ISOPERIMETRIC PROBLEMS FOR SPACELIKE DOMAINS IN GENERALIZED ROBERTSON-WALKER SPACES

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ABSTRACT. We use a locally constrained mean curvature flow to prove the isoperimetric inequality for spacelike domains in generalized Robertson-Walker spaces satisfying the null convergence condition.

1. INTRODUCTION

For a bounded domain Ω of the two-dimensional Euclidean, hyperbolic or spherical space the isoperimetric inequality is given by

$$L^2 \geq 4\pi A, \quad L^2 \geq 4\pi A + A^2, \quad L^2 \geq 4\pi A - A^2,$$

respectively, where L is the boundary length of Ω and A the area of Ω. Equality holds precisely when Ω is a metric ball. For open sets Ω of $(n+1)$ dimensional Euclidean space this becomes

$$(1.1) \qquad |\partial\Omega| \geq c_n|\Omega|^{\frac{n}{n+1}},$$

with an explicit dimensional constant c_n, where again, equality holds precisely on balls. Here $|\cdot|$ is the Hausdorff measure of a submanifold of the appropriate dimension. In higher dimensional hyperbolic and spherical spaces no explicit form like (1.1) is available. However, the isoperimetric problem is solved, [18, 19, 20]: For any bounded domain Ω in hyperbolic or spherical space and any geodesic ball B_r with $|\Omega| = |B_r|$ there holds

$$|\partial\Omega| \geq |\partial B_r|$$

with equality precisely when $\Omega = B_r$. We can describe this result in an alternative way: Define

$$f_0(r) = |B_r|, \quad f_1(r) = |\partial B_r|,$$

then

$$|\partial \Omega| \geq |\partial B_r| = f_1(r) = f_1 \circ f_0^{-1}(|B_r|) \equiv \varphi(|\Omega|),$$

which is an implicit form of the isoperimetric inequality. In more general Riemann-ian warped product spaces such an implicit isoperimetric inequality was deduced in [13], while also on other Riemannian spaces isoperimetric inequalities have received lots of attention. We do not provide a full bibliography but mention [6, 7, 8, 14, 22].

Much less seems to be known about the comparison of the volume of a domain in Lorentzian spaces which is bounded by spacelike hypersurfaces. Here the question is, which hypersurfaces maximize area under volume constraint. Some results are available in Minkowski space [4], on two-dimensional Lorentzian surfaces satisfying a curvature bound [3] and in warped product spaces, such as in a certain class of Friedman-Robertson-Walker spaces [1].

The goal of this paper is to solve the isoperimetric problem for spacelike domains in a large class of Lorentzian warped product manifolds, which we describe in the following.

A spacetime

$$N = (a,b) \times \mathcal{S}_0, \quad \bar{g} = -dr^2 + \vartheta^2(r)\hat{g},$$

with $a < b$ real numbers, a compact n-dimensional Riemannian manifold $(\mathcal{S}_0, \hat{g})$ and positive warping factor $\vartheta \in C^\infty([a,b))$ is called *generalized Robertson-Walker space*.

In this paper we use the *locally constrained mean curvature flow*, that is, a time dependent family of spacelike parametrisations $x : [0,T) \times \mathcal{S}_0 \to N$ such that

$$\tag{1.2} \dot{x} = (\Delta_{\Sigma_t} \Theta)\nu,$$

to solve an isoperimetric problem in N. Here $\Theta'(r) = \vartheta(r)$, Θ is understood to be defined on the ambient manifold and ν is the future directed timelike (i.e. $\bar{g}(\nu, \nu) = -1$) normal vector to the flow hypersurfaces Σ_t. The terminology used for this flow stems from the fact that (1.2) may equivalently be defined by

$$\tag{1.3} \dot{x} = (uH - n\vartheta'(\rho))\nu,$$

where H is the mean curvature of the flow hypersurfaces with respect to $-\nu$, $\rho(t, \cdot) = r_{|\Sigma_t}$ and u is the support function

$$u = -\bar{g}(\vartheta \partial_r, \nu).$$

The key observation is that, if N satisfies the null convergence condition below, (1.3) deforms a hypersurface so as to keep the volume enclosed by a hypersurface constant while increasing the area of the hypersurface, and the flow converges to a totally umbilic hypersurface as $t \to \infty$. This idea and the suitable adaption of (1.3) to the Riemannian setting was introduced in [12] and further studied in [13].

