ISOTROPIC FUNCTIONS REVISITED

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Abstract. To a real n-dimensional vector space V and a smooth, symmetric function f defined on the n-dimensional Euclidean space we assign an associated operator function F defined on linear transformations of V. F shall have the property that, for each inner product g on V, its restriction F_g to the subspace of g-selfadjoint operators is the isotropic function associated to f. This means that it acts on these operators via f acting on their eigenvalues. We generalize some well known relations between the derivatives of f and each F_g to relations between f and F, while also providing new elementary proofs of the known results. By means of an example we show that well known regularity properties of F_g do not carry over to F.

1. Introduction

Consider a function $f \in C^\infty(\mathbb{R}^n)$ which is *symmetric*, i.e.

$$f(\kappa_1, \dots, \kappa_n) = f(\kappa_{\pi(1)}, \dots, \kappa_{\pi(n)}) \quad \forall \pi \in \mathcal{P}_n,$$

where $\mathcal{P}_n$ is the permutation group on n elements. Let V be a real, n-dimensional vector space and $\mathcal{L}(V)$ be the vector space of linear operators on V. If V carries an inner product g, on the vector subspace $\Sigma_g(V) \subset \mathcal{L}(V)$ of g-selfadjoint operators one can define a map

$$\begin{aligned} F_g \colon \Sigma_g(V) &\to \mathbb{R} \\ A &\mapsto f(\mathrm{EV}(A)), \end{aligned}$$

where $\mathrm{EV}(A) = (\kappa_1, \dots, \kappa_n)$ denotes the ordered n-tuple of real eigenvalues of A. In [2] J. Ball proved that if $f \in C^r(\mathbb{R}^n)$, $r = 1, 2, \infty$, the function F_g is also of class C^r. Furthermore, using Schauder theory, he showed that if $f \in C^{r,\alpha}(\mathbb{R}^n)$, $r \in \mathbb{N}$, $0 < \alpha < 1$, then also F is in the respective function class. Also compare [11, Sec. 2.1] for a detailed proof of these regularity results. For $r \geq 3$, the implication

$$f \in C^r(\mathbb{R}^n) \quad \Rightarrow \quad F_g \in C^r(\Sigma_g(V))$$

was proven in [19].

In these results one always starts with an inner product space (V, g). In many applications one has to deal with a whole family of such spaces, where g may vary. For example in geometric curvature problems one is often faced with a map F being evaluated on the Weingarten tensor $\mathcal{W}$, an endomorphism field with values in the tensor bundle of linear transformations of the tangent spaces. From point to point, these linear maps $\mathcal{W}(x)$ are self-adjoint with respect to different metrics, so one has to be careful with the domain of F.

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One may observe, that for the most natural symmetric functions, e.g.

$$s_1 = \sum_{i=1}^n \kappa_i \quad \text{or} \quad s_n = \prod_{i=1}^n \kappa_i$$

there is no ambiguity about how to define F even on the whole space $\mathcal{L}(V)$ and not only on some $\Sigma_g(V)$. Namely for s_1 just set

$$F(A) = S_1(A) = \operatorname{tr}(A)$$

and for s_n set

$$F(A) = S_n(A) = \det(A).$$

The functions s_1 and s_n are special cases of the *elementary symmetric polynomials* s_k, $1 \leq k \leq n$, cf. Definition 2.1, to which we associate

$$S_k(A) = \frac{1}{k!}\frac{d^k}{dt^k}\det(I + tA)_{|t=0}.$$

It is true that every symmetric function $f \in C^\infty(\Gamma)$ on a symmetric open set $\Gamma \subset \mathbb{R}^n$ can be written as a function of the s_i,

$$f = \rho(s_1, \ldots, s_n),$$

where $\rho \in C^\infty(U)$ for some open $U \subset \mathbb{R}^n$, cf. [12]. In case $f \in C^r(\Gamma)$, ρ will in general have less regularity, cf. [3]. In both cases the function

$$F = \rho(S_1, \ldots, S_n)$$

is defined on an open set $\Omega \subset \mathcal{L}(V)$ and satisfies

$$F(A) = f(\mathrm{EV}(A))$$

for all $\mathbb{R}$-diagonalisable $A \in \mathcal{L}(V)$ with eigenvalues in Γ. Hence F can be differentiated in all directions of $\mathcal{L}(V)$.

