Generating sequences from the sums of binomial coefficients in a residue class modulo q.

David M. Humphreys
School of Mathematics, Cardiff University

Summary

For non-negative integers r we examine four families of alternating and non-alternating sign closed form binomial sums, $\mathcal{F}_{s;ab}(r,t,q)$, in a generalised congruence modulo q. We explore sums of squares and divisibility properties such as those determined by Weisman (and Fleck). Extending r to all integers we express the sequences in terms of closed form roots of unity and subsequently cosines.

By a renumbering of these sequences we build eight new "diagonalised" sequences, $\mathcal{L}_{s;abc}(r,t,q)$, and construct equivalent closed forms and sums of squares relations.

We modify Fibonacci type polynomials to construct order m recurrence polynomials that satisfy these diagonalised sequences. These recurrence polynomial sequences are shown to satisfy second order differential equations and exhibit orthogonal relations. From these latter relations we establish three term recurrence relations both between and within sequences.

By the application of the reciprocal recurrence polynomial and hypergeometric functions, generating functions for these renumbered sequences are determined. Then employing these latter functions, we establish theorems that enable us to express each of the new sequences in terms of a Minor Corner Layered (MCL) determinant.

When r is a negative integer and $q = 2m+b$ is unspecified, the MCL determinants produce sequences of polynomials in m. For particular sequences we truncate these polynomials to contain only the leading coefficient and find that the truncated polynomial is equal to that of a Dirichlet series of the form zeta, lambda, beta or eta. From this relationship, recurrence polynomials for these latter functions are established

Finally we develop a congruence for the denominator of the uncancelled modified Bernoulli numbers of the first kind, $B_n/n!$, and consequently a similar congruence for the zeta function at positive even valued integers. Furthermore we determine that these congruences obey the Fleck congruence.

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Notation

Table 1: A list of standard notation used in the thesis

Symbol	Notation	See Section:
$\lfloor x \rfloor$	lower floor function	2.2
$\binom{r}{q}$	binomial coefficient	2.2
$\|x\|$	absolute value function	3.2
$\mathbb{Z}$	integers	1.1.2
$\mathbb{N}_{\geq 0}$	0,1,2,3,...	1.1.2
$\mathbb{N}$	1,2,3,4...	1.1.2
$\Re x$	real part of x	4.3
$\Im x$	imaginary part of x	4.3
$ord_p(x)$	order of p in x	4.5
(a,b)	highest common factor (of a and b)	4.5.1
$\prod_d^*$	product over d relatively prime to modulus (q)	4.5.1
$\mathbb{Q}(\zeta_q)$	cyclotomic field	4.5.1
$F_Q(x)$	Q-th Fibonacci polynomial	5.1.1
$L_Q(x)$	Q-th Lucas polynomial	5.1.2
$T_Q(x)$	Q-th Chebyshev polynomial of the first kind	5.3
$U_Q(x)$	Q-th Chebyshev polynomial of the second kind	5.3
$C_Q(x)$	Q-th monic Chebyshev polynomial of the first kind	5.3.1
$S_Q(x)$	Q-th monic Chebyshev polynomial of the second kind	5.3.1
$\delta_{m,n}$	Kronecker delta function	7.1
$mFn(a_i;b_j;x)$	generalised hypergeometric function	8.3
$x^{\overline{m}}$	rising factorial	8.3
$x^{\underline{m}}$	falling factorial	8.3
$\Gamma(x)$	Gamma function	8.3
$\zeta(s)$	Riemann zeta function	9.5.3
$\eta(s)$	Dirichlet eta function	9.5.3
$\beta(s)$	Dirichlet beta function	9.5.3
$\lambda(s)$	Dirichlet lambda function	9.5.3
B_n	n-th Bernoulli number: first kind	10.1
b_n	n-th Bernoulli number: second kind	10.2

Table 2: Specialized notation used

Symbol	Notation	See Section:
γ	sign oscillator	2.1
λ	sign type alternator	2.1
t_q	the smallest residue of $t \bmod q$	2.1
$\mathcal{F}_{s;ab}(r,t,q)$	generalized Fleck sum	2.2
$f_{ab}(r,t,q)$	generalized Fleck sum $s=0$ case	2.2
$F_{ab}(r,t,q)$	generalized Fleck sum $s=1$ case	2.2
$\mathcal{L}_{s;abc}(r,t,q)$	renumbered Fleck sum	3.1
$l_{abc}(r,t,q)$	renumbered Fleck sum $s=0$ case	3.1
$L_{abc}(r,t,q)$	renumbered Fleck sum $s=1$ case	3.1
$\mathcal{L}_s$	shift operator	3.1
ζ_Q, ζ	primitive Q-th root of unity	4.1
ω	primitive $4q$-th root of unity	4.3
$ord_p(F)$	highest exponent of p in F	4.5
$A_{2M+e}(x)$	generalized Fibonacci polynomial	5
$A_{2M+e}^r(x)$	amended ("square rooted") form of $A_{2M+e}(x)$	5
$\mathcal{A}_{s;ab}(x,Q)$	generalised Fibonacci polynomial	5.4
$\mathcal{A}_{s;ab}^r(x,Q)$	amended ("square rooted") form of $\mathcal{A}_{s;ab}(x,Q)$	5.4
$J_Q(x)$	Q-th Jacobsthal polynomial	5.2.1
$J_Q^{(2)}(x)$	Q-th Jacobsthal polynomial (Horadam)	5.2.1
$j_Q(x)$	Q-th Jacobsthal-Lucas polynomial	5.2.2
$j_Q^{(2)}(x)$	Q-th Jacobsthal-Lucas polynomial (Horadam)	5.2.2
$\mathcal{R}_{s;ab}(x,m)$	linear recurrence polynomial of function $\mathcal{L}_{s;abc}$	5.5
$\mathcal{G}$	generating function of following function	8.2
$\Delta_r(\vec{\mathbf{h}})$	MCL determinant (Lettington)	9.1
$\Psi_r(\vec{\mathbf{h}}, \vec{\mathbf{H}})$	half-weighted MCL determinant (Lettington)	9.1
$\Delta_r^\rho(\vec{\mathbf{a}_n})$	signed MCL determinant	9.1
$\Psi_r^\rho(\vec{\mathbf{a}_n}, \vec{\mathbf{A}_{N,0}})$	signed half-weighted MCL determinant	9.1
$P^\rho(r,T,N,n)$		9.2
$\mathcal{L}_{s;abc}^-(r,t,q)$	$\mathcal{L}_{s;abc}(-r,t,q)$	9.4.1
$\mathcal{L}_{s;abc}^{T-}(r,t,q)$	leading (truncated) coefficient of $\mathcal{L}_{s;abc}(-r,t,q)$	9.4.2
B_n^+	n-th Bernoulli number: first kind with $B_1 = 1/2$	10.1
$\mathcal{B}_n$	n-th "modified" Bernoulli number: first kind	10.1
$\mathcal{B}_n^+$	n-th "modified" Bernoulli number: first kind	10.1
b_n^-	n-th Bernoulli number: second kind	10.2

Chapter 1

Introduction

1.1 A brief history of the Fleck Congruence and associated sums

1.1.1 The Fleck congruence

The Fleck numbers are attributed to A. Fleck in 1913 [17], who showed (by utilising a primitive p-th root of unity, ϵ,) that for non-negative integer variables r (the term number of the sequence), t (the specific sequence or residue class), $p \geq 2$ (the prime modulus) and some integers a and b such that $a + b \equiv 0 \pmod{p}$, we have the congruence notation

$$\sum_{\substack{k=0 \\ k \equiv t \pmod{p}}}^{r} \binom{r}{k} a^{r-k} b^k = \frac{1}{p} \sum_{\epsilon} (a + b\epsilon)^r \epsilon^{p-t} \equiv 0 \pmod{p^\alpha}, \quad \text{where } \alpha = \left\lfloor \frac{r-1}{p-1} \right\rfloor.$$

$$(1.1.1)$$

This generalised a special case brought to his attention that for $a = -b = 1$, (1.1.1) simplifies to (what we denote as)

$$F(r, t, p) = \sum_{\substack{k \equiv t \pmod{p}}} (-1)^k \binom{r}{k} \equiv 0 \pmod{p^\alpha}, \quad \text{where } \alpha = \left\lfloor \frac{r-1}{p-1} \right\rfloor, \qquad (1.1.2)$$

the so called Fleck congruence.

In 1977 Weisman [44], independently derived (1.1.2) and extended the congruence relation to

$$F(r, t, p^e) = \sum_{\substack{k \equiv t \pmod{p^e}}} (-1)^k \binom{r}{k} \equiv 0 \pmod{p^\alpha}, \quad \text{where } \alpha = \left\lfloor \frac{r - p^{e-1}}{\phi(p^e)} \right\rfloor, \qquad (1.1.3)$$

where ϕ is Euler's totient function.

1.1.2 Fleck type sums

In 1992 Z.H. Sun [38] considered the nonalternating form of (1.1.2) to obtain the sum

$$T^r_{t(q)} = \sum_{\substack{k=0 \\ k\equiv t \pmod q}}^{r} \binom{r}{k} = \frac{2^r}{q} \sum_{d=0}^{q-1} \cos^r \frac{\pi d}{q} \cos \frac{\pi d(r-2t)}{q}$$

$$= \frac{2^r}{q} \left(1 + \sum_{d=1}^{q-1} \cos^r \frac{\pi d}{q} \cos \frac{\pi d(r-2t)}{q} \right). \qquad (1.1.4)$$

A decade later Z.W. Sun [39] considered both the nonalternating and an alternating sum of (1.1.2), notating them as

$$\begin{bmatrix} r \\ t \end{bmatrix}_q = \sum_{\substack{k=0 \\ k\equiv t \pmod q}}^{r} \binom{r}{k}, \quad \text{and} \quad \left\{ \begin{matrix} r \\ t \end{matrix} \right\}_q = \sum_{\substack{k=0 \\ k\equiv t \pmod q}}^{r} (-1)^{\frac{k-t}{q}} \binom{r}{k} \qquad (1.1.5)$$

respectively, and related by the identity

$$\begin{bmatrix} r \\ t \end{bmatrix}_q + \left\{ \begin{matrix} r \\ t \end{matrix} \right\}_q = 2 \begin{bmatrix} r \\ t \end{bmatrix}_{2q}.$$

The two forms of (1.1.5) can also be expressed by

$$\sum_{\substack{0\le k\le r \\ k\equiv t \pmod q}} \binom{r}{k} a^k = \sum_{k=0}^{r} \binom{r}{k} \frac{a^k}{q} \sum_{\zeta^q=1} \zeta^{k-t} = \frac{1}{q} \sum_{\zeta^q=1} \zeta^{-t} (1+a\zeta)^r, \qquad (1.1.6)$$

where $a = 1$ or $a = -1$ respectively, and $\zeta = e^{2\pi i/q}$ is a primitive q-th root of unity. However, we note that the second form needs to be multiplied by $(-1)^{\lfloor t/q \rfloor}$ to achieve this.

1.1.3 The renumbered Fleck sums.

Also considered in [38] was the amended sum

$$\Delta_q(t, R) = \begin{cases} qT^R_{R/2+t\,(q)} - 2^R & \text{if } 2 \nmid q \\ qT^R_{[R/2]+t\,(q)} - 2^R & \text{if } 2 \mid q. \end{cases} \qquad (1.1.7)$$

The function Δ_q eliminates the denominator q and the single term 2^R at $d = 0$; moreover, it realigns the residue class t.

If we let $q = 2m + b$ and $R = 2r + c$, where $b \in \{0, 1\}$ and $c \in \{0, 1\}$ represent the parity of q and R respectively, then (1.1.7) can be alternatively expressed as

$$\Delta_q(t, R) = qT^{2r+c}_{r+bc(m+1)+t\,(q)} - 2^{2r+c}$$

$$= \sum_{\substack{0\le k\le 2r+c \\ k\equiv r+bc(m+1)+t \pmod q}} q\binom{2r+c}{k} - 2^{2r+c}$$

$$= (-1)^{bc} 2^{2r+c+1} \sum_{d=1}^{\lfloor (q-1)/2 \rfloor} \cos \frac{\pi(2^b d - b)(2t - (1-b)c)}{q} \left(\cos \frac{\pi(2^b d - b)}{q} \right)^{2r+c}.$$

$$(1.1.8)$$

The cosine form given in (1.1.8) summarizes the three separate forms provided by Z.H. Sun.

In developing the roots of unity identity (1.1.6), Z.W. Sun [39] employed (in our own notation) the expression $2 + \zeta + \zeta^{-1}$, that acts as a shift (or renumbering) of the Fleck sums, such that the two forms of (1.1.5) become transformed to

$$\left[\begin{array}{c} 2r \\ t+r \end{array} \right]_q = \frac{1}{q} \sum_{\zeta^q=1} \zeta^t (2 + \zeta + \zeta^{-1})^r, \quad \text{and} \quad \left\{ \begin{array}{c} 2r \\ t+r \end{array} \right\}_q = \frac{1}{q} \sum_{\zeta^q=-1} \zeta^t (2 + \zeta + \zeta^{-1})^r \quad (1.1.9)$$

respectively. Also of relevance is an oscillation of the sign of the sum. This does not affect the absolute value of the sum, but if we consider the sequence of terms generated by either form of (1.1.9), we find that on varying r, (but fixing q and t), consecutive terms will oscillate in sign. The author details this as

$$\sum_{\zeta^q=\epsilon} \zeta^t (2 - \zeta - \zeta^{-1})^r = \sum_{\zeta^q=(-1)^q \epsilon} (-\zeta)^t (2 + \zeta + \zeta^{-1})^r = (-1)^{t+r} q \times \begin{cases} \left[\begin{array}{c} 2r \\ t+r \end{array} \right]_q & \text{if } \epsilon = (-1)^q \\ \left\{ \begin{array}{c} 2r \\ t+r \end{array} \right\}_q & \text{otherwise.} \end{cases}$$

A different way of perceiving the oscillation is by consideration of both forms of (1.1.5). If we vary t (and fix r and q), one notes that all the terms of the residue class t are multiplied by $(-1)^t$.

1.1.4 Recurrence relations satisfying the renumbered Fleck sums

Z.H. Sun [38] studying the even modulus $q = 2m$, established a recurrence polynomial, termed $Q_{m-1}(x)$, that recursively produce the values, $\Delta_q(k, 2r + c)$. This was expressed in the form

$$\sum_{k=0}^m (-1)^{m-k} \binom{m+1+k}{m-k} \Delta_{2m+2}(t, 2r + c + 2k) = 0, \qquad r \in \mathbb{N}_{\geq 0}, \quad c \in \{0, 1\}, \quad (1.1.10)$$

and illustrates the fact that there exist two separate recurrence sequences, determined by the parity of the term $R = 2r + c$. The roots of (1.1.10) were identified as

$$x = 2 + 2 \cos (2\pi d/q), \qquad 1 \leq d \leq m - 1,$$

making it apparent that the signed (binomial) coefficients of Q_n correspond to those of the monic Chebyshev polynomial of the second kind, $S_{2n+1}(x)$.

Also considered was the polynomial $G_m(x)$ defined as

$$G_m(x) = \prod_{d=1}^m \left(x + 2 \cos \frac{(2d-1)\pi}{2m+1} \right),$$

that satisfies the linear $m + 1$ term recurrence relation

$$\sum_{k=0}^m (-1)^{\left[\frac{m-k}{2} \right]} \binom{\left[\frac{m+k}{2} \right]}{k} \Delta_{2m+1}(k, r + k) = 0, \qquad (r = 0, 1, 2, \ldots).$$

We observe that the function Δ_{2m+1} is being considered as a single sequence (without consideration to the parity of r).

Z.W. Sun [39] employed the function, notated by D_R, and defined such that $D_0 = 2$, and thereafter

$$D_R(x) = \sum_{i=0}^{\lfloor R/2 \rfloor} (-1)^i \frac{R}{R-i} \binom{R-i}{i} x^{\lfloor R/2 \rfloor - i}, \qquad R \in \mathbb{N}.$$

For $R \geq 0$, the polynomial $D_R(x)$ is related to the Chebyshev polynomial of the first kind, $T_R(x)$, by the identity

$$2T_R(x) = (2x)^\epsilon D_R(4x^2), \qquad \text{where} \quad \epsilon = \begin{cases} 0 & \text{if } 2 \mid R \\ 1 & \text{if } 2 \nmid R, \end{cases}$$

and it forms a recurrence polynomial for the sums (1.1.9).

1.1.5 Connections between the Fleck sums and the Riemann zeta function

The types of binomial sum sequences detailed above, considered by Fleck, Weisman, Z.H. Sun and Z.W. Sun, are contained in more generality by the family of eight binomial sum sequences obtained from (1.1.1) by putting $a = 1$, $b = \pm 1$; r is either odd or even and the sum is taken over the congruence modulo n, where n is either odd or even.

The fundamental categorisations of these eight sequences are not immediately obvious and only become apparent when each sequence is renumbered using a diagonal approach similar to that of Z.H. Sun and Z.W. Sun. For example, the renumbered Fleck sequences, as considered by Lettington in [30] split into two binomial sum sequence categories:

$$F_1(r, t, 2m+1) = (2m+1) \sum_{k \equiv r+t \pmod{2m+1}} (-1)^k \binom{2r+1}{k}, \tag{1.1.11}$$

and

$$F_2(r, t, 2m+1) = (2m+1) \sum_{k \equiv r+t+1 \pmod{2m+1}} (-1)^k \binom{2r+2}{k}. \tag{1.1.12}$$

These two integer sequences each satisfy an $(m+1)$-th term recurrence relation and by construction satisfy Fleck's congruence when $p = 2m+1$ is a prime number.

Lettington went on to show that if the two sequences are run in reverse by using the same initial values but the reciprocal recurrence relation, then one has in effect two bi-infinite sequences where the negative term sequence values are rational numbers whose denominator prior to cancellation are powers of $(2m+1)$. These negative index sequence terms can be generated by determinants yielding polynomial expressions (see Chapter 6 of [30]) and Lettington demonstrated that the leading terms of the polynomial expressions yield the recurrence relation for the special values of the Riemann zeta function (i.e. at positive even integer arguments)

$$\zeta(2j) = (-1)^{j+1} \left(\frac{j\pi^{2j}}{(2j+1)!} + \sum_{k=1}^{j-1} \frac{(-1)^k \pi^{2j-2k}}{(2j-2k+1)!} \zeta(2k) \right), \text{ where } \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ for } \Re s > 1.$$

Variations of $\zeta(s)$ are given by

$$\lambda(s) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s}, \quad \beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \quad \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad \text{and} \quad \phi(s) = \sum_{n=1}^{\infty} \frac{1}{(2n)^s},$$

and these are respectively known as Dirichlet's lambda, beta, eta and phi function. In Lettington (2013) [31], recurrence relations for these functions were also given without reference to binomial sum sequences, where links to Toeplitz determinants were established. Such recurrence relations for the special values of these Riemann zeta type functions have recently been of considerable interest as illustrated in publications by Merca (2017) [33], Coffey (2018) [13], and Hu and Kim (2019) [25].

1.2   Overview of main results

Our main results are centred around a comprehensive classification of four families of binomial sum sequences and eight families of renumbered binomial sum sequences. The former sequences are considered in terms of various closed form expressions and divisibility properties; whereas the latter are expressed, in addition to equivalent closed form expressions, in terms of recurrence relations, generating functions, Toeplitz determinants, and recurrences for the Riemann zeta function at positive even integer arguments.

