QUANTITATIVE OSCILLATION ESTIMATES FOR ALMOST-UMBILICAL CLOSED HYPERSURFACES IN EUCLIDEAN SPACE

JULIAN SCHEUER

ABSTRACT. We prove ϵ-closeness of hypersurfaces to a sphere in Euclidean space under the assumption that the traceless second fundamental form is δ-small compared to the mean curvature. We give the explicit dependence of δ on ϵ within the class of uniformly convex hypersurfaces with bounded volume.

CONTENTS

1. INTRODUCTION

In this article we investigate the potential of the traceless second fundamental form, also called the *umbilicity tensor*,

$$\mathring{A} = A - \frac{\operatorname{tr}(A)}{n} g \tag{1.1}$$

of a hypersurface embedded in the Euclidean space to pinch other geometric quantities of the hypersurface. Questions like this arise from the well known fact, that $\mathring{A} = 0$ implies that the hypersurface must be a sphere. Then it is natural to ask if this behavior is a kind of continuous, in the sense that a small traceless second fundamental form implies closeness to a sphere. During the last decade, substantial progress has been made towards a better understanding of this question. In 2005 an article by Camillo de Lellis and Stefan Müller appeared, [7], where the estimate

$$\inf_{\lambda\in\mathbb{R}} \|A - \lambda g\|_{L^2(M)} \leq C\|\mathring{A}\|_{L^2(M)} \tag{1.2}$$

was proven for hypersurfaces $M \subset \mathbb{R}^3$. From this, the authors deduced $W^{2,2}$-closeness to a sphere. One year later, in [8], the authors made a step towards uniform closeness and showed that in addition the metric is C^0-close to the standard sphere metric. In 2011, one of de Lellis' PhD students, Daniel Perez, proved in the

Date: January 16, 2015.
2010 *Mathematics Subject Classification.* 53C20, 53C21, 53C24, 58C40.
Key words and phrases. Pinching, almost-umbilical, almost-Einstein, convex hypersurface.
This work is being supported by the DFG.

class of hypersurfaces with volume 1 and bounded second fundamental form, that for given $\epsilon > 0$ there exists $\delta > 0$, such that a δ-small traceless second fundamental form yields ϵ-closeness to a sphere, compare [11, Cor. 1.2]. He used an argument via contradiction and it does not seem possible to extract the ϵ-dependence of δ along his proof. In [11, p. xvi] the author posed the derivation of a quantitative dependence as an open problem. In this article we tackle this problem and prove the following theorem.

1.1. **Theorem.** *Let $n \geq 2$ and $X : M^n \hookrightarrow \mathbb{R}^{n+1}$ be the smooth, isometric embedding of a closed, connected, orientable and strictly mean-convex hypersurface. Let $0 < \alpha < 1$. Then there exists $c > 0$, such that whenever we have $\epsilon < c|M|^{\frac{1}{n}}$ and the pointwise estimate*

$$\|\mathring{A}\| \leq H|M|^{-\frac{2+\alpha}{n}}\epsilon^{2+\alpha} \tag{1.3}$$

holds, then M is strictly convex and

$$M \subset B_{\sqrt{\frac{n}{\lambda_1(M)}}+\epsilon}(x_0) \backslash B_{\sqrt{\frac{n}{\lambda_1(M)}}-\epsilon}(x_0). \tag{1.4}$$

The constant c depends on n, α, $\|\tilde{A}\|_\infty$ and $\|\tilde{A}^{-1}\|_\infty$, where $|M| = \mathrm{vol}(M)$, $\tilde{A} = |M|^{\frac{1}{n}}A$, $\lambda_1(M)$ is the first nonzero eigenvalue of the Laplace-Beltrami operator on M and x_0 is the center of mass of M.

A more detailed description of the notation involved here will be presented in section 2. Thus in the class of uniformly convex hypersurfaces of unit volume we obtain ϵ-closeness to a sphere, if $\mathring{A}$ is of order $\epsilon^{2+\alpha}$ and ϵ is sufficiently small.

Note that recently a similar result by Julien Roth appeared, cf. [13]. In more general ambient spaces he proves quasi-isometry of hypersurfaces to the sphere under certain assumptions, including smallness of the gradient of the second fundamental form.

The author's motivation to find a quantitative dependence like this arose from his work on inverse curvature flows in the Euclidean space. In [15, Appendix A] Oliver Schnürer derived a pinching estimate of the traceless second fundamental form for hypersurfaces evolving by the inverse Gauss curvature flow in $\mathbb{R}^3$. Ben Andrews applied estimates like this to bound the difference between circumradius r_+ and inradius r_- of the surface in [1, Section 4]. However, we are not aware whether those methods may be transferred to higher dimensions. Clearly, Theorem 1.1 provides an estimate of $r_+ - r_-$ in terms of $\mathring{A}$. Indeed, we are going to apply this estimate to prove asymptotical roundness of hypersurfaces solving an inverse curvature flow equation in $\mathbb{R}^{n+1}$, cf. [14].