The aim of this short note is a transfer of some well known and often used relations between derivatives of F and f to the new situation, that F can be differentiated in all of $\mathcal{L}(V)$. In previous treatments of this, only the relation between f and F_g was studied for some fixed metric g, compare for example [1, 2, 4, 8, 9, 11, 13, 16, 18, 19]. Our approach is by direct calculation of the proposed relations for the elementary symmetric polynomials and then to transfer them to general functions. Note that this approach also provides a new, quite elementary proof of the corresponding results for the pair (f, F_g) with fixed inner product g, as obtained in [1, Thm. 5.1] and [11, Lemma 2.1.14].

The motivation to write this note came up during the preparation of [7], where we had to apply derivatives of F_g to some non-g-selfadjoint operators, so the need for a globally defined F was apparent. For illustration, have a look at the following simple example:

1.1. *Example.* Let f be the second power sum,

$$f(\kappa) = \sum_{i=1}^n \kappa_i^2, \quad F(A) = \operatorname{tr}(A^2),$$

then F is clearly the associated operator function for f and F is defined on whole $\mathcal{L}(V)$. f is a convex function of the κ_i. However,

$$F \colon \mathcal{L}(V) \to \mathbb{R}$$

is *not* convex: Indeed there holds

$$dF(A)B = 2\,\mathrm{tr}(A \circ B),$$

$$d^2F(A)(B, C) = 2\,\mathrm{tr}(B \circ C)$$

and hence

$$d^2F(A)(\eta, \eta) = 2\,\mathrm{tr}(\eta^2) < 0$$

for a nonzero skew-symmetric (with respect to a basis of eigenvectors of A) η.

The fact that F is in general not convex, when considered as a function on $\mathcal{L}(V)$, caused trouble in the preparation of [7], where we had to estimate the term $d^2F(\dot{\mathcal{W}}, \dot{\mathcal{W}})$ along some curvature flow. Here $\dot{\mathcal{W}}$ is the evolution of the Weingarten tensor. For the particular flow considered in [7], we could not prove the symmetry of $\dot{\mathcal{W}}$. This trouble was the main motivation to write this note and to extend the formulas for derivatives of F, as in Proposition 2.8.

2. Symmetric functions and associated operator functions

For an n-dimensional, real vector space V, the aim of this section is to deduce relations between the derivatives of the functions f and F as described in the introduction. First we fix some definitions and notation.

2.1. **Definition.** On $\mathbb{R}^n$ we denote the elementary symmetric polynomials for $1 \leq k \leq n$ by s_k,

$$s_k(\kappa_1, \ldots, \kappa_n) := \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} \kappa_{i_j}$$

and the power sums for all $k \in \mathbb{N}$ by p_k,

$$p_k(\kappa) = \sum_{i=1}^{n} \kappa_i^k.$$

2.2. **Definition.** (i) Let V be an n-dimensional real vector space and $\mathcal{D}(V) \subset \mathcal{L}(V)$ be the set of diagonalisable endomorphisms. Then we denote by EV the eigenvalue map, i.e.

$$\mathrm{EV}\colon \mathcal{D}(V) \to \mathbb{R}^n/\mathcal{P}_n$$
$$A \mapsto (\kappa_1, \ldots, \kappa_n),$$

where $\kappa_1, \ldots, \kappa_n$ denote the eigenvalues of A and $\mathcal{P}_n$ is the permutation group of n elements.

(ii) Let $\Gamma \subset \mathbb{R}^n$ be open and symmetric, then we define

$$\mathcal{D}_\Gamma(V) = \mathrm{EV}^{-1}(\Gamma/\mathcal{P}_n).$$

2.3. *Remark.* Note that EV is continuous, compare [21].

2.4. **Lemma.** *Let V be an n-dimensional real vector space. Then for all $k \in \mathbb{N}$ there exists a function $P_k \in C^\infty(\mathcal{L}(V))$ with*

$$P_k(A) = p_k \circ \mathrm{EV}(A) \quad \forall A \in \mathcal{D}(V).$$

Proof. Simply set

$$P_k(A) = \operatorname{tr}(A^k).$$

Then there holds

$$P_k(A) = p_k(\mathrm{EV}(A)) \quad \forall A \in \mathcal{D}(V).$$

□

Since the P_k are smooth, we want to investigate the structure of their derivatives.