In Definition 2.2.1 we generalise the sum $F(r, t, p)$ given in (1.1.2) to that of

$$\mathcal{F}_{s;ab}(r, t, q) = (-1)^{st} \sum_{\substack{k=0 \\ k \equiv t \pmod{q}}}^{r} (-1)^{\frac{a(k-t)}{q}} \binom{r}{k}. \tag{1.2.1}$$

Here we introduce three parameters s, a, $b \in \{0, 1\}$, that respectively represent the sign "oscillator", the alternation and the base of the modulus. (We note that these parameters are not related to the integers a and b employed in (1.1.1)). The parameters a and b produce four different sequence families and the parameter s determines whether, for each family, the odd residue classes t are multiplied by -1 (or not). We also replace the prime p with a general positive integer $q = 2m + b \geq 1$, and we find it fruitful to consider each sequence term as a sum with modulus $2q$. Then from Theorem 2.2.5 we write (1.2.1) as

$$\mathcal{F}_{s;ab}(r, t, q) = \sum_{k \equiv t \pmod{2q}} (-1)^{sk} \binom{r}{k} + (-1)^{a+sb} \sum_{k \equiv t+q \pmod{2q}} (-1)^{sk} \binom{r}{k}. \tag{1.2.2}$$

In Section 2.3 we examine three term recurrence relations satisfied by $\mathcal{F}_{s;ab}$. Commencing with Lemma 2.3.1 that states

$$\mathcal{F}_{s;ab}(r+1, t, q) = \mathcal{F}_{s;ab}(r, t, q) + \gamma \mathcal{F}_{s;ab}(r, t-1, q),$$

where $\gamma = (-1)^s$, we establish in Lemma 2.3.2 a sum of products rule

$$\mathcal{F}_{s;ab}(r+k, t, q) = \sum_{j=0}^{q-1} \mathcal{F}_{s;ab}(k, j, q) \mathcal{F}_{s;ab}(r, t-j, q),$$

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that then yields in Lemma 2.3.3 the sums of squares relation

$$\mathcal{F}_{s;ab}(2r,r,q) = \gamma^r \sum_{j=0}^{q-1} \mathcal{F}_{s;ab}(r,j,q)^2. \tag{1.2.3}$$

Using the notation $\mathcal{L}_{s;abc}$, where the additional parameter $c \in \{0,1\}$ represents the corresponding row parity, we extend the forms (1.1.11) and (1.1.12) considered by Coffey et al. [9]. By a renumbering of (1.2.1) we obtain

$$\mathcal{L}_{s;abc}(r,t,q) = \mathcal{F}_{s;ab}(2r+2-c, t+r+1-c, q)$$

$$= \sum_{k \equiv T \pmod{2q}} (-1)^{sk} \binom{2r+2-c}{k} + (-1)^{a+sb} \sum_{k \equiv T+q \pmod{2q}} (-1)^{sk} \binom{2r+2-c}{k}, \tag{1.2.4}$$

where $r \geq 0$, $q = 2m+b \geq 1$, $c \leq t \leq m$ and $T = t+r+1-c$. The parameters a, b and c represent each of the eight family sequences and, following the renumbering of the terms $\mathcal{F}_{s;ab}(r,t,q)$ in (1.2.4), the parameter s, now determines whether for **each** residue class t, the sequence of numbers generated oscillate in sign (or not). In Theorem 3.2.1 we apply the relation (1.2.4) to equation (1.2.3) to obtain the sums of squares identity

$$\mathcal{L}_{s;ab1}(2r,1,q) = \frac{\gamma}{2} \sum_{j=0}^{q-1} \mathcal{L}_{s;ab1}(r,j,q)^2.$$

In Chapter 4, using the binomial sum forms (1.2.1) and (1.2.4), we develop, in terms of primitive $2q$-th roots of unity, alternative closed form expressions that are given respectively in Theorem 4.1.3 stating

$$\mathcal{F}_{s;ab}(r,t,q) = \frac{1}{q} \sum_{d=\epsilon}^{q+\epsilon-1} \left(\zeta^{2d-\epsilon}\right)^{-t} \left(1 + \gamma\zeta^{2d-\epsilon}\right)^r, \tag{1.2.5}$$

and Theorem 4.2.2 that states

$$\mathcal{L}_{s;abc}(r,t,q) = \mathcal{F}_{s;ab}(2r+2-c, t+r+1-c, q)$$

$$= \frac{1}{q} \sum_{d=\epsilon}^{q+\epsilon-1} \left(\zeta^{2d-\epsilon}\right)^{-(t+r+1-c)} \left(1 + \gamma\zeta^{2d-\epsilon}\right)^{2r+2-c}, \tag{1.2.6}$$

where $\epsilon \equiv a + sb \pmod 2$. Furthermore, by appropriate consideration of either the real or imaginary parts, we transform (1.2.5) and (1.2.6) into expressions involving cosines (or shifted cosines). These are given respectively by Theorem 4.3.3 that states

$$\mathcal{F}_{s;ab}(r,t,q) = \frac{\gamma^{\lfloor r/2 \rfloor} 2^r}{q} \sum_{d=\epsilon}^{q+\epsilon-1} \cos\left(\frac{\pi(r-2t)(2d-\epsilon-scq)}{2q}\right) \left(\cos\frac{\pi(2d-\epsilon-sq)}{2q}\right)^r,$$

and Theorem 4.4.2 stating

$$\mathcal{L}_{s;abc}(r,t,q) = \frac{\gamma^{r+1-c} 2^{2r+3-c}}{q} \times$$

$$\left(\frac{(a-1)st}{2} + \sum_{d=1}^{\lfloor (q+a-1)/2 \rfloor} \cos\left(\frac{\pi(c-2t)(2d-\epsilon-scq)}{2q}\right) \left(\cos\left(\frac{\pi(2d-\epsilon-sq)}{2q}\right)\right)^{2r+2-c}\right).$$

Using the closed forms developed in Theorems 2.2.5 and 4.1.3 we also establish
Theorem 4.5.10 which states that the only expected divisibility properties of the sums $\mathcal{F}_{s;ab}(r, t, q)$
are:

1.
$$ord_p(\mathcal{F}_{s;11}(r, t, p^e)) \geq \left\lfloor \frac{r - p^{e-1}}{p^{e-1}(p-1)} \right\rfloor,$$

and when $r = j2^e + l$, where $j \geq 1$ and $0 \leq l < 2^e$,

2.
$$ord_2(\mathcal{F}_{s;00}(r, t, 2^e)) \geq \left\lfloor \frac{r - 2^{e-1}}{2^{e-1}} \right\rfloor = \begin{cases} 2j - 1 & \text{if } l = 0 \\ 2j & \text{if } 1 \leq l \leq 2^e - 1, \end{cases}$$

and

3.
$$ord_2(\mathcal{F}_{s;10}(r, t, 2^e)) \geq j.$$

As indicated in Section 2.3 the Fibonacci type polynomials, (that also include the monic
Chebyshev polynomials), are intimately connected to the recurrence polynomials of the functions $\mathcal{L}_{s;abc}$. Denoting these recurrence polynomials as $\mathcal{R}_{s;ab}(x, q)$, in the Corollary to
Theorem 5.6.1 we demonstrate that

$$\mathcal{R}_{s;ab}(x, m) = \begin{cases} (\sqrt{x})^{b-1}(x - 4)S_{q-1}(\sqrt{x}) & \text{if } s = 0, \ a = 0 \\ (\sqrt{x})^{-b}C_q(\sqrt{x}) & \text{if } s = 0, \ a = 1 \\ \\ (\sqrt{x})^{b-1}(x + 4)F_q(\sqrt{x}) & \text{if } s = 1, \ a = 0 \\ (\sqrt{x})^{-b}L_q(\sqrt{x}) & \text{if } s = 1, \ a = 1. \end{cases}$$

Exploiting known properties of the Fibonacci type polynomials we develop theorems involving
second order differential equations and orthogonality relations. More specifically, from
Theorem 6.2.2 we have that the polynomials $\mathcal{R}_{s;1b}(u, m)$ satisfy the second order differential
equation

$$4(u - 4\gamma)u\mathcal{R}''_{s;1b}(u, m) + 4\left((b+1)u - 2\gamma(2b+1)\right)\mathcal{R}'_{s1b}(u, m) - (q^2 - b)\mathcal{R}_{s;1b}(u, m) = 0,$$

and Theorem 6.3.5, that the polynomials $\mathcal{R}_{s;0b}(u, m)$ satisfy the second order differential
equation

$$4u(u - 4\gamma)^2\mathcal{R}''_{s;0b}(u, m) + 4(u - 4\gamma)\left((1-b)u - 2\gamma(3 - 2b)\right)\mathcal{R}'_{s0b}(u, m)$$
$$- \left((q^2 - b)u - 4\gamma(q^2 + 2 - b)\right)\mathcal{R}_{s;0b}(u, m) = 0.$$

Next Theorem 7.1.7 states that the polynomials $\mathcal{R}_{s;ab}(u, m)$ form an orthogonal polynomial
sequence, with respect to the weight factor,

$$w_{s;ab}(u) = \frac{(\sqrt{\gamma u})^{\lambda(1-2b)}}{(u - 4\gamma)^{2+\lambda}}, \qquad \text{where} \qquad \lambda = (-1)^a,$$

that satisfies the integral equation

$$\int_0^{4\gamma} \mathcal{R}_{s;ab}(u,m)\mathcal{R}_{s;ab}(u,k)w_{s;ab}(u)\mathrm{d}u = \begin{cases} 0 & \text{if } m \neq k \\ -4\pi a\imath^{1-s} & \text{if } m = k \text{ and } q = 0 \\ 2\pi\lambda\imath^{1+\lambda s} & \text{if } m = k \text{ and } q \geq 1. \end{cases}$$

Furthermore, we are able to utilise these orthogonality relations to develop in Theorem 7.2.4 a three term intra sequence recurrence that takes the form

$$\mathcal{R}_{s;ab}(u,m+2) = (u-2\gamma)\mathcal{R}_{s;ab}(u,m+1) - \mathcal{R}_{s;ab}(u,m),$$

and in Theorem 7.2.5 a three term inter sequence recurrence of the form

$$\mathcal{R}_{s;ab}(u,m+1) = u^{|b-a|}\mathcal{R}_{s;ab'}(u,m+b) - \gamma\mathcal{R}_{s;ab}(u,m).$$

In Definitions 5.2.2 and 5.2.6 we introduce the Jacobsthal polynomial, $J_N(x)$, and the Jacobsthal-Lucas polynomial, $j_n(x)$ respectively. Employing these polynomials in conjunction with the use of hypergeometric functions, we present in Theorems 8.3.7 and 8.5.6 that, for $r \geq 0$, the generating functions for the renumbered sequences are given respectively as

$$\mathcal{GL}_{s;1bc}(x,t,q) = \begin{cases} \dfrac{2\gamma J_{2(m-1)+1+b}(-\gamma x)}{j_{2m+b}(-\gamma x)} & \text{if } c = t = 0 \\ \dfrac{x^{t-1}J_{2(m-1-t)+2}(-\gamma x)}{j_{2m}(-\gamma x)} & \text{if } b = c = 0 \text{ and } t \geq 1 \\ \dfrac{\gamma^c x^{t-1}J_{2(m-t)+b+c}(-\gamma x)}{j_{2m+b}(-\gamma x)} & \text{otherwise,} \end{cases}$$

and

$$\mathcal{GL}_{s;0bc}(x,t,q) = \begin{cases} \dfrac{2\gamma j_{2M+1-b}(-\gamma x)}{(1-4\gamma x)J_{2M+2-b}(-\gamma x)} & \text{if } c = t = 0 \\ \dfrac{\gamma^c x^{t-1}j_{2(m+bc-t)+b+c-2bc}(-\gamma x)}{(1-4\gamma x)J_{2M+2-b}(-\gamma x)} & \text{otherwise,} \end{cases}$$

where $c \leq t \leq M$ and $M = m - 1 + b$. Subsequently in Theorems 9.4.3 and 9.4.4, we develop equivalent generating functions for the sequences $\mathcal{L}_{s;abc}^-(r,t,q)$, where $\mathcal{L}_{s;abc}^-(r,t,q) = \mathcal{L}_{s;abc}(-r,t,q)$ and $r \geq 0$.

Utilising the generating function, we develop Theorems 9.3.1 and 9.3.2, which express the terms $\mathcal{L}_{s;abc}(r,t,q)$ as Toeplitz determinants. When we consider the sequences $\mathcal{L}_{s;abc}^-(r,t,q)$ and $q = 2m + b$ is unspecified, these determinants produce sequences of polynomials in m. In Theorem 9.4.8 and Theorem 9.4.9, for the particular sequences $t = 1$ and $t = m$, we truncate these polynomials, such that only the leading term remains. We denote these truncated sequences of polynomials by $\mathcal{L}_{s;abc}^{T-}(r,t,q)$. We then demonstrate that the sequences of $\mathcal{L}_{s;abc}^{T-}(r,t,q)$ are equal to (a fixed multiple of) a Dirichlet series of the form $\eta(2r)$, $\zeta(2r)$, $\lambda(2r)$ or $\beta(2r+1)$.

From this relationship, with $\eta(0) = 1/2$, $\zeta(0) = -1/2$, we are able in Theorem 9.5.15 to demonstrate the linear recurrence relations

$$\eta(2r) = -\sum_{k=0}^{r-1} \frac{(-1)^{r-k}\pi^{2(r-k)}}{(2r-2k+1)!}\eta(2k),$$

$$\zeta(2r) = -\sum_{k=0}^{r} \frac{(-1)^{r-k}\pi^{2(r-k)}}{(2r-2k)!}\eta(2k) = -2\sum_{k=0}^{r-1} \frac{(-1)^{r-k}\pi^{2(r-k)}(r-k)}{(2r-2k+1)!}\eta(2k),$$

and

$$\zeta(2r) = \frac{(-1)^{r+1}\pi^{2r}}{2(2r)!} - \sum_{k=0}^{r-1} \frac{(-1)^{r-k}\pi^{2(r-k)}}{(2r-2k+1)!}\zeta(2k).$$

Similarly with $\beta(1) = \pi/4$, $\lambda(0) = 0$, in Theorem 9.5.16 we demonstrate the linear recurrence relations

$$\beta(2r+1) = -\sum_{k=0}^{r-1} \frac{(-1)^{r-k}\pi^{2(r-k)}}{4^{r-k}(2r-2k)!}\beta(2k+1),$$

$$\lambda(2r) = \frac{\pi}{2}\sum_{k=0}^{r} \frac{(-1)^{r-k}\pi^{2(r-k)}}{4^{r-k}(2r-2k+1)!}\beta(2k+1) = -\frac{1}{\pi}\sum_{k=0}^{r-1} \frac{(-1)^{r-k}(r-k)\pi^{2(r-k)}}{4^{r-k-1}(2r-2k)!}\beta(2k+1),$$

and

$$\lambda(2r) = \frac{(-1)^{r-1}\pi^{2r}}{2.4^{r}(2r-1)!} - \sum_{k=1}^{r-1} \frac{(-1)^{r-k}\pi^{2(r-k)}}{4^{r-k}(2r-2k)!}\lambda(2k).$$

In the final chapter we apply our established methods to derive uncancelled denominator theorems for the Bernoulli numbers of the "modified" first ($\mathcal{B}_n = B_n/n!$) and second (b_n) kinds as detailed in Theorems 10.3.9 and 10.3.3 respectively. These both state (with one minor modification) that the exponent of each prime p occurring in the product of the denominator of the n-th Bernoulli number is that of the Fleck quotient, given as:

$$\prod_{p \leq n+1} p^{\lfloor \frac{n}{p-1} \rfloor}. \tag{1.2.7}$$

The minor modification concerns the exponent of the prime $p = 2$ occurring in the uncancelled denominator of the modified Bernoulli numbers of the first kind so that (1.2.7) becomes

$$2^{1-D} \prod_{p \leq n+1} p^{\lfloor \frac{n}{p-1} \rfloor}, \tag{1.2.8}$$

where D is the sum of the digits of $n = 2r$, (when $n \geq 2$), expressed in the scale of 2. Furthermore, due to the established result [26]

$$\zeta(2r) = \frac{(2\pi)^{2r}}{2(2r)!}|B_{2r}| \tag{1.2.9}$$

an immediate corollary of (1.2.8) concerns the uncancelled denominator of (1.2.9) that we express as

$$2^{2-D} \prod_{3 \leq p \leq 2r+1} p^{\lfloor \frac{2r}{p-1} \rfloor}.$$

This gives some understanding of the connection that exists between the Fleck quotient and the Riemann zeta function at integer arguments.

1.3 Organisation of the thesis

In Chapter 2 we consider the function $\mathcal{F}_{s;ab}$ that expresses eight variations of the closed form binomial sums under investigation. We also develop sum of squares relations satisfied by these functions.

This then enables us in Chapter 3 to create the "renumbering" function $\mathcal{L}_{s;abc}$ (with sixteen different forms) in terms of closed form binomial sums. Sums of squares relations that satisfy these functions are similarly established.

In Chapter 4 we express the sums $\mathcal{F}_{s;ab}(r,t,q)$ and $\mathcal{L}_{s;abc}(r,t,q)$ in terms of closed form primitive $2q$-th roots of unity and cosine forms. Expected divisibility properties for the sums $\mathcal{F}_{s;ab}(r,t,q)$ are also established.

Chapter 5 opens with a discussion of the Fibonacci type (that is Fibonacci, Lucas and monic Chebyshev of the first and second kind) polynomials. From these polynomials we develop recurrence relation polynomials, notated $\mathcal{R}_{s;ab}(x,q)$, for the respective functions $\mathcal{L}_{s;abc}$. These polynomials are expressed both in sum and product form.

Chapters 6 and 7 explore properties of the recurrence polynomials, $\mathcal{R}_{s;ab}(x,q)$. The former determines second order differential equations satisified by the sequences of these polynomials; whilst the latter examines orthogonal relations of the sequences. From the orthogonality properties we establish two types of three term recurrence relations.

In Chapters 8 and 9 we return to examining methods of producing the terms $\mathcal{L}_{s;abc}(r,t,q)$. The former chapter determines the generating function for each of these terms, whilst the latter employs Toeplitz determinants that (as in [30] and [31]) will be referred to as minor corner layered (MCL) determinants. We determine the sequences $\mathcal{L}_{s;abc}^{-}(r,t,q) = \mathcal{L}_{s;abc}(-r,t,q)$ and subsequently those of $\mathcal{L}_{s;abc}^{T-}(r,t,q)$. Methods are established that connect these latter sequences to Dirichlet functions and consequently to the development of linear recurrence relations involving $\eta(2r)$, $\zeta(2r)$, $\beta(2r+1)$ and $\lambda(2r)$.

Finally in Chapter 10 we study the (modified) Bernoulli numbers of the first and second kind, and in addition to examining various ways of producing these numbers, we also investigate their uncancelled denominators.

Chapter 2

Types of sums

We now introduce some notation that will be employed throughout this thesis. This is followed in Section 2.2 by the definition of a function $\mathcal{F}_{s;ab}$ to encompass eight types of sums derived from (1.1.2). Also in this section, (Subsection 2.2.5), we demonstrate the equivalence of the functions $\mathcal{F}_{s;ab}$ and $\mathcal{H}_{s;ab}$. This latter function offers an alternative way of presenting the binomial sums that will be of great value in subsequent chapters. Finally in Section 2.3 we develop various recursive relations including in Lemma 2.3.3 an elegant sums of squares relation.

2.1   Notation.

To aid us in our deliberation we are motivated to develop some notation. We consider a general function $\mathcal{D}_{s;ab}$ or $\mathcal{D}_{s;abc}$ according to the parameters s, a, b, $c \in \{0,1\}$.

1. The oscillation of the sign.
$$\gamma = (-1)^s.$$

2(i). The alternation (sum type).
$$\lambda = (-1)^a.$$

2(ii). The base of the modulus q (modulo 2).
$$q \equiv b \pmod 2.$$

2(iii). The corresponding row of Pascal's triangle (row parity).

$$\text{row parity} \equiv c \pmod 2 \tag{2.1.1}$$

Moreover, we denote the specific cases $s = 0$ and $s = 1$, by

$$\mathcal{D}_{s;ab} = \begin{cases} d_{ab} & \text{if } s = 0 \\ D_{ab} & \text{if } s = 1. \end{cases} \tag{2.1.2}$$

For non-negative integer t, we employ the following definition.

