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ON THE VOLUME OF HYPERPLANE SECTIONS OF A d -CUBE

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ABSTRACT. We obtain an optimal upper bound for the normalised volume of a hyperplane section of an origin-symmetric d -dimensional cube. This confirms a conjecture posed by Imre Bárány and Péter Frankl.

1. STATEMENT OF THE RESULTS

Let $\mathcal{C}^d = [-1/2, 1/2]^d$ be the d -dimensional unit cube. Throughout this paper we assume that $d \geq 2$. For a nonzero vector $\mathbf{v} \in \mathbb{R}^d$ we will denote by $\mathbf{v}^\perp$ the hyperplane orthogonal to $\mathbf{v}$ and consider the section $\mathcal{C}^d \cap \mathbf{v}^\perp$ of the cube $\mathcal{C}^d$. Let further $\|\cdot\|_1$ and $\|\cdot\|_2$ denote ℓ_1 and ℓ_2 norms, respectively. In this paper we will be interested in the quantity

$$(1) \quad V_d = \max_{\mathbf{v} \in \mathbb{R}^d} \frac{\|\mathbf{v}\|_1}{\|\mathbf{v}\|_2} \cdot \text{vol}_{d-1}(\mathcal{C}^d \cap \mathbf{v}^\perp),$$

where $\text{vol}_{d-1}(\cdot)$ stands for the $(d-1)$ -volume. Imre Bárány and Péter Frankl [3] conjectured that the maximum in (1) is given by the vector $\mathbf{v} = \mathbf{1}_d := (1, \dots, 1)$. Our main result confirms this conjecture.

Theorem 1. *We have*

$$(2) \quad V_d = \sqrt{d} \cdot \text{vol}_{d-1}(\mathcal{C}^d \cap \mathbf{1}_d^\perp).$$

It is known that

$$\lim_{d \rightarrow \infty} \text{vol}_{d-1}(\mathcal{C}^d \cap \mathbf{1}_d^\perp) = \sqrt{\frac{6}{\pi}}$$

(see [9], [11] and e. g. [5]). The expression (2) also allows finding the exact value of V_d in the following way. Let $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{S}^{d-1}$ be a unit vector. Then

$$(3) \quad \text{vol}_{d-1}(\mathcal{C}^d \cap \mathbf{s}^\perp) = \frac{2}{\pi} \int_0^\infty \prod_{i=1}^d \frac{\sin s_i t}{s_i t} dt$$

(see e. g. [2]). Consider the *sinc* integral [4]

$$\sigma_d = \frac{2}{\pi} \int_0^\infty \left(\frac{\sin t}{t} \right)^d dt.$$

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In view of (2) and (3) we have

$$(4) \quad V_d = \frac{2\sqrt{d}}{\pi} \int_0^\infty \left(\frac{\sin \frac{t}{\sqrt{d}}}{\frac{t}{\sqrt{d}}} \right)^d dt = d \sigma_d.$$

Further

$$\sigma_d = \frac{d}{2^{d-1}} \sum_{0 \leq r < d/2, r \in \mathbb{Z}} \frac{(-1)^r (d-2r)^{d-1}}{r!(d-r)!}$$

(see e. g. [10]). The sequences of numerators and denominators of $\sigma_d/2$ can be found in [12].

Theorem 1 and (3) immediately imply the following lower bound for sinc integrals.

Corollary 2. *For any unit vector $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{S}^{d-1}$*

$$\frac{2\|\mathbf{s}\|_1}{\pi d} \int_0^\infty \prod_{i=1}^d \frac{\sin s_i t}{s_i t} dt \leq \sigma_d.$$

It is known that $0 < \sigma_{d+1}/\sigma_d < 1$ (see e. g. [1, Lemma 1]). Theorem 1 also implies the following lower bound for the ratio of consecutive sinc integrals.