2.5. **Proposition.** *Let V be an n-dimensional real vector space. Let $U \subset \mathbb{R}^m$ be open and $\psi \in C^r(U)$, $r \geq 1$. Then the function*

$$f = \psi(p_1, \dots, p_m)$$

is defined on an open symmetric set $\Gamma \subset \mathbb{R}^n$ and the function $F = \psi(P_1, \dots, P_m)$ is defined on an open set $\Omega \subset \mathcal{L}(V)$. There holds

$$F_{|\mathcal{D}_\Gamma(V)} = f \circ \mathrm{EV}_{|\mathcal{D}_\Gamma(V)}$$

and the derivatives of F evaluated at a fixed $A \in \Omega$ are given by

$$dF(A)B = \operatorname{tr}(F'(A) \circ B) = \sum_{l=1}^{m} l \frac{\partial \psi}{\partial P_l} \operatorname{tr}\left(A^{l-1} \circ B\right) \quad \forall B \in \mathcal{L}(V), \tag{2.1}$$

where

$$F'(A) = \sum_{l=1}^{m} l \frac{\partial \psi}{\partial P_l} A^{l-1}. \tag{2.2}$$

Proof. Only the formula for dF has to be checked, while all other statements are obvious. The function $P_1(A) = \operatorname{tr}(A)$ is linear and hence

$$dP_1(A)B = \operatorname{tr}(B) \quad \forall A, B \in \mathcal{L}(V).$$

Furthermore by the chain rule there holds

$$dP_k(A)B = d(P_1(A^k))(A)B = k \operatorname{tr}(A^{k-1} \circ B) \quad \forall A, B \in \mathcal{L}(V). \tag{2.3}$$

Thus

$$dF(A)B = \sum_{l=1}^{m} \frac{\partial \psi}{\partial P_l} dP_l(A)B = \operatorname{tr}(F'(A) \circ B) \tag{2.4}$$

and hence the proof is complete. □

2.6. *Remark.* It is well known that the elementary symmetric polynomials s_k are functions of the p_k, cf. [17], and hence Proposition 2.5 also applies to these.

2.7. **Corollary.** *Let V be an n-dimensional real vector space and let f and F be as in Proposition 2.5. Suppose $A \in \mathcal{D}_\Gamma(V)$. Then the endomorphisms $F'(A)$ and A are simultaneously diagonalisable. For a basis $(e_1, \dots, e_n)$ of eigenvectors for A with eigenvalues $\kappa = (\kappa_1, \dots, \kappa_n)$, the eigenvalue F^i of $F'(A)$ with eigenvector e_i is given by*

$$F^i(A) = \frac{\partial f}{\partial \kappa_i}(\kappa). \tag{2.5}$$

Proof. That $F'(A)$ and A are simultaneously diagonalisable follows from (2.2) immediately. Let (κ_i) be the eigenvalues of A. The eigenvalues of F' can be read off (2.2). They are

$$F^i = \sum_{l=1}^{m} l \frac{\partial \psi}{\partial p_l} \kappa_i^{l-1} = \frac{\partial f}{\partial \kappa_i},$$

due to the chain rule. □

There also follows a representation for the second derivatives of the function F. Proofs for the case that F is defined on the subspace of selfadjoint operators with respect to a fixed metric can be found in [1, Thm. 5.1], [11, Lemma 2.1.14] and [19], where in the latter even higher derivatives are treated. The proof presented here is by direct differentiation of (2.4). It extends similar proofs used in the context of tensor valued functions in [4, 5, 8, 9] to the case $n > 3$ and diagonalisable A. There are several other very recent results [15], which address similar questions in the context of operator monotone functions and k-isotropic functions. Also compare the comprehensive thesis [14], as well as [16] and [18].

2.8. **Proposition.** *Let V be an n-dimensional real vector space and let F and f be as in Proposition 2.5 with $r \geq 2$. Let $A \in \mathcal{D}_\Gamma(V)$ and let (η^i_j) be a matrix representation of some $\eta \in \mathcal{L}(V)$ with respect to a basis of eigenvectors of A. Then there holds*

$$d^2F(A)(\eta,\eta) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial \kappa_i \partial \kappa_j} \eta^i_i \eta^j_j + \sum_{i \neq j}^{n} \frac{\frac{\partial f}{\partial \kappa_i} - \frac{\partial f}{\partial \kappa_j}}{\kappa_i - \kappa_j} \eta^i_j \eta^j_i, \tag{2.6}$$

where f is evaluated at the n-tuple (κ_i) of corresponding eigenvalues. The latter quotient is also well defined in case $\kappa_i = \kappa_j$ for some $i \neq j$.

