A GEOMETRIC INEQUALITY FOR CONVEX FREE BOUNDARY HYPERSURFACES IN THE UNIT BALL

BEN LAMBERT AND JULIAN SCHEUER

Abstract. We use the inverse mean curvature flow with a free boundary perpendicular to the sphere to prove a geometric inequality involving the Willmore energy for convex hypersurfaces of dimension $n \geq 3$ with boundary on the sphere.

1. Introduction

In [16] we considered the inverse mean curvature flow (IMCF) perpendicular to the sphere, namely a family of embeddings

$$X\colon \mathbb{D} \times [0, T^*) \to \mathbb{R}^{n+1}, \tag{1.1}$$

where $\mathbb{D}$ denotes the n-dimensional unit disk, which satisfy the Neumann boundary value problem

$$\dot{X} = \frac{1}{H} N, \tag{1.2a}$$

$$X(\partial\mathbb{D}) = \partial X(\mathbb{D}) \subset \mathbb{S}^n, \tag{1.2b}$$

$$0 = \left\langle N_{|\partial\mathbb{D}}, \tilde{N}(X_{|\partial\mathbb{D}}) \right\rangle, \tag{1.2c}$$

$$\left\langle \dot{\gamma}(0), \tilde{N} \right\rangle \geq 0 \quad \forall \gamma \in C^1((-\epsilon, 0], M_t)\colon \gamma(0) \in \partial X(\mathbb{D}) \tag{1.2d}$$

with initial embedding X_0 of a strictly convex hypersurface M_0, also satisfying the conditions (1.2b), (1.2c) and (1.2d). Here $\tilde{N}$ denotes the outward unit normal of $\mathbb{S}^n$. In the following we will refer to these three conditions by saying that M_0 is *perpendicular to the sphere from the inside.*

In [16, Thm. 1] we proved that (1.2) with strictly convex initial data exists smoothly up to a maximal time T^*, preserves the strict convexity as well as the perpendicularity condition up to T^* and that T^* is characterised by the $C^{1,\alpha}$-convergence of the embeddings $X(t, \cdot)$ to the embedding of a flat disk bisecting the unit ball, where $\alpha < 1$ is arbitrary; also compare [16, Rem. 1]. Note that the proof of this convergence result heavily depends on the assumption of strict convexity for the initial embedding X_0. This is due to the fact that we obtained the final flat limiting shape at time T^* by applying a rigidity result for weakly convex bodies in the sphere $\mathbb{S}^n$, which was deduced in [19]. In order to arrive at a situation where this rigidity result holds, we needed the strict convexity of X_0. We are not aware of a proof which avoids this assumption and in fact it is an interesting open problem

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to obtain convergence results for the IMCF perpendicular to the sphere under the assumption of initial mean-convexity, rather than strict convexity.

However, the object of this paper is different, namely to apply this convergence result to prove a Li-Yau type inequality (cf. [18]) for convex hypersurfaces with boundary in any dimension $n \geq 3$.

1.1. **Theorem.** *Let $n \geq 3$ and $M^n \subset \mathbb{R}^{n+1}$ be a smoothly embedded n-disk, such that M^n is a convex hypersurface perpendicular to $\mathbb{S}^n$ from the inside. Then there holds*

$$\frac{1}{2}|M|^{\frac{2-n}{n}}\int_M H^2 + \omega_n^{\frac{2-n}{n}}|\partial M| \geq \omega_n^{\frac{2-n}{n}}|\mathbb{S}^{n-1}| \tag{1.3}$$

and equality holds if and only if M is a perpendicularly intersecting hyperplane.

Here $|\cdot|$ denotes the respective surface measures of M, ∂M and $\mathbb{S}^{n-1}$ as inherited from $\mathbb{R}^{n+1}$ and ω_n is the volume of the n-dimensional unit ball. We call a hypersurface M *convex*, if there exists a choice of a unit normal vector field, such that all the principal curvatures at any point are non-negative and *strictly convex*, if they are all positive throughout M. Note that convex or strictly convex hypersurfaces with boundary may be way more complicated than in the boundaryless case. In particular the well known supporting hyperplane property in the boundaryless case is not valid without further assumptions if M has nonempty boundary, compare for example the nice treatment of these issues in [11].

