ALEXANDROV-FENCHEL INEQUALITIES FOR CONVEX HYPERSURFACES WITH FREE BOUNDARY IN A BALL

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Abstract

In this paper we first introduce quermassintegrals for free boundary hypersurfaces in the $(n + 1)$-dimensional Euclidean unit ball. Then we solve some related isoperimetric type problems for convex free boundary hypersurfaces, which lead to new Alexandrov-Fenchel inequalities. In particular, for $n = 2$ we obtain a Minkowski-type inequality and for $n = 3$ we obtain an optimal Willmore-type inequality. To prove these estimates, we employ a specifically designed locally constrained inverse harmonic mean curvature flow with free boundary.

1. Introduction

The study of free boundary surfaces or hypersurfaces has a very long history. It goes back at least to Courant [6]. Since then, there have been a lot research activities on this topic, see for example [18, 17, 38, 42]. Due to recent developments inspired by the proof of the Willmore conjecture by Marques-Neves [33] and Fraser-Schoen's work on the first Steklov eigenvalue and minimal free boundary surfaces [10, 11], there are many new interesting results on the existence of minimal free boundary surfaces. We are interested in the study of hypersurfaces in the unit ball with free boundary in the unit sphere. Such hypersurfaces share many properties with closed ones in the Euclidean space or the sphere. This naturally leads to questions, such as if a certain result is true for hypersurfaces of the sphere, is it also true for hypersurfaces with free boundary on the sphere. One good example is a result of Fraser-Schoen [11]:

> If Σ is a free boundary minimal annulus given by first Steklov eigenfunctions, then Σ is a critical catenoid.

This is a free boundary version of a result of Montiel-Ros [**34**] for closed minimal surfaces in the unit sphere. However, the proof is much more delicate. In fact, in many cases the free boundary version is more difficult or even left open. For example, a free boundary version of the Lawson conjecture solved by Brendle [**2**] is still open [**9**, **35**]: Is the critical catenoid the unique embedded annulus? One can also ask if the critical catenoid has the least area among minimal annuli. Its counterpart in the unit sphere was solved by Marques-Neves [**33**] and plays a key role in the resolution of the Willmore conjecture. We remark that the critical catenoid does not minimize the Willmore functional (a conformally invariant functional with a correction term involving the geodesic curvature of the boundary), but the piece of the Clifford torus intersecting the unit sphere does. The difficulties arise from two different geometries. Unlike the theory of closed hypersurfaces, a free boundary hypersurface lies in two different geometries: the Euclidean geometry for its interior and the spherical geometry for its boundary. One might have to deal with these two geometries simultaneously. In our recent work, we have used a vector field which is conformal Killing in the Euclidean space and is also conformal Killing on the unit sphere after restriction. This vector field helped us to solve the stability problem for CMC free boundary hypersurfaces in [**48**]. It will also play an important role in this paper. The objective of this paper is to find a counterpart of quermassintegrals (cross-section integrals) for free boundary hypersurfaces and consider related isoperimetric type problems. We will see in this paper that the quermassintegrals we find involve not only geometric integrals in the Euclidean space, but also geometric integrals in the spherical space. We hope that our research provides a deeper understanding of free boundary hypersurfaces and leads to more interesting questions, for example:

Is there an integral geometry for free boundary hypersurfaces in a Euclidean ball?

This article is mainly about convex hypersurfaces with boundary $(\Sigma, \partial\Sigma)$ in the closed $(n + 1)$-dimensional Euclidean unit ball $\bar{\mathbb{B}}^{n+1}$. Hereby we mean that the second fundamental form is semi-definite. At the boundary $\partial\Sigma$ the hypersurface is supposed to meet the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ perpendicularly. We will label this property via the following terminology.

1.1. Definition. Let $n \geq 2$ and $\Sigma \subset \bar{\mathbb{B}}^{n+1}$ be a smooth compact, embedded topological closed n-ball, given by an immersion

$$x\colon \bar{\mathbb{B}}^n \to \Sigma \subset \bar{\mathbb{B}}^{n+1}.$$

We say that Σ has *free boundary in the unit ball*, if

$$x(\mathbb{B}^n) \subset \mathbb{B}^{n+1}$$

$$\partial\Sigma \subset \mathbb{S}^n$$

$$\langle \bar{N}, \nu \rangle_{|\partial\mathbb{B}^n} = 0,$$

where ν is a smooth choice of a unit normal field on Σ and $\bar{N}$ is the position vector field in $\mathbb{R}^{n+1}$, i.e. its restriction to $\mathbb{S}^n$ is the outward normal on $\mathbb{S}^n$. We also identify, without ambiguity, the outward pointing conormal μ of $\partial\Sigma \subset \Sigma$ with $\bar{N}$.

We are interested in geometric inequalities for free boundary hypersurfaces in the unit ball. The most prominent and best studied one is the relative isoperimetric inequality, which holds in a broad sense, cf. [5, Thm. 18.1.3]. Here we want to prove higher order generalisations of this, which are classically called Alexandrov-Fenchel inequalities. In the case of closed convex hypersurfaces of the Euclidean space $\mathbb{R}^{n+1}$ these are

$$\int_\Sigma H_k \geq \omega_n^{\frac{k-l}{n-l}} \left(\int_\Sigma H_l \right)^{\frac{n-k}{n-l}}, \quad 0 \leq l < k \leq n,$$

with equality precisely at spheres. ω_n denotes the surface area of $\mathbb{S}^n$ and H_k the normalized k-th mean curvatures of Σ with the convention $H_0 = 1$ and $H_{n+1} = 0$, see [37]. There are generalizations to other space forms, such as the hyperbolic space [12, 13, 29, 47] and the sphere [31, 49]. The aim of the present paper is to find the correct extensions to hypersurfaces with free boundary in the unit ball.

In order to state the main theorem, we have to introduce some more notation. If Σ is strictly convex, then $\partial\Sigma \subset \mathbb{S}^n$ is a strictly convex hypersurface of the unit sphere and bounds a strictly convex body in $\mathbb{S}^n$, which we denote by $\widehat{\partial\Sigma}$, cf. [8]. Denote by $\widehat{\Sigma}$ the domain enclosed by Σ in $\mathbb{B}^{n+1}$, which contains $\widehat{\partial\Sigma}$. Let σ_k denote the k-th elementary symmetric polynomial, evaluated at the principal curvatures of the hypersurface Σ,

$$\sigma_k = \binom{n}{k} H_k, \quad 0 \leq k \leq n, \quad \sigma_{n+1} = H_{n+1} = 0.$$

Then we define the following geometric functionals, which we expect to be the correct counterparts to the quermassintegrals for closed convex hypersurfaces.

(1.1)
$$W_0(\widehat{\Sigma}) = |\widehat{\Sigma}|$$

$$W_1(\widehat{\Sigma}) = \frac{1}{n+1}|\Sigma|$$

$$W_k(\widehat{\Sigma}) = \frac{1}{n+1}\int_{\Sigma} H_{k-1} + \frac{k-1}{(n+1)(n-k+2)}W_{k-2}^{\mathbb{S}^n}(\widehat{\partial\Sigma}), \quad 2 \le k \le n+1,$$

where for a k-dimensional submanifold $M \subset \mathbb{R}^{n+1}$ (with or without boundary), $|M|$ always denotes the k-dimensional Hausdorff measure of M. $W_{k-2}^{\mathbb{S}^n}(\widehat{\partial\Sigma})$ denotes the standard $(k-2)$-th quermassintegral of the closed convex hypersurface $\partial\Sigma \subset \mathbb{S}^n$. We refer to section 2 for a description via curvature integrals and for more information. Furthermore we define the *spherical caps of radius R around $e \in \mathbb{S}^n$* by

$$C_R(e) = \{x \in \bar{\mathbb{B}}^{n+1} : |x - (R^2+1)^{\frac{1}{2}}e| = R\}, \quad R < \infty,$$

where we will drop the argument e, in cases where it is not relevant. Let H denote the mean curvature of a hypersurface, i.e. $H = \sigma_1$.

The main results we will present are geometric inequalities relating some particular quantities from (1.1) of convex free boundary hypersurfaces in the unit ball:

1.2. Theorem. *Let $n \ge 2$ and $\Sigma \subset \bar{\mathbb{B}}^{n+1}$ be a convex free boundary hypersurface in the unit ball. Then*

$$W_{n+1}(\widehat{\Sigma}) = \frac{\omega_n}{2(n+1)}$$

and for $0 \le k \le n-1$ there holds

$$W_n(\widehat{\Sigma}) \ge (f_n \circ f_k^{-1})(W_k(\widehat{\Sigma})),$$

where $f_k = f_k(r)$ is the strictly increasing real function

$$f_k(r) = W_k(\widehat{C_r}).$$

Equality holds if and only if Σ is a spherical cap or a flat disk.