Let us give an overview over the main ingredients involved in the proof. Certainly we need a result, which somehow yields the transition from *qualitative to quantitative.* We found the following result due to Julien Roth. We formulate a special case and only the statements which are of interest to our proof.

1.2. **Theorem.** [12, Thm. 1] *Let (M^n, g) be a compact, connected and oriented Riemannian manifold without boundary isometrically immersed in $\mathbb{R}^{n+1}$. Assume that $|M| = 1$ and $H_2 > 0$. Then for any $p \geq 2$ and $\epsilon > 0$ there exists a constant $C_\epsilon = C_\epsilon(n, \|H\|_\infty, \|H_2\|_{2p})$, such that if*

$$\lambda_1(M)\left(\int_M H\right)^2 - n\|H_2\|^2_{2p} > -C_\epsilon \tag{1.5}$$

is satisfied, then

$$M \subset B_{\sqrt{\frac{n}{\lambda_1}}+\epsilon}(x_0) \backslash B_{\sqrt{\frac{n}{\lambda_1}}-\epsilon}(x_0), \tag{1.6}$$

where x_0 is the center of mass of M and H_2 is the second normalized elementary symmetric polynomial.

This theorem is a generalization of [6] to higher k-th mean curvatures. There are generalizations to ambient spaces of bounded sectional curvature, cf. [10], as well. At first glance, it does not seem to be a quantitative result, but a rather tedious scanning of the proof shows, that C_ϵ can be chosen to be of order ϵ^2, compare section 3.

Certainly, this ϵ^2 gives insight into the question, where the order $\epsilon^{2+\alpha}$ comes from in Theorem 1.1. It is an interesting question, whether, and if how, this could be improved.

Thus we have to derive (1.5) from (1.3). Firstly, we need to relate the first eigenvalue of the Laplacian to the traceless second fundamental form. This transition has another stop at the Ricci tensor. The following result, due to Erwann Aubry, relates the Ricci tensor to λ_1. It was proven in [3], but is accessible more easily in [4, Thm. 1.6]. Again, we only cite the aspects, which are relevant to our work.

1.3. **Theorem.** [4, Thm. 1.6] *For any $p > \frac{n}{2}$ there exists $C(n,p)$, such that if M^n is a complete manifold with*

$$\int_M (\underline{\mathrm{Ric}} - (n-1))_-^p < \frac{|M|}{C(n,p)}, \tag{1.7}$$

then M is compact and satisfies

$$\lambda_1(M) \geq n\left(1 - C\left(\frac{1}{|M|}\int_M (\underline{\mathrm{Ric}} - (n-1))_-^p\right)^{\frac{1}{p}}\right). \tag{1.8}$$

Here, $\underline{\mathrm{Ric}} = \underline{\mathrm{Ric}}(x)$ denotes the smallest eigenvalue of the Ricci tensor at $x \in M$ and for $y \in \mathbb{R}$ we set $y_- = \max(0, -y)$.

The other quantities in (1.5) can be controlled with the help of (1.3) quite easily. Thus the only ingredient, which is left, is to control the Ricci tensor in (1.7). The following result, due to Daniel Perez, [11] and also to De Lellis and Müller for $n = 2$, [7], is helpful.

1.4. **Theorem.** [11, Thm. 1.1] *Let $n \geq 2$, $p \in (n, \infty)$ and $c_0 > 0$. Then there exists $C(n,p,c_0) > 0$, such that for any smooth, closed and connected hypersurface $M \subset \mathbb{R}^{n+1}$ with*

$$|M| = 1 \tag{1.9}$$

and

$$\|A\|_p \leq c_0 \tag{1.10}$$

we have

$$\min_{\mu\in\mathbb{R}} \|A - \mu g\|_p \leq C\|\mathring{A}\|_p. \tag{1.11}$$

This result will enable to move, via the Ricci tensor, to an estimate on λ_1 and to finally provide the estimate (1.5). Then the result follows. There largest technical difficulty is, that we finally need L^∞ bounds, where the theorems 1.3 and 1.4 only make statements on L^p norms. We will present the way to handle this in section 4.

Note, that we will not need to know the explicit value of μ_0 in (1.11), where the minimum is attained. However, this is another interesting question with some history. According to [11, p. 50], it was Gerhard Huisken to suggest an inverse mean curvature flow approach to prove, that the minimum is attained at

$$\mu = \frac{1}{|M|}\int_M H. \tag{1.12}$$

In [11, p. 52, Ch. 3.4] this is proven for $n \geq 2$, $p = 2$ and for closed convex hypersurfaces. Unfortunately, the case $p = 2$ is not enough in our case. Hence we have to deal with the little technical difficulty, that μ_0 is not explicitly known.