In the case of surfaces, $n = 2$, inequalities similar to (1.3) have attracted a lot of attention. In this situation an even sharper version of (1.3) was shown in broader generality than in the restricted class of convex surfaces, and was even demonstrated in higher codimension. Namely, replacing the leading factor $1/2$ in (1.3) by $1/4$, Volkmann proved the inequality without the convexity assumption in [24]. In the case of higher dimensions less is known, let us only mention a result by Brendle on minimal surfaces, [1]. We refer to the extensive bibliography in [23] for a broader overview over the topic. To our knowledge, the inequality (1.3) has not previously been treated in the higher dimensional hypersurface case.

Let us discuss the well established method of proof of geometric inequalities as in (1.3) using curvature flows. (1.3) makes a statement about a certain class of hypersurfaces M, here smooth and convex ones. To prove this inequality with the help of a specific curvature flow, three things have to be satisfied: First of all M must be an admissible initial hypersurface for the flow, i.e. one has short-time existence with sufficient regularity up to M. Then one has to show that the functional Q, here the left hand side of (1.3), is monotone during the evolution of the flow. Finally we need a convergence result for the flow to a limiting shape in a sufficiently smooth manner. Then we deduce the desired inequality due to the monotonicity of the functional, which yields

$$Q(M) = Q(0) \geq Q(\text{limiting shape}). \tag{1.4}$$

Using this strategy, several geometric inequalities which might have or have not previously been known for convex hypersurfaces could be generalised to a broader class. For example the well known Minkowski inequality for closed convex surfaces in $\mathbb{R}^3$,

$$\frac{1}{\sqrt{|M|}}\int_M H \geq 4\sqrt{\pi} \tag{1.5}$$

with equality if and only if M is a round sphere, was generalised to closed, star-shaped and mean-convex surfaces in [12]. This was possible since the inverse mean curvature flow in $\mathbb{R}^{n+1}$ allows such more general hypersurfaces as initial data and the left hand side of (1.5) is decreasing under this flow and constant if and only if it is a flow of spheres. The relevant convergence result for the IMCF in $\mathbb{R}^{n+1}$ was established independently by Gerhardt in [7] and Urbas in [22]. They show that for such initial hypersurfaces the flow expands to infinity and to a round sphere after rescaling. Due to the scale-invariance of the left hand side of (1.5) the Minkowski inequality follows. Note that once the convergence result is settled, the proof of the inequality is incredibly easy. The same method, also using other flows than the IMCF, was successfully used to prove various kinds of geometric inequalities such as those of Alexandrov-Fenchel type and inequalities for quermassintegrals of convex bodies. Compare [15] for a related inequality in $\mathbb{R}^{n+1}$, [19] and [25] for Alexandrov-Fenchel inequalities in the sphere, [3], [5], [17] and [25] for related results in the hyperbolic space, as well as [2] and [6] in other Riemannian manifolds. The relevant convergence result for the flow in hyperbolic space was established by Gerhardt in [9]. The IMCF in the sphere was treated by Gerhardt in [10] and with different methods by Makowski and the second author in [19]. The probably most famous result in this direction is the proof of the Riemannian Penrose inequality in [14] by Huisken and Ilmanen, which additionally faced the difficulty of singularity formation under the evolution. They overcame this by using the weak notion of IMCF.

In the proof of our proposed inequality (1.3) we try to adapt this method. However, a thorough look at the statement of the theorem and the flow result reveals that the flow result is not available for the whole class of hypersurfaces for which we want to prove the inequality. Namely the flow result requires strict convexity while the inequality is supposed to hold for convex hypersurfaces. This becomes most obvious when looking at the limiting case: A flat disk is certainly a singularity for IMCF and hence there is now way to start the IMCF from it. This introduces an additional complication. The standard proof only works for strictly convex hypersurfaces, the case of which we will treat in section 2. We will resolve the general issue using approximation by strictly convex hypersurfaces. Here the main technical difficulty is that we need an approximation which preserves the perpendicularity to the sphere at boundary points. Fortunately the mean curvature flow serves as a way out, as we will see in section 3. In section 4 we put everything together for the final proof.

