be the solution of (3.33) and suppose, that there exists $0 < \gamma \leq 1$, such that

$$D^k \varphi = \mathcal{O}^{-\gamma} \quad \forall 1 \leq k \leq m-1 \tag{5.34}$$

and set

$$\tilde{g}_{ij} = \varphi_i\varphi_j + \sigma_{ij}, \tag{5.35}$$

then

$$\tilde{g} = \mathcal{O}^0, \tag{5.36}$$

$$D^k\tilde{g} = \mathcal{O}^{-2\gamma} \quad \forall 1 \leq k \leq m-2, \tag{5.37}$$

$$D^{m-1}\tilde{g} = \mathcal{O}^{-2\gamma} + D^m\varphi \star \mathcal{O}^{-\gamma}, \tag{5.38}$$

$$D^m\tilde{g} = \mathcal{O}^{-2\gamma} + D^m\varphi \star \mathcal{O}^{-\gamma} + D^{m+1}\varphi \star \mathcal{O}^{-\gamma}, \tag{5.39}$$

$$D^k(\tilde{h}^i_j) = \mathcal{O}^{1+k(1-\gamma)} \quad \forall 0 \leq k \leq m-3, \tag{5.40}$$

$$D^{m-2}(\tilde{h}^i_j) = \mathcal{O}^{1+(m-2)(1-\gamma)} + D^m\varphi \star \mathcal{O}^0, \tag{5.41}$$

$$D^{m-1}(\tilde{h}^i_j) = \mathcal{O}^{1+(m-1)(1-\gamma)} + D^m\varphi \star \mathcal{O}^{1-\gamma} + D^{m+1}\varphi \star \mathcal{O}^0, \tag{5.42}$$

and

$$\begin{aligned} \tilde{h}^l_{a;i_1\dots i_m} = &-v^{-1}\tilde{g}^{lr}\varphi_{ra;i_1\dots i_m} + v^{-1}\vartheta^2\varphi_{i_1\dots i_m}\delta^l_a \\ &+ \mathcal{O}^{1+m(1-\gamma)} + D^m\varphi \star \mathcal{O}^{1-\gamma} + D^{m+1}\varphi \star \mathcal{O}^{1-\gamma} \\ &+ D^m\varphi \star D^m\varphi \star \mathcal{O}^{-\gamma}. \end{aligned} \tag{5.43}$$

Proof. We have

$$\tilde{g}^{lr} = \sigma^{lr} - v^{-2}\varphi^l\varphi^r = \mathcal{O}^0. \tag{5.44}$$

For all $1 \leq k \leq m$ we deduce

$$\begin{aligned} D^k(\tilde{g}^{lr}) = -\Big(&D^k(v^{-2})\varphi^l\varphi^r + v^{-2}D^k(\varphi^l\varphi^r) \\ &+ \sum_{\substack{s+t=k \\ s,t\geq 1}} D^s(v^{-2}) \star D^t(\varphi^l\varphi^r)\Big), \end{aligned} \tag{5.45}$$

from which (5.37)-(5.39) follow by Lemma 5.3. There holds

$$\tilde{h}^l_a = -v^{-1}(\tilde{g}^{lr}\varphi_{ra} - \dot{\vartheta}\delta^l_a). \tag{5.46}$$

For all $1 \leq k \leq m$ we have

$$\begin{aligned} D^k(\tilde{h}^l_a) = &-(D^k(v^{-1})\tilde{g}^{lr}\varphi_{ra} + v^{-1}D^k(\tilde{g}^{lr})\varphi_{ra} + v^{-1}\tilde{g}^{lr}D^k(\varphi_{ra})) \\ &+ D^k(v^{-1})\dot{\vartheta}\delta^l_a + v^{-1}D^k(\dot{\vartheta})\delta^l_a \\ &+ \sum_{\substack{j_1+j_2+j_3=k \\ \exists i_1\neq i_2:\, j_{i_1},j_{i_2}\geq 1}} D^{j_1}(v^{-1}) \star D^{j_2}(\tilde{g}^{lr}) \star D^{j_3}(\varphi_{ra}) \\ &+ \sum_{\substack{j_1+j_2=k \\ j_1,j_2\geq 1}} D^{j_1}(v^{-1}) \star D^{j_2}(\dot{\vartheta})\delta^l_a. \end{aligned} \tag{5.47}$$

In order to prove (5.40)-(5.43), we examine (5.47) and use Lemma 5.3, Lemma 5.5 and (5.37)-(5.39). First let $1 \le k \le m-3$. Then

$$D^k\tilde{h}^l_a = \mathcal{O}^{-3\gamma} + \mathcal{O}^{-\gamma} + \mathcal{O}^{1-2\gamma} + \mathcal{O}^{1+k(1-\gamma)} = \mathcal{O}^{1+k(1-\gamma)}, \tag{5.48}$$

$$\begin{aligned} D^{m-2}\tilde{h}^l_a &= \mathcal{O}^{-3\gamma} + D^m\varphi \star \mathcal{O}^0 + \mathcal{O}^{1-2\gamma} + \mathcal{O}^{1+(m-2)(1-\gamma)} \\ &= \mathcal{O}^{1+(m-2)(1-\gamma)} + \mathcal{O}^0 \star D^m\varphi, \end{aligned} \tag{5.49}$$

$$\begin{aligned} D^{m-1}\tilde{h}^l_a &= \mathcal{O}^{-3\gamma} + D^m\varphi \star \mathcal{O}^{-2\gamma} + D^{m+1}\varphi \star \mathcal{O}^0 + \mathcal{O}^{1-2\gamma} \\ &\quad + D^m\varphi \star \mathcal{O}^{1-\gamma} + \mathcal{O}^{1+(m-1)(1-\gamma)} \\ &= D^m\varphi \star \mathcal{O}^{1-\gamma} + D^{m+1}\phi \star \mathcal{O}^0 + \mathcal{O}^{1+(m-1)(1-\gamma)} \end{aligned} \tag{5.50}$$

and finally

$$\begin{aligned} \tilde{h}^l_{a;i_1\dots i_m} &= \mathcal{O}^{-3\gamma} + D^m\varphi \star \mathcal{O}^{-2\gamma} + D^{m+1}\varphi \star \mathcal{O}^{-2\gamma} \\ &\quad - v^{-1}\tilde{g}^{lr}\varphi_{ra;i_1\dots i_m} + \mathcal{O}^{1-2\gamma} + D^m\varphi \star \mathcal{O}^{1-\gamma} \\ &\quad + D^{m+1}\varphi \star \mathcal{O}^{1-\gamma} + \mathcal{O}^{1+m(1-\gamma)} + v^{-1}\vartheta^2\varphi_{i_1\dots i_m}\delta^l_a \\ &\quad + D^m\varphi \star D^m\varphi \star \mathcal{O}^{-\gamma} \\ &= \mathcal{O}^{1+m(1-\gamma)} + D^m\varphi \star \mathcal{O}^{1-\gamma} + D^{m+1}\varphi \star \mathcal{O}^{1-\gamma} \\ &\quad + D^m\varphi \star D^m\varphi \star \mathcal{O}^{-\gamma} - v^{-1}\tilde{g}^{lr}\varphi_{ra;i_1\dots i_m} \\ &\quad + v^{-1}\vartheta^2\varphi_{i_1\dots i_m}\delta^l_a. \end{aligned} \tag{5.51}$$

□

5.7. **Lemma.** Let φ be the solution of (3.33) and suppose, that there exists $0 < \gamma \le 1$, such that

$$D^k\varphi = \mathcal{O}^{-\gamma} \quad \forall 1 \le k \le m-1, \tag{5.52}$$

then

$$\mathcal{D}^\alpha\Phi = \mathcal{O}^{-(p+\alpha)} \quad \forall 0 \le \alpha \le m, \tag{5.53}$$

where $\mathcal{D}^\alpha$ denotes an arbitrary derivative of order α with respect to the argument $\tilde{h}^l_a$.