Definition 2.1.1. *Let t_q be the standard residue of t (mod q), with $0 \le t_q \le q - 1$, and $t \equiv t_q$ (mod q).*

t_q will be employed throughout this work when it is required to distinguish between the integer t and its residue, t_q modulo q.

2.2   The generalised Fleck function.

We wish to generalise the Fleck function according to the parameters s, a, $b \in \{0, 1\}$.

Definition 2.2.1. *We define the generalised Fleck function, $\mathcal{F}_{s;ab}$, with non-negative integer variables r (the term number of the sequence), t (the specific sequence or residue class) and $q \ge 1$ (the modulus) as*

$$\mathcal{F}_{s;ab}(r, t, q) = \gamma^t \sum_{\substack{k=0 \\ k \equiv t \pmod{q}}}^{r} \lambda^{\frac{k-t}{q}} \binom{r}{k} = \gamma^t \lambda^{\lfloor t/q \rfloor} \sum_{d=0}^{\lfloor (r - t_q)/q \rfloor} \lambda^d \binom{r}{t_q + dq}$$

where $\gamma = (-1)^s$ and $\lambda = (-1)^a$.

It will be shown in Section 2.2.4 that fixing q provides us with $2q$ sequences (residue classes); then if we fix the residue class t, we find that for each $r \ge 0$, the sum $\mathcal{F}_{s;ab}(r, t, q)$ provides us with a term of that particular sequence. For fixed variables r, t and q we can affect this sum by changing either the parameter s or a. We also find that b, the parity of q, has an impact on the nature of the sum.

2.2.1   The sign parameter, s.

In (2.2.2) the influence of the parameter s is exercised by the term γ and determines whether the sums (terms) comprising a sequence are negated (for odd t) or not. Denoting the specific cases of the function $\mathcal{F}_{s;ab}$ as

$$\mathcal{F}_{s;ab} = \begin{cases} f_{ab} & \text{if } s = 0 \\ F_{ab} & \text{if } s = 1, \end{cases}$$

we then have

$$f_{ab}(r, t, q) = \sum_{\substack{k=0 \\ k \equiv t \pmod{q}}}^{r} \lambda^{\frac{k-t}{q}} \binom{r}{k} = \lambda^{\lfloor t/q \rfloor} \sum_{d=0}^{\lfloor (r - t_q)/q \rfloor} \lambda^d \binom{r}{t_q + dq}, \tag{2.2.1}$$

and

$$F_{ab}(r, t, q) = (-1)^t \sum_{\substack{k=0 \\ k \equiv t \pmod{q}}}^{r} \lambda^{\frac{k-t}{q}} \binom{r}{k} = (-1)^t \lambda^{\lfloor t/q \rfloor} \sum_{d=0}^{\lfloor (r - t_q)/q \rfloor} \lambda^d \binom{r}{t_q + dq}. \tag{2.2.2}$$

2.2.2 The alternating parameter, a

In (2.2.1) this parameter is controlled by the term λ and determines whether consecutive terms of the sum alternate in sign or not.

Nonalternating sums, $\mathcal{F}_{s;0b}(r, t, q)$.

Definition 2.2.2. *We define the nonalternating sum, $\mathcal{F}_{s;0b}(r, t, q)$, with integers $r \geq 0$ and $q \geq 1$ as*

$$\mathcal{F}_{s;0b}(r, t, q) = \gamma^t \sum_{\substack{k=0 \\ k \equiv t \pmod{q}}}^{r} \binom{r}{k} = \gamma^t \sum_{d=0}^{\lfloor (r-t_q)/q \rfloor} \binom{r}{t_q + dq}. \tag{2.2.3}$$

Alternating sums, $\mathcal{F}_{s;1b}(r, t, q)$.

Definition 2.2.3. *We define the alternating sum, $\mathcal{F}_{s;1b}(r, t, q)$, with integers $r \geq 0$ and $q \geq 1$ as*

$$\mathcal{F}_{s;1b}(r, t, q) = \gamma^t \sum_{\substack{k=0 \\ k \equiv t \pmod{q}}}^{r} (-1)^{\frac{k-t}{q}} \binom{r}{k} = \gamma^t (-1)^{\lfloor t/q \rfloor} \sum_{d=0}^{\lfloor (r-t_q)/q \rfloor} (-1)^d \binom{r}{t_q + dq}. \tag{2.2.4}$$

2.2.3 The effect of the parity of the modulus, q

This will become more apparent in Subsection 2.2.5. Presently we will limit ourselves to an illustration of the effect of the parity of the modulus, $q = 2m + b$ on the sum $\mathcal{F}_{s;ab}(r, t, q)$ by consideration of the Fleck sum that we recall from (1.1.2) is

$$F(r, t, 2m + b) = \sum_{k \equiv t \pmod{q}} (-1)^k \binom{r}{k}.$$

It is straight forward to verify that for q even we have

$$F(r, t, 2m) = \sum_{k \equiv t \pmod{q}} (-1)^k \binom{r}{k} = \sum_{k \equiv t \pmod{q}} (-1)^t \binom{r}{k} = (-1)^t \sum_{k \equiv t \pmod{q}} \binom{r}{k}$$

$$= \mathcal{F}_{1;00}(r, t, 2m) = \mathcal{F}_{1;00}(r, t_q, q), \tag{2.2.5}$$

and for q odd,

$$F(r, t, 2m + 1) = \sum_{k \equiv t \pmod{2m+1}} (-1)^k \binom{r}{k} = \sum_{k \equiv t \pmod{q}} (-1)^{t + \frac{k-t}{q}} \binom{r}{k}$$

$$= (-1)^t \sum_{k \equiv t \pmod{q}} (-1)^{\frac{k-t}{q}} \binom{r}{k}$$

$$= \mathcal{F}_{1;11}(r, t, 2m + 1). \tag{2.2.6}$$

In other words, for sign parameter $s = 1$, the Fleck function yields a *non-alternating* sum with an *even* modulus and an *alternating* sum with an *odd* modulus. To verify (2.2.6) we first require a lemma.

LEMMA 2.2.1 (alternating sign for odd modulus). *For positive integers k and t, with $k \equiv t$ (mod q), and q some positive odd integer, we have*

$$k \equiv \frac{k-t}{q} + t \pmod 2. \tag{2.2.7}$$

Proof. We note firstly that $(k-t)/q$ is an integer.

Then we observe that for $q \equiv 1 \pmod 2$ we have

$$k \equiv \begin{cases} \frac{k-t}{q} \pmod 2 & \text{if } t \equiv 0 \pmod 2 \\ \frac{k-t}{q} + 1 \pmod 2 & \text{if } t \equiv 1 \pmod 2, \end{cases}$$

and so conclude that the equivalence of (2.2.7) holds. $\square$

2.2.4 The requirement of $2q$ residue classes

It is not immediately obvious that the generalised Fleck function $\mathcal{F}_{s;ab}$ possesses $2q$ residue classses. However, we note that the exponent $\lfloor t/q \rfloor$ in (2.2.4) is the correction between t and t_q and this alternates between $0 \pmod 2$ and $1 \pmod 2$ with period $2q$. Therefore, we have

$$\mathcal{F}_{s;1b}(r,t,q) = \begin{cases} \mathcal{F}_{s;1b}(r,t_q,q) & \text{if } \lfloor t/q \rfloor \equiv 0 \pmod 2 \\ -\mathcal{F}_{s;1b}(r,t_q,q) & \text{if } \lfloor t/q \rfloor \equiv 1 \pmod 2. \end{cases} \tag{2.2.8}$$

2.2.5 Use of a modulus function of $2q$ to determine the sums $\mathcal{F}_{s;ab}(r,t,q)$.

Prompted by a suggestion of M.N. Huxley, the author appreciated that a more elegant way of presenting the function $\mathcal{F}_{s;ab}$, so that the effects of each of three parameters and the requirement of $2q$ residue classes are clearer, is by the development of a function operating over a modulus of $2q$. For parameters s, a and b defined as previously, and for non-negative integers r, t and $q \geq 1$, let us consider the sum

$$\mathcal{H}_{s;ab}(r,t,q) = \sum_{k \equiv t \pmod{2q}} (-1)^{sk} \binom{r}{k} + (-1)^{a+sb} \sum_{k \equiv t+q \pmod{2q}} (-1)^{sk} \binom{r}{k} \tag{2.2.9}$$

that can also be written as

$$\mathcal{H}_{s;ab}(r,t,q) = \sum_{k \equiv t \pmod{2q}} \gamma^k \binom{r}{k} + \lambda \gamma^b \sum_{k \equiv t+q \pmod{2q}} \gamma^k \binom{r}{k}, \tag{2.2.10}$$

where $\gamma = (-1)^s$ and $\lambda = (-1)^a$. Furthermore, we also write

$$\mathcal{H}_{s;ab}(r,t,q) = \begin{cases} h_{ab}(r,t,q) & \text{if } s = 0 \\ H_{ab}(r,t,q) & \text{if } s = 1. \end{cases}$$

We claim that $\mathcal{F}_{s;ab}(r,t,q) = \mathcal{H}_{s;ab}(r,t,q)$. Let us employ some lemmas.

LEMMA 2.2.2 (case $a = 0$). *With $s = a = 0$ in (2.2.9) we have that*

$$f_{0b}(r,t,q) = h_{0b}(r,t,q) = \sum_{k \equiv t \pmod{2q}} \binom{r}{k} + \sum_{k \equiv t+q \pmod{2q}} \binom{r}{k}.$$

Proof. From (2.2.3) we have

$$f_{0b}(r,t,q) = \sum_{\substack{k=0 \\ k \equiv t \pmod{q}}}^{r} \binom{r}{k} = \sum_{\substack{k=0 \\ k \equiv t \pmod{2q}}}^{r} \left[\binom{r}{k} + \binom{r}{k+q} \right] = \sum_{\substack{k=0 \\ k \equiv t \pmod{2q}}}^{r} \binom{r}{k} + \sum_{\substack{k=0 \\ k \equiv t+q \pmod{2q}}}^{r} \binom{r}{k}.$$

$\square$

LEMMA 2.2.3 (case $a = 1$). *With $s = 0$ and $a = 1$ in (2.2.9), we have that*

$$f_{1b}(r,t,q) = h_{1b}(r,t,q) = \sum_{k \equiv t \pmod{2q}} \binom{r}{k} - \sum_{k \equiv t+q \pmod{2q}} \binom{r}{k}.$$

Proof. From (2.2.8) we have

$$f_{1b}(r,t,q) = \sum_{\substack{k=0 \\ k \equiv t \pmod{q}}}^{r} (-1)^{\frac{k-t}{q}} \binom{r}{k} = \sum_{\substack{k=0 \\ k \equiv t \pmod{2q}}}^{r} \left[\binom{r}{k} - \binom{r}{k+q} \right]$$

$$= \sum_{\substack{k=0 \\ k \equiv t \pmod{2q}}}^{r} \binom{r}{k} - \sum_{\substack{k=0 \\ k \equiv t+q \pmod{2q}}}^{r} \binom{r}{k}.$$

$\square$

LEMMA 2.2.4 (case $s = 1$). *With $s = 1$ in (2.2.9) we have that*

$$F_{ab}(r,t,q) = H_{ab}(r,t,q)$$

$$= \sum_{k \equiv t \pmod{2q}} (-1)^k \binom{r}{k} + (-1)^{a+b} \sum_{k \equiv t+q \pmod{2q}} (-1)^k \binom{r}{k}.$$

Proof. Using Lemmata 2.2.2 and 2.2.3 we have

$$F_{ab}(r,t,q) = (-1)^t f_{ab} = (-1)^t h_{ab}$$

$$= (-1)^t \left(\sum_{\substack{k=0 \\ k \equiv t \pmod{2q}}}^{r} \binom{r}{k} + (-1)^a \sum_{\substack{k=0 \\ k \equiv t+q \pmod{2q}}}^{r} \binom{r}{k} \right)$$

$$= \sum_{\substack{k=0 \\ k \equiv t \pmod{2q}}}^{r} (-1)^t \binom{r}{k} + (-1)^a \sum_{\substack{k=0 \\ k \equiv t+q \pmod{2q}}}^{r} (-1)^t \binom{r}{k}$$

$$= \sum_{\substack{k=0 \\ k \equiv t \pmod{2q}}}^{r} (-1)^k \binom{r}{k} + (-1)^{a+b} \sum_{\substack{k=0 \\ k \equiv t+q \pmod{2q}}}^{r} (-1)^k \binom{r}{k} = H_{ab}(r,t,q).$$

$\square$

We are now in a position to fulfil our claim on the equivalence of these two functions.

THEOREM 2.2.5 (equivalence of $\mathcal{F}_{s;ab}$ and $\mathcal{H}_{s;ab}$). *With $\mathcal{F}_{s;ab}$ and $\mathcal{H}_{s;ab}$ defined as in* (2.1.2) *and* (2.2.9) *respectively, we have that*

$$\mathcal{F}_{s;ab}(r,t,q) = \mathcal{H}_{s;ab}(r,t,q) = \sum_{k\equiv t \pmod{2q}} (-1)^{sk}\binom{r}{k} + (-1)^{a+sb} \sum_{k\equiv t+q \pmod{2q}} (-1)^{sk}\binom{r}{k}.$$

Proof. We have from Lemmata 2.2.2 and 2.2.3 that

$$f_{ab}(r,t,q) = h_{ab}(r,t,q) = \sum_{k\equiv t \pmod{2q}} \binom{r}{k} + (-1)^a \sum_{k\equiv t+q \pmod{2q}} \binom{r}{k}, \qquad (2.2.11)$$

and from Lemma 2.2.4 that

$$F_{ab}(r,t,q) = H_{ab}(r,t,q) = \sum_{k\equiv t \pmod{2q}} (-1)^k \binom{r}{k} + (-1)^{a+b} \sum_{k\equiv t+q \pmod{2q}} (-1)^k \binom{r}{k},$$
$$(2.2.12)$$

and so on combining (2.2.11) and (2.2.12) the result is established. $\qquad\square$

For clarity we will maintain the use of the form $\mathcal{F}_{s;ab}$.

Remark 1. We note in (2.2.11) that when $k = t$ the binomial term is positive and furthermore, since the modulus is $2q$ every k congruent to t is an even multiple of t. Conversely when $k = t + q$ is negative then similarly every k congruent to $t + q$ will also be negative.

Remark 2. The difference between equations (2.2.11) and (2.2.12) is the inclusion of the $(-1)^k$ terms. However, there is a second more subtle difference and that is the requirement of the $(-1)^b$ term to neutralise an added $(-1)^q$ term in the second sum term of (2.2.12). Therefore, when expressing the function F_{ab} (relative to the function f_{ab}) this sign creation will have no effect when the modulus is even but will impact on the function when the modulus is odd.

For examples of the values generated by each of the different functions, f_{ab} and F_{ab}, with $6 \le q \le 7$, the reader is referred to Appendix $E.1$.

2.3 Recurrence relations

2.3.1 Three term recurrence relation

From the above definition we obtain the following three term binomial recurrence relation.

LEMMA 2.3.1 (three term relation). *With $\mathcal{F}_{s;ab}$ defined as in Definition* 2.2.1, *we have*

$$\mathcal{F}_{s;ab}(r+1,t,q) = \mathcal{F}_{s;ab}(r,t,q) + \gamma \mathcal{F}_{s;ab}(r,t-1,q),$$

Proof. From Theorem 2.2.5 we have

$$\mathcal{F}_{s;ab}(r,t,q) = \sum_{k\equiv t \pmod{2q}} (-1)^{sk}\binom{r}{k} + (-1)^{a+sb}\sum_{k\equiv t+q \pmod{2q}} (-1)^{sk}\binom{r}{k}.$$

Using the three term binomial relation, known as Pascal's identity,

$$\binom{r+1}{t} = \binom{r}{t} + \binom{r}{t-1}$$

we have

$$\sum_{k\equiv t \pmod{2q}} (-1)^{sk}\binom{r}{k} + (-1)^{a+sb}\sum_{k\equiv t+q \pmod{2q}} (-1)^{sk}\binom{r}{k}$$

$$+(-1)^s\left(\sum_{k\equiv t-1 \pmod{2q}} (-1)^{sk}\binom{r}{k} + (-1)^{a+sb}\sum_{k\equiv t-1+q \pmod{2q}} (-1)^{sk}\binom{r}{k}\right)$$

$$= \sum_{k\equiv t \pmod{2q}} (-1)^{sk}\left(\binom{r}{k}+\binom{r}{k-1}\right) + (-1)^{a+sb}\sum_{k\equiv t+q \pmod{2q}} (-1)^{sk}\left(\binom{r}{k}+\binom{r}{k-1}\right)$$

$$= \sum_{k\equiv t \pmod{2q}} (-1)^{sk}\binom{r+1}{k} + (-1)^{a+sb}\sum_{k\equiv t+q \pmod{2q}} (-1)^{sk}\binom{r+1}{k}$$

$$= \sum_{k\equiv t \pmod{2q}} \gamma^k\binom{r+1}{k} + \lambda\gamma^b\sum_{k\equiv t+q \pmod{2q}} \gamma^k\binom{r+1}{k}$$

$$= \mathcal{F}_{s;ab}(r+1,t,q).$$

□

We illustrate the function $\mathcal{F}_{s;ab}$ and Lemma 2.3.1 with two examples. Example 2 also demonstrates a divisibility property that will be explained in Section 4.5.

Example 1.

$$f_{01}(25,9,5) = \left[\binom{25}{4}+\binom{25}{9}+\binom{25}{14}+\binom{25}{19}+\binom{25}{24}\right] \qquad = 6690150,$$

$$f_{01}(24,9,5) = \left[\binom{24}{4}+\binom{24}{9}+\binom{24}{14}+\binom{24}{19}+\binom{24}{24}\right] \qquad = 3321891,$$

$$f_{01}(24,8,5) = \left[\binom{24}{3}+\binom{24}{8}+\binom{24}{13}+\binom{24}{18}+\binom{24}{23}\right] \qquad = 3368259.$$

Example 2.

$$F_{11}(25,9,5) = (-1)^9\left[-\binom{25}{4}+\binom{25}{9}-\binom{25}{14}+\binom{25}{19}-\binom{25}{24}\right] \qquad = 2250000 = 5^6\times 144,$$

$$F_{11}(24,9,5) = (-1)^9\left[-\binom{24}{4}+\binom{24}{9}-\binom{24}{14}+\binom{24}{19}-\binom{24}{24}\right] \qquad = 621875 = 5^5\times 199,$$

$$F_{11}(24,8,5) = (-1)^8\left[-\binom{24}{3}+\binom{24}{8}-\binom{24}{13}+\binom{24}{18}-\binom{24}{23}\right] \qquad = -1628125 = -5^5\times 521.$$

If we are given the rth term of each of the sequences $\mathcal{F}_{s;ab}(r, t, q)$ with $0 \leq t \leq q - 1$, then we are not limited to establishing the term $\mathcal{F}_{s;ab}(r + 1, t, q)$, as in Lemma 2.3.1. We show this by the next lemma.

