INVERSE CURVATURE FLOWS IN RIEMANNIAN WARPED PRODUCTS

JULIAN SCHEUER

Abstract. The long-time existence and umbilicity estimates for compact, graphical solutions to expanding curvature flows are deduced in Riemannian warped products of a real interval with a compact fibre. Notably we do not assume the ambient manifold to be rotationally symmetric, nor the radial curvature to converge, nor a lower bound on the ambient sectional curvature. The inverse speeds are given by powers $0 < p \leq 1$ of a curvature function satisfying few common properties.

1. Introduction

This paper deals with expanding curvature flows of the form

$$\dot{x} = \frac{1}{F^p}\nu, \quad 0 < p \leq 1, \tag{1.1}$$

where

$$x\colon [0, T^*) \times M^n \to N^{n+1}, \quad n \geq 2,$$

is a family of embeddings of a smooth, orientable, compact manifold M^n and $N = N^{n+1}$ is a product

$$N = (R_0, \infty) \times \mathcal{S}_0$$

with metric

$$\bar{g} = dr^2 + \vartheta^2(r)\sigma.$$

Here $\vartheta \in C^\infty((R_0, \infty))$ satisfies $\vartheta' > 0$, $\vartheta'' \geq 0$ and $(\mathcal{S}_0, \sigma)$ is a compact Riemannian manifold. In (1.1), F is a function evaluated at the Weingarten operator $\mathcal{W}$ of the flow hypersurfaces $M_t = x(t, M)$ at the respective point x and ν is the outward pointing normal, i.e.

$$\bar{g}(\nu, \partial_r) > 0.$$

The detailed assumptions on the curvature function F and on N are the following.

1.1. **Assumption.** Let $\Gamma \subset \mathbb{R}^n$ be an open, symmetric and convex cone containing the positive cone

$$\Gamma_+ = \{(\kappa_i) \in \mathbb{R}^n\colon \kappa_i > 0 \quad \forall 1 \leq i \leq n\}$$

and suppose $f \in C^\infty(\Gamma)$ is a positive, symmetric, strictly monotone, 1-homogeneous and concave function with

$$f(1, \dots, 1) = n, \quad f_{|\partial\Gamma} = 0$$

and associated curvature function $F = F(\mathcal{W})$, cf. section 2.2.

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Particular examples of curvature functions satisfying these assumptions are roots or quotients of elementary symmetric polynomials,

$$F = nH_k^{\frac{1}{k}}, \quad F = n\frac{H_{k+1}}{H_k}$$

and many more, cf. [4].

In order to obtain good asymptotics we will make the following assumption on the warping function. This assumption will not be needed for the long-time existence.

1.2. **Assumption.** Assume the warping function $\vartheta \in C^\infty((R_0,\infty))$ to satisfy

$$\limsup_{r\to\infty} \frac{\vartheta''\vartheta}{\vartheta'^2} < \infty \quad \text{and} \quad \limsup_{\substack{r\to\infty \\ \vartheta''(r)>0}} \frac{\vartheta'''\vartheta}{\vartheta'\vartheta''} < \infty.$$

In the following theorem $\widehat{\underline{\mathrm{Rc}}}$ denotes the smallest eigenvalue of the Ricci tensor of σ and H_k denotes the curvature function determined by the k-th normalized elementary symmetric polynomial of the principal curvatures, compare section 2.2 for further information. In this paper we aim to prove the following theorem.

1.3. **Theorem.** *Let $(\mathcal{S}_0,\sigma)$ be a smooth, compact and orientable Riemannian manifold of dimension $n\geq 2$, $R_0>0$, $N=(R_0,\infty)\times\mathcal{S}_0$ and define a warped product metric on N,*

$$\bar{g} = dr^2 + \vartheta^2(r)\sigma,$$

with $\vartheta\in C^\infty((R_0,\infty))$, $\vartheta''\geq 0$ and $\vartheta'>0$. Let $0<p\leq 1$ and F satisfy Assumption 1.1. Let

$$x_0\colon M \hookrightarrow N$$

be the embedding of a hypersurface M_0, which is graphical over $\mathcal{S}_0$, i.e. there exists $u\in C^\infty(\mathcal{S}_0,(R_0,\infty))$ such that

$$M_0 = \{(u(y),y)\colon y\in\mathcal{S}_0\},$$

and such that all its n-tuples of principal curvatures belong to Γ.

(i) Assume either of the following properties to hold:
(a) σ has non-negative sectional curvature.
(b) $F = n\frac{H_{k+1}}{H_k}$, $\quad 0\leq k\leq n-1$.
Then there exists a unique immortal solution

$$x\colon [0,\infty)\times M \to N$$

of

$$\begin{aligned} \dot{x} &= \frac{1}{F^p}\nu \\ x(0,\cdot) &= x_0, \end{aligned} \tag{1.2}$$

which is also graphical over $\mathcal{S}_0$, i.e. $\langle\nu,\partial_r\rangle>0$.

(ii) Assume σ has non-negative sectional curvature and that all of the following three items hold:
(A) Assumption 1.2 holds.
(B) In case that $\sup_{r>0}\vartheta'(r)<\infty$ and $p=1$, we assume

$$\widehat{\underline{\mathrm{Rc}}} > 0 \text{ and } F = n\frac{H_{k+1}}{H_k}, \quad 0\leq k\leq n-1.$$

(C) In case that $\sup_{r>0}\vartheta'(r)=\infty$ and $p=1$, we assume

$$\liminf_{r\to\infty}\frac{\vartheta''\vartheta}{\vartheta'^2} > 0.$$

Then the flow hypersurfaces become umbilical with the rate

$$\left|h^i_j - \frac{\vartheta'}{\vartheta}\delta^i_j\right| \leq ct\frac{\vartheta'^{1-p(p+1)}}{\vartheta}, \tag{1.3}$$

where the t-factor may be dropped in case $p < 1$ or bounded ϑ' and may even be replaced by $e^{-\alpha t}$ for some positive α if ϑ' is bounded and $p = 1$.

Let us make some remarks on the technical assumptions made in Theorem 1.3.

1.4. *Remark.* (i) The assumptions in statement (i) of Theorem 1.3 are optimal in the sense, that for example in a spherical ambient space with $\vartheta'' < 0$ the inverse mean curvature flow only exists for a finite time, cf. [26, 51] and for $p > 1$ the maximal existence is finite if $N = \mathbb{R}^{n+1}$, cf. [25].

(ii) The assumption on the sectional curvature of σ can be relaxed. The crucial point, where we use this assumption is in the first gradient estimates, especially in estimate (3.4), where we throw away the term involving $\widehat{\mathrm{Rm}}$, if F is general. However, under a further suitable technical assumption we could also absorb it into the first line of this equation. For the special case of the inverse mean curvature flow in the Reissner-Nordström manifolds this has been accomplished in the recent preprint [9]. However, in order to avoid too many technical assumptions, we will not improve the main result in this direction here, except that we prove the long-time existence in general, provided that F is a quotient of the H_k. For the IMCF this was also accomplished in [43, 71].

(iii) The rates of convergence in this theorem can be improved, if the ambient sectional curvatures approach each other at infinity. Such results have been accomplished for example in [10, 49, 60] in case $p = 1$ and in [58] in case $p < 1$ in the hyperbolic space. Since the main aim of this work is to deal with spaces in which the limits of the quantities in Assumption 1.2 do not exist (if σ is the round metric this implies that N is not asymptotically a spaceform), we will not pursue these optimal estimates here and stick to the best we could accomplish in general ambient spaces. To the best of my knowledge, the only result in such general spaces is the analogous result for the inverse mean curvature flow proven in [60].

(iv) The question, whether (1.3) implies that the flow hypersurfaces do become almost umbilical, depends on the ambient space N and on p. However, if $p = 1$, the analysis in [60, Prop. 3.1] implies that ϑ grows exponentially. Hence in this case we obtain exponential decay of $\mathcal{W} - \frac{\vartheta'}{\vartheta}\,\mathrm{id}$.

(v) In case $p = 1$, the gradient decay estimates obtained in Lemma 4.8 are optimal even if the ambient space is asymptotically a spaceform. Compare the explanation in [60, Rem. 1.5].

(vi) In case $p = 1$ the estimate (1.3) turned out to be strong enough to obtain geometric inequalities, for example in [5, 21, 50, 67]. We are optimistic that Theorem 1.3 will be helpful with such applications as well.

The motivation to analyse the behaviour of inverse curvature flows has mostly been driven by their power to deduce geometric inequalities for hypersurfaces. The most prominent example is the proof of the Riemannian Penrose inequality due to Huisken/Ilmanen [33], building on the observation made by Geroch [27] and Jang/Wald [38] that the Hawking mass of a connected surface is non-decreasing under the inverse mean curvature flow (IMCF) with $F = H$ and $p = 1$, if the ambient scalar curvature is non-negative. Since for general initial data the IMCF may develop singularities, Huisken and Ilmanen defined a notion of a weak solution for this flow, maintaining the Geroch monotonicity. This enabled them to prove the

Riemannian Penrose inequality. For a short outline of their procedure also compare [32].

Also the classical solution to IMCF has lead to very interesting applications. A crucial feature of this flow in $\mathbb{R}^{n+1}$ is that one does not need to require convexity of the initial hypersurface to avoid finite time singularities. Namely, Gerhardt [22] and Urbas [65] proved the long-time existence even for more general flows in $\mathbb{R}^{n+1}$, with F satisfying Assumption 1.1, $p = 1$ and a starshaped initial hypersurface M_0 with $F_{|M_0} > 0$. Furthermore, after exponential rescaling, the flow converges to a sphere smoothly. This result, with $F = nH_{k+1}/H_k$, was later exploited by Guan/Li [29] to generalise the Alexandrov-Fenchel quermassintegral inequalities from the convex setting to the starshaped and H_{k+1}-convex setting. Since then a cascade of similar results followed by the same method (monotone quantity plus some convergence result) in various ambient spaces. The tough parts are to find the monotone quantity and to prove a sufficient convergence result. Examples of other results in this direction are a generalised Minkowski-type inequality in the anti-de Sitter-Schwarzschild manifold due to Brendle/Hung/Wang [5], Alexandrov-Fenchel-type inequalities in the hyperbolic space [16, 20, 31, 44, 69] and in the sphere [28, 51, 69]. Further similar applications can be found in [21, 41, 50, 67].

In many of these papers, there was a need to investigate the asymptotical behaviour of the corresponding inverse curvature flow separately, since a unified treatment had not been present. Hence, a branch of research solely dealing with inverse curvature flows has developed within the community, where the main aims are to generalise the convergence results in various directions (concerning flow speed and ambient space). A step towards generalising the ambient space was made by the author with the paper [60], where the IMCF was considered in rotationally symmetric warped products under assumptions similar to Assumption 1.2. Before (and after) that, some more special ambient spaces were treated, which, to the best of my knowledge, all assumed convergence of the quantities in Assumption 1.2. Instead of giving a description of the available results verbally, the following table is supposed to give an overview as broad as I could accomplish over the previous results on *smooth, inverse curvature flows of closed hypersurfaces in Riemannian warped products with* $\vartheta'' \geq 0$. The topics they cover are for example long-time existence, asymptotic behaviour, solitons and others. We point out that, in order to keep things manageable, we leave aside treatments of contracting flows, weak solutions, flows in Lorentzian manifolds, flows of entire graphs, flows with boundary conditions, anisotropic flows and flows with constraints (e.g. volume preserving flows).

N/F		$F = n\frac{H_{k+1}}{H_k}$	F more general and $p = 1$	$p \neq 1$ or non-hom. speed
CSC	$\mathbb{R}^{n+1}$	[8, 11, 18] [34]	[15, 22, 45] [64, 65, 66]	[2, 3, 7, 12, 13] [14, 25, 36, 37] [40, 42, 46, 47] [59, 63, 68]
	$\mathbb{H}^{n+1}$	[17, 35]	[24, 48, 70]	[42, 57, 58, 68]
Asympt. CSC	$\mathbb{R}^{n+1}$	[17, 43, 50]		
	$\mathbb{H}^{n+1}$	[5, 49, 53]	[10]	
More general	$\frac{\vartheta''\vartheta}{\vartheta'^2}$ converges	[9, 43, 52, 71]		
	1.2	[60]		

Note that a reference only appears in the most general slot it can be placed. Also note that there are few works on the inverse mean curvature flow in ambient spaces which are not warped products, [1, 39, 55, 56]. For completeness, we also mention that there are some results for inverse flows in the sphere, cf. [6, 7, 26, 48, 51, 68]. This paper aims to fill some gaps in this table, especially in the two bottom rows, and is organised as follows. Section 2 collects some notation, conventions, basic facts about curvature functions and the relevant evolution equations. In section 3 we treat the long-time existence and in section 4 we analyse the asymptotic behaviour and finish the proof of Theorem 1.3.

2. Preliminaries

2.1. **Notation and conventions.** In this paper we deal with embedded hypersurfaces

$$x\colon M \hookrightarrow N$$

of a smooth, closed and orientable manifold M^n into an ambient Riemannian manifold $(N^{n+1}, \bar{g})$. All geometric quantities of N will be furnished with an overbar, e.g. $\bar{g} = (\bar{g}_{\alpha\beta})$ for the metric, $\bar{\nabla}$ for its Levi-Civita connection etc. In coordinate expressions, greek indices run from 0 to n. For the quantities induced by the embedding x, we use latin indices running from 1 to n, e.g. for the induced metric $g = (g_{ij})$ with Levi-Civita connection ∇. For a (k,l) tensor field T on M, its covariant derivative ∇T is a $(k, l+1)$ tensor field given by

$$\begin{aligned}
&(\nabla T)(Y^1, \dots, Y^k, X_1, \dots, X_l, X)\\
&= (\nabla_X T)(Y^1, \dots, Y^k, X_1, \dots, X_l)\\
&= X(T(Y^1, \dots, Y^k, X_1, \dots, X_l)) - T(\nabla_X Y^1, Y^2, \dots, Y^k, X_1, \dots, X_l) - \dots\\
&\quad - T(Y^1, \dots, Y^k, X_1, \dots, X_{l-1}\nabla_X X_l),
\end{aligned}$$

the coordinate expression of which is denoted by

$$\nabla T = \left(T^{i_1\dots i_k}_{j_1\dots j_l ; j_{l+1}}\right).$$

The index appearing after the semicolon indicates the derivative index.

Our convention for the $(1,3)$-Riemannian curvature tensor Rm of a connection ∇ is

$$\mathrm{Rm}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

where X, Y, Z are vector fields and where $[X,Y]$ is the Lie-bracket

$$[X,Y]\varphi = X(Y\varphi) - Y(X\varphi) \quad \forall \varphi \in C^\infty(M).$$

The purely covariant Riemannian curvature tensor is defined by lowering to the fourth slot:

$$\mathrm{Rm}(X,Y,Z,W) = g(\mathrm{Rm}(X,Y)Z, W).$$

Finally the Ricci curvature is

$$\mathrm{Rc}(X,Y) = \mathrm{tr}\left(\mathrm{Rm}(\cdot, X)Y\right).$$

For metrics (g_{ij}) we always denote its dual by (g^{ij}), i.e.

$$\delta^i_j = g^{ik}g_{kj}.$$

The induced geometry of M is governed by the following relations. The second fundamental form $h = (h_{ij})$ is defined by the Gaussian formula

$$\bar{\nabla}_X Y = \nabla_X Y - h(X,Y)\nu, \tag{2.1}$$

where ν is a normal field. The Weingarten endomorphism $\mathcal{W} = (h^i_j)$ is defined by $h^i_j = g^{ki}h_{kj}$ and we have the Weingarten equation

$$\bar\nabla_X \nu = \mathcal{W}(X). \tag{2.2}$$

We also have the Codazzi equation

$$\nabla_Z h(X,Y) - \nabla_Y h(X,Z) = -\overline{\mathrm{Rm}}(\nu, X, Y, Z).$$

Let us record this equation is coordinates:

$$h_{ij;k} - h_{ik;j} = -\overline{\mathrm{Rm}}(\nu, x_{;i}, x_{;j}, x_{;k}). \tag{2.3}$$

The Gauss equation states

$$\mathrm{Rm}(W,X,Y,Z) = \overline{\mathrm{Rm}}(W,X,Y,Z) + h(W,Z)h(X,Y) - h(W,Y)h(X,Z) \tag{2.4}$$

or in coordinates

$$R_{ijkl} = \overline{\mathrm{Rm}}(x_{;i}, x_{;j}, x_{;k}, x_{;l}) + h_{il}h_{jk} - h_{ik}h_{jl}.$$

Warped products. Throughout this paper we assume that the ambient manifold is a *warped product* of the form

$$(N, \bar g) = (I \times \mathcal{S}_0, \bar g),$$

where $I = (R_0, \infty)$, $(\mathcal{S}_0, \sigma)$ is an n-dimensional compact Riemannian manifold and

$$\bar g = dr^2 + \vartheta^2(r)\sigma \tag{2.5}$$

with $\vartheta \in C^\infty((R_0,\infty))$. We will need to know how the curvature tensor of $\bar g$ arises from the curvature tensors of dr^2 and σ. The relevant formulae can be found in [54, Ch. 7, Prop. 42]. We state them here for further use, but adapted to our curvature convention, which differs from the one in op. cit. We denote by $\mathscr{L}(\mathbb{R})$ and $\mathscr{L}(\mathcal{S}_0)$ the space of all vector field on $\mathbb{R}$ resp. $\mathcal{S}_0$ lifted to N.

2.1. **Lemma.** ([54, Ch. 7, Prop. 42]) *Let N be given as above. If $X, Y, Z \in \mathscr{L}(\mathbb{R})$ and $U, V, W \in \mathscr{L}(\mathcal{S}_0)$, then the Riemannian curvature tensor of N is given by*

(i) $\overline{\mathrm{Rm}}(X,Y)Z = 0$,
(ii) $\overline{\mathrm{Rm}}(V,X)Y = -\frac{\bar\nabla^2\vartheta(X,Y)}{\vartheta}V = -\frac{\vartheta''}{\vartheta}\bar g(X,Y)V$,
(iii) $\overline{\mathrm{Rm}}(X,Y)V = \overline{\mathrm{Rm}}(V,W)X = 0$,
(iv) $\overline{\mathrm{Rm}}(X,V)W = -\frac{\vartheta''}{\vartheta}\bar g(V,W)X$
(v) $\overline{\mathrm{Rm}}(V,W)U = \widetilde{\mathrm{Rm}}(V,W)U - \frac{\vartheta'^2}{\vartheta^2}(\bar g(W,U)V - \bar g(V,U)W)$,

where $\widetilde{\mathrm{Rm}}$ is the lift of the Riemann tensor of the fibre $(\mathcal{S}_0, \vartheta^2(r)\sigma)$ under the projection $\pi\colon N \to \mathcal{S}_0$.

It will turn out to be convenient to have a closed coordinate expression for $\overline{\mathrm{Rm}}$, which follows easily from checking all of the five cases.

2.2. **Lemma.** *In coordinates the Riemannian curvature tensor of the warped product*

$$(N, \bar g) = (I \times \mathcal{S}_0, dr^2 + \vartheta^2(r)\sigma)$$

is given by

$$\bar R^\epsilon_{\alpha\beta\gamma} = \left(\left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)\bar S_{\alpha'\beta'\gamma'\delta'} + \widetilde R_{\alpha'\beta'\gamma'\delta'}\right) P^{\alpha'}_\alpha P^{\beta'}_\beta P^{\gamma'}_\gamma P^{\delta'\epsilon} - \frac{\vartheta''}{\vartheta}\bar S^\epsilon_{\alpha\beta\gamma}, \tag{2.6}$$

where

$$\bar S^\epsilon_{\alpha\beta\gamma} = \bar g_{\beta\gamma}\delta^\epsilon_\alpha - \bar g_{\alpha\gamma}\delta^\epsilon_\beta$$

and

$$P = \mathrm{id} - \frac{\partial}{\partial r} \otimes dr.$$

Hence we obtain a formula for the derivative of $\overline{\mathrm{Rm}}$.

