PINCHING AND ASYMPTOTICAL ROUNDNESS FOR INVERSE CURVATURE FLOWS IN EUCLIDEAN SPACE

JULIAN SCHEUER

ABSTRACT. We consider inverse curvature flows in the $(n+1)$-dimensional Euclidean space, $n \geq 2$, expanding by arbitrary negative powers of a 1-homogeneous, monotone curvature function F with some concavity properties. We obtain asymptotical roundness, meaning that circumradius minus inradius of the flow hypersurfaces decays to zero and that the flow becomes close to a flow of spheres.

CONTENTS

1. INTRODUCTION

We consider inverse curvature flows in Euclidean space $\mathbb{R}^{n+1}$, $n \geq 2$,

$$\dot{x} = F^{-p}\nu, \ 0 < p < \infty, \tag{1.1}$$

where F is a symmetric, monotone, homogeneous of degree 1 and concave curvature function, which is defined in an open, convex cone of $\mathbb{R}^n$, such that F vanishes on its boundary, where in case $p > 1$ we assume $\Gamma = \Gamma_+$. Here ν is the outward normal to the flow hypersurfaces of the flow

$$x \colon [0, T^*) \times M \to \mathbb{R}^{n+1} \tag{1.2}$$

with starshaped initial hypersurface M_0, where in case $p > 1$ we assume that M_0 is strictly convex.

In [9] the same flow was considered. Here it is shown for any $0 < p < \infty$ that the maximal time of existence T^* is characterized by the property

$$\inf |x| \to \infty, \ t \to T^*, \tag{1.3}$$

and that the rescaled surfaces

$$\tilde{M}_t = \Theta^{-1} M_t, \tag{1.4}$$

---

where $\Theta = \Theta(t)$ is the radius of a suitable expanding sphere, converge to the unit sphere in C^∞, cf. [9, Thm. 1.1, Thm. 1.2].

The goal of this paper is the improvement of the asymptotical behavior of the flow. Let us explain our motivation to do this. To our knowledge, almost all of the existing results on classical smooth inverse curvature flows, cf. the end of this introduction for an overview, assume that the initial hypersurface may be written as a graph over a sphere. Of course, there are many possibilities to do this. For example take a sphere $\mathcal{S}$ centered at $q \in \mathbb{R}^{n+1}$ as initial hypersurface of the flow. It follows, that is evolves through expanding spheres centered at q as well. Now take another point $z \neq q$ in the interior of the ball enclosed by $\mathcal{S}$ and write $\mathcal{S}$ as a graph over a sphere $\mathcal{S}_z$ around z,

$$(1.5) \qquad\qquad \mathcal{S} = \{(u(x), x) \colon x \in \mathcal{S}_z\}.$$

Let $\mathcal{S}$ evolve and let $u(t, \cdot)$ denote the corresponding graph functions over $\mathcal{S}_z$. From the previous observations it is clear that the oscillation of u can not decay to zero as $t \to T^*$, even though circumradius minus inradius of the hypersurfaces *is* constantly zero. Thus the choice of the sphere $\mathcal{S}_z$ is not optimal and u does not reflect the nice spherical shape of the evolving surfaces. The optimal sphere would be one around q. In this paper we are going to show that such an optimal sphere exists in the sense that the flow hypersurfaces will fit to a flow of spheres arbitrarily close. We are going to achieve this for $0 < p < \infty$ and for $n \geq 2$. The main ingredient in the proof is an estimate of the oscillation of the support function as it appears in [1, Prop. 4, Lemma 5], also cf. [17, Prop. 7.3]. These results hold for the case $n = 2$. In higher dimensions there is a generalization of these results, which provides closeness to a sphere in terms of the difference of the principal radii of a convex hypersurface, cf. [12, Thm. 1.4].

In a recent paper Hung and Wang came up with a counterexample to such an asymptotical roundness for hypersurfaces expanding by the inverse mean curvature flow in the hyperbolic space, cf. [10, Thm. 1]. This shows the impossibility of proving results like ours in $\mathbb{H}^{n+1}$.

Before we give an overview over previous results on expanding flows in Euclidean space, let us state the main result of this paper. We require the following assumptions on F.

1.1. **Assumption.** Let $\Gamma \subset \mathbb{R}^n$ be an open, convex and symmetric cone containing the positive cone

$$(1.6) \qquad\qquad \Gamma_+ = \{(\kappa_i) \in \mathbb{R}^n \colon \kappa_i > 0, \ 1 \leq i \leq n\}.$$

Let F be a positive, monotone, symmetric and concave curvature function, normalized to $F(1, \ldots, 1) = n$, such that

  (i) in case $0 < p \leq 1$ we have $F \in C^\infty(\Gamma)$ and $F_{|\partial\Gamma} = 0$,

  (ii) in case $p > 1$ we additionally have $\Gamma = \Gamma_+$.

Recall, that a hypersurface M is called F-*admissable*, if $F((\kappa_i)(x))$ is well-defined for all $x \in M$, where $\kappa_i(x)$ are the principal curvatures of M at x with respect to the inward unit normal.

The main result of this paper is the following one.