1.3. REMARK. The monotonicity of the functions f_k is proven in section 4.

We obtain the two following important special cases immediately:

1.4. Corollary. *Under the assumptions of Theorem 1.2, for $n = 2$ there holds the Minkowski type inequality*

$$\frac{1}{6}\left(\int_{\Sigma} H + |\widehat{\partial\Sigma}|\right) \ge f_2 \circ f_1^{-1}\left(\frac{1}{3}|\Sigma|\right)$$

and for $n = 3$ we obtain a Willmore type inequality

(1.2)
$$\frac{1}{12}\left(\frac{1}{3}\int_{\Sigma} H^2 + |\partial\Sigma|\right) \ge f_3 \circ f_1^{-1}\left(\frac{1}{4}|\Sigma|\right),$$

where in each case equality holds precisely on spherical caps or flat disks.

Note that in [45] Volkmann obtained an inequality similar to (1.2) (with equality on caps and disks) in case $n = 2$ for a much broader class of hypersurfaces using methods from geometric measure theory. In arbitrary higher dimensions, a Willmore type estimate for the convex case was deduced in [28], however with equality only on flat disks. Hence (1.2) is an extension of this result in dimension $n = 3$.

The method of proof follows the nowadays classical method for proving geometric inequalities by employing monotonicity properties along and convergence of a suitable curvature flow. In the case of closed hypersurfaces, such methods were used in [19] to deduce Alexandrov-Fenchel inequalities and in [26] to prove the Riemannian Penrose inequality. In other ambient spaces there is a zoo of variants of such inequalities, e.g. [1, 7, 12, 25, 47, 49] in the hyperbolic space, [16, 31, 49] in the sphere and [4, 14] in more general ambient spaces.

The most suitable flows for this strategy are flows that decrease or increase a certain quantity, while preserving another one. Together with the knowledge that the flow converges to a well understood limit object, this implies the inequality.

In the paper [20], Guan and Li constructed the mean curvature type flow

$$(1.3) \qquad \dot{x} = (n - \sigma_1 \langle x, \nu \rangle)\nu$$

for closed and starshaped hypersurfaces in $\mathbb{R}^{n+1}$. The Minkowski identity

$$\int_M n = \int_M \sigma_1 \langle x, \nu \rangle$$

shows that the enclosed volume is preserved, while the second Minkowski identity

$$\int_M n\sigma_1 = \int_M \sigma_2 \langle x, \nu \rangle$$

shows that the surface area is decreasing. Together with suitable a priori estimates this gave a new flow approach to the isoperimetric inequality for starshaped hypersurfaces. Also in [20], a similar flow was considered in any space forms. Further generalizations of such flows were treated in [21, 22, 36]. A suitably modified version of (1.3) for free boundary hypersurfaces in the unit ball will be treated in [46]. Possible generalizations also contain inverse type flows

$$\dot{x} = \left(\frac{c_{n,k}}{\frac{\sigma_k}{\sigma_{k-1}}} - \langle x, \nu \rangle \right) \nu,$$

which have similar monotonicity properties due to the higher order Minkowski identities, cf. [3].

In the recent preprint [**48**, Prop. 5.1], GW and CX have proven new Minkowski identities for immersed hypersurfaces Σ with free boundary in the unit ball:

$$\int_{\Sigma} \sigma_{k-1} \langle x, e \rangle = \frac{k}{n+1-k} \int_{\Sigma} \sigma_k \langle X_e, \nu \rangle, \quad 1 \le k \le n.$$

Here $e \in \mathbb{S}^n$ is fixed and X_e is the conformal Killing field

$$X_e = \langle x, e \rangle x - \frac{1}{2}(|x|^2 + 1)e.$$

The advantage of these Minkowski identities is that they do not contain any boundary integrals.

Motivated by these identities as well as the original flow (1.3) due to Guan and Li, it is natural to consider the following inverse type flow for strictly convex hypersurfaces with free boundary in the unit ball:

$$\dot{x} = \left(\frac{\langle x, e \rangle}{n \frac{\sigma_n}{\sigma_{n-1}}} - \langle X_e, \nu \rangle \right) \nu$$

with a suitably chosen direction $e \in \mathbb{S}^n$. We will prove that this curvature flow (with e to be chosen in dependence of the initial data) will drive the strictly convex initial data in an infinite amount of time to a spherical cap. On the other hand we will show that the quantities in (1.1) have certain monotonicity properties, which will allow us to conclude the proof of Theorem 1.2. The restriction to σ_n is due to the technical obstruction, that for the general curvature function σ_k/σ_{k-1} we could not prove the preservation of convexity along this flow. However, we still conjecture that all the lower order Alexandrov-Fenchel inequalities are true for convex free boundary hypersurfaces.

Finally we mention some previous results on curvature flows with free boundary in the unit ball. The classical mean curvature flow was considered in [**40**, **41**]. The inverse mean curvature flow was treated in [**27**], where it was shown that strictly convex initial data are driven to a flat perpendicular disk. We will also use several estimates for strictly convex hypersurfaces with free boundary in the unit ball, which were proven in this paper.

Our exposition is organized as follows. In section 2 we collect our conventions on geometric objects and we recall the definition of the quermassintegrals. In section 3 we prove the relevant properties of our flow and in section 4 we finish the proof of Theorem 1.2.

2. Quermassintegrals

Notation and basic definitions. In this section we state the basic conventions used in this paper. Let M be a smooth manifold and g be a Riemannian metric on M with Levi-Civita connection ∇. Our

convention for the Riemannian curvature tensor Rm is

$$\text{Rm}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and the purely covariant version is

$$\text{Rm}(X,Y,Z,W) = g(\text{Rm}(X,Y)Z,W).$$

Let

$$x \colon M \to \mathbb{R}^{n+1}$$

be the smooth embedding of an n-dimensional closed and connected manifold. The induced metric of $x(M)$ is given by the pullback of the ambient Euclidean metric $\bar{g}$ by x. The second fundamental form h of the embedding x is given by the Gaussian formula

$$\bar{\nabla}_X Y = \nabla_X Y - h(X,Y)\nu.$$

The Weingarten operator is defined via

$$g(\mathcal{W}(X),Y) = h(X,Y)$$

and the Weingarten equation says that

$$\bar{\nabla}_X \nu = \mathcal{W}(X).$$

Finally, we have Gauss equation,

$$\text{Rm}(W,X,Y,Z) = \left(h(W,Z)h(X,Y) - h(W,Y)h(X,Z) \right).$$

2.1. REMARK. We will simplify the notation by using the following shortcuts occasionally:

(i) When dealing with complicated evolution equations of tensors, we will use a local frame to express tensors with the help of their components, i.e. for a tensor field $T \in \mathcal{T}^{k,l}(M)$, the expression $T^{i_1 \dots i_k}_{j_1 \dots j_l}$ denotes

$$T^{i_1 \dots i_k}_{j_1 \dots j_l} = T(e_{j_1}, \dots, e_{j_l}, \epsilon^{i_1}, \dots \epsilon^{i_k}),$$

where (e_i) is a local frame and (ϵ^i) its dual coframe.

(ii) The coordinate expression for the m-th covariant derivative of a (k,l)-tensor field T is

$$\nabla^m T = \left(T^{i_1 \dots i_k}_{j_1 \dots j_l; j_{l+1} \dots j_{l+m}} \right),$$

where indices appearing after the semi-colon denote the covariant derivatives.

(iii) For convenience the components of the Weingarten map $\mathcal{W}$ are denoted by (h^i_j).

Quermassintegrals in $\mathbb{R}^{n+1}$. In order to compare our new geometric integrals for free boundary hypersurfaces with the quermassintegrals for closed hypersurfaces in a space form, we start to introduce the quermassintegrals in $\mathbb{R}^{n+1}$. Let Ω be a convex body (non-empty, compact, convex set) in $\mathbb{R}^{n+1}$. The quermassintegrals $W_k^{\mathbb{R}^{n+1}}$ are defined by

$$W_0^{\mathbb{R}^{n+1}}(\Omega) = |\Omega|, \quad W_1^{\mathbb{R}^{n+1}}(\Omega) = \frac{1}{n+1}|\partial\Omega|,$$

$$W_k^{\mathbb{R}^{n+1}}(\Omega) = \frac{1}{n+1}\binom{n}{k-1}^{-1}\int_{\partial\Omega}\sigma_{k-1}dA$$

$$= \frac{1}{n+1}\int_{\partial\Omega}H_{k-1}dA, \qquad 2 \leq k \leq n+1.$$

In particular,

$$W_{n+1}^{\mathbb{R}^{n+1}}(\Omega) = \frac{\omega_n}{n+1}.$$

The first variation formula of $W_k^{\mathbb{R}^{n+1}}$ is as follows. Consider a family of bounded convex bodies $\{\Omega_t\}$ in $\mathbb{R}^{n+1}$ whose boundary $\partial\Omega_t$ evolving by a normal variation with speed function f, then

$$(2.1)\frac{d}{dt}W_k^{\mathbb{R}^{n+1}}(\Omega_t) = \frac{n+1-k}{n+1}\int_{\partial K_t}H_k f dA_t, \quad 0 \leq k \leq n+1.$$