Corollary 3. *We have*

$$\frac{d}{d+1} \leq \frac{\sigma_{d+1}}{\sigma_d}.$$

2. INTERSECTION BODY OF $\mathcal{C}^d$

We can associate with each star body $\mathcal{L}$ the *distance function* $f_{\mathcal{L}}(\mathbf{x}) = \inf\{\lambda > 0 : \mathbf{x} \in \lambda \mathcal{L}\}$. The *intersection body* $I\mathcal{L}$ of a star body $\mathcal{L} \subset \mathbb{R}^d$ (recall that we assume $d \geq 2$) is defined as the $\mathbf{0}$ -symmetric star body with distance function

$$f_{I\mathcal{L}}(\mathbf{x}) = \frac{\|\mathbf{x}\|_2}{\text{vol}_{d-1}(\mathcal{L} \cap \mathbf{x}^\perp)}.$$

The Busemann theorem (see e. g. [6], Chapter 8) states that if $\mathcal{L}$ is $\mathbf{0}$ -symmetric and convex, then $I\mathcal{L}$ is a convex set. For more details on intersection bodies we refer the reader to [7, 8].

For convenience, in what follows we will work with normalised cube

$$\mathcal{Q}^d = \frac{1}{\text{vol}_{d-1}(\mathcal{C}^d \cap \mathbf{1}_d^\perp)^{1/(d-1)}} \cdot \mathcal{C}^d.$$

Then, in particular,

$$(5) \quad \text{vol}_{d-1}(\mathcal{Q}^d \cap \mathbf{1}_d^\perp) = 1.$$

Lemma 4. *The affine hyperplane*

$$\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^d : x_1 + \dots + x_d = \sqrt{d}\}$$

is a supporting hyperplane of $I\mathcal{Q}^d$.

Proof. Let $f = f_{I\mathcal{Q}^d}$ denote the distance function of $I\mathcal{Q}^d$, so that $I\mathcal{Q}^d = \{\mathbf{x} \in \mathbb{R}^d : f(\mathbf{x}) \leq 1\}$. By (5), for the point

$$(6) \quad \mathbf{h} := \frac{1}{\sqrt{d}} \mathbf{1}_d = \left(\frac{1}{\sqrt{d}}, \dots, \frac{1}{\sqrt{d}} \right)$$

we have $f(\mathbf{h}) = 1$. Therefore $\mathbf{h}$ is on the boundary of $I\mathcal{Q}^d$.

Suppose, to derive a contradiction, that $\mathcal{H}$ is not a supporting hyperplane of $I\mathcal{Q}^d$. Observe that $\mathbf{h} \in \mathcal{H} \cap I\mathcal{Q}^d$. Hence for any $\epsilon > 0$ there exists a point $\mathbf{p} = (p_1, \dots, p_d)$ in the interior of $I\mathcal{Q}^d$ with

$$(7) \quad \|\mathbf{h} - \mathbf{p}\|_2 < \epsilon$$

and $p_1 + \dots + p_d > \sqrt{d}$.

By (7) we may assume that $\mathbf{p} \in \mathbb{R}_{>0}^d$. Further, as the point $\mathbf{p}$ is in the interior of $I\mathcal{Q}^d$ we may assume, for simplicity, that the entries of $\mathbf{p}$ are pairwise distinct: $p_i \neq p_j$ for $i \neq j$. Consider d points

$$\begin{aligned} \mathbf{p}_1 &= (p_1, \dots, p_{d-1}, p_d) \\ \mathbf{p}_2 &= (p_2, \dots, p_d, p_1) \\ &\vdots \\ \mathbf{p}_d &= (p_d, \dots, p_{d-2}, p_{d-1}). \end{aligned}$$

For each i , the section $\mathcal{Q}^d \cap \mathbf{p}_i^\perp$ is the image of the section $\mathcal{Q}^d \cap \mathbf{p}_1^\perp$ under an orthogonal transformation defined by a permutation matrix. Therefore $\mathbf{p}_i \in I\mathcal{Q}^d$. Set

$$\mathbf{y} = \frac{1}{d}(\mathbf{p}_1 + \dots + \mathbf{p}_d) = \frac{\sum_{i=1}^d p_i}{\sqrt{d}} \mathbf{h}.$$