Proof. Starting from (2.4) we can calculate for all $A \in \Omega \subset \mathcal{L}(V)$ and $B, C \in \mathcal{L}(V)$, that

$$\begin{aligned} d^2F(A)(B,C) &= \sum_{k,l=1}^{m} \frac{\partial^2 \psi}{\partial P_l \partial P_k} (dP_l(A)B)(dP_k(A)C) \\ &\quad + \sum_{k=1}^{m} \frac{\partial \psi}{\partial P_k} d^2P_k(A)(B,C). \end{aligned} \tag{2.7}$$

From (2.3) we obtain, already inserting $B = C = \eta = \hat{\eta} + \tilde{\eta}$, where $\hat{\eta}$ is the diagonal part of η in a basis of eigenvectors for A and $\tilde{\eta}$ is the corresponding off-diagonal part $\tilde{\eta} = \eta - \hat{\eta}$,

$$\begin{aligned} d^2P_k(A)(\eta,\eta) &= k \sum_{l=1}^{k-1} \operatorname{tr}(A^{l-1} \circ \eta \circ A^{k-1-l} \circ \eta) \\ &= k \sum_{l=1}^{k-1} \big(\operatorname{tr}(A^{l-1} \circ \hat{\eta} \circ A^{k-1-l} \circ \hat{\eta}) \\ &\qquad + \operatorname{tr}(A^{l-1} \circ \tilde{\eta} \circ A^{k-1-l} \circ \tilde{\eta}) \big). \end{aligned} \tag{2.8}$$

Using the specific basis of eigenvectors we get

$$
\begin{aligned}
d^2P_k(A)(\eta,\eta) &= k(k-1)\sum_{i=1}^{n}\kappa_i^{k-2}(\eta_i^i)^2 + k\sum_{l=1}^{k-1}\sum_{i,j=1}^{n}\kappa_i^{l-1}\kappa_j^{k-1-l}\tilde{\eta}_j^i\tilde{\eta}_i^j \\
&= \sum_{i,j=1}^{n}\frac{\partial^2 p_k}{\partial\kappa_i\partial\kappa_j}\eta_i^i\eta_j^j + \sum_{i\neq j} k\frac{\kappa_i^{k-1}-\kappa_j^{k-1}}{\kappa_i-\kappa_j}\eta_j^i\eta_i^j \qquad (2.9)\\
&= \sum_{i,j=1}^{n}\frac{\partial^2 p_k}{\partial\kappa_i\partial\kappa_j}\eta_i^i\eta_j^j + \sum_{i\neq j}\frac{\frac{\partial p_k}{\partial\kappa_i}-\frac{\partial p_k}{\partial\kappa_j}}{\kappa_i-\kappa_j}\eta_j^i\eta_i^j.
\end{aligned}
$$

Hence the claimed result holds for the power sums. Returning to (2.7) we obtain, also using Corollary 2.7,

$$
\begin{aligned}
d^2F(A)(\eta,\eta) &= \sum_{k,l=1}^{m}\frac{\partial^2\psi}{\partial P_l\partial P_k}(dP_l(A)\hat{\eta})(dP_k(A)\hat{\eta}) + \sum_{k=1}^{m}\frac{\partial\psi}{\partial P_k}d^2P_k(A)(\eta,\eta)\\
&= \sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial\kappa_i\partial\kappa_j}\eta_i^i\eta_j^j + \sum_{k=1}^{m}\frac{\partial\psi}{\partial P_k}\sum_{i\neq j}\frac{\frac{\partial p_k}{\partial\kappa_i}-\frac{\partial p_k}{\partial\kappa_j}}{\kappa_i-\kappa_j}\eta_j^i\eta_i^j,
\end{aligned}
$$

from which the claim follows due to the chain rule. Also in this formula, the quotient makes sense even if $\kappa_i = \kappa_j$, since the singularity in this fraction is removable, as can be seen from (2.9). □

2.9. *Remark.* The representation formulae (2.5) and (2.6) are only valid a diagonalisable A, since their expressions make use of a particular basis of eigenvectors. Formulae which are valid for arbitrary $A \in \Omega$ are given, though a little less explicit, in (2.1) and (2.8). They are still easy enough to serve as a computational tool, particularly in low dimensions.

Although in the previous proof we have already seen an explicit expression for the quotient term in (2.6), we want to at least mention another representation. It appeared in [11, Lemma 2.1.14] and [19], also compare [10, Lemma 2]. The proof is similar to these references.

2.10. **Lemma.** *Let f be as in Proposition 2.5 with $r \geq 2$ and suppose that Γ is convex. Then there holds*

$$\frac{\frac{\partial f}{\partial\kappa_i}-\frac{\partial f}{\partial\kappa_j}}{\kappa_i-\kappa_j} = \frac{1}{2}\int_0^1\left(\frac{\partial^2 f}{\partial\kappa_i^2} - 2\frac{\partial^2 f}{\partial\kappa_i\partial\kappa_j} + \frac{\partial^2 f}{\partial\kappa_j^2}\right),$$

where the integrand is evaluated along the line segment

$$\sigma(t) = \kappa + t\frac{\kappa_j-\kappa_i}{2}(e_i - e_j).$$

An alternative proof. Let us have a look at a second nice proof of Proposition 2.8, the idea of which appeared in [19, Lemma 3.2]. I owe thanks to the anonymous referee for the observation that this method can also be applied in our situation. It is based on the fact that the function F, as given in Proposition 2.5, is $\mathrm{Gl_n}(V)$-invariant:

$$F(SAS^{-1}) = F(A) \quad \forall A \in \mathcal{L}(V)\ \forall S \in \mathrm{Gl_n}(V). \qquad (2.10)$$