We want to mention as well, that there is literature on spherical closeness in terms of lower bounds on the principal curvatures, cf. [5], which is sort of a different issue, since we want to provide arbitrary closeness.

Now we start with the detailed analysis of the problem at hand and start with an explanation of our notation.

2. Notation and preliminaries

In this article we consider closed embedded hypersurfaces $M^n \subset \mathbb{R}^{n+1}$. We follow the notation as it appears in the references as closely as possible.

$g = (g_{ij})$ denotes the induced metric of M^n, $A = (h_{ij})$ the second fundamental form and κ_i, $i = 1, \dots, n$, the principal curvatures ordered pointwise,

$$\kappa_1 \leq \cdots \leq \kappa_n. \tag{2.1}$$

The volume of M is

$$|M| = \int_M 1 \; d\mu, \tag{2.2}$$

where μ is the canonical surface measure associated to g.

$\lambda_1(M)$ denotes the first nonzero eigenvalue of $-\Delta$, where Δ is the Laplace-Beltrami operator on (M, g).

For $k = 1, \dots, n$ we define

$$H_k = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \kappa_{i_1} \cdots \kappa_{i_k}. \tag{2.3}$$

This includes the definition of the mean curvature,

$$H = \frac{1}{n}\sum_{i=1}^{n} \kappa_i, \tag{2.4}$$

which deviates from some of the references. It corresponds to the notation in [12]. Thus the traceless second fundamental form is

$$\mathring{A} = A - Hg. \tag{2.5}$$

For smooth tensor fields on M, $T = (t^{i_1\dots i_k}_{j_1\dots j_l})$, we define the pointwise norms to be

$$\|T\| = \sqrt{t^{i_1\dots i_k}_{j_1\dots j_l} t^{j_1\dots j_l}_{i_1\dots i_k}}, \tag{2.6}$$

where indices are lowered or lifted with respect to the induced metric of the hypersurface the tensor field is defined on. With the help of this definition we may define L^p-norms on a subset $\Omega \subset M$ to be

$$\|T\|_{p,\Omega} = \left(\int_\Omega \|T\|^p\right)^{\frac{1}{p}}, \tag{2.7}$$

where the surface measure to be used is implicitly included in the set of integration Ω. Analogously we set

$$\|T\|_{\infty,\Omega} = \sup_{\Omega} \|T\|. \tag{2.8}$$

The tensor $\mathrm{Ric} = (R_{ij})$ is the Ricci tensor and $R = \mathrm{tr}(\mathrm{Ric}) = R^i_i$ the scalar curvature. $\underline{\mathrm{Ric}}(x)$ denotes the smallest eigenvalue of the Ricci tensor at $x \in M$.

For M^n the symbol $\tilde{M}^n$ always denotes the normalized manifold

$$\tilde{M} = |M|^{-\frac{1}{n}} M \hookrightarrow \mathbb{R}^{n+1} \tag{2.9}$$

with $|\tilde{M}| = 1$. The corresponding rescaled geometric quantities are denoted with a tilde as well, e.g.

$$\tilde{g} = (\tilde{g}_{ij}),\ \tilde{A} = (\tilde{h}_{ij}) \tag{2.10}$$

etc.

Finally

$$B_r(x_0) \subset \mathbb{R}^{n+1} \tag{2.11}$$

denotes an $(n+1)$-dimensional ball in $\mathbb{R}^{n+1}$ with radius r and center x_0.

3. Qualitative closeness revisited

In this section we turn our attention to the result, which connects λ_1 with closeness to a sphere, Theorem 1.2. We state, how the constant C_ϵ involved here depends on ϵ, whereafter we indicate, how this can be deduced from the corresponding sequence of lemmata in [12]. We prove the following

3.1. **Proposition.** *In the situation of Theorem 1.2 let* $0 < \epsilon < \frac{2}{3\|H\|_\infty}$. *If (1.5) holds for*

$$C_\epsilon = \frac{1}{2} \min\left(L\sqrt{\frac{n}{\lambda_1(M)}}\epsilon^2, L \right), \tag{3.1}$$

where L is bounded and uniformly positive whenever $\|H\|_\infty$ and $\|H_2\|_{2p}$ range in compact subsets of $(0,\infty)$, then we have

$$M \subset B_{\sqrt{\frac{n}{\lambda_1(M)}}+\epsilon}(x_0) \backslash B_{\sqrt{\frac{n}{\lambda_1(M)}}-\epsilon}(x_0). \tag{3.2}$$