We remark that with an improved result on IMCF perpendicular to the sphere which is valid for more general initial hypersurfaces, (1.3) should also be generalisable away from the convex setting.

2. The case of strictly convex hypersurfaces

In order to prove Theorem 1.1 in the strictly convex case we will use the strategy described in the introduction to show that the left hand side of (1.3) is decreasing under the flow and then use the convergence result for the flow to show that it limits into the right hand side of (1.3). Let us first recollect some essential facts proven in [16] which we need in this section.

2.1. *Remark.* (i) In [16] we proved that the IMCF perpendicular to the sphere drives strictly convex initial hypersurfaces M_0 in finite time T^* to a flat disk in

$C^{1,\alpha}$. Hence the boundaries $\partial M_t \subset \mathbb{S}^n$ are driven uniformly to an equator $\mathcal{S}$. Let $\mathcal{H}(e_0)$ be the closed hemisphere with center $e_0 \in \mathbb{S}^n$ that contains all the ∂M_t. Then for t close enough to T^* we have

$$\operatorname{dist}(e_0, \partial M_t) \geq c \geq 0 \tag{2.1}$$

and hence by the result in [16, Lemma 11] we have

$$\langle N, e_0 \rangle \leq c_0 < 0 \tag{2.2}$$

for t sufficiently close to T^*. Here $-N$ denotes the unit normal field with respect to which the flow hypersurfaces are strictly convex.

(ii) The next crucial fact, previously applied to give $C^{1,\alpha}$-convergence in [16], is a bound on the principal curvatures (and hence a C^2-estimate). Due to the convexity this is equivalent to a bound on the mean curvature, which was obtained by a standard maximum principle argument: The interior evolution of the negative speed

$$\Phi = -\frac{1}{H} \tag{2.3}$$

along IMCF is given by

$$\dot{\Phi} - \frac{1}{H^2}\Delta\Phi = \frac{\|A\|^2}{H^2}\Phi, \tag{2.4}$$

where Δ is the Laplace-Beltrami operator of the induced metric and $\|A\|$ is the norm of the second fundamental form, cf. [8, Lemma 2.3.4]. Thus the mean curvature satisfies

$$\dot{H} - \frac{1}{H^2}\Delta H \leq -\frac{\|A\|^2}{H^2}H. \tag{2.5}$$

The boundary derivative in our case is

$$\left\langle \nabla H, \tilde{N} \right\rangle = -H, \tag{2.6}$$

compare [16, Lemma 1]. The parabolic maximum principle yields

$$H \leq \max_{M_0} H \tag{2.7}$$

for all times $t < T^*$.

(iii) We will also need that the height

$$w = \langle X, e_0 \rangle \tag{2.8}$$

satisfies

$$\Delta w = -H \langle N, e_0 \rangle \geq -c_0 H, \tag{2.9}$$

where the equality is due to the Gaussian formula and the inequality holds due to (2.2). At the boundary the height function satisfies

$$\left\langle \nabla w, \tilde{N} \right\rangle = w, \tag{2.10}$$

cf. [16, Lemma 5] for a proof.

(iv) The embeddings $X(t, \cdot)\colon \mathbb{D} \to \mathbb{R}^{n+1}$ restrict to embeddings

$$y_t \colon \partial\mathbb{D} \to \mathbb{S}^n. \tag{2.11}$$

As the embeddings $X(t, \cdot)$ give strictly convex hypersurfaces, the y_t yield strictly convex hypersurfaces of the sphere $\mathbb{S}^n$, cf. [16, Lemma 4] for the simple proof. Since the flow of $X(t,\cdot)$ is smooth up to the boundary by standard regularity theory, the y_t themselves satisfy a curvature flow equation in the sphere, namely

$$\dot{y} = \frac{1}{H}\nu, \tag{2.12}$$

where H is the full mean curvature of M_t restricted to $\partial\mathbb{D}$ and ν is the pullback of the normal N along the embedding $x\colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$, cf. [16, equ. (20)] for a detailed derivation.

We can now prove the monotonicity of the curvature functional. For this purpose we need control on the L^2-norm of H.