By construction, $\mathbf{y}$ is a convex combination of the points $\mathbf{p}_1, \dots, \mathbf{p}_d$. Since $I\mathcal{Q}^d$ is convex, $\mathbf{y} = (y_1, \dots, y_d) \in I\mathcal{Q}^d$. Further

$$y_1 + \dots + y_d = \sum_{i=1}^d p_i > \sqrt{d}.$$

Therefore the point $\mathbf{h}$ must be in the interior of $I\mathcal{Q}^d$. The derived contradiction completes the proof. $\square$

3. PROOF OF THEOREM 1

It is sufficient to show that for any unit vector $\mathbf{v} \in \mathbb{S}^{d-1}$ the inequality

$$(8) \quad \|\mathbf{v}\|_1 \cdot \text{vol}_{d-1}(\mathcal{Q}^d \cap \mathbf{v}^\perp) \leq \sqrt{d} \cdot \text{vol}_{d-1}(\mathcal{Q}^d \cap \mathbf{1}_d^\perp) = \sqrt{d}$$

holds.

In view of symmetry of $\mathcal{Q}^d$ we may assume without loss of generality that $\mathbf{v} \in \mathbb{R}_{\geq 0}^d$. Consider the plane $\mathcal{P}$ spanned by the vector $\mathbf{h}$, defined by (6), and the vector $\mathbf{v}$ and let α be the angle between these two vectors with $\cos(\alpha) = \mathbf{h} \cdot \mathbf{v}$ (Figure 1). It is not difficult to see that $\cos(\alpha) \geq 1/\sqrt{d}$ and, consequently, $\alpha < \pi/2$.

Notice that $\mathbf{h}$ is orthogonal to the line $\mathcal{H} \cap \mathcal{P}$. Let $\mathbf{u}$ denote the intersection point of the line spanned by $\mathbf{v}$ and $\mathcal{H} \cap \mathcal{P}$. Further, let $\mathbf{w}$ be the orthogonal projection of $\mathbf{v}$ onto the line spanned by $\mathbf{h}$.

Then we have

$$(9) \quad \cos(\alpha) = \|\mathbf{w}\|_2 = \frac{1}{\|\mathbf{u}\|_2}.$$

FIGURE 1. Geometric argument on the plane $\mathcal{P}$

Since $\mathbf{h} \in \mathcal{H}$, all points $\mathbf{x}$ on the line passing through the points $\mathbf{v}$ and $\mathbf{w}$ have $x_1 + \dots + x_d = \sqrt{d} \|\mathbf{w}\|_2$. Therefore, we have $\|\mathbf{v}\|_1 = \sqrt{d} \|\mathbf{w}\|_2$. It was shown in Lemma 4 that $\mathcal{H}$ is a supporting hyperplane of $I\mathcal{Q}^d$. Hence we have

$$(10) \quad \text{vol}_{d-1}(\mathcal{Q}^d \cap \mathbf{v}^\perp) \leq \|\mathbf{u}\|_2.$$

Finally, using (9) and (10), we have

$$\|\mathbf{v}\|_1 \cdot \text{vol}_{d-1}(\mathcal{Q}^d \cap \mathbf{v}^\perp) \leq \sqrt{d} \|\mathbf{w}\|_2 \|\mathbf{u}\|_2 = \sqrt{d},$$

that confirms (8).

4. PROOF OF COROLLARY 3

It was observed in [3] that the sequence $\{V_d\}_{d=1}^\infty$ is increasing: $V_d \leq V_{d+1}$ for all $d \geq 2$. It is now sufficient to note that, by Theorem 1 (see (4)), we can write $V_d = d\sigma_d$.

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