A Lorentzian manifold N is said to satisfy the *null convergence condition* if for all lightlike vectors X,

$$\tag{1.4} \overline{\mathrm{Rc}}(X, X) \geq 0.$$

We also say that the null convergence condition is satisfied *strictly*, if (1.4) holds with strict inequality for all nonzero lightlike X. We observe that this condition is implied by the more commonly used timelike convergence condition which is well

known to be important in prescribed mean curvature equations, see for example the work of Bartnik [5], Ecker and Huisken [10] and Ecker [9]. However, it is also valid on any Einstein manifold, while the timelike convergence condition is not.

We prove the following isoperimetric inequality for domains bounded by a time-slice and a closed spacelike hypersurface. For its formulation we first require some further terminology. A hypersurface $\Sigma \subset N$ is called spacelike, if the induced metric is positive definite. $\hat{\Sigma}$ is the region enclosed by $\{a\} \times \mathcal{S}_0$ and Σ, see Section 2. Such a spacelike hypersurface is called achronal if no timelike curve meets Σ more than once (see [17, p. 425] for more details). Also note that this is automatically satisfied if N is simply connected, [17, p. 427].

1.1. **Theorem.** *Let $n \geq 2$ and N^{n+1} be a generalized Robertson-Walker space which satisfies the null convergence condition. Let $\Sigma \subset N$ be a spacelike, compact, achronal and connected hypersurface. Then there holds*

$$\varphi(\mathrm{vol}(\hat{\Sigma})) \geq |\Sigma|,$$

where $\varphi \colon [0, \mathrm{vol}(N)) \to \mathbb{R}_+$ is the function which gives equality on the coordinate slices. Furthermore:

(i) *If equality holds, then Σ is totally umbilic.*
(ii) *If N satisfies the null convergence condition strictly, then equality is attained precisely on the timeslices of N.*

2. Conventions and some hypersurface geometry

2.1. **Basic notation.** Throughout this paper we use the curvature conventions from [17], in particular the Riemann tensor of a semi-Riemannian manifold with metric $\bar{g}$ and Levi-Civita connection $\overline{\nabla}$ is defined by

$$\overline{\mathrm{Rm}}(X,Y)Z = \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_{[Y,X]} Z$$

for all vector fields X, Y, Z on N.

Given any orthonormal frame $E_1, \ldots, E_{n+1}$ where E_{n+1} is timelike, define the Ricci curvature by

$$\overline{\mathrm{Rc}}(X,Y) = \bar{g}(\overline{\mathrm{Rm}}(E_i, X)E^i, Y) - \bar{g}(\overline{\mathrm{Rm}}(E_{n+1}, X)E_{n+1}, Y).$$

Here the summation has been chosen so that the Ricci curvature of the Lorentz product metric on $\mathbb{S}^n \times \mathbb{R}$ is non-negative.

Spacelike hypersurfaces. Let $\Sigma \subset N$ be a spacelike, compact, connected and achronal hypersurface given by an embedding x. The manifold N is globally hyperbolic [11, Thm. 1.4.2] and $\mathcal{S}_0$ is a Cauchy hypersurface. Thus Σ is a graph over $\mathcal{S}_0$,

$$\Sigma = \{(\rho(x^i), x^i) \colon (x^i) \in \mathcal{S}_0\},$$

see [11, Prop. 1.6.3]. Latin indices range between 1 and n and greek indices range from 0 to n. Sometime we will write

$$x^0 = r.$$

We state the relations between the geometric quantities of Σ and the graph function ρ. Details can be found in [11, Sec. 1.6]. We use the coordinate based notation, e.g. the induced metric g is

$$g_{ij} = g(\partial_i, \partial_j)$$

and we denote its Levi-Civita connection by $\nabla_i = \nabla_{\partial_i}$. We also write

$$x_i := \partial_i x.$$

Let ν be the future directed timelike normal, i.e.