2.3. **Lemma.** *The coordinate functions of the covariant derivative of the* $(0,4)$*-curvature tensor are given by*

$$\begin{aligned}\bar{R}_{\alpha\beta\gamma\delta;\epsilon} &= -\left(\frac{\vartheta''}{\vartheta}\right)' r_{;\epsilon}\bar{S}_{\alpha\beta\gamma\delta} + \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)' r_{;\epsilon}\bar{S}_{\alpha'\beta'\gamma'\delta'}P_\alpha^{\alpha'}P_\beta^{\beta'}P_\gamma^{\gamma'}P_\delta^{\delta'} \\ &\quad + \widetilde{R}_{\alpha'\beta'\gamma'\delta';\epsilon}P_\alpha^{\alpha'}P_\beta^{\beta'}P_\gamma^{\gamma'}P_\delta^{\delta'} - \frac{\vartheta'}{\vartheta}r_{;\alpha}\bar{T}_{\alpha'\beta'\gamma'\delta'}P_\epsilon^{\alpha'}P_\beta^{\beta'}P_\gamma^{\gamma'}P_\delta^{\delta'} \\ &\quad - \frac{\vartheta'}{\vartheta}r_{;\beta}\bar{T}_{\alpha'\beta'\gamma'\delta'}P_\alpha^{\alpha'}P_\epsilon^{\beta'}P_\gamma^{\gamma'}P_\delta^{\delta'} - \frac{\vartheta'}{\vartheta}r_{;\gamma}\bar{T}_{\alpha'\beta'\gamma'\delta'}P_\alpha^{\alpha'}P_\beta^{\beta'}P_\epsilon^{\gamma'}P_\delta^{\delta'} \\ &\quad - \frac{\vartheta'}{\vartheta}r_{;\delta}\bar{T}_{\alpha'\beta'\gamma'\delta'}P_\alpha^{\alpha'}P_\beta^{\beta'}P_\gamma^{\gamma'}P_\epsilon^{\delta'},\end{aligned} \tag{2.7}$$

where

$$\bar{T}_{\alpha'\beta'\gamma'\delta'} = \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)\bar{S}_{\alpha'\beta'\gamma'\delta'} + \tilde{R}_{\alpha'\beta'\gamma'\delta'}.$$

Proof. Denote by $\bar{\Gamma}_{\alpha\beta}^{\gamma}$ the Christoffel symbols, i.e.

$$\bar{\Gamma}_{\alpha\beta}^{\gamma}\frac{\partial}{\partial x^\gamma} = \bar{\nabla}_{\frac{\partial}{\partial x^\alpha}}\frac{\partial}{\partial x^\beta}$$

and

$$\bar{\Gamma}_{\alpha\beta}^{\gamma} = \frac{1}{2}\bar{g}^{\gamma\delta}\left(\frac{\partial}{\partial x^\beta}\bar{g}_{\alpha\delta} + \frac{\partial}{\partial x^\alpha}\bar{g}_{\beta\delta} - \frac{\partial}{\partial x^\delta}\bar{g}_{\alpha\beta}\right).$$

Using the definition of the metric we see

$$\bar{\Gamma}_{\alpha\epsilon}^{0} = -\frac{\vartheta'}{\vartheta}\bar{g}_{\alpha'\beta'}P_\alpha^{\alpha'}P_\epsilon^{\beta'} = -\frac{\vartheta'}{\vartheta}\bar{g}_{\alpha'\epsilon}P_\alpha^{\alpha'}, \quad \bar{\Gamma}_{0\epsilon}^{\alpha'} = \frac{\vartheta'}{\vartheta}P_\epsilon^{\alpha'}$$

and hence there holds

$$\begin{aligned}P_{\alpha;\epsilon}^{\alpha'} = -r_{;\epsilon}^{\alpha'}r_\alpha - r_{;}^{\alpha'}r_{;\alpha\epsilon} &= -\bar{\Gamma}_{0\epsilon}^{\alpha'}r_{;\alpha} + r_{;}^{\alpha'}\bar{\Gamma}_{\alpha\epsilon}^{0} \\ &= -\frac{\vartheta'}{\vartheta}P_\epsilon^{\alpha'}r_{;\alpha} - \frac{\vartheta'}{\vartheta}r_{;}^{\alpha'}\bar{g}_{\gamma'\epsilon}P_\alpha^{\gamma'}.\end{aligned}$$

There holds

$$\bar{T}_{\alpha'\beta'\gamma'\delta'}r_{;}^{\alpha'}P_{\epsilon\alpha}P_\beta^{\beta'}P_\gamma^{\gamma'}P_\delta^{\delta'} = 0$$

and hence differentiation of (2.6) gives

$$\begin{aligned}\bar{R}_{\alpha\beta\gamma\delta;\epsilon} &= -\left(\frac{\vartheta''}{\vartheta}\right)' r_{;\epsilon}\bar{S}_{\alpha\beta\gamma\delta} + \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)' r_{;\epsilon}\bar{S}_{\alpha'\beta'\gamma'\delta'}P_\alpha^{\alpha'}P_\beta^{\beta'}P_\gamma^{\gamma'}P_\delta^{\delta'} \\ &\quad + \tilde{R}_{\alpha'\beta'\gamma'\delta';\epsilon}P_\alpha^{\alpha'}P_\beta^{\beta'}P_\gamma^{\gamma'}P_\delta^{\delta'} - \frac{\vartheta'}{\vartheta}r_{;\alpha}\bar{T}_{\alpha'\beta'\gamma'\delta'}P_\epsilon^{\alpha'}P_\beta^{\beta'}P_\gamma^{\gamma'}P_\delta^{\delta'} \\ &\quad - \frac{\vartheta'}{\vartheta}r_{;\beta}\bar{T}_{\alpha'\beta'\gamma'\delta'}P_\alpha^{\alpha'}P_\epsilon^{\beta'}P_\gamma^{\gamma'}P_\delta^{\delta'} - \frac{\vartheta'}{\vartheta}r_{;\gamma}\bar{T}_{\alpha'\beta'\gamma'\delta'}P_\alpha^{\alpha'}P_\beta^{\beta'}P_\epsilon^{\gamma'}P_\delta^{\delta'} \\ &\quad - \frac{\vartheta'}{\vartheta}r_{;\delta}\bar{T}_{\alpha'\beta'\gamma'\delta'}P_\alpha^{\alpha'}P_\beta^{\beta'}P_\gamma^{\gamma'}P_\epsilon^{\delta'},\end{aligned}$$

which is the claimed formula. □

We will later have to deal with the $\tilde{R}_{\alpha'\beta'\gamma'\delta';\epsilon}$-term in (2.7).

2.4. **Lemma.** *For every* $r_0 > R_0$ *there exists a constant* c *such that*

$$\|\bar{\nabla}\widetilde{\mathrm{Rm}}\| \leq c\frac{\vartheta'}{\vartheta^3}.$$

Proof. We define a $\bar{g}$-orthonormal frame $(\tilde{e}_\alpha)_{0\leq\alpha\leq n}$ as follows:

$$e_0 = \tilde{e}_0 = \partial_r$$

and, given a σ-orthonormal frame $(e_i)_{1\leq i\leq n}$ on $\mathcal{S}_0$ we put

$$\tilde{e}_i = \vartheta^{-1}e_i.$$

Then clearly

$$\bar{g}(\tilde{e}_\alpha, \tilde{e}_\beta) = \delta_{\alpha\beta}, \quad 0 \leq \alpha, \beta \leq n.$$

To prove the lemma, it suffices to estimate the components of $\bar{\nabla}\widetilde{\mathrm{Rm}}$ accordingly with respect to this frame. There holds

$$\begin{aligned}\bar{\nabla}_{\tilde{e}_\epsilon}\widetilde{\mathrm{Rm}}(\tilde{e}_\alpha, \tilde{e}_\beta, \tilde{e}_\gamma, \tilde{e}_\delta) &= \tilde{e}_\epsilon\left(\widetilde{\mathrm{Rm}}(\tilde{e}_\alpha, \tilde{e}_\beta, \tilde{e}_\gamma, \tilde{e}_\delta)\right) - \widetilde{\mathrm{Rm}}(\bar{\nabla}_{\tilde{e}_\epsilon}\tilde{e}_\alpha, \tilde{e}_\beta, \tilde{e}_\gamma, \tilde{e}_\delta)\\ &\quad - \widetilde{\mathrm{Rm}}(\tilde{e}_\alpha, \bar{\nabla}_{\tilde{e}_\epsilon}\tilde{e}_\beta, \tilde{e}_\gamma, \tilde{e}_\delta) - \widetilde{\mathrm{Rm}}(\tilde{e}_\alpha, \tilde{e}_\beta, \bar{\nabla}_{\tilde{e}_\epsilon}\tilde{e}_\gamma, \tilde{e}_\delta)\\ &\quad - \widetilde{\mathrm{Rm}}(\tilde{e}_\alpha, \tilde{e}_\beta, \tilde{e}_\gamma, \bar{\nabla}_{\tilde{e}_\epsilon}\tilde{e}_\delta).\end{aligned} \tag{2.8}$$

There holds

$$\widetilde{\mathrm{Rm}}(\tilde{e}_\alpha, \tilde{e}_\beta, \tilde{e}_\gamma, \tilde{e}_\delta) = \vartheta^{-2}\widehat{\mathrm{Rm}}(\pi_* e_\alpha, \pi_* e_\beta, \pi_* e_\gamma, \pi_* e_\delta),$$

where $\widehat{\mathrm{Rm}}$ is the Riemann tensor of σ. Hence

$$\tilde{e}_\epsilon\left(\widetilde{\mathrm{Rm}}(\tilde{e}_\alpha, \tilde{e}_\beta, \tilde{e}_\gamma, \tilde{e}_\delta)\right) = \begin{cases} -\frac{2\vartheta'}{\vartheta^3}\widehat{\mathrm{Rm}}(\pi_* e_\alpha, \pi_* e_\beta, \pi_* e_\gamma, \pi_* e_\delta), & \epsilon = 0\\ \vartheta^{-3} e_\epsilon\left(\widehat{\mathrm{Rm}}(\pi_* e_\alpha, \pi_* e_\beta, \pi_* e_\gamma, \pi_* e_\delta)\right), & \epsilon \neq 0. \end{cases} \tag{2.9}$$

From [54, Ch. 7, Prop. 35] we obtain

$$\pi_*\bar{\nabla}_{\tilde{e}_\epsilon}\tilde{e}_\alpha = \begin{cases} \frac{\vartheta'}{\vartheta}\pi_*\tilde{e}_\alpha, & \epsilon = 0\\ \hat{\nabla}_{\tilde{e}_\epsilon}\tilde{e}_\alpha, & \epsilon \neq 0, \alpha \neq 0\\ \frac{\vartheta'}{\vartheta}\tilde{e}_\epsilon, & \epsilon \neq 0, \alpha = 0, \end{cases} \tag{2.10}$$

where $\hat{\nabla}$ is the Levi-Civita connection of σ. In case $\epsilon \neq 0$, $\alpha \neq 0$ we have

$$\hat{\nabla}_{\tilde{e}_\epsilon}\tilde{e}_\alpha = \vartheta^{-1}\hat{\nabla}_{e_\epsilon}(\vartheta^{-1}e_\alpha) = \vartheta^{-2}\hat{\nabla}_{e_\epsilon}e_\alpha$$

and

$$\widetilde{\mathrm{Rm}}(\bar{\nabla}_{\tilde{e}_\epsilon}\tilde{e}_\alpha, \tilde{e}_\beta, \tilde{e}_\gamma, \tilde{e}_\delta) = \vartheta^{-3}\widehat{\mathrm{Rm}}(\hat{\nabla}_{e_\epsilon}e_\alpha, e_\beta, e_\gamma, e_\delta).$$

Using (2.9) and (2.10) in (2.8) in any of the cases, we obtain the desired estimate, since $\vartheta' \geq c_{r_0} > 0$ on every interval $[r_0, \infty)$, giving the estimate

$$\vartheta^{-3} \leq c\frac{\vartheta'}{\vartheta^3}.$$

□

2.5. *Remark.* For example, if σ is the round metric on $\mathcal{S}_0 = \mathbb{S}^n$, then

$$\tilde{R}_{\alpha\beta\gamma\delta} = \frac{1}{\vartheta^2}\bar{S}_{\alpha'\beta'\gamma'\delta'}P^{\alpha'}_{\alpha}P^{\beta'}_{\beta}P^{\gamma'}_{\gamma}P^{\delta'}_{\delta}.$$

Graphs in warped products. The hypersurfaces

$$x\colon M \hookrightarrow N$$

we deal with in this paper will all be graphs over $\mathcal{S}_0$,

$$x(M) = \{(u(y), y)\colon y \in \mathcal{S}_0\} = \{(u(y(\xi)), y(\xi))\colon \xi \in M\},$$

where

$$u\colon \mathcal{S}_0 \to (R_0, \infty)$$

is smooth. Along M we will always use the *outward* pointing normal

$$\nu = v^{-1}(1, -\vartheta^{-2}\sigma^{ik}u_{;k}),$$

where

$$v^2 = 1 + \vartheta^{-2}\sigma^{ij}u_{;i}u_{;j},$$

and use this normal in the Gaussian formula (2.1). The support function of M is defined by

$$s = \bar{g}(\vartheta\partial_r, \nu) = \frac{\vartheta}{v}. \tag{2.11}$$

There is a relation between the second fundamental form and the graph function on the hypersurface. Let

$$\bar{h} = \vartheta'\vartheta\sigma,$$

then there holds

$$v^{-1}h_{ij} = -u_{;ij} + \bar{h}_{ij},$$

cf. [23, equ. (1.5.10)]. The induced metric is given by

$$g_{ij} = u_{;i}u_{;j} + \vartheta^2\sigma_{ij}$$

and hence

$$v^{-1}h_{ij} = -u_{;ij} + \frac{\vartheta'}{\vartheta}g_{ij} - \frac{\vartheta'}{\vartheta}u_{;i}u_{;j}. \tag{2.12}$$

In order to deduce the gradient estimates, it has proven to be useful to consider the function

$$\varphi\colon \mathcal{S}_0 \to \mathbb{R}$$

$$\varphi(y) = \int_{\inf u_0}^{u(y)} \frac{1}{\vartheta(s)}\, ds. \tag{2.13}$$

There holds

$$h_i^j = \frac{\vartheta'}{\vartheta v}\delta_i^j - \frac{1}{\vartheta v}\tilde{g}^{jk}\varphi_{:ki}, \tag{2.14}$$

where

$$\tilde{g}^{ij} = \sigma^{ij} - \frac{\varphi_{:}^{\ i}\varphi_{:}^{\ j}}{v^2}$$

and the covariant derivative and index raising is performed with respect to σ, cf. [24, equ. (3.26)]. We will use $\hat{\nabla}$ to denote the covariant derivative on $\mathcal{S}_0$ throughout this paper.

2.2. **Curvature functions.** Let $\Gamma \subset \mathbb{R}^n$ be an open and symmetric cone. In Assumption 1.1 the symmetric function $f \in C^\infty(\Gamma)$ is supposed to be evaluated at the principal curvatures of the flow hypersurfaces. This gives rise to an *associated curvature function* F, acting on diagonalisable endomorphisms A of an arbitrary real vector space V via

$$F(A) = f(\mathrm{EV}(A)),$$

where $\mathrm{EV}(A)$ is the unordered n-tuple of eigenvalues of A.

However, when using this definition, F is not defined on the whole space of endomorphisms, but only on the diagonalisable operators. Hence it appears reasonable to view F as defined on bilinear forms,

$$\hat{F}(g,h) := F\left(\frac{1}{2}g^{ik}(h_{kj} + h_{jk})\right)$$

for all positive definite $g = (g_{ij})$ and all bilinear forms $h = (h_{ij}) \in T_p^{0,2}M$. Then

$$\hat{F}^{ij} = \frac{\partial F}{\partial h_{ij}}$$

is a $(2,0)$-tensor and we also write

$$\hat{F}^{ij,kl} = \frac{\partial F}{\partial h_{ij}\partial h_{kl}}.$$

Furthermore, if $F = F(\kappa_i)$ is strictly monotone, then $\hat{F}^{ij}$ is strictly elliptic. If F is concave, then

$$\hat{F}^{ij,kl}\eta_{ij}\eta_{kl} \leq 0$$

for all symmetric (η_{ij}). We refer to [4], [23, Ch. 2] and [61] for more details on curvature functions.

Furthermore we will abuse notation and also write F for $\hat{F}$, since no confusion will be possible. E.g., when writing F^{ij}, we can only mean $\hat{F}^{ij}$, since there are two contravariant indices.

We will also use the special curvature functions H_k, associated to the k-th normalised elementary symmetric polynomial σ_k defined on Γ_k, the connected component of $\{\sigma_k > 0\}$ which contains the point $(1, \dots, 1)$.

2.3. **Evolution equations.** The following evolution equations for (1.1) are well known and can be found in several places, for example in [23, Sec. 2.3, Sec. 2.4]. Note that, compared to this reference, we use a different convention on the Riemann tensor.

2.6. **Lemma.** *Denote $\mathcal{F} = -F^{-p}$. Along (1.1) there hold:*

(i) The induced metric g satisfies

$$\dot{g} = -2\mathcal{F}h.$$

(ii) The normal vector field satisfies

$$\frac{\bar{\nabla}}{dt}\nu = \operatorname{grad}\mathcal{F},$$

where $\frac{\bar{\nabla}}{dt}$ is the covariant time derivative along the curve $x(\cdot, \xi)$ for fixed $\xi \in M$.