Quermassintegrals in $\mathbb{S}^n$. For a convex body (non-empty, compact, convex set) $K \subset \mathbb{S}^n$ with smooth boundary, the quermassintegrals $W_k^{\mathbb{S}^n}$ are inductively defined by

$$W_0^{\mathbb{S}^n}(K) = |K|, \quad W_1^{\mathbb{S}^n}(K) = \frac{1}{n}|\partial K|,$$

$$W_k^{\mathbb{S}^n}(K) = \frac{1}{n}\binom{n-1}{k-1}^{-1}\int_{\partial K}\sigma_{k-1}^{\mathbb{S}^n}dA + \frac{k-1}{n-k+2}W_{k-2}^{\mathbb{S}^n}(K)$$

$$= \frac{1}{n}\int_{\partial K}H_{k-1}^{\mathbb{S}^n}dA + \frac{k-1}{n-k+2}W_{k-2}^{\mathbb{S}^n}(K), \qquad 2 \leq k \leq n,$$

where by $\sigma_k^{\mathbb{S}^n}$ we denote the k-th elementary symmetric polynomial, evaluated at the $n-1$ principal curvatures of the hypersurface $\partial K \subset \mathbb{S}^n$ and

$$H_k^{\mathbb{S}^n} = \frac{1}{\binom{n-1}{k}}\sigma_k^{\mathbb{S}^n}, \quad 0 \leq k \leq n-1, \quad H_n^{\mathbb{S}^n} = \sigma_n^{\mathbb{S}^n} = 0.$$

In particular,

$$W_n^{\mathbb{S}^n}(K) = \frac{\omega_{n-1}}{n}$$

due to the spherical Gauss-Bonnet-Chern's Theorem, cf. [39].

The first variation formula of $W_k^{\mathbb{S}^n}$ is as follows. Consider a family of bounded convex bodies $\{K_t\}$ in $\mathbb{S}^n$ whose boundary ∂K_t evolves by a

normal variation with speed function f, we have similarly
(2.2)
$$\frac{d}{dt}W_k^{\mathbb{S}^n}(K_t) = \frac{1}{\binom{n}{k}}\int_{\partial K_t}\sigma_k^{\mathbb{S}^n}f dA_t = \frac{n-k}{n}\int_{\partial K_t}H_k^{\mathbb{S}^n}f dA_t, \quad 0 \le k \le n.$$

This can be proved by a simple induction argument; see for example [47, Prop. 3.1] for a similar deduction in the hyperbolic case.

Related quantities for free boundary hypersurfaces in $\bar{\mathbb{B}}^{n+1}$.
Let $\Sigma \subset \bar{\mathbb{B}}^{n+1}$ be a smooth convex, embedded hypersurface with free boundary $\partial\Sigma \subset \mathbb{S}^n$. Denote by $\widehat{\partial\Sigma}$ the convex body in $\mathbb{S}^n$ enclosed by $\partial\Sigma$. Denote by $\widehat{\Sigma}$ the enclosed convex domain in $\bar{\mathbb{B}}^{n+1}$ which contains $\widehat{\partial\Sigma}$.

We define W_k for $\widehat{\Sigma} \subset \bar{\mathbb{B}}^{n+1}$ as follows.
(2.3)
$$W_0(\widehat{\Sigma}) = |\widehat{\Sigma}|, \quad W_1(\widehat{\Sigma}) = \frac{1}{n+1}|\Sigma|,$$

$$W_k(\widehat{\Sigma}) = \frac{1}{n+1}\binom{n}{k-1}^{-1}\left\{\int_\Sigma \sigma_{k-1}dA + \binom{n}{k-2}W_{k-2}^{\mathbb{S}^n}(\widehat{\partial\Sigma})\right\}$$

$$= \frac{1}{n+1}\int_\Sigma H_{k-1}dA + \frac{1}{n+1}\frac{k-1}{n-k+2}W_{k-2}^{\mathbb{S}^n}(\widehat{\partial\Sigma}), \quad 2 \le k \le n+1.$$

It is the following variational formula which motivates our definition of the W_k.

2.2. Proposition. *Let $\{\Sigma_t\}$ be a family of smooth, embedded hypersurfaces with free boundary in $\bar{\mathbb{B}}^{n+1}$, given by the embeddings $x(\cdot,t): \bar{\mathbb{B}}^n \to \bar{\mathbb{B}}^{n+1}$, which evolve by*

$$\partial_t x(\cdot,t) = f(\cdot,t)\nu(\cdot,t).$$

Then there holds

$$(2.4) \qquad \partial_t W_k(\widehat{\Sigma_t}) = \frac{n+1-k}{n+1}\int_{\Sigma_t}H_k f\ dA_t, \qquad \forall 0 \le k \le n+1.$$

In particular,

$$(2.5) \qquad\qquad\qquad \partial_t W_{n+1}(\widehat{\Sigma_t}) = 0.$$

Proof. The following general evolution equations are valid

$$\dot{g}_{ij} = 2f h_{ij},$$

$$\dot{h}_j^i = -f_{;j}^{\ i} - f h_m^i h_j^m.$$

The variational formulas for W_0 and W_1 are well-known. Consider the case $k \ge 2$. We prove the following equivalent formula of (2.4).

$$(2.6) \qquad \partial_t\left\{\int_\Sigma \sigma_{k-1}dA + \binom{n}{k-2}W_{k-2}^{\mathbb{S}^n}(\widehat{\partial\Sigma})\right\} = \int_{\Sigma_t}k\sigma_k f\ dA_t.$$

We calculate, using

$$\frac{\partial \sigma_{k-1}}{\partial h_i^j} h_m^i h_j^m = \sigma_1 \sigma_{k-1} - k\sigma_k, \quad 2 \leq k \leq n,$$

$$\frac{\partial \sigma_n}{\partial h_i^j} h_m^i h_j^m = \sigma_1 \sigma_n,$$

that

$$\partial_t \int_{\Sigma_t} \sigma_{k-1} \, dA_t = \int_{\Sigma_t} \sigma_{k-1} f \sigma_1 \, dA_t - \int_{\Sigma_t} \frac{\partial \sigma_{k-1}}{\partial h_i^j} f_{;j}^{\;\;i} \, dA_t$$

$$- \int_{\Sigma_t} \frac{\partial \sigma_{k-1}}{\partial h_j^j} h_m^i h_j^m f \, dA_t$$

$$= \begin{cases} \int_{\Sigma_t} k\sigma_k f \, dA_t - \int_{\partial \Sigma_t} \frac{\partial \sigma_{k-1}}{\partial h_i^j} f_{;j} \mu^i \, dA_t, & 2 \leq k \leq n \\ - \int_{\partial \Sigma_t} \frac{\partial \sigma_{k-1}}{\partial h_i^j} f_{;j} \mu^i \, dA_t, & k = n+1, \end{cases}$$

where we have used that the σ_k are divergence-free in $\mathbb{R}^{n+1}$. Let $\{e_I\}_{2 \leq I \leq n}$ be an orthonormal frame for $\partial\Sigma$, so that $\{e_1 = \mu, (e_I)_{2 \leq I \leq n}\}$ is an orthonormal frame for Σ along $\partial\Sigma$. Since μ is a principal direction due to the free boundary condition, there holds

$$\frac{\partial \sigma_{k-1}}{\partial h_\mu^J} = 0, \quad 2 \leq J \leq n.$$

Differentiation of $\langle \nu, \mu \rangle = 0$ with respect to time gives

$$\langle -\nabla f, \mu \rangle + f h_{\nu\nu}^{\mathbb{S}^n} = 0,$$

which implies

(2.7) $$\nabla_\mu f = f.$$

It follows that

$$\int_{\partial \Sigma_t} \frac{\partial \sigma_{k-1}}{\partial h_i^j} f_{;j} \mu^i \, dA_t = \int_{\partial \Sigma_t} \frac{\partial \sigma_{k-1}}{\partial h_\mu^\mu} f \, dA_t.$$

On the other hand, the flow of $x(\cdot, t)$ induces a normal flow in $\mathbb{S}^n$ with speed f. It follows from (2.2) that

$$\partial_t W_{k-2}^{\mathbb{S}^n}(\widehat{\partial \Sigma_t}) = \frac{1}{\binom{n}{k-2}} \int_{\partial \Sigma_t} \sigma_{k-2}^{\mathbb{S}^n} f \, dA_t.$$

Combining these equalities, also having in mind that

$$\frac{\partial \sigma_{k-1}}{\partial h_\mu^\mu} = \sigma_{k-2}^{\mathbb{S}^n},$$

gives (2.6), and hence (2.4). q.e.d.

2.3. Remark. The variation formulas of geometric integrals W_k, which we define for free boundary hypersurfaces,

$$\partial_t W_k(\widehat{\Sigma_t}) = \frac{n+1-k}{n+1} \int_{\Sigma_t} H_k f \ dA_t \quad \forall 1 \le k \le n,$$

are similar to those for quermassintegrals for closed hypersurfaces in a space form. Therefore we believe that these geometric integrals are correct counterparts to the classical quermassintegrals. In fact, for W_{n+1} we also have a very similar form, see the next Proposition. It is an interesting question whether one can establish an integral geometric theory for free boundary hypersurfaces in a ball.