$$\bar{g}(\partial_r, \nu) < 0,$$

and define the shape operator $S = (h_j^i)$ of Σ with respect to $-\nu$. Then we call

$$A = h_{ij} := g_{ik} h_j^k = -\bar{g}\left(\mathrm{II}\left(\partial_i, \partial_j\right), \nu\right) = \bar{g}(\overline{\nabla}_i \nu, x_j).$$

the second fundamental form of Σ, which has eigenvalues with respect to g ordered by

$$\kappa_1 \leq \cdots \leq \kappa_n.$$

For any spacelike hypersurface the Codazzi–Mainardi equations may be written [17, Prop. 33, p. 115]

$$\bar{g}(\overline{\mathrm{Rm}}(X, Y)Z, \nu) = \nabla_X h(Y, Z) - \nabla_Y h(X, Z).$$

The second fundamental form of the slices $\{x^0 = r\}$ is

$$\bar{h}_{ij} := \frac{\vartheta'(r)}{\vartheta(r)} \bar{g}_{ij},$$

[11, (1.6.13)], while the induced metric is

$$g_{ij} = -\partial_i \rho \partial_j \rho + \vartheta^2(\rho) \hat{g}_{ij} = -\partial_i \rho \partial_j \rho + \bar{g}_{ij}.$$

With the definition

$$v^2 = 1 - \vartheta^{-2} \hat{g}^{ij} \partial_i \rho \partial_j \rho,$$

the second fundamental form satisfies

$$v^{-1} h_{ij} = \nabla_{ij} \rho + \bar{h}_{ij},$$

[11, (1.6.11)]. Note that in this reference the past directed normal is used.

Suppose Θ solves $\Theta'(r) = \vartheta(r)$, and by abuse of notation we identify $\Theta = \Theta(r)$ so that $\Theta : N \to \mathbb{R}$. As a function on N we have

$$\overline{\nabla}^2_{\alpha\beta} \Theta = -\vartheta' \overline{g}_{\alpha\beta},$$

while on Σ_t,

$$\nabla_{ij} \Theta(\rho) = \vartheta' \partial_i \rho \partial_j \rho + \vartheta \nabla_{ij} \rho = -\vartheta' g_{ij} + \frac{\vartheta}{v} h_{ij}.$$

In particular we observe that (1.2) and (1.3) are the same flows.

We define the support function

$$u := \frac{\vartheta}{v} = -\bar{g}(\vartheta \partial_r, \nu) = \overline{g}(\overline{\nabla}\Theta, \nu),$$

and observe that this is related to $\nabla\Theta$ by the identity

$$|\nabla\Theta|^2 \equiv g^{ij} \partial_i \Theta \partial_j \Theta = u^2 - \vartheta^2.$$

We will use the following important inequality in several places.

2.1. **Lemma.** *If N satisfies the null convergence condition and $\Sigma \subset N$ is a spacelike graph as above, then*

$$(2.1) \qquad\qquad\qquad \overline{\mathrm{Rc}}(\nabla\Theta, \nu) \geq 0.$$

Proof. In a GRW space $\overline{\nabla}\Theta$ is an eigenvector of $\overline{\mathrm{Rc}}$, [17, Cor. 43, p. 211], and so

$$
\begin{aligned}
\overline{\mathrm{Rc}}(\nu, \nabla\Theta) &= \overline{\mathrm{Rc}}(\nu, \overline{\nabla}\Theta + u\nu) \\
&= u\overline{\mathrm{Rc}}(\nu, \nu) + \overline{\mathrm{Rc}}(\nu, \overline{\nabla}\Theta) \\
&= u\overline{\mathrm{Rc}}\left(V - \frac{\overline{\nabla}\Theta}{\vartheta}\frac{u}{\vartheta}, V - \frac{\overline{\nabla}\Theta}{\vartheta}\frac{u}{\vartheta}\right) + \overline{\mathrm{Rc}}\left(V - \frac{\overline{\nabla}\Theta}{\vartheta}\frac{u}{\vartheta}, \overline{\nabla}\Theta\right) \\
&= u\left[\left(\frac{u^2}{\vartheta^2} - 1\right)\overline{\mathrm{Rc}}\left(\frac{\overline{\nabla}\Theta}{\vartheta}, \frac{\overline{\nabla}\Theta}{\vartheta}\right) + \overline{\mathrm{Rc}}\left(V, V\right)\right] \\
&= u\overline{\mathrm{Rc}}(W, W),
\end{aligned}
\tag{2.2}
$$

where V is the projection of ν onto $(\overline{\nabla}\Theta)^\perp$ and

$$
W = V + \sqrt{\frac{u^2}{\vartheta^2} - 1}\,\frac{\overline{\nabla}\Theta}{\vartheta}.
\tag{2.3}
$$