Proof. Di Bruno's formula, (5.7), gives

$$\mathcal{D}^\alpha\Phi = \sum_{k_1+\ldots+\alpha k_\alpha=\alpha} c_\alpha \Phi^{(\sum_{i=1}^\alpha k_i)}(F)\prod_{i=1}^\alpha\left(\frac{\mathcal{D}^iF}{i!}\right)^{k_i}, \tag{5.54}$$

where $\Phi^{(r)} = \frac{d^r}{ds^r}\Phi(s)$. In view of

$$F = F(\tilde{h}^l_a) = F(\vartheta h^l_a), \tag{5.55}$$

we have

$$F = \mathcal{O}^1 \tag{5.56}$$

and by homogeneity

$$\mathcal{D}^iF = \mathcal{O}^{1-i}, \tag{5.57}$$

as well as

$$\Phi^{(r)}(F) = \mathcal{O}^{-(p+r)}. \tag{5.58}$$

Thus

$$\prod_{i=1}^{\alpha}\left(\frac{\mathcal{D}^i F}{i!}\right)^{k_i} = \prod_{i=1}^{\alpha}\frac{\mathcal{O}^{k_i - ik_i}}{i!} = \mathcal{O}^{\sum_{i=1}^{\alpha}k_i - \sum_{i=1}^{\alpha} ik_i}, \tag{5.59}$$

which implies

$$\mathcal{D}^\alpha \Phi = \sum_{k_1+\ldots+\alpha k_\alpha = \alpha} \mathcal{O}^{-(p+\sum_{i=1}^{\alpha}k_i)} \star \mathcal{O}^{\sum_{i=1}^{\alpha}k_i - \sum_{i=1}^{\alpha} ik_i} = \mathcal{O}^{-(p+\alpha)}. \tag{5.60}$$

□

5.8. **Lemma.** Let φ be the solution of (3.33) and suppose, that there exists $0 < \gamma \leq 1$, such that

$$D^k\varphi = \mathcal{O}^{-\gamma} \quad \forall 1 \leq k \leq m-1, \tag{5.61}$$

then

$$D^k\Phi = \mathcal{O}^{k(1-\gamma)-p} \quad \forall 0 \leq k \leq m-3, \tag{5.62}$$

$$D^{m-2}\Phi = \mathcal{O}^{(m-2)(1-\gamma)-p} + D^m\varphi \star \mathcal{O}^{-(p+1)}, \tag{5.63}$$

$$\begin{aligned} D^{m-1}\Phi &= \mathcal{O}^{(m-1)(1-\gamma)-p} + D^m\varphi \star \mathcal{O}^{-(p+\gamma)} \\ &\quad + D^m\varphi \star D^m\varphi \star \mathcal{O}^{-(p+2)} + D^{m+1}\varphi \star \mathcal{O}^{-(p+1)} \end{aligned} \tag{5.64}$$

and

$$\begin{aligned} \Phi_{i_1\ldots i_m} &= -\dot\Phi v^{-1}F^a_l \tilde g^{lr}\varphi_{ra;i_1\ldots i_m} + \dot\Phi v^{-1}F^a_l \vartheta^2 \varphi_{i_1\ldots i_m}\delta^l_a \\ &\quad + \mathcal{O}^{m(1-\gamma)-p} + D^m\varphi \star \mathcal{O}^{1-2\gamma-p} \\ &\quad + D^m\varphi \star D^m\varphi \star \mathcal{O}^{-(p+1+\gamma)} + D^{m+1}\varphi \star \mathcal{O}^{-(p+\gamma)} \\ &\quad + D^m\varphi \star D^{m+1}\varphi \star \mathcal{O}^{-(p+2)} \\ &\quad + D^m\varphi \star D^m\varphi \star D^m\varphi \star \mathcal{O}^{-(p+3)}. \end{aligned} \tag{5.65}$$

Proof. We consider $\Phi(\tilde h^i_j) \equiv \Phi(F(\tilde h^i_j))$.

$$D^\beta \Phi = \sum_{k_1+\ldots+\beta k_\beta = \beta} c_\beta \mathcal{D}^{\sum_{i=1}^{\beta}k_i}\Phi(\tilde h^l_a)\prod_{i=1}^{\beta}\left(\frac{D^i \tilde h^l_a}{i!}\right)^{k_i}. \tag{5.66}$$

We consider the different cases separately and use Lemma 5.6 and Lemma 5.7 to obtain, that if $\beta \leq m-3$, then

$$D^\beta\Phi = \mathcal{O}^{-(p+\sum_{i=1}^{\beta}k_i)} \star \mathcal{O}^{\sum_{i=1}^{\beta}k_i + \sum_{i=1}^{\beta} ik_i(1-\gamma)} = \mathcal{O}^{\beta(1-\gamma)-p}. \tag{5.67}$$

If $\beta = m-2$, then

$$(5.68)\quad \begin{aligned} D^\beta\Phi = &\sum_{k_1+\ldots\beta k_\beta=\beta} \mathcal{O}^{-(p+\sum_{i=1}^{\beta} k_i)} \star \prod_{i=1}^{\beta-1}\left(\frac{D^i\tilde{h}^l_a}{i!}\right)^{k_i} \\ &\star (\mathcal{O}^{1+(m-2)(1-\gamma)} + D^m\varphi\star\mathcal{O}^0)^{k_{m-2}} \\ = &\,\mathcal{O}^{\beta(1-\gamma)-p} + D^m\varphi\star\mathcal{O}^{-(p+1)}. \end{aligned}$$

For $\beta = m-1$ we get

$$(5.69)\quad \begin{aligned} D^\beta\Phi = &\sum_{k_1+\ldots+\beta k_\beta=\beta} \mathcal{O}^{-(p+\sum_{i=1}^{\beta} k_i)} \star \prod_{i=1}^{\beta-2}\left(\frac{D^i\tilde{h}^l_a}{i!}\right)^{k_i} \\ &\star (\mathcal{O}^{1+(m-2)(1-\gamma)} + D^m\varphi\star\mathcal{O}^0)^{k_{m-2}} \\ &\star (\mathcal{O}^{1+(m-1)(1-\gamma)} + D^m\varphi\star\mathcal{O}^{1-\gamma} + D^{m+1}\varphi\star\mathcal{O}^0)^{k_{m-1}} \\ = &\,\mathcal{O}^{\beta(1-\gamma)-p} + D^m\varphi\star\mathcal{O}^{-(p+\gamma)} + D^m\varphi\star D^m\varphi\star\mathcal{O}^{-(p+2)} \\ &+ D^{m+1}\varphi\star\mathcal{O}^{-(p+1)}. \end{aligned}$$

In order to prove (5.65), we calculate

$$(5.70)\quad \begin{aligned} \Phi_{i_1\ldots i_m} = &\sum_{k_1+\ldots mk_m=m} \frac{m!}{k_1!\cdots k_m!}\Phi^{(\sum_{i=1}^m k_i)} \prod_{i=1}^{m-3}\left(\frac{D^i\tilde{h}^l_a}{i!}\right)^{k_i} \\ &\star (\mathcal{O}^{1+(m-2)(1-\gamma)} + D^m\varphi\star\mathcal{O}^0)^{k_{m-2}} \\ &\star (\mathcal{O}^{1+(m-1)(1-\gamma)} + D^m\varphi\star\mathcal{O}^{1-\gamma} + D^{m+1}\varphi\star\mathcal{O}^0)^{k_{m-1}} \\ &\star \Big(-\frac{1}{m!}v^{-1}\tilde{g}^{lr}\varphi_{ra;i_1\ldots i_m} + v^{-1}\vartheta^2\varphi_{i_1\ldots i_m}\delta^l_a \\ &+ \mathcal{O}^{1+m(1-\gamma)} + D^m\varphi\star\mathcal{O}^{1-\gamma} + D^{m+1}\varphi\star\mathcal{O}^{1-\gamma} \\ &+ D^m\varphi\star D^m\varphi\star\mathcal{O}^{-\gamma}\Big)^{k_m}. \end{aligned}$$