In [19, Lemma 3.2] this property held for all orthogonal transformations S of a subspace of self-adjoint operators, but the proof basically carries over. Let us repeat it quickly here.

We suppose that all eigenvalues of A are mutually different. The general case can then be treated by approximation as in [19]. Differentiating the relation (2.10) with respect to A in direction of an arbitrary $\eta \in \mathcal{L}(V)$ we obtain for all $S \in \mathrm{Gl_n}(V)$, that

$$dF(SAS^{-1})(S\eta S^{-1}) = dF(A)(\eta). \tag{2.11}$$

In particular, choosing $S = e^{tW}$ for arbitrary $W \in \mathcal{L}(V)$, $t \in \mathbb{R}$, and differentiating (2.11) with respect to t at $t = 0$ gives

$$d^2F(A)(WA - AW, \eta) = dF(A)(\eta W - W\eta).$$

On the other hand, writing

$$\eta = \hat{\eta} + \tilde{\eta},$$

with diagonal $\hat{\eta}$ and off-diagonal $\tilde{\eta}$, we have

$$d^2F(A)(\eta,\eta) = d^2F(A)(\hat{\eta},\hat{\eta}) + 2d^2F(A)(\hat{\eta},\tilde{\eta}) + d^2F(A)(\tilde{\eta},\tilde{\eta}).$$

With respect to a basis of eigenvectors for A and $F'(A)$ we define

$$W^i_j = \frac{\tilde{\eta}^i_j}{\kappa_j - \kappa_i},$$

which implies

$$W^i_k A^k_j - A^i_k W^k_j = \tilde{\eta}^i_j$$

and hence

$$d^2F(A)(\hat{\eta},\tilde{\eta}) = dF(A)(\hat{\eta}W - W\hat{\eta}) = 0$$

and

$$d^2F(A)(\tilde{\eta},\tilde{\eta}) = DF(A)(\tilde{\eta}W - W\tilde{\eta}) = \sum_{i\neq j} \frac{\frac{\partial f}{\partial \kappa_i} - \frac{\partial f}{\partial \kappa_j}}{\kappa_i - \kappa_j} \eta^i_j \eta^j_i.$$

Finally, since A and $\hat{\eta}$ are simultaneously diagonal, we have

$$\begin{aligned} d^2F(A)(\hat{\eta},\hat{\eta}) &= \frac{d}{dt}\left(dF(A + t\hat{\eta})(\hat{\eta})\right)_{|t=0} \\ &= \frac{d}{dt}\left(\frac{\partial f}{\partial \kappa_i}(\kappa + t(\eta^i_i))\eta^i_i\right)_{|t=0} \\ &= \frac{\partial^2 f}{\partial \kappa_i \partial \kappa_j}(\kappa)\eta^i_i \eta^j_j \end{aligned}$$

and Proposition 2.8 follows. □

There is a slight advantage of the first proof of Proposition 2.8, namely that the calculation in (2.9) gives a precise description of why the term involving $\kappa_i - \kappa_j$ in the denominator also makes sense in case of coalescing eigenvalues.

3. Functions on bilinear forms

There is a useful relation of our maps $F\colon \Omega \subset \mathcal{L}(V) \to \mathbb{R}$ to maps which are defined on bilinear forms. First we need several definitions.

3.1. **Definition.** Let V be a finite dimensional real vector space.

(i) We denote the vector space of bilinear forms on V by $\mathcal{B}(V)$. The space of bilinear forms on the dual space V^* is denoted by $\mathcal{B}^*(V)$. The respective subsets of symmetric and positive definite forms will be denoted by $\mathcal{B}_+(V)$ and $\mathcal{B}^*_+(V)$.