Since $|V|^2 = u^2\vartheta^{-2} - 1$, W is a lightlike vector and the result follows. $\qquad\square$

Area and volume calculations. Let Σ be graphical as above. We define the integral of a function $f \in C^\infty(\Sigma)$ by

$$
\int_\Sigma f := \int_{\mathcal{S}_0} f\, d\omega_g.
$$

Here $d\omega_g$ is the Riemannian volume form on Σ. For

$$
\hat{\Sigma} := \{(r, \xi) \in N : a \le r \le \rho(\xi),\, \xi \in \mathcal{S}_0\}
$$

we define the *enclosed volume* (see [17, p. 194]) by

$$
\mathrm{vol}(\hat{\Sigma}) := \int_{\mathcal{S}_0} \int_a^{\rho(\cdot)} \frac{\sqrt{\det(\bar{g}_{ij}(s, \cdot))}}{\sqrt{\det(\bar{g}_{ij}(a, \cdot))}}\, ds\, d\omega_a = \int_{\hat{\Sigma}} d\,\mathrm{vol}_{\bar{g}},
$$

where $d\omega_a$ is the volume form on the time slice $\{a\} \times \mathcal{S}_0$ and locally

$$
d\,\mathrm{vol}_{\bar{g}} = \sqrt{|\det(\bar{g}_{\alpha\beta})|}.
$$

The surface area of Σ is

$$
|\Sigma| = \int_\Sigma 1.
$$

Suppose $\hat{\Sigma} \subset N$ is open with compact closure such that $\partial\hat{\Sigma}$ may be written as a union of smooth spacelike hypersurfaces with *outward pointing* normal ν. If X is a smooth vector field on $\hat{\Sigma}$ then

$$
\int_{\hat{\Sigma}} \overline{\mathrm{div}}X\, d\,\mathrm{vol} = -\int_{\partial\hat{\Sigma}} \langle \nu, X \rangle.
$$

This follows from Stoke's theorem and [17, Lemma 21, p. 195].

We now suppose that $\hat{\Sigma} \subset N$ is a time dependent set which is bounded by spacelike hypersurfaces Σ_0 and Σ, where Σ varies with time and Σ_0 is fixed. Let x be a time dependent parametrization of Σ then the above divergence theorem and [17, Lemma 21, p. 195] imply that

$$
\partial_t\,\mathrm{vol}(\hat{\Sigma}_t) = -\int_\Sigma \langle \dot{x}, \nu \rangle.
\tag{2.4}
$$

Throughout the subsequent computations of this paper, differential operators ∇ and Δ are always those induced on the flow hypersurfaces Σ_t. We suppress the

subscript Σ_t for the sake of a shorter notation. Differential operators of the ambient space are always furnished with an overbar.

2.2. Lemma. *Let $n \geq 2$. Along the flow* (1.2)

(i) *the volume* $\mathrm{vol}(\hat{\Sigma})$ *is preserved and*
(ii) *the surface area increases, provided $(N, \bar{g})$ satisfies the timelike convergence condition.*

If $\partial_t |\Sigma_t| = 0$, then Σ is umbilic.