(iii) The second fundamental form satisfies

$$\dot{h}_{ij} = \mathcal{F}_{;ij} - \mathcal{F}h_{ik}h^k_j + \mathcal{F}\overline{\mathrm{Rm}}(x_{;i}, \nu, \nu, x_{;j}). \tag{2.15}$$

(iv) The flow speed $\mathcal{F}$ satisfies

$$\dot{\mathcal{F}} - \mathcal{F}^{ij}F_{;ij} = \mathcal{F}^{ij}h_{ik}h^k_j\mathcal{F} + \mathcal{F}^{ij}\overline{\mathrm{Rm}}(x_{;i}, \nu, \nu, x_{;j})\mathcal{F}. \tag{2.16}$$

2.7. **Lemma.** *Under the flow (1.1) with $\mathcal{F} = -F^{-p}$ the second fundamental form evolves by*

$$\begin{aligned}
\dot{h}^i_j - \mathcal{F}^{kl}h^i_{j;kl} = {} & \mathcal{F}^{kl,rs}h_{kl;}{}^i h_{rs;j} + \mathcal{F}^{kl}h_{rk}h^r_l h^i_j - (\mathcal{F}^{kl}h_{kl} - \mathcal{F})h^i_r h^r_j \\
& + \mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta}\left(x^\alpha_{;l}x^\beta_{;j}x^\gamma_{;k}x^\delta_{;m}h^{im} + x^\alpha_{;l}x^\beta_{;r}x^\gamma_{;k}x^\delta_{;m}h^m_j g^{ri}\right) \\
& + 2\mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;r}x^\beta_{;m}x^\gamma_{;k}x^\delta_{;j}h^m_l g^{ri} - \mathcal{F}^{kl}h_{kl}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;r}\nu^\beta\nu^\gamma x^\delta_{;j}g^{ri} \\
& + \mathcal{F}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;r}\nu^\beta\nu^\gamma x^\delta_j g^{ri} + \mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;k}\nu^\beta\nu^\gamma x^\delta_{;l}h^i_j \\
& + \mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{;k}x^\gamma_{;r}x^\delta_{;l}x^\epsilon_{;j}g^{ri} + \mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{;r}x^\gamma_{;j}x^\delta_{;k}x^\epsilon_{;l}g^{ri}.
\end{aligned}$$

Proof. Basically this is [23, Lemma 2.4.1]. For convenience we deduce it again, since the proof in that reference is a little rough and we use another convention for the Riemann tensor. There hold

$$\mathcal{F}_{;i} = \mathcal{F}^{kl}h_{kl;i}$$

and

$$\mathcal{F}_{;ij} = \mathcal{F}^{kl,rs}h_{kl;i}h_{rs;j} + \mathcal{F}^{kl}h_{kl;ij}.$$

We differentiate the Codazzi equation (2.3) to replace the second term on the right hand side. First we differentiate the Codazzi equation with respect to ∂_j, then

use the Ricci identities and then differentiate the Codazzi equation with respect to ∂_l. We also use the Weingarten equation (2.2) and the Gauss equation (2.4).

$$\begin{aligned}
h_{kl;ij} &= h_{ki;lj} - \left(\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;k} x^\gamma_{;l} x^\delta_{;i}\right)_{;j} \\
&= h_{ki;jl} + R_{ljk}{}^a h_{ai} + R_{lji}{}^a h_{ka} - \left(\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;k} x^\gamma_{;l} x^\delta_{;i}\right)_{;j} \\
&= R_{ljk}{}^a h_{ai} + R_{lji}{}^a h_{ka} - \left(\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;k} x^\gamma_{;l} x^\delta_{;i}\right)_{;j} \\
&\quad + h_{ij;kl} - \left(\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;i} x^\gamma_{;k} x^\delta_{;j}\right)_{;l} \\
&= h_{ij;kl} + (h_{la}h_{jk} - h_{lk}h_{ja} + \bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;l}x^\beta_{;j}x^\gamma_{;k}x^\delta_{;a})h^a_i \\
&\quad + (h_{la}h_{ji} - h_{li}h_{ja} + \bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;l}x^\beta_{;j}x^\gamma_{;i}x^\delta_{;a})h^a_k \\
&\quad - \bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{;k}x^\gamma_{;l}x^\delta_{;i}x^\epsilon_{;j} - \bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;m}x^\beta_{;k}x^\gamma_{;l}x^\delta_{;i}h^m_j + \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;k}\nu^\gamma x^\delta_{;i}h_{lj} \\
&\quad + \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;k}x^\gamma_{;l}\nu^\delta h_{ij} - \bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{;i}x^\gamma_{;k}x^\delta_{;j}x^\epsilon_{;l} - \bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;m}x^\beta_{;i}x^\gamma_{;k}x^\delta_{;j}h^m_l \\
&\quad + h_{kl}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;i}\nu^\gamma x^\delta_{;j} + \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;i}x^\gamma_{;k}\nu^\delta h_{jl}.
\end{aligned}$$

Recall that h satisfies (2.15):

$$\dot{h}_{ij} = \mathcal{F}_{;ij} - \mathcal{F}h_{ik}h^k_j + \mathcal{F}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_i\nu^\beta\nu^\gamma x^\delta_j$$

and hence

$$\begin{aligned}
\dot{h}_{ij} - \mathcal{F}^{kl}h_{ij;kl} &= \mathcal{F}^{kl,rs}h_{kl;i}h_{rs;j} - \mathcal{F}^{kl}h_{kl}h^r_i h_{rj} + \mathcal{F}^{kl}h_{rk}h^r_l h_{ij} \\
&\quad + \mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta}\left(x^\alpha_{;l}x^\beta_{;j}x^\gamma_{;k}x^\delta_{;m}h^m_i + x^\alpha_{;l}x^\beta_{;i}x^\gamma_{;k}x^\delta_{;m}h^m_j\right) \\
&\quad + \mathcal{F}^{kl}h_{kl}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;i}\nu^\gamma x^\delta_{;j} + \mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;k}x^\gamma_{;l}\nu^\delta h_{ij} \\
&\quad + 2\mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;i}x^\beta_{;m}x^\gamma_{;k}x^\delta_{;j}h^m_l - \mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{;k}x^\gamma_{;l}x^\delta_{;i}x^\epsilon_{;j} \\
&\quad - \mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{;i}x^\gamma_{;k}x^\delta_{;j}x^\epsilon_{;l} - \mathcal{F}h_{ik}h^k_j + \mathcal{F}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_i\nu^\beta\nu^\gamma x^\delta_j.
\end{aligned}$$

The result follows after reverting to the mixed representation. □

Graphical hypersurfaces. Given the flow (1.1) of graphs

$$M_t = \{(u(t, y(t,\xi)), y(t,\xi))\colon \xi \in M\}$$

in a warped product with metric of the form (2.5), we first of all deduce from (2.12) that

$$\dot{u} - \mathcal{F}^{ij}u_{;ij} = \frac{p+1}{F^p}v^{-1} - \frac{\vartheta'}{\vartheta}\frac{p}{F^{p+1}}F^{ij}g_{ij} + \frac{\vartheta'}{\vartheta}\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}. \tag{2.17}$$

Now we deduce the evolution of the quantity

$$w = \frac{1}{\vartheta^2(u)}|du|^2_\sigma = |d\varphi|^2_\sigma,$$

where φ was defined in (2.13). The function φ is better suited to these estimates than u itself, since the representation of the second fundamental form is simpler and so the differentiation of the speed $\mathcal{F}$ is easier to perform. This trick was also used in [22], [65] and in subsequent treatments of graphical expanding flows. Note that φ satisfies

$$\partial_t\varphi = -\mathcal{F}s^{-1}, \tag{2.18}$$

where s is the support function defined in (2.11). In the next lemma we derive the evolution equation for w. We simplify notation: Putting lower indices to a function means covariant differentiation with respect to σ.

2.8. **Lemma.** *Under the flow (2.18) in a warped product of the form (2.5) the gradient function*

$$|\hat\nabla\varphi|^2_\sigma = \varphi_i\varphi^i$$

satisfies

$$\begin{aligned}
&\left(\frac{d}{dt} - \frac{1}{\vartheta^2}\mathcal{F}^k_l\tilde g^{lr}\hat\nabla_{kr}\right)|\hat\nabla\varphi|^2\\
&= 2\mathcal{F}\frac{s_i}{s^2}\varphi^i - 2\mathcal{F}^k_l h^l_k\frac{s_i}{s^2}\varphi^i + 4\vartheta'\mathcal{F}^k_l h^l_k s^{-1}|\hat\nabla\varphi|^2 - 2\frac{\vartheta''}{\vartheta}\mathcal{F}^k_k|\hat\nabla\varphi|^2\\
&-\frac{1}{2v^2\vartheta^2}\mathcal{F}^k_l\sigma^{lm}|\hat\nabla\varphi|^2_m|\hat\nabla\varphi|^2_k - \frac{1}{v^2\vartheta^2}\mathcal{F}^k_l\varphi^l|\hat\nabla\varphi|^2_r\varphi^r_k\\
&+\frac{1}{v^4\vartheta^2}\mathcal{F}^k_l\varphi^l|\hat\nabla\varphi|^2_k|\hat\nabla\varphi|^2_i\varphi^i - \frac{2}{\vartheta^2}\mathcal{F}^k_l\tilde g^{lr}\varphi_{ir}\varphi^i_k - \frac{2}{\vartheta^2}\mathcal{F}^k_l\tilde g^{lr}\hat R_{ikrm}\varphi^i\varphi^m.
\end{aligned}$$

Proof. From (2.18) we get

$$\frac{d}{dt}|\hat\nabla\varphi|^2 = 2\dot\varphi_i\varphi^i = \frac{2}{s^2}\mathcal{F}s_i\varphi^i - 2\mathcal{F}^k_l\hat\nabla_i h^l_k\varphi^i s^{-1}.$$

Due to (2.14) there holds

$$\begin{aligned}
\hat\nabla_i h^l_k &= -\frac{v_i\vartheta + v\vartheta'\vartheta\varphi_i}{v^2\vartheta^2}\left(\vartheta'\delta^l_k - \tilde g^{lr}\varphi_{rk}\right) + \frac{1}{v\vartheta}\left(\vartheta''\vartheta\varphi_i\delta^l_k - \hat\nabla_i\tilde g^{lr}\varphi_{rk} - \tilde g^{lr}\varphi_{rki}\right)\\
&= -\frac{v_i}{v}h^l_k - \vartheta'\varphi_i h^l_k + \frac{\vartheta''}{v}\delta^l_k\varphi_i + \frac{\varphi^l_i\varphi^r + \varphi^l\varphi^r_i}{v^3\vartheta}\varphi_{rk} - \frac{2}{v^4\vartheta}v_i\varphi^l\varphi^r\varphi_{rk}\\
&\quad - \frac{1}{v\vartheta}\tilde g^{lr}\varphi_{rki}\\
&= \frac{s_i}{s}h^l_k - 2\vartheta'\varphi_i h^l_k + \frac{\vartheta''}{v}\delta^l_k\varphi_i + \frac{\varphi^l_i\varphi^r + \varphi^l\varphi^r_i}{v^3\vartheta}\varphi_{rk}\\
&\quad - \frac{1}{v^5\vartheta}|\hat\nabla\varphi|^2_i\varphi^l\varphi^r\varphi_{rk} - \frac{1}{v\vartheta}\tilde g^{lr}\varphi_{rik} + \frac{1}{v\vartheta}\tilde g^{lr}\hat R^m_{ikr}\varphi_m,
\end{aligned}$$

where we used the definition of the Riemann tensor of σ. Using

$$|\hat\nabla\varphi|^2_{rk} = 2\varphi_{irk}\varphi^i + 2\varphi_{ir}\varphi^i_k,$$

we combine these two equalities to get

$$\begin{aligned}
&\left(\frac{d}{dt} - \frac{1}{\vartheta^2}\mathcal{F}^k_l\tilde g^{lr}\hat\nabla_{kr}\right)|\hat\nabla\varphi|^2\\
&= 2\mathcal{F}\frac{s_i}{s^2}\varphi^i - 2\mathcal{F}^k_l h^l_k\frac{s_i}{s^2}\varphi^i + 4\vartheta'\mathcal{F}^k_l h^l_k s^{-1}|\hat\nabla\varphi|^2 - 2\frac{\vartheta''}{\vartheta}F^k_k|\hat\nabla\varphi|^2\\
&-\frac{2}{v^2\vartheta^2}\mathcal{F}^k_l(\varphi^l_i\varphi^r + \varphi^l\varphi^r_i)\varphi^i\varphi_{rk} + \frac{2}{v^4\vartheta^2}\mathcal{F}^k_l\varphi^l\varphi^r\varphi_{rk}|\hat\nabla\varphi|^2_i\varphi^i\\
&-\frac{2}{\vartheta^2}\mathcal{F}^k_l\tilde g^{lr}\varphi_{ir}\varphi^i_k - \frac{2}{\vartheta^2}\mathcal{F}^k_l\tilde g^{lr}\hat R^m_{ikr}\varphi^i\varphi_m
\end{aligned}$$

and hence the result. □

The support function satisfies the following evolution.

2.9. **Lemma.** *Along (1.1) in a warped product with metric (2.5), the support function*

$$s = \vartheta(u)\bar g(\partial_r, \nu)$$

satisfies

$$\dot s - \mathcal{F}^{ij}s_{;ij} = \mathcal{F}^{ij}h_{ik}h^k_j s - \vartheta'\frac{p-1}{F^p} + \bar g(\vartheta\partial_r, \nabla\mathcal{F}) - \mathcal{F}^{ij}(\bar g(\vartheta\partial_r, x_{;k}h^k_{i;j})). \tag{2.19}$$

Proof. The vector field $\vartheta\partial_r$ is conformal,

$$\bar\nabla_{\bar X}(\vartheta\partial_r) = \vartheta'\bar X \quad \forall \bar X \in T^{1,0}(N).$$

Hence

$$\dot s = \bar g(\vartheta'\dot x, \nu) + \bar g(\vartheta\partial_r, \bar\nabla_{\dot x}\nu) = -\vartheta'\mathcal{F} + \bar g(\vartheta\partial r, \nabla\mathcal{F}),$$

$$Xs = \bar g(\vartheta\partial_r, \mathcal{W}(X)) \tag{2.20}$$

and

$$\begin{aligned}\nabla^2 s(X,Y) &= Y(Xs) - (\nabla_Y X)s\\ &= \vartheta' h(X,Y) - h(X,\mathcal{W}(Y))s + \bar g(\vartheta\partial_r, \nabla_Y\mathcal{W}(X)).\end{aligned}$$

The result follows from combining these equalities. □

We will also make use of the evolution of $\dot\varphi$. This method was used in [5, Prop. 3.4] and [60, Lemma 3.5].

2.10. **Lemma.** *Under the flow (2.18) in a warped product of the form (2.5) the speed $\dot\varphi$ satisfies*

$$\partial_t\dot\varphi - \frac{\partial\dot\varphi}{\partial\varphi_{ij}}\dot\varphi_{ij} - \frac{\partial\dot\varphi}{\partial\varphi_i}\dot\varphi_i = \frac{\vartheta'}{\vartheta}v\mathcal{F}^i_j h^j_i\dot\varphi - \frac{\vartheta''}{\vartheta}\mathcal{F}^k_k\dot\varphi + \frac{\vartheta'}{\vartheta}v\mathcal{F}\dot\varphi. \tag{2.21}$$

Proof. Differentiating

$$\dot\varphi = -\mathcal{F}s^{-1}$$

gives

$$\begin{aligned}\partial_t\dot\varphi - \frac{\partial\dot\varphi}{\partial\varphi_{ij}}\dot\varphi_{ij} - \frac{\partial\dot\varphi}{\partial\varphi_i}\dot\varphi_i &= \frac{\partial\dot\varphi}{\partial\varphi}\dot\varphi\\ &= -\mathcal{F}^j_i\frac{h^i_j}{\partial\varphi}s^{-1}\dot\varphi + s^{-1}\mathcal{F}\vartheta'\dot\varphi.\end{aligned}$$

From (2.14) we get

$$\frac{\partial h^i_j}{\partial\varphi} = -\frac{\vartheta'}{v\vartheta}\left(\vartheta'\delta^i_j - \tilde g^{ik}\varphi_{kj}\right) + \frac{\vartheta''}{v}\delta^i_j$$

and inserting this gives the result. □

3. Long-time existence

3.1. Barriers.

3.1. **Lemma.** *Let $\vartheta \in C^2((R_0,\infty))$ with $\vartheta' > 0$ and $\vartheta'' \geq 0$, $r_0 > R_0$ and $0 < p \leq 1$. Let $r(t, r_0)$ be the unique solution of the initial value problem*

$$\begin{aligned}\dot r &= \frac{\vartheta^p(r)}{n^p\vartheta'^p(r)}\\ r(0) &= r_0.\end{aligned} \tag{3.1}$$

Then r is defined for all times and

$$r(t, r_0) \to \infty, \quad t \to \infty.$$

Consequently, for x_0 as in Theorem 1.3 with associated graph function u_0, we have

$$\inf_M u(t,\cdot) \to \infty, \quad t \to \infty,$$

provided the flow (1.2) exists for all times.

Proof. Due to (3.1) we have

$$\dot r \leq 1 + \frac{\vartheta(r)}{\vartheta'(r)},$$

where the right hand side grows at most linearly in r due to $\vartheta'' \geq 0$. Hence r is defined for all times. Suppose r does not converge to infinity. Due to its monotonicity it converges to some $r_1 < \infty$. From

$$\frac{\vartheta}{\vartheta'}(r) > 0 \quad \forall r \in [r_0, r_1]$$

we obtain $\dot r \geq c > 0$ and reach a contradiction. The second claim follows from the maximum principle which gives

$$r(t, \inf u_0) \leq u(t, \cdot) \leq r(t, \sup u_0).$$

□

3.2. **Gradient estimates.** Let us first prove some rough gradient estimates which will suffice to get the long-time existence. In the a priori estimates that appear in the rest of the paper, generic constants will be allowed to depend on the data of the problem, namely N, p, M_0 unless otherwise stated.

First we need a bound on F from below:

3.2. **Lemma.** *Under the assumptions of Theorem 1.3 (i), along (1.1) the spatial maxima of the quantity*

$$\dot\varphi = \frac{1}{F^p}\frac{v}{\vartheta}$$

are non-increasing.

Proof. According to (2.21) we have

$$\partial_t \dot\varphi - \frac{\partial \dot\varphi}{\partial \varphi_{ij}}\dot\varphi_{ij} - \frac{\partial \dot\varphi}{\partial \varphi_i}\dot\varphi_i = \frac{(p-1)\vartheta'}{\vartheta F^p} v\dot\varphi - \frac{\vartheta'' p}{\vartheta F^{p+1}} F^i_i \dot\varphi \leq 0.$$

The result follows from the maximum principle. □

Now we prove some very general gradient estimates for inverse curvature flows in warped products. We use the notation from Lemma 2.8.

3.3. **Lemma.** *Under the assumptions of Theorem 1.3 (i), along (1.1) the function $|\hat\nabla\varphi|^2$ is bounded on every finite time interval. Furthermore, under the assumptions of Theorem 1.3 (ii), there exists a positive constant γ, such that the spatial maxima of*

$$\hat z = |\hat\nabla\varphi|^2 \vartheta^\gamma$$

are non-increasing, provided $p < 1$, and such that the spatial maxima of

$$\tilde z = |\hat\nabla\varphi|^2 \vartheta'^\gamma$$

are non-increasing, regardless the value of $0 < p \leq 1$.

Proof. We want to calculate the evolution equations of $\hat z$ and $\tilde z$. Hence we need one for u, which makes use of the parabolic operator with respect to the metric σ. Note that in (2.17) we use covariant derivatives of the metric induced by u, hence we need to rewrite this. The covariant derivatives with respect to σ and g are related by

$$\begin{aligned}\hat\nabla^2 u &= v^2\nabla^2 u + \frac{\vartheta'}{\vartheta}\left(2du\otimes du - \vartheta^2(v^2-1)\sigma\right)\\ &= -vh + v^2\frac{\vartheta'}{\vartheta}g - v^2\frac{\vartheta'}{\vartheta}du\otimes du + 2\frac{\vartheta'}{\vartheta}du\otimes du - \vartheta'\vartheta(v^2-1)\sigma,\end{aligned}$$

cf. [60, equ. (71)] and (2.12). We obtain

$$\begin{aligned}\left(\partial_t - \frac{1}{\vartheta^2}\frac{p}{F^{p+1}}F_l^k\tilde{g}^{lr}\hat{\nabla}^2_{kr}\right)u &= \frac{v}{F^p} + \frac{pv}{F^p} - \frac{p}{F^{p+1}}\frac{\vartheta'}{\vartheta}F_k^k v^2\\ &\quad + \frac{p}{F^{p+1}}\frac{\vartheta'}{\vartheta}F^{kr}u_{;k}u_{;r}v^2 - \frac{2p}{F^{p+1}}\frac{\vartheta'}{\vartheta}F^{kr}u_{;k}u_{;r}\\ &\quad + \frac{p}{F^{p+1}}\frac{\vartheta'}{\vartheta}F^{kr}(g_{kr} - u_{;k}u_{;r})(v^2-1)\\ &= \frac{p+1}{F^p}v - \frac{p}{F^{p+1}}\frac{\vartheta'}{\vartheta}F_k^k - \frac{p}{F^{p+1}}\frac{\vartheta'}{\vartheta}F^{kr}u_{;k}u_{;r}.\end{aligned}$$