Thus

$$(5.71)\quad \begin{aligned} \Phi_{i_1\ldots i_m} = &-\dot{\Phi}F^a_l v^{-1}\tilde{g}^{lr}\varphi_{ra;i_1\ldots i_m} + \dot{\Phi}F^a_l v^{-1}\vartheta^2\varphi_{i_1\ldots i_m}\delta^l_a \\ &+ \mathcal{O}^{m(1-\gamma)-p} + D^m\varphi\star\mathcal{O}^{-(p+\gamma)} + D^{m+1}\varphi\star\mathcal{O}^{-(p+\gamma)} \\ &+ D^m\varphi\star D^m\varphi\star\mathcal{O}^{-(p+1+\gamma)} + \mathcal{O}^{m(1-\gamma)-p} \\ &+ D^m\varphi\star\mathcal{O}^{1-2\gamma-p} + D^{m+1}\varphi\star\mathcal{O}^{-(p+\gamma)} \\ &+ D^m\varphi\star D^m\varphi\star\mathcal{O}^{-(p+1+\gamma)} + D^m\varphi\star D^{m+1}\varphi\star\mathcal{O}^{-(p+2)} \\ &+ \mathcal{O}^{m(1-\gamma)-p} + D^m\varphi\star\mathcal{O}^{1-2\gamma-p} + D^m\varphi\star D^m\varphi\star\mathcal{O}^{-(p+2)} \\ &+ \mathcal{O}^{m(1-\gamma)-p} + D^m\varphi\star\mathcal{O}^{1-2\gamma-p} + \mathcal{O}^{m(1-\gamma)-p} \\ &+ D^m\varphi\star\mathcal{O}^{1-2\gamma-p} + D^m\varphi\star D^m\varphi\star\mathcal{O}^{-(p+1+\gamma)} \\ &+ D^m\varphi\star D^m\varphi\star D^m\varphi\star\mathcal{O}^{-(p+3)}, \end{aligned}$$

so that finally

$$\begin{aligned}\Phi_{i_1\dots i_m} &= -\dot{\Phi}F^a_l v^{-1}\tilde{g}^{lr}\varphi_{ra;i_1\dots i_m} + \dot{\Phi}F^a_l v^{-1}\vartheta^2\varphi_{i_1\dots i_m}\delta^l_a \\ &\quad + \mathcal{O}^{m(1-\gamma)-p} + D^m\varphi \star \mathcal{O}^{1-2\gamma-p} \\ &\quad + D^m\varphi\star D^m\varphi\star\mathcal{O}^{-(p+1+\gamma)} + D^{m+1}\varphi\star\mathcal{O}^{-(p+\gamma)} \\ &\quad + D^m\varphi\star D^{m+1}\varphi\star\mathcal{O}^{-(p+2)} \\ &\quad + D^m\varphi\star D^m\varphi\star D^m\varphi\star\mathcal{O}^{-(p+3)}.\end{aligned} \tag{5.72}$$

□

5.9. **Lemma.** For a function

$$\varphi\colon \mathbb{S}^n \to \mathbb{R} \tag{5.73}$$

there holds

$$\varphi_{rk;i_1\dots i_m} = \varphi_{i_1\dots i_m;rk} + D^m\varphi\star\mathcal{O}^0. \tag{5.74}$$

Proof. We shift i_j into the j-th position inductively. For $j=1$ we have

$$\begin{aligned}\varphi_{rk;i_1\dots i_m} &= \varphi_{ri_1;ki_2\dots i_m} + ({R^s}_{rki_1}\varphi_s)_{;i_2\dots i_m} \\ &= \varphi_{ri_1ki_2\dots i_m} + (\delta^s_k\sigma_{ri_1} - \delta^s_{i_1}\sigma_{rk})\varphi_{si_2\dots i_m} \\ &= \varphi_{i_1rki_2\dots i_m} + D^m\varphi\star\mathcal{O}^0.\end{aligned} \tag{5.75}$$

Suppose inductively

$$\varphi_{rk;i_1\dots i_m} = \varphi_{i_1\dots i_jrki_{j+1}\dots i_m} + D^m\varphi\star\mathcal{O}^0, \tag{5.76}$$

then

$$\begin{aligned}\varphi_{rki_1\dots i_m} &= \varphi_{i_1\dots i_jri_{j+1}ki_{j+2}\dots i_m} \\ &\quad + \left(\sum_{l=1}^{j} {R^{s_l}}_{i_lki_{j+1}}\varphi_{i_1\dots i_{l-1}s_li_{l+1}\dots i_jr}\right)_{;i_{j+2}\dots i_m} \\ &\quad + ({R^s}_{rki_{j+1}}\varphi_{i_1\dots i_js})_{;i_{j+2}\dots i_m} + D^m\varphi\star\mathcal{O}^0 \\ &= \varphi_{i_1\dots i_jri_{j+1}ki_{j+2}\dots i_m} + D^m\varphi\star\mathcal{O}^0\end{aligned} \tag{5.77}$$

and analogously for exchanging r and i_{j+1}. □

5.10. **Lemma.** Let φ be the solution of (3.33) and suppose, that there exists $0<\gamma\le 1$, such that

$$D^k\varphi = \mathcal{O}^{-\gamma} \quad \forall 1\le k\le m-1, \tag{5.78}$$

then the functions

$$z = \frac{1}{2}|D^{m-1}\varphi|^2 = \frac{1}{2}\varphi_{i_1\dots i_{m-1}}\varphi^{i_1\dots i_{m-1}} \tag{5.79}$$

and

$$w = \frac{1}{2}|D^m\varphi|^2 = \frac{1}{2}\varphi_{i_1\dots i_m}\varphi^{i_1\dots i_m} \tag{5.80}$$

satisfy

$$\begin{aligned}\dot z - \vartheta^{p-1}\dot\Phi F^{ar} z_{ar} = &-\vartheta^{p-1}\dot\Phi F^{ar}\varphi_{i_1\ldots i_{m-1};r}\varphi^{i_1\ldots i_{m-1}}{}_{;a}\\ &+\mathcal{O}^{-(1+\gamma)+(m-1)(1-\gamma)}\\ &+D^m\varphi\star\mathcal{O}^{-(1+2\gamma)}+D^m\varphi\star D^m\varphi\star\mathcal{O}^{-(3+\gamma)}\end{aligned} \tag{5.81}$$

and

$$\begin{aligned}\dot w - \vartheta^{p-1}\dot\Phi F^{ar} w_{ar} = &-2(p-1)\vartheta^{p-1}\dot\vartheta v\Phi w - 2\vartheta^{p+1}\dot\Phi F^a_a w\\ &-\vartheta^{p-1}\dot\Phi F^{ar}\varphi_{i_1\ldots i_m;r}\varphi^{i_1\ldots i_m}{}_{;a}\\ &+D^m\varphi\star\mathcal{O}^{-1+m(1-\gamma)}+D^m\varphi\star D^m\varphi\star\mathcal{O}^{-2\gamma}\\ &+D^m\varphi\star D^m\varphi\star D^m\varphi\star\mathcal{O}^{-(2+\gamma)}\\ &+D^m\varphi\star D^m\varphi\star D^m\varphi\star D^m\varphi\star\mathcal{O}^{-4}\\ &+D^m\varphi\star D^{m+1}\varphi\star\mathcal{O}^{-(1+\gamma)}\\ &+D^m\varphi\star D^m\varphi\star D^{m+1}\varphi\star\mathcal{O}^{-3}.\end{aligned} \tag{5.82}$$