Proof. By equations (2.4) and (1.2) we have that

$$\partial_t \mathrm{vol}(\hat{\Sigma}_t) = \int_{\Sigma_t} \Delta\Theta = 0.$$

We recall that

$$\sigma_2 = \frac{1}{2}\left(H^2 - |A|^2\right), \qquad \sigma_2^{ij} = Hg^{ij} - h^{ij},$$

so

$$\nabla_i \sigma_2^{ij} = \nabla^j H - \nabla_i h^{ij} = -\bar{g}(\overline{\mathrm{Rm}}(x_i, x^j)x^i, \nu) = -\overline{\mathrm{Rc}}(x^j, \nu).$$

Therefore, by the divergence theorem

$$(2.5) \qquad \int_{\Sigma_t} (2\sigma_2 u - (n-1)\vartheta' H) = \int_{\Sigma_t} \sigma_2^{ij} \nabla_{ij}^2 \Theta = \int_{\Sigma_t} \overline{\mathrm{Rc}}(\nu, \nabla\Theta).$$

Using Lemma 3.2 we get

$$\partial_t |\Sigma_t| = \int_{\Sigma_t} (H^2 u - n\vartheta' H)$$

$$(2.6) \qquad = \int \left(H^2 - \frac{2n}{n-1}\sigma_2 + \frac{n}{u(n-1)}\overline{\mathrm{Rc}}(\nabla\Theta, \nu)\right) u$$

$$\geq \frac{n}{n-1} \int_{\Sigma_t} |\mathring{A}|^2 u.$$

where we used (2.5) on the second line and (2.1) on the last line. $\qquad\square$

3. Evolution equations

We now calculate several required evolution equations.

3.1. Lemma. *On Σ_t the function Θ satisfies*

$$\dot{\Theta} - u\Delta\Theta = 0.$$

Proof. We calculate that

$$\dot{\Theta} = \bar{g}(\overline{\nabla}\Theta, \dot{x}) = \bar{g}(\overline{\nabla}\Theta, \nu)\Delta\Theta = u\Delta\Theta.$$

$\qquad\square$

3.2. Lemma. *On Σ_t the induced metric g_{ij} satisfies*

$$\dot{g}_{ij} = 2h_{ij}\Delta\Theta$$

Proof.

$$\partial_t \bar{g}(x_i, x_j) = \bar{g}\left(\frac{\partial}{\partial x^i}(\Delta\Theta\nu), x_j\right) + \bar{g}\left(\frac{\partial}{\partial x^j}(\Delta\Theta\nu), x_i\right) = 2h_{ij}\Delta\Theta.$$

$\square$

3.3. Lemma. *On Σ_t the future oriented normal ν satisfies*

$$\overline{\nabla}_{\dot{x}}\nu = \nabla\Delta\Theta = u\nabla H + H\nabla u - n\frac{\vartheta''}{\vartheta}\nabla\Theta.$$

Proof. We have

$$\bar{g}\left(\overline{\nabla}_{\dot{x}}\nu, x_i\right) = -\bar{g}\left(\nu, \overline{\nabla}_{\dot{x}}x_i\right) = -\bar{g}\left(\nu, \overline{\nabla}_{x_i}\dot{x}\right) = -\bar{g}\left(\nu, \overline{\nabla}_{x_i}(\Delta\Theta\nu)\right) = \nabla_i\Delta\Theta$$

and observe

$$\nabla_i(-n\vartheta') = -n\vartheta''\bar{g}(\overline{\nabla}r, x_i) = -n\frac{\vartheta''}{\vartheta}\bar{g}(\overline{\nabla}\Theta, x_i).$$

$\square$

3.4. Lemma. *On Σ_t the function u satisfies*

$$\dot{u} - u\Delta u = -|\mathring{A}|^2 u^2 - \left(\frac{Hu}{\sqrt{n}} - \sqrt{n}\vartheta'\right)^2 - n\frac{\vartheta''}{\vartheta}(u^2 - \vartheta^2)$$
$$- u\overline{\mathrm{Rc}}(\nabla\Theta, \nu) + H\bar{g}(\nabla u, \nabla\Theta).$$

Proof. We calculate

$$\dot{u} = \bar{g}\left(\overline{\nabla}_{\dot{x}}\nu, \overline{\nabla}\Theta\right) + \bar{g}\left(\nu, \overline{\nabla}_{\dot{x}}\overline{\nabla}\Theta\right)$$
$$= u\bar{g}(\nabla H, \nabla\Theta) + H\bar{g}(\nabla u, \nabla\Theta) - n\frac{\vartheta''}{\vartheta}|\nabla\Theta|^2 + \Delta\Theta\overline{\nabla}^2_{\nu\nu}\Theta.$$