LEMMA 2.3.2 (sums of products relation). *We have*

$$\mathcal{F}_{s;ab}(r + k, t, q) = \sum_{j=0}^{q-1} \mathcal{F}_{s;ab}(k, j, q)\mathcal{F}_{s;ab}(r, t - j, q).$$

Proof. From (repeated application of) Lemma 2.3.1 we can write

$$\mathcal{F}_{s;ab}(r + k, t, q) = \sum_{i=0}^{k} \gamma^i \binom{k}{i} \mathcal{F}_{s;ab}(r, t - i, q). \tag{2.3.1}$$

Expanding the second member of (2.3.1) we have

$$\mathcal{F}_{s;ab}(r + k, t, q) =$$
$$\mathcal{F}_{s;ab}(r, t, q) \left[\gamma^0 \binom{k}{0} + \gamma^{2q} \binom{k}{2q} + \gamma^{4q} \binom{k}{4q} + \ldots + \gamma^{2ql_0} \binom{k}{2ql_0} \right]$$
$$+ \mathcal{F}_{s;ab}(r, t - 1, q) \left[\gamma^1 \binom{k}{1} + \gamma^{2q+1} \binom{k}{2q + 1} + \gamma^{4q+1} \binom{k}{4q + 1} + \ldots + \gamma^{2ql_1+1} \binom{k}{2ql_1 + 1} \right]$$
$$+ \mathcal{F}_{s;ab}(r, t - 2, q) \left[\gamma^2 \binom{k}{2} + \gamma^{2q+2} \binom{k}{2q + 2} + \gamma^{4q+2} \binom{k}{4q + 2} + \ldots + \gamma^{2ql_2+2} \binom{k}{2ql_2 + 2} \right]$$
$$\vdots$$
$$+ \mathcal{F}_{s;ab}(r, t - q, q) \left[\gamma^q \binom{k}{q} + \gamma^{3q} \binom{k}{3q} + \gamma^{5q} \binom{k}{5q} + \ldots + \gamma^{2ql_q+q} \binom{k}{2ql_q + q} \right]$$
$$\vdots$$
$$+ \mathcal{F}_{s;ab}(r, t - 2q + 1, q) \left[\gamma^{2q-1} \binom{k}{2q - 1} + \ldots + \gamma^{2q(l_{2q-1}+1)-1} \binom{k}{2q(l_{2q-1} + 1) - 1} \right], \tag{2.3.2}$$

where $2ql_j + j \leq k$, $0 \leq j \leq 2q - 1$ and $l_j \in \mathbb{N}_{\geq 0}$.

Now pairing the terms $\mathcal{F}_{s;ab}(r, t - j, q)$ and $\mathcal{F}_{s;ab}(r, t - j - q, q)$, where $0 \leq j \leq q - 1$, and using Theorem 2.2.5 that demonstrates that $\mathcal{F}_{s;ab}(r, t, q) = (-1)^{a+sb}\mathcal{F}_{s;ab}(r, t - q, q)$, we can

express (2.3.2) as

$$\mathcal{F}_{s;ab}(r+k,t,q) =$$

$$\mathcal{F}_{s;ab}(r,t,q) \left[\sum_{i \equiv 0 \pmod{2q}} \gamma^i \binom{k}{i} + (-1)^{a+sb} \sum_{i \equiv q \pmod{2q}} \gamma^i \binom{k}{i} \right]$$

$$+ \mathcal{F}_{s;ab}(r,t-1,q) \left[\sum_{i \equiv 1 \pmod{2q}} \gamma^i \binom{k}{i} + (-1)^{a+sb} \sum_{i \equiv 1+q \pmod{2q}} \gamma^i \binom{k}{i} \right]$$

$$+ \mathcal{F}_{s;ab}(r,t-2,q) \left[\sum_{i \equiv 2 \pmod{2q}} \gamma^i \binom{k}{i} + (-1)^{a+sb} \sum_{i \equiv 2+q \pmod{2q}} \gamma^i \binom{k}{i} \right]$$

$$\vdots$$

$$+ \mathcal{F}_{s;ab}(r,t-q+1,q) \left[\sum_{i \equiv q-1 \pmod{2q}} \gamma^i \binom{k}{i} + (-1)^{a+sb} \sum_{i \equiv 2q-1 \pmod{2q}} \gamma^i \binom{k}{i} \right] \quad (2.3.3)$$

In turn we can now simplify (2.3.3) to

$$\mathcal{F}_{s;ab}(r+k,t,q) = \sum_{j=0}^{q-1} \mathcal{F}_{s;ab}(r,t-j,q) \left(\sum_{i \equiv j \pmod{2q}} \gamma^i \binom{k}{i} + (-1)^{a+sb} \sum_{i \equiv j+q \pmod{2q}} \gamma^i \binom{k}{i} \right)$$

$$= \sum_{j=0}^{q-1} \mathcal{F}_{s;ab}(k,j,q) \mathcal{F}_{s;ab}(r,t-j,q).$$

$$\square$$

Using the symmetry of the binomial coefficients of Pascal's triangle and more generally that $f_{ab}(N, N-t, q) = f_{ab}(N, t, q)$, an application to Lemma 2.3.2 is a sums of squares relation given by the next two lemmas.

LEMMA 2.3.3 (duplication of squares). *When $k = r$ and $t_{2q} = r$, we have*

$$\mathcal{F}_{s;ab}(2r, r, q) = \gamma^r \sum_{j=0}^{q-1} \mathcal{F}_{s;ab}(r, j, q)^2.$$

Proof. Considering the two cases of the parameter s separately, we substitute $k = r$ and $t_{2q} = r$ into Lemma 2.3.2. Then when $s = 0$, we obtain

$$f_{ab}(2r, r, q) = \sum_{j=0}^{q-1} f_{ab}(r, j, q) f_{ab}(r, r-j, q) = \sum_{j=0}^{q-1} f_{ab}(r, j, q) f_{ab}(r, j, q) = \sum_{j=0}^{q-1} f_{ab}(r, j, q)^2,$$

and when $s = 1$, we have

$$F_{ab}(2r, r, q) = \sum_{j=0}^{q-1} F_{ab}(r, j, q) F_{ab}(r, r-j, q) = \sum_{j=0}^{q-1} F_{ab}(r, j, q) (-1)^{r-2j} F_{ab}(r, j, q)$$

$$= (-1)^r \sum_{j=0}^{q-1} F_{ab}(r, j, q)^2 = \gamma^r \sum_{j=0}^{q-1} F_{ab}(r, j, q)^2.$$

□

Example. When the parameters $s = a = b = 1$, and the variables $r = 8$ and $q = 5$, we have

$$\mathcal{F}_{1;11}(16, 8, 5) = (-1)^8 \sum_{j=0}^{4} \mathcal{F}_{1;11}(8, j, 5)^2$$

$$= F_{11}(8, 0, 5)^2 + F_{11}(8, 1, 5)^2 + F_{11}(8, 2, 5)^2 + F_{11}(8, 3, 5)^2 + F_{11}(8, 4, 5)^2$$

$$= 3025 + 400 + 400 + 3025 + 4900 = 11,750.$$

A slight variation of Lemma 2.3.3 occurs on replacing the residue class r with $r - q$. We state this as a corollary.

COROLLARY. *We have*

$$\mathcal{F}_{s;ab}(2r, r - q, q) = \gamma^r \lambda \sum_{j=0}^{q-1} \mathcal{F}_{s;ab}(r, j, q)^2.$$

Proof. On replacing the residue class r with $r - q$ in Lemma 2.3.3, we obtain

$$f_{ab}(r, r - q - j, q) = f_{ab}(r, j - q, q) = f_{ab}(r, j + q, q) = (-1)^a f_{ab}(r, j_{2q}, q) = \lambda f_{ab}(r, j_{2q}, q).$$

□

Example. When the parameters $s = b = 0$ and $a = 1$, and the variables $r = 9$ and $q = 6$, we have

$$\mathcal{F}_{0;10}(18, 3, 6) = (-1) \sum_{j=0}^{5} \mathcal{F}_{0;10}(9, j, 6)^2$$

$$= -\left(f_{10}(9, 0, 6)^2 + f_{10}(9, 1, 6)^2 + f_{10}(9, 2, 6)^2 + f_{10}(9, 3, 6)^2 + f_{10}(9, 3, 6)^2 + f_{10}(9, 5, 6)^2\right)$$

$$= -(6889 + 729 + 729 + 6889 + 15876 + 15876) = -46,988.$$

Chapter 3

The function $\mathcal{L}_{s;abc}$

The works of [30], [9] and [39] examined sequences created by a diagonal renumbering of particular cases of the sequences $\mathcal{F}_{s;ab}$. This chapter considers all possible cases, and these sequences are notated as $\mathcal{L}_{s;abc}$. In Section 3.1, we define (in Definition 3.1.2) the function $\mathcal{L}_{s;abc}$ in terms of a shift of $\mathcal{F}_{s;ab}$ and with Equation (3.1.5) provide a generalised binomial representation of the sum $\mathcal{L}_{s;abc}(r, t, q)$; (whilst Appendix $A.1$ gives individual representations for the parameters a, b and c). Then in Section 3.2, with Theorem 3.2.1 and Theorem 3.2.2, we create sums of squares relations for the sums $\mathcal{L}_{s;abc}(r, t, q)$ that utilise the relations of the sums $\mathcal{F}_{s;ab}(r, t, q)$ developed in Lemmata 2.3.3 and 2.3.1.

3.1 Introduction

In this introductory section we state our main results, deferring proofs until the next chapter which employs primitive $2q$-th roots of unity.

We take the two families (determined by parameter s) of four types of sequences (determined by parameters a and b), whose values are denoted by $\mathcal{F}_{s;ab}(r, t, q)$, each one generating either q or $2q$ unique sequences of integers. Then by a shift of these functions we create two new families of eight types of sequences (determined by parameters a, b and c), written $\mathcal{L}_{s;abc}(r, t, q)$, where due to the repetition of sequences, we generally consider the restriction $c \leq t \leq m + bc$. Here a third parameter $c \in \{0, 1\}$, defined as in (2.1.1), is introduced to indicate the parity of the sequence number r of the term $\mathcal{F}_{s;ab}(r, t, q)$, from which the new sequences are derived, so that $r \equiv c \pmod 2$.

Then similar to (2.1.2) we write

$$\mathcal{L}_{s;abc} = \begin{cases} l_{abc} & \text{if } s = 0 \\ L_{abc} & \text{if } s = 1. \end{cases} \tag{3.1.1}$$

Now to relate the sequences $\mathcal{L}_{s;abc}(r, t, q)$ to those of $\mathcal{F}_{s;ab}(r', t, q)$ we introduce a shift operator $\mathcal{L}_s^r$ whose action on $\mathcal{F}_{s;ab}(r', t, q)$ is given in the following way.

Definition 3.1.1. *For integer r and non-negative integer r', we define the shift operator $\mathcal{L}_s^r$ by its action on $\mathcal{F}_{s;ab}(r', t, q)$ as*

$$\mathcal{L}_s^r \mathcal{F}_{s;ab}(r', t, q) = \mathcal{F}_{s;ab}(r' + 2r, t + r, q). \tag{3.1.2}$$

Now putting $r' = 1$ in (3.1.2) we obtain

$$\mathcal{L}_s^r \mathcal{F}_{s;ab}(1, t, q) = \mathcal{F}_{s;ab}(2r + 1, t + r, q) = \mathcal{L}_{s;ab1}(r, t, q), \tag{3.1.3}$$

and putting $r' = 2$ and replacing t with $t + 1$ in (3.1.2) we have

$$\mathcal{L}_s^r \mathcal{F}_{s;ab}(2, t + 1, q) = \mathcal{F}_{s;ab}(2r + 2, t + r + 1, q) = \mathcal{L}_{s;ab0}(r, t, q). \tag{3.1.4}$$

Then using (3.1.2) and (the reversed forms of) (3.1.3) and (3.1.4) we obtain the following definition.

Definition 3.1.2. *For integers $r \geq 0$, $q = 2m + b \geq 1$ and $c \leq t \leq m$, where $b, c \in \{0, 1\}$, we have*

$$\mathcal{L}_{s;abc}(r, t, q) = \mathcal{L}_s^r \mathcal{F}_{s;ab}(2 - c, t + 1 - c, q) = \mathcal{F}_{s;ab}(2r + 2 - c, t + r + 1 - c, q).$$

In words, starting with the $(2 - c)$-th term (corresponding row) and the t-th sequence of $\mathcal{F}_{s;ab}(r, t, q)$, we obtain the r-th term of the sequence $\mathcal{L}_{s;abc}(r, t, q)$ by the r-th application of the shift operator $\mathcal{L}_s$.

Now employing Definitions 2.2.1 and 3.1.2, and Theorem 1.2.1, we put $R = 2r + 2 - c$ and $T = r + t + 1 - c$, and then we obtain

$$\mathcal{L}_{s;abc}(r, t, q) = \mathcal{F}_{s;ab}(R, T, q) = \sum_{k \equiv T \pmod{2q}} \gamma^k \binom{R}{k} + \lambda \gamma^b \sum_{k \equiv T + q \pmod{2q}} \gamma^k \binom{R}{k}$$

$$= \gamma^T \lambda^{\lfloor T/q \rfloor} \sum_{d=0}^{\lfloor (R - T_q)/q \rfloor} \lambda^d \binom{R}{T_q + dq}. \tag{3.1.5}$$

Remark. For the first $q = 2m + b$ residue classes of the alternating sum $f_{1b}(r', t, q)$, the sequence starts with $q + 2t$ non-negative values. It then alternates between negative and positive integers with period $4q$. For the second q residue classes the absolute values are identical to the first q, but the signs are reversed. On application of the shift operator $\mathcal{L}_s^r$ to $f_{1b}(r', t, q)$, the result is a new sequence generated by moving diagonally along the $2q$ residue classes of (2.2.11) and so producing values that are all positive. See Appendix $E.1$, Tables $E.3$ and $E.4$.

3.2 Sum of squares relation

We can apply the identities involving the sums of squares relations developed in Section 2.3 to also create similar relations for our functions $\mathcal{L}_{s;abc}$.

THEOREM 3.2.1. *For $r \geq 0$ we have*

$$\mathcal{L}_{s;ab1}(2r, 1, q) = \frac{\gamma}{2} \sum_{j=0}^{q-1} \mathcal{L}_{s;ab1}(r, j, q)^2 \tag{3.2.1}$$

Proof. From Lemma 2.3.3, we recall that

$$\mathcal{F}_{s;ab}(2R, R, q) = \gamma^R \sum_{j=0}^{q-1} \mathcal{F}_{s;ab}(R, j, q)^2. \tag{3.2.2}$$

We put $R = 2r + 1$ to obtain

$$\mathcal{F}_{s;ab}(4r + 2, 2r + 1, q) = \gamma \sum_{j=0}^{q-1} \mathcal{F}_{s;ab}(2r + 1, j, q)^2. \tag{3.2.3}$$

From Lemma 2.3.1 we have the three term relation

$$\mathcal{F}_{s;ab}(4r + 2, 2r + 1, q) = \mathcal{F}_{s;ab}(4r + 1, 2r + 1, q) + \gamma \mathcal{F}_{s;ab}(4r + 1, 2r, q),$$

but from the symmetry of Pascal's triangle and (due to their formation as repeated steps of the modulus) the generalised Fleck numbers, we have
$\mathcal{F}_{s;ab}(R, t, q) = \gamma \mathcal{F}_{s;ab}(R, R - t, q)$, or with $R = 4r + 1$,

$$\mathcal{F}_{s;ab}(4r + 1, 2r + 1, q) = \gamma \mathcal{F}_{s;ab}(4r + 1, 2r, q),$$

and so

$$\mathcal{F}_{s;ab}(4r + 2, 2r + 1, q) = 2\mathcal{F}_{s;ab}(4r + 1, 2r + 1, q) = 2\gamma \mathcal{F}_{s;ab}(4r + 1, 2r, q).$$

Therefore, (3.2.3) becomes

$$2\mathcal{F}_{s;ab}(4r + 1, 2r + 1, q) = \gamma \sum_{j=0}^{q-1} \mathcal{F}_{s;ab}(2r + 1, j, q)^2,$$

and so we have

$$\mathcal{L}_{s;ab1}(2r, 1, q) = \frac{\gamma}{2} \sum_{j=0}^{q-1} \mathcal{L}_{s;ab1}(r, j - r, q)^2 = \frac{\gamma}{2} \sum_{j=0}^{q-1} \mathcal{L}_{s;ab1}(r, j, q)^2.$$

$\square$

The squares of the terms $\mathcal{L}_{s;abc}(r, j, q)$ in (3.2.1) are not all unique, and we find that when $c = 1$ and either $a = 1$ or $a = b = 0$, there are exactly m pairs of non-zero terms. we state this as a corollary to Theorem 3.2.1.

COROLLARY. *When $c = 1$, and either $a = 1$, or $a = b = 0$, we have*

$$\mathcal{L}_{s;ab1}(2r, 1, 2m + b) = \gamma \sum_{j=1}^{m} \mathcal{L}_{s;ab1}(r, j, 2m + b)^2. \tag{3.2.4}$$

Proof. We will see in the Corollary of Lemma 4.5.3 that when $R = (2n+1)q + 2t$, $(n \in \mathbb{N}_{\geq 0})$, that $\mathcal{F}_{s;1b}(R, t, q) = 0$. This implies that $\mathcal{F}_{s;10}(R, t, 2m) = 0$, when R is even, and $\mathcal{F}_{s;11}(R, t, 2m+1) = 0$, when R is odd.

Let $R = 2nq + k$, where $0 \leq k \leq 2q - 1$, and, without loss of generality, let $0 \leq t \leq q - 1$. We then have

$$\mathcal{F}_{s;ab}(R, t, q) = \gamma^R \mathcal{F}_{s;ab}(R, R-t, q) = \gamma^k \mathcal{F}_{s;ab}(R, 2nq+k-t, q) = \gamma^k \mathcal{F}_{s;ab}(R, k-t, q) = \gamma^k \mathcal{F}_{s;ab}(R, v, q),$$

where $-q + 1 \leq v \leq 2q - 1$. Moreover,

$$\mathcal{F}_{s;ab}(R, v, q) = \begin{cases} \mathcal{F}_{s;ab}(R, v_q, q) & \text{if } 0 \leq v \leq q - 1 \\ (-1)^a \mathcal{F}_{s;ab}(R, v_q, q) & \text{if } v < 0, \text{ or } v \geq q. \end{cases}$$

We are interested in the squares of the terms, and so we require $|\mathcal{F}_{s;ab}(R, t_q, q)| = |\mathcal{F}_{s;ab}(R, v_q, q)|$. If we consider the transformation of $\mathcal{F}_{s;ab}(R, t_q, q)$ to $\mathcal{F}_{s;ab}(R, v_q, q)$ as a mapping, then since R (or k) is fixed, the mapping is bijective.

Now when $c = 1$, we are employing the odd rows, and we consider two cases.

Case 1. When the parameters $a = b = 1$, $q = 2m+1$, and $t_q = v_q$ only when $R = 2nq+q+2t$, and then $\mathcal{F}_{s;11}(R, t, 2m+1) = 0$. Otherwise we have

$$|\mathcal{F}_{s;ab}(R, t_q, q)| = |\mathcal{F}_{s;ab}(R, v_q, q)|, \qquad \text{where } t_q \neq v_q, \tag{3.2.5}$$

and (3.2.5) holds for precisely m different values and when we put $R = 2r + 1$ then (3.2.4) follows.

Case 2. When the parameter $b = 0$, the argument is independent of the parameter a, and since $q = 2m$ is even, $t_q = v_q$ has no solutions, and we only have (3.2.5), for exactly m different values. So with $R = 2r + 1$ we obtain (3.2.4) and, therefore, we establish the result. $\qquad \square$

Example. When the parameters $s = a = c = 1$ and $b = 0$, and the variables $r = 7$ and $q = 6$, we have

$$\mathcal{L}_{1;101}(14, 1, 6) = (-1)^7 \sum_{j=1}^{3} \mathcal{L}_{1;101}(7, j, 6)^2 = -\left(L_{101}(7, 1, 6)^2 + L_{101}(7, 2, 6)^2 + L_{101}(7, 3, 6)^2 \right)$$
$$= -(39,879,225 + 20,693,401 + 2,683,044) = -63,255,670.$$

In [31] the result of this corollary is applied in the particular cases of the function $\mathcal{L}_{1;11c}$, with $c \in \{0, 1\}$.