Proof. φ satisfies

$$\dot\varphi = -\vartheta^{p-1}v\Phi \text{ on } [0,\infty)\times\mathbb{S}^n. \tag{5.83}$$

Differentiating covariantly with respect to σ_{ij} gives

$$\begin{aligned}\dot\varphi_{i_1\ldots i_k} = &-(\vartheta^{p-1})_{i_1\ldots i_k}v\Phi - \vartheta^{p-1}v_{i_1\ldots i_k}\Phi - \vartheta^{p-1}v\Phi_{i_1\ldots i_k}\\ &+\sum_{\substack{j_1+j_2+j_3=k\\ \exists s\neq t:\ j_s,j_t\neq 0}} D^{j_1}(\vartheta^{p-1})\star D^{j_2}v\star D^{j_3}\Phi.\end{aligned} \tag{5.84}$$

In order to prove (5.81), we consider $k=m-1$ and obtain

$$\begin{aligned}\dot\varphi_{i_1\ldots i_{m-1}} = &\,\mathcal{O}^{-1+(m-1)(1-\gamma)}+D^m\varphi\star\mathcal{O}^{-(1+\gamma)}\\ &+D^m\varphi\star D^m\varphi\star\mathcal{O}^{-3}+\vartheta^{p-1}\dot\Phi F^a_l\tilde g^{lr}\varphi_{ra;i_1\ldots i_{m-1}}.\end{aligned} \tag{5.85}$$

There holds

$$z_{ra} = \varphi_{i_1\ldots i_{m-1}ra}\varphi^{i_1\ldots i_{m-1}} + \varphi_{i_1\ldots i_{m-1}r}\varphi^{i_1\ldots i_{m-1}}{}_{;a} \tag{5.86}$$

and thus

$$\begin{aligned}\varphi_{ra;i_1\ldots i_{m-1}}\varphi^{i_1\ldots i_{m-1}} &= \varphi_{i_1\ldots i_{m-1};ra}\varphi^{i_1\ldots i_{m-1}}\\ &\quad+D^{m-1}\varphi\star D^{m-1}\varphi\star\mathcal{O}^0\\ &= z_{ra}-\varphi_{i_1\ldots i_{m-1};r}\varphi^{i_1\ldots i_{m-1}}{}_{;a}\\ &\quad+D^{m-1}\varphi\star D^{m-1}\varphi\star\mathcal{O}^0.\end{aligned} \tag{5.87}$$

We conclude, that

$$\begin{aligned}\dot z - \vartheta^{p-1}\dot\Phi F^a_l\tilde g^{lr}z_{ra} = &-\vartheta^{p-1}\dot\Phi F^a_l\tilde g^{lr}\varphi_{i_1\ldots i_{m-1};r}\varphi^{i_1\ldots i_{m-1}}{}_{;a}\\ &+\mathcal{O}^{-(1+\gamma)+(m-1)(1-\gamma)}+D^m\varphi\star\mathcal{O}^{-(1+2\gamma)}\\ &+D^m\varphi\star D^m\varphi\star\mathcal{O}^{-(3+\gamma)}.\end{aligned} \tag{5.88}$$

To prove (5.82), set $k=m$ to obtain

$$\begin{aligned}\dot\varphi_{i_1\dots i_m} &= -(p-1)\vartheta^{p-1}\dot\vartheta\varphi_{i_1\dots i_m}v\Phi - \vartheta^{p-1}v^{-1}\varphi_{ai_1\dots i_m}\varphi^a\Phi\\ &\quad + \mathcal{O}^{-1+m(1-\gamma)} + D^m\varphi\star\mathcal{O}^{-2\gamma} + \vartheta^{p-1}\dot\Phi F^{ar}\varphi_{ra;i_1\dots i_m}\\ &\quad - \vartheta^{p+1}\dot\Phi F^a_a\varphi_{i_1\dots i_m} + D^m\varphi\star D^m\varphi\star\mathcal{O}^{-(2+\gamma)}\\ &\quad + D^{m+1}\varphi\star\mathcal{O}^{-(1+\gamma)} + D^m\varphi\star D^{m+1}\varphi\star\mathcal{O}^{-3}\\ &\quad + D^m\varphi\star D^m\varphi\star D^m\varphi\star\mathcal{O}^{-4}.\end{aligned} \tag{5.89}$$

As above we have

$$w_{ra} = \varphi_{i_1\dots i_m;ra}\varphi^{i_1\dots i_m} + \varphi_{i_1\dots i_m;r}\varphi^{i_1\dots i_m}{}_{;a}, \tag{5.90}$$

$$\begin{aligned}\varphi_{ra;i_1\dots i_m}\varphi^{i_1\dots i_m} &= \varphi_{i_1\dots i_m;ra}\varphi^{i_1\dots i_m} + D^m\varphi\star D^m\varphi\star\mathcal{O}^0\\ &= w_{ra} - \varphi_{i_1\dots i_m;r}\varphi^{i_1\dots i_m}{}_{;a} + D^m\varphi\star D^m\varphi\star\mathcal{O}^0\end{aligned} \tag{5.91}$$

and thus

$$\begin{aligned}\dot w - \vartheta^{p-1}\dot\Phi F^{ar}w_{ar} &= -2(p-1)\vartheta^{p-1}\dot\vartheta v\Phi w + D^m\varphi\star\mathcal{O}^{-1+m(1-\gamma)}\\ &\quad + D^m\varphi\star D^{m+1}\varphi\star\mathcal{O}^{-(1+\gamma)}\\ &\quad + D^m\varphi\star D^m\varphi\star\mathcal{O}^{-2\gamma}\\ &\quad - \vartheta^{p-1}\dot\Phi F^{ar}\varphi_{i_1\dots i_m;r}\varphi^{i_1\dots i_m}{}_{;a} - 2\vartheta^{p+1}\dot\Phi F^a_a w\\ &\quad + D^m\varphi\star D^m\varphi\star D^m\varphi\star\mathcal{O}^{-(2+\gamma)}\\ &\quad + D^m\varphi\star D^m\varphi\star D^{m+1}\varphi\star\mathcal{O}^{-3}\\ &\quad + D^m\varphi\star D^m\varphi\star D^m\varphi\star D^m\varphi\star\mathcal{O}^{-4}.\end{aligned} \tag{5.92}$$

□

5.11. **Theorem.** Let φ be the solution of (3.33), then

$$D^m\varphi = \mathcal{O}^{-\gamma}\quad\forall m\in\mathbb{N}^*\quad\forall 0\le\gamma<1. \tag{5.93}$$

Proof. We use a method similar to the proof of [5, Lemma 6.10].

For $m=1,2$ this has been proven for $\gamma=1$, cf. Theorem 4.5 and Theorem 4.6. Thus let the conclusion hold for $1\le k\le m-1$, $m\ge 3$. Let