(ii) For $a \in \mathcal{B}(V)$ and $b \in \mathcal{B}^*(V)$ we set

$$\begin{aligned} a_*\colon V &\to V^* \\ v &\mapsto a(v,\cdot) \end{aligned}$$

and

$$\begin{aligned} b^*\colon V^* &\to V \\ \phi &\mapsto J^{-1}\left(b(\phi,\cdot)\right), \end{aligned}$$

where $J\colon V \to V^{**}$ is the canonical identification given by

$$v \mapsto (\phi \mapsto \phi(v)).$$

(iii) Let $a \in \mathcal{B}(V)$ and $b \in \mathcal{B}^*(V)$, then we define $b * a \in \mathcal{L}(V)$ by contraction, i.e.

$$b * a = b^* \circ a_*.$$

(iv) For $g \in \mathcal{B}_+(V)$ we define $g^{-1} \in \mathcal{B}^*_+(V)$ by requiring

$$g^{-1} * g = \mathrm{id}.$$

(v) For $a \in \mathcal{B}(V)$ and $g \in \mathcal{B}_+(V)$ we define the operator $a^{\sharp_g} \in \mathcal{L}(V)$ by

$$a^{\sharp_g} = g^{-1} * a$$

(vi) For any bilinear form a on either V or V^* we denote by $\hat{a}$ the symmetrisation, i.e.

$$\hat{a}(v,w) = \frac{1}{2}\left(a(v,w) + a(w,v)\right).$$

3.2. *Remark.* For $a \in \mathcal{B}(V)$ and $g \in \mathcal{B}_+(V)$ we have

$$a(v,w) = g(a^{\sharp_g}(v), w) \quad \forall v, w \in V.$$

The following construction is very useful.

3.3. **Proposition.** *Let V be an n-dimensional real vector space, $\Omega \subset \mathcal{L}(V)$ open and F be as in Proposition 2.5. Define*

$$\begin{aligned} \Phi\colon \Lambda \subset \mathcal{B}_+(V) \times \mathcal{B}(V) &\to \mathbb{R} \\ (g,h) &\mapsto F(g^{-1} * \hat{h}), \end{aligned}$$

where Λ is the open subset such that $g^{-1} * \hat{h} \in \Omega$ for all $(g,h) \in \Lambda$. Then Φ is as smooth as F and the partial derivative of Φ at (g,h) with respect to h can be regarded as a symmetric bilinear form,

$$\frac{\partial \Phi}{\partial h}(g,h) \in \mathcal{B}^*(V).$$

Furthermore the derivatives of F and Φ are related by

$$\frac{\partial\Phi}{\partial h}(g,h)a = \operatorname{tr}(F'(g^{-1}*\hat{h})\circ \hat{a}^{\sharp_g}) = dF(g^{-1}*\hat{h})\hat{a}^{\sharp_g}. \tag{3.1}$$

Proof. Since the map $h\mapsto g^{-1}*\hat{h}$ is linear, we obtain

$$\frac{\partial\Phi}{\partial h}(g,h)a = \operatorname{tr}\left(F'\circ (g^{-1}*\hat{a})\right)$$

and it can be regarded as a symmetric bilinear form acting on pairs (ξ,ζ) via letting it act on $\xi\otimes\zeta$. $\square$

4. Properties of symmetric functions

We investigate some special properties associated to symmetric functions, which are particularly related to applications in geometric flows. The most crucial one, the monotonicity, usually ensures that a flow is parabolic. Define

$$\Gamma_+ = \{(\kappa_i)\in\mathbb{R}^n\colon \kappa_i>0\quad \forall 1\leq i\leq n\}.$$

4.1. **Definition.** Let $\Gamma\subset\mathbb{R}^n$ open and symmetric, $r\geq 1$ and let $f\in C^r(\Gamma)$ be symmetric.

(i) f is called *strictly monotone*, if

$$\frac{\partial f}{\partial\kappa_i}(\kappa)>0\quad \forall\kappa\in\Gamma\ \forall 1\leq i\leq n.$$

(ii) Let Γ in addition be a cone, then f is called *homogeneous of degree* $p\in\mathbb{R}$ if

$$f(\lambda\kappa)=\lambda^p f(\kappa)\quad \forall\lambda>0\ \forall\kappa\in\Gamma.$$

(iii) A nowhere vanishing function $f\in C^r(\Gamma_+)$, $r\geq 2$, is called *inverse concave (inverse convex)*, if the so-called *inverse symmetric function* $\tilde{f}\in C^r(\Gamma_+)$, defined by

$$\tilde{f}(\kappa_i)=\frac{1}{f(\kappa_i^{-1})},$$

is concave (convex).

These properties carry over to the function F from Proposition 2.5 in the following sense.