1.2. **Theorem.** *Let $n \geq 2$, $0 < p < \infty$ and let F satisfy Assumption 1.1. Let*

$$(1.7) \qquad\qquad x_0 \colon M \hookrightarrow M_0 \subset \mathbb{R}^{n+1}$$

be the smooth embedding of a closed, orientable, connected and F-admissable hypersurface, which can be written as a graph over a sphere $\mathbb{S}^n$,

(1.8) $$M_0 = \{(u(0,x), x) \colon x \in \mathbb{S}^n\}.$$

Then

  (i) *there exists a unique smooth solution on a maximal time interval*

(1.9) $$x \colon [0, T^*) \times M \hookrightarrow \mathbb{R}^{n+1},$$

    which satisfies the flow equation

(1.10) $$\dot{x} = \frac{1}{F^p}\nu$$
$$x(0, \xi) = x_0(\xi),$$

    where $\nu = \nu(t, \xi)$ is the outward unit normal to $M_t = x(t, M)$ at $x(t, \xi)$ and F is evaluated at the principal curvatures of M_t at $x(t, \xi)$.

 (ii) *There exists a point $Q \in \mathbb{R}^{n+1}$ and a sphere $S^* = S_{R^*}(Q)$ around Q with radius R^*, such that the spherical solutions S_t with radii R_t of (1.10) with $M_0 = S_{R^*}$ satisfy*

(1.11) $$\operatorname{dist}(M_t, S_t) \leq cR_t^{-\frac{p}{2}} \quad \forall t \in [0, T^*),$$

    $c = c(p, M_0, F)$. *Here* dist *denotes the Hausdorff distance of compact sets.*

Statement (i) is just the existence of a solution on a maximal time interval. This result is not new, holds in even more general situations and a proof can be found in [7, Thm. 2.5.19, Lemma 2.6.1]. We stated it for convenience. Also note that the statement in (ii) indeed says that the flow becomes close to a flow of spheres. This is due to the fact that the radii of spheres which satisfy (1.10) with a sphere as initial hypersurface do converge to infinity during the maximal time of existence, cf. [9, Rem. 3.1] and [6, Thm. 0.1].

Indeed, (1.11) also allows to improve the rate of convergence of $\tilde{M}_t$ by choosing the *optimal* geodesic sphere to rescale. We will not carry this out here, but refer to [17, Sec. 7] for a rough outline of the arguments involved.

Now we give a brief overview over the state of the art in classical expanding curvature flows. We leave aside the theory of contracting flows, weak solutions, flows with boundary conditions, other ambient spaces and evolving curves, due to the tremendous amount of literature, which is not of direct interest with respect to our results.

For smooth, expanding flows in Euclidean space usually, except for [16], the asymptotic behavior of the flow hypersurfaces is described via the rescaling

(1.12) $$\tilde{M}_t = \Theta^{-1} M_t,$$

where $\Theta(t)$ is the radius of the evolution of an *arbitrary* geodesic sphere, such that $\tilde{M}_t$ is bounded below and above. In those works, the authors show that $\tilde{M}_t$ converges to a sphere smoothly, which is less than (1.11) on the C^0-level.

Convergence of the rescaled surfaces $\tilde{M}_t$ was proven, for example, in the papers by Gerhardt, [6], and Urbas, [19], in the case $p = 1$ under similar assumptions on F as we do impose them. Results like these in the case $0 < p \leq 1$ and general F were derived in [20] and for more general, but still concave functions of the inverse Gauss curvature or the principal radii, compare the works by Chow and Tsai, [5] and [4] respectively, as well as [11]. Results for $p > 1$ have been accomplished for

$n = p = 2$ and $F = 2K^{\frac{1}{2}}$, the classical inverse Gauss curvature flow in $\mathbb{R}^3$, by Schnürer, [16], and for $n = 2$, $1 < p < 2$ by Li, [13]. Probably the most general existing paper on classical inverse curvature flows in $\mathbb{R}^{n+1}$ is [9]. Besides those convergence results, Smoczyk has found an explicit representation for the solution of the inverse harmonic mean curvature flow, [18].

To our knowledge, the only situation, in which statement (ii) is proven, is the case $n = p = 2$ and $F = 2K^{\frac{1}{2}}$, where K is the Gaussian curvature, cf. [16]. We are not aware of the existence of a convergence result of type (1.11) in case of the other parameters.

2. NOTATION AND DEFINITIONS

In this article we consider closed, embedded and oriented hypersurfaces $M \hookrightarrow \mathbb{R}^{n+1}$, which can be written as graphs over a sphere $\mathbb{S}^n$,

$$(2.1) \qquad\qquad M = \{(u(x), x)\colon x \in \mathbb{S}^n\}.$$

The coordinate representation $(u(x), x)$ is to be understood in polar coordinates, in which the Euclidean metric reads

$$(2.2) \qquad\qquad d\bar{s}^2 = dr^2 + r^2\sigma_{ij}dx^i dx^j \equiv dr^2 + \bar{g}_{ij}dx^i dx^j.$$

We prefer the coordinate based notation for tensors.

Note that sometimes we use the slightly ambiguous notation to write x for an element $x \in \mathbb{S}^n$, where it is to be understood as $x = (x^i)$, latin indices ranging between 1 and n, or to write x as an element $x \in \mathbb{R}^{n+1}$, where it is then to be understood as $(x^\alpha) = (x^0, (x^i))$, greek indices ranging from 0 to n. Then x^0 denotes the radial component r.