$$z = \frac{1}{2}|D^{m-1}\varphi|^2 \tag{5.94}$$

and

$$w = \frac{1}{2}|D^m\varphi|^2, \tag{5.95}$$

as well as

$$\tilde w = we^{\frac{2\lambda}{n^p}t},\ 0\le\lambda\le 1. \tag{5.96}$$

Set

$$\zeta = \log\tilde w + z. \tag{5.97}$$

Then by 5.10 we have

$$
\begin{aligned}
\dot{\zeta} - \vartheta^{p-1}\dot{\Phi}F^a_l\tilde{g}^{lr}\zeta_{ra} &= \tilde{w}^{-1}(\dot{\tilde{w}} - \vartheta^{p-1}\dot{\Phi}F^a_l\tilde{g}^{lr}\tilde{w}_{ar}) \\
&\quad + \vartheta^{p-1}\dot{\Phi}F^a_l\tilde{g}^{lr}(\log\tilde{w})_a(\log\tilde{w})_r \\
&\quad + \dot{z} - \vartheta^{p-1}\dot{\Phi}F^a_l\tilde{g}^{lr}z_{ra} \\
&= w^{-1}(\dot{w} - \vartheta^{p-1}\dot{\Phi}F^a_l\tilde{g}^{lr}w_{ra}) + \frac{2}{n^p}\lambda \\
&\quad + \vartheta^{p-1}\dot{\Phi}F^a_l\tilde{g}^{lr}(\log w)_a(\log w)_r \\
&\quad + \dot{z} - \vartheta^{p-1}\dot{\Phi}F^a_l\tilde{g}^{lr}z_{ar} \\
&= -2(p-1)\vartheta^{p-1}\dot{\vartheta}v\Phi - 2\vartheta^{p+1}\dot{\Phi}F^a_a \\
&\quad - \vartheta^{p-1}\dot{\Phi}|D^{m+1}\varphi|^2w^{-1} \\
&\quad + \vartheta^{p-1}\dot{\Phi}(\sigma^{ar} - F^{ar})\varphi_{i_1\dots i_m;r}\varphi^{i_1\dots i_m}{}_{;a}w^{-1} \\
&\quad + (\mathcal{O}^{-1+m(1-\gamma)} \star D^m\varphi)w^{-1} \\
&\quad + (\mathcal{O}^{-2\gamma} \star D^m\varphi \star D^m\varphi)w^{-1} \\
&\quad + (\mathcal{O}^{-(2+\gamma)} \star D^m\varphi \star D^m\varphi \star D^m\varphi)w^{-1} \\
&\quad + (\mathcal{O}^{-4} \star D^m\varphi \star D^m\varphi \star D^m\varphi \star D^m\varphi)w^{-1} \\
&\quad + (\mathcal{O}^{-(1+\gamma)} \star D^m\varphi \star D^{m+1}\varphi)w^{-1} \\
&\quad + (\mathcal{O}^{-3} \star D^m\varphi \star D^m\varphi \star D^{m+1}\varphi)w^{-1} \\
&\quad - \vartheta^{p-1}\dot{\Phi}|D^m\varphi|^2 \\
&\quad + \vartheta^{p-1}\dot{\Phi}(\sigma^{ar} - F^{ar})\varphi_{i_1\dots i_{m-1};r}\varphi^{i_1\dots i_{m-1}}{}_{;a} \\
&\quad + \mathcal{O}^{-(1+\gamma)+(m-1)(1-\gamma)} + \mathcal{O}^{-(1+2\gamma)} \star D^m\varphi \\
&\quad + \mathcal{O}^{-(3+\gamma)} \star D^m\varphi \star D^m\varphi \\
&\quad + \vartheta^{p-1}\dot{\Phi}F^{ar}(\log w)_a(\log w)_r + \frac{2}{n^p}\lambda.
\end{aligned} \tag{5.98}
$$

We want to bound ζ. Thus, fix $0 < T < \infty$ and suppose that

$$\sup_{[0,T]\times\mathbb{S}^n} \zeta = \zeta(t_0, x_0),\ t_0 > 0. \tag{5.99}$$

At this point we have

$$-z_a = (\log w)_a \tag{5.100}$$

and thus

$$
\begin{aligned}
&\vartheta^{p-1}\dot{\Phi}F^a_l\tilde{g}^{lr}(\log w)_a(\log w)_r \\
=&\vartheta^{p-1}\dot{\Phi}F^a_l\tilde{g}^{lr}\varphi_{i_1\dots i_{m-1};a}\varphi^{i_1\dots i_{m-1}}\varphi_{j_1\dots j_{m-1};r}\varphi^{j_1\dots j_{m-1}} \\
=&\mathcal{O}^{-2(1+\gamma)} \star D^m\varphi \star D^m\varphi.
\end{aligned} \tag{5.101}
$$

Thus, at (t_0, x_0), also supposing that

$$|D^m\varphi|e^{\frac{\lambda}{n^p}t} \geq 1, \tag{5.102}$$

$$\begin{aligned}
0 \le\ & 2\vartheta^{p-1}\dot{\vartheta} v\Phi - 2(\vartheta^{p+1}\dot{\Phi}F^a_a + p\vartheta^{p-1}\dot{\vartheta}v\Phi) \\
& + w^{-1}(-\vartheta^{p-1}\dot{\Phi}|D^{m+1}\varphi|^2 \\
& + \vartheta^{p-1}\dot{\Phi}(\sigma^{ar} - F^{ar})\varphi_{i_1\dots i_m;r}\varphi^{i_1\dots i_m}{}_{;a} \\
& + c_m e^{-\frac{1+\gamma}{n^p}t}|D^m\varphi||D^{m+1}\varphi| + c_m e^{-\frac{3}{n^p}t}|D^m\varphi|^2|D^{m+1}\varphi|) \\
& + (-\vartheta^{p-1}\dot{\Phi}|D^m\varphi|^2 + \vartheta^{p-1}\dot{\Phi}(\sigma^{ar} - F^{ar})\varphi_{i_1\dots i_{m-1};r}\varphi^{i_1\dots i_{m-1}}{}_{;a} \\
& + c_m e^{\frac{m(1-\gamma)-1}{n^p}t}|D^m\varphi|^{-1} + c_m e^{\frac{(m-1)(1-\gamma)-(1+\gamma)}{n^p}t} \\
& + c_m e^{-\frac{1+2\gamma}{n^p}t}|D^m\varphi| + c_m e^{-\frac{2(1+\gamma)}{n^p}t}|D^m\varphi|^2) + \frac{2}{n^p}\lambda \\
\le\ & -2\vartheta^{-1}\dot{\vartheta}vF^{-p}(h^l_a) + 2pF^{-p}(h^l_a)(\vartheta^{-1}\dot{\vartheta}v - F^{-1}F^a_a) \\
& + w^{-1}(-\vartheta^{p-1}\dot{\Phi}|D^{m+1}\varphi|^2 \\
& + \vartheta^{p-1}\dot{\Phi}(\sigma^{ar} - F^{ar})\varphi_{i_1\dots i_m;r}\varphi^{i_1\dots i_m}{}_{;a} \\
& + ce^{-\frac{\gamma}{n^p}t}|D^m\varphi|^2 + ce^{-\frac{2+\gamma}{n^p}t}|D^{m+1}\varphi|^2 \\
& + ce^{-\frac{3}{n^p}t}(|D^m\varphi|^4 + |D^{m+1}\varphi|^2)) \\
& + (-\vartheta^{p-1}\dot{\Phi}|D^m\varphi|^2 + \vartheta^{p-1}\dot{\Phi}(\sigma^{ar} - F^{ar})\varphi_{i_1\dots i_{m-1};r}\varphi^{i_1\dots i_{m-1}}{}_{;a} \\
& + ce^{\frac{\lambda+m(1-\gamma)-1}{n^p}t} + ce^{\frac{(m-1)(1-\gamma)-(1+\gamma)}{n^p}t} + ce^{-\frac{1+2\gamma}{n^p}t}|D^m\varphi|^2) \\
& + \frac{2}{n^p}\lambda,
\end{aligned} \tag{5.103}$$

where we used

$$ab \le \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2 \tag{5.104}$$

with $a = |D^m\varphi|$, $b = |D^{m+1}\varphi|$ and $\epsilon = e^{\frac{t}{n^p}}$, as well as with $a = 1$, $b = |D^m\varphi|$ and $\epsilon = 1$.
From the C^1 and C^2 estimates we know

$$-2\vartheta^{-1}\dot{\vartheta}vF^{-p} \to -\frac{2}{n^p}, \tag{5.105}$$

$$\limsup_{t\to\infty} 2pF^{-p}(\vartheta^{-1}\dot{\vartheta}v - F^{-1}F^a_a) \le 0 \tag{5.106}$$

and

$$|\sigma^{ar} - F^{ar}| \to 0. \tag{5.107}$$

In view of

$$\vartheta^{p-1}\dot{\Phi} = p\vartheta^{-2}F^{-(p+1)}(h^l_a) \ge ce^{-\frac{2}{n^p}t}, \tag{5.108}$$

cf. Corollary 3.6, we may absorb any bad term by the good terms

$$-\vartheta^{p-1}\dot{\Phi}|D^k\varphi|^2,\ k = m, m+1, \tag{5.109}$$

if t_0 is supposed to be large enough and $0 \le \lambda < 1$. Thus, for large t_0 and $\lambda < 1$ we obtain a contradiction and conclude

$$|D^m\varphi|e^{\frac{\lambda}{n^p}t} \le c = c(n, p, M_0, m, \lambda) \quad \forall 0 \le \lambda < 1, \tag{5.110}$$

which means

$$D^m\varphi = \mathcal{O}^{-\gamma} \quad \forall 0 \le \gamma < 1. \tag{5.111}$$