4.2. **Proposition.** *Let V be an n-dimensional real vector space, $\Gamma\subset\mathbb{R}^n$ open and symmetric, $r\geq 1$ and let $f\in C^r(\Gamma)$ and $F\in C^r(\Omega)$ be as in Proposition 2.5. Then there hold:*

(i) *If f is strictly monotone, then $F'(A)$ only has positive eigenvalues at all $A\in\mathcal{D}_\Gamma(V)$ and the bilinear form $\frac{\partial\Phi}{\partial h}$ from Proposition 3.3 is positive definite at all (g,h) with $g^{-1}*\hat{h}\in\mathcal{D}_\Gamma(V)$.*

(ii) *If Γ is a cone and f is homogeneous of degree p, then $\mathcal{D}_\Gamma(V)$ is a cone and $F_{|\mathcal{D}_\Gamma(V)}$ is homogeneous of degree p.*

(iii) *If $r\geq 2$, Γ is convex and f is concave, then F satisfies*

$$d^2F(A)(\eta,\eta)\leq 0$$

for all η having a symmetric matrix representation with respect to a basis of eigenvectors of A. The reverse inequality holds if f is convex.

Proof. (i) $F'(A)$ has positive eigenvalues due to Corollary 2.7. From (3.1) we obtain (omitting the arguments) for $0 \neq \xi \in V$,

$$\frac{\partial \Phi}{\partial h}(\xi,\xi) = \frac{\partial \Phi}{\partial h}(\xi \otimes \xi) = dF(\xi \otimes \xi)^{\sharp_g} > 0.$$

(ii) Let $A \in \mathcal{D}_\Gamma(V)$ and $\lambda > 0$. Then the claim follows from $\mathrm{EV}(\lambda A) = \lambda \mathrm{EV}(A)$.
(iii) Follows immediately from (2.8) and Lemma 2.10. □

In Proposition 4.2, item (iii), the restriction to symmetric η is indeed necessary, as can be seen from Example 1.1

The following estimates for 1-homogeneous resp. inverse concave curvature functions are very useful and are also needed in [7]. The idea for the first statement comes from [1, Thm. 2.3] and also appeared in a similar form in [6, Lemma 14]. The proof for the second statement, however appearing in a slightly different form, can be found in [20, p. 112].

4.3. **Proposition.** *Let V be an n-dimensional real vector space and $r \geq 1$. Let $f \in C^r(\Gamma_+)$ and $F \in C^r(\Omega)$ be as in Proposition 2.5 with f symmetric, positive, strictly monotone and homogeneous of degree one. Then there hold:*

(i) *For every pair $A \in \mathcal{D}_{\Gamma_+}(V)$ and $g \in \mathcal{B}_+(V)$ such that A is self-adjoint with respect to g, there holds for all $\eta \in \mathcal{L}(V)$ that*

$$dF(A)(\mathrm{ad}_g(\eta) \circ A^{-1} \circ \eta) \geq F^{-1}\left(dF(A)\eta\right)^2,$$

where $\mathrm{ad}_g(\eta)$ is the adjoint of η with respect to g.

(ii) *If f is inverse concave, then for every pair $A \in \mathcal{D}_{\Gamma_+}(V)$ and $g \in \mathcal{B}_+(V)$ such that A is self-adjoint with respect to g, there holds*

$$d^2F(A)(\eta,\eta) + 2dF(A)(\eta \circ A^{-1} \circ \eta) \geq 2F^{-1}\left(dF(A)\eta\right)^2,$$

for all g-selfadjoint η.

Proof. (i) Note that for each $A \in \mathcal{D}_{\Gamma_+}(V)$ the kernel S of the map

$$dF(A)\colon \mathcal{L}(V) \to \mathbb{R}$$

has dimension $n^2 - 1$, due to the homogeneity which implies

$$dF(A)A = F(A) > 0.$$

Now let $\eta \in \mathcal{L}(V)$, then there exists a decomposition

$$\eta = aA + \xi,$$

where $\xi \in S$. Hence, omitting the argument A of F,

$$\begin{aligned} dF(\mathrm{ad}_g(\eta) \circ A^{-1} \circ \eta) &= adF(\eta) + adF(\mathrm{ad}_g(\xi)) + dF(\mathrm{ad}_g(\xi) \circ A^{-1} \circ \xi) \\ &\geq adF(\eta), \end{aligned}$$

since F' and A can be diagonalised simultaneously. The result follows from $F = dF(A) = a^{-1}dF(\eta)$.