Proof. We will spot and note the relevant formulae in [12], always denoting in which way they depend on the geometric quantities and on ϵ. There is a sequence of constants, C_ϵ is combined by. We start with [12, p. 297, Lemma 2.1]. First of all, it is required, that

$$C_\epsilon < \frac{n}{2}\|H_2\|^2_{2p}. \tag{3.3}$$

Equation (5) yields

$$A_1 = \frac{2\|H\|^2_\infty}{\|H_2\|^2_{2p}}. \tag{3.4}$$

[12, p. 298, Lemma 2.2] yields

$$A_2 = \frac{A_1}{n\|H_2\|^2_{2p}}. \tag{3.5}$$

The proofs of [12, Lemma 2.4, Lemma 2.5] imply, that A_3 and A_4 are of a similar form. Finally, the author cites a lemma implying an L^∞-estimate on the function

$$\varphi = |X|\left(|X| - \sqrt{\frac{n}{\lambda_1(M)}}\right)^2, \tag{3.6}$$

where X is the position vector field with respect to the center of mass of M, x_0. The lemma is, cf. [12, Lemma 3.1],

For $p \geq 2$ and any $\eta > 0$, there exists $K_\eta(n, \|H\|_\infty, \|H_2\|_{2p})$, such that if (1.5) holds with $C_\epsilon = K_\eta$, then $\|\varphi\|_\infty \leq \eta$.

Essentially, the proof of this lemma is given in [6, p. 188, pf. of Lemma 3.1], also compare [12, Sec. 6]. Here one sees, that this K_η can be chosen to be

$$K_\eta = \min\left(\frac{\eta}{(L'A_4)^4}, c_n\right) > 0, \tag{3.7}$$

where L' is just of the same form as A_4. Now, in [12, p. 301] the author defines

$$\begin{aligned}\eta(\epsilon) &= \min\left(\left(\sqrt{\frac{n}{\lambda_1(M)}} - \epsilon\right)\epsilon^2, \frac{1}{27\|H\|_\infty^3}\right) \\ &\geq \min\left(\frac{1}{3}\sqrt{\frac{n}{\lambda_1(M)}}\epsilon^2, \frac{1}{27\|H\|_\infty^3}\right),\end{aligned} \tag{3.8}$$

since $\epsilon < \frac{2}{3\|H\|_\infty}$ and

$$\lambda_1(M) \leq \frac{1}{n-1}\|R\|_\infty \leq n\|H\|_\infty^2, \tag{3.9}$$

compare [9, Thm. 3.1]. He concludes

$$M \subset B_{\sqrt{\frac{n}{\lambda_1(M)}}+\epsilon}(x_0) \backslash B_{\sqrt{\frac{n}{\lambda_1(M)}}-\epsilon}(x_0) \tag{3.10}$$

under the assumption (1.5) with

$$C_\epsilon = \frac{1}{2}\min\left(\frac{n}{2}\|H_2\|_{2p}^2, c_n, \frac{1}{3(L'A_4)^4}\sqrt{\frac{n}{\lambda_1(M)}}\epsilon^2, \frac{1}{27(L'A_4)^4\|H\|_\infty^3}\right), \tag{3.11}$$

which has the form claimed in the proposition. □

4. Quantitative spherical closeness

Now we come to the proof of the main result. Let us state it again for a better readability.

4.1. **Theorem.** *Let $n \geq 2$ and $X\colon M^n \hookrightarrow \mathbb{R}^{n+1}$ be the smooth, isometric embedding of a closed, connected, orientable and mean-convex hypersurface. Let $0 < \alpha < 1$. Then there exists $c > 0$, such that whenever we have $\epsilon < c|M|^{\frac{1}{n}}$ and the pointwise estimate*

$$\|\mathring{A}\| \leq H|M|^{-\frac{2+\alpha}{n}}\epsilon^{2+\alpha} \tag{4.1}$$

holds, then M is strictly convex and

$$X(M) \subset B_{\sqrt{\frac{n}{\lambda_1(M)}}+\epsilon}(x_0) \backslash B_{\sqrt{\frac{n}{\lambda_1(M)}}-\epsilon}(x_0). \tag{4.2}$$

The constant c depends on n, α, $\|\tilde{A}\|_\infty$ and $\|\tilde{A}^{-1}\|_\infty$, where $|M| = \operatorname{vol}(M)$, $\tilde{A} = |M|^{\frac{1}{n}}A$, $\lambda_1(M)$ is the first nonzero eigenvalue of the Laplace-Beltrami operator on M and x_0 is the center of mass of M.