2.2. **Lemma.** *Let the family (M_t) of strictly convex hypersurfaces evolve by (1.2). Then for all $1 \leq p < \infty$ there holds*

$$\lim_{t\to T^*}\int_{M_t} H^p(\cdot,t) = 0. \tag{2.13}$$

Proof. Combine (2.2), (2.9) and (2.10) to deduce

$$\int_{M_t} H \leq -c_0^{-1}\int_{M_t}\Delta w = -c_0^{-1}\int_{\partial M_t} w \to 0, \quad t\to T^*, \tag{2.14}$$

where the latter convergence follows since the boundaries $\partial M_t \subset \mathbb{S}^n$ converge to the equator in C^1. The complete result follows due to the boundedness of H, (2.7), and interpolation. $\square$

Now we can prove Theorem 1.1 in the special case of a strictly convex hypersurface, which will also be needed in the proof of the limiting case.

2.3. **Lemma.** *Let $n \geq 2$ and $M \subset \mathbb{R}^{n+1}$ be a smooth and strictly convex hypersurface perpendicular to $\mathbb{S}^n$ from the inside. Then there holds*

$$\frac{1}{2}|M|^{\frac{2-n}{n}}\int_M H^2 + \omega_n^{\frac{2-n}{n}}|\partial M| > \omega_n^{\frac{2-n}{n}}|\mathbb{S}^{n-1}|. \tag{2.15}$$

Proof. Rewriting (2.4) gives

$$\dot{H} = \Delta\left(-\frac{1}{H}\right) - \frac{\|A\|^2}{H} \tag{2.16}$$

and [8, Lemma 2.3.1] yields the evolution of the volume element

$$\frac{d}{dt}\sqrt{\det(g_{ij})} = \sqrt{\det(g_{ij})}. \tag{2.17}$$

Thus

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_{M_t} H^2 d\mu_t\right) &= \int_{M_t} H\Delta\left(-\frac{1}{H}\right)d\mu_t - \int_{M_t}\|A\|^2 d\mu_t + \frac{1}{2}\int_{M_t} H^2 d\mu_t\\ &= -\int_{M_t}\frac{\|\nabla H\|^2}{H^2}d\mu_t - \int_{M_t}\|A\|^2 d\mu_t + \frac{1}{2}\int_{M_t} H^2 d\mu_t\\ &\quad - |\partial M_t|,\end{aligned} \tag{2.18}$$

where we used the divergence theorem and (2.6). Since

$$\|A\|^2 = \|\mathring{A}\|^2 + \frac{1}{n}H^2, \tag{2.19}$$

we have

$$\frac{1}{2}H^2 - \|A\|^2 = \frac{n-2}{2n}H^2 - \|\mathring{A}\|^2 \tag{2.20}$$

and thus

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_{M_t} H^2 d\mu_t\right) &= -\int_{M_t} \frac{\|\nabla H\|^2}{H^2} d\mu_t - \int_{M_t} \|\mathring{A}\|^2 d\mu_t \\ &\quad + \frac{n-2}{2n}\int_{M_t} H^2 d\mu_t - |\partial M_t|.\end{aligned} \tag{2.21}$$

Furthermore, due to (2.12) the volume elements of the induced hypersurfaces

$$y_t \colon \partial\mathbb{D} \to \mathbb{S}^n \tag{2.22}$$

satisfy

$$\frac{d}{dt}\sqrt{\det(\gamma_{IJ})} = \frac{\gamma^{IJ}\eta_{IJ}}{H}\sqrt{\det(\gamma_{IJ})} < \sqrt{\det(\gamma_{IJ})}, \tag{2.23}$$

where γ_{IJ} and η_{IJ} denotes the metric and the second fundamental form of these hypersurfaces respectively. Define

$$Q(t) = \frac{1}{2}|M_t|^{\frac{2-n}{n}}\int_{M_t} H^2 + \omega_n^{\frac{2-n}{n}}|\partial M_t|. \tag{2.24}$$