THEOREM 3.2.2. *We have*

$$\mathcal{L}_{s;ab0}(2r+1,0,q) = \sum_{j=0}^{q-1} \mathcal{L}_{s;ab1}(r,j,q)^2 \qquad (3.2.6)$$

Proof. Once more we take as our starting point, Lemma 2.3.3,

$$\mathcal{F}_{s;ab}(2R,R,q) = \gamma^R \sum_{j=0}^{q-1} \mathcal{F}_{s;ab}(R,j,q)^2.$$

Now we put $R = 2r+2$ to obtain

$$\mathcal{F}_{s;ab}(4r+4,2r+2,q) = \gamma^2 \sum_{j=0}^{q-1} \mathcal{F}_{s;ab}(2r+2,j,q)^2. \qquad (3.2.7)$$

We can then write (3.2.7) as

$$\mathcal{L}_{s;ab0}(2r+1,0,q) = \sum_{j=0}^{q-1} \mathcal{L}_{s;ab0}(r,j-r-1,q)^2 = \sum_{j=0}^{q-1} \mathcal{L}_{s;ab0}(r,j,q)^2,$$

$\square$

The equations (3.2.1) and (3.2.6) is equivalently stated using binomial coefficients as

$$\gamma\lambda^{\lfloor(2r+1)/q\rfloor} \sum_{d=0}^{\lfloor(4r+1-V_q)/q\rfloor} \lambda^d \binom{4r+1}{V_q+dq} = \frac{\gamma}{2}\sum_{j=0}^{q-1} \left(\sum_{d=0}^{\lfloor(2r+1-W_q)/q\rfloor} \lambda^d \binom{2r+1}{W_q+dq} \right)^2,$$

where $V_q \equiv 2r+1 \pmod q$, $W_q \equiv r+j \pmod q$, and

$$\lambda^{\lfloor(2r+2)/q\rfloor} \sum_{d=0}^{\lfloor(4r+4-V_q)/q\rfloor} \lambda^d \binom{4r+4}{V_q+dq} = \sum_{j=0}^{q-1} \left(\sum_{d=0}^{\lfloor(2r+2-W_q)/q\rfloor} \lambda^d \binom{2r+2}{W_q+dq} \right)^2,$$

where $V_q \equiv 2r+2 \pmod q$ and $W_q \equiv r+1+j \pmod q$, and we recall that $\gamma = (-1)^s$ and $\lambda = (-1)^a$.

Chapter 4

Roots of unity closed forms

This chapter is primarily concerned with the expression of the sums $\mathcal{F}_{s;ab}(r,t,q)$ and $\mathcal{L}_{s;abc}(r,t,q)$ in terms of their primitive $2q$-th roots of unity, and then secondly in terms of cosines. We commence in Section 4.1 with Theorem 4.1.3 with an expression for the sums $\mathcal{F}_{s;ab}(r,t,q)$ in terms of their primitive $2q$-th roots of unity, and then in Section 4.2 with Theorem 4.2.2 giving an expression for the sums $\mathcal{L}_{s;abc}(r,t,q)$. These two theorems are then developed in Sections 4.3 and 4.4 to obtain respectively, Theorems 4.3.3 and 4.4.2, that express each of the two sets of sums in terms of cosines. Finally in Section 4.5 we examine expected divisibility properties of the sequences $\mathcal{F}_{s;ab}(r,t,q)$. These are summarised in Theorem 4.5.10.

4.1 Expressing $\mathcal{F}_{s;ab}$ in terms of the primitive $2q$-th roots of unity

We commence by defining a primitive Q-th root of unity.

Definition 4.1.1 (primitive Q-th root of unity). *For a given positive integer Q we define a primitive Q-th root of unity as a primitive solution $x = \zeta_Q$ to the equation*

$$x^Q - 1 = 0, \tag{4.1.1}$$

when we map the (complex) number ζ_Q to $e^{2\pi\imath/Q}$. To enhance the readability of the work, the subscript Q is omitted when there is no ambiguity as to its value.

In Chapter 2 we introduced the function $\mathcal{F}_{s;ab}$ where we recall that the parameter s is a sign indicator, and that the parameters a and b represent the sum type and the modulus respectively. Here we express $\mathcal{F}_{s;ab}(r,t,q)$ in terms of $\zeta = e^{2\pi\imath/2q}$.

To enable us to achieve this we first require some lemmas.

LEMMA 4.1.1. *For $r \geq 0$ and ζ a primitive $2q$-th root of unity, we have*

$$\mathcal{F}_{0;ab}(r,t,q) = f_{ab}(r,t,q) = \frac{1}{2q} \sum_{d=0}^{2q-1} \left(\zeta^d\right)^{-t} \left(1 + \zeta^d\right)^r \left(1 + (-1)^{a+d}\right). \tag{4.1.2}$$

Proof. Using (2.2.9) with $s = 0$, we have

$$f_{ab}(r, t, q) = \sum_{k \equiv t \pmod{2q}} \binom{r}{k} + (-1)^a \sum_{k \equiv t+q \pmod{2q}} \binom{r}{k}$$

$$= \frac{1}{2q} \left(\sum_{k=0}^{r} \sum_{d \pmod{2q}} \zeta^{-td} \zeta^{dk} \binom{r}{k} + (-1)^a \sum_{k=0}^{r} \sum_{d \pmod{2q}} \zeta^{-(t+q)d} \zeta^{dk} \binom{r}{k} \right)$$

$$= \frac{1}{2q} \left(\sum_{d \pmod{2q}} \zeta^{-td} (1 + \zeta^d)^r + (-1)^a \sum_{d \pmod{2q}} \zeta^{-(t+q)d} (1 + \zeta^d)^r \right)$$

$$= \frac{1}{2q} \left(\sum_{d=0}^{2q-1} \zeta^{-td} (1 + \zeta^d)^r + (-1)^a \zeta^{-(t+q)d} (1 + \zeta^d)^r \right)$$

$$= \frac{1}{2q} \sum_{d=0}^{2q-1} \zeta^{-td} (1 + \zeta^d)^r \left(1 + (-1)^a (\zeta^q)^{-d} \right). \tag{4.1.3}$$

For a primitive $2q$-th root of unity, we note that $\zeta^q = -1$, and that $-d \equiv d \pmod 2$, so we obtain

$$f_{ab}(r, t, q) = \frac{1}{2q} \sum_{d=0}^{2q-1} \zeta^{-td} (1 + \zeta^d)^r \left(1 + (-1)^a (-1)^d \right) = \frac{1}{2q} \sum_{d=0}^{2q-1} \left(\zeta^d \right)^{-t} (1 + \zeta^d)^r \left(1 + (-1)^{a+d} \right).$$

$\square$

LEMMA 4.1.2. *For $r \geq 0$ and ζ a primitive $2q$-th root of unity, we have*

$$\mathcal{F}_{1;ab}(r, t, q) = F_{ab}(r, t, q) = \frac{1}{2q} \sum_{d=0}^{2q-1} \left(\zeta^d \right)^{-t} \left(1 - \zeta^d \right)^r \left(1 + (-1)^{a+b+d} \right). \tag{4.1.4}$$

Proof. Using (2.2.9) with $s = 1$ and developing as in Lemma 4.1.1 we have

$$F_{ab}(r, t, q) = (-1)^t f_{ab}(r, t, q)$$

$$= \sum_{k \equiv t \pmod{2q}} (-1)^k \binom{r}{k} + (-1)^{a+b} \sum_{k \equiv t+q \pmod{2q}} (-1)^k \binom{r}{k}$$

$$= \frac{1}{2q} \left(\sum_{k=0}^{r} \sum_{d \pmod{2q}} \zeta^{-td} \zeta^{dk} (-1)^k \binom{r}{k} + (-1)^{a+b} \sum_{k=0}^{r} \sum_{d \pmod{2q}} \zeta^{-(t+q)d} \zeta^{dk} (-1)^k \binom{r}{k} \right)$$

$$= \frac{1}{2q} \left(\sum_{d \pmod{2q}} \zeta^{-td} (1 - \zeta^d)^r + (-1)^{a+b} \sum_{d \pmod{2q}} \zeta^{-(t+q)d} (1 - \zeta^d)^r \right)$$

$$= \frac{1}{2q} \sum_{d=0}^{2q-1} \left(\zeta^d \right)^{-t} (1 - \zeta^d)^r \left(1 + (-1)^{a+b+d} \right).$$

$\square$

THEOREM 4.1.3 (roots of unity expression of the functions $\mathcal{F}_{s;ab}$). *For $r \geq 0$ and ζ a primitive $2q$-th root of unity, we have*

$$\mathcal{F}_{s;ab}(r,t,q) = \frac{1}{q} \sum_{d=\epsilon}^{q+\epsilon-1} \left(\zeta^{2d-\epsilon}\right)^{-t} \left(1 + \gamma\zeta^{2d-\epsilon}\right)^r, \qquad (4.1.5)$$

where $\epsilon \equiv a + sb \pmod{2}$.

Proof. Combining Lemmata 4.1.1 and 4.1.2 we obtain

$$\mathcal{F}_{s;ab}(r,t,q) = \frac{1}{2q} \sum_{d=0}^{2q-1} \left(\zeta^d\right)^{-t} \left(1 + \gamma\zeta^d\right)^r \left(1 + (-1)^{a+sb+d}\right). \qquad (4.1.6)$$

We consider the even case $\epsilon = 0$ and the odd case $\epsilon = 1$ separately. When $\epsilon = 0$, then $a = sb$, and we have

$$\mathcal{F}_{s;ab}(r,t,q) = \frac{2}{2q} \sum_{\substack{d=0 \\ d\,\text{even}}}^{2q-1} \left(\zeta^d\right)^{-t} \left(1 + \gamma\zeta^d\right)^r = \frac{1}{q} \sum_{g=0}^{q-1} \left(\zeta^{2g}\right)^{-t} \left(1 + \gamma\zeta^{2g}\right)^r. \qquad (4.1.7)$$

Secondly when $\epsilon = 1$, then $a + sb$ is odd and we have

$$\mathcal{F}_{s;ab}(r,t,q) = \frac{2}{2q} \sum_{\substack{d=0 \\ d\,\text{odd}}}^{2q-1} \left(\zeta^d\right)^{-t} \left(1 + \gamma\zeta^d\right)^r = \frac{1}{q} \sum_{g=1}^{q} \left(\zeta^{2g-1}\right)^{-t} \left(1 + \gamma\zeta^{2g-1}\right)^r. \qquad (4.1.8)$$

Combining (4.1.7) and (4.1.8) produces the result. $\qquad\square$

Isolating the individual cases we have the following corollary.

COROLLARY. *For ζ a primitive $2q$-th root of unity, and ζ_q a primitive q-th root of unity, the sequences $\mathcal{F}_{s;ab}(r,t,q)$ simplify to*

$$f_{00}(r,t,q) = f_{01}(r,t,q) = \frac{1}{q}\left(\sum_{d=0}^{q-1}(\zeta_q^d)^{-t}(1 + \zeta_q^d)^r\right),$$

$$f_{10}(r,t,q) = f_{11}(r,t,q) = \frac{1}{q}\sum_{d=1}^{q}\zeta^{-(2d-1)t}(1 + \zeta^{2d-1})^r,$$

$$F_{00}(r,t,q) = F_{11}(r,t,q) = \frac{1}{q}\sum_{d=0}^{q-1}(\zeta_q^d)^{-t}(1 - \zeta_q^d)^r,$$

and

$$F_{01}(r,t,q) = F_{10}(r,t,q) = \frac{1}{q}\sum_{d=1}^{q}(\zeta^{2d-1})^{-t}(1 - \zeta^{2d-1})^r.$$

Proof. From Theorem 4.1.3, we observe that $\epsilon = 0$, either when $s = a = 0$, or when $s = 1$ and $a = b$; otherwise $\epsilon = 1$. We substitute into (4.1.5) according to the parameter s, and the value of $\epsilon \equiv a + sb \pmod{2}$. $\qquad\square$

Remark. We note that $\mathcal{F}_{s;01}(r,0,1) = 2^r$ and $\mathcal{F}_{s;11}(r,0,1) = 0$; $\mathcal{F}_{s;00}(r,0,2) = 2^{r-1}$ and

$$\mathcal{F}_{s;10}(r,0,2) = \begin{cases} (-4)^{r/4} & \text{if } r \equiv 0 \pmod{4} \\ (-4)^{(r-1)/4} & \text{if } r \equiv 1 \pmod{4} \\ 0 & \text{if } r \equiv 2 \pmod{4} \\ \imath(-4)^{(r-1)/4} & \text{if } r \equiv 3 \pmod{4}. \end{cases}$$

4.2   The function $\mathcal{L}_{s;abc}$ expressed in terms of ζ_{2q}

Recalling the notation employed in (3.1.1), we have

$$\mathcal{L}_{s;abc}(r,t,q) = \begin{cases} l_{abc}(r,t,q) & \text{if } s = 0 \\ L_{abc}(r,t,q) & \text{if } s = 1, \end{cases}$$

and from Definition 3.1.2, that

$$\mathcal{L}_{s;abc}(r,t,q) = \mathcal{L}_s^r \mathcal{F}_{s;ab}(2-c, t+1-c, q) = \mathcal{F}_{s;ab}(2r+2-c, t+r+1-c, q). \tag{4.2.1}$$

Here the shift operator, $\mathcal{L}_s^r$, is given by Definition 3.1.1 as

$$\mathcal{L}_s^r \mathcal{F}_{s;ab}(r', t, q) = \mathcal{F}_{s;ab}(r' + 2r, t+r, q).$$

Moreoever, by Theorem 4.1.3, $\mathcal{F}_{s;ab}$ expressed in terms of ζ a primitive $2q$-th root of unity, is

$$\mathcal{F}_{s;ab}(r,t,q) = \frac{1}{2q} \sum_{d=0}^{2q-1} \left(\zeta^d\right)^{-t} \left(1 + \gamma\zeta^d\right)^r \left(1 + \gamma^b(-1)^{a+d}\right) = \frac{1}{q} \sum_{d=\epsilon}^{q+\epsilon-1} \left(\zeta^{2d-\epsilon}\right)^{-t} \left(1 + \gamma\zeta^{2d-\epsilon}\right)^r,$$
$$\tag{4.2.2}$$

where $\epsilon \equiv a + sb \pmod{2}$. If we make the substitution $2r+2-c$ for r, and $t+1-c$ for t in (4.2.2), we obtain the corresponding expression

$$\mathcal{L}_{s;abc}(r,t,q) = \frac{1}{q} \sum_{d=\epsilon}^{q+\epsilon-1} \left(\zeta^{2d-\epsilon}\right)^{-(t+r+1-c)} \left(1 + \gamma\zeta^{2d-\epsilon}\right)^{2r+2-c}. \tag{4.2.3}$$

However, this simple substitution gives no justification of the shift operator $\mathcal{L}_s^r$. To fulfil this objective we introduce the function, z_s^r, given as follows.

Definition 4.2.1. *For sign s, integers r and d with $0 \le d \le 2q-1$, and ζ a primitive $2q$-th root of unity, we have*

$$z_s^r(d, 2q) = \left(\zeta^d + \zeta^{-d} + 2\gamma\right)^r. \tag{4.2.4}$$

This then leads us to the following more informative definition of the shift operator, $\mathcal{L}_s^r$.

Definition 4.2.2 (shift of $\mathcal{F}_{s;ab}$). *For $z_s^r(d, 2q)$ given by Definition 4.2.1, and ζ a primitive $2q$-th root of unity, we have*

$$\mathcal{L}_s^r \mathcal{F}_{s;ab}(r', t, q) = \frac{1}{2q} \sum_{d=0}^{2q-1} \left(1 + (-1)^{a+sb+d}\right) \left(\zeta^d\right)^{-t} \left(1 + \gamma\zeta^d\right)^{r'} z_s^r(d, 2q). \tag{4.2.5}$$

When incorporated into the function $\mathcal{F}_{s;ab}(r', t, q)$ in this manner, $z_s^r(d, 2q)$ produces a "shift" demonstrated by Lemma 4.2.1.

LEMMA 4.2.1 (term shift). *For ζ a primitive $2q$-th root of unity, we have*

$$\mathcal{L}_s^r \mathcal{F}_{s;ab}(r', t, q) = \frac{1}{2q} \sum_{d=0}^{2q-1} \left(1 + (-1)^{a+sb+d}\right) \left(\zeta^d\right)^{-(t+r)} \left(1 + \gamma\zeta^d\right)^{r'+2r}. \tag{4.2.6}$$

Proof. From Definitions 4.2.1 and 4.2.2 we have

$$\begin{aligned}
\mathcal{L}_s^r \mathcal{F}_{s;ab}(r', t, q) &= \frac{1}{2q} \sum_{d=0}^{2q-1} \left(1 + \gamma^b(-1)^{a+d}\right) \left(\zeta^d\right)^{-t} \left(1 + \gamma\zeta^d\right)^{r'} \left(\zeta^d + \zeta^{-d} + 2\gamma\right)^r \\
&= \frac{1}{2q} \sum_{d=0}^{2q-1} \left(1 + (-1)^{a+sb+d}\right) \left(\zeta^d\right)^{-t-r} \left(1 + \gamma\zeta^d\right)^{r'} (\zeta^{2d} + 1 + 2\gamma\zeta^d)^r \\
&= \frac{1}{2q} \sum_{d=0}^{2q-1} \left(1 + (-1)^{a+sb+d}\right) \left(\zeta^d\right)^{-(t+r)} \left(1 + \gamma\zeta^d\right)^{r'+2r}. \tag{4.2.7}
\end{aligned}$$

$\square$

We now state (4.2.3) as a theorem.

THEOREM 4.2.2 (roots of unity expression of the functions $\mathcal{L}_{s;abc}$). *For ζ a primitive $2q$-th root of unity, we have*

$$\mathcal{L}_{s;abc}(r, t, q) = \frac{1}{q} \sum_{d=\epsilon}^{q+\epsilon-1} \left(\zeta^{2d-\epsilon}\right)^{-(t+r+1-c)} \left(1 + \gamma\zeta^{2d-\epsilon}\right)^{2r+2-c}, \tag{4.2.8}$$

where $\epsilon \equiv a + sb \pmod 2$.

Proof. From Lemma 4.2.1 and then Theorem 4.1.3 we have

$$\begin{aligned}
\mathcal{L}_s^r \mathcal{F}_{s;ab}(r', t, q) = \mathcal{F}_{s;ab}(r' + 2r, t + r, q) &= \frac{1}{2q} \sum_{d=0}^{2q-1} \left(\zeta^d\right)^{-(t+r)} \left(1 + \gamma\zeta^d\right)^{r'+2r} \left(1 + \gamma^b(-1)^{a+d}\right) \\
&= \frac{1}{q} \sum_{d=\epsilon}^{q+\epsilon-1} \left(\zeta^{2d-\epsilon}\right)^{-(t+r)} \left(1 + \gamma\zeta^{2d-\epsilon}\right)^{r'+2r}. \tag{4.2.9}
\end{aligned}$$

Now we recall from Definition 3.1.2 that

$$\mathcal{L}_{s;ab0}(r, t, q) = \mathcal{F}_{s;ab}(2r+2, t+r+1, q), \quad and \quad \mathcal{L}_{s;ab1}(r, t, q) = \mathcal{F}_{s;ab}(2r+1, t+r, q). \tag{4.2.10}$$

Combining both forms of (4.2.10), we let $r' = 2 - c$, and replace t with $t + 1 - c$, in (4.2.9) to obtain

$$\begin{aligned}
\mathcal{L}_s^r \mathcal{F}_{s;ab}(2 - c, t + 1 - c, q) &= \frac{1}{q} \sum_{d=\epsilon}^{q+\epsilon-1} \left(\zeta^{2d-\epsilon}\right)^{-(t+r+1-c)} \left(1 + \gamma\zeta^{2d-\epsilon}\right)^{2r+2-c} \\
&= \mathcal{F}_{s;ab}(2r + 2 - c, t + r + 1 - c, q) \\
&= \mathcal{L}_{s;abc}(r, t, q).
\end{aligned}$$

$\square$

To express each sign case of $\mathcal{L}_{s;abc}$ explicitly we have the following corollary.