We also see that

$$\nabla_i u = h_i^k \nabla_k\Theta + \overline{\nabla}^2_{i\nu}\Theta = h_i^k\nabla_k\Theta$$

and

$$\nabla^2_{ij}u = \nabla_j h_{ik}\nabla^k\Theta + h_i^k h_{kj}u + h_i^k\overline{\nabla}^2_{x_k x_j}\Theta = \nabla_j h_{ik}\nabla^k\Theta + h_i^k h_{kj}u - \vartheta' h_{ij}.$$

Taking a trace, and applying Codazzi Mainardi on the first term we see that

$$\Delta u = \bar{g}(\nabla H, \nabla\Theta) + \bar{g}(\overline{\mathrm{Rm}}(x_j, \nabla\Theta)x^j, \nu) + |A|^2 u - \vartheta' H$$
$$= \bar{g}(\nabla H, \nabla\Theta) + |A|^2 u + \overline{\mathrm{Rc}}(\nabla\Theta, \nu) - \vartheta' H.$$

Overall we have

$$\dot{u} - u\Delta u$$
$$= -|A|^2 u^2 + H\bar{g}(\nabla u, \nabla\Theta) - n\frac{\vartheta''}{\vartheta}(u^2 - \vartheta^2) + \vartheta'\Delta\Theta - u\overline{\mathrm{Rc}}(\nabla\Theta, \nu) + \vartheta' uH$$
$$= -|A|^2 u^2 - u\overline{\mathrm{Rc}}(\nabla\Theta, \nu) + 2\vartheta' uH - n\frac{\vartheta''}{\vartheta}u^2 + n(\vartheta''\vartheta - (\vartheta')^2) + H\bar{g}(\nabla u, \nabla\Theta)$$
$$= -|A|^2 u^2 + \frac{1}{n}H^2 u^2 - \frac{1}{n}H^2 u^2 + 2\vartheta' Hu - n\vartheta'^2 - n\frac{\vartheta''}{\vartheta}u^2 + n\vartheta''\vartheta$$
$$- u\overline{\mathrm{Rc}}(\nabla\Theta, \nu) + H\bar{g}(\nabla u, \nabla\Theta)$$
$$= -|\mathring{A}|^2 u^2 - \left(\frac{Hu}{\sqrt{n}} - \sqrt{n}\vartheta'\right)^2 - n\frac{\vartheta''}{\vartheta}u^2 + n\vartheta''\vartheta - u\overline{\mathrm{Rc}}(\nabla\Theta, \nu) + H\bar{g}(\nabla u, \nabla\Theta).$$

$\square$

4. Gradient estimate

4.1. Lemma. *There exist uniform bounds*

$$\inf_{\Sigma_0} \Theta \le \Theta(p,t) \le \sup_{\Sigma_0} \Theta$$

Proof. This follows directly from Lemma 3.1 and the maximum principle. $\square$

4.2. Lemma (Gradient bound)**.** *The support function is uniformly bounded along the flow.*

Proof. Define

$$w = u + \Theta^2.$$

Then w satisfies

$$\begin{aligned}
\dot{w} - u\Delta w &\le -\frac{H^2 u^2}{n} + 2Hu\vartheta' + H\bar{g}(\nabla u, \nabla\Theta) - 2u|\nabla\Theta|^2 + c(u^2+1) \\
&\le -\frac{H^2 u^2}{n} + 2Hu\vartheta' - 2H\Theta(u^2 - \vartheta^2) - 2u^3 \\
&\quad + c(u^2+1) + H\bar{g}(\nabla w, \nabla\Theta) \\
&\le \left(\epsilon - \frac{1}{n}\right) H^2 u^2 - 2u^3 + c_\epsilon(u^2+1) + H\bar{g}(\nabla w, \nabla\Theta),
\end{aligned}$$

where we estimated using Young's inequality on the final line. At a large maximal point of w, u must also be very large, as ρ is bounded. Setting $\epsilon = \frac{1}{n}$, the result follows from the maximum principle.