By the previous calculations we have

$$\begin{aligned}\dot{Q}(t) &< \frac{2-n}{2n}|M_t|^{\frac{2-n}{n}}\int_{M_t} H^2 + \frac{n-2}{2n}|M_t|^{\frac{2-n}{n}}\int_{M_t} H^2 - |M_t|^{\frac{2-n}{n}}|\partial M_t| \\ &\quad + \omega_n^{\frac{2-n}{n}}|\partial M_t| \\ &= \left(\omega_n^{\frac{2-n}{n}} - |M_t|^{\frac{2-n}{n}}\right)|\partial M_t| \\ &\le 0,\end{aligned} \tag{2.25}$$

since we already know by [16, Thm. 1] that $|M_t|$ is increasingly converging to ω_n. Furthermore we know by Lemma 2.2 that

$$\int_{M_t} H^2 \to 0 \tag{2.26}$$

and thus we obtain

$$Q(0) > Q(T^*) = \omega_n^{\frac{2-n}{n}}|\mathbb{S}^{n-1}|. \tag{2.27}$$

□

We also need the following exact description of the maximal time of existence of a smooth solution to (1.2).

2.4. **Lemma** (Exact existence time)**.** *Suppose the initial data M_0 to (1.2) is strictly convex. Then the maximal time of existence T^* is*

$$T^* = \log\left(\frac{\omega_n}{|M_0|}\right). \tag{2.28}$$

In particular we obtain the volume estimate

$$|M_0| < \omega_n. \tag{2.29}$$

Proof. Using (2.17), we see that $\frac{d}{dt}|M_t| = |M_t|$ and so

$$|M_t| = e^t |M_0|. \tag{2.30}$$

Since we know that the maximal time is when the flow becomes a flat disk and the flow converges in $C^{1,\beta}$, we know $\omega_n = e^{T^*}|M_0|$ and the equation follows. $\square$

3. Approximation of weakly convex hypersurfaces

One of the main difficulties in proving Theorem 1.1 is the lack of information about the IMCF for weakly convex hypersurfaces. The proof of the result in [16] makes essential use of the strict convexity. Hence it is not straightforward to obtain the limiting case in Theorem 1.1. We will use approximation by strictly convex hypersurfaces to overcome this obstacle. To do this we use the mean curvature flow with the same Neumann boundary condition. More specifically, we still assume M_0 is parametrised by $X_0 : \mathbb{D} \to \mathbb{R}^{n+1}$. Contrary to our previous solution X of the inverse mean curvature flow, we now consider the solution $F : \mathbb{D} \times [0, T) \to \mathbb{R}^{n+1}$ of the mean curvature flow with Neumann boundary condition, i.e.

$$\dot{F} = -HN, \tag{3.1a}$$

$$X(\partial\mathbb{D}) = \partial X(\mathbb{D}) \subset \mathbb{S}^n, \tag{3.1b}$$

$$0 = \left\langle N_{|\partial\mathbb{D}}, \tilde{N}(X_{|\partial\mathbb{D}}) \right\rangle, \tag{3.1c}$$

$$\left\langle \dot{\gamma}(0), \tilde{N} \right\rangle \geq 0 \quad \forall \gamma \in C^1((-\epsilon, 0], M_t) \colon \gamma(0) \in \partial X(\mathbb{D}) \tag{3.1d}$$

with initial embedding X_0.

Properties of such mean curvature flows with boundary conditions were studied by A. Stahl in [20] and [21]. Now we use Stahl's short time existence result [20, Thm. 2.1] in conjunction with the following strong maximum principle statement to obtain strictly convex approximating hypersurfaces arbitrarily close to M_0 in $C^{2,\alpha}$. First we need a lemma to ensure that a nontrivial M has a strictly convex point.

3.1. **Lemma.** *Let $M \subset \mathbb{R}^{n+1}$ be a smooth and weakly convex hypersurface perpendicular to $\mathbb{S}^n$ from the inside with embedding vector X. Then either ∂M is an equator of the sphere or there exists $x \in \mathbb{D}$ such that the second fundamental form of M at x is positive definite.*