(ii) For the inverse symmetric function $\tilde{f}$ the corresponding $\tilde{F}$ has the property

$$\tilde{F}(A) = \frac{1}{F(A^{-1})} \quad \forall A \in \mathcal{D}_{\Gamma_+}(V).$$

Thus we may differentiate $\tilde{F}$ using this formula, if we restrict to directions B which are self-adjoint with respect to g. Hence for all g-selfadjoint $A \in \mathcal{D}_{\Gamma_+}(V)$ we get

$$d\tilde{F}(A)B = \tilde{F}^2 dF(A^{-1})(A^{-1} \circ B \circ A^{-1})$$

and, omitting arguments,

$$\begin{aligned} d^2\tilde{F}(B,B) = 2\tilde{F}^3 \left(dF(A^{-1} \circ B \circ A^{-1})\right)^2 \\ - \tilde{F}^2 d^2F(A^{-1} \circ B \circ A^{-1}, A^{-1} \circ B \circ A^{-1}) \\ - 2\tilde{F}^2 dF(A^{-1} \circ B \circ A^{-1} \circ B \circ A^{-1}), \end{aligned}$$

where $\tilde{F} = \tilde{F}(A)$ and $F = F(A^{-1})$. Since $\tilde{f}$ is inverse concave, there holds

$$d^2\tilde{F}(B,B) \leq 0$$

for all g-selfadjoint B. For some g-selfadjoint η set

$$B = A \circ \eta \circ A$$

to obtain

$$d^2F(\eta,\eta) + 2dF(\eta \circ A \circ \eta) \geq 2F^{-1} \left(dF(\eta)\right)^2,$$

where we again have in mind $F = F(A^{-1})$. The result follows. □

5. Examples

Let us have a look at some familiar symmetric functions, their corresponding associated operator functions and their properties. The most important examples are the elementary symmetric polynomials satisfying

$$s_k \circ \mathrm{EV}(A) = \frac{1}{k!}\frac{d^k}{dt^k}\det(I + tA)_{|t=0},$$

compare [11, equ. (2.1.31)]. s_k is strictly monotone on the set

$$\Gamma_k = \{\kappa \in \mathbb{R}^n \colon s_1(\kappa) > 0, \dots, s_k(\kappa) > 0\},$$

which is equal to the connected component of the set $\{s_k > 0\}$ containing Γ_+, compare [13, Prop. 2.6]. Obviously s_1 is also concave and convex.

Define the quotients

$$q_k \colon \Gamma_{k-1} \to \mathbb{R}$$
$$q_k = \frac{s_k}{s_{k-1}}.$$

These are homogeneous of degree one and concave, cf. [13, Thm. 2.5]. On Γ_+ the q_k are also strictly monotone and inverse concave, cf. [1, Thm. 2.6]. Also the functions

$$f = \left(\frac{s_k}{s_l}\right)^{\frac{1}{k-l}}, \quad 0 \leq l < k \leq n,$$

share all these properties on Γ_+, [1, p. 23]. More examples of such curvature functions can be found in [1].

6. Loss of regularity

In this final section we discuss the regularity properties of the associated operator function F and show be means of an example that the loss of regularity from f to ψ in the correspondence

$$f = \psi(p_1, \dots, p_m)$$

also leads, in general, to the same loss of regularity from f to

$$F \colon \Omega \to \mathbb{R}$$

in the relation

$$F_{|\mathcal{D}_\Gamma(V)} = f \circ \mathrm{EV}_{|\mathcal{D}_\Gamma(V)}. \tag{6.1}$$

Consider the following example:

$$f(\kappa_1, \kappa_2) = (\kappa_1^2 + \kappa_2^2)^{\frac{3}{2}}.$$

Then $f \in C^2(\mathbb{R}^2)$. Since F is required to satisfy (6.1) and the *open* domain Ω of F has to contain the zero matrix, we must use

$$\psi(x_1, \dots, x_m) = |x_2|^{\frac{3}{2}}$$

to connect to f (note that $P_2(A)$ can be negative). Hence

$$\begin{aligned} F \colon \mathcal{L}(V) &\to \mathbb{R} \\ F(A) = \psi(P_2(A)) &= |\operatorname{tr}(A^2)|^{\frac{3}{2}} \end{aligned}$$

is an associated operator function. Writing, with respect to a basis,

$$A = \begin{pmatrix} w & x \\ y & z \end{pmatrix},$$

we see that

$$F(A) = F(w, x, y, z) = |w^2 + 2xy + z^2|^{\frac{3}{2}},$$

which is not C^2, since its restriction to the straight line

$$x \mapsto (0, x, 1, 0) \tag{6.2}$$

is not C^2. It is in fact only as smooth as ψ. This is in sharp contrast to the regularity of the restriction to a subspace of g-selfadjoint operators,

$$F \colon \Sigma_g(V) \to \mathbb{R},$$

which has the same regularity as f, cf. [2, 19]. The crucial difference is that the variations in (6.2) are not allowed, since one must remain within the class of symmetric matrices.

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