Now we use

$$\frac{s_i}{s} = \vartheta'\varphi_i - \frac{v_i}{v} = \vartheta'\varphi_i - \frac{1}{2}\frac{|\hat{\nabla}\varphi|^2_i}{v^2}$$

to deduce from Lemma 2.8:

$$\begin{aligned}&\left(\partial_t - \frac{1}{\vartheta^2}\frac{p}{F^{p+1}}F_l^k\tilde{g}^{lr}\hat{\nabla}_{kr}\right)|\hat{\nabla}\varphi|^2\\ &= -\frac{2}{F^p}\frac{\vartheta'}{\vartheta}v|\hat{\nabla}\varphi|^2 + \frac{1}{\vartheta v F^p}|\hat{\nabla}\varphi|^2_i\varphi^i - \frac{2p}{F^p}\frac{\vartheta'}{\vartheta}v|\hat{\nabla}\varphi|^2 + \frac{p}{\vartheta v F^p}|\hat{\nabla}\varphi|^2_i\varphi^i\\ &\quad + \frac{4p}{F^p}\frac{\vartheta'}{\vartheta}v|\hat{\nabla}\varphi|^2 - \frac{2p}{F^{p+1}}\frac{\vartheta''}{\vartheta}F_k^k|\hat{\nabla}\varphi|^2 - \frac{1}{2v^2\vartheta^2}\frac{p}{F^{p+1}}F_l^k\sigma^{lm}|\hat{\nabla}\varphi|^2_m|\hat{\nabla}\varphi|^2_k\\ &\quad - \frac{1}{v^2\vartheta^2}\frac{p}{F^{p+1}}F_l^k\varphi^l|\hat{\nabla}\varphi|^2_r\varphi^r_k + \frac{1}{v^4\vartheta^2}\frac{p}{F^{p+1}}F_l^k\varphi^l|\hat{\nabla}\varphi|^2_k|\hat{\nabla}\varphi|^2_i\varphi^i\\ &\quad - \frac{2}{\vartheta^2}\frac{p}{F^{p+1}}F_l^k\tilde{g}^{lr}\varphi_{ir}\varphi^i_k - \frac{2}{\vartheta^2}\frac{p}{F^{p+1}}F_l^k\tilde{g}^{lr}\hat{R}^m_{ikr}\varphi^i\varphi_m\\ &= \frac{2(p-1)}{F^p}\frac{\vartheta'}{\vartheta}v|\hat{\nabla}\varphi|^2 - \frac{2p}{F^{p+1}}\frac{\vartheta''}{\vartheta}F_k^k|\hat{\nabla}\varphi|^2 - \frac{2}{\vartheta^2}\frac{p}{F^{p+1}}F_l^k\tilde{g}^{lr}\hat{R}^m_{ikr}\varphi^i\varphi_m\\ &\quad + \frac{p+1}{F^p}\frac{1}{v\vartheta}|\hat{\nabla}\varphi|^2_i\varphi^i - \frac{2p}{F^{p+1}}\frac{1}{\vartheta^2}F_l^k\tilde{g}^{lr}\varphi_{ir}\varphi^i_k - \frac{1}{2v^2\vartheta^2}\frac{p}{F^{p+1}}F_l^k\sigma^{lm}|\hat{\nabla}\varphi|^2_k|\hat{\nabla}\varphi|^2_m\\ &\quad + \frac{1}{v^4\vartheta^2}\frac{p}{F^{p+1}}F_l^k\varphi^l|\hat{\nabla}\varphi|^2_k|\hat{\nabla}\varphi|^2_i\varphi^i - \frac{1}{v^2\vartheta^2}\frac{p}{F^{p+1}}F_l^k\varphi^l|\hat{\nabla}\varphi|^2_r\varphi^r_k.\end{aligned}$$

Now first generally put

$$z = f(u)|\hat{\nabla}\varphi|^2.$$

With the help of the previous calculations we get at a maximal point of z, where

$$|\hat{\nabla}\varphi|^2_i = -\frac{f'}{f}\vartheta|\hat{\nabla}\varphi|^2\varphi_i,$$

$$
\begin{aligned}
&\mathcal{L}z \equiv \left(\partial_t - \frac{1}{\vartheta^2}\frac{p}{F^{p+1}}F_l^k\tilde{g}^{lr}\hat{\nabla}_{kr}\right)z\\
&= f\mathcal{L}|\hat{\nabla}\varphi|^2 + z\frac{f'}{f}\mathcal{L}u - \frac{f''}{f}\frac{p}{\vartheta^2F^{p+1}}F_l^k\tilde{g}^{lr}u_ku_rz - \frac{2p}{\vartheta^2F^{p+1}}F_l^k\tilde{g}^{lr}|\hat{\nabla}\varphi|_k^2f_r\\
&= \frac{2(p-1)}{F^p}\frac{\vartheta'}{\vartheta}vz - \frac{2p}{F^{p+1}}\frac{\vartheta''}{\vartheta}F_k^kz - \frac{2f}{\vartheta^2}\frac{p}{F^{p+1}}F_l^k\tilde{g}^{lr}\hat{R}^m_{ikr}\varphi^i\varphi_m\\
&-\frac{p+1}{F^p}\frac{f'}{f}\frac{1}{v}|\hat{\nabla}\varphi|^2z - \frac{2p}{F^{p+1}}\frac{1}{\vartheta^2}F_l^k\tilde{g}^{lr}\varphi_{ir}\varphi_k^if\\
&-\frac{1}{2v^2}\frac{f'^2}{f^2}\frac{p}{F^{p+1}}F_l^k\varphi^l\varphi_k|\hat{\nabla}\varphi|^2z + \frac{1}{v^4}\frac{f'^2}{f^2}\frac{p}{F^{p+1}}F_l^k\varphi^l\varphi_k|\hat{\nabla}\varphi|^4z\\
&-\frac{1}{2v^2}\frac{f'^2}{f^2}\frac{p}{F^{p+1}}F_l^k\varphi^l\varphi_k|\hat{\nabla}\varphi|^2z + \frac{p+1}{F^p}\frac{f'}{f}vz - \frac{p}{F^{p+1}}\frac{f'}{f}\frac{\vartheta'}{\vartheta}F_k^kz\\
&-\frac{p}{F^{p+1}}\frac{f'}{f}\frac{\vartheta'}{\vartheta}F^{kr}u_{;k}u_{;r}z + \frac{2p}{F^{p+1}}\frac{f'^2}{f^2}F_l^k\tilde{g}^{lr}\varphi_k\varphi_rz - \frac{p}{F^{p+1}}\frac{f''}{f}F^{kr}u_ku_rz\\
&= \frac{2(p-1)}{F^p}\frac{\vartheta'}{\vartheta}vz - \frac{2p}{F^{p+1}}\frac{\vartheta''}{\vartheta}F_k^kz - \frac{2f}{\vartheta^2}\frac{p}{F^{p+1}}F_l^k\tilde{g}^{lr}\hat{R}^m_{ikr}\varphi^i\varphi_m\\
&+\frac{p+1}{F^p}\frac{f'}{f}\frac{1}{v}z - \frac{2p}{F^{p+1}}\frac{1}{\vartheta^2}F_l^k\tilde{g}^{lr}\varphi_{ir}\varphi_k^if - \frac{f'^2}{f^2}\frac{p}{F^{p+1}}F_l^k\varphi^l\varphi_k\frac{|\hat{\nabla}\varphi|^2}{v^4}z\\
&-\frac{p}{F^{p+1}}\frac{f'}{f}\frac{\vartheta'}{\vartheta}F_k^kz + \frac{p}{F^{p+1}}F^{kr}u_ku_rz\left(2\frac{f'^2}{f^2} - \frac{f''}{f} - \frac{f'}{f}\frac{\vartheta'}{\vartheta}\right).
\end{aligned}
\tag{3.2}
$$

In order to prove the first claim under the assumptions of Theorem 1.3 (i), we first put $f = 1$ and obtain at critical points of z:

$$
\begin{aligned}
\mathcal{L}z = &\frac{2(p-1)}{F^p}\frac{\vartheta'}{\vartheta}vz - \frac{2p}{F^{p+1}}\frac{\vartheta''}{\vartheta}F_k^kz - \frac{2}{\vartheta^2}\frac{p}{F^{p+1}}F_l^k\tilde{g}^{lr}\hat{R}^m_{ikr}\varphi^i\varphi_m\\
&- \frac{2p}{F^{p+1}}\frac{1}{\vartheta^2}F_l^k\tilde{g}^{lr}\varphi_{ir}\varphi_k^i.
\end{aligned}
\tag{3.3}
$$

According to Theorem 1.3 (i), we have to distinguish two cases. In case (a), when σ has non-negative sectional curvature, each term on the right hand side of (3.3) is non-positive and z is bounded. In case (b), when

$$
F = n\frac{H_{k+1}}{H_k},
$$

we pick coordinates such that F_l^k and $\tilde{g}^{lr}$ are diagonal, use Lemma 3.2 and $F_k^k \le c$, cf. [49, Lemma 2.7], to estimate

$$
\left|\frac{2}{\vartheta^2}\frac{p}{F^{p+1}}F_l^k\tilde{g}^{lr}\hat{R}^m_{ikr}\varphi^i\varphi_m\right| \le c|\hat{\nabla}\varphi|^2 = cz,
$$

where c depends on the initial data and a bound on the Riemann tensor $\widehat{\mathrm{Rm}}$ of σ. Hence in this case there holds $\mathcal{L}z \le cz$ and on finite time intervals we obtain a bound on z. This completes the proof of the first statement.

Now we focus on the two further statements, for which we may use the assumptions in Theorem 1.3 (ii). Putting $f(u) = \vartheta^\gamma(u)$ in (3.2) with $p < 1$[1], we obtain

$$\begin{aligned}\mathcal{L}\hat{z} &\leq \frac{2(p-1)}{F^p}\frac{\vartheta'}{\vartheta}\hat{z} - \frac{2p}{F^{p+1}}\frac{\vartheta''}{\vartheta}F^k_k\hat{z} + \frac{p+1}{F^p}\gamma\frac{\vartheta'}{\vartheta}\frac{1}{v}\hat{z} - \frac{p\gamma}{F^{p+1}}\frac{\vartheta'^2}{\vartheta^2}F^k_k\hat{z}\\ &\quad + \frac{p}{F^{p+1}}F^{kr}u_ku_r\hat{z}\left(2\gamma^2\frac{\vartheta'^2}{\vartheta^2} - \gamma(\gamma-1)\frac{\vartheta'^2}{\vartheta^2} - \gamma\frac{\vartheta''}{\vartheta} - \gamma\frac{\vartheta'^2}{\vartheta^2}\right)\\ &\leq \frac{2(p-1)}{F^p}\frac{\vartheta'}{\vartheta}\hat{z} - \frac{2p}{F^{p+1}}\frac{\vartheta''}{\vartheta}F^k_k\hat{z} + \frac{p+1}{F^p}\gamma\frac{\vartheta'}{\vartheta}\frac{1}{v}\hat{z} - \frac{p\gamma}{F^{p+1}}\frac{\vartheta'^2}{\vartheta^2}F^k_k\hat{z}\\ &\quad + \frac{p}{F^{p+1}}F^{kr}u_ku_r\hat{z}\left(\gamma^2\frac{\vartheta'^2}{\vartheta^2} - \gamma\frac{\vartheta''}{\vartheta}\right).\end{aligned}$$

Since

$$F^{kr}u_ku_r \leq cF^k_kg^{ij}u_{;i}u_{;j} \leq cF^k_k,$$

we obtain $\mathcal{L}\hat{z} \leq 0$ if $\gamma > 0$ is small enough. This proves the claim about $\hat{z}$. If $p = 1$, the same is true with $\gamma = 0$, i.e. $|\hat{\nabla}\varphi|$ is uniformly bounded in time under the non-negative sectional curvature assumption.

To prove the statement about $\tilde{z}$, we first note that due to (2.14) we have

$$F = F^k_lh^l_k = \frac{\vartheta'}{v\vartheta}F^k_k - \frac{1}{v\vartheta}F^k_l\tilde{g}^{lr}\varphi_{rk}$$

and hence

$$\begin{aligned}&\frac{p+1}{F^p}\frac{f'}{f}\frac{1}{v}z - \frac{2p}{F^{p+1}}\frac{1}{\vartheta^2}F^k_l\tilde{g}^{lr}\varphi_{ir}\varphi^i_kf\\ &= \frac{p+1}{F^{p+1}}\frac{f'}{f}\frac{\vartheta'}{\vartheta}\frac{1}{v^2}F^k_kz - \frac{p}{F^{p+1}}\frac{p+1}{p}\frac{f'}{f}\frac{1}{v^2\vartheta}F^k_l\tilde{g}^{lr}\varphi_{rk}z - \frac{2p}{F^{p+1}}\frac{1}{\vartheta^2}F^k_l\tilde{g}^{lr}\varphi_{ir}\varphi^i_kf\\ &= \frac{p+1}{F^{p+1}}\frac{f'}{f}\frac{\vartheta'}{\vartheta}\frac{1}{v^2}F^k_kz + \frac{p}{F^{p+1}}\frac{(p+1)^2}{8p}\frac{f'^2}{f^2}\frac{|\hat{\nabla}\varphi|^2}{v^4}F^k_l\tilde{g}^{lr}\sigma_{rk}z\\ &\quad - \frac{p}{F^{p+1}}\frac{f}{2}F^k_l\tilde{g}^{lr}\left(\frac{p+1}{2p}\frac{f'}{f^2}\frac{1}{v^2}z\sigma_{ir} + \frac{2}{\vartheta}\varphi_{ir}\right)\left(\frac{p+1}{2p}\frac{f'}{f^2}\frac{1}{v^2}z\delta^i_k + \frac{2}{\vartheta}\varphi^i_k\right).\end{aligned}$$

Hence at a maximal point of z we get for $f = \vartheta'^\gamma$

$$\begin{aligned}\mathcal{L}\tilde{z} &\leq -\frac{2p}{F^{p+1}}\left(\frac{\vartheta''}{\vartheta} + \frac{1}{2}\frac{f'}{f}\frac{\vartheta'}{\vartheta} - \frac{(p+1)^2}{16p}\frac{f'^2}{f^2}|\hat{\nabla}\varphi|^2 - \frac{p+1}{2p}\frac{f'}{f}\frac{\vartheta'}{\vartheta}\right)F^k_k\tilde{z}\\ &\quad + \frac{2(p-1)}{F^p}\frac{\vartheta'}{\vartheta}v\tilde{z} - \frac{2f}{\vartheta^2}\frac{p}{F^{p+1}}F^k_l\tilde{g}^{lr}\hat{R}^m_{ikr}\varphi^i\varphi_m\\ &\quad + \frac{p}{F^{p+1}}F^{kr}u_ku_r\tilde{z}\left(2\frac{f'^2}{f^2} - \frac{f''}{f} - \frac{f'}{f}\frac{\vartheta'}{\vartheta}\right)\\ &= -\frac{2p}{F^{p+1}}\left(\frac{\vartheta''}{\vartheta} + \frac{\gamma}{2}\frac{\vartheta''}{\vartheta} - \frac{(p+1)^2\gamma^2}{16p}\frac{\vartheta''^2}{\vartheta'^2}|\hat{\nabla}\varphi|^2 - \frac{\gamma(p+1)}{2p}\frac{\vartheta''}{\vartheta}\right)F^k_k\tilde{z}\\ &\quad + \frac{2(p-1)}{F^p}\frac{\vartheta'}{\vartheta}v\tilde{z} - \frac{2f}{\vartheta^2}\frac{p}{F^{p+1}}F^k_l\tilde{g}^{lr}\hat{R}^m_{ikr}\varphi^i\varphi_m\\ &\quad + \frac{p}{F^{p+1}}F^{kr}u_ku_r\tilde{z}\left(\gamma^2\frac{\vartheta''^2}{\vartheta'^2} + \gamma\frac{\vartheta''^2}{\vartheta'^2} - \gamma\frac{\vartheta'''}{\vartheta'} - \gamma\frac{\vartheta''}{\vartheta}\right).\end{aligned} \tag{3.4}$$

The middle line is non-positive. Due to Assumption 1.2, we can estimate

$$\frac{\vartheta''^2}{\vartheta'^2} + \frac{\vartheta'''}{\vartheta'} \leq c\frac{\vartheta''}{\vartheta}$$

[1]i.e. we are only allowed to use the non-negative sectional curvature and Assumption 1.2 from the assumptions in Theorem 1.3 (ii). However, for this statement we only need the non-negative sectional curvature.

and hence all terms which are controlled by $\gamma\vartheta''$ can be absorbed into the good term

$$-\frac{2p}{F^{p+1}}\frac{\vartheta''}{\vartheta}F^k_k\tilde{z}$$

in the first line. Hence $\mathcal{L}\tilde{z}\leq 0$, if $0<\gamma$ is small enough. □

3.3. **Curvature estimates.** We prove that along (1.1) all principal curvatures are bounded as long as the flow remains in a compact subset of N. Due to all previous a priori estimates this will imply uniform C^2 estimates on each finite time interval, as well as a uniformly elliptic operator $F^{-(p+1)}F^{ij}$. Hence the regularity estimates by Krylov and Safonov apply to get $C^{2,\alpha}$ estimates. With the linear Schauder estimates we obtain uniform C^∞-bounds on each finite interval. We may extend the solution beyond any finite T, completing the proof of item (i) of Theorem 1.3.

3.4. **Proposition.** *Under the assumptions of Theorem 1.3 (i), on every finite interval $[0,T]$ there exists a compact set $\Lambda\subset\Gamma$ such that along the flow (1.1) the principal curvatures κ_i satisfy*

$$(\kappa_i)\in\Lambda\quad\forall 0\leq t\leq T.$$

Proof. In this proof the generic constant c is allowed to depend on T. We proceed similarly as in [24, 62]. First we simplify the evolution of the second fundamental form, cf. Lemma 2.7. We have the following estimate in normal coordinates:

$$\begin{aligned}\dot{h}^n_n-\mathcal{F}^{kl}h^n_{n;kl}\leq{}&\mathcal{F}^{kl,rs}h_{kl;n}h_{rs;}{}^n+\frac{p}{F^{p+1}}F^{kl}h_{rk}h^r_lh^n_n-\frac{p+1}{F^p}(h^n_n)^2\\&+\frac{c}{F^{p+1}}F^{ij}g_{ij}(h^n_n+1)+\frac{c}{F^p}.\end{aligned}$$

According to (2.19) the support function, which is bounded from below due to the gradient estimates Lemma 3.3, satisfies

$$\begin{aligned}\dot{s}-\mathcal{F}^{kl}s_{;kl}={}&\frac{p}{F^{p+1}}F^{kl}h_{rk}h^r_ls-\frac{(p-1)\vartheta'}{F^p}\\&+\frac{p\vartheta}{F^{p+1}}F^{kl}u_{;}{}^m\overline{\mathrm{Rm}}(\nu,x_{;k},x_{;m},x_{;l}).\end{aligned}\tag{3.5}$$

Due to a well known trick, e.g. see the proof of [24, Lemma 4.4], it suffices to bound the evolution equation of the function

$$w=\log h^n_n+f(s)+\alpha u,$$

α to be determined, at a maximal point of w in which normal coordinates are given,

$$g_{ij}=\delta_{ij},\quad h_{ij}=\kappa_i\delta_{ij},\quad \kappa_1\leq\dots\leq\kappa_n.$$

For small $\beta>0$ set

$$f(s)=-\log(s-\beta).$$

Also using (2.17), we see that w satisfies

$$\begin{aligned}\dot{w}-\mathcal{F}^{kl}w_{;kl}\leq{}&\frac{p}{F^{p+1}}F^{kl}h_{rk}h^r_l(1+f's)-\frac{p+1}{F^p}h^n_n\\&+\frac{c}{F^{p+1}}F^k_k(1+(h^n_n)^{-1})+\frac{c}{F^p}(1+(h^n_n)^{-1}+\alpha)\\&+\frac{p}{F^{p+1}}F^{kl,rs}h_{kl;n}h_{rs;}{}^n(h^n_n)^{-1}+\frac{p}{F^{p+1}}F^{ij}(\log h^n_n)_{;i}(\log h^n_n)_{;j}\\&-f''\frac{p}{F^{p+1}}F^{ij}s_{;i}s_{;j}-\alpha\frac{\vartheta'}{\vartheta}\frac{p}{F^{p+1}}F^{ij}(g_{ij}-u_{;i}u_{;j}).\end{aligned}\tag{3.6}$$

We employ a trick already used in [19]. Due to the concavity of F there holds

$$F^{nn}\leq\dots\leq F^{11}\quad\text{and}\quad F^{kl,rs}\eta_{kl}\eta_{rs}\leq\frac{2}{\kappa_n-\kappa_1}\sum_{k=1}^n(F^{nn}-F^{kk})\eta^2_{nk}$$

for all symmetric (η_{ij}). It is possible to exploit this term in order to estimate (3.6).