2.4. Proposition. *Let $n \ge 2$ and $\Sigma \subset \bar{\mathbb{B}}^{n+1}$ be a convex free boundary hypersurface in the unit ball. Then*

$$W_{n+1}(\widehat{\Sigma}) = \frac{1}{2} \frac{\omega_n}{n+1}.$$

The proof will be given in Appendix A. In fact, we will give a precise formula for W_{n+1} for any free boundary hypersurface in terms of the Euler characteristic.

2.5. Remark. For $n = 1$ there holds

$$W_2(\widehat{\Sigma}) = \frac{1}{2} \left(\int_\Sigma \kappa ds + |\widehat{\partial \Sigma}| \right) = \frac{\pi}{2}.$$

For $n = 2$ we have

$$W_2(\widehat{\Sigma}) = \frac{1}{6} \left(\int_\Sigma H dA + |\widehat{\partial \Sigma}| \right)$$

and

$$W_3(\widehat{\Sigma}) = \frac{1}{3} \left(\int_\Sigma \sigma_2 dA + |\partial \Sigma| \right) = \frac{2\pi}{3},$$

3. The curvature flow

Throughout this section Σ is as in Theorem 1.2 and in addition strictly convex. In order to prove the proposed geometric inequality, we use the following curvature flow with free perpendicularity condition on $\mathbb{S}^n$,

(3.1)
$$\dot{x} = f\nu \equiv \left(\frac{\langle x, e \rangle}{F} - \langle X_e, \nu \rangle \right) \nu$$
$$x(0, \bar{\mathbb{B}}^n) = \Sigma_0 := \Sigma,$$

where $e \in \widehat{\partial \Sigma}$ is chosen such that $\widehat{\partial \Sigma}$ lies in the open hemisphere

$$\mathcal{H} = \{ x \in \mathbb{S}^n \colon \langle x, e \rangle > 0 \}$$

and where X_e is the conformal Killing field

$$X_e = X_e(x) = \langle x, e \rangle \, x - \frac{1}{2}(|x|^2 + 1)e.$$

In particular we will use the curvature function

$$(3.2) \qquad\qquad F = n \frac{\sigma_n}{\sigma_{n-1}}$$

and denote the flow hypersurfaces by $\Sigma_t = x(t, \bar{\mathbb{B}}^n)$. The normalization is chosen such that

$$F(1, \ldots, 1) = 1.$$

Monotonicity. The main motivation to study the flow (3.1) comes from its monotonicity properties. From Proposition 2.2 we obtain:

3.1. Proposition. *Along the flow* (3.1), *the quantity* $W_n(\widehat{\Sigma_t})$ *is preserved and* $W_k(\widehat{\Sigma_t})$ *is increasing for all* $0 \leq k \leq n - 1$. *In case* $0 \leq k \leq n - 1$, $W_k(\widehat{\Sigma_t})$ *is constant if and only if* Σ_t *is a spherical cap.*

Proof. Expressing (2.4) for $1 \leq k \leq n$ explicitly gives

$$
\begin{aligned}
(n+1)\binom{n}{k-1}\partial_t W_k(\widehat{\Sigma_t}) &= \frac{k}{n}\int_{\Sigma_t}\left(\frac{\sigma_k \sigma_{n-1}}{\sigma_n}\langle x, e \rangle - n\sigma_k \langle X_e, \nu \rangle\right) \\
&\geq k\int_{\Sigma_t}\left(\frac{n-k+1}{k}\sigma_{k-1}\langle x, e \rangle - \sigma_k \langle X_e, \nu \rangle\right) \\
&= 0,
\end{aligned}
$$

due to Newton's inequality and [48, Prop. 5.1]. If $k = n$, we have equality. For $k = 0$ we have

$$\partial_t W_0(\widehat{\Sigma_t}) = \int_{\Sigma_t}\left(\frac{\langle x, e \rangle}{\frac{n\sigma_n}{\sigma_{n-1}}} - \langle X_e, \nu \rangle\right) \geq \int_{\Sigma_t}\left(\frac{n\langle x, e \rangle}{H} - \langle X_e, \nu \rangle\right) \geq 0,$$

due to the Heintze-Karcher-type inequality in [48, Thm. 5.2]. This proves the first claim, while the second claim is due to the equality case of Newton's inequality.   q.e.d.

In the next part of this section we prove the smooth convergence of the solution of (3.1) to a spherical cap. The previous monotonicity properties will then finish the proof of Theorem 1.2.

Barriers. Contrary to a purely expanding inverse curvature flow, as it was considered in [27] and applied in [28] to give an estimate for a Willmore type quantity for convex free boundary hypersurfaces, our flow (3.1) will stay away from the minimal disk and hence, at least in principle, natural singularities are avoided. This nice feature is due to the constraining terms $\langle x, e \rangle$ and $\langle X_e, \nu \rangle$, which force the flow to remain between two spherical barriers.

First we have the short-time existence for the flow. The convexity of Σ_0 allows us to transform the flow equation to be a scalar parabolic Neumann problem by using the Möbius coordinate transformation, see [**27**] or (3.9) below. Thus the short-time existence is established by standard parabolic PDE theory, since due to our choice of e there holds $\langle x, e \rangle_{|\Sigma_0} > 0$ and hence the scalar parabolic equation is uniformly parabolic, compare [**32**] for instance.

Let T^* denote the maximal time of smooth existence of a solution to (3.1). This implies that all flow hypersurfaces are strictly convex up to T^* due to the positivity of F up to T^*. The family of spherical caps lying entirely in the closed half-ball $\bar{\mathbb{B}}^{n+1}_+$ is given by

$$C_R(e) = \{x \in \bar{\mathbb{B}}^{n+1} : |x - (R^2 + 1)^{\frac{1}{2}} e| = R\}, \quad R < \infty.$$

These form the natural barriers of (3.1):

3.2. Lemma. *Suppose for two radii $R_1 < R_2$, Σ_0 satisfies*

$$\Sigma_0 \subset \widehat{C_{R_2}(e)} \backslash \widehat{C_{R_1}(e)},$$

then this is preserved along the flow. In particular the height function $\langle x, e \rangle$ satisfies a priori bounds

$$\epsilon \leq \langle x, e \rangle \leq 1 - \epsilon$$

and the normal vector points uniformly downwards,

$$\langle \nu(t, \xi), e \rangle \leq -\tilde{\epsilon} \quad \forall (t, \xi) \in [0, T^*) \times \bar{\mathbb{B}}^n,$$

for some suitable $\epsilon, \tilde{\epsilon} \in (0, 1)$, which only depend on the initial datum.

Proof. An elementary calculation shows that the spherical caps $C_R(e)$ are static solutions of (3.1), i.e. $f = 0$ along $C_R(e)$. A simple comparison principle shows that the caps are barriers, since the avoidance principle holds due to the free boundary condition. The height estimate follows immediately. The final claim is the statement of [**27**, Lemma 11].
<div align="right">q.e.d.</div>

Evolution equations. We collect the relevant evolution equations, which will allow us to handle the curvature flow. In order to avoid confusion with tensor indices, here we abbreviate

$$X = X_e,$$

keeping in mind that X depends on e. We need the following lemma.

3.3. Lemma. *There hold*

$$\langle X, \nu \rangle_{;i} = \langle x_{;i}, e \rangle \langle x, \nu \rangle - \langle x_{;i}, x \rangle \langle e, \nu \rangle + h_i^k \langle X, x_{;k} \rangle$$

and

$$\begin{aligned}
\langle X, \nu \rangle_{;ij} = {} & h_{i;j}^k \langle X, x_{;k} \rangle + \langle x, e \rangle h_{ij} - h_i^k h_{kj} \langle X, \nu \rangle - g_{ij} \langle e, \nu \rangle \\
& + h_j^k \left(\langle x_{;i}, e \rangle \langle x, x_{;k} \rangle - \langle x_{;i}, x \rangle \langle e, x_{;k} \rangle \right) \\
& + h_i^k \left(\langle x_{;j}, e \rangle \langle x, x_{;k} \rangle - \langle x_{;j}, x \rangle \langle e, x_{;k} \rangle \right)
\end{aligned}$$

Proof.

$$\langle X, \nu \rangle_{;i} = \langle X_{;i}, \nu \rangle + \langle X, \nu_{;i} \rangle$$
$$= \langle x_{;i}, e \rangle \langle x, \nu \rangle - \langle x_{;i}, x \rangle \langle e, \nu \rangle + h_i^k \langle X, x_{;k} \rangle .$$

$$\langle X, \nu \rangle_{;ij} = -h_{ij} \langle e, \nu \rangle \langle x, \nu \rangle + \langle x_{;i}, e \rangle h_j^k \langle x, x_{;k} \rangle$$
$$+ h_{ij} \langle \nu, x \rangle \langle e, \nu \rangle - g_{ij} \langle e, \nu \rangle - \langle x_{;i}, x \rangle h_j^k \langle e, x_{;k} \rangle$$
$$+ h_{i;j}^k \langle X, x_{;k} \rangle + h_i^k \left(\langle x_{;j}, e \rangle x + \langle x, e \rangle x_{;j} - \langle x_{;j}, x \rangle e, x_{;k} \right) - h_i^k h_{kj} \langle X, \nu \rangle$$
$$= h_j^k \left(\langle x_{;i}, e \rangle \langle x, x_{;k} \rangle - \langle x_{;i}, x \rangle \langle e, x_{;k} \rangle \right) - g_{ij} \langle e, \nu \rangle + h_{i;j}^k \langle X, x_{;k} \rangle$$
$$+ h_i^k \left(\langle x_{;j}, e \rangle \langle x, x_{;k} \rangle - \langle x_{;j}, x \rangle \langle e, x_{;k} \rangle \right) + h_{ij} \langle x, e \rangle - h_i^k h_{kj} \langle X, \nu \rangle .$$

q.e.d.