Proof. In this proof, $\tilde{C}_i$, $i \in \mathbb{N}$, always denote generic constants which depend on n, α, $\|\tilde{A}\|_\infty$ and $\|\tilde{A}^{-1}\|_\infty$ at most. Set

$$p = n + 1 \tag{4.3}$$

and let

$$k = \frac{6}{\alpha}. \tag{4.4}$$

For the rescaled surfaces

$$\tilde{M} = |M|^{-\frac{1}{n}} M \tag{4.5}$$

we find from Theorem 1.4, that

$$\|\tilde{A} - \mu_0 \tilde{g}\|_{kp} \le \tilde{C}_1 \|\mathring{\tilde{A}}\|_{kp}, \tag{4.6}$$

where $\mu_0 = \mu_0(n, \alpha, \|\tilde{A}\|_\infty, \|\tilde{A}^{-1}\|_\infty)$.

The first condition we put on the constant c is to satisfy

$$c < \left(\frac{1}{\sqrt{n(n-1)}}\right)^{\frac{1}{2+\alpha}}. \tag{4.7}$$

Then (4.1) yields the strict convexity of $\tilde{M}$, due to [2, Lemma 2.2]. μ_0 is strictly positive, since obviously we have

$$\inf_{\tilde{M}} \tilde{\kappa}_1 \le \mu_0 \le \sup_{\tilde{M}} \tilde{\kappa}_n. \tag{4.8}$$

Define

$$\hat{M} = \mu_0 \tilde{M}. \tag{4.9}$$

Then

$$\|\hat{A} - \hat{g}\|_{kp} = \left(\int_{\hat{M}} \mu_0^{-kp} \|\tilde{A} - \mu_0 \tilde{g}\|^{kp}\right)^{\frac{1}{kp}} = \mu_0^{\frac{n}{kp}-1} \|\tilde{A} - \mu_0 \tilde{g}\|_{kp}. \tag{4.10}$$

Define the set

$$\hat{P} = \{\hat{x} \in \hat{M} \colon \|\hat{A}(\hat{x}) - \hat{g}(\hat{x})\| < 1\}. \tag{4.11}$$

Then its complement has volume

$$|\hat{P}^c| \le \int_{\hat{P}^c} \|\hat{A} - \hat{g}\|^{kp} \le \mu_0^{n-kp} \|\tilde{A} - \mu_0 \tilde{g}\|_{kp}^{kp}. \tag{4.12}$$

In order to apply Theorem 1.3, we need an estimate on the Ricci tensor $\hat{\mathrm{Ric}} = (\hat{R}_{ij})$. By the Gaussian formula there holds

$$\hat{R}_{ij} = n\hat{H}\hat{h}_{ij} - \hat{h}_{ik}\hat{h}^k_j. \tag{4.13}$$

Let $\hat{x} \in \hat{P}$ and $\xi \in T_{\hat{x}}\hat{M}$. Then

$$\begin{aligned} \hat{R}_{ij}\xi^i\xi^j &= n\hat{H}\hat{h}_{ij}\xi^i\xi^j - \hat{h}_{ik}\hat{h}^k_j\xi^i\xi^j \\ &= n(\hat{H} - 1)(\hat{h}_{ij} - \hat{g}_{ij})\xi^i\xi^j + n(\hat{h}_{ij} - \hat{g}_{ij})\xi^i\xi^j \\ &\quad + n(\hat{H} - 1)\|\xi\|^2 + (n-1)\|\xi\|^2 - 2(\hat{h}_{ij} - \hat{g}_{ij})\xi^i\xi^j \\ &\quad - (\hat{h}_{ik} - \hat{g}_{ik})(\hat{h}^k_j - \delta^k_j)\xi^i\xi^j, \end{aligned} \tag{4.14}$$

from which we obtain at $\hat{x}$

$$\|\hat{\mathrm{Ric}} - (n-1)\hat{g}\| \le \tilde{C}_2 \|\hat{A} - \hat{g}\|, \tag{4.15}$$

since $\|\hat{A}-\hat{g}\|<1$. In the notation of Theorem 1.3 we obtain

$$\begin{aligned}\int_{\hat{M}}(\underline{\hat{\mathrm{Ric}}}-(n-1))_-^{kp} &\leq \int_{\hat{P}}\tilde{C}_2^{kp}\|\hat{A}-\hat{g}\|^{kp}+\int_{\hat{P}^c}(\underline{\hat{\mathrm{Ric}}}-(n-1))_-^{kp}\\ &\leq \left(\tilde{C}_2^{kp}\mu_0^{n-kp}+(n-1)^{kp}\mu_0^{n-kp}\right)\|\tilde{A}-\mu_0\tilde{g}\|_{kp}^{kp}\\ &=\tilde{C}_3\|\tilde{A}-\mu_0\tilde{g}\|_{kp}^{kp}.\end{aligned}\tag{4.16}$$