COROLLARY. *For ζ a primitive $2q$-th root of unity, and ζ_q a primitive q-th root of unity, the sequences $\mathcal{L}_{s;abc}(r,t,q)$ simplify to*

$$l_{000}(r,t,q) = l_{010}(r,t,q) = \frac{1}{q}\sum_{d=0}^{q-1}\left(\zeta_q^d\right)^{-(t+r+1)}\left(1+\zeta_q^d\right)^{2r+2},$$

$$l_{001}(r,t,q) = l_{011}(r,t,q) = \frac{1}{q}\sum_{d=0}^{q-1}\left(\zeta_q^d\right)^{-(t+r)}\left(1+\zeta_q^d\right)^{2r+1},$$

$$l_{100}(r,t,q) = l_{110}(r,t,q) = \frac{1}{q}\sum_{d=1}^{q}\left(\zeta^{2d-1}\right)^{-(t+r+1)}\left(1+\zeta^{2d-1}\right)^{2r+2},$$

$$l_{101}(r,t,q) = l_{111}(r,t,q) = \frac{1}{q}\sum_{d=1}^{q}\left(\zeta^{2d-1}\right)^{-(t+r)}\left(1+\zeta^{2d-1}\right)^{2r+1},$$

$$L_{000}(r,t,q) = L_{110}(r,t,q) = \frac{1}{q}\sum_{d=0}^{q-1}\left(\zeta_q^d\right)^{-(t+r+1)}\left(1-\zeta_q^d\right)^{2r+2},$$

$$L_{001}(r,t,q) = L_{111}(r,t,q) = \frac{1}{q}\sum_{d=0}^{q-1}\left(\zeta_q^d\right)^{-(t+r)}\left(1-\zeta_q^d\right)^{2r+1},$$

$$L_{010}(r,t,q) = L_{100}(r,t,q) = \frac{1}{q}\sum_{d=1}^{q}\left(\zeta^{2d-1}\right)^{-(t+r+1)}\left(1-\zeta^{2d-1}\right)^{2r+2},$$

and

$$L_{011}(r,t,q) = L_{101}(r,t,q) = \frac{1}{q}\sum_{d=1}^{q}\left(\zeta^{2d-1}\right)^{-(t+r)}\left(1-\zeta^{2d-1}\right)^{2r+1}.$$

Proof. The result follows on substitution of the paramers s, a, b and c into (4.2.8), according to $\epsilon \equiv a + sb \pmod 2$. $\qquad\square$

Remark. We note that $\mathcal{L}_{s;01c}(r,0,1) = \gamma^{r+1-c}2^{2r+2-c}$ and $\mathcal{L}_{s;11c}(r,0,1) = 0$; $\mathcal{L}_{s;00c}(r,c,2) = \gamma^{r+1}2^{2r+1-c}$ and $\mathcal{L}_{s;10c}(r,c,2) = \gamma^{r+1}2^{r+1-c}$.

4.3 Expressing the sums $\mathcal{F}_{s;ab}(r,t,q)$ in terms of cosines

To obtain the sums $\mathcal{F}_{s;ab}(r,t,q)$ in terms of cosines we utilize the expressions developed in the previous section but also introduce Lemmata 4.3.1 and 4.3.2. The first involves the expression of z_s^r in terms of a cosine.

LEMMA 4.3.1 (cosine form of z_s^r). *We have*

$$z_s^r(d,2q) = \gamma^r\left(2\cos\left(\frac{\pi(d-sq)}{2q}\right)\right)^{2r}. \tag{4.3.1}$$

Proof. We show that when $s = 0$, (4.3.1) is equivalent to

$$z_0^r(d, 2q) = \left(2 \cos \left(\frac{\pi d}{2q} \right) \right)^{2r},$$

and that when $s = 1$, we have

$$z_1^r(d, 2q) = (-1)^r \left(2 \cos \left(\frac{\pi (d - q)}{2q} \right) \right)^{2r}.$$

We first note that

$$\zeta^d + \zeta^{-d} = 2 \cos \left(\frac{\pi d}{q} \right).$$

Then separating the $s = 0$ and $s = 1$ cases of the function z_s^r, we have respectively

$$z_0^r(d, 2q) = \left(\zeta^d + \zeta^{-d} + 2 \right)^r = \left(2 \left(\cos \frac{\pi d}{q} + 1 \right) \right)^r$$

$$= \left(2 \left(2 \cos^2 \frac{\pi d}{2q} - 1 + 1 \right) \right)^r = \left(2 \cos \frac{\pi d}{2q} \right)^{2r},$$

and

$$z_1^r(d, 2q) = \left(\zeta^d + \zeta^{-d} - 2 \right)^r = \left(2 \left(\cos \frac{\pi d}{q} - 1 \right) \right)^r$$

$$= \left(2 \left(1 - 2 \sin^2 \frac{\pi d}{2q} - 1 \right) \right)^r = \left(4 \left(- \sin^2 \frac{\pi d}{2q} \right) \right)^r$$

$$= (-1)^r \left(2 \sin \frac{\pi d}{2q} \right)^{2r} = (-1)^r \left(2 \cos \left(\frac{\pi (d - q)}{2q} \right) \right)^{2r}.$$

$\square$

Secondly we derive a cosine form of the expression $1 + \gamma \zeta^d$. To achieve this we introduce a primitive $4q$-th root of unity ω so that $\omega^2 = \zeta$.

LEMMA 4.3.2 (cosine form of $1 + \gamma \zeta^d$). *With $\omega^2 = \zeta$, we have*

$$1 + \gamma \zeta^d = 2\omega^{d+3qs} \left(\cos \left(\frac{\pi (d - sq)}{2q} \right) \right).$$

Proof. We have

$$1 + \gamma \zeta^d = 1 + \gamma \omega^{2d} = \omega^d \left(\omega^{-d} + \gamma \omega^d \right). \tag{4.3.2}$$

Let us consider each case of s in (4.3.2) separately. Then when $s = 0$, we write

$$\omega^d \times 2 \Re \omega^d = 2\omega^d \cos \left(\frac{2\pi d}{4q} \right) = 2\omega^d \cos \left(\frac{\pi d}{2q} \right). \tag{4.3.3}$$

When $s = 1$, (4.3.2) becomes

$$\omega^d \times (-2\imath) \Im \omega^d = -2\imath \omega^d \sin \left(\frac{2\pi d}{4q} \right) = 2\omega^{2q} \omega^q \omega^d \sin \left(\frac{\pi d}{2q} \right) = 2\omega^{d+3q} \sin \left(\frac{\pi d}{2q} \right). \tag{4.3.4}$$

On combining expressions (4.3.3) and (4.3.4) we obtain the result. $\square$

We are now in a position to state the following theorem.

THEOREM 4.3.3. *With $r' = 2n + c \in \mathbb{Z}$, we have*

$$
\mathcal{F}_{s;ab}(r', t, q) = \frac{\gamma^{\lfloor r'/2 \rfloor} 2^{r'}}{q} \sum_{d=\epsilon}^{q+\epsilon-1} \cos\left(\frac{\pi(r' - 2t)(2d - \epsilon - scq)}{2q} \right) \left(\cos \frac{\pi(2d - \epsilon - sq)}{2q} \right)^{r'},
$$

where $\epsilon \equiv a + sb \pmod 2$.

Proof. Recalling from Theorem 4.1.3 the expression for $\mathcal{F}_{s;ab}(r', t, q)$, in terms of ζ a primitive $2q$-th root of unity, we have

$$
\mathcal{F}_{s;ab}(r', t, q) = \frac{1}{q} \sum_{d=\epsilon}^{q+\epsilon-1} \left(\zeta^{2d-\epsilon} \right)^{-t} \left(1 + \gamma \zeta^{2d-\epsilon} \right)^{r'}. \tag{4.3.5}
$$

We use Lemma 4.2.1, and (putting $r = n$) the function z_s^n, given by Definition 4.2.1, to express $\left(1 + \gamma \zeta^{2d-\epsilon} \right)^{r'}$ in terms of a cosine. Then to similarly express the other ζ term in (4.3.5) as a cosine we consider its real or imaginary part as appropriate.

From Definition 4.2.1 and, with d replaced by $2d - \epsilon$, then Lemma 4.3.1, the function z_s^n, expressed in terms of a cosine is

$$
z_s^n(2d - \epsilon, 2q) = \left(\zeta^{2d-\epsilon} + \zeta^{\epsilon-2d} + 2\gamma \right)^n = \gamma^n \left(2 \cos\left(\frac{\pi(2d - \epsilon - sq)}{2q} \right) \right)^{2n}.
$$

So with $r' = 2n + c$, if $c = 0$, Lemma 4.3.1 is sufficient to express the term $\left(1 + \gamma \zeta^{2d-\epsilon} \right)^n$ as a cosine. However, if $c = 1$, a single $\left(1 + \gamma \zeta^{2d-\epsilon} \right)$ term remains. From Lemma 4.2.1 we write (4.3.5) as

$$
\mathcal{F}_{s;ab}(2n + c, t, q) = \frac{1}{q} \sum_{d=\epsilon}^{q+\epsilon-1} \left(\zeta^{2d-\epsilon} \right)^{-(t-n)} \left(1 + \gamma \zeta^{2d-\epsilon} \right)^c \left(\zeta^{2d-\epsilon} + \zeta^{\epsilon-2d} + 2\gamma \right)^n. \tag{4.3.6}
$$

Then from Lemmata 4.3.1 and 4.3.2, equation (4.3.6) becomes

$$
\frac{\gamma^n \omega^{3qsc} 2^{2n+c}}{q} \sum_{d=\epsilon}^{q+\epsilon-1} \omega^{(2d-\epsilon)(2n+c-2t)} \cos\left(\frac{\pi c(2d - \epsilon - sq)}{2q} \right) \left(\cos\left(\frac{\pi(2d - \epsilon - sq)}{2q} \right) \right)^{2n}. \tag{4.3.7}
$$

By consideration of the imaginary part of the ω term when $s = c = 1$, (and then we have $\imath^{sc} = \omega^{qsc}$ - a fourth root of unity), and the real part otherwise, we obtain

$$
\frac{\gamma^n \omega^{4qsc} 2^{2n+c}}{q} \sum_{d=\epsilon}^{q+\epsilon-1} \cos\left(\frac{\pi(2n + c - 2t)(2d - \epsilon - scq)}{2q} \right) \left(\cos\left(\frac{\pi(2d - \epsilon - sq)}{2q} \right) \right)^{2n+c}
$$

$$
= \frac{\gamma^{\lfloor r'/2 \rfloor} 2^{r'}}{q} \sum_{d=\epsilon}^{q+\epsilon-1} \cos\left(\frac{\pi(r' - 2t)(2d - \epsilon - scq)}{2q} \right) \left(\cos\left(\frac{\pi(2d - \epsilon - sq)}{2q} \right) \right)^{r'}.
$$

$\square$

COROLLARY. *In terms of cosines the sequences $\mathcal{F}_{s;ab}(r,t,q)$ can be expressed as*

$$f_{00}(r,t,q) = f_{01}(r,t,q) = \frac{2^r}{q} \sum_{d=0}^{q-1} \cos\left(\frac{\pi(r-2t)d}{q}\right)\left(\cos\frac{\pi d}{q}\right)^r,$$

$$f_{10}(r,t,q) = f_{11}(r,t,q) = \frac{2^r}{q} \sum_{d=1}^{q} \cos\left(\frac{\pi(r-2t)(2d-1)}{2q}\right)\left(\cos\left(\frac{\pi(2d-1)}{2q}\right)\right)^r,$$

$$F_{00}(r,t,q) = F_{11}(r,t,q) = \frac{(-1)^{\frac{(r-c)}{2}} 2^{r'}}{q} \sum_{d=0}^{q-1} \cos\left(\frac{\pi(r-2t)(2d-cq)}{2q}\right)\left(\sin\frac{\pi d}{q}\right)^r,$$

and

$$F_{01}(r,t,q) = F_{10}(r,t,q) = \frac{(-1)^{\frac{(r-c)}{2}} 2^{r}}{q} \sum_{d=1}^{q} \cos\left(\frac{\pi(r-2t)(2d-1-cq)}{2q}\right)\left(\sin\frac{\pi(2d-1)}{2q}\right)^r.$$

Proof. The result follows on substitution of each set of parameters into Theorem 4.3.3, and expressing shifted cosines as sines when applicable. $\qquad\square$

4.4   Expressing the sums $\mathcal{L}_{s;abc}(r,t,q)$ in terms of cosines

To express $\mathcal{L}_{s;abc}(r,t,q)$ in terms of cosines we employ Theorem 4.3.3 and the following lemma.

LEMMA 4.4.1. *We have that*

$$\cos\left(\frac{\pi(c-2t)(D-scq)}{2q}\right)\left(\cos\left(\frac{\pi(D-sq)}{2q}\right)\right)^{2-c}$$

$$= \cos\left(\frac{\pi(c-2t)(2q-D-scq)}{2q}\right)\left(\cos\left(\frac{\pi(2q-D-sq)}{2q}\right)\right)^{2-c}.$$

Proof. We consider the four cases arising from the two parameters s and c. Employing standard trigonometric identities and when applicable, writing shifted cosines as sines, and putting $T = 1 - 2t$, we have (commencing with the case $s = c = 0$)

$$\cos\frac{2\pi t(2q-D)}{2q}\cos^2\frac{\pi(2q-D)}{2q} = \cos\frac{2\pi tD}{2q}(-1)^2\cos^2\frac{\pi D}{2q} = \cos\frac{2\pi tD}{2q}\cos^2\frac{\pi D}{2q},$$

$$\cos\frac{\pi T(2q-D)}{2q}\cos\frac{\pi(2q-D)}{2q} = (-1)\cos\frac{TD}{2q}(-1)\cos\frac{\pi D}{2q} = \cos\frac{\pi TD}{2q}\cos\frac{\pi D}{2q},$$

$$\cos\frac{2\pi t(2q-D)}{2q}\sin^2\frac{\pi(2q-D)}{2q} = \cos\frac{2\pi tD}{2q}(-1)^2\sin^2\frac{\pi D}{2q} = \cos\frac{2\pi tD}{2q}\sin^2\frac{\pi D}{2q},$$

and

$$\sin\frac{\pi T(2q-D)}{2q}\sin\frac{\pi(2q-D)}{2q} = (-1)\sin\frac{2\pi tD}{2q}(-1)\sin\frac{\pi D}{2q} = \cos\frac{2\pi tD}{2q}\sin\frac{\pi D}{2q}.$$

$\qquad\square$

We now state Theorem 4.4.2.

THEOREM 4.4.2. *We have for $q \geq 2$*

$$\mathcal{L}_{s;abc}(r,t,q) = \frac{\gamma^{r+1-c}2^{2r+3-c}}{q} \times$$

$$\left(\frac{(a-1)^{st}}{2} + \sum_{d=1}^{\lfloor (q+\epsilon-1)/2 \rfloor} \cos\left(\frac{\pi(c-2t)(2d-\epsilon-scq)}{2q} \right) \left(\cos\left(\frac{\pi(2d-\epsilon-sq)}{2q} \right) \right)^{2r+2-c} \right),$$

where $\epsilon \equiv a + sb \pmod 2$.

Proof. From Theorem 4.3.3 we have

$$\mathcal{F}_{s;ab}(r,t,q) = \frac{\gamma^{\lfloor r/2 \rfloor}2^r}{q} \sum_{d=\epsilon}^{q+\epsilon-1} \cos\left(\frac{\pi(r-2t)(2d-\epsilon-scq)}{2q} \right) \left(\cos\frac{\pi(2d-\epsilon-sq)}{2q} \right)^r.$$

Replacing r with $2r+2-c$ and t with $t+r+1-c$ and simplifying we obtain

$$\mathcal{F}_{s;ab}(2r+2-c, t+r+1-c, q)$$

$$= \frac{\gamma^{r+1-c}2^{2r+2-c}}{q} \sum_{d=\epsilon}^{q+\epsilon-1} \cos\left(\frac{\pi(c-2t)(2d-\epsilon-scq)}{2q} \right) \left(\cos\frac{\pi(2d-\epsilon-sq)}{2q} \right)^{2r+2-c}. \quad (4.4.1)$$

From the Corollary to Theorem 4.1.3, it is observed that the sequences $\mathcal{F}_{0:0b}(r,t,q)$ (when $\epsilon = 0$), has a single term (with value 1) at $d = 0$. The corresponding term, (with value $(-1)^t$), in the sequences $\mathcal{F}_{1:ab}(r,t,q)$, occurs at $d = q/2$ when $a = b = \epsilon = 0$, and at $d = (q+1)/2$ when $a = 0$ and $b = \epsilon = 1$. Then from Lemma 4.4.1, for either value of ϵ, each of terms for $1 \leq d \leq \lfloor (q-1)/2 \rfloor$, can be paired to another equal in value, in the upper half of the summation interval. Hence, following separation of the single term, we halve the upper summation limit in (4.4.1), and consequently scale the expression by a factor of 2, therefore, producing the result. $\square$

COROLLARY. *In terms of cosines the sequences $\mathcal{L}_{s;abc}(r,t,q)$, where $q \geq 2$, can be expressed as*

$$l_{0b0}(r,t,q) = \frac{2^{2r+3}}{q} \left(\frac{1}{2} + \sum_{d=1}^{\lfloor (q-1)/2 \rfloor} \cos\left(\frac{2\pi dt}{q} \right) \left(\cos\frac{\pi d}{q} \right)^{2r+2} \right),$$

$$l_{0b1}(r,t,q) = \frac{2^{2r+2}}{q} \left(\frac{1}{2} + \sum_{d=1}^{\lfloor (q-1)/2 \rfloor} \cos\left(\frac{\pi d(1-2t)}{q} \right) \left(\cos\frac{\pi d}{q} \right)^{2r+1} \right),$$

$$l_{1b0}(r,t,q) = \frac{2^{2r+3}}{q} \sum_{d=1}^{\lfloor q/2 \rfloor} \cos\left(\frac{\pi(2d-1)t}{q} \right) \left(\cos\left(\frac{\pi(2d-1)}{2q} \right) \right)^{2r+2},$$

$$l_{1b1}(r,t,q) = \frac{2^{2r+2}}{q} \sum_{d=1}^{\lfloor q/2 \rfloor} \cos\left(\frac{\pi(2d-1)(1-2t)}{2q} \right) \left(\cos\left(\frac{\pi(2d-1)}{2q} \right) \right)^{2r+1},$$

$$L_{000}(r,t,q) = \frac{(-1)^{r+1}2^{2r+3}}{q} \left(\frac{(-1)^t}{2} + \sum_{d=1}^{(q-2)/2} \cos\left(\frac{2\pi dt}{q}\right) \left(\sin\frac{\pi d}{q}\right)^{2r+2} \right),$$

$$L_{001}(r,t,q) = \frac{(-1)^r 2^{2r+2}}{q} \left(\frac{(-1)^t}{2} + \sum_{d=1}^{(q-2)/2} \sin\left(\frac{\pi d(1-2t)}{q}\right) \left(\sin\frac{\pi d}{q}\right)^{2r+1} \right),$$

$$L_{010}(r,t,q) = \frac{(-1)^{r+1}2^{2r+3}}{q} \left(\frac{(-1)^t}{2} + \sum_{d=1}^{(q-1)/2} \cos\left(\frac{\pi(2d-1)t}{q}\right) \left(\sin\left(\frac{\pi(2d-1)}{2q}\right)\right)^{2r+2} \right),$$

$$L_{011}(r,t,q) = \frac{(-1)^r 2^{2r+2}}{q} \left(\frac{(-1)^t}{2} + \sum_{d=1}^{(q-1)/2} \sin\left(\frac{\pi(2d-1)(1-2t)}{2q}\right) \left(\sin\left(\frac{\pi(2d-1)}{2q}\right)\right)^{2r+1} \right),$$

$$L_{100}(r,t,q) = \frac{(-1)^{r+1}2^{2r+3}}{q} \sum_{d=1}^{q/2} \cos\left(\frac{\pi(2d-1)t}{q}\right) \left(\sin\left(\frac{\pi(2d-1)}{2q}\right)\right)^{2r+2},$$

$$L_{101}(r,t,q) = \frac{(-1)^r 2^{2r+2}}{q} \sum_{d=1}^{q/2} \sin\left(\frac{\pi(2d-1)(1-2t)}{2q}\right) \left(\sin\left(\frac{\pi(2d-1)}{2q}\right)\right)^{2r+1},$$

$$L_{110}(r,t,q) = \frac{(-1)^{r+1}2^{2r+3}}{q} \sum_{d=1}^{(q-1)/2} \cos\left(\frac{2\pi dt}{q}\right) \left(\sin\frac{\pi d}{q}\right)^{2r+2},$$

and

$$L_{111}(r,t,q) = \frac{(-1)^r 2^{2r+2}}{q} \sum_{d=1}^{(q-1)/2} \sin\left(\frac{\pi d(1-2t)}{q}\right) \left(\sin\frac{\pi d}{q}\right)^{2r+1}.$$

Proof. The result follows on substitution of each set of parameters into Theorem 4.4.2, and expressing shifted cosines as sines when applicable. $\square$

Remark. When the parameter $c = 1$ and the variable $t = 1$, that is for the sequences $\mathcal{L}_{s;ab1}(r,1,q)$, we find that the sums conveniently simplify and can, therefore, be expressed concisely as the sum of $(r+1)$-th powers of $4\gamma\cos^2(\pi X/2q)$, where X is a sum involving d, a, b, s and q. This is elaborated on with some examples in Appendix $A.2$.