□

6. The conformally flat parametrization and convergence to a sphere

In order to complete the proof of Theorem 1.2, we use the conformally flat parametrization and consider the flow in $\mathbb{R}^{n+1}$. From now on, we distinguish quantities in $\mathbb{H}^{n+1}$ from those in $\mathbb{R}^{n+1}$ by an additional brève, e.g. $\breve{u}, \breve{g}_{ij}$, etc., compare [5, ch. 5]. For a flow hypersurface

$$M = \operatorname{graph} \breve{u} = \operatorname{graph} u \tag{6.1}$$

we then have

$$\breve{u} = \log(2+u) - \log(2-u) \tag{6.2}$$

and

$$|D\breve{u}|^2 = u^{-2}\sigma^{ij}u_iu_j \equiv |Du|^2. \tag{6.3}$$

Note that

$$\begin{aligned} d\bar{s}^2 &= \frac{1}{(1-\frac{1}{4}r^2)^2}(dr^2 + r^2\sigma_{ij}dx^idx^j) \\ &= e^{2\psi}(dr^2 + r^2\sigma_{ij}dx^idx^j). \end{aligned} \tag{6.4}$$

Let

$$\tilde{\vartheta} = \frac{1}{2}\frac{r}{1-\frac{1}{4}r^2}, \tag{6.5}$$

then the second fundamental forms $\breve{h}^i_j$ and h^i_j satisfy the relation

$$e^{\psi}\breve{h}^i_j = h^i_j + v^{-1}\tilde{\vartheta}\delta^i_j \equiv \check{h}^i_j, \tag{6.6}$$

cf. [5, (5.11), (5.13)]. Set

$$g_{ij} = u_iu_j + u^2\sigma_{ij} \tag{6.7}$$

and

$$\check{h}_{ij} = g_{ik}\check{h}^k_j, \tag{6.8}$$

then the flow in $\mathbb{H}^{n+1}$,

$$\dot{x} = F^{-p}\breve{\nu}, \; F = F(\breve{h}^i_j), \tag{6.9}$$

now reads in $\mathbb{R}^{n+1}$

$$\dot{x} = F^{-p}e^{(p-1)\psi}\nu, \tag{6.10}$$

where

$$F = F(\check{h}_{ij}) = F(\check{h}^i_j). \tag{6.11}$$

Using

$$h_{ij}v^{-1} = -u_{;ij} + \bar{h}_{ij} \tag{6.12}$$

and the homogeneity of $F = F(\check{h}_{ij})$, we obtain

$$\dot{u} - \dot{\Phi} F^{ij} u_{;ij} = -e^{(p-1)\psi} v\Phi + v^{-1}\dot{\Phi} F - \dot{\Phi} v^{-2}\tilde{\vartheta} F^{ij} g_{ij} - \dot{\Phi} F^{ij}\bar{h}_{ij}. \tag{6.13}$$

Here and in the following, $u_{;ij}$ denotes covariant differentiation with respect to g_{ij}, where merely indices, u_{ij}, denote derivatives with respect to σ_{ij} and $\dot{u} = \frac{\partial u}{\partial t}$ is a partial derivative. We want to use coordinates (x^i).

6.1. **Lemma.** Let u be the scalar solution of (6.10). Then

$$D^m u = \mathcal{O}^{-1+\epsilon} \quad \forall m \in \mathbb{N}^* \quad \forall \epsilon > 0. \tag{6.14}$$

Proof. We have

$$\check{u}_i = \frac{u_i}{1 - \frac{1}{4}u^2}. \tag{6.15}$$

In view of (6.2) there holds

$$2 - u = (2+u)e^{-\check{u}} \tag{6.16}$$

and thus

$$(2-u)^\beta = \mathcal{O}^{-\beta} \quad \forall \beta \in \mathbb{R}, \tag{6.17}$$

using Lemma 3.5. Set

$$g(u) = \frac{1}{1 - \frac{1}{4}u^2} \equiv \frac{1}{f(u)}. \tag{6.18}$$

Then

$$D^m g = \sum_{k_1 + \ldots + m k_m = m} c_m f^{-(\sum_{i=1}^m k_i + 1)} \prod_{i=1}^m \left(\frac{D^i f}{i!}\right)^{k_i}, \tag{6.19}$$

and

$$D^i f = \sum_{s+r=i} D^s u \star D^r u. \tag{6.20}$$

Taking $|D\check{u}|$ with respect to the spherical norm, we see that the claim holds for $m = 1$, by Lemma 5.4. Suppose the claim to be true for $1 \le k \le m-1$. Then

$$D^m \check{u} = g D^m u + Du \star D^{m-1} g + \sum_{\substack{s+r=m-1 \\ s,r \ge 1}} D^{r+1} u \star D^s g \tag{6.21}$$

so that

$$\begin{aligned} D^m u &= g^{-1} D^m \check{u} + g^{-1} Du \star D^{m-1} g + g^{-1} \sum_{\substack{s+r=m-1 \\ s,r \ge 1}} D^{r+1} u \star D^s g \\ &= \mathcal{O}^{m(1-\gamma)-1} + \mathcal{O}^{\epsilon m - 1} \ \forall \gamma < 1 \ \forall \epsilon > 0. \end{aligned} \tag{6.22}$$

□

6.2. **Lemma.** For all $m\in\mathbb{N}^*$ and for all $\epsilon>0$ there hold

$$D^m v^\beta = \mathcal{O}^{-2+\epsilon} \quad \forall\beta\in\mathbb{R}, \tag{6.23}$$

$$D^m\left(\frac{2u}{2+u}\right)=\mathcal{O}^{-1+\epsilon}=D^m\left(\left(\frac{4}{2+u}\right)^{p-1}\right), \tag{6.24}$$

$$D^m((2-u)^\beta)=\mathcal{O}^{-\beta+\epsilon m}, \tag{6.25}$$

$$D^m(\check{h}^i_j(2-u))=\mathcal{O}^{-1+\epsilon} \tag{6.26}$$

and

$$D^m(\check{h}_{ij}(2-u))=\mathcal{O}^{-1+\epsilon}. \tag{6.27}$$

Proof. We consider

$$v=\sqrt{1+u^{-2}\sigma^{ij}u_iu_j}. \tag{6.28}$$

Differentiation gives

$$\begin{aligned} v_{i_1}&=\frac{1}{2v}(2u^{-2}\sigma^{kl}u_{ki_1}u_l-2u^{-3}\sigma^{kl}u_ku_lu_{i_1})\\ &=v^{-1}(u^{-2}\sigma^{kl}u_{ki_1}u_l-u^{-3}\sigma^{kl}u_ku_lu_{i_1})\\ &=\mathcal{O}^{-2+\epsilon}\ \forall\epsilon>0. \end{aligned} \tag{6.29}$$

Thus $D(v^\beta)=\beta v^{\beta-1}Dv=\mathcal{O}^{-2+\epsilon}$ $\forall\epsilon>0$. Let the claim hold for $1\le k\le m-1$. Then

$$\begin{aligned} D^m v&=\sum_{s+r=m-1}D^s(v^{-1})\star D^r(u^{-2}\sigma^{kl}u_{ki_1}u_l-u^{-3}\sigma^{kl}u_ku_lu_{i_1})\\ &=\mathcal{O}^{-2+\epsilon}, \end{aligned} \tag{6.30}$$

so that

$$D^m(v^\beta)=\sum_{k_1+\ldots+mk_m=m}c_m\star\mathcal{O}^0\star\prod_{i=1}^m\left(\frac{D^iv}{i!}\right)^{k_i}=\mathcal{O}^{-2+\epsilon}. \tag{6.31}$$

Thus (6.23) is true.