Thus Theorem 1.3 will be applicable under condition (4.1), if we choose c small enough to ensure the last of the following inequalities (note that in the first inequality we use (4.6)).

$$\begin{aligned}\tilde{C}_3\|\tilde{A}-\mu_0\tilde{g}\|_{kp}^{kp} &\leq \tilde{C}_3\tilde{C}_1^{kp}\|\mathring{\tilde{A}}\|_{kp}^{kp}=\tilde{C}_3\tilde{C}_1^{kp}|M|^{\frac{kp}{n}-1}\|\mathring{A}\|_{kp}^{kp}\\ &\leq \tilde{C}_3\tilde{C}_1^{kp}|M|^{-\frac{(1+\alpha)kp+n}{n}}\epsilon^{(2+\alpha)kp}\|H\|_{kp}^{kp}\\ &=\tilde{C}_3\tilde{C}_1^{kp}|M|^{-\frac{(2+\alpha)kp}{n}}\epsilon^{(2+\alpha)kp}\|\tilde{H}\|_{kp}^{kp}\\ &<\tilde{C}_3\tilde{C}_1^{kp}c^{(2+\alpha)kp}\|\tilde{H}\|_{kp}^{kp}\\ &\overset{!}{<}\frac{|\hat{M}|}{C(n,kp)}=\frac{\mu_0^n}{C(n,kp)},\end{aligned}\tag{4.17}$$

where $C(n,kp)$ is the constant from Theorem 1.3. Thus $c=c(n,\alpha,\|\tilde{A}\|_\infty,\|\tilde{A}^{-1}\|_\infty)$ is additionally choosable, such that this chain of inequalities is true. We may apply Theorem 1.3 to conclude

$$\begin{aligned}\lambda_1(\hat{M}) &\geq n\left(1-C(n,kp)\left(\frac{1}{|\hat{M}|}\int_{\hat{M}}(\underline{\hat{\mathrm{Ric}}})-(n-1))_-^{kp}\right)^{\frac{1}{kp}}\right)\\ &\geq n\left(1-C(n,kp)\mu_0^{-\frac{n}{kp}}\tilde{C}_1\tilde{C}_3^{\frac{1}{kp}}\|\tilde{H}\|_{kp}\tilde{\epsilon}^{2+\alpha}\right),\end{aligned}\tag{4.18}$$

where $\tilde{\epsilon}=|M|^{-\frac{1}{n}}\epsilon$. We obtain

$$\lambda_1(\tilde{M})\geq \mu_0^2 n(1-\tilde{C}_4\tilde{\epsilon}^{2+\alpha}),\tag{4.19}$$

with a new constant $\tilde{C}_4$.

Now we want to apply Theorem 1.2. Therefore we need estimates of the curvature integrals. First note, that

$$\tilde{H}_2=\frac{1}{n(n-1)}\tilde{R}.\tag{4.20}$$

A similar calculation as (4.14) shows, that at any point

$$\tilde{x}\in\tilde{P}_\gamma=\{\tilde{x}\in\tilde{M}\colon\|\tilde{A}-\mu_0\tilde{g}\|<\gamma\},\ 0<\gamma<1,\tag{4.21}$$

we have

$$\|\tilde{R}_{ij}-\mu_0^2(n-1)\tilde{g}_{ij}\|\leq\tilde{C}_5(n,\mu_0)\|\tilde{A}-\mu_0\tilde{g}\|.\tag{4.22}$$

Furthermore there holds

$$|\tilde{P}_\gamma^c|\gamma^{kp}\leq\int_{\tilde{P}_\gamma^c}\|\tilde{A}-\mu_0\tilde{g}\|^{kp}\leq\tilde{C}_1^{kp}\|\mathring{\tilde{A}}\|_{kp}^{kp}\leq\tilde{C}_6\tilde{\epsilon}^{(2+\alpha)kp}\tag{4.23}$$

and thus

$$|\tilde{P}_\gamma^c|\leq\tilde{C}_6\left(\frac{\tilde{\epsilon}^{2+\alpha}}{\gamma}\right)^{kp}.\tag{4.24}$$

We estimate

$$\begin{aligned}\left(\int_{\tilde M}\tilde H_2^{2p}\right)^{\frac1p}&=\left(\int_{\tilde P_\gamma}\left(\frac{\tilde R}{n(n-1)}\right)^{2p}+\int_{\tilde P_\gamma^c}\left(\frac{\tilde R}{n(n-1)}\right)^{2p}\right)^{\frac1p}\\&\le\left\|\frac{\tilde R}{n(n-1)}\right\|^2_{2p,\tilde P_\gamma}+\left\|\frac{\tilde R}{n(n-1)}\right\|^2_{2p,\tilde P_\gamma^c}\\&\le\left(\mu_0^2+\tilde C_5\|\tilde A-\mu_0\tilde g\|_{2p,\tilde P_\gamma}\right)^2+|\tilde P_\gamma^c|^{\frac1p}\|\tilde H\|_\infty^4,\end{aligned}\tag{4.25}$$

where we used $\tilde H_2^{\frac12}\le\tilde H$ and (4.22).