Proof. As mentioned in Remark 2.1, item (iv), $\partial M \subset \mathbb{S}^n$ is a convex hypersurface of the sphere which is either an equator or strictly contained in an open hemisphere by the classical results in [4]. In the first case we are done. In the second case we pick a point $e_0 \in \operatorname{conv}(\partial M) \subset \mathbb{S}^n$, where the latter denotes the spherical convex body bounded by ∂M, such that also

$$\partial M \subset \operatorname{int}(\mathcal{H}(e_0)), \tag{3.2}$$

where $\mathcal{H}(e_0)$ denotes the closed hemisphere with center e_0. By (2.10) the height

$$w = \langle X, e_0 \rangle \tag{3.3}$$

over the hyperplane $e_0^\perp$ attains its global minimum in the interior of $\mathbb{D}$. By attaching a large supporting sphere to M from below we find the existence of a strictly convex point. $\square$

Now we can prove the approximation result. A similar technique was used in [13].

3.2. **Theorem.** *Suppose* $F\colon \mathbb{D}\times[0,T)\to\mathbb{R}^{n+1}$ *is a solution to* (3.1) *with initial hypersurface* M_0 *being weakly convex and perpendicular to the sphere from the inside. Then either* ∂M_0 *is an equator of the sphere or* $(h_{ij})>0$ *for* $t>0$.

Proof. If ∂M_0 is not an equator, then due to Lemma 3.1 there exists a strictly convex point.

Let

$$\chi(x,t)=\min_{|V|=1}h_{ij}V^iV^j. \tag{3.4}$$

Due to the smoothness of h_{ij}, $\chi(x,t)$ is Lipschitz continuous in space and therefore by a simple cut-off function argument we find a smooth function $\phi_0 : M^n\to\mathbb{R}$ so that $0\le\phi_0\le\chi(x,0)$ and there exists $y\in M^n$ so that $\phi_0(y)>0$. We now extend this function to $\phi:\mathbb{D}^n\times[0,\delta)\to\mathbb{R}$ by a heat flow,

$$\begin{cases}\left(\frac{\partial}{\partial t}-\Delta\right)\phi=0 & \text{on } \mathrm{int}(\mathbb{D})\times[0,\tau)\\ \nabla_\mu\phi=0 & \text{on } \partial\mathbb{D}\times[0,\tau)\\ \phi(\cdot,0)=\phi_0(\cdot), & \end{cases} \tag{3.5}$$

where Δ is the time dependent Laplace-Beltrami operator of the metrics induced by the solution F of (3.1). This is a linear parabolic PDE and so by standard theory a solution exists for a short time $\tau>0$. By the strong maximum principle (e.g. [20, Cor. 3.2]), for $t>0$ we have $\phi(\cdot,t)>0$ in $\mathbb{D}$.

We now consider

$$M_{ij}=h_{ij}-\phi g_{ij} \tag{3.6}$$

as long both the MCF and the heat flow exist, say for $0\le t<\tau$. We know that at time $t=0$ we have $M_{ij}\ge 0$ by construction of ϕ. We now aim to apply the weak maximum principle with Neumann boundary conditions, [20, Thm. 3.3, Lemma 3.4].

Using the evolution equations in [21, p. 432], we have that on the flowing manifold

$$\left(\frac{\partial}{\partial t}-\Delta\right)M_{ij}=|A|^2h_{ij}-2Hh_i^kh_{kj}+2\phi Hh_{ij}=:N_{ij}. \tag{3.7}$$

We see that for a unit vector v such that

$$M_{ij}v^i=h_{ij}v^i-\phi g_{ij}v^i=0, \tag{3.8}$$

we obtain

$$N_{ij}v^iv^j=|A|^2\phi-2H\phi^2+2H\phi^2=|A|^2\phi\ge 0, \tag{3.9}$$

that is, the evolution of M_{ij} satisfies a null eigenvector condition.