4.5 Divisibility properties of the sequences $\mathcal{F}_{s;ab}(r,t,q)$

We seek to determine to what extent the known divisibility properties established by Fleck [17] and Weisman [44] can be extended to the generalised Fleck sums $\mathcal{F}_{s;ab}(r,t,q)$.

Definition 4.5.1. *We denote the highest exponent of the prime p in the nonzero integer F by $\mathrm{ord}_p(F)$. This is referred to as the p-adic valuation of F. If $F = 0$ we write $\mathrm{ord}_p(F) = \infty$.*

We state Weisman's result [44] as our Lemma 4.5.1.

LEMMA 4.5.1. *When $q = p^e$, where p is an odd prime and e is a positive integer, the following conditions hold:*

$$ord_p(\mathcal{F}_{s;11}(r,t,p^e)) \geq \left\lfloor \frac{r - p^{e-1}}{p^{e-1}(p-1)} \right\rfloor, \text{ and } ord_2(\mathcal{F}_{s;00}(r,t,2^e)) \geq \left\lfloor \frac{r - 2^{e-1}}{2^{e-1}} \right\rfloor. \quad (4.5.1)$$

Proof. We recall the result (1.1.3) of Weisman.

$$\sum_{k \equiv t \pmod{p^e}} (-1)^k \binom{r}{k} \equiv 0 \pmod{p^\alpha}, \quad \text{where } \alpha = \left\lfloor \frac{r - p^{e-1}}{\phi(p^e)} \right\rfloor. \quad (4.5.2)$$

From (2.2.6) and (2.2.5) we are able to apply (4.5.2) directly to the sums $\mathcal{F}_{s;11}(r,t,p^e)$ and $\mathcal{F}_{s;00}(r,t,2^e)$ respectively. $\qquad \square$

The second inequality of (4.5.1) can take the succinct form as given in Corollary 1 below.

COROLLARY 1. *When $q = 2^e$ and $r = j2^e + l$, where $j \geq 1$ and $0 \leq l < 2^e$, we have*

$$ord_2(\mathcal{F}_{s;00}(r,t,2^e)) \geq \begin{cases} 2j - 1 & \text{if } l = 0 \\ 2j & \text{if } 1 \leq l \leq 2^e - 1. \end{cases} \quad (4.5.3)$$

Proof. The result follows upon substitution and simplification of the second inequality of (4.5.1). $\qquad \square$

When $e = 1$, then $q = p$ and Lemma 4.5.1 simplifies to (1.1.2), that is, the result of Fleck [17], producing Corollary 2 (below).

COROLLARY 2. *For $q = p$, an odd prime, we have the following inequalities*

$$ord_p(\mathcal{F}_{s;11}(r,t,p)) \geq \left\lfloor \frac{r-1}{p-1} \right\rfloor \text{ and } ord_2(\mathcal{F}_{s;00}(r,t,2)) \geq r - 1.$$

Proof. This is how the result of Lemma 4.5.1 simplifies when $e = 1$. $\qquad \square$

Remark. In actual fact $ord_2(\mathcal{F}_{s;00}(r,t,2)) = r - 1$.

Lemma 4.5.1 establishes divisibility properties for certain types of Fleck sums when $q = p^e$ is a power of a prime p. We next employ some further lemmas to examine relations between the various types of sums $\mathcal{F}_{s;ab}(r,t,q)$.

LEMMA 4.5.2. *When $r = 2nq + q + 2t$, where $n \geq 0$, we have*

$$\mathcal{F}_{s;0b}(r,t,q) = 2\mathcal{F}_{s;00}(r,t,2q).$$

Proof. When $r = 2nq + q + 2t$ and the modulus is $2q$ by symmetry we can manipulate the residue class t of the sum $\mathcal{F}_{s;00}(r,t,2q)$ as

$$\mathcal{F}_{s;00}(r,t,2q) = \gamma^r \mathcal{F}_{s;00}(r,r-t,2q) = \gamma^q \mathcal{F}_{s;00}(r,2nq+q+t,2q) = \gamma^q \mathcal{F}_{s;00}(r,t+q,2q)$$

and so

$$\mathcal{F}_{s;00}(2nq + q + 2t, t, 2q) = \gamma^q \mathcal{F}_{s;00}(2nq + q + 2t, t + q, 2q).$$

However, from Theorem 2.2.5 we also have

$$\mathcal{F}_{s;0b}(2nq + q + 2t, t, q) = \mathcal{F}_{s;00}(2nq + q + 2t, t, 2q) + \gamma^q \mathcal{F}_{s;00}(2nq + q + 2t, t + q, 2q)$$
$$= 2\mathcal{F}_{s;00}(2nq + q + 2t, t, 2q)$$

as required. $\qquad\square$

LEMMA 4.5.3. *We have the following relations:*

$$\mathcal{F}_{s;11}(r, t, q) + \mathcal{F}_{s;01}(r, t, q) = 2\mathcal{F}_{s;00}(r, t, 2q), \qquad (4.5.4)$$

and

$$\mathcal{F}_{s;00}(r, t, q) + \mathcal{F}_{s;10}(r, t, q) = 2\mathcal{F}_{s;00}(r, t, 2q). \qquad (4.5.5)$$

Proof. From Theorem 2.2.5 we have

$$\mathcal{F}_{s;ab}(r, t, q) = \sum_{k \equiv t \pmod{2q}} (-1)^{sk} \binom{r}{k} + (-1)^{a+sb} \sum_{k \equiv t+q \pmod{2q}} (-1)^{sk} \binom{r}{k}, \qquad (4.5.6)$$

and so from (2.2.6),

$$\mathcal{F}_{s;11}(r, t, q) + \mathcal{F}_{s;01}(r, t, q)$$
$$= \sum_{k \equiv t \pmod{2q}} (-1)^{sk} \binom{r}{k} + (-1)^{1+s} \sum_{k \equiv t+q \pmod{2q}} (-1)^{sk} \binom{r}{k}$$
$$+ \sum_{k \equiv t \pmod{2q}} (-1)^{sk} \binom{r}{k} + (-1)^{s} \sum_{k \equiv t+q \pmod{2q}} (-1)^{sk} \binom{r}{k}$$
$$= 2 \sum_{k \equiv t \pmod{2q}} (-1)^{sk} \binom{r}{k} = 2\mathcal{F}_{s;00}(r, t, 2q),$$

and from (2.2.5),

$$\mathcal{F}_{s;00}(r, t, q) + \mathcal{F}_{s;10}(r, t, q)$$
$$= \sum_{k \equiv t \pmod{2q}} (-1)^{sk} \binom{r}{k} + \sum_{k \equiv t+q \pmod{2q}} (-1)^{sk} \binom{r}{k}$$
$$+ \sum_{k \equiv t \pmod{2q}} (-1)^{sk} \binom{r}{k} - \sum_{k \equiv t+q \pmod{2q}} (-1)^{sk} \binom{r}{k}$$
$$= 2 \sum_{k \equiv t \pmod{2q}} (-1)^{sk} \binom{r}{k} = 2\mathcal{F}_{s;00}(r, t, 2q).$$

$$\square$$

COROLLARY. *When $r = 2nq + q + 2t$, where n is a nonnegative integer we have*

$$\mathcal{F}_{s;1b}(2nq + q + 2t, t, q) = 0.$$

Proof. We substitute the result of Lemma 4.5.2 into (4.5.4) and (4.5.5) of Lemma 4.5.3. □

Equation (4.5.5) indicates a divisibility property of the sum $\mathcal{F}_{s;10}(r, t, 2^e)$ that we explore in Lemma 4.5.4.

LEMMA 4.5.4. *With $q = 2^e$ and $r = j2^e + l$, where $j \geq 1$ and $0 \leq l < 2^e$, we have*

$$ord_2(\mathcal{F}_{s;10}(r, t, 2^e)) = ord_2(\mathcal{F}_{s;00}(r, t, 2^{e+1})) \geq j.$$

Proof. Rearranging (4.5.5), we have

$$\mathcal{F}_{s;10}(r, t, q) = 2\mathcal{F}_{s;00}(r, t, 2q) - \mathcal{F}_{s;00}(r, t, q) \tag{4.5.7}$$

and so

$$
\begin{aligned}
ord_2(\mathcal{F}_{s;10}(r, t, 2^e)) &= \min\left(ord_2(2\mathcal{F}_{s;00}(r, t, 2^{e+1})), ord_2(\mathcal{F}_{s;00}(r, t, 2^e))\right) \\
&\geq \min\left(ord_2\left(2^{\left\lfloor \frac{r-2^e}{2^e} \right\rfloor + 1}\right), ord_2\left(2^{\left\lfloor \frac{r-2^{e-1}}{2^{e-1}} \right\rfloor}\right)\right) \\
&\geq \min\left(ord_2\left(2^{\left\lfloor \frac{j2^e + l - 2^e}{2^e} \right\rfloor + 1}\right), ord_2\left(2^{\left\lfloor \frac{j2^{e+1} + 2l - 2^e}{2^e} \right\rfloor}\right)\right) \\
&\geq \min(j, 2j - 1) \geq j
\end{aligned}
$$

since $l \leq 2^e - 1$ and $j \geq 1$.

When $r = m2^{e+1} + 2^e + 2t$, $(m \geq 0)$, then from the Corollary of Lemma 4.5.3, we have that $\mathcal{F}_{s;10}(r, t, q) = 0$. However, by definition $ord_2(0) = \infty > j$ as required. □

4.5.1 The primality of the term $1 - \zeta_q$

At this point we are yet to state any divisibility properties of the sum $\mathcal{F}_{s;01}(r, t, q)$, or to the more general case of each of the functions $\mathcal{F}_{s;ab}$ when the modulus q has more than one distinct prime factor. We next seek to address these questions, but this first requires a brief examination of prime ideal factors in the cyclotomic field $\mathbb{Q}(\zeta_q)$ that comprise the terms $\mathcal{F}_{s;ab}(r, t, p^e)$.

LEMMA 4.5.5. *When q is a prime power p^e and ζ_q is a primitive q-th root of unity, the term $1 - \zeta_q$ is prime in $\mathbb{Q}(\zeta_q)$.*

Proof. Since $q = p^e$ and ζ_q is a primitive q-th root of unity, we have $\zeta_q^q = 1$, but $\zeta_q^{q/p} \neq 1$, and so ζ_q is a root of the equation

$$0 = \frac{x^q - 1}{x^{q/p} - 1} = 1 + x^{q/p} + x^{2q/p} + \ldots + x^{(p-1)q/p}, \tag{4.5.8}$$

and the degree of the field $\mathbb{Q}(\zeta_q)$ is $f = \phi(q) = p^{e-1}(p-1)$.

Now let $\lambda = 1 - \zeta_q$, so that on rearrangement $\zeta_q = 1 - \lambda$. With $y = 1 - x$, the minimal polynomial of ζ_q on the right hand side of (4.5.8) becomes

$$f(y) = 1 + (1-y)^{q/p} + (1-y)^{2q/p} + \ldots + (1-y)^{(p-1)q/p}. \tag{4.5.9}$$

On expansion of (4.5.9) with $f = (p-1)q/p$, (since there are p terms) we obtain the simplified minimal polynomial of λ:

$$f(y) = y^f + a_{f-1}y^{f-1} + \ldots + a_1 y + p. \tag{4.5.10}$$

In $\mathbb{Q}(\zeta_p)$, (4.5.10) then factorises as

$$f(y) = (y - \lambda_1)(y - \lambda_2)\ldots(y - \lambda_f), \tag{4.5.11}$$

where $\lambda_i = (1 - \zeta_q^{k_i})$ for $1 \le i \le f$ and $(k_i, q) = 1$.

From (4.5.10) and (4.5.11) we now have [1]

$$\mathrm{Norm}(\lambda_i) = \prod_{i=1}^{f} \lambda_i = \prod_{k_i}^{*}(1 - \zeta_q^{k_i}) = p, \tag{4.5.12}$$

and since p is a rational prime integer, we see from (4.5.12) that each of the (field) integers λ_i are also prime. $\qquad\square$

LEMMA 4.5.6. *Each of the numbers $\lambda_i = (1-\zeta_q^{k_i})$ with $1 \le i \le f$ and $(k_i, p^e) = 1$ generates the same principal degree 1 prime ideal in $\mathbb{Q}(\zeta_q)$. Furthermore p is a totally ramified prime in $\mathbb{Q}(\zeta_q)$.*

Proof. From (4.5.12) of Lemma 4.5.5 we have that as (field) integers in $\mathbb{Q}(\zeta_q)$, $1 - \zeta_q^{k_i}$ is prime when $(k_i, q) = 1$. So that as ideals we have

$$\langle 1 - \zeta_q^{k_i} \rangle \mid \langle p \rangle.$$

However,

$$1 - \zeta_q^{k_i} = (1 - \zeta_q)(1 + \zeta_q + \zeta_q^2 + \ldots + \zeta_q^{k_i - 1})$$

and so similarly as ideals

$$\langle 1 - \zeta_q^{k_i} \rangle = \langle 1 - \zeta_q \rangle \langle 1 + \zeta_q + \zeta_q^2 + \ldots + \zeta_q^{k_i - 1} \rangle. \tag{4.5.13}$$

We denote $N(\langle \lambda \rangle)$ as the *norm* of the ideal $\langle \lambda \rangle$. Now since it is a totally multiplicative function, we take the *norm* of both sides of (4.5.13) and using (4.5.12) we have

$$p = N(\langle 1 - \zeta_q^{k_i} \rangle) = N(\langle 1 - \zeta_q \rangle)N(\langle 1 + \zeta_q + \zeta_q^2 + \ldots + \zeta_q^{k_i - 1} \rangle)$$
$$= pN(\langle 1 + \zeta_q + \zeta_q^2 + \ldots + \zeta_q^{k_i - 1} \rangle)$$
$$= p.1.$$

---

[1]* indicates that the product is taken over the set of values, where k_i is relatively prime to and less than q.

That is, the ideal $\langle 1 + \zeta_q + \zeta_q^2 + \ldots + \zeta_q^{k_i - 1} \rangle$ is a unit and each of the ideals $\langle \lambda_i \rangle = \langle 1 - \zeta_q^{k_i} \rangle$ is the same prime ideal $\langle \lambda \rangle$. Therefore, in $\mathbb{Q}(\zeta_q)$ the prime $\langle \lambda \rangle$ is a principal ideal number, so that we have as ideals

$$\langle \prod_{i=1}^{f} \lambda_i \rangle = \prod_{i=1}^{f} \langle \lambda_i \rangle = \langle \lambda_i \rangle^f = \langle p \rangle, \tag{4.5.14}$$

demonstrating that p is a totally ramified prime in $\mathbb{Q}(\zeta_q)$. $\qquad \square$

In the sums $\mathcal{F}_{0;ab}(r, t, p^e)$, we consider the terms $1 + \zeta_{2q}$ instead of those of $1 - \zeta_q$, so its seems pertinent in the next two lemmas to demonstrate the equivalence of the term $1 + \zeta_{2q}^g$, where $(g, 2q) = 1$, to that of a corresponding term $1 - \zeta_q^d$, where $(d, q) = 1$.

LEMMA 4.5.7. *With modulus $q = p^e$, where $p > 2$ is prime and ζ_q a primitive q-th root of unity we have*

$$\prod_{d}^{*} (1 - \zeta_q^d) = \prod_{g}^{*} (1 + \zeta_{2q}^g). \tag{4.5.15}$$

Proof. When $(d, p) = 1$, then

$$-\zeta_q^d = -e^{\frac{2\pi i d}{q}} = e^{\frac{2\pi i q}{2q}} e^{\frac{2\pi i 2d}{2q}} = e^{\frac{2\pi i (2d+q)}{2q}} = e^{\frac{2\pi i g}{2q}} = \zeta_{2q}^g.$$

Therefore,

$$1 - \zeta_q^d = 1 + \zeta_{2q}^g,$$

and with $g = 2d + q$ (reduced mod $2q$ if necessary), since $(2d, q) = 1$, we have $(g, 2q) = 1$ so that the mapping is injective. Moreover, $\phi(q) = \phi(2q) = p^{e-1}(p-1)$ and so it is also surjective. Consequently we obtain (4.5.15) as required. $\qquad \square$

LEMMA 4.5.8. *With modulus $q = 2^e$ and ζ_q a primitive q-th root of unity we have*

$$\prod_{d}^{*} (1 - \zeta_q^d) = \prod_{d}^{*} (1 + \zeta_q^d). \tag{4.5.16}$$

Proof. Let $q = 2^e$ and $(d, 2) = 1$, then

$$-\zeta_q^d = -e^{\frac{2\pi i d}{q}} = e^{\frac{2\pi i q}{2q}} e^{\frac{2\pi i 2d}{2q}} = e^{\frac{2\pi i (2d+2^e)}{2^{e+1}}} = e^{\frac{2\pi i (d+2^{e-1})}{2^e}} = e^{\frac{2\pi i g}{q}} = \zeta_q^g.$$

Therefore,

$$1 - \zeta_q^d = 1 + \zeta_q^g,$$

and with $g = d + q/2$ (reduced mod q if necessary), since $(d, 2^{e-1}) = 1$, we have $(g, q) = 1$ so that the mapping is injective. Moreover, it is clearly also surjective. Consequently each value of g corresponds to one value of d and so we obtain (4.5.16) as required. $\qquad \square$

Now in Lemma 4.5.9 and its Corollary, we are in a position to examine the more general case: when the modulus q has more than one distinct prime factor.

LEMMA 4.5.9 (Huxley's Disappointing Lemma). *When q has two distinct prime factors p and p', then $1 - \zeta_q$ is a unit.*

Proof. Let $q = pp'r$, $\omega = \zeta_q^{p'r}$, $\omega' = \zeta_q^{pr}$. Then ω is a primitive p-th root of unity, and ω' is a primitive p'-th root of unity, so that

$$\langle 1 - \zeta_q \rangle \mid \langle 1 - \omega \rangle \mid \langle p \rangle, \text{ and } \langle 1 - \zeta_q \rangle \mid \langle 1 - \omega' \rangle \mid \langle p' \rangle$$

and, therefore,

$$\langle 1 - \zeta_q \rangle \mid \langle p, p' \rangle.$$

However, $\langle p, p' \rangle = \langle 1 \rangle$, and therefore, $\langle 1 - \zeta_q \rangle$ is a unit. $\qquad\square$

COROLLARY. *When $q = pp'r$, where p and p' are two distinct prime factors, then the term $\mathcal{F}_{s;ab}(r, t, q)$ possesses no divisibility properties.*

Proof. From Lemma 4.5.9 when the modulus q has more than one distinct prime factor, the ideal $\langle 1 - \zeta_q \rangle$ is a unit. Therefore, from (4.5.14) of Lemma 4.5.6 we will have

$$\langle 1 - \zeta_q \rangle^f = \langle 1 \rangle,$$

and we cannot expect to observe regular powers of the prime factors, comprising the modulus q, to appear in the sums $\mathcal{F}_{s;ab}(r, t, q)$. $\qquad\square$

Finally we summarise the divisibility properties contained in the above Lemmas with Theorem 4.5.10.