$\square$

4.3. Corollary. *Along* (1.3) *we have uniform C^m-estimates for every m and long-time existence of the flow.*

Proof. Under (1.3), the graph function $\rho(\cdot,t)$ satisfies a quasi-linear parabolic equation which, by Lemmas 4.1 and 4.2, is uniformly parabolic. Lemmas 4.1 and 4.2 and standard application of the Nash–Moser–De Giorgi theorem [16, Ch. XII] provides uniform $C^{1+\alpha;\frac{1+\alpha}{2}}$ bounds on ρ, and then Schauder theory [15, Thm. IV.10.1, p. 351-352] implies uniform estimates to all orders. Standard parabolic existence theory completes the proof. $\square$

5. Completion of the proof

Proof. We have to prove that the flow converges to a coordinate slice $\{r = \text{const}\}$ and finish the proof of Theorem 1.1. We will prove that the function Θ on the flowing hypersurface converges to a constant as $t \to \infty$ using similar methods to [2, Thm. 3.1] and [21, Sec. 6.2]. Recall that Θ satisfies

$$\dot{\Theta} - u\Delta\Theta = 0,$$

where $u\Delta$ is uniformly elliptic due to the support function estimates. Hence Θ enjoys the validity of the strong maximum principle for parabolic operators and hence the oscillation of Θ,

$$\omega(t) = \operatorname{osc}\Theta(t,\cdot) = \max\Theta(t,\cdot) - \min\Theta(t,\cdot)$$

is strictly decreasing, unless Θ is constant at some (and hence all) $t > 0$, in which case we would be done.

Suppose that ω does not converge to zero as $t \to \infty$. Then it converges to another value $\omega_\infty > 0$. Define a sequence of flows by

$$x_i(t,\xi) = x(t+i,\xi)$$

and the corresponding functions Θ_i. Due to Corollary 4.3, on a given time interval $[0,T]$ we can apply Arzéla-Ascoli and obtain smooth convergence of a subsequence of x_i to a limit flow

$$x_\infty\colon [0,T] \times \mathcal{S}_0 \to N,$$

which solves the same flow equation (1.2). By construction, the oscillation of the associated limiting function Θ_∞ is $\omega_\infty > 0$ constantly, which is a contradiction to the strong maximum principle, which holds for Θ_∞ as well. We conclude that

$$\lim_{t\to\infty} \omega(t) = 0$$

and hence every subsequential limit of the original flow x must be a time-slice of the spacetime N. By the barrier estimates in Lemma 4.1, this timeslice is unique and we obtain that the whole flow x converges to a timeslice.

We conclude the proof by showing that the isoperimetric inequality holds. Hence let Σ satisfy the assumption of Theorem 1.1 and evolve Σ by the flow (1.2). Define

$$S_R = \{r = R\}, \quad f_0(R) = \operatorname{vol}(\hat{S}_R), \quad f_1(R) = |S_R|.$$

Clearly f_0 is monotonically increasing in R, and $\varphi = f_1 \circ f_0^{-1}$. As $\operatorname{vol}(\Sigma_t)$ is fixed, this defines a unique slice S_{R_∞} to which the flow must converge with area $\varphi(\operatorname{vol}(S_{R_\infty}))$. By the monotonicity properties of Lemma 2.2 the claimed isoperimetric inequality holds.

If equality holds and Σ was not umbilic, then equation (2.6) implies that variations of Σ along (1.2) would violate this inequality. Hence in the equality case Σ must be umbilic.

It remains to prove item (ii) of Theorem 1.1. On a time slice equality holds by construction. Hence assume equality holds on Σ and evolve Σ by (1.2). The variation formula for the area (2.6) and (2.2) show that

$$\partial_t|\Sigma_t| \geq \frac{n}{n-1}\int_{\Sigma_t} u\overline{\operatorname{Rc}}(W,W).$$

Hence we must have

$$\overline{\operatorname{Rc}}(W,W) = 0,$$

for otherwise we would reach a contradiction to what we have already proved. Due to the strict null convergence condition we obtain $W = 0$ and from (2.3) we deduce

$$0 = \sqrt{\frac{u^2}{\vartheta^2} - 1} = \sqrt{\frac{1}{v^2} - 1},$$

hence $v = 1$ and $\nabla\Theta = 0$. This shows that Σ is a timeslice. $\qquad\square$

Acknowledgments

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