For a better comparability to the results in [20] and [21] we switch to Stahl's notation, so that for $p\in\mathbb{S}^n$ write $\mu\in T_pM$ for the outward pointing normal to $\mathbb{S}^n$. Due to [21, Thm. 4.3 (i)], at a point $p\in\partial M$ for basis tangent vectors $\partial_I\in T_pM\cap T_p\mathbb{S}$, the basis

$$\mathcal{B}=(\mu,\partial_I)_{2\le I\le n} \tag{3.10}$$

induces the coordinate representation $M_{I\mu}=0$. That is μ is both an eigenvector of M_{ij} and a principal direction at the boundary. We now demonstrate that the conditions of [20, Lemma 3.4] hold. For $\partial_I, \partial_J \in T_pM \cap T_p\mathbb{S}^n$, [21, Thm. 4.3 (ii), (iii)] give

$$\nabla_\mu M_{IJ} = h_{\mu\mu}\delta_{IJ} - h_{IJ}, \qquad \nabla_\mu M_{\mu\mu} = 2H - nh_{\mu\mu}. \tag{3.11}$$

We suppose first that $V \in T_p(\partial M)$ is a minimal eigenvector with eigenvalue $\lambda \in (-\delta, 0]$, that is

$$M_{ij}V^i = \lambda g_{ij}V^i. \tag{3.12}$$

We see that V is also a minimal eigenvector of h_{ij}, and therefore

$$h_{ij}V^iV^j \le h_{\mu\mu}. \tag{3.13}$$

Equation (3.11) now implies $\nabla_\mu M_{IJ}V^IV^J \ge 0$.

Now suppose that μ is a minimal eigenvector with eigenvalue $\lambda \in (-\delta, 0]$. Again minimality of μ implies that for all $W \in T_p(\partial M)$ there holds

$$h_{ij}W^iW^j \ge h_{\mu\mu}. \tag{3.14}$$

In particular this implies $H \ge nh_{\mu\mu}$, and so $\nabla_\mu M_{\mu\mu} \ge H \ge 0$, where we used [21, Thm. 3.1].

We may now apply [20, Thm. 3.3, Lemma 3.4], to give that $M_{ij} \ge 0$. Since $\phi > 0$ for $t > 0$, $h_{ij} > 0$ for $\tau > t > 0$. This then holds for all time that the flow exists by applying [21, Prop. 4.5] to the mean curvature flow defined by $F(x, t - \frac{\tau}{2})$. ☐

3.3. **Corollary.** *Suppose M is a weakly convex hypersurface perpendicular to the sphere from the inside, such that ∂M is not an equator. Then there exists an $\epsilon > 0$ such that for $0 \le t < \epsilon$ there are smooth and strictly convex hypersurfaces perpendicular to the sphere from the inside and satisfy*

$$\int_{M_t} H^2 \to \int_M H^2, \quad |M_t| \to |M|, \quad |\partial M_t| \to |\partial M|$$

as $t \to 0$.

Proof. By [20, Thm. 2.1] there exists a solution to equation (3.1) for $F \in C^\infty(\mathbb{D} \times (0, \epsilon)) \cap C^{2+\alpha;1+\frac{\alpha}{2}}(\mathbb{D} \times [0, \epsilon))$. The convergence then follows due to the regularity of the flow at $t = 0$. ☐

Now we can prove a crucial estimate for the volume.

3.4. **Lemma** (Volume estimate)**.** *Let M be a weakly convex hypersurface perpendicular to the sphere from the inside such that ∂M is not an equator. Then there holds*

$$|M| \le \omega_n - c_{\partial M}, \tag{3.15}$$

where $c_{\partial M} > 0$ is a constant only depending on the outer radius of $\partial M \subset \mathbb{S}^n$, in the sense that it tends to zero only if the outer radius tends to $\pi/2$.

Proof. ∂M is a convex hypersurface of the sphere. Since it is not an equator, it is strictly contained in an open hemisphere $\mathrm{int}(\mathcal{H}(e_0))$ with $e_0 \in \mathrm{conv}_{\mathbb{S}^n}(\partial M)$ by [4]. Pick a geodesic ball B_R of radius $R < \pi/2$ around e_0 such that

$$\partial M \subset B_R \tag{3.16}$$

and denote $S_R = \partial B_R$. Use Corollary 3.3 to obtain a strictly convex hypersurface $\tilde{M} \subset \mathbb{R}^{n+1}$, such that $\partial\tilde{M} \subset B_R$. From [10] or also [19] we know that the IMCF for strictly convex closed hypersurfaces of the unit sphere converges in finite time to an equator. By the avoidance principle the IMCF starting at $\partial\tilde{M}$ exists longer that the one starting from S_R. The existence time T_R of the latter flow however can be calculated explicitly in terms of $\pi/2 - R$. Since the volume element also grows exponentially along the IMCF in the sphere and limits to the volume of the equator, we must have