Furthermore we obtain from (4.22), that

$$\begin{aligned}\left(\int_{\tilde M}\tilde H\right)^2\ge\left(\int_{\tilde P_\gamma}\left(\frac{\tilde R}{n(n-1)}\right)^{\frac12}\right)^2&\ge\left(|\tilde P_\gamma|\sqrt{\mu_0^2-\tilde C_5\gamma}\right)^2\\&=|\tilde P_\gamma|^2\mu_0^2-|\tilde P_\gamma|^2\tilde C_5\gamma\end{aligned}\tag{4.26}$$

for all

$$0<\gamma<\frac{\mu_0^2}{\tilde C_5}.\tag{4.27}$$

From (4.19), (4.25) and (4.26) we obtain

$$\begin{aligned}&\lambda_1(\tilde M)\left(\int_{\tilde M}\tilde H\right)^2-n\|\tilde H_2\|_{2p}^2\\\ge&(\mu_0^2n-\mu_0^2n\tilde C_4\tilde\epsilon^{2+\alpha})(|\tilde P_\gamma|^2\mu_0^2-|\tilde P_\gamma|^2\tilde C_5\gamma)\\&\quad-n\mu_0^4-n\tilde C_5^2\gamma^2-2n\mu_0^2\tilde C_5\gamma-n|\tilde P_\gamma^c|^{\frac1p}\|\tilde H\|_\infty^4\\\ge&-\tilde C_7|\tilde P_\gamma^c|-\tilde C_7\gamma-\tilde C_7\tilde\epsilon^{2+\alpha}-\tilde C_7\left(\frac{\tilde\epsilon^{2+\alpha}}{\gamma}\right)^k,\end{aligned}\tag{4.28}$$

where $\tilde C_7$ is a new constant. According to Theorem 1.2 and Proposition 3.1 there exists $C_{\tilde\epsilon}$, which can be chosen as

$$C_{\tilde\epsilon}=\frac12\min\left(L\sqrt{\frac{n}{\lambda_1(\tilde M)}}\tilde\epsilon^2,L\right),\tag{4.29}$$

such that whenever $\tilde\epsilon<\frac{2}{3\|\tilde H\|_\infty}$ and

$$\lambda_1(\tilde M)\left(\int_{\tilde M}\tilde H\right)^2-n\|\tilde H_2\|_{2p}^2>-C_{\tilde\epsilon},\tag{4.30}$$

we could conclude

$$\tilde M\subset B_{\sqrt{\frac{n}{\lambda_1(\tilde M)}}+\tilde\epsilon}(\tilde x_0)\backslash B_{\sqrt{\frac{n}{\lambda_1(\tilde M)}}-\tilde\epsilon}(\tilde x_0).\tag{4.31}$$

Now define

$$\gamma=\tilde\epsilon^{2+\frac\alpha2}.\tag{4.32}$$

Then

$$\begin{aligned}
&\tilde{C}_7\left(\left(\frac{\tilde{\epsilon}^{2+\alpha}}{\gamma}\right)^{kp}+\left(\frac{\tilde{\epsilon}^{2+\alpha}}{\gamma}\right)^{k}+\gamma+\tilde{\epsilon}^{2+\alpha}\right)\\
\leq&\tilde{C}_7\left(\tilde{\epsilon}^{\frac{\alpha kp}{2}}+\tilde{\epsilon}^{\frac{\alpha k}{2}}+\tilde{\epsilon}^{2+\frac{\alpha}{2}}+\tilde{\epsilon}^{2+\alpha}\right)\\
=&\tilde{C}_7\left(\tilde{\epsilon}^{3p}+\tilde{\epsilon}^{3}+\tilde{\epsilon}^{2+\frac{\alpha}{2}}+\tilde{\epsilon}^{2+\alpha}\right)\\
<&\frac{1}{2}\min\left(L\sqrt{\frac{n}{\lambda_1(\tilde{M})}}\tilde{\epsilon}^2, L\right),
\end{aligned}\tag{4.33}$$

for all $0<\tilde{\epsilon}<c$, if c is small enough in dependence of n, α, $\|\tilde{A}\|_\infty$ and $\|\tilde{A}^{-1}\|_\infty$, such that the requirements for γ, namely

$$\gamma<\min\left(1,\frac{\mu_0^2}{\tilde{C}_5}\right),\tag{4.34}$$

are fulfilled as well.