Case 1: $\kappa_1 < -\epsilon_1\kappa_n$, $0 < \epsilon_1 < \frac{1}{2}$. There hold

$$F^{ij}h_{ik}h^k_j \geq F^{11}\kappa_1^2 \geq \frac{1}{n}F^{ij}g_{ij}\kappa_1^2 \geq \frac{1}{n}F^{ij}g_{ij}\epsilon_1^2\kappa_n^2,$$

$$\begin{aligned}\frac{p}{F^{p+1}}F^{ij}(\log h^n_n)_{;i}(\log h^n_n)_{;j} &= f'^2\frac{p}{F^{p+1}}F^{ij}s_{;i}s_{;j} + f'\frac{2\alpha p}{F^{p+1}}F^{ij}s_{;i}u_{;j}\\ &\quad + \frac{\alpha^2 p}{F^{p+1}}F^{ij}u_{;i}u_{;j},\end{aligned}$$

due to $\nabla w = 0$. Using (2.20), if κ_n is large, (3.6) with any α becomes

$$\begin{aligned}0 &\leq \frac{p}{nF^{p+1}}F^{ij}g_{ij}\left(\epsilon_1^2\kappa_n^2(1+f's) + c + c\alpha|f'|\kappa_n + c\alpha^2\right)\\ &\quad + \frac{p+1}{F^p}\left(c + c\alpha - \kappa_n\right) - \frac{p}{F^{p+1}}F^{ij}s_{;i}s_{;j}(f'' - f'^2)\\ &< 0\end{aligned}$$

and hence w is bounded in this case, due to $1 + f's \leq c < 0$ and $f'' = f'^2$.

Case 2: $\kappa_1 \geq -\epsilon_1\kappa_n$. Then

$$\begin{aligned}&\frac{2}{\kappa_n - \kappa_1}\sum_{k=1}^n (F^{nn} - F^{kk})(h_{nk;n})^2(h^n_n)^{-1}\\ &\leq \frac{2}{1+\epsilon_1}\sum_{k=1}^n (F^{nn} - F^{kk})(h_{nk;n})^2(h^n_n)^{-2}\\ &\leq \frac{2}{1+\epsilon_1}\sum_{k=1}^n (F^{nn} - F^{kk})(h_{nn;k})^2(h^n_n)^{-2} + c(\epsilon_1)\sum_{k=1}^n (F^{kk} - F^{nn})\kappa_n^{-2}\\ &\quad + \frac{4}{1+\epsilon_1}\sum_{k=1}^n (F^{nn} - F^{kk})h_{nn;k}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;n}x^\gamma_{;n}x^\delta_{;k}(h^n_n)^{-2}\\ &\leq \frac{2}{1+2\epsilon_1}\sum_{k=1}^n (F^{nn} - F^{kk})(h_{nn;k})^2(h^n_n)^{-2} + c(\epsilon_1)\sum_{k=1}^n (F^{kk} - F^{nn})\kappa_n^{-2},\end{aligned}$$

where we used the Codazzi equation (2.3) and Cauchy-Bunjakowski-Schwarz. We deduce further:

$$\begin{aligned}&F^{ij}(\log h^n_n)_{;i}(\log h^n_n)_{;j} + \frac{2}{\kappa_n - \kappa_1}\sum_{k=1}^n (F^{nn} - F^{kk})(h_{nk;n})^2(h^n_n)^{-1}\\ &\leq \frac{2}{1+2\epsilon_1}\sum_{k=1}^n F^{nn}(\log h^n_n)^2_{;k} - \frac{1-2\epsilon_1}{1+2\epsilon_1}\sum_{k=1}^n F^{kk}(\log h^n_n)^2_{;k} + c(\epsilon_1)F^{ij}g_{ij}\kappa_n^{-2}\\ &\leq \sum_{k=1}^n F^{nn}(\log h^n_n)^2_{;k} + c(\epsilon_1)F^{ij}g_{ij}\kappa_n^{-2}\\ &= c(\epsilon_1)F^{ij}g_{ij}\kappa_n^{-2} + f'^2F^{nn}\|\nabla s\|^2 + 2\alpha f'F^{nn}\langle\nabla s, \nabla u\rangle + \alpha^2F^{nn}\|\nabla u\|^2.\end{aligned}$$

Hence we can estimate (3.6), using $F^{ij}\bar{g}_{ij} \geq c_0F^{ij}g_{ij}$,

$$\begin{aligned}0 \leq{}& \frac{p}{F^{p+1}}F^{nn}\left(\kappa_n^2(1+f's) + \alpha c\kappa_n + c\alpha^2\right) + \frac{p+1}{F^p}\left(c + \alpha - \kappa_n\right)\\ &+ \frac{p}{F^{p+1}}F^{ij}g_{ij}\left(\frac{c(\epsilon_1)}{\kappa_n^2} + c - c_0\alpha\frac{\vartheta'}{\vartheta}\right) + \frac{p}{F^{p+1}}\left(f'^2F^{nn}\|\nabla s\|^2 - f''F^{ij}s_{;i}s_{;j}\right).\end{aligned}$$

Due to the barrier estimates, on every finite interval $[0,T]$ there holds $\vartheta \leq c(T)$. Picking α large enough, we see that κ_n is bounded on $[0,T]$ and the proof is complete. □

4. Asymptotics

4.1. **Global bounds.** In order to study the long-time behaviour of (1.1), we need to investigate the evolution of the second fundamental form in greater detail. Therefore we need a more detailed version of its evolution equation.

4.1. **Lemma.** *Along* (1.1) *the Weingarten operator evolves according to*

$$
\begin{aligned}
&\dot h^i_j - \frac{p}{F^{p+1}}F^{kl}h^i_{j;kl}\\
&= -\frac{p(p+1)}{F^{p+2}}F_{;j}F_{;}^{\ i} + \frac{p}{F^{p+1}}F^{kl,rs}h_{kl;j}h_{rs;}^{\ \ \ i} + \frac{p}{F^{p+1}}F^{kl}h_{rk}h^r_l h^i_j - \frac{p+1}{F^p}h^i_r h^r_j\\
&+ \frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F^k_k h^i_j - \frac{\vartheta'^2}{\vartheta^2}\frac{p-1}{F^p}\delta^i_j - \left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)v^{-2}\frac{p}{F^{p+1}}F^k_k h^i_j\\
&+ \left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)v^{-2}\frac{p+1}{F^p}\delta^i_j + \left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}F^{kl}\left(u_{;k}u_{;l}h^i_j - 2h^m_l u_{;m}u_{;k}\delta^i_j\right)\\
&+ \left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)\frac{1-p}{F^p}u_{;j}u_{;}^{\ i} + \left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}F^k_k\left(h^i_m u_{;}^{\ m}u_{;j} + h^m_j u_{;m}u_{;}^{\ i}\right)\\
&+ \left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}\left(F^{il}u_{;m}(h^m_l u_{;j} - h^m_j u_{;l}) + F^k_l h^l_j u_{;}^{\ i}u_{;k} - F^l_j h^{im}u_{;l}u_{;m}\right)\\
&+ \frac{p}{F^{p+1}}F^{kl}\left(\widetilde{\mathrm{Rm}}(x_{;l},x_{;j},x_{;k},x_{;m})h^{im} + \widetilde{\mathrm{Rm}}(x_{;l},x_{;r},x_{;k},x_{;m})h^m_j g^{ri}\right)\\
&+ \frac{2p}{F^{p+1}}F^{kl}\widetilde{\mathrm{Rm}}(x_{;r},x_{;m},x_{;k},x_{;j})h^m_l g^{ri} - \frac{p+1}{F^p}\widetilde{\mathrm{Rm}}(x_{;r},\nu,\nu,x_{;j})g^{ri}\\
&+ \frac{p}{F^{p+1}}F^{kl}\widetilde{\mathrm{Rm}}(x_{;k},\nu,\nu,x_{;l})h^i_j\\
&+ \frac{p}{F^{p+1}}F^{kl}\left(\bar\nabla\overline{\mathrm{Rm}}(\nu,x_{;k},x_{;r},x_{;l},x_{;j}) + \bar\nabla\overline{\mathrm{Rm}}(\nu,x_{;r},x_{;j},x_{;k},x_{;l})\right)g^{ri}.
\end{aligned}
$$

Proof. In Lemma 2.7 we rewrite the terms involving the Riemann tensor employing (2.6):

$$
\begin{aligned}
&\bar R_{\alpha\beta\gamma\delta}x^\alpha_{;l}x^\beta_{;j}x^\gamma_{;k}x^\delta_{;m}\\
&= -\frac{\vartheta''}{\vartheta}(g_{lm}g_{jk}-g_{lk}g_{jm}) + \left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)(\bar g_{lm}\bar g_{jk}-\bar g_{lk}\bar g_{jm}) + \tilde R_{\alpha\beta\gamma\delta}x^\alpha_{;l}x^\beta_{;j}x^\gamma_{;k}x^\delta_{;m}\\
&= -\frac{\vartheta'^2}{\vartheta^2}(g_{lm}g_{jk}-g_{lk}g_{jm}) + \tilde R_{\alpha\beta\gamma\delta}x^\alpha_{;l}x^\beta_{;j}x^\gamma_{;k}x^\delta_{;m}\\
&- \left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)(u_{;l}u_{;m}g_{jk}+u_{;j}u_{;k}g_{lm}-u_{;l}u_{;k}g_{jm}-u_{;j}u_{;m}g_{lk}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\mathcal{F}^{kl}\bar R_{\alpha\beta\gamma\delta}\left(x^\alpha_{;l}x^\beta_{;j}x^\gamma_{;k}x^\delta_{;m}h^{im} + x^\alpha_{;l}x^\beta_{;r}x^\gamma_{;k}x^\delta_{;m}h^m_j g^{ri}\right)\\
&= -\frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F^{kl}h^i_l g_{jk} + \frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F^k_k h^i_j - \frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F^{il}h_{jl} + \frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F^k_k h^i_j\\
&+ \left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}F^{kl}(u_{;k}u_{;l}h^i_j + g_{kl}h^{im}u_{;m}u_{;j} - h^i_l u_{;j}u_{;k} - g_{kj}h^{im}u_{;l}u_{;m})\\
&+ \left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}(F^{kl}(u_{;k}u_{;l}h^i_j + g_{kl}h^m_j u_{;m}u_{;}^{\ i}) - F^{il}h^m_j u_{;l}u_{;m} - F^k_m h^m_j u_{;}^{\ i}u_{;k})\\
&+ \frac{p}{F^{p+1}}F^{kl}\left(\tilde R_{\alpha\beta\gamma\delta}x^\alpha_{;l}x^\beta_{;j}x^\gamma_{;k}x^\delta_{;m}h^{im} + \tilde R_{\alpha\beta\gamma\delta}x^\alpha_{;l}x^\beta_{;r}x^\gamma_{;k}x^\delta_{;m}h^m_j g^{ri}\right)
\end{aligned}
$$

and thus the following two equations hold:

$$\begin{aligned}
&\mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta}\left(x^\alpha_{;l}x^\beta_{;j}x^\gamma_{;k}x^\delta_{;m}h^{im}+x^\alpha_{;l}x^\beta_{;r}x^\gamma_{;k}x^\delta_{;m}h^m_j g^{ri}\right)\\
=&\ -2\frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}(F^i_l h^l_j-F^{kl}g_{kl}h^i_j)\\
&+\left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}(2F^{kl}u_{;k}u_{;l}h^i_j+F^{kl}g_{kl}(h^{im}u_{;m}u_{;j}+h^m_j u_{;m}u_{;}{}^i)\\
&\qquad\qquad -F^l_j h^{im}u_{;l}u_{;m}-F^{il}h^m_j u_{;l}u_{;m}-F^{kl}h^i_l u_{;j}u_{;k}-F^k_m h^m_j u_{;}{}^i u_{;k})\\
&+\frac{p}{F^{p+1}}F^{kl}\left(\tilde{R}_{\alpha\beta\gamma\delta}x^\alpha_{;l}x^\beta_{;j}x^\gamma_{;k}x^\delta_{;m}h^{im}+\tilde{R}_{\alpha\beta\gamma\delta}x^\alpha_{;l}x^\beta_{;r}x^\gamma_{;k}x^\delta_{;m}h^m_j g^{ri}\right),\\
&2\mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;r}x^\beta_{;m}x^\gamma_{;k}x^\delta_{;j}h^m_l g^{ri}\\
=&\ -2\frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}(F\delta^i_j-F^i_l h^l_j)+\frac{2p}{F^{p+1}}F^{kl}\tilde{R}_{\alpha\beta\gamma\delta}x^\alpha_{;r}x^\beta_{;m}x^\gamma_{;k}x^\delta_{;j}h^m_l g^{ri}\\
&+2\left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}(F^k_l h^l_j u_{;}{}^i u_{;k}+F^{il}h^m_l u_{;j}u_{;m}-Fu_{;}{}^i u_{;j}-F^{kl}h^m_l u_{;m}u_{;k}\delta^i_j).
\end{aligned}$$

Adding up, also using $F^i_k h^k_j = h^i_k F^k_j$, gives

$$\begin{aligned}
&\mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta}\left(x^\alpha_{;l}x^\beta_{;j}x^\gamma_{;k}x^\delta_{;m}h^{im}+x^\alpha_{;l}x^\beta_{;r}x^\gamma_{;k}x^\delta_{;m}h^m_j g^{ri}\right)\\
&+2\mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;r}x^\beta_{;m}x^\gamma_{;k}x^\delta_{;j}h^m_l g^{ri}\\
&=2\frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}(F^k_j h^i_j-F\delta^i_j)\\
&+\left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}(2F^{kl}u_{;k}u_{;l}h^i_j+F^k_k(h^i_m u_{;}{}^m u_{;j}+h^m_j u_{;m}u_{;}{}^i)\\
&\qquad -F^l_j h^{im}u_{;l}u_{;m}-F^{il}h^m_j u_{;l}u_{;m}+F^k_l h^l_j u_{;}{}^i u_{;k}\\
&\qquad +F^{il}h^m_l u_{;j}u_{;m}-2Fu_{;}{}^i u_{;j}-2F^{kl}h^m_l u_{;m}u_{;k}\delta^i_j)\\
&+\frac{p}{F^{p+1}}F^{kl}\tilde{R}_{\alpha\beta\gamma\delta}\left(x^\alpha_{;l}x^\beta_{;j}x^\gamma_{;k}x^\delta_{;m}h^{im}+x^\alpha_{;l}x^\beta_{;r}x^\gamma_{;k}x^\delta_{;m}h^m_j g^{ri}\right)\\
&+\frac{2p}{F^{p+1}}F^{kl}\tilde{R}_{\alpha\beta\gamma\delta}x^\alpha_{;r}x^\beta_{;m}x^\gamma_{;k}x^\delta_{;j}h^m_l g^{ri}.
\end{aligned}\tag{4.1}$$

Using $\nu = v^{-1}(1,-\bar{g}^{ik}u_{;k})$ and $v^{-2}\bar{g}^{ij}u_{;i}u_{;j}=\|\nabla u\|^2$, we get

$$\begin{aligned}
\overline{\mathrm{Rm}}(x_{;i},\nu,\nu,x_{;j}) &= -\frac{\vartheta''}{\vartheta}g_{ij}+\left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)\left(\|\nabla u\|^2\bar{g}_{ij}-v^{-2}u_{;i}u_{;j}\right)\\
&\quad+\widetilde{\mathrm{Rm}}(x_{;i},\nu,\nu,x_{;j})\\
&= -\frac{\vartheta''}{\vartheta}g_{ij}+\left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)\left(\|\nabla u\|^2 g_{ij}-u_{;i}u_{;j}\right)\\
&\quad+\widetilde{\mathrm{Rm}}(x_{;i},\nu,\nu,x_{;j}).
\end{aligned}\tag{4.2}$$

Thus

$$\begin{aligned}
&(\mathcal{F}-\mathcal{F}^{kl}h_{kl})\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;r}\nu^\beta\nu^\gamma x^\delta_j g^{ri}+\mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;k}\nu^\beta\nu^\gamma x^\delta_{;l}h^i_j\\
&=\frac{p+1}{F^p}\left(\frac{\vartheta''}{\vartheta}\delta^i_j-\left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)(\|\nabla u\|^2\delta^i_j-u_{;j}u_{;}{}^i)\right)\\
&-\frac{p}{F^{p+1}}F^{kl}\left(\frac{\vartheta''}{\vartheta}g_{kl}-\left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)(\|\nabla u\|^2 g_{kl}-u_{;l}u_{;k})\right)h^i_j\\
&-\frac{p+1}{F^p}\widetilde{\mathrm{Rm}}(x_{;r},\nu,\nu,x_{;j})g^{ri}+\frac{p}{F^{p+1}}F^{kl}\widetilde{\mathrm{Rm}}(x_{;k},\nu,\nu,x_{;l})h^i_j.
\end{aligned}\tag{4.3}$$

Adding up (4.1) and (4.3) and inserting the result into Lemma 2.7 gives the claimed formula. □

4.2. **Lemma.** *Along* (1.1) *the function* $v = \vartheta s^{-1}$ *satisfies the evolution equation*

$$\begin{aligned}
&\dot v - \frac{p}{F^{p+1}} F^{ij} v_{;ij} \\
&= -\frac{p}{F^{p+1}} F^{ij} h_{ik} h^k_j v - \frac{\vartheta'^2}{\vartheta^2} \frac{p}{F^{p+1}} F^{ij} g_{ij} v + \frac{\vartheta'}{\vartheta} \frac{p+1}{F^p} + \frac{\vartheta'}{\vartheta} \frac{p-1}{F^p} v^2 \\
&+ \left(\frac{\vartheta'^2}{\vartheta^2} - \frac{\vartheta''}{\vartheta} \right) \frac{p}{F^{p+1}} F^{ij} u_{;i} u_{;j} v \\
&+ \frac{p}{F^{p+1}} F^{kl} \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2} \right) (u_{;l} u_{;k} - \|\nabla u\|^2 g_{kl}) v \\
&- \frac{p}{F^{p+1}} F^{kl} \widetilde{\mathrm{Rm}}(x_{;k}, \nu, \nu, x_{;l}) v + 2 \frac{\vartheta'}{\vartheta} \frac{p}{F^{p+1}} F^{ij} u_{;i} v_{;j} - \frac{2}{v} \frac{p}{F^{p+1}} F^{ij} v_{;i} v_{;j}.
\end{aligned} \tag{4.4}$$

Proof. Due to (2.17) and (3.5) we have

$$\begin{aligned}
&\dot v - \frac{p}{F^{p+1}} F^{ij} v_{;ij} \\
&= \frac{\vartheta'}{\vartheta} v \left(\dot u - \frac{p}{F^{p+1}} F^{ij} u_{;ij} \right) - \frac{\vartheta''}{\vartheta} v \frac{p}{F^{p+1}} F^{ij} u_{;i} u_{;j} - \frac{v}{s} \left(\dot s - \frac{p}{F^{p+1}} F^{ij} s_{;ij} \right) \\
&- \frac{2\vartheta}{s^3} \frac{p}{F^{p+1}} F^{ij} s_{;i} s_{;j} - 2 \frac{p}{F^{p+1}} F^{ij} \vartheta_{;i} \left(\frac{1}{s} \right)_{;j} \\
&= \frac{\vartheta'}{\vartheta} \frac{p+1}{F^p} - \frac{\vartheta'^2}{\vartheta^2} \frac{p}{F^{p+1}} F^{ij} g_{ij} v + \left(\frac{\vartheta'^2}{\vartheta^2} - \frac{\vartheta''}{\vartheta} \right) \frac{p}{F^{p+1}} F^{ij} u_{;i} u_{;j} v \\
&- \frac{p}{F^{p+1}} F^{ij} h_{ik} h^k_j v + \frac{\vartheta'}{\vartheta} \frac{p-1}{F^p} v^2 - \frac{p}{F^{p+1}} F^{kl} \overline{\mathrm{Rm}}(\nu, x_{;k}, x_{;m}, x_{;l}) u_{;}{}^m v^2 \\
&- \frac{2}{v} \frac{p}{F^{p+1}} F^{ij} v_{;i} v_{;j} - 2v \frac{\vartheta'^2}{\vartheta^2} \frac{p}{F^{p+1}} F^{ij} u_{;i} u_{;j} + 4 \frac{\vartheta'}{\vartheta} \frac{p}{F^{p+1}} F^{ij} u_{;i} v_{;j} \\
&- 2 \frac{\vartheta'}{\vartheta} \frac{p}{F^{p+1}} F^{ij} u_{;i} v_{;j} + 2 \frac{\vartheta'^2}{\vartheta^2} \frac{p}{F^{p+1}} F^{ij} u_{;i} u_{;j} v,
\end{aligned}$$

which is the claimed formula up to rewriting the term involving $\overline{\mathrm{Rm}}$. However, we use (2.6) to deduce

$$\begin{aligned}
\overline{\mathrm{Rm}}(\nu, x_{;k}, x_{;m}, x_{;l}) u_{;}{}^m = &- \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2} \right) v^{-1} (u_{;l} u_{;k} - \|\nabla u\|^2 g_{kl}) \\
&+ \widetilde{\mathrm{Rm}}(\nu, x_{;k}, x_{;m}, x_{;l}) u_{;}{}^m
\end{aligned}$$

and hence

$$\begin{aligned}
\overline{\mathrm{Rm}}(\nu, x_{;k}, x_{;m}, x_{;l}) u_{;}{}^m v^2 = &- \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2} \right) (u_{;l} u_{;k} - \|\nabla u\|^2 g_{kl}) v \\
&+ \widetilde{\mathrm{Rm}}(x_{;k}, \nu, \nu, x_{;l}) v,
\end{aligned}$$

where we have used

$$u_{;}{}^m = v^{-2} \vartheta^{-2} \sigma^{mk} u_{;k}.$$

Inserting gives the result. □

We start the investigation of the long-time behavior of (1.2) under the assumptions in item (ii) of Theorem 1.3 by proving a lower bound on the curvature function.