Let
$$\mathcal{L} = \partial_t - \frac{\langle x, e \rangle}{F^2} F^{ij} \nabla_{ij}^2 - \langle X, \nabla \rangle ,$$

where F may either be understood to depend on the Weingarten operator, $F = F(h_j^i)$ or on the second fundamental form and the metric, $F = F(h_{ij}, g_{ij})$. There holds

$$F_j^i \equiv \frac{\partial F}{\partial h_i^j} = g_{kj} F^{ki} \equiv g_{kj} \frac{\partial F}{\partial h_{ki}},$$

cf. [**15**, Ch. 2]. The following specific evolution equations are valid.

3.4. Proposition. *For general 1-homogeneous F, along* (3.1) *there hold*

(i)

$$\mathcal{L} \langle x, e \rangle = \frac{2}{F} \langle x, e \rangle \langle \nu, e \rangle - \langle X, e \rangle .$$

(3.3) $$\nabla_\mu \langle x, e \rangle = \langle x, e \rangle \quad \text{along } \partial \Sigma_t.$$

(ii)

$$\mathcal{L} F = -\frac{2 \langle x, e \rangle}{F^3} F^{ij} F_{;i} F_{;j} + \frac{2}{F^2} F^{ij} \langle x_{;i}, e \rangle F_{;j}$$

(3.4)
$$+ \left(1 - F^{ij} g_{ij} \right) \langle e, \nu \rangle + F \langle x, e \rangle \left(1 - \frac{F^{ij} h_i^k h_{kj}}{F^2} \right) .$$

(3.5) $$\nabla_\mu F = 0 \quad \text{along } \partial \Sigma_t.$$

(iii)
(3.6)
$$\mathcal{L} H = \frac{\langle x, e \rangle}{F^2} g^{ij} F^{kl,rs} h_{kl;i} h_{rs;j} + \left(\frac{F^{kl} h_{km} h_l^m}{F^2} + 1 \right) \langle x, e \rangle H - \frac{2 \langle x, e \rangle}{F} \|\mathcal{W}\|^2$$
$$+ \left(\frac{H}{F} - n \right) \langle \nu, e \rangle + \frac{2}{F^2} g^{ij} \langle x_{;i}, e \rangle F_{;j} - \frac{2 \langle x, e \rangle}{F^3} \|\nabla F\|^2.$$

(3.7) $$\nabla_\mu H \leq 0 \quad along \ \partial\Sigma_t,$$

provided F is concave.

Proof. (i) (a)

$$\partial_t \langle x, e \rangle - \frac{\langle x, e \rangle}{F^2} F^{ij} \langle x, e \rangle_{;ij} = \frac{\langle x, e \rangle \langle \nu, e \rangle}{F} - \langle X, \nu \rangle \langle \nu, e \rangle + \frac{\langle x, e \rangle \langle \nu, e \rangle}{F}$$

$$= \frac{2}{F} \langle x, e \rangle \langle \nu, e \rangle - \langle X, \nu \rangle \langle \nu, e \rangle$$

$$= \frac{2}{F} \langle x, e \rangle \langle \nu, e \rangle - \langle X, e \rangle + \langle x, e \rangle_{;}^{\ i} \langle X, x_{;i} \rangle .$$

(b) was deduced in [**27**, Lemma 5].

(ii) (a)

$$\dot{F} = F_i^j \dot{h}_j^i$$

$$= -F_i^j \left(\frac{\langle x, e \rangle}{F} - \langle X, \nu \rangle \right)_{;j}^{\ i} - F_i^j \left(\frac{\langle x, e \rangle}{F} - \langle X, \nu \rangle \right) h_k^i h_j^k$$

$$= -F^{ij} \left(\frac{-h_{ij} \langle \nu, e \rangle}{F} - \frac{\langle x, e \rangle}{F^2} F_{;ij} + \frac{2 \langle x, e \rangle}{F^3} F_{;i} F_{;j} - \frac{2 \langle x_{;i}, e \rangle}{F^2} F_{;j} \right)$$

$$+ F_{;}^{\ k} \langle X, x_{;k} \rangle + \langle x, e \rangle F - F^{ij} h_i^k h_{kj} \langle X, \nu \rangle - F^{ij} g_{ij} \langle e, \nu \rangle$$

$$- \frac{\langle x, e \rangle}{F} F^{ij} h_i^k h_{kj} + F^{ij} h_i^k h_{kj} \langle X, \nu \rangle$$

$$= \frac{\langle x, e \rangle}{F^2} F^{ij} F_{;ij} - \frac{2 \langle x, e \rangle}{F^3} F^{ij} F_{;i} F_{;j} + \frac{2}{F^2} F^{ij} \langle x_{;i}, e \rangle F_{;j} + F_{;}^{\ k} \langle X, x_{;k} \rangle$$

$$+ \left(1 - F^{ij} g_{ij} \right) \langle e, \nu \rangle + F \langle x, e \rangle \left(1 - \frac{F^{ij} h_i^k h_{kj}}{F^2} \right) .$$

(b) There holds, using an orthonormal frame $(e_I)_{2 \leq I \leq n}$ for $T_x(\partial\Sigma_t)$,

$$\nabla_\mu \langle X, \nu \rangle_{|\partial\Sigma_t} = \nabla_\mu \left(\langle x, e \rangle \langle x, \nu \rangle - \frac{1}{2} (|x|^2 + 1) \langle e, \nu \rangle \right)_{|\partial\Sigma_t}$$

$$= \langle \mu, e \rangle \nabla_\mu \langle x, \nu \rangle - \langle e, \nu \rangle - \nabla_\mu \langle e, \nu \rangle$$

$$= - \langle e, \nu \rangle + \langle \langle \mu, e \rangle \mu - e, \mathcal{W}(\mu) \rangle$$

$$= - \langle e, \nu \rangle + \left\langle \langle \mu, e \rangle \mu - e, \langle \mathcal{W}(\mu), \mu \rangle \mu + \sum_{I=2}^{n} \langle \mathcal{W}(\mu), e_I \rangle e_I \right\rangle$$

$$= - \langle e, \nu \rangle$$

$$= \langle X, \nu \rangle_{|\partial\Sigma_t} .$$

From (3.3) and (2.7) we obtain

$$\nabla_\mu F = 0.$$

(iii) (a)
(3.8)

$$\dot{H} = -g^{ij} f_{;ij} - f\|\mathcal{W}\|^2$$

$$= g^{ij}\left(\langle X, \nu\rangle - \frac{\langle x, e\rangle}{F}\right)_{;ij} - f\|\mathcal{W}\|^2$$

$$= H_{;}^{\ k}\langle X, x_{;k}\rangle + H\langle x, e\rangle - \|\mathcal{W}\|^2\langle X, \nu\rangle - n\langle e, \nu\rangle + \frac{H}{F}\langle \nu, e\rangle$$

$$+ \frac{2}{F^2} g^{ij}\langle x_{;i}, e\rangle F_{;j} + \frac{\langle x, e\rangle}{F^2} g^{ij} F_{;ij} - \frac{2\langle x, e\rangle}{F^3}\|\nabla F\|^2 - f\|\mathcal{W}\|^2$$

$$= H_{;}^{\ k}\langle X, x_{;k}\rangle + H\langle x, e\rangle - \frac{\langle x, e\rangle}{F}\|\mathcal{W}\|^2 - n\langle e, \nu\rangle + \frac{H}{F}\langle \nu, e\rangle$$

$$+ \frac{2}{F^2} g^{ij}\langle x_{;i}, e\rangle F_{;j} + \frac{\langle x, e\rangle}{F^2} g^{ij} F^{kl} h_{kl;ij} + \frac{\langle x, e\rangle}{F^2} g^{ij} F^{kl,rs} h_{kl;i} h_{rs;j}$$

$$- \frac{2\langle x, e\rangle}{F^3}\|\nabla F\|^2.$$

Now we have to interchange the pairs (h, k) with (i, j) in $h_{kl;ij}$. Due to the Codazzi equations, the Ricci identities and the Gauss equation there holds

$$h_{kl;ij} = h_{ki;lj}$$
$$= h_{ki;jl} + R_{ljk}{}^a h_{ai} + R_{lji}{}^a h_{ka}$$
$$= h_{ij;kl} + R_{ljk}{}^a h_{ai} + R_{lji}{}^a h_{ka}$$
$$= h_{ij;kl} + (h_{la}h_{jk} - h_{lk}h_{ja})h_i^a + (h_{la}h_{ji} - h_{li}h_{ja})h_k^a$$

and thus

$$g^{ij} F^{kl} h_{kl;ij} = F^{kl} H_{;kl} + F^{kl} h_{la} h_k^a H - F\|\mathcal{W}\|^2.$$

Inserting into (3.8) gives the proposed evolution equation.