We conclude, rescaling again,

$$M\subset B_{\sqrt{\frac{n}{\lambda_1(M)}}+\epsilon}(x_0)\backslash B_{\sqrt{\frac{n}{\lambda_1(M)}}-\epsilon}(x_0),\tag{4.35}$$

the desired result. □

4.2. **Remark.** The previous result is easier to comprehend, if one restricts to the class of hypersurfaces of bounded volume and modulus of convexity, namely

$$0<c\leq|M|\leq C\tag{4.36}$$

and

$$0<cg\leq A\leq Cg.\tag{4.37}$$

Then, in order to prove ϵ-closeness, one has to find constants $c>0$ and $\beta>0$, such that

$$\|A-Hg\|\leq cH\epsilon^{2+\beta},\tag{4.38}$$

where c must not depend on ϵ. Then applying Theorem 4.1 with $\alpha=\frac{\beta}{2}$, one concludes ϵ-closeness for small $0<\epsilon<\epsilon_0$.

5. Concluding remarks and open questions

We must not hesitate to remark, that this result is only a first step towards a better understanding of the stability problem. It helps to control the order of the sufficient δ with respect to ϵ, which is sufficient for first applications in geometric flows, compare [14].

However, two things will be desirable in this context. Firstly, there would be direct applications to geometric flows, if one could improve the order $\epsilon^{2+\alpha}$. We are not aware of the existence of such a result. Secondly, pinching results for the first eigenvalue of the Laplacian are known in other ambient space, cf. [10]. It would be interesting, with immediate applications to curvature flows in those spaces, whether results like ours could be deduced in those settings as well.

References

[1] Ben Andrews, *Gauss curvature flow: the fate of the rolling stones*, Invent. Math. **138** (1999), no. 1, 151–161.

[2] Ben Andrews and James McCoy, *Convex hypersurfaces with pinched principal curvatures and flow of convex hypersurfaces by high powers of curvature*, Trans. Am. Math. Soc. **364** (2012), no. 7, 3427–3447.

[3] Erwann Aubry, *Finiteness of π_1 and geometric inequalities in almost positive Ricci curvature*, Ann. Sci. l'Ecole Norm. Supér. **40** (2007), no. 4, 675–695.

[4] ———, *Diameter pinching in almost positve Ricci curvature*, Comment. Math. Helv. **84** (2009), no. 2, 223–233.

[5] Alexandr Borisenko and Konstiantyn Drach, *Closeness to spheres of hypersurfaces with normal curvature bounded below*, Sbornik: Math. **204** (2013), no. 11, 1565–1583.

[6] Bruno Colbois and Jean-Francois Grosjean, *A pinching theorem for the first eigenvalue of the laplacian on hypersurfaces of the Euclidean space*, Comment. Math. Helv. **82** (2007), no. 1, 175–195.

[7] Camillo De Lellis and Stefan Müller, *Optimal rigidity estimates for nearly umbilical surfaces*, J. Differ. Geom. **69** (2005), 75–110.

[8] ———, *A C^0-estimate for nearly umbilical surfaces*, Calc. Var. Partial Differ. Equ. **26** (2006), no. 3, 283–296.

[9] Jean-Francois Grosjean, *Upper bounds for the first eigenvalue of the Laplacian on compact submanifolds*, Pac. J. Math. **206** (2002), no. 1, 93–112.

[10] Jean-Francois Grosjean and Julien Roth, *Eigenvalue pinching and application to the stability and the almost umbilicity of hypersurfaces*, Math. Z. **271** (2012), no. 1-2, 469–488.

[11] Daniel Perez, *On nearly umbilical hypersurfaces*, Ph.D. thesis, Zuerich, 2011.

[12] Julien Roth, *Pinching of the first eigenvalue of the Laplacian and almost-Einstein hypersurfaces of the Euclidean space*, Ann. Glob. Anal. Geom. **33** (2008), no. 3, 293–306.

[13] ———, *A new result about almost umbilical hypersurfaces of real space forms*, Bull. Aust. Math. Soc. **First view** (2014), 1–10.

[14] Julian Scheuer, *Pinching and asymptotical roundness for inverse curvature flows in Euclidean space*, preprint, arxiv:1404.2719v3, 2014.

[15] Oliver C. Schnürer, *Surfaces expanding by the inverse Gauß curvature flow*, J. Reine Angew. Math. **600** (2006), 117–134.

Dr. Julian Scheuer, Ruprecht-Karls-Universität, Institut für Angewandte Mathematik, Im Neuenheimer Feld 294, 69120 Heidelberg, Germany

E-mail address: scheuer@math.uni-heidelberg.de