4.3. **Lemma.** *Under the assumptions of Theorem 1.3 (ii), along* (1.1) *there exists a constant* c*, such that*

$$\frac{\vartheta' v}{\vartheta F} \le c.$$

Proof. If $p = 1$ and ϑ' is bounded , the result follows from Lemma 3.2 immediately. If $p < 1$ or ϑ' is unbounded, Lemma 3.3 says that $v \to 1$. Due (2.16), (2.17), (4.2) and (4.4) the function

$$w = \log\left(\frac{1}{F^p}\right) + f(v) + p\log\vartheta' - p\log\vartheta,$$

where f with $f' \geq 0$ is yet to be determined, satisfies

$$\begin{aligned}
&\dot{w} - \frac{p}{F^{p+1}}F^{ij}w_{;ij}\\
&= \frac{p}{F^{p+1}}F^{ij}h_{ik}h^k_j - \frac{\vartheta''}{\vartheta}\frac{p}{F^{p+1}}F^{ij}g_{ij} + \frac{p}{F^{p+1}}F^{ij}\widetilde{\mathrm{Rm}}(x_{;i},\nu,\nu,x_{;j})\\
&+ \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}F^{ij}(\|\nabla u\|^2 g_{ij} - u_{;i}u_{;j})\\
&+ \frac{p}{F^{p+1}}F^{ij}\left(\log\frac{1}{F^p}\right)_{;i}\left(\log\frac{1}{F^p}\right)_{;j} - \frac{p}{F^{p+1}}F^{ij}h_{ik}h^k_j f'v - \frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F^{ij}g_{ij}f'v\\
&+ \frac{\vartheta'}{\vartheta}\frac{p+1}{F^p}f' + \frac{\vartheta'}{\vartheta}\frac{p-1}{F^p}f'v^2 + \left(\frac{\vartheta'^2}{\vartheta^2} - \frac{\vartheta''}{\vartheta}\right)\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}f'v\\
&+ \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}F^{ij}(u_{;i}u_{;j} - \|\nabla u\|^2 g_{ij})f'v\\
&- \frac{p}{F^{p+1}}F^{ij}\widetilde{\mathrm{Rm}}(x_{;i},\nu,\nu,x_{;j})f'v + 2f'\frac{\vartheta'}{\vartheta}\frac{p}{F^{p+1}}F^{ij}u_{;i}v_{;j} - \frac{2}{v}f'\frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j}\\
&- f''\frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j} + (p+1)\left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta}\right)\frac{p}{F^{p+1}}v^{-1}F\\
&- p\left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}F^{ij}g_{ij} + p\left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}\\
&- p\left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta}\right)'\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}.
\end{aligned}$$

Sorting the terms appropriately and replacing $(\log 1/F^p)_{;i}$ we get

$$\begin{aligned}
&\dot{w} - \frac{p}{F^{p+1}}F^{ij}w_{;ij}\\
&\leq \frac{p}{F^{p+1}}F^{ij}h_{ik}h^k_j(1 - f'v) + \frac{p}{F^{p+1}}F^{ij}\widetilde{\mathrm{Rm}}(x_{;i},\nu,\nu,x_{;j})(1 - f'v)\\
&+ \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}F^{ij}(\|\nabla u\|^2 g_{ij} - u_{;i}u_{;j})(1 - f'v)\\
&+ \frac{p(p+1)}{F^{p+1}}\frac{\vartheta''}{\vartheta}\left(-F^{ij}g_{ij} + \frac{\vartheta}{\vartheta'}Fv^{-1} + c(f'v+1)F^{ij}u_{;i}u_{;j}\right)\\
&+ \frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}\Big(-F^{ij}g_{ij}f'v + \frac{\vartheta}{\vartheta'}\frac{p+1}{p}f'F + \frac{\vartheta}{\vartheta'}\frac{p-1}{p}f'Fv^2\\
&- (p+1)\frac{\vartheta}{\vartheta'}Fv^{-1} + pF^{ij}g_{ij}\Big) + \frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}(f'v - 1)\\
&+ (p-1)^2\frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j} + \frac{p}{F^{p+1}}F^{ij}w_{;i}w_{;j} - 2f'\frac{p}{F^{p+1}}F^{ij}w_{;i}v_{;j}\\
&- 2p\left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta}\right)\frac{p}{F^{p+1}}F^{ij}w_{;i}u_{;j} + f'^2\frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j}\\
&- 2pf'\left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta}\right)\frac{p}{F^{p+1}}F^{ij}v_{;i}u_{;j} + 2f'\frac{\vartheta'}{\vartheta}\frac{p}{F^{p+1}}F^{ij}u_{;i}v_{;j}\\
&- \frac{2}{v}f'\frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j} - f''\frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j}.
\end{aligned} \tag{4.5}$$

Now choose

$$f(v) = -\log\left(v^{-\frac{1}{2}} - \frac{3}{4}\right)$$

where in the sequel we only consider sufficiently large times where $v^{\frac{1}{2}} < \frac{4}{3}$. Then

$$f' = \frac{1}{2}\frac{v^{-\frac{3}{2}}}{v^{-\frac{1}{2}} - \frac{3}{4}}, \quad f'' = -\frac{3}{4}\frac{v^{-\frac{5}{2}}}{v^{-\frac{1}{2}} - \frac{3}{4}} + \frac{1}{4}\frac{v^{-3}}{\left(v^{-\frac{1}{2}} - \frac{3}{4}\right)^2}, \quad f'v = \frac{1}{2}\frac{v^{-\frac{1}{2}}}{v^{-\frac{1}{2}} - \frac{3}{4}} \geq \frac{3}{2}$$

and

$$\begin{aligned} f'^2 - \frac{2}{v}f' - f'' &= -\frac{v^{-\frac{5}{2}}}{v^{-\frac{1}{2}} - \frac{3}{4}} + \frac{3}{4}\frac{v^{-\frac{5}{2}}}{v^{-\frac{1}{2}} - \frac{3}{4}} \\ &= -\frac{1}{4}\frac{v^{-\frac{5}{2}}}{v^{-\frac{1}{2}} - \frac{3}{4}} \\ &\leq -\frac{3}{4v^2}. \end{aligned}$$

Hence, using Cauchy-Schwarz on $F^{ij}u_{;i}v_{;j}$,

$$2F^{ij}u_{;i}v_{;j} \leq \frac{\epsilon\vartheta}{v\vartheta'}F^{ij}v_{;i}v_{;j} + \frac{v\vartheta'}{\epsilon\vartheta}F^{ij}u_{;i}u_{;j},$$

with the help of (1.2) we can estimate at maximal points of w:

$$\begin{aligned} &\dot{w} - \frac{p}{F^{p+1}}F^{ij}w_{;ij} \\ &\leq \frac{p(p+1)}{F^{p+1}}\frac{\vartheta''}{\vartheta}\left(-F^{ij}g_{ij} + \frac{\vartheta}{\vartheta'}Fv^{-1} + c_\epsilon(f'v+1)F^{ij}u_{;i}u_{;j}\right) \\ &+ \frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}\Big(pF^{ij}g_{ij} - F^{ij}g_{ij}f'v + 2\frac{\vartheta}{\vartheta'}f'F - (p+1)\frac{\vartheta}{\vartheta'}Fv^{-1} + F^{ij}g_{ij}\|\nabla u\|^2 f'v \\ &+ c_\epsilon(f'v+1)F^{ij}u_{;i}u_{;j}\Big) + \frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j}\left(-\frac{3}{4v^2} + \frac{\epsilon c}{v}f'\right). \end{aligned}$$

The last bracket is negative for small ϵ due to the boundedness of v. To estimate the first big bracket we calculate with the help of $F^{ij}g_{ij} \geq n$:

$$-F^{ij}g_{ij} + \frac{\vartheta}{\vartheta'}Fv^{-1} + c_\epsilon(f'v+1)F^{ij}u_{;i}u_{;j} \leq F^{ij}\left(cu_{;i}u_{;j} - \frac{1}{2}g_{ij}\right) - \frac{n}{2} + \frac{\vartheta}{\vartheta'}Fv^{-1}.$$

After some well-determined time $T > 0$ when the gradient of u is small enough due to $v \to 1$, this expression will be negative if w becomes too large[2]. A very similar argument applies to the second big bracket, due to $f'v \geq 3/2$. Hence w is bounded. □

We need a similar estimate of the rescaled principal curvatures.

4.4. **Lemma.** *Under the assumptions of Theorem 1.3 (ii), along (1.1) there exists a constant c, such that*

$$\kappa_n \leq c\frac{\vartheta'}{\vartheta}.$$

Proof. Define

$$z = \log h_n^n + \log\frac{\vartheta}{\vartheta'} + f(v).$$

[2]equivalently, $\frac{\vartheta}{\vartheta'}Fv^{-1}$ is small

We estimate the evolution of z directly from (2.17), Lemma 4.1 and (4.4) and, as in the proof of Proposition 3.4, from the start calculate in a maximal point of z in coordinates such that

$$g_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_i \delta_{ij}, \quad \kappa_1 \leq \cdots \leq \kappa_n.$$

First of all there holds

$$\begin{aligned}
&F^{kl}\widetilde{\mathrm{Rm}}(x_{;l}, x_{;j}, x_{;k}, x_{;m})h^{im} + F^{kl}\widetilde{\mathrm{Rm}}(x_{;l}, x_{;r}, x_{;k}, x_{;m})h^m_j g^{ri} \\
&+ 2F^{kl}\widetilde{\mathrm{Rm}}(x_{;r}, x_{;m}, x_{;k}, x_{;j})h^m_l g^{ri} \\
&= 2F^{kk}\widetilde{\mathrm{Rm}}(x_{;n}, x_{;k}, x_{;k}, x_{;n})(\kappa_k - \kappa_n) \\
&\leq 0,
\end{aligned}$$

since the sectional curvatures of σ are non-negative.

Due to Lemma 2.4 we have

$$\|\widetilde{\mathrm{Rm}}\| \leq \frac{c}{\vartheta^2}, \quad \|\bar{\nabla}\widetilde{\mathrm{Rm}}\| \leq c\frac{\vartheta'}{\vartheta^3}$$

and we get

$$\begin{aligned}
&F^{kl}\bar{\nabla}\overline{\mathrm{Rm}}(\nu, x_{;k}, x_{;r}, x_{;l}, x_{;j})g^{ri} + F^{kl}\bar{\nabla}\overline{\mathrm{Rm}}(\nu, x_{;r}, x_{;j}, x_{;k}, x_{;l})g^{ri} \\
&\leq c\frac{\vartheta'^3}{\vartheta^3}\|\nabla u\|^2 F^k_k + c\frac{\vartheta'}{\vartheta^3}\|\nabla u\|F^k_k,
\end{aligned}$$

where we have used that the terms in (2.7) involving $r_{;\alpha}$ are cancelled, since $\bar{T}$ carries the symmetries of a curvature tensor.

Hence

$$\begin{aligned}
&\dot{z} - \frac{p}{F^{p+1}}F^{ij}z_{;ij} \\
&\leq \frac{p}{F^{p+1}}F^{ij}(\log h^n_n)_{;i}(\log h^n_n)_{;j} + \frac{p}{F^{p+1}}F^{ij}h_{ik}h^k_j(1 - f'v) - \frac{p+1}{F^p}h^n_n \\
&+ \frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F^k_k(1 - f'v) - \frac{\vartheta'^2}{\vartheta^2}\frac{p-1}{F^p}\kappa_n^{-1} - \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{v^2F^{p+1}}F^k_k \\
&+ \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)\frac{p+1}{v^2F^p}\kappa_n^{-1} + c(1 + |f'|)\frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F^k_k\|\nabla u\|^2 \\
&+ c\frac{\vartheta'^2}{\vartheta^2}\frac{1}{F^p}\|\nabla u\|^2\kappa_n^{-1} + \frac{c}{F^{p+1}}\frac{\vartheta'^3}{\vartheta^3}\|\nabla u\|^2\kappa_n^{-1}F^k_k + \frac{c}{F^{p+1}}\frac{\vartheta'}{\vartheta^3}\|\nabla u\|\kappa_n^{-1}F^k_k \\
&+ f'\frac{\vartheta'}{\vartheta}\frac{p+1}{F^p} + f'\frac{\vartheta'}{\vartheta}\frac{p-1}{F^p}v^2 + 2f'\frac{\vartheta'}{\vartheta}\frac{p}{F^{p+1}}F^{ij}u_{;i}v_{;j} - \frac{2f'}{v}\frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j} \\
&- f''\frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j} + \left(\frac{\vartheta'}{\vartheta} - \frac{\vartheta''}{\vartheta'}\right)\frac{p+1}{F^p}v^{-1} - \left(\frac{\vartheta'^2}{\vartheta^2} - \frac{\vartheta''}{\vartheta}\right)\frac{p}{F^{p+1}}F^k_k.
\end{aligned}$$

Hence

$$
\begin{aligned}
&\dot z - \frac{p}{F^{p+1}}F^{ij}z_{;ij}\\
&\le \frac{p}{F^{p+1}}F^{ij}(\log h_n^n)_{;i}(\log h_n^n)_{;j} + \frac{p}{F^{p+1}}F^{ij}h_{ik}h_j^k(1-f'v)\\
&+ \frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F_k^k(1-f'v) - \frac{p+1}{F^p}\left(\kappa_n - \frac{\vartheta'}{\vartheta}f'\right)\\
&+ \left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta}\right)\frac{p+1}{F^p}v^{-1}\left(v^{-1}\kappa_n^{-1}\frac{\vartheta'}{\vartheta} - 1\right) + \frac{\vartheta'}{\vartheta}\frac{p-1}{F^p}\left(f'v^2 - \frac{\vartheta'}{\vartheta}\kappa_n^{-1}\right)\\
&+ \frac{\vartheta'^2}{\vartheta^2}\left(\frac{\vartheta'}{\vartheta}\kappa_n^{-1}+1\right)\frac{c(1+|f'|)}{F^{p+1}}F_k^k\|\nabla u\|^2 + \frac{\vartheta'^2}{\vartheta^2}\frac{c}{F^p}\|\nabla u\|^2\kappa_n^{-1}\\
&+ \frac{c}{F^{p+1}}\frac{\vartheta'}{\vartheta^3}\|\nabla u\|\kappa_n^{-1}F_k^k + 2f'\frac{\vartheta'}{\vartheta}\frac{p}{F^{p+1}}F^{ij}u_{;i}v_{;j} - \frac{2f'}{v}\frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j}\\
&- f''\frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j}.
\end{aligned}
\tag{4.6}
$$

Furthermore we insert

$$(\log h_n^n)_{;i} = z_{;i} - \left(\frac{\vartheta'}{\vartheta} - \frac{\vartheta''}{\vartheta'}\right)u_{;i} - f'v_{;i}$$

and hence

$$
\begin{aligned}
&\dot z - \frac{p}{F^{p+1}}F^{ij}z_{;ij}\\
&\le \frac{p}{F^{p+1}}F^{ij}h_{ik}h_j^k(1-f'v) + \frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F_k^k(1-f'v) - \frac{p+1}{F^p}\left(\kappa_n - \frac{\vartheta'}{\vartheta}f'\right)\\
&+ \left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta}\right)\frac{p+1}{F^p}v^{-1}\left(v^{-1}\kappa_n^{-1}\frac{\vartheta'}{\vartheta} - 1\right) + \frac{p-1}{F^p}\left(\frac{\vartheta'}{\vartheta}f'v^2 - \frac{\vartheta'^2}{\vartheta^2}\kappa_n^{-1}\right)\\
&+ \frac{\vartheta'^2}{\vartheta^2}\left(\frac{\vartheta'}{\vartheta}\kappa_n^{-1}+1\right)\frac{c(1+|f'|)}{F^{p+1}}F_k^k\|\nabla u\|^2 + \frac{\vartheta'^2}{\vartheta^2}\frac{c}{F^p}\|\nabla u\|^2\kappa_n^{-1}\\
&+ \frac{c}{F^{p+1}}\frac{\vartheta'}{\vartheta^3}\|\nabla u\|\kappa_n^{-1}F_k^k + 2f'\frac{\vartheta'}{\vartheta}\frac{p}{F^{p+1}}F^{ij}u_{;i}v_{;j}\\
&+ \frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j}\left(f'^2 - \frac{2f'}{v} - f''\right) + 2\left(\frac{\vartheta'}{\vartheta} - \frac{\vartheta''}{\vartheta'}\right)f'\frac{p}{F^{p+1}}F^{ij}u_{;i}v_{;j}\\
&+ \frac{p}{F^{p+1}}F^{ij}z_{;i}z_{;j} - 2\frac{p}{F^{p+1}}F^{ij}z_{;i}\left(f'v_{;j} + \left(\frac{\vartheta'}{\vartheta} - \frac{\vartheta''}{\vartheta'}\right)u_{;j}\right).
\end{aligned}
$$

Pick

$$f(v) = -\log(v^{-\alpha} - \beta),$$

where

$$0 < \beta < \frac{1}{2v}, \quad 1 - \frac{\beta}{2} < \alpha < 1.$$

Then

$$1 - f'v = \frac{(1-\alpha)v^{-\alpha} - \beta}{v^{-\alpha} - \beta} \le \frac{\beta(\frac{v^{-\alpha}}{2} - 1)}{v^{-\alpha} - \beta} \le -\frac{\beta}{2} < 0$$

and

$$f'^2 - \frac{2}{v}f' - f'' = \frac{\alpha v^{-(\alpha+2)}}{v^{-\alpha} - \beta}(\alpha - 1) \le \frac{3}{4}\frac{\alpha - 1}{v^2} < 0.$$

Hence at a maximal point of z there holds

$$\begin{aligned}
&\dot z - \frac{p}{F^{p+1}}F^{ij}z_{;ij}\\
&\leq \frac{1}{F^p}\left(-(p+1)h_n^n + c\frac{\vartheta'}{\vartheta} + c\frac{\vartheta'^2}{\vartheta^2}(h_n^n)^{-1}\right)\\
&+ \frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F_k^k\left(-\frac{\beta}{2} + c_\epsilon\left(1+\frac{\vartheta'}{\vartheta}\kappa_n^{-1}\right)\|\nabla u\|^2 + c\frac{\vartheta'}{\vartheta}\frac{\|\nabla u\|}{\vartheta'^2}\kappa_n^{-1}\right)\\
&+ \left(\frac{3}{4}\frac{\alpha-1}{v^2} + \epsilon c f'\right)\frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j},
\end{aligned}$$

where we used

$$2|F^{ij}u_{;i}v_{;j}| \leq \frac{\epsilon\vartheta}{\vartheta'}F^{ij}v_{;i}v_{;j} + \frac{\vartheta'}{\epsilon\vartheta}F^{ij}u_{;i}u_{;j}$$

with sufficiently small ϵ. In case $\sup_{r>0}\vartheta'(r) = \infty$ or $p<1$, we have $\|\nabla u\|^2 \to 0$ and hence the result follows from the maximum principle. If $\vartheta' \leq c$ and $p=1$ we supposed that

$$F = n\frac{H_{k+1}}{H_k}$$

which implies $F_k^k \leq c$, cf. [49, Lemma 2.7], and hence

$$\frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F_k^k \leq c\frac{\vartheta'}{\vartheta}\frac{1}{F^p}$$

due to Lemma 4.3. Hence the term $-(p+1)h_n^n$ dominates the whole evolution and we also obtain the bound on z in this case. □

4.5. *Remark.* Lemma 4.4 is the only place where we need that F has this special form in case of bounded ϑ'. Of course the Euclidean case is excluded from this restriction, since the error terms involving $\|\nabla u\|^2$ will not appear here. However, the Euclidean case has already been settled in [22].