To prove (6.24), suppose that f is smooth, then

$$D^m(f\circ u)=\sum_{k_1+\ldots+mk_m=m}c_mf^{(k)}(u)\prod_{i=1}^m\left(\frac{D^iu}{i!}\right)^{k_i}=\mathcal{O}^{-1+\epsilon}, \tag{6.32}$$

$k=\sum_{i=1}^m k_i$, since in case

$$f(x)=\frac{2x}{2+x} \tag{6.33}$$

or

$$f(x)=\left(\frac{4}{2+x}\right)^{p-1} \tag{6.34}$$

we have

$$D^k f \in C^\infty(u([0,\infty)\times\mathbb{S}^n)). \tag{6.35}$$

In case of (6.25) we have

$$f(x) = (2-x)^\beta \tag{6.36}$$

such that

$$f^{(k)}(x) = \prod_{i=0}^{k-1}(\beta - i)(2-x)^{\beta-k}(-1)^k, \tag{6.37}$$

implying

$$f^{(k)}(u) = \mathcal{O}^{k-\beta}. \tag{6.38}$$

Thus

$$D^m(f\circ u) = \mathcal{O}^{-\beta+\epsilon m}. \tag{6.39}$$

In order to show (6.26), first observe that there holds, according to (3.112),

$$h^i_j = \frac{1}{vu}\delta^i_j + \frac{1}{v^3u^3}u^iu_j - \frac{\sigma^{ik} - v^{-2}u^{-2}u^iu^k}{vu^2}u_{kj}. \tag{6.40}$$

Have in mind, that now $\vartheta(u) = u$, $u^i = \sigma^{ik}u_k$ and derivatives are taken with respect to σ_{ij}. Thus

$$\begin{aligned} D^m(\check{h}^i_j(2-u)) &= D^m\Big(\frac{2-u}{vu}\delta^i_j + \frac{2-u}{v^3u^3}u^iu_j \\ &\quad - (2-u)\frac{\sigma^{ik} - v^{-2}u^{-2}u^iu^k}{vu^2}u_{kj} + v^{-1}\frac{2u}{2+u}\delta^i_j\Big) \\ &= \mathcal{O}^{-1+\epsilon} + \mathcal{O}^{-3+\epsilon} + \mathcal{O}^{-2+\epsilon} \\ &= \mathcal{O}^{-1+\epsilon}. \end{aligned} \tag{6.41}$$

(6.27) follows from

$$D^m(g_{ij}) = D^m(u_iu_j + u^2\sigma_{ij}) = \mathcal{O}^{-1+\epsilon}. \tag{6.42}$$

□

6.3. **Theorem.** Let u be the scalar solution of (6.10), then

$$D^m u = \mathcal{O}^{-1} \quad \forall m \in \mathbb{N}^*. \tag{6.43}$$

Proof. We follow the corresponding proof in [5, Thm. 6.11].

Define

$$\phi = (2-u)^{-1},\ \tilde{\phi} = \phi e^{-\frac{t}{np}} \tag{6.44}$$

and

$$\tilde{F} = F(\check{h}^k_l(2-u)),\ \tilde{\Phi} = \Phi(\tilde{F}). \tag{6.45}$$

There holds, having in mind that $h^i_j \to \frac{1}{2}\delta^i_j$, and using (6.17) as well as Theorem 4.5, that

$$|\check{h}^k_l(2-u) - \delta^k_l| \le |h^k_l(2-u)| + \left|\left(v^{-1}\frac{2u}{2+u} - 1\right)\delta^k_l\right| \le ce^{-\frac{t}{n^p}}. \tag{6.46}$$

We have

$$\frac{\partial\tilde{\phi}}{\partial t} = \dot{\tilde{\phi}} = \left(\frac{\dot{u}}{2-u} - \frac{1}{n^p}\right)\tilde{\phi}, \tag{6.47}$$

$$\tilde{\phi}_{ij} = \frac{u_{ij}}{2-u}\tilde{\phi} + \frac{2u_i u_j}{(2-u)^2}\tilde{\phi} \tag{6.48}$$

and thus

$$\begin{aligned} &\dot{\tilde{\phi}} - v^{-2}\dot{\tilde{\Phi}}\tilde{\phi}^{-(p+1)}e^{-\frac{p+1}{n^p}t}\tilde{F}^{ij}\tilde{\phi}_{ij} \\ =&\frac{\tilde{\phi}}{2-u}\Big(\dot{u} - v^{-2}\dot{\tilde{\Phi}}\tilde{\phi}^{-(p+1)}e^{-\frac{p+1}{n^p}t}\tilde{F}^{ij}u_{ij} \\ &\quad - v^{-2}\frac{2}{2-u}\dot{\tilde{\Phi}}\tilde{\phi}^{-(p+1)}e^{-\frac{p+1}{n^p}t}\tilde{F}^{ij}u_i u_j - \frac{2-u}{n^p}\Big). \end{aligned} \tag{6.49}$$

An easy calculation shows

$$\begin{aligned} u_{ij} &= v^2 u_{;ij} - u^{-1}(\sigma^{kl}u_k u_l \sigma_{ij} - 2u_i u_j) \\ &= -vh_{ij} + v^2\bar{h}_{ij} - u^{-1}(\sigma^{kl}u_k u_l\sigma_{ij} - 2u_i u_j) \\ &= -v\check{h}_{ij} + \tilde{\vartheta}g_{ij} + v^2\bar{h}_{ij} - u^{-1}(\sigma^{kl}u_k u_l\sigma_{ij} - 2u_i u_j). \end{aligned} \tag{6.50}$$

Thus we conclude

$$\begin{aligned} &\dot{\tilde{\phi}} - v^{-2}\dot{\tilde{\Phi}}\tilde{\phi}^{-(p+1)}e^{-\frac{p+1}{n^p}t}\tilde{F}^{ij}\tilde{\phi}_{ij} \\ =&\frac{\tilde{\phi}}{2-u}\Big(\dot{u} - \dot{\Phi}F^{ij}u_{;ij} + u^{-1}v^{-2}\dot{\tilde{\Phi}}\tilde{\phi}^{-(p+1)}e^{-\frac{p+1}{n^p}t}\tilde{F}^{ij}(\sigma^{kl}u_k u_l\sigma_{ij} \\ &\quad - 2u_i u_j) - v^{-2}\frac{2}{2-u}\dot{\tilde{\Phi}}\tilde{\phi}^{-(p+1)}e^{-\frac{p+1}{n^p}t}\tilde{F}^{ij}u_i u_j - \frac{2-u}{n^p}\Big), \end{aligned} \tag{6.51}$$