We denote the standard induced metric of $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ by (σ_{ij}). The geometric quantities of M are denoted via the following notation. The induced metric is denoted by $g = (g_{ij})$ with inverse $g^{-1} = (g^{ij})$ and the second fundamental form with respect to the inward normal is $A = (h_{ij})$. Tensor indices of tensor fields on M are always lowered or lifted via g, unless stated otherwise, e.g.

$$(2.3) \qquad\qquad h^i_j = g^{ik}h_{kj}.$$

Covariant derivatives with respect to the induced metric will simply be denoted by indices, e.g. u_i for a function $u\colon M \to \mathbb{R}$, or by a semicolon, if ambiguities are possible, e.g. $h_{ij;k}$.

The outward normal vector field to M is given by

$$(2.4) \qquad\qquad (\nu^\alpha) = v^{-1}(1, -\check{u}^i),$$

where $\check{u}^i = \bar{g}^{ik}u_k$, $(\bar{g}^{ik}) = (\bar{g}_{ik})^{-1}$ and

$$(2.5) \qquad\qquad v^2 = 1 + \bar{g}^{ij}u_i u_j \equiv 1 + |Du|^2.$$

For a tensor field $T = (t^{i_1,\dots,i_k}_{j_1,\dots,j_l})$ on M the pointwise norm $\|T\|$ is always defined with respect to the induced metric

$$(2.6) \qquad\qquad \|T\|^2 = t^{i_1,\dots,i_k}_{j_1,\dots,j_l} t^{j_1,\dots,j_l}_{i_1,\dots,i_k}.$$

A dot over a function or a tensor always denotes a total time derivative, e.g.

$$(2.7) \qquad\qquad \dot{u} = \frac{d}{dt}u,$$

whereas a prime denotes differentiation with respect to a direct argument. If for example $f = f(u)$, then

$$(2.8) \qquad f' = \frac{d}{du} f$$

and

$$(2.9) \qquad \dot{f} = f' \dot{u}.$$

Note, that this notation partially deviates from those in [7] and [9].

Curvature functions.

The formulation of our assumptions on the curvature function F has used cones in $\mathbb{R}^n$, i.e. F depends smoothly on the principal curvatures,

$$(2.10) \qquad F = F(\kappa_i).$$

However, as it is shown in [7, Ch. 2.1] and the references therein, it is also possible to consider F as a smooth function of the second fundamental form and the metric,

$$(2.11) \qquad F = F(h_{ij}, g_{ij}),$$

or, as well, as a function defined on the mixed tensor (h_j^i),

$$(2.12) \qquad F = F(h_j^i).$$

Those formulations are basically equivalent. In the formulation of evolution equations we will use the second of those three. Note, that

$$(2.13) \qquad F^{kl} = \frac{\partial F}{\partial h_{kl}}$$

defines a tensor field on M of two contravariant indices.

Evolution equations.

2.1. **Remark.** The existence of a solution to (1.10) on a maximal time interval $[0, T^*)$ is well-known. We refer to [7, Thm. 2.5.19, Lemma 2.6.1]. Furthermore, the solution x exists at least as long as the solution

$$(2.14) \qquad u \colon [0, \bar{T}) \times \mathbb{S}^n \to \mathbb{R}$$

of the scalar flow equation

$$(2.15) \qquad \begin{aligned} \frac{\partial}{\partial t} u &= \frac{v}{F^p} \\ u(0, \cdot) &= u_0, \end{aligned}$$

where u_0 is the graph representation of the initial hypersurface, also compare [7, Thm. 2.5.17] and [7, p. 98-99]. Note as well that under Assumption 1.1 we have $\bar{T} = T^*$, cf. [9, Thm. 1.1, Thm. 1.2].

For real numbers $r > 0$ define

$$(2.16) \qquad \Phi(r) = -r^{-p}.$$

The relevant evolution equations involved in the curvature flow are the following. The second fundamental form in mixed form satisfies

$$(2.17) \qquad \dot{h}_j^i - \Phi' h_{j;kl}^i = \Phi' F^{kl} h_{rk} h_l^r h_j^i - (\Phi' F - \Phi) h_k^i h_j^k + \Phi^{kl,rs} h_{kl;j} h_{rs;}{}^i,$$

cf. [7, Lemma 2.4.1]. The curvature function Φ satisfies

$$(2.18) \qquad \dot{\Phi} - \Phi' F^{kl} \Phi_{kl} = \Phi' F^{kl} h_{rk} h_l^r \Phi,$$

cf. [7, Lemma 2.3.4]. In the sequel we will need two other derived evolution equations, namely for the mean curvature $H = h_i^i$,

$$(2.19) \qquad \dot{H} - \Phi' F^{kl} H_{kl} = \Phi' F^{kl} h_{rk} h_l^r H - (\Phi' F - \Phi) \|A\|^2 + \Phi^{kl,rs} h_{kl;i} h_{rs;}{}^i$$

and for $\|A\|^2 = h_j^i h_i^j$,

$$(2.20) \qquad \frac{d}{dt} \|A\|^2 - \Phi' F^{kl} (\|A\|)_{kl} = 2\Phi' F^{kl} h_{rk} h_l^r \|A\|^2 - 2(\Phi' F - \Phi) h_k^i h_j^k h_i^j$$
$$+ 2\Phi^{kl,rs} h_{kl;i} h_{rs;}{}^j h_j^i - 2\Phi' F^{kl} h_{j;k}^i h_{i;l}^j.$$

2.2. **Remark.** For better readability of the subsequent results, we stick to the convention that whenever we claim the existence of constants, c, γ etc., they are allowed and understood to depend on n, p, M_0 and F as given data of the initial value problem, without mentioning this over and over again.