(b) For an orthonormal frame $(e_I)_{2 \le I \le n}$ of $T_x(\partial\Sigma_t)$, such that the second fundamental form of Σ is diagonal with respect to $(\nu, e_I)_{2 \le I \le n}$, along $\partial\Sigma_t$ we calculated in [27, Lemma 2], that

$$h_{IJ;\mu} = -h_{IJ} + h_{\mu\mu}\langle e_I, e_J\rangle.$$

From (3.5) we obtain

$$0 = \nabla_\mu F = \frac{\partial F}{\partial h}(\nabla_\mu h) = \sum_{I=2}^{n} F^{II} h_{II;\mu} + F^{\mu\mu} h_{\mu\mu;\mu}$$

and thus

$$
\begin{aligned}
\nabla_\mu H &= h_{\mu\mu;\mu} + \sum_{I=2}^{n} h_{II;\mu} \\
&= -\sum_{I=2}^{n} \frac{F^{II}}{F^{\mu\mu}} h_{II;\mu} - \sum_{I=2}^{n} h_{II} + (n-1)h_{\mu\mu} \\
&= \sum_{I=2}^{n} \frac{F^{II}}{F^{\mu\mu}} h_{II} - \sum_{I=2}^{n} \frac{F^{II}}{F^{\mu\mu}} h_{\mu\mu} - \sum_{I=2}^{n} h_{II} + (n-1)h_{\mu\mu} \\
&= \sum_{I=2}^{n} \frac{1}{F^{\mu\mu}} \left(F^{II} - F^{\mu\mu} \right) \left(h_{II} - h_{\mu\mu} \right),
\end{aligned}
$$

which is non-positive due to the concavity of F.   q.e.d.

Curvature estimates. We continue the a priori estimates for (3.1).

3.5. Proposition. *Let F be given by* (3.2). *Then along* (3.1) *there hold:*

(i) *The function*
$$
\varphi(t) = \min_{\bar{\mathbb{B}}^n} F(t, \cdot)
$$
is non-decreasing and hence
$$
F(t, \xi) \geq \min_{\bar{\mathbb{B}}^n} F(0, \cdot) \quad \forall (t, \xi) \in [0, T^*) \times \bar{\mathbb{B}}^n.
$$

(ii) *There exists a constant $c > 0$, depending only on the initial data, such that*
$$
F \leq c.
$$

Proof. (i) Considering F, and also σ_k, as a function of the principal curvatures κ_i and denoting
$$
F^i = \frac{\partial F}{\partial \kappa_i},
$$
we compute
$$
F^i = n \frac{\sigma_n^i}{\sigma_{n-1}} - n \frac{\sigma_n \sigma_{n-1}^i}{\sigma_{n-1}^2}.
$$
Due to the relation
$$
\sigma_k = \sigma_k^i \kappa_i + \sigma_{k+1}^i \quad \forall 0 \leq k \leq n
$$
we obtain
$$
\sum_{i=1}^{n} \sigma_k^i = \sum_{i=1}^{n} \left(\sigma_{k-1} - \sigma_{k-1}^i \kappa_i \right) = (n - k + 1)\sigma_{k-1}.
$$
Hence
$$
\sum_{i=1}^{n} F^i = n - 2n \frac{\sigma_n \sigma_{n-2}}{\sigma_{n-1}^2} \geq 1
$$

due to the Newton-Maclaurin inequalities. Furthermore there holds for $1 \leq k \leq n$:

$$\sigma_1 \sigma_k - (k+1)\sigma_{k+1} = \sum_{i=1}^{n} \left(\kappa_i \sigma_k - \kappa_i \sigma_{k+1}^i \right) = \sum_{i=1}^{n} \kappa_i^2 \sigma_k^i$$

and thus we compute

$$\sum_{i=1}^{n} F^i \kappa_i^2 = \frac{n}{\sigma_{n-1}} \sigma_1 \sigma_n - n \frac{\sigma_n}{\sigma_{n-1}^2} (\sigma_1 \sigma_{n-1} - n\sigma_n) = n^2 \frac{\sigma_n^2}{\sigma_{n-1}^2} = F^2.$$

At the boundary there holds

$$\nabla_\mu F = 0$$

and hence at minimizing points for F, the right hand side of its evolution (3.4) is non-negative and thus φ is non-decreasing, taking Lemma 3.2 into account.

(ii) To estimate F from above we define

$$\varphi = \log F - \alpha \langle x, e \rangle$$

for some positive α which will be determined later. We calculate the evolution equation

$$\mathcal{L}\varphi = \frac{1}{F} \mathcal{L} F + \frac{\langle x, e \rangle}{F^2} F^{ij} (\log F)_{;i} (\log F_{;j}) - \alpha \mathcal{L} \langle x, e \rangle$$

$$= -\frac{\langle x, e \rangle}{F^2} F^{ij} (\log F)_{;i} (\log F)_{;j} + \frac{2}{F^2} F^{ij} \langle x, e \rangle_{;i} (\log F)_{;j}$$

$$+ \frac{1}{F} \left(1 - F^{ij} g_{ij} \right) \langle e, \nu \rangle - \alpha \left(\frac{2}{F} \langle x, e \rangle \langle \nu, e \rangle - \langle X, e \rangle \right).$$

On the boundary there holds

$$\nabla_\mu \varphi = -\alpha \langle x, e \rangle < 0.$$

Hence maximal values of φ are attained in the interior and at such we have

$$0 \leq \mathcal{L}\varphi \leq \frac{1}{F^2} \left(-\alpha^2 \langle x, e \rangle + 2\alpha \right) F^{ij} \langle x, e \rangle_{;i} \langle x, e \rangle_{;j} + \frac{c + 2\alpha}{F} + \alpha \langle X, e \rangle.$$

The quantity

$$\langle X, e \rangle = \langle x, e \rangle^2 - \frac{1}{2}(|x|^2 + 1) = -\frac{1}{2} \left(|x|^2 - \langle x, e \rangle^2 \right) - \frac{1}{2} \left(1 - \langle x, e \rangle^2 \right) \leq -c_\epsilon,$$

due to Lemma 3.2. Hence, picking α large enough in dependence of the previous a priori bounds, we obtain a contradiction if F is too large. q.e.d.

To finish the curvature estimates, we use the evolution of the mean curvature $H = \sigma_1$ to prove that it is bounded. This is sufficient due to the convexity.

3.6. Proposition. *If F is given by* (3.2)*, then the mean curvature, and hence all principal curvatures, are a priori bounded by the initial data.*

Proof. In the evolution equation (3.6) we use the Cauchy-Schwarz inequality to obtain

$$\frac{2}{F^2} g^{ij} \langle x_{;i}, e \rangle F_{;j} - \frac{2 \langle x, e \rangle}{F^3} \|\nabla F\|^2 \leq \mathrm{const}.$$

Hence, due to the concavity of F, the bounds on F and the lower bound on $\langle x, e \rangle$, the term involving $\|\mathcal{W}\|^2$ is the leading order term with a negative sign. Due to the boundary condition (3.7), we obtain the result.

q.e.d.

We obtain the long-time existence of the solution to (3.1).