4.2. **Decay.** The global bounds from Lemma 4.3 and Lemma 4.4 as well as

$$F_{|\partial\Gamma} = 0$$

imply that the rescaled principal curvatures

$$\tilde\kappa_i = \frac{\vartheta}{\vartheta'}\kappa_i$$

range in a compact subset of Γ and hence the elliptic operator d_hF is uniformly bounded,

$$c\|\xi\|^2 \leq d_hF(\xi,\xi) \leq C\|\xi\|^2.$$

The aim of this final section is to show that all $\tilde\kappa_i$ actually behave according to the convergence rates described in Theorem 1.3. The following two lemmata prepare this result. Throughout this whole section, the procedure is similar to the one in [60].

4.6. **Lemma.** *Under the assumptions of Theorem 1.3 (ii), along (1.1) there exist constants μ and c, such that*

$$\frac{\vartheta'}{\vartheta}\frac{1}{F} - \frac{1}{n} \leq \frac{c}{\vartheta'^\mu}$$

Proof. We only have to consider the case that ϑ' is unbounded. Come back to the proof of Lemma 4.3 and consider (4.5) with

$$f(v) = \log v.$$

Hence from (4.5) we deduce that

$$z = \log\left(\frac{1}{F^p}\right) + \log v + p\log\vartheta' - p\log\vartheta + p\log n$$

satisfies

$$\begin{aligned}
&\dot z - \frac{p}{F^{p+1}}F^{ij}z_{;ij}\\
&\le \frac{p(p+1)}{F^{p+1}}\frac{\vartheta''}{\vartheta}\left(-F^{ij}g_{ij} + \frac{\vartheta}{\vartheta'}Fv^{-1} + c\|\nabla u\|^2\right)\\
&+ \frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}\left((p-1)F^{ij}g_{ij} + \frac{1-p^2}{p}\frac{\vartheta}{\vartheta'}Fv^{-1} + \frac{\vartheta}{\vartheta'}\frac{p-1}{p}Fv + c\|\nabla u\|^2\right)\\
&+ \frac{p}{F^{p+1}}F^{ij}z_{;i}z_{;j} - \frac{2}{v}\frac{p}{F^{p+1}}F^{ij}z_{;i}v_{;j} - 2p\left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta}\right)\frac{p}{F^{p+1}}F^{ij}z_{;i}u_{;j}\\
&\le \frac{np(p+1)}{F^{p+1}}\frac{\vartheta''}{\vartheta}\left(-1 + e^{-\frac{z}{p}} + c\|\nabla u\|^2\right) + \frac{np(1-p)}{F^{p+1}}\frac{\vartheta'^2}{\vartheta^2}\left(-1 + e^{-\frac{z}{p}} + c\|\nabla u\|^2\right)\\
&+ \frac{p}{F^{p+1}}F^{ij}z_{;i}z_{;j} - \frac{2}{v}\frac{p}{F^{p+1}}F^{ij}z_{;i}v_{;j} - 2p\left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta}\right)\frac{p}{F^{p+1}}F^{ij}z_{;i}u_{;j}.
\end{aligned}$$

For $\mu \ge 0$ define

$$\rho = (e^z - 1)\vartheta'^{\mu}.$$

Then

$$\begin{aligned}
&\dot\rho - \frac{p}{F^{p+1}}F^{ij}\rho_{;ij}\\
&= \left(\dot z - \frac{p}{F^{p+1}}z_{;ij}\right)e^z\vartheta'^{\mu} - \frac{p}{F^{p+1}}F^{ij}z_{;i}z_{;j}e^z\vartheta'^{\mu} + \mu\frac{\vartheta''}{\vartheta'}\left(\dot u - \frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}\right)\rho\\
&- \left(\mu(\mu-1)\frac{\vartheta'^2}{\vartheta'^2} + \mu\frac{\vartheta'''}{\vartheta'}\right)\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}\rho\\
&\le \frac{np(p+1)}{F^{p+1}}\frac{\vartheta''}{\vartheta}e^{\frac{p-1}{p}z}\Big((1-e^{\frac{z}{p}})\vartheta'^{\mu} + \frac{\mu}{np}F\frac{\vartheta}{\vartheta'}v^{-1}e^{\frac{1-p}{p}z}\rho\\
&\qquad\qquad - \frac{\mu}{n(p+1)}F^{ij}g_{ij}e^{\frac{1-p}{p}z}\rho + c_\mu\|\nabla u\|^2\vartheta'^{\mu}\Big)\\
&+ \frac{np(1-p)}{F^{p+1}}\frac{\vartheta'^2}{\vartheta^2}e^{\frac{p-1}{p}z}\left(-e^{\frac{z}{p}} + 1 + c\|\nabla u\|^2\right)\vartheta'^{\mu} - \frac{2}{v}\frac{p}{F^{p+1}}F^{ij}z_{;i}v_{;j}e^z\vartheta'^{\mu}\\
&- 2p\left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta}\right)\frac{p}{F^{p+1}}F^{ij}z_{;i}u_{;j}e^z\vartheta'^{\mu}.
\end{aligned}$$

Now we estimate at maximal points of ρ and thus may assume $z > 0$. Then, also using

$$0 = \vartheta'^{-\mu}\rho_{;i} = z_{;i}e^z + \mu\frac{\vartheta''}{\vartheta'}(e^z - 1)u_{;i},$$

we obtain

$$\begin{aligned}
&\dot\rho - \frac{p}{F^{p+1}}F^{ij}\rho_{;ij}\\
&\le \frac{np(p+1)}{F^{p+1}}\frac{\vartheta''}{\vartheta}e^{\frac{p-1}{p}z}\Big(-\rho + \frac{\mu}{np}F\frac{\vartheta}{\vartheta'}v^{-1}e^{\frac{1-p}{p}z}\rho\\
&\qquad\qquad - \frac{\mu}{n(p+1)}F^{ij}g_{ij}e^{\frac{1-p}{p}z}\rho + c\|\nabla u\|^2\vartheta'^{\mu}\Big)\\
&+ \frac{np(1-p)}{F^{p+1}}\frac{\vartheta'^2}{\vartheta^2}e^{\frac{p-1}{p}z}\left(-\rho + c\|\nabla u\|^2\vartheta'^{\mu}\right),
\end{aligned} \tag{4.7}$$

which is negative for sufficiently small μ and large times, due to Lemma 3.3 and the remarks at the beginning of this section. The proof is complete. □

4.7. **Lemma.** *Under the assumptions of Theorem 1.3 (ii), along (1.1) the i-th rescaled principal curvature converges uniformly to 1,*

$$\left| v\kappa_i \frac{\vartheta}{\vartheta'} - 1 \right| \to 0,$$

provided ϑ' is unbounded.

Proof. Using (4.6) with $f(v) = \log v$ we obtain that

$$z = \log h_n^n + \log \frac{\vartheta}{\vartheta'} + \log v$$

satisfies

$$\begin{aligned}
&\dot z - \frac{p}{F^{p+1}} F^{ij} z_{;ij} \\
&\leq \frac{p}{F^{p+1}} F^{ij} (\log h_n^n)_{;i} (\log h_n^n)_{;j} + c \frac{\vartheta'^2}{\vartheta^2} \frac{1}{F^{p+1}} \|\nabla u\|^2 - \frac{p+1}{F^p} \frac{\vartheta'}{\vartheta} v^{-1} \left(e^z - 1\right) \\
&\quad + \left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta} \right) \frac{p+1}{F^p} v^{-1} \left(e^{-z} - 1\right) + \frac{\vartheta'}{\vartheta} \frac{p-1}{F^p} v \left(1 - e^{-z}\right) + \frac{c}{F^{p+1}} \frac{1}{\vartheta^2} \|\nabla u\| \\
&= \frac{p}{F^{p+1}} F^{ij} (\log h_n^n)_{;i} (\log h_n^n)_{;j} - \frac{\vartheta''}{\vartheta'} \frac{p+1}{F^p} v^{-1} (1 - e^{-z}) + c \frac{\vartheta'^2}{\vartheta^2} \frac{1}{F^{p+1}} \|\nabla u\|^2 \\
&\quad - \frac{\vartheta'}{\vartheta} \frac{p+1}{F^p} v^{-1} e^{-z} (e^z - 1)^2 + \frac{\vartheta'}{\vartheta} \frac{p-1}{F^p} v \left(1 - e^{-z}\right) + \frac{c}{F^{p+1}} \frac{1}{\vartheta^2} \|\nabla u\|.
\end{aligned}$$

Define

$$\rho = (e^z - 1) \vartheta'^{\mu},$$

with $\mu \geq 0$. ρ satisfies

$$\begin{aligned}
&\dot\rho - \frac{p}{F^{p+1}} F^{ij} \rho_{;ij} \\
&= \left(\dot z - \frac{p}{F^{p+1}} z_{;ij} \right) e^z \vartheta'^{\mu} - \frac{p}{F^{p+1}} F^{ij} z_{;i} z_{;j} e^z \vartheta'^{\mu} + \mu \frac{\vartheta''}{\vartheta'} \left(\dot u - \frac{p}{F^{p+1}} F^{ij} u_{;i} u_{;j} \right) \rho \\
&\quad - \left(\mu(\mu - 1) \frac{\vartheta''^2}{\vartheta'^2} + \mu \frac{\vartheta'''}{\vartheta'} \right) \frac{p}{F^{p+1}} F^{ij} u_{;i} u_{;j} \rho \\
&\leq - \frac{\vartheta''}{\vartheta'} \frac{p+1}{F^p} v^{-1} \rho - \frac{\vartheta'}{\vartheta} \frac{p+1}{F^p} v^{-1} (e^z - 1) \rho - \frac{\vartheta'}{\vartheta} \frac{1-p}{F^p} v \rho + \frac{\vartheta'^2}{\vartheta^2} \frac{c}{F^{p+1}} \|\nabla u\|^2 \vartheta'^{\mu} \\
&\quad + \frac{c}{F^{p+1}} \frac{1}{\vartheta^2} \|\nabla u\| \vartheta'^{\mu} + \frac{p}{F^{p+1}} F^{ij} (\log h_n^n)_{;i} (\log h_n^n)_{;j} e^z \vartheta'^{\mu} \\
&\quad - \frac{p}{F^{p+1}} F^{ij} z_{;i} z_{;j} e^z \vartheta'^{\mu} + \mu \frac{\vartheta''}{\vartheta'} \frac{p+1}{F^p} v^{-1} \rho - \mu \frac{\vartheta''}{\vartheta} \frac{p}{F^{p+1}} F^{ij} g_{ij} \rho.
\end{aligned}$$

At spatial maxima of ρ we have

$$\begin{aligned}
0 = \vartheta'^{-\mu} \rho_{;i} &= z_{;i} e^z + \mu \frac{\vartheta''}{\vartheta'} (e^z - 1) u_{;i} \\
&= h_{n;i}^n v \frac{\vartheta}{\vartheta'} + h_n^n \left(\frac{\vartheta}{\vartheta'} \right)_{;i} v + h_n^n \frac{\vartheta}{\vartheta'} v_{;i} + \mu \frac{\vartheta''}{\vartheta'} (e^z - 1) u_{;i}
\end{aligned}$$

and hence

$$F^{ij} (\log h_n^n)_{;i} (\log h_n^n)_{;j} \leq c \frac{\vartheta'^2}{\vartheta^2} \|\nabla u\|^2.$$

We obtain at a maximal point where $\rho > 0$

$$\begin{aligned}
&\dot\rho - \frac{p}{F^{p+1}}F^{ij}\rho_{;ij}\\
\leq\ & -\frac{\vartheta'}{\vartheta}\frac{p+1}{F^p}v^{-1}(e^z-1)\rho - \frac{\vartheta'}{\vartheta}\frac{1-p}{F^p}v\rho\\
&+ \frac{n}{F^p}\frac{\vartheta''}{\vartheta'}v^{-1}\rho\left(-\mu p\frac{v}{F}\frac{\vartheta'}{\vartheta} + (p+1)\frac{\mu-1}{n}\right) + \frac{\vartheta'^2}{\vartheta^2}\frac{c}{F^{p+1}}\|\nabla u\|^2\vartheta'^\mu\\
&+ \frac{c}{F^{p+1}}\frac{1}{\vartheta^2}\|\nabla u\|\vartheta'^\mu\\
\leq\ & \frac{\vartheta'}{\vartheta}\frac{p+1}{vF^p}\left(-\frac{1-p}{p+1}\rho - (e^z-1)\rho + c\|\nabla u\|^2\vartheta'^\mu + \frac{c\vartheta'^\mu}{\vartheta'^2}\|\nabla u\|\right)\\
&+ \frac{n}{F^p}\frac{\vartheta''}{\vartheta'}v^{-1}\rho\left(-\mu p\frac{v}{F}\frac{\vartheta'}{\vartheta} + (p+1)\frac{\mu-1}{n}\right).
\end{aligned} \tag{4.8}$$

Set

$$\tilde\rho(t) = \max_M \rho(t,\cdot).$$

Note that $\tilde\rho$ is Lipschitz continuous, hence differentiable almost everywhere in $(0,\infty)$ and at points of differentiability there holds

$$\dot{\tilde\rho}(t) = \frac{\partial\rho}{\partial t}(t,x_t),$$

where

$$\rho(t,x_t) = \tilde\rho(t),$$

cf. [23, Lemma 6.3.2]. The original idea of this useful fact goes back to Hamilton [30, Lemma 3.5]. Choosing $\mu > 0$ small enough, $\|\nabla u\|^2\vartheta'^\mu$ converges to zero due to Lemma 3.3 and we obtain that for sufficiently large t,

$$\dot{\tilde\rho}(t) \leq 0$$

on the set $\{\tilde\rho \geq 1\}$, provided $p < 1$. Hence in this case ρ is bounded. In case $p = 1$ we set $\mu = 0$ and obtain that for all $\epsilon > 0$ there exist $\delta_\epsilon > 0$ and T_ϵ, such that for all $t \geq T_\epsilon$ where $\tilde\rho$ is differentiable, there holds

$$\tilde\rho(t) \geq \epsilon \quad\Rightarrow\quad \dot{\tilde\rho}(t) < -\delta_\epsilon.$$

[58, Lemma 4.2] implies $\limsup_{t\to\infty}\tilde\rho \leq 0$. Hence

$$\limsup_{t\to\infty} v\kappa_n\frac{\vartheta}{\vartheta'} \leq 1$$

in both cases. Now

$$\sum_{i=1}^n \frac{\left(1 - v\kappa_i\frac{\vartheta}{\vartheta'}\right)}{nvF\frac{\vartheta}{\vartheta'}} = \frac{n - vH\frac{\vartheta}{\vartheta'}}{nvF\frac{\vartheta}{\vartheta'}} \leq \frac{n - vF\frac{\vartheta}{\vartheta'}}{nvF\frac{\vartheta}{\vartheta'}} \leq \frac{\vartheta'}{F\vartheta} - \frac{1}{n} \leq c\vartheta'^{-\mu}$$

and hence

$$1 - v\kappa_1\frac{\vartheta}{\vartheta'} \leq c\vartheta'^{-\mu} + \sum_{i=2}^n\left(v\kappa_i\frac{\vartheta}{\vartheta'} - 1\right). \tag{4.9}$$

The proof is complete. □

Now we are in the position to optimise the decay estimates. We start with the gradient.

4.8. **Lemma.** *Under the assumption of Theorem 1.3 (ii) the function*

$$\tilde{z} = |\hat{\nabla}\varphi|^2 \vartheta'^{2p}$$

is uniformly bounded. If $p = 1$, then additionally there exist constants c and α such that

$$|\hat{\nabla}\varphi|^2 \leq ce^{-\alpha t}.$$

Proof. If ϑ' is unbounded, using Lemma 4.7 we can rewrite the evolution of

$$z = f(u)|\hat{\nabla}\varphi|^2$$

from (3.2) with

$$f(u) = \vartheta'^{\gamma}(u)$$

at maximal points as

$$\begin{aligned}
\mathcal{L}z \leq{}& \frac{2z}{F^{p+1}}\Big((p-1)\frac{\vartheta'}{\vartheta}vF - p\frac{\vartheta''}{\vartheta}F^k_k + \gamma\frac{p+1}{2}\frac{\vartheta''}{\vartheta'}\frac{F}{v} - \frac{\gamma np}{2}\frac{\vartheta''}{\vartheta}\Big)\\
&- \frac{2f}{\vartheta^2}\frac{p}{F^{p+1}}F^k_l\tilde{g}^{lr}\hat{R}^m_{ikr}\varphi^i\varphi_m + \frac{p}{F^{p+1}}F^{kr}u_k u_r z\left(2\frac{f'^2}{f^2} - \frac{f''}{f} - \frac{f'}{f}\frac{\vartheta'}{\vartheta}\right)\\
\leq{}& \frac{2z}{F^{p+1}}\left(o(1)\frac{\vartheta'^2}{\vartheta^2} + n(p-1)\frac{\vartheta'^2}{\vartheta^2} - np\frac{\vartheta''}{\vartheta} + \frac{\gamma n}{2}\frac{\vartheta''}{\vartheta}\right)\\
&- \frac{2f}{\vartheta^2}\frac{p}{F^{p+1}}F^k_l\tilde{g}^{lr}\hat{R}^m_{ikr}\varphi^i\varphi_m.
\end{aligned}$$

Since we want to bound z, it suffices to consider spatial maxima at which z is positive. At such there holds

$$\begin{aligned}
\mathcal{L}z \leq \frac{2z}{F^{p+1}}\frac{\vartheta'^2}{\vartheta^2}\Big(o(1) + n(p-1) - n\left(p - \frac{\gamma}{2}\right)\frac{\vartheta''\vartheta}{\vartheta'^2}\\
- \frac{p}{\vartheta'^2}\widehat{\mathrm{Rc}}\left(\frac{\hat{\nabla}\varphi}{|\hat{\nabla}\varphi|}, \frac{\hat{\nabla}\varphi}{|\hat{\nabla}\varphi|}\right)\Big).
\end{aligned} \tag{4.10}$$

In case $p < 1$ with $\gamma = 2p$, the right hand side is eventually negative for large t, since only the case of unbounded ϑ' has to be considered to prove the first statement. In case $p = 1$ we put $\gamma = 0$ and regardless whether ϑ' is bounded or unbounded, we use (4.10) to get

$$\mathcal{L}z \leq -\delta z$$

for some δ and large times. The exponential decay follows. To prove the remaining claim, we evaluate (3.4) with

$$f = \vartheta'^2, \quad p = 1$$

and see

$$\mathcal{L}z \leq -\frac{2}{F^2}\frac{\vartheta''}{\vartheta}\left(1 + 1 - ce^{-\alpha t} - 2\right)F^k_k z + \frac{c}{F^2}\frac{\vartheta'^2}{\vartheta^2}e^{-\alpha t}z.$$

Hence the function

$$\bar{z}(t) = \max_{\mathcal{S}_0} z(t, \cdot)$$

satisfies

$$\dot{\bar{z}} \leq ce^{-\alpha t}\bar{z}$$

and is thus bounded. □

We optimise the convergence rate of the rescaled principal curvatures.

4.9. **Lemma.** *Under the assumptions of Theorem 1.3 (ii), along (1.1) there exists a constant c, such that for all $1 \leq i \leq n$, the i-th rescaled principal curvature satisfies*

$$\left| v\kappa_i \frac{\vartheta}{\vartheta'} - 1 \right| \leq \frac{ct}{\vartheta'^{p(p+1)}},$$

where we may drop the t-factor if $p < 1$ or if ϑ' is bounded.