3. Pinching estimates

In this section we successively improve the pinching estimates. First we need to revisit some results from [9].

Let $\Theta = \Theta(t, r)$ denote a geodesic sphere with initial radius r, that exists as long as the solution x of (1.10) and for which

$$(3.1) \qquad \tilde{u} = u\Theta^{-1}$$

is bounded below and above by positive constants, compare [9, Lemma 3.3-3.5] and [6, (4.8)]. The following proposition holds.

3.1. **Proposition.** *Let x be the solution of (1.10) under Assumption 1.1. Then the rescaled principal curvatures*

$$(3.2) \qquad \tilde{\kappa}_i = \kappa_i \Theta$$

satisfy

$$(3.3) \qquad 0 < c^{-1} \le \tilde{\kappa}_i \le c \quad \forall t \in [0, T^*),$$

if $p > 1$, and in case $p \le 1$ the $\tilde{\kappa}_i$ stay within a compact subset of Γ.

Proof. For $p < 1$ this is [9, Lemma 4.11, Cor. 4.12] and for $p = 1$ we refer to [6, (4.9)]. In case $p > 1$ we refer to [9, Lemmata 3.10, 4.7, 4.9]. $\square$

Furthermore, we obtain decay estimates for the gradient.

3.2. **Proposition.** *Let x be the solution of (1.10) under Assumption 1.1, $0 < p < \infty$. Then there exist positive constants c and γ, such that the function*

$$(3.4) \qquad \varphi = \log u$$

satisfies the gradient estimate

$$(3.5) \qquad |D\varphi| = \sigma^{ij} \varphi_i \varphi_j \le c\Theta^{-\gamma} \quad \forall t \in [0, T^*).$$

Proof. In case $p > 1$ this holds with $\gamma = \frac{1}{2}$, cf. [9, Lemma 3.7]. In case $p = 1$ this follows from [6, Lemma 2.5]. In case $p < 1$ this follows from the formula [9, (3.45)], where one should also note [9, (3.41)], as well as the spherical growth

$$(3.6) \qquad \Theta(t, r) = \left(\frac{1-p}{n^p} t + r^{1-p} \right)^{\frac{1}{1-p}},$$

cf. [9, (3.11)]. $\square$

From these observations we deduce that the pinching of the hypersurfaces actually improves.

3.3. Proposition. *Let x be the solution of (1.10) under Assumption 1.1, $0 < p < \infty$. Then there exist positive constants c and γ, such that the principal curvatures κ_i of the flow hypersurfaces satisfy the pointwise estimate*

$$(3.7) \qquad (\kappa_i - \kappa_j)^2 \leq cH^{2+\gamma} \quad \forall t \in [0, T^*).$$

Proof. For this proof norms of tensors are formed with respect to σ_{ij}.

The function $\log \tilde{u}$ is bounded in $C^\infty(\mathbb{S}^n)$, cf. [9, Lemma 5.1, Thm. 5.2]. Thus, via interpolation, compare [8, Lemma 6.1] and Proposition 3.2, we obtain

$$(3.8) \qquad |D^2\varphi|^2 \leq c|D\varphi||\varphi|_{C^3} \leq c\Theta^{-\gamma} \quad \forall t \in [0, T^*).$$

The rescaled second fundamental form $\tilde{h}^i_j = \Theta h^i_j$ satisfies

$$(3.9) \qquad \tilde{h}^i_j = v^{-1}\tilde{u}^{-1}(\delta^i_j - (\sigma^{ik} - v^{-2}\varphi^i\varphi^k)\varphi_{kj}),$$

cf. [9, (5.2)] and [7, Lemma 2.7.6]. Thus

$$(3.10) \qquad |\tilde{h}^i_j - \lambda\delta^i_j| \leq c\Theta^{-\frac{\gamma}{2}},$$

where we used Proposition 3.2 and the fact that $\tilde{u}$ converges to some constant λ^{-1}. Rescaling backwards yields

$$(3.11) \qquad |\kappa_i - \lambda\Theta^{-1}| \leq c\Theta^{-\left(1+\frac{\gamma}{2}\right)} \quad \forall 1 \leq i \leq n,$$

hence we obtain the result in view of

$$(3.12) \qquad 0 < c^{-1} \leq \Theta H \leq c.$$

$\square$

The final pinching improvement will allow us to derive asymptotical roundness of the flow hypersurfaces.

3.4. Proposition. *Let x be the solution of (1.10) under Assumption 1.1, $0 < p < \infty$. Then for all $\delta < 4+2p$ there exists a constant $c = c_\delta > 0$, such that the principal curvatures κ_i satisfy the pointwise estimate*

$$(3.13) \qquad (\kappa_i - \kappa_j)^2 \leq cH^\delta \quad \forall t \in [0, T^*).$$