3.7. Theorem. *Let*

$$x_0 \colon \bar{\mathbb{B}}^n \hookrightarrow \Sigma \subset \bar{\mathbb{B}}^{n+1}$$

be the embedding of a strictly convex free boundary hypersurface in the unit ball. Then the maximal solution of (3.1) *exists for all times with uniform C^∞-estimates.*

Proof. In order to apply parabolic regularity theory, it is convenient to transform the flow to a scalar parabolic Neumann problem. This was already completely performed in [27]. Our previous a priori estimates allow us to use the same conclusion in that paper. For convenience we sketch the main ingredients again. We use a coordinate transformation, which we call *Möbius coordinates* for the upper half-ball. Namely, consider the map

$$\varphi \colon \mathbb{B}^n \times [1, \infty) \to \bar{\mathbb{B}}^{n+1}_+$$

$$\varphi(x, \lambda) = \frac{4\lambda x + (1 + |x|^2)(\lambda^2 - 1)e}{(1 + \lambda)^2 + (1 - \lambda)^2 |x|^2}.$$

This is a conformal map from the Euclidean cylinder $\bar{\mathbb{B}}^n \times [1, \infty)$ to $\bar{\mathbb{B}}^{n+1}_+$ and in [27, Prop 1] it was shown that any convex hypersurface of $\bar{\mathbb{B}}^{n+1}$ with perpendicular Neumann condition can be written as a graph over $\bar{\mathbb{B}}^n$ in Moebius coordinates with uniform gradient estimates, as long as λ stays away from infinity. Hence all flow hypersurfaces can be written as graphs

$$\Sigma_t = \{(x(t, z), u(t, x(t, z))) \colon (t, z) \in [0, T^*) \times \mathbb{B}^n\},$$

where u solves the Neumann problem

(3.9)
$$\begin{aligned}
\dot{u} &= -\frac{v}{e^\psi F} && \text{in } (0, T^*) \times \mathbb{B}^n \\
\langle Du, N \rangle &= 0 && \text{on } [0, T^*) \times \partial \mathbb{B}^n \\
u(0, \cdot) &= u_0 && \text{on } \{0\} \times \mathbb{B}^n,
\end{aligned}$$

where $v^2 = 1 + |Du|^2$, N is the outward normal to $\mathbb{B}^n$ and $e^{2\psi}$ is the conformal factor arising from the Möbius coordinates, compare [**27**, Cor. 4]. As in [**27**, Lemma 15], we can conclude the uniform C^∞-estimates and the long-time existence from [**30**, Thm. 14.23] or [**44**, Thm. 4, Thm. 5]. Note that we have also used that due to $F \geq c > 0$ and $H \leq c$ the operator is uniformly parabolic.   q.e.d.

We finish the convergence result by proving that any subsequential limit is a spherical cap of a uniquely determined curvature.

3.8. Proposition. *The flow* (3.1) *converges smoothly to a uniquely determined spherical cap around e.*

Proof.

$$\partial_t W_0(\widehat{\Sigma_t}) = \int_{\Sigma_t} f \geq \int_{\Sigma_t} \left(\frac{n \langle x, e \rangle}{H} - \langle X, \nu \rangle \right) \geq 0.$$

Furthermore

$$W_0(\widehat{\Sigma_t}) = |\widehat{\Sigma_t}|$$

is bounded and thus due to the C^∞-estimates we must have

$$\int_{\Sigma_t} \left(\frac{n \langle x, e \rangle}{H} - \langle X, \nu \rangle \right) \to 0$$

as $t \to \infty$. Hence any limit satisfies the Heintze-Karcher type inequality with equality and thus must be a spherical cap due to [**48**, Thm. 5.2]. We claim that this cap is uniquely determined, and hence the flow converges. The uniqueness is proved as follows.

The above showed that $x(t, \cdot)$ subconverges smoothly to a free boundary spherical cap $C_{\rho_\infty}(e_\infty)$. It is important to note that the radius ρ_∞ is independent of the choice of a subsequence of t ($t \to \infty$). This follows from the constancy of W_n along the flow (3.1) and the fact that W_n is strictly increasing with the radius. Now we show $e_\infty = e$, and hence the uniqueness.

Define $\rho(t, \xi)$ to be the radius of the unique spherical cap around e passing through the point $x(t, \xi)$. By the barrier estimates, the function

$$\rho_{max}(t) = \max \rho(t, \cdot) = \rho(t, \xi_t)$$

is non-increasing. Hence

$$\lim_{t \to \infty} \rho_{max}(t) = A$$

exists. Since we have all uniform estimates, we know

(3.10) $$\lim_{t \to \infty} \frac{d}{dt} \rho_{max} = 0.$$

It is clear that $\rho_{max} \geq \rho_\infty$. Next we prove that

(3.11) $$\lim_{t \to \infty} \rho_{max} = \rho_\infty.$$

Assume not, then there exists some $\epsilon > 0$ such that

$$(3.12) \qquad \rho_{max}(t) > \rho_\infty + \epsilon, \forall t.$$

It is easy to see that

$$\sqrt{\rho^2 + 1} = \frac{\frac{1}{2}(|x|^2 + 1)}{\langle x, e \rangle}.$$

Thus ρ is a smooth function and one can check that ρ satisfies

$$
\begin{aligned}
\frac{\rho}{\rho^2 + 1}\partial_t \rho &= f\left(\frac{\langle x, \nu \rangle}{\frac{1}{2}(|x|^2 + 1)} - \frac{\langle \nu, e \rangle}{\langle x, e \rangle}\right) \\
(3.13) \qquad &= \left(\frac{\langle x, e \rangle}{n\frac{\sigma_n}{\sigma_{n-1}}} - \langle X_e, \nu \rangle\right)\frac{\langle X_e, \nu \rangle}{\frac{1}{2}(|x|^2 + 1)\langle x, e \rangle}.
\end{aligned}
$$

Recall that $C_{\rho_{max}(t)}(e)$ is the spherical cap of radius $\rho_{max}(t)$ around e. Then one sees that the flow hypersurface $x(t, \cdot)$ is tangential to $C_{\rho_{max}(t)}(e)$ at $x(t, \xi_t)$ by the definition of $\rho_{max}(t)$. Note that $x(t, \cdot)$ and $C_{\rho_{max}(t)}(e)$ has the same position vector and normal vector at $x(t, \xi_t)$. On the other hand, for $C_{\rho_{max}(t)}(e)$, the speed function

$$f = \frac{\langle x, e \rangle}{n\frac{\sigma_n}{\sigma_{n-1}}} - \langle X_e, \nu \rangle \equiv 0,$$

which yields

$$\left.\frac{\langle X_e, \nu \rangle}{\langle x, e \rangle}\right|_{C_{\rho_{max}(t)}(e)} = \rho_{max}(t).$$

Therefore

$$\left.\frac{\langle X_e, \nu \rangle}{\langle x, e \rangle}\right|_{x(t,\xi_t)} = \rho_{max}(t).$$

We deduce from (3.13) that

$$(3.14) \quad \frac{\rho_{max}}{\rho_{max}^2 + 1}\frac{d}{dt}\rho_{max} = \left(\frac{1}{n\frac{\sigma_n}{\sigma_{n-1}}} - \rho_{max}(t)\right)\frac{\langle X_e, \nu \rangle}{\frac{1}{2}(|x|^2 + 1)}.$$

Since $x(t, \cdot)$ converges smoothly to $C_{\rho_\infty}(e_\infty)$ and ρ_∞ is unique, the curvature term

$$n\frac{\sigma_n}{\sigma_{n-1}} \to \frac{1}{\rho_\infty} \text{ uniformly.}$$

Thus, there exists $T > 0$, depending on ϵ, such that for $t > T$,

$$\frac{1}{n\frac{\sigma_n}{\sigma_{n-1}}} - \rho_\infty < \frac{\epsilon}{2}.$$

Combining with (3.12), we see for $t > T$,

$$\frac{1}{n\frac{\sigma_n}{\sigma_{n-1}}} - \rho_{max}(t) < -\frac{\epsilon}{2}.$$

Since $\frac{\langle X_e, \nu \rangle}{\frac{1}{2}(|x|^2+1)}$ is uniformly bounded below, it follows from (3.14) that for $t > T$,

$$\frac{d}{dt}\rho_{max}(t) \leq -C\epsilon.$$

This is a contradiction to (3.10). We get (3.11). Similarly, for $\rho_{min}(t) = \min \rho(t, \cdot)$, we can show

(3.15) $$\lim_{t \to \infty} \rho_{min} = \rho_\infty.$$

From (3.11) and (3.15), We conclude that $\lim_{t \to \infty} \rho(t, \cdot)$ is a constant. That means the limit hypersurface is the spherical cap around e, the claimed uniqueness. q.e.d.

4. Proof of the main results

Proof of Theorem 1.2. (i) First we prove that the functions

$$f_k(r) = W_k(\widehat{C_r(e)})$$

are strictly increasing. This can be seen from the flow

$$\dot{y} = X_e$$
$$y(0, \bar{\mathbb{B}}^n) = C_r(e).$$

The flow hypersurfaces of this flow are $(C_s)_{s \geq r}$, where $s = r(t)$ is some increasing function and there holds

$$(n+1)\binom{n}{k-1}\partial_t W_k(\widehat{C_{r(t)}}) = \int_{C_{r(t)}} k\sigma_k \langle X_e, \nu \rangle > 0$$

due to the strict convexity of the caps.

(ii) We prove Theorem 1.2 in the strictly convex case. Start the flow (3.1) from Σ and denote the limit cap by C_{R_0}. Due to Proposition 3.1 there hold

$$W_n(\widehat{\Sigma}) = W_n(\widehat{C_{R_0}}) = f_n(R_0) = f_n \circ f_k^{-1}(f_k(R_0)) \geq f_n \circ f_k^{-1}(W_k(\widehat{\Sigma})).$$

This inequality is strict, unless Σ is a spherical cap, due to the final statement in Proposition 3.1.