Proof. Only the case that ϑ' is unbounded has to be considered.

(i) First we optimise the decay in Lemma 4.6. Using the optimal gradient estimates Lemma 4.8, we see from (4.7) that Lemma 4.6 holds with any $\mu < p(p+1)$, if c is allowed to depend on a lower bound of $p(p+1) - \mu$.

Now consider the function ρ defined in the proof of Lemma 4.7 and obtain from (4.8) with $\mu < p(p+1)$ at points where $\rho \geq 1$ that

$$\begin{aligned}
&\dot\rho - \frac{p}{F^{p+1}} F^{ij}\rho_{;ij} \\
&\leq \frac{\vartheta'}{\vartheta}\frac{p+1}{vF^p}\left(-\frac{1-p}{p+1}\rho + c\|\nabla u\|^2\vartheta'^\mu + \frac{c\vartheta'^\mu}{\vartheta'^2}\|\nabla u\|\right) \\
&\quad + \frac{n}{F^p}\frac{\vartheta''}{\vartheta'}v^{-1}\rho\left(\frac{-p\mu}{n} + o(1) + (p+1)\frac{\mu-1}{n}\right) \\
&= \frac{\vartheta'}{\vartheta}\frac{p+1}{vF^p}\left(-\frac{1-p}{p+1}\rho + c\|\nabla u\|^2\vartheta'^\mu + \frac{c\vartheta'^\mu}{\vartheta'^2}\|\nabla u\|\right) \\
&\quad + \frac{1}{F^p}\frac{\vartheta''}{\vartheta'}v^{-1}\rho\left(\mu - (p+1) + o(1)\right) \\
&< 0
\end{aligned}$$

in case $p < 1$ for large times. In case $p = 1$ the right hand side of this inequality eventually decays exponentially and thus

$$\rho \leq c$$

in both cases. Hence for any $\mu < p(p+1)$ we have

$$v\kappa_n \frac{\vartheta}{\vartheta'} - 1 \leq \frac{c_\mu}{\vartheta'^\mu}.$$

Now putting $\mu = p(p+1)$ in (4.7) we see that the function ρ defined in the proof of Lemma 4.6 satisfies at positive maximal points with $\rho \geq 1$

$$\begin{aligned}
&\dot\rho - \frac{p}{F^{p+1}} F^{ij}\rho_{;ij} \\
&\leq \frac{np(p+1)}{F^{p+1}}\frac{\vartheta''}{\vartheta}e^{\frac{p-1}{p}z}\Big(-\rho + (p+1)\frac{F}{n}\frac{\vartheta}{\vartheta'}v^{-1}e^{\frac{1-p}{p}z}\rho - p\rho + c\|\nabla u\|^2\vartheta'^{p(p+1)}\Big) \\
&\quad + \frac{np(1-p)}{F^{p+1}}\frac{\vartheta'^2}{\vartheta^2}e^{\frac{p-1}{p}z}\left(-\rho + c\|\nabla u\|^2\vartheta'^{p(p+1)}\right) \\
&< \frac{np(p+1)}{F^{p+1}}\frac{\vartheta''}{\vartheta}e^{\frac{p-1}{p}z}\left(c\vartheta'^{-p}\rho + c\|\nabla u\|^2\vartheta'^{p(p+1)}\right) \\
&\quad + \frac{np(1-p)}{F^{p+1}}\frac{\vartheta'^2}{\vartheta^2}e^{\frac{p-1}{p}z}\left(-\rho + c\|\nabla u\|^2\vartheta'^{p(p+1)}\right).
\end{aligned}$$

In case $p < 1$ we use

$$\frac{\vartheta''}{\vartheta} \leq c\frac{\vartheta'^2}{\vartheta^2}$$

to absorb every decaying term into $-\rho$ in the second line. In case $p = 1$ we use

$$\vartheta'^{-p}\rho \leq c$$

to conclude

$$\dot\rho - \frac{p}{F^{p+1}} F^{ij}\rho_{;ij} \leq c.$$

Hence we obtain

$$\frac{\vartheta'}{\vartheta}\frac{1}{F}-\frac{1}{n}\leq\frac{ct}{\vartheta'^{p(p+1)}},$$

and the same without the t-factor in case $p<1$.

(ii) In the second step we optimise the convergence rate in Lemma 4.7. Therefore we consider the function ρ defined in that proof and obtain from (4.8) with $\mu=p(p+1)$ at points where $\rho\geq 1$ that

$$\begin{aligned}
&\dot\rho-\frac{p}{F^{p+1}}F^{ij}\rho_{;ij}\\
&\leq\frac{\vartheta'}{\vartheta}\frac{p+1}{vF^p}\left(-\frac{1-p}{p+1}\rho+c\|\nabla u\|^2\vartheta'^{p(p+1)}+\frac{c\vartheta'^{p(p+1)}}{\vartheta'^2}\|\nabla u\|\right)\\
&\quad+\frac{n}{F^p}\frac{\vartheta''}{\vartheta'}v^{-1}\rho\left(\frac{-p^2(p+1)}{n}+o(1)+(p+1)\frac{p(p+1)-1}{n}\right)\\
&=\frac{\vartheta'}{\vartheta}\frac{p+1}{vF^p}\left(-\frac{1-p}{p+1}\rho+c\|\nabla u\|^2\vartheta'^{p(p+1)}+\frac{c\vartheta'^{p(p+1)}}{\vartheta'^2}\|\nabla u\|\right)\\
&\quad+\frac{1}{F^p}\frac{\vartheta''}{\vartheta'}v^{-1}\rho\left(p^2-1+o(1)\right)\\
&\leq 0
\end{aligned}$$

in case $p<1$ for large times. In case $p=1$ the right hand side of this inequality is bounded and thus

$$\rho\leq ct$$

in this case. Estimating (4.9) with the optimised bounds completes the proof. □

We finish the proof of Theorem 1.3 by proving the final statement about the exponential decay in item (ii). The function

$$z=\log\vartheta(u)-\frac{t}{n}$$

defined on $[0,\infty)\times\mathcal{S}_0$ satisfies

$$\dot z=\frac{v\vartheta'}{\vartheta F(\mathcal{W})}-\frac{1}{n}=\frac{v}{F\left(\frac{1}{v}\delta^i_j+\frac{1}{v^3\vartheta^2}u^iu_j-\frac{1}{v\vartheta'\vartheta}\tilde g^{ik}u_{kj}\right)}-\frac{1}{n}=G(y,z,\hat\nabla z,\hat\nabla^2 z).$$

Hence

$$\frac{\partial G}{\partial z_{ij}}=\frac{\vartheta'^{-2}}{F^2\left(\frac{\vartheta}{\vartheta'}\mathcal{W}\right)}F^i_k\tilde g^{kj},$$

which is uniformly elliptic, since ϑ' is globally bounded. z is uniformly bounded, as can be seen similarly as in [60, Prop. 3.1, Lemma 3.2]. Furthermore

$$|\hat\nabla z|\leq c|\hat\nabla\varphi|\leq ce^{-\alpha t}$$

and

$$|\hat\nabla^2\varphi|\leq c.$$

Applying the regularity results of Krylov and Safonov as well as Schauder theory, we obtain uniform C^m-bounds for z. Due to interpolation we get

$$|\hat\nabla^2\varphi|\leq ce^{-\alpha t},$$

which implies (1.3) with t replaced by $e^{-\alpha t}$.

4.10. *Remark.* The previous argument is precisely the way to deduce a uniform bound on the rescaled principal curvatures for the inverse mean curvature flow, when ϑ' is bounded, as it was performed in [52, 71]. The crucial point is here, that

one does not need curvature estimates to have F^{ij} uniformly elliptic. One only needs a bound on the rescaled speed

$$\tilde{H} = \frac{\vartheta}{\vartheta'} H.$$

Then the above argumentation applies.

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References

1. Brian Allen, *Long time existence of IMCF on metrics conformal to warped product manifolds*, preprint, arxiv:1708.02535, 2017.
2. Ben Andrews, *Harnack inequalities for evolving hypersurfaces*, Math. Z. **217** (1994), no. 1, 179–197.
3. ———, *Evolving convex curves*, Calc. Var. Partial Differ. Equ. **7** (1998), no. 4, 315–371.
4. ———, *Pinching estimates and motion of hypersurfaces by curvature functions*, J. Reine Angew. Math. **608** (2007), 17–33.
5. Simon Brendle, Pei-Ken Hung, and Mu Tao Wang, *A Minkowski inequality for hypersurfaces in the anti-de Sitter-Schwarzschild manifold*, Commun. Pure Appl. Math. **69** (2016), no. 1, 124–144.
6. Paul Bryan, Mohammad N. Ivaki, and Julian Scheuer, *Harnack inequalities for evolving hypersurfaces on the sphere*, to appear in Commun. Anal. Geom., arxiv:1512.03374, 2015.
7. ———, *Harnack inequalities for curvature flows in Riemannian and Lorentzian manifolds*, preprint, arxiv:1703.07493, 2017.
8. Ildefonso Castro and Ana Lerma, *Lagrangian homothetic solitons for the inverse mean curvature flow*, Result. Math. **71** (2017), no. 3–4, 1109–1125.
9. Daguang Chen, Haizhong Li, and Tailong Zhou, *A Penrose type inequality for graphs over Reissner-Nordström-anti-deSitter manifold*, preprint, arxiv:1710.02340, 2017.
10. Li Chen and Jing Mao, *Non-parametric inverse curvature flows in the AdS-Schwarzschild manifold*, J. Geom. Anal. **28** (2018), no. 2, 921–949.
11. Bennett Chow, Lii-Perng Liou, and Dong-Ho Tsai, *Expansion of embedded curves with turning angle greater than $-\pi$*, Invent. Math. **123** (1996), no. 3, 415–429.
12. Bennett Chow and Dong-Ho Tsai, *Geometric expansion of convex plane curves*, J. Differ. Geom. **44** (1996), no. 2, 312–330.
13. ———, *Expansion of convex hypersurfaces by nonhomogeneous functions of curvature*, Asian J. Math. **1** (1997), no. 4, 769–784.
14. ———, *Nonhomogeneous Gauss curvature flows*, Indiana Univ. Math. J. **47** (1998), no. 3, 965–994.
15. Tzs-Kiu Aaron Chow, Ka-Wing Chow, and Frederick Tsz-Ho Fong, *Self-expanders to inverse curvature flows by homogeneous functions*, to appear in Commun. Anal. Geom., arxiv:1701.03995, 2017.
16. Levi Lopes De Lima and Frederico Girao, *An Alexandrov-Fenchel-type inequality in hyperbolic space with an application to a Penrose inequality*, Ann. Henri Poincaré **17** (2016), no. 4, 979–1002.
17. Qi Ding, *The inverse mean curvature flow in rotationally symmetric spaces*, Chin. Ann. Math., Ser. B **32** (2011), no. 1, 27–44.
18. Gregory Drugan, Hoojoo Lee, and Glen Wheeler, *Solitons for the inverse mean curvature flow*, Pac. J. Math. **284** (2016), no. 2, 309–326.
19. Christian Enz, *The scalar curvature flow in Lorentzian manifolds*, Adv. Calc. Var. **1** (2008), no. 3, 323–343.
20. Yuxin Ge, Guofang Wang, and Jie Wu, *Hyperbolic Alexandrov-Fenchel quermassintegral inequalities 2*, J. Differ. Geom. **98** (2014), no. 2, 237–260.
21. Yuxin Ge, Guofang Wang, Jie Wu, and Chao Xia, *A Penrose inequality for graphs over Kottler space*, Calc. Var. Partial Differ. Equ. **52** (2015), no. 3, 755–782.
22. Claus Gerhardt, *Flow of nonconvex hypersurfaces into spheres*, J. Differ. Geom. **32** (1990), no. 1, 299–314.
23. ———, *Curvature problems*, Series in Geometry and Topology, vol. 39, International Press of Boston Inc., Sommerville, 2006.

24. ———, *Inverse curvature flows in hyperbolic space*, J. Differ. Geom. **89** (2011), no. 3, 487–527.
25. ———, *Non-scale-invariant inverse curvature flows in Euclidean space*, Calc. Var. Partial Differ. Equ. **49** (2014), no. 1-2, 471–489.
26. ———, *Curvature flows in the sphere*, J. Differ. Geom. **100** (2015), no. 2, 301–347.
27. Robert Geroch, *Energy extraction*, Ann. New York Acad. Sci. **224** (1973), 108–117.
28. Frederico Girao and Neilha Pinheiro, *An Alexandrov-Fenchel-type inequality for hypersurfaces in the sphere*, Ann. Glob. Anal. Geom. **52** (2017), no. 4, 413–424.
29. Pengfei Guan and Junfang Li, *The quermassintegral inequalities for k-convex starshaped domains*, Adv. Math. **221** (2009), no. 5, 1725–1732.
30. Richard Hamilton, *Four-manifolds with positive curvature operator*, J. Differ. Geom. **24** (1986), no. 2, 153–197.
31. Yingxiang Hu, *Willmore inequality on hypersurfaces in hyperbolic space*, Proc. Am. Math. Soc. **146** (2018), no. 6, 2679–2688.
32. Gerhard Huisken and Tom Ilmanen, *The Riemannian Penrose inequality*, Int. Math. Res. Not. **1997** (1997), no. 20, 1045–1058.
33. ———, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differ. Geom. **59** (2001), no. 3, 353–437.
34. ———, *Higher regularity of the inverse mean curvature flow*, J. Differ. Geom. **80** (2008), no. 3, 433–451.
35. Pei-Ken Hung and Mu Tao Wang, *Inverse mean curvature flows in the hyperbolic 3-space revisited*, Calc. Var. Partial Differ. Equ. **54** (2015), no. 1, 119–126.
36. Mohammad N. Ivaki, *An application of dual convex bodies to the inverse Gauss curvature flow*, Proc. Am. Math. Soc. **143** (2015), no. 3, 1257–1271.
37. Nina Ivochkina, Thomas Nehring, and Friedrich Tomi, *Evolution of starshaped hypersurfaces by nonhomogeneous curvature functions*, Algebra i Anal. **12** (2000), no. 1, 185–203.
38. Pong Jang and Robert Wald, *The positive energy conjecture and the cosmic censor hypothesis*, J. Math. Phys. **18** (1977), no. 1, 41–44.
39. Naoyuki Koike and Yusuke Sakai, *The inverse mean curvature flow in rank one symmetric spaces of non-compact type*, Kyushu J. Math. **69** (2015), 259–284.
40. Heiko Kröner and Julian Scheuer, *Expansion of pinched hypersurfaces of the Euclidean and hyperbolic space by high powers of curvature*, preprint, arxiv:1703.07087, 2017.
41. Kwok-Kun Kwong and Pengzi Miao, *A new monotone quantity along the inverse mean curvature flow in* $\mathbb{R}^n$, Pac. J. Math. **267** (2014), no. 2, 417–422.
42. Haizhong Li, Xianfeng Wang, and Yong Wei, *Surfaces expanding by non-concave curvature functions*, Ann. Glob. Anal. Geom. (2018).
43. Haizhong Li and Yong Wei, *On inverse mean curvature flow in Schwarzschild space and Kottler space*, Calc. Var. Partial Differ. Equ. **56** (2017), no. 3, 62.
44. Haizhong Li, Yong Wei, and Changwei Xiong, *A geometric inequality on hypersurface in hyperbolic space*, Adv. Math. **253** (2014), 152–162.
45. Qi-Rui Li, *Surfaces expanding by the power of the Gauss curvature flow*, Proc. Am. Math. Soc. **138** (2010), no. 11, 4089–4102.
46. Yi Li, *Harnack inequality for the negative power Gaussian curvature flow*, Proc. Am. Math. Soc. **139** (2011), no. 10, 3707–3717.
47. Yu-Chu Lin and Dong-Ho Tsai, *Using Aleksandrov reflection to estimate the location of the center of expansion*, Proc. Am. Math. Soc. **138** (2010), no. 2, 557–565.
48. Yan-nan Liu, *Inscribed radius estimates for inverse curvature flow in the sphere and hyperbolic space*, Nonlinear Anal. **155** (2017), 198–206.
49. Siyuan Lu, *Inverse curvature flow in anti-de Sitter-Schwarzschild manifold*, To appear in Commun. Anal. Geom., arxiv:1609.09733, 2016.
50. Siyuan Lu and Pengzi Miao, *Minimal hypersurfaces and boundary behavior of compact manifolds with nonnegative scalar curvature*, to appear in J. Differ. Geom., arxiv:1703.08164, 2017.
51. Matthias Makowski and Julian Scheuer, *Rigidity results, inverse curvature flows and Alexandrov-Fenchel type inequalities in the sphere*, Asian J. Math. **20** (2016), no. 5, 869–892.
52. Thomas Mullins, *On the inverse mean curvature flow in warped product manifolds*, preprint, arxiv:1610.05234, 2016.
53. Andre Neves, *Insufficient convergence of inverse mean curvature flow on asymptotically hyperbolic manifolds*, J. Differ. Geom. **84** (2010), no. 1, 191–229.
54. Barrett O'Neill, *Semi-Riemannian geometry with applications to relativity*, Pure and applied mathematics, vol. 103, Academic Press, London, 1983.

55. Giuseppe Pipoli, *Inverse mean curvature flow in complex hyperbolic space*, to appean in Ann. Sci. Éc. Norm. Supér., arxiv:1610.01886, 2016.
56. ———, *Inverse mean curvature flow in quaternionic hyperbolic space*, Rend. Lincei Mat. Appl. **29** (2017), no. 1, 153–171.
57. Julian Scheuer, *Gradient estimates for inverse curvature flows in hyperbolic space*, Geom. Flows **1** (2015), no. 1, 11–16.
58. ———, *Non-scale-invariant inverse curvature flows in hyperbolic space*, Calc. Var. Partial Differ. Equ. **53** (2015), no. 1, 91–123.
59. ———, *Pinching and asymptotical roundness for inverse curvature flows in Euclidean space*, J. Geom. Anal. **26** (2016), no. 3, 2265–2281.
60. ———, *The inverse mean curvature flow in warped cylinders of non-positive radial curvature*, Adv. Math. **306** (2017), 1130–1163.
61. ———, *Isotropic functions revisited*, Arch. Math. **110** (2018), no. 6, 591–604.
62. Julian Scheuer and Chao Xia, *Locally constrained inverse curvature flows*, preprint, arxiv:1708.06125, 2017.
63. Oliver C. Schnürer, *Surfaces expanding by the inverse Gauß curvature flow*, J. Reine Angew. Math. **600** (2006), 117–134.
64. Knut Smoczyk, *A representation formula for the inverse harmonic mean curvature flow*, Elem. Math. **60** (2005), no. 2, 57–65.
65. John Urbas, *On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures*, Math. Z. **205** (1990), no. 1, 355–372.
66. ———, *An expansion of convex hypersurfaces*, J. Differ. Geom. **33** (1991), no. 1, 91–125.
67. Zhuhai Wang, *A Minkowski-type inequality for hypersurfaces in the Reissner-Nordström-anti-DeSitter manifold*, Ph.D. thesis, Columbia University, 2015.
68. Yong Wei, *New pinching estimate for inverse curvature flow in space form*, J. Geom. Anal. (2018), doi:10.1007/s12220-018-0051-1.
69. Yong Wei and Changwei Xiong, *Inequalities of Alexandrov-Fenchel type for convex hypersurfaces in hyperbolic space and in the sphere*, Pac. J. Math. **277** (2015), no. 1, 219–239.
70. Hao Yu, *Dual flows in hyperbolic space and de Sitter space*, Ph.D. thesis, University of Heidelberg, 2017.
71. Hengyu Zhou, *Inverse mean curvature flows in warped product manifolds*, J. Geom. Anal. **28** (2018), no. 2, 1749–1772.

Julian Scheuer, Albert-Ludwigs-Universität, Mathematisches Institut, Ernst-Zermelo-Str. 1, 79104 Freiburg, Germany
E-mail address: julian.scheuer@math.uni-freiburg.de