Proof. Using Proposition 3.3 it suffices to show this for $\delta > 2$. From (2.19) and (2.20) we obtain that

$$(3.14) \qquad w = \|A\|^2 - \frac{1}{n}H^2$$

satisfies

$$(3.15) \qquad \begin{aligned} \dot{w} - \Phi'F^{kl}w_{kl} = {}&2\Phi'F^{kl}h_{rk}h^r_l w - 2(\Phi'F - \Phi)\left(h^i_k h^k_j h^j_i - \frac{1}{n}\|A\|^2 H\right) \\ &+ 2\Phi^{kl,rs}h_{kl;i}h_{rs;}{}^j\left(h^i_j - \frac{1}{n}H\delta^i_j\right) \\ &- 2\Phi'F^{kl}\left(h^i_{j;k}h^j_{i;l} - \frac{1}{n}H_k H_l\right) \end{aligned}$$

and thus, for positive constants c and $\delta > 2$ yet to be chosen, we obtain the evolution equation for

$$(3.16) \qquad z = w - cH^\delta,$$

namely

$$\dot{z} - \Phi' F^{kl} z_{kl} = 2\Phi' F^{kl} h_{rk} h_l^r z - c(\delta - 2)\Phi' F^{kl} h_{rk} h_l^r H^\delta$$

$$- 2(\Phi' F - \Phi)\left(h_k^i h_j^k h_i^j - \frac{1}{n}\|A\|^2 H - \frac{c\delta}{2} H^{\delta-1}\|A\|^2\right)$$

(3.17)
$$+ 2\Phi^{kl,rs} h_{kl;i} h_{rs;}^{\ j}\left(h_j^i - \frac{1}{n} H\delta_j^i - \frac{c\delta}{2} H^{\delta-1}\delta_j^i\right)$$

$$- 2\Phi' F^{kl}\left(h_{j;k}^i h_{i;l}^j - \frac{1}{n} H_k H_l\right)$$

$$+ c\delta(\delta - 1) H^{\delta-2}\Phi' F^{kl} H_k H_l.$$

From Proposition 3.3 we find $0 < t_0 < T^*$, such that

(3.18)
$$\max\left(\frac{w}{H^2}, \kappa_n\right) < \sigma := \frac{1}{\delta^\alpha n(n-1)} \quad \forall t \in [t_0, T^*),$$

where $\alpha_\delta >> 1$ will be chosen appropriately later. Thus t_0 will only depend on δ, p and M_0 as well and thus we may define

(3.19)
$$c = \frac{\sigma}{\inf_{M_{t_0}} H^{\delta-2}}.$$

Note that due to [2, Lemma 2.2] the M_t are strictly convex for $t \geq t_0$. On M_{t_0} we have

(3.20)
$$z = w - cH^\delta = H^2\left(\frac{w}{H^2} - cH^{\delta-2}\right) < 0.$$

We wish to show that this remains valid up to T^*. Thus suppose $t_1 > t_0$ to be the first time, such that there exists $\xi_1 \in M_{t_1}$ with the property

(3.21)
$$z(t_1, \xi_1) = 0.$$

Define

(3.22)
$$\epsilon := cH^{\delta-2}(t_1, \xi_1),$$

then there holds

(3.23)
$$0 < \epsilon < \sigma,$$

due to (3.18). From (3.17) we obtain at (t_1, ξ_1) that

$$0 \leq -c(\delta - 2)\Phi' F^{kl} h_{rk} h_l^r H^\delta$$

$$- 2(\Phi' F - \Phi)\left(h_k^i h_j^k h_i^j - \frac{1}{n}\|A\|^2 H - \frac{c\delta}{2} H^{\delta-1}\|A\|^2\right)$$

(3.24)
$$+ 2\Phi^{kl,rs} h_{kl;i} h_{rs;}^{\ j}\left(h_j^i - \frac{1}{n} H\delta_j^i - \frac{c\delta}{2} H^{\delta-1}\delta_j^i\right)$$

$$- 2\Phi' F^{kl}\left(h_{j;k}^i h_{i;l}^j - \frac{1}{n} H_k H_l\right) + c\delta(\delta - 1) H^{\delta-2}\Phi' F^{kl} H_k H_l.$$

From [2, Lemma 2.3] we have at (t_1, ξ_1), note that $w = \epsilon H^2$,

(3.25)
$$h_k^i h_j^k h_i^j - \left(\frac{1}{n} + \epsilon\right) H\|A\|^2 \geq \epsilon\left(\frac{1}{n} + \epsilon\right)(1 - \sqrt{n(n-1)\epsilon})H^3$$

$$= \epsilon\left(1 - \sqrt{n(n-1)\epsilon}\right)\|A\|^2 H$$

$$> 0$$

and we have

$$g^{kl}\left(h^i_{j;k}h^j_{i;l} - \frac{1}{n}H_k H_l\right) = \left\|D\left(A - \frac{1}{n}Hg\right)\right\|^2$$

(3.26)
$$\geq \frac{2(n-1)}{3n}\|DA\|^2$$

$$\geq \frac{2(n-1)}{n(n+2)}\|DH\|^2,$$

cf. [2, Lemma 2.1]. In view of the concavity of F we have $F \leq H$, [7, Lemma 2.2.20], and thus from (3.24) we obtain

(3.27)
$$0 \leq -2(\Phi' F - \Phi)\left(\epsilon\left(2 - \sqrt{n(n-1)\epsilon}\right)\|A\|^2 H - \frac{\epsilon\delta}{2}\|A\|^2 H\right)$$
$$- \epsilon(\delta - 2)\frac{p}{p+1}(\Phi' F - \Phi)\|A\|^2 H$$
$$- \epsilon(\delta - 2)\frac{p}{p+1}(\Phi' F - \Phi)(F^{kl} - g^{kl})h_{rk}h^r_l H$$
$$+ 2H\Phi^{kl,rs}h_{kl;i}h_{rs;}{}^j\left(H^{-1}h^i_j - \frac{1}{n}\delta^i_j - \frac{\epsilon\delta}{2}\delta^i_j\right) - \frac{2(n-1)}{n(n+2)}\Phi'\|DH\|^2$$
$$- \frac{2(n-1)}{3n}\Phi'\|DA\|^2 - 2\Phi'(F^{kl} - g^{kl})\left(h^i_{j;k}h^j_{i;l} - \frac{1}{n}H_k H_l\right)$$
$$+ \epsilon\delta(\delta - 1)\Phi'\|DH\|^2 + \epsilon\delta(\delta - 1)\Phi'(F^{kl} - g^{kl})H_k H_l.$$