which is

$$\begin{aligned} &v^{-1}\dot{\tilde{\Phi}}\tilde{F}(2-u)^{p-1}\tilde{\phi} - v\tilde{\Phi}(e^{\psi}(2-u))^{p-1}\tilde{\phi} \\ &- v^{-2}\dot{\tilde{\Phi}}(2-u)^{p-1}\frac{2u}{2+u}F^{ij}g_{ij}\tilde{\phi} - \dot{\tilde{\Phi}}F^{ij}\bar{h}_{ij}(2-u)^{p-1}e^{-\frac{t}{n^p}} \\ &- \left(2u^{-1} + \frac{2}{2-u}\right)(v^{-2}\dot{\tilde{\Phi}}(2-u)^p\tilde{F}^{ij}u_i u_j)\tilde{\phi} \\ &+ u^{-1}v^{-2}\dot{\tilde{\Phi}}(2-u)^p\tilde{F}^{ij}\sigma^{kl}u_k u_l\sigma_{ij}\tilde{\phi} - \frac{1}{n^p}\tilde{\phi}, \end{aligned} \tag{6.52}$$

being equal to

$$\begin{aligned}
&\left(v^{-1}\dot{\tilde{\Phi}}\tilde{F} - v^{-2}\dot{\tilde{\Phi}}\frac{2u}{2+u}F^{ij}g_{ij}\right)(2-u)^{p-1}\tilde{\phi} \\
&- \left(v\tilde{\Phi}\left(\frac{4}{2+u}\right)^{p-1} + \frac{1}{n^p}\right)\tilde{\phi} - \dot{\tilde{\Phi}}F^{ij}\bar{h}_{ij}(2-u)^{p-1}e^{-\frac{t}{n^p}} \\
&- \left(2u^{-1} + \frac{2}{2-u}\right)(v^{-2}\dot{\tilde{\Phi}}(2-u)^p\tilde{F}^{ij}u_iu_j)\tilde{\phi} \\
&+ u^{-1}v^{-2}\dot{\tilde{\Phi}}(2-u)^p\tilde{F}^{ij}\sigma_{ij}\sigma^{kl}u_ku_l\tilde{\phi}.
\end{aligned} \tag{6.53}$$

Set

$$w = \frac{1}{2}|D^m\tilde{\phi}|^2. \tag{6.54}$$

Then by Lemma 6.2 we have

$$D^m\tilde{\phi} = \mathcal{O}^{\epsilon m} \quad \forall m \in \mathbb{N}^* \quad \forall \epsilon > 0. \tag{6.55}$$

Differentiating the equation for $\tilde{\phi}$ covariantly with respect to σ_{ij} m times, we obtain

$$\begin{aligned}
\dot{w} - v^{-2}\dot{\tilde{\Phi}}\tilde{\phi}^{-(p+1)}e^{-\frac{p+1}{n^p}t}\tilde{F}^{ij}w_{ij} &= \mathcal{O}^{-p} \star w + \mathcal{O}^{\epsilon-p+3\epsilon m} \\
&\quad + \mathcal{O}^{-1+\epsilon+\epsilon m} + \mathcal{O}^{-p+\epsilon m} \\
&\quad + \mathcal{O}^{-(p+1)} \star D^{m+1}\tilde{\phi} \star D^{m+1}\tilde{\phi} \\
&= \mathcal{O}^{-\delta}, \ \delta > 0,
\end{aligned} \tag{6.56}$$

where first ϵ has to be chosen in dependence of p and m. Thus

$$\tilde{w} = \sup_{x\in\mathbb{S}^n} w(\cdot, x) \tag{6.57}$$

satisfies

$$\dot{\tilde{w}} \le c_{m,\delta}e^{-\delta t} \tag{6.58}$$

and is bounded.

Thus

$$D^m\phi = \mathcal{O}^1 \quad \forall m \in \mathbb{N}. \tag{6.59}$$

This yields

$$Du = (2-u)^2D\phi = \mathcal{O}^{-1}. \tag{6.60}$$

If

$$D^ku = \mathcal{O}^{-1} \quad \forall 1 \le k \le m-1, \tag{6.61}$$

then

$$\begin{aligned}
D^m\phi &= \sum_{k_1+\ldots+mk_m=m}\frac{m!}{k_1!\cdots k_m!}\frac{1}{(2-u)^{1+k}}\prod_{i=1}^{m-1}\left(\frac{D^iu}{i!}\right)^{k_i}\left(\frac{D^mu}{m!}\right)^{k_m} \\
&= \frac{D^mu}{(2-u)^2} + \mathcal{O}^1,
\end{aligned}$$

which implies

$$D^m u = \mathcal{O}^{-2} \star D^m \phi + \mathcal{O}^{-1} = \mathcal{O}^{-1}. \tag{6.62}$$

□

6.4. **Corollary.** The rescaled functions

$$\tilde{u} = (u-2)e^{\frac{t}{n^p}} \text{ in } \mathbb{R}^{n+1} \tag{6.63}$$

and

$$\tilde{\breve{u}} = \breve{u} - \frac{t}{n^p} \text{ in } \mathbb{H}^{n+1} \tag{6.64}$$

are uniformly bounded in $C^m(\mathbb{S}^n)$ for all $m \in \mathbb{N}$ and converge in $C^\infty(\mathbb{S}^n)$ to a uniquely determined limit $\tilde{u}$ or $\tilde{\breve{u}}$ respectively.

Proof. We follow the proof of [5, Thm. 6.11].

Because of the boundedness we only have to show, that the pointwise limit

$$\lim_{t\to\infty}(u(t,x)-2)e^{\frac{t}{n^p}} \tag{6.65}$$

exists for all $x \in \mathbb{S}^n$. We have

$$\begin{aligned} \dot{\tilde{u}} = \frac{\partial \tilde{u}}{\partial t} &= e^{(p-1)\psi}\frac{v}{F^p}e^{\frac{t}{n^p}} + \frac{1}{n^p}\tilde{u} \\ &= \frac{u+2}{4}(2-u)e^{\frac{t}{n^p}}v\frac{4^p}{(2+u)^p}\tilde{F}^{-p} + \frac{1}{n^p}\tilde{u} \\ &= \left(\frac{1}{n^p} - \frac{4^{p-1}}{(2+u)^{p-1}}v\tilde{F}^{-p}\right)\tilde{u} \\ &\geq -ce^{-\frac{t}{n^p}}. \end{aligned} \tag{6.66}$$

Thus

$$(\tilde{u} - n^p c e^{-\frac{t}{n^p}})' \geq 0, \tag{6.67}$$

which implies the result. □

6.5. **Corollary.** The limit function

$$\tilde{\breve{u}} = \lim_{t\to\infty}\frac{\breve{u}}{t} \tag{6.68}$$

is constant in $\mathbb{H}^{n+1}$.

References

[1] Qi Ding, *The inverse mean curvature flow in rotationally symmetric spaces*, Chin. Ann. of Math. - Ser. B, 1–18, (2010).
[2] Claus Gerhardt, *Flow of nonconvex hypersurfaces into spheres*, J. Differ. Geom. **32**, 299–314, (1990).
[3] Claus Gerhardt, *Closed Weingarten hypersurfaces in space forms*, Geom. Anal. and the Calc. of Var. (Jürgen Jost, ed.), International Press, Boston, (1996).
[4] Claus Gerhardt, *Curvature Problems*, Ser. in Geom. and Topol., vol. 39, International Press, Somerville, MA, (2006).
[5] Claus Gerhardt, *Inverse curvature flows in hyperbolic space*, J. Differ. Geom. **89**, 487 – 527, (2011).

[6] Claus Gerhardt, *Non-scale-invariant inverse curvature flows in Euclidean space*, Calc. of Var. and Partial Differ. Equ., (2012). doi: 10.1007/s00526-012-0589-x
[7] Gerhard Huisken, *Flow by mean curvature of convex surfaces into spheres.*, J. Differ. Geom. **20**, 237–266, (1984).
[8] Steven G. Krantz, Harold R. Parks, *A primer of real analytic functions*, 2. ed., Birkhäuser, Boston, MA, (2002).
[9] Nikolai V. Krylov, *Nonlinear elliptic and parabolic equations of the second order*, Reidel, Dordrecht, (1987).

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