$$|\mathbb{S}^{n-1}| - |\partial\tilde{M}| \geq c_R > 0. \tag{3.17}$$

Now start the IMCF perpendicular to the sphere from $\tilde{M}$. Due to (2.23) the boundary measures $|\partial\tilde{M}|$ grow less than exponentially, but they must still limit to $|\mathbb{S}^{n-1}|$ in finite time. Hence the existence time of this flow is uniformly bounded below in terms of R and in turn we must have

$$|\tilde{M}| \leq \omega_n - \tilde{c}_R \tag{3.18}$$

with a new positive constant $\tilde{c}_R$ due to the exponential growth of the area measure and the convergence result. Taking $\tilde{M}$ arbitrarily close to M yields the result. $\square$

4. Proof of Theorem 1.1

If ∂M is an equator, (1.3) is trivial. Due to Corollary 3.3 we see that (1.3) now also holds for weakly convex hypersurfaces. So all we have to prove is the characterisation of the limit. So suppose that (1.3) holds with equality. If ∂M is an equator, then M must be a convex minimal surface, hence totally umbilic and hence a hyperplane. So we may suppose that ∂M is not an equator, which in particular implies that

$$|M| \leq \omega_n - c_{\partial M}, \tag{4.1}$$

where we used Lemma 3.4.

Due to Corollary 3.3 for every $\epsilon > 0$ there exists a strictly convex hypersurface perpendicular to the sphere from the inside M^ϵ such that

$$Q(M^\epsilon) \leq Q(M) + \epsilon, \tag{4.2}$$

where $Q(M)$ is the quantity in (2.24) evaluated at the hypersurface M. Starting the flow (1.2) with initial hypersurface M_ϵ, flow hypersurfaces M_t^ϵ and maximal existence time

$$T_\epsilon^* = \log\left(\frac{\omega_n}{|M^\epsilon|}\right), \tag{4.3}$$

in view of (2.25) the corresponding quantities $Q^\epsilon(t)$ satisfy

$$\begin{aligned} \dot{Q}^\epsilon(t) &\leq \left(\omega_n^{\frac{2-n}{n}} - |M_t^\epsilon|^{\frac{2-n}{n}}\right)|\partial M_t^\epsilon| \\ &= \omega_n^{\frac{2-n}{n}}\left(1 - e^{\frac{n-2}{n}(T_\epsilon^* - t)}\right)|\partial M_t^\epsilon|. \end{aligned} \tag{4.4}$$

Due to Lemma 3.4 and Corollary 3.3 there exists a positive time T which only depends on $|M|$ and is independent of ϵ, such that

$$T_\epsilon^* \geq 2T > 0. \tag{4.5}$$

Hence for all ϵ and all $0 \leq t \leq T$ there holds

$$\dot{Q}^\epsilon(t) \leq -c\left(1 - e^{\frac{n-2}{n}T}\right) \equiv -c, \tag{4.6}$$

where $c > 0$ only depends on n, $|M|$ and $|\partial M|$. Using the the strict convexity of M_ϵ and Lemma 2.3, we obtain that

$$\begin{aligned} \omega_n^{\frac{2-n}{n}}|\mathbb{S}^{n-1}| < Q^\epsilon(T) &= Q(M_\epsilon) + \int_0^T \dot{Q}^\epsilon(s)\, ds \\ &\leq Q(M) + \epsilon - cT \\ &= \omega_n^{\frac{2-n}{n}}|\mathbb{S}^{n-1}| + \epsilon - cT, \end{aligned} \tag{4.7}$$

giving a contradiction for small ϵ and completing the proof.

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Ben Lambert, University of Konstanz, Zukunftskolleg, Box 216, 78457 Konstanz, Germany

E-mail address: `benjamin.lambert@uni-konstanz.de`

Julian Scheuer, Albert-Ludwigs-Universität, Mathematisches Institut, Eckerstr. 1, 79104 Freiburg, Germany

E-mail address: `julian.scheuer@math.uni-freiburg.de`