In view of (3.18) and due to to the fact that

(3.28)
$$\|F^{kl} - g^{kl}\| \to 0,$$

we may without loss of generality enlarge t_0 and α, such that the terms involving curvature derivatives are absorbed by the terms

(3.29)
$$-\frac{2(n-1)}{n(n+2)}\Phi'\|DH\|^2 \text{ and } -\frac{2(n-1)}{3n}\Phi'\|DA\|^2.$$

In this case at (t_1, ξ_1) we obtain

(3.30)
$$0 \leq -2(\Phi' F - \Phi)\epsilon\|A\|^2 H\left(\frac{p+2}{p+1} - \delta^{-\frac{\alpha}{2}} - \frac{\delta}{2(p+1)}\right)$$
$$- \epsilon(\delta - 2)\frac{p}{p+1}(\Phi' F - \Phi)(F^{kl} - g^{kl})h_{rk}h^r_l H$$
$$< 0,$$

if we choose

(3.31)
$$\delta < 4 + 2p,$$

$\alpha = \alpha(p, \delta)$ large enough and again t_0 larger to ensure that the term involving $\|F^{kl} - g^{kl}\|$ can be absorbed by the strictly negative, remaining part of the first line in (3.30). This contradiction shows that z will remain negative up to the time T^*, which yields the result. $\qquad\square$

4. OSCILLATION DECAY

Let $\rho_+(t)$ denote the *circumradius* of M_t, i.e. the radius of the smallest ball in $\mathbb{R}^{n+1}$ enclosing M_t and analogously $\rho_-(t)$ the *inradius* of M_t, the radius of the largest ball in $\mathbb{R}^{n+1}$ enclosed by M_t. In this section we prove (ii) of Theorem 1.2, namely

that $\rho_+ - \rho_-$ converges to zero and that we find an expanding family of geodesic spheres S_t with radii R_t the flow hypersurfaces M_t fit themselves to,

$$(4.1) \qquad \operatorname{dist}(M_t, S_t) < cR_t^{-\frac{p}{2}}.$$

Note that the R_t represent a suitable choice of the $\Theta(t)$.

Let $\hat{M}_t$ denote the convex body enclosed by M_t, which is well defined for large t. For an interior point $y \in \operatorname{int}(M_t)$ let u_y denote the graph representation of M_t over the standard sphere centered in y,

$$(4.2) \qquad M_t = \{(u_y(x), x) \colon x \in \mathbb{S}^n(y)\}.$$

4.1. Proposition. *Let x be the solution of (1.10) under Assumption 1.1, $0 < p < \infty$. Then there holds*

$$(4.3) \qquad \rho_+(t) - \rho_-(t) \leq \operatorname{osc} u_{y_t} \leq c\Theta^{-\frac{p}{2}}(t) \quad \forall t \in [0, T^*),$$

where y_t is a suitable oscillation minimizing center of $\hat{M}_t$.

Proof. First note that Θ is still an arbitrary rescaling factor as in Proposition 3.1.

From (3.13) we deduce that principal curvatures satisfy

$$(4.4) \qquad (\kappa_i - \kappa_j)^2 \leq cH^\delta \leq c\Theta^{-\delta}$$

for $\delta = 4 + p$. Since for large t the M_t are strictly convex we may consider the difference of the largest and smallest principal radius of curvature,

$$(4.5) \qquad \left| \frac{1}{\kappa_1} - \frac{1}{\kappa_n} \right| = \left| \frac{\kappa_n - \kappa_1}{\kappa_1 \kappa_n} \right| \leq c\Theta^{2 - \frac{\delta}{2}}.$$

Applying [12, Thm. 1.4] we obtain

$$(4.6) \qquad \operatorname{dist}(M_t, S(y_t)) \leq c\Theta^{2 - \frac{\delta}{2}},$$

from which the claim follows. Note that if a y_t happens not to minimize the oscillation, we can adjust it to do so. A simple geometric argument also shows that then the y_t must lie in the convex body $\hat{M}_t$. $\qquad \square$

In order to find an optimally fitted spherical flow, we need the centers y_t from Proposition 4.1 to converge.

4.2. Lemma. *The centers y_t from Proposition 4.1 converge in $\mathbb{R}^{n+1}$. In particular we have*

$$(4.7) \qquad |y_t - Q| \leq c\Theta^{-\frac{p}{2}}(t)$$

for the limit $Q \in \mathbb{R}^{n+1}$.