(iii) We consider the convex case, where we may suppose that Σ is not a flat disk, otherwise the statement is trivially true. By [28, Cor. 3.3] we can approximate Σ by strictly convex hypersurfaces with free boundary in the unit ball in the $C^{2,\alpha}$-norm. Hence the inequality holds for Σ.

To prove the limiting case, we employ an argument previously used in [19]. Suppose Σ is convex and

$$W_n(\widehat{\Sigma}) = f_n \circ f_k^{-1}(W_k(\widehat{\Sigma})).$$

Due to [28, Lemma 3.1] the set

$$\Sigma_+ = \{\xi \in \mathbb{B}^n \colon (h_{ij}(\xi)) > 0\}$$

is not empty, since Σ must have a strictly convex point in the interior. Σ_+ is obviously open. We prove that Σ_+ is also closed, by showing that

$$h_{ij|\Sigma_+} \geq c g_{ij|\Sigma_+}$$

for some constant c. So let $\xi_0 \in \Sigma_+$, $U \subset \mathbb{B}^n$ open with

$$\xi_0 \in U \subset \bar{U} \subset \mathbb{B}^n,$$

$\eta \in C_c^\infty(U)$ and consider the normal variation

$$x_s = x_0 + s\eta\nu,$$

where x_0 is the embedding of Σ. Obviously, for small s the corresponding hypersurfaces Σ_s are convex with free boundary in the unit ball and hence

$$W_n(\widehat{\Sigma_s}) \geq f_n \circ f_k^{-1}(W_k(\widehat{\Sigma_s})).$$

Thus

$$0 = n(n+1)\frac{d}{ds}_{|s=0}\left(W_n(\widehat{\Sigma_s}) - f_n \circ f_k^{-1}(W_k(\widehat{\Sigma_s}))\right) = \int_\Sigma \left(n\sigma_n - c_1\sigma_k\right)\eta$$

with a suitable positive c_1. Since η was arbitrary, we have on all of $\mathbb{B}^n$:

$$n\sigma_n = c_1\sigma_k,$$

where Since

$$\sigma_n \leq c_{n,k}\sigma_k^{\frac{n}{k}},$$

we obtain

$$\sigma_n = \frac{c_1}{n}\sigma_k \geq \frac{c_1}{n}\left(\frac{\sigma_n}{c_{n,k}}\right)^{\frac{k}{n}}.$$

Hence σ_n has a positive lower bound on Σ_+, which implies that also the second fundamental form has a positive lower bound on Σ_+. Thus Σ_+ is closed. Hence Σ is strictly convex and by the first case it is a spherical cap.

5. Appendix A. Quermassintegrals and the Euler characteristic

In this Appendix, we show that W_{n+1} is a topological constant for a general free boundary hypersurface in the unit ball. First we can easily extend the definition of W_{n+1} as follows:

(i) If n is even and Σ immersed, the definition of quermassintegrals given in (2.3) carries over, since it only depends on the area of Σ and curvature integrals. We still denote it by $W_{n+1}(\widehat{\Sigma})$, though $\widehat{\Sigma}$ is not well-defined.

(ii) If n is odd, we consider an embedded hypersurface $\Sigma \subset \bar{\mathbb{B}}^{n+1}$ with an embedded free boundary $\partial\Sigma \subset \mathbb{S}^n$. Since $\partial\Sigma \subset \mathbb{S}^n$ is an embedding, it divides the sphere into 2 pieces. Let us choose one of these 2 pieces and denote it as before by $\widehat{\partial\Sigma}$. Then $\widehat{\Sigma}$ is the

domain in $\bar{\mathbb{B}}^{n+1}$ enclosed by $\widehat{\partial\Sigma}$ and Σ and $W_{n+1}(\widehat{\Sigma})$ is defined as in (2.3).[1]

Moreover, the first variation formula of W_{n+1} is the same as the one proved Proposition 2.2, namely

$$\partial_t W_{n+1}(\widehat{\Sigma}) = 0$$

for any smooth deformation. This formula implies the following Gauss-Bonnet-Chern theorem.

5.1.   Theorem. *Let $\Sigma \subset \bar{\mathbb{B}}^{n+1}$ be immersed hypersurface with free boundary $\partial\Sigma \subset \mathbb{S}^n$. If n is even, then*

$$\frac{\omega_n}{2}\chi(\Sigma) = (n+1)W_{n+1}(\widehat{\Sigma})$$

$$= \int_\Sigma H_n + \int_{\partial\Sigma} H_{n-2}^{\mathbb{S}^n} + \frac{n-2}{3}\int_{\partial\Sigma} H_{n-4}^{\mathbb{S}^n} + \cdots$$

$$+ \frac{n-2}{3}\cdots\frac{4}{n-3}\int_{\partial\Sigma} H_2^{\mathbb{S}^n} + \frac{n-2}{3}\cdots\frac{2}{n-1}|\partial\Sigma|.$$

If n is odd and $\Sigma \subset \bar{\mathbb{B}}^{n+1}$ is embedded with an embedded boundary $\partial\Sigma$, then

$$\frac{\omega_n}{2}\chi(\widehat{\Sigma}) = (n+1)W_{n+1}(\widehat{\Sigma})$$

$$= \int_\Sigma H_n + \int_{\partial\Sigma} H_{n-2}^{\mathbb{S}^n} \quad + \frac{n-2}{3}\int_{\partial\Sigma} H_{n-4}^{\mathbb{S}^n} + \cdots + \frac{n-2}{3}\int_{\partial\Sigma} H_1^{\mathbb{S}^n} + |\widehat{\partial\Sigma}|.$$

Here H_k is the (normalized) k-th mean curvature of Σ in $\mathbb{R}^{n+1}$ and $H_j^{\mathbb{S}^n}$ is the (normalized) j-th mean curvature of $\partial\Sigma$ in $\mathbb{S}^n$.

Note that in the second case $\chi(\widehat{\Sigma}) = \frac{1}{2}\chi(\Sigma \cup \widehat{\partial\Sigma})$.

Proof. The second equality follows from the very definition. We need to prove the first, which follows from the variation formula (2.5) of W_{n+1} by a standard argument as follows. One deforms the hypersurface Σ to a very small free boundary hypersurface $\widetilde{\Sigma}$. (2.5) implies that both Σ and $\widetilde{\Sigma}$ have the same W_{n+1}. It is easy to see that except the leading order term, all other terms of W_{n+1} become small, when $\widetilde{\Sigma}$ shrinks to a point. In this case, $\widetilde{\Sigma}$ looks more and more like a free boundary hypersurface in the half-space $\mathbb{R}_+^{n+1}$ with the boundary perpendicular on $\partial\mathbb{R}_+^{n+1}$. Hence the first equality follows from the Gauss-Bonnet theorem for closed hypersurfaces in $\mathbb{R}^{n+1}$.   q.e.d.

We would like to compare the above Gauss-Bonnet-Chern theorem to the Gauss-Bonnet theorem for closed hypersurfaces in $\mathbb{R}^{n+1}$ by Hopf

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[1]Note that if we choose the other piece, we have to flip the normal in the definition of the principal curvatures.

[**23**, **24**] and in $\mathbb{S}^n$ by Teufel [**43**]. For the Gauss-Bonnet theorem for closed hypersurfaces in the hyperbolic space, we refer to [**39**].

If n is even and Σ is an immersed closed hypersurface in $\mathbb{R}^{n+1}$, then by [**23**], we have

$$\frac{\omega_n}{2}\chi(\Sigma) = (n+1)W_{n+1}^{\mathbb{R}^{n+1}} = \int_\Sigma H_n.$$

If n is odd and Σ is an embedded closed hypersurface in $\mathbb{R}^{n+1}$, then by [**24**], we have

$$\omega_n\chi(\Omega) = (n+1)W_{n+1}^{\mathbb{R}^{n+1}} = \int_\Sigma H_n,$$

where Ω is the domain enclosed by Σ. Note that in this case $\chi(\Omega) = \frac{1}{2}\chi(\Sigma)$.

For a closed hypersurfaces in $\mathbb{S}^n$, Teufel [**43**] obtained: If n is odd and Σ is immersed, then

$$\frac{\omega_{n-1}}{2}\chi(\Sigma) = nW_n^{\mathbb{S}^n}(\Sigma) = c_{n-1}\int_\Sigma H_{n-1} + c_{n-3}\int_\Sigma H_{n-3} + \cdots + c_2\int_\Sigma H_2 + |\partial\Sigma|.$$

If n is even and Σ is embedded with the enclosed domain K, then

$$\omega_{n-1}\chi(K) = nW_n^{\mathbb{S}^n}(K) = c_{n-1}\int_\Sigma H_{n-1} + c_{n-3}\int_\Sigma H_{n-3} + \cdots + c_1\int_\Sigma H_1 + |K|.$$

For the constants c_j, we refer to [**43**].

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