Proof. For a point $q \in \mathbb{R}^{n+1}$ let

$$(4.8) \qquad \bar{u}_q = \langle x - q, \nu \rangle$$

denote the support function with respect to q. For t close to T^* the M_t are strictly convex and hence we may apply a gradient estimate for convex hypersurfaces, [7, Lemma 2.7.10], to conclude

$$(4.9) \qquad v \leq e^{\bar{\kappa} \operatorname{osc} u_{y_t}},$$

where $\bar{\kappa}$ is an upper for the principal curvatures of the slices $\{x^0 = \operatorname{const}\}$, which in our case can be estimated:

$$(4.10) \qquad \bar{\kappa} \leq \max_{M_t} \frac{1}{u_{y_t}}.$$

Thus

(4.11)
$$v - 1 \leq e^{c\bar{\kappa}\Theta^{-\frac{p}{2}}} - 1 \leq c\bar{\kappa}\Theta^{-\frac{p}{2}}.$$

Since

(4.12)
$$\frac{u_{y_t}}{v} = \bar{u}_{y_t},$$

we obtain

(4.13)
$$|\bar{u}_{y_t} - u_{y_t}| \leq u_{y_t}\frac{v-1}{v} \leq c\Theta^{-\frac{p}{2}}$$

and hence

(4.14)
$$\max \bar{u}_{y_t} - \min \bar{u}_{y_t} \leq \max u_{y_t} - \min u_{y_t} + c\Theta^{-\frac{p}{2}} \leq c\Theta^{-\frac{p}{2}}(t).$$

Using that the oscillation of the support function with respect to a fixed point is decreasing, cf. [14, Thm. 3.1], originally proved by Chow and Gulliver, [3], we obtain

(4.15)
$$\operatorname{osc} u_{y_{t_2}} \leq \operatorname{osc} \bar{u}_{y_{t_2}} \leq \operatorname{osc} \bar{u}_{y_{t_1}} \leq c\Theta^{-\frac{p}{2}}(t_1) \quad \forall t_2 > t_1$$

and since y_{t_2} is oscillation minimizing we must have

(4.16)
$$|y_{t_1} - y_{t_2}| \leq c\Theta^{-\frac{p}{2}}(t_1) \quad \forall t_2 > t_1.$$

Letting $t_1 \to T^*$ we obtain the limit Q and then for fixed t_1 letting $t_2 \to T^*$, we obtain the desired estimate. $\qquad\square$

We finish this paper by showing that the flow actually becomes close to a flow of spheres.

4.3. **Theorem.** *Let x be the solution of (1.10) under Assumption 1.1. Then there exists a geodesic sphere $S_{R^*}(Q)$, where Q is the limit from Lemma 4.2, such that the spherical leaves S_t of the initial value problem*

(4.17)
$$\dot{y} = \frac{1}{F^p}\nu$$
$$y(0, M) = S_{R^*}(Q)$$

and the flow hypersurfaces M_t of the flow x satisfy

(4.18)
$$\operatorname{dist}(M_t, S_t) < cR_t^{-\frac{p}{2}},$$

where R_t is the radius of S_t.

Proof. In case $p > 1$, R^* is determined by the requirement, that the spherical flow exists as long as the flow x. Then for all large t we must have

(4.19)
$$S_t \cap M_t \neq \emptyset,$$

compare the arguments at the end of the proof of [16, Lemma 5.1]. The result follows from the propositions 4.1 and 4.2.

In case $p \leq 1$ we need a different argument to show that there exists an expanding flow of spheres with the property (4.19), since we have $T^* = \infty$ and R^* is not determined clearly. Recall (3.6), that for initial radius r the radius R of a sphere evolves according to

(4.20)
$$R(t, r) = \left(\frac{1-p}{n^p}t + r^{1-p}\right)^{\frac{1}{1-p}},$$

if $p < 1$ and

(4.21)
$$R(t, r) = re^{\frac{t}{n}},$$

if $p = 1$. We see that in any case

$$(4.22) \qquad\qquad R_t = R(t, \cdot)$$

is an increasing diffeomorphism from $(0, \infty)$ onto its image $\left(\left(\frac{1-p}{n^p}t\right)^{\frac{1}{1-p}}, \infty\right)$ in case $p < 1$ and onto $(0, \infty)$ in case $p = 1$. Now let $u = u_Q$ and define sequences

$$(4.23) \qquad\qquad \bar{R}^k = R_k^{-1}(\sup u(k, \cdot)),$$

$$(4.24) \qquad\qquad \bar{R}_k = R_k^{-1}(\inf u(k, \cdot))$$

and

$$(4.25) \qquad\qquad R^k = \frac{1}{2}(\bar{R}^k + \bar{R}_k).$$

By the maximum principle $\bar{R}^k$ is non-increasing. There holds

$$(4.26) \qquad\qquad S_k^k \cap M_k \neq \emptyset,$$

where S_k^k denotes the spherical leave at time k, which has started with initial radius R^k. Since in case $p < 1$

$$(4.27) \qquad\qquad \frac{d}{dr} R_k(r) = \left(\frac{1-p}{n^p} k + r^{1-p}\right)^{\frac{p}{1-p}} r^{-p}$$

and in case $p = 1$

$$(4.28) \qquad\qquad \frac{d}{dr} R_k(r) = e^{\frac{k}{n}},$$

$\frac{d}{dr} R_k$ is uniformly bounded from below and since

$$(4.29) \qquad\qquad \operatorname{osc} u \to 0,$$

we obtain that $\bar{R}^k$, $\bar{R}_k$ and R^k all converge to the same limit R^*. We claim, that the initial sphere S_{R^*} around Q leads to a spherical flow satisfying (4.19).

Otherwise there existed a time k_0, such that without loss of generality

$$(4.30) \qquad\qquad R(k_0, R^*) < \inf u(k_0, \cdot).$$

By continuity of ODE orbits with respect to initial values on compact intervals, there is $\tilde{R} > R^*$, such that

$$(4.31) \qquad\qquad R(k, R^*) < R(k, \tilde{R}) < \inf u(k, \cdot) \quad \forall k \geq k_0,$$

where we also used the maximum principle. Applying R_k^{-1} we find

$$(4.32) \qquad\qquad R^* < \tilde{R} < \bar{R}_k \quad \forall k \geq k_0,$$

since R_k^{-1} is increasing with respect to r. This is a contradiction to $\bar{R}_k \to R^*$. So the spherical leaves of the flow with initial value $S_{R^*}(Q)$ intersect the M_t for all large times and due to the oscillation estimates we obtain the desired result. $\qquad\square$

5. Concluding remarks

The pinching estimates, Proposition 3.4, turned out to improve, whenever p becomes larger. This fact is somehow surprising, since the evolution equation of the gradient function

$$(5.1) \qquad\qquad v^2 = 1 + |Du|^2$$

does not allow to apply the classical maximum principle, also compare the proof of [9, Lemma 3.6], so one could expect that it should become harder to control oscillations. As we have seen, however, the equation for the traceless second fundamental form serves as a way out.

Also note that in a further work we applied a method similar to the one in section 4 to prove that there can not be an estimate of the form

$$(5.2) \qquad \operatorname{dist}(M, S_R) \le c\|\mathring{A}\|^{\alpha}, \quad \alpha > 1,$$

in the class of uniformly convex hypersurfaces with a universal constant. The idea is that otherwise we could use a similar proof as in section 4 to prove asymptotical roundness of the inverse mean curvature flow in the hyperbolic space, which has shown not to be true in [10]. See [15] for a preprint version and a detailed description of this result.

References

1. Ben Andrews, *Gauss curvature flow: the fate of the rolling stones*, Invent. Math. **138** (1999), no. 1, 151–161.
2. Ben Andrews and James McCoy, *Convex hypersurfaces with pinched principal curvatures and flow of convex hypersurfaces by high powers of curvature*, Trans. Am. Math. Soc. **364** (2012), no. 7, 3427–3447.
3. Bennett Chow and Robert Gulliver, *Aleksandrov reflection and nonlinear evolution equations, i: The n-sphere and n-ball*, Calc. Var. Partial Differ. Equ. **4** (1996), no. 3, 249–264.
4. Bennett Chow and Dong-Ho Tsai, *Expansion of convex hypersurfaces by nonhomogeneous functions of curvature*, Asian J. Math. **1** (1997), no. 4, 769–784.
5. _____, *Nonhomogeneous Gauss curvature flows*, Indiana Univ. Math. J. **47** (1998), no. 3, 965–994.
6. Claus Gerhardt, *Flow of nonconvex hypersurfaces into spheres*, J. Differ. Geom. **32** (1990), no. 1, 299–314.
7. _____, *Curvature problems*, Series in Geometry and Topology, vol. 39, International Press of Boston Inc., 2006.
8. _____, *Inverse curvature flows in hyperbolic space*, J. Differ. Geom. **89** (2011), no. 3, 487–527.
9. _____, *Non-scale-invariant inverse curvature flows in Euclidean space*, Calc. Var. Partial Differ. Equ. **49** (2014), no. 1-2, 471–489.
10. Pei-Ken Hung and Mu Tao Wang, *Inverse mean curvature flows in the hyperbolic 3-space revisited*, Calc. Var. Partial Differ. Equ. (2014), DOI:10.1007/s00526-014-0780-3.
11. Nina Ivochkina, Thomas Nehring, and Friedrich Tomi, *Evolution of starshaped hypersurfaces by nonhomogeneous curvature functions*, Algebra i Anal. **12** (2000), no. 1, 185–203.
12. Kurt Leichtweiß, *Nearly umbilical ovaloids in the n-space are close to spheres*, Result. Math. **36** (1999), no. 1-2, 102–109.
13. Qi-Rui Li, *Surfaces expanding by the power of the Gauss curvature flow*, Proc. Amer. Math. Soc. **138** (2010), no. 11, 4089–4102.
14. James McCoy, *The surface area preserving mean curvature flow*, Asian J. Math. **7** (2003), no. 1, 7–30.
15. Julien Roth and Julian Scheuer, *Explicit rigidity of almost-umbilical hypersurfaces*, preprint available at arxiv:1504.05749, 2015.
16. Oliver C. Schnürer, *Surfaces expanding by the inverse Gauß curvature flow*, J. Reine Angew. Math. **600** (2006), 117–134.
17. _____, *Surfaces expanding by the inverse Gauss curvature flow*, 2006, arxiv:0412297v2.
18. Knut Smoczyk, *A representation formula for the inverse harmonic mean curvature flow*, Elem. Math. **60** (2005), no. 2, 57–65.
19. John Urbas, *On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures*, Math. Z. **205** (1990), no. 1, 355–372.
20. _____, *An expansion of convex hypersurfaces*, J. Differ. Geom. **33** (1991), no. 1, 91–125.

Dr. Julian Scheuer, Albert-Ludwigs-Universität, Mathematisches Institut, Abteilung für Reine Mathematik, Eckerstr. 1, 79104 Freiburg, Germany

E-mail address: julian.scheuer@math.uni-freiburg.de