THEOREM 4.5.10. *The only expected divisibility properties of the sums $\mathcal{F}_{s;ab}(r, t, q)$ are:*

1.

$$ord_p(\mathcal{F}_{s;11}(r, t, p^e)) \geq \left\lfloor \frac{r - p^{e-1}}{p^{e-1}(p-1)} \right\rfloor,$$

and when $r = j2^e + l$, where $j \geq 1$ and $0 \leq l < 2^e$,

2.

$$ord_2(\mathcal{F}_{s;00}(r, t, 2^e)) \geq \left\lfloor \frac{r - 2^{e-1}}{2^{e-1}} \right\rfloor = \begin{cases} 2j - 1 & \text{if } l = 0 \\ 2j & \text{if } 1 \leq l \leq 2^e - 1, \end{cases}$$

and

3.

$$ord_2(\mathcal{F}_{s;10}(r, t, 2^e)) \geq j.$$

Proof. Divisibility property 1 follows from Lemma 4.5.1; inequality 2 from Lemma 4.5.1 and Corollary 1 of this lemma and finally inequality 3 from Lemma 4.5.4. The fact that these particular sums are the only ones to be expected to possess these divisibility properties follow from Lemma 4.5.9 and its Corollary, that prevents the sums $\mathcal{F}_{s;ab}(r, t, q)$ with moduli q of more than one distinct prime factor from containing a regular power of these prime factors $p_1, p_2, \ldots, p_n$. $\qquad\square$

Chapter 5

Recurrences

In this chapter our primary interest lies in the establishment of an order n linear recurrence relation involving $n+1$ consecutive terms for each sequence t of the sums $\mathcal{L}_{s;abc}(r,t,2m+b)$. We denote this linear recurrence polynomial by $\mathcal{R}_{s;ab}(x,m)$ and unless $a = 0$ and $b = 1$, we find that $n = m$. In this individual case we have that $n = m+1$. Previous studies ([9], [38] and [39]) have determined that the Fibonacci ($F_Q(x)$) and Lucas ($L_Q(x)$) polynomials and the (monic) Chebyshev polynomials of the first ($C_Q(x)$) and second ($S_Q(x)$) kind are central to these recurrence relation polynomials and consequently we devote a section to the development of each of the four types. In Theorem 5.4.1 and the Corollary to it, we combine these four types into a single polynomial that we denote as $\mathcal{A}^r_{ab}(x,Q)$. Finally in Theorem 5.6.1 we then express $\mathcal{R}_{ab}(x,m)$ in terms of $\mathcal{A}^r_{ab}(x,Q)$.

The Jacobsthal and Jacobsthal-Lucas polynomials are employed in Chapter 8 to express the generating function of the functions $\mathcal{L}_{s;abc}$, but due to their close respective relations to the Fibonacci and Lucas polynomials they are also discussed alongside these latter poynomials. We start by considering a polynomial $\mathcal{A}_Q(x)$ that encompasses the four polynomials $F_Q(x)$, $L_Q(x)$, $C_Q(x)$ and $S_Q(x)$, which we refer to collectively as "Fibonacci type" polynomials.

Definition 5.0.1. *With $Q = 2M + e$, where $M \geq 0$ and $0 \leq e \leq 2$ are integers, we denote by $A_{2M+e}(x)$ a polynomial that represents the Qth Fibonacci, Lucas or (monic) Chebyshev polynomial of the first or second kind. Accordingly we write*

$$A_{2M+e}(x) = \sum_{k=0}^{M}(-\gamma)^{M-k}B(k,M)x^{2k+e-f}$$

as the generalised sum form of the polynomial. Here f is determined by

$$f = \begin{cases} 0 & \text{if } A_Q(x) \neq F_Q(x) \\ 1 & \text{if } A_Q(x) = F_Q(x), \end{cases}$$

where $F_Q(x)$ is the Qth Fibonacci polynomial; γ is given by

$$\gamma \text{ is } \begin{cases} -1 & \text{if } A_Q(x) \text{ is a Fibonacci or Lucas polynomial} \\ 1 & \text{if } A_Q(x) \text{ is a (monic) Chebyshev polynomial,} \end{cases}$$

and finally $B(k, M)$ is a function of one or binomial coefficients.

Remark. Defining Q in this manner puts emphasis on the fact that the function summed up to M.

Definition 5.0.2. *For function A_{2M+e} defined as in Definition 5.0.1 and parameter $r \in \{0, 1\}$ we also have the modified polynomial $A_{2M+e}^r(x)$ such that*

$$A_{2M+e}^r(x) = (\sqrt{x})^{f-e} A_{2M+e}(\sqrt{x}) = \sum_{k=0}^{M} (-\gamma)^{M-k} B(k, M) x^k.$$

Remark. Here the parameter r represents a "(square) rooting" of the variable x of the original function, A_Q.

The notation employed in Definitions 5.0.1 and 5.0.2 have been selected to mirror as closely as possible the current formats of each of the two aforementioned groups of polynomials.

5.1 Fibonacci, Lucas and Jacobsthal polynomials

The Fibonacci and Lucas ploynomials are well documented and for a detailed exposition of these polynomials the reader is directed to [29]. The main purpose of their inclusion is to examine their polynomial representation as binomial sums that will be pertinent to our current work in terms of the recurrence polynomials and the generating functions. To this end the Jacobsthal polynomials are also important to us. However, on researching these latter polynomials in the literature this author feels that there is some ambuigity as to the precise definition of these polynomials (and numbers) and so a little work has been done to hopefully help clarify this matter.

5.1.1 The Fibonacci polynomials, $F_n(x)$

With $F_0(x) = 0$ and $F_1(x) = 1$, the Fibonacci polynomials are defined by the recurrence formula

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x). \tag{5.1.1}$$

With $x = 1$, we have that $F_n(1) = F_n$, the n-th Fibonacci number. These polynomials are generated more efficiently, either (for $n \geq 2$) by a product formula, or (for $n \geq 1$) by a binomial sum, given as

$$F_n(x) = \prod_{k=1}^{n-1} \left(x - 2\imath \cos \frac{k\pi}{n} \right) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{k} x^{n-2k-1}.$$

Of particular interest to us in this present study are considering separately the specific cases of $n = 2m + b$, being either an even ($n = 2m = 2(m-1) + 2$), or an odd ($n = 2m + 1$) number. These even and odd forms make reference to the fact that the upper limit of the

sum is $M = m - (1 - b)$.

Therefore, replacing n with $2M + 2 - b$, according as to the parity of n, we write the nth Fibonacci polynomial as

$$F_{2M+2-b}(x) = \sum_{k=0}^{M} \binom{2M + 1 - b - k}{k} x^{2M+1-b-2k}, \tag{5.1.2}$$

or on reversing the summation

$$\sum_{k=0}^{M} \binom{M + k + (1 - b)}{2k + (1 - b)} x^{2k+(1-b)}.$$

From our current perspective, it is of value to us to express (5.1.2) as a polynomial where the power of each term decreases uniformly by 1. We achieve this quite simply by making the substitution $\sqrt{x}$ for x, and multiplying through by $(\sqrt{x})^{b-1}$. Now employing the notation of Definition 5.0.2 we have

$$F_{2M+2-b}^{r}(x) = (\sqrt{x})^{b-1} F_{2M+2-b}(\sqrt{x}) = \sum_{k=0}^{M} \binom{2M + 1 - b - k}{k} x^{M-k} = \sum_{k=0}^{M} \binom{M + k + 1 - b}{2k + 1 - b} x^{k}. \tag{5.1.3}$$

When $b = 0$, (5.1.3) becomes

$$F_{2M+2}^{r}(x) = (\sqrt{x})^{-1} F_{2M+2}(\sqrt{x}) = \sum_{k=0}^{M} \binom{2M + 1 - k}{k} x^{M-k} = \sum_{k=0}^{M} \binom{M + k + 1}{2k + 1} x^{k}, \tag{5.1.4}$$

and when $b = 1$,

$$F_{2M+1}^{r}(x) = F_{2M+1}(\sqrt{x}) = \sum_{k=0}^{M} \binom{2M - k}{k} x^{M-k} = \sum_{k=0}^{M} \binom{M + k}{2k} x^{k}. \tag{5.1.5}$$

Table 5.1: The Fibonacci polynomials $F_n(x)$ and the modified Fibonacci polynomials $F_n^r(x)$, for $1 \leq n \leq 8$, with $n = 2m + b = 2M + 2 - b$, where $M = m - (1 - b)$ and b is the parity of n.

n	m	M	b	$F_n(x)$	$F_n^r(x)$
1	0	0	1	1	1
2	1	0	0	x	1
3	1	1	1	$x^2 + 1$	$x + 1$
4	2	1	0	$x^3 + 2x$	$x + 2$
5	2	2	1	$x^4 + 3x^2 + 1$	$x^2 + 3x + 1$
6	3	2	0	$x^5 + 4x^3 + 3x$	$x^2 + 4x + 3$
7	3	3	1	$x^6 + 5x^4 + 6x^2 + 1$	$x^3 + 5x^2 + 6x + 1$
8	4	3	0	$x^7 + 6x^5 + 10x^3 + 4x$	$x^3 + 6x^2 + 10x + 4$

5.1.2 The Lucas polynomials, $L_n(x)$

Employing the standard convention, we denote by $L_n(x)$, the n-th Lucas polynomial. With the initial values $L_0(x) = 2$ and $L_1(x) = x$, these polynomials obey the same recurrence relation as (5.1.1), so that

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x).$$

The product formula is, with $n \geq 1$, given by (see [29])

$$L_n(x) = \prod_{k=0}^{n-1} \left(x - 2\imath \cos \frac{(2k+1)\pi}{2n} \right).$$

However, the simple amendment of $L_0(x) = 2$, as opposed to $F_0(x) = 0$, has implications for the (binomial) sum. For now two binomial terms generate the n-th polynomial, given by

$$L_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] x^{n-2k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}. \tag{5.1.6}$$

On putting $n = 2m + b$, (5.1.6) becomes

$$L_{2m+b}(x) = \sum_{k=0}^{m} \frac{2m+b}{2m+b-k} \binom{2m+b-k}{k} x^{2m+b-2k} = \sum_{k=0}^{m} \frac{2m+b}{m+b+k} \binom{m+k+b}{2k+b} x^{2k+b}.$$

Remark. Since $M = m$, for the purpose of clarity it is preferable to use m rather than M.

Once more, we replace x with $\sqrt{x}$, and multiply through by $(\sqrt{x})^{-b}$, and using the notation of Definition 5.0.2, we have

$$L_{2m+b}^r(x) = (\sqrt{x})^{-b} L_{2m+b}(\sqrt{x}) = \sum_{k=0}^{m} \frac{2m+b}{2m+b-k} \binom{2m+b-k}{k} x^{m-k}$$

$$= \sum_{k=0}^{m} \frac{2m+b}{m+k+b} \binom{m+k+b}{2k+b} x^k. \tag{5.1.7}$$

Then when $b = 0$, (5.1.7) becomes

$$L_{2m}^r(x) = L_{2m}(\sqrt{x}) = \sum_{k=0}^{m} \frac{2m}{2m-k} \binom{2m-k}{k} x^{m-k} = \sum_{k=0}^{m} \frac{2m}{m+k} \binom{m+k}{2k} x^k, \tag{5.1.8}$$

and when $b = 1$, we have

$$L_{2m+1}^r(x) = (\sqrt{x})^{-1} L_{2m+1}(\sqrt{x}) = \sum_{k=0}^{m} \frac{2m+1}{2m+1-k} \binom{2m+1-k}{k} x^{m-k}$$

$$= \sum_{k=0}^{m} \frac{2m+1}{m+k+1} \binom{m+k+1}{2k+1} x^k. \tag{5.1.9}$$

Remark. An important point is that the substitution of $-x$ for x shifts the zeroes of the polynomial from the imaginary axis to the real axis.

Table 5.2: The Lucas polynomials, $L_n(x)$, and the modified Lucas polynomials, $L_n^r(x)$, for $1 \leq n \leq 8$ with $n = 2m + b$ where b is the parity of n.

n	m	b	$L_n(x)$	$L_n^r(x)$
1	0	1	x	1
2	1	0	$x^2 + 2$	$x + 2$
3	1	1	$x^3 + 3x$	$x + 3$
4	2	0	$x^4 + 4x^2 + 2$	$x^2 + 4x + 2$
5	2	1	$x^5 + 5x^3 + 5x$	$x^2 + 5x + 5$
6	3	0	$x^6 + 6x^4 + 9x^2 + 2$	$x^3 + 6x^2 + 9x + 2$
7	3	1	$x^7 + 7x^5 + 14x^3 + 7x$	$x^3 + 7x^2 + 14x + 7$
8	4	0	$x^8 + 8x^6 + 20x^4 + 16x^2 + 2$	$x^4 + 8x^3 + 20x^2 + 16x + 2$

5.2 The Jacobsthal and Jacobsthal-Lucas Polynomials

5.2.1 The Jacobsthal Polynomials

In 1919, Jacobsthal [27] defined (using later notation) the Jacobsthal polynomials (for $n \geq 2$) by the recurrence relation

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x). \tag{5.2.1}$$

As in the case of the Fibonacci polynomials we have initial conditions $J_0 = 0$ and $J_1 = 1$. From this relation we establish the Jacobsthal polynomial, $J_n(x)$, written as a sum is

$$J_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{k} x^k. \tag{5.2.2}$$

With $n = 2M + 2 - b$, where $M = m - (1 - b)$, (5.2.2) becomes either

$$J_{2(m-1)+2}(x) = \sum_{k=0}^{m-1} \binom{2(m-1)-k+1}{k} x^k \quad \text{or} \quad J_{2m+1}(x) = \sum_{k=0}^{m} \binom{2m-k}{k} x^k. \tag{5.2.3}$$

Remark. On examination one sees that the Jacobsthal polynomials have identical binomial coefficients to the Fibonacci polynomials but differ in the exponent of the variable x. On reversing the order of these polynomial coefficients we thus obtain our modified Fibonacci functions F_{2M+2-b}^r.

Setting $x = 1$, we have $J_n(1) = F_n(1) = F_n$, the nth Fibonacci number. For application of these polynomials as defined in (5.2.2) we refer the reader to such works as Bergram and associates [6], Hoggatt Jr. and Bicknell-Johnson [21] and Koshy [29].

In a series of three papers commencing in 1988, [22], [23] and [24], Horadam introduced an additional factor of 2 into the recurrence (5.2.1) producing the altered relation (using, to avoid confusion, our own notation)

$$J_n^{(2)}(x) = J_{n-1}^{(2)}(x) + 2xJ_{n-2}^{(2)}(x), \tag{5.2.4}$$

with corresponding sum

$$J_n^{(2)}(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{k}(2x)^k. \tag{5.2.5}$$

His rational reflected a desire to mirror the Pell polynomials that he was also working on at the time. A point that he alluded to in the latter of these papers. However, subsequent authors such as Swamy (1999) [43] and Djordjevic (2000) [16] have taken (5.2.4) as the standard definition of a Jacobsthal polynomial (also using the notation $J_n(x)$) and as a consequence this now appears as the definition on sites such OEIS, Wikipedia and Wolfram Mathworld.

Of course, as is evident by comparing (5.2.2) and (5.2.5), we can easily relate the polynomials $J_n^{(2)}(x)$ and $J_n(x)$ by

$$J_n(x) = J_n^{(2)}(x/2).$$

Table 5.3: The Jacobsthal polynomials $J_n^{(2)}(x)$ and the polynomials $J_n(x)$ and $J_n(-x)$, for $1 \le n \le 8$ with $n = 2m + b = 2M + 2 - b$, where b is the parity of n.

n	m	M	b	$J_n^{(2)}(x)$	$J_n(x)$	$J_n(-x)$
1	0	0	1	1	1	1
2	1	0	0	1	1	1
3	1	1	1	$1 + 2x$	$1 + x$	$1 - x$
4	2	1	0	$1 + 4x$	$1 + 2x$	$1 - 2x$
5	2	2	1	$1 + 6x + 4x^2$	$1 + 3x + x^2$	$1 - 3x + x^2$
6	3	2	0	$1 + 8x + 12x^2$	$1 + 4x + 3x^2$	$1 - 4x + 3x^2$
7	3	3	1	$1 + 10x + 24x^2 + 8x^3$	$1 + 5x + 6x^2 + x^3$	$1 - 5x + 6x^2 - x^3$
8	4	3	0	$1 + 12x + 40x^2 + 32x^3$	$1 + 6x + 10x^2 + 4x^3$	$1 - 6x + 10x^2 - 4x^3$

5.2.2 The Jacobsthal-Lucas polynomials

As the Lucas ploynomials are a sister sequence to the Fibonacci polynomials, it would seem only natural to have a parallel companion to the Jacobsthal polynomials. Surprisingly it appears that this does not seem to have been documented until Horadam (1997) [24]. Horadam used the notation $j_n(x)$ for the nth Jacobsthal-Lucas polynomial.

It seems apt to employ the notation of Horadam, but with the caveat that

$$j_n(x) = j_{n-1}(x) + xj_{n-2}(x),$$

with the initial values $j_0(x) = 2$ and $j_1(x) = 1$ This produces for the nth polynomial

$$j_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} x^k. \tag{5.2.6}$$

Then with $n = 2m$, and $n = 2m + 1$ respectively, we have

$$j_{2m}(x) = \sum_{k=0}^{m} \frac{2m}{2m-k} \binom{2m-k}{k} x^k, \qquad \text{and} \qquad j_{2m+1}(x) = \sum_{k=0}^{m} \frac{2m+1}{2m+1-k} \binom{2m+1-k}{k} x^k. \tag{5.2.7}$$

Again if we employ the notation $j_n^{(2)}(x)$ for Horadam's Jacobsthal-Lucas polynomials we have the relation

$$j_n(x) = j_n^{(2)}(x/2).$$

Table 5.4: The Jacobsthal-Lucas polynomials $j_n^{(2)}(x)$ and the polynomials $j_n(x)$ and $j_n(-x)$, for $1 \le n \le 8$, with $n = 2m + b$, where b is the parity of n.

n	m	b	$j_n^{(2)}(x)$	$j_n(x)$	$j_n(-x)$
1	0	1	1	1	1
2	1	0	$1 + 4x$	$1 + 2x$	$1 - 2x$
3	1	1	$1 + 6x$	$1 + 3x$	$1 - 3x$
4	2	0	$1 + 8x + 8x^2$	$1 + 4x + 2x^2$	$1 - 4x + 2x^2$
5	2	1	$1 + 10x + 20x^2$	$1 + 5x + 5x^2$	$1 - 5x + 5x^2$
6	3	0	$1 + 12x + 36x^2 + 16x^3$	$1 + 6x + 9x^2 + 2x^3$	$1 - 6x + 9x^2 - 2x^3$
7	3	1	$1 + 14x + 56x^2 + 56x^3$	$1 + 7x + 14x^2 + 7x^3$	$1 - 7x + 14x^2 - 7x^3$

To complete this section we will complete two lemmas that illustrate the inter-relationship between the Fibonacci and Jacobsthal polynomials and then the Lucas and Jacobsthal-Lucas polynomials.

LEMMA 5.2.1 (Jacobsthal reciprocal)**.**

$$x^M F_{2M+2-b}^r(1/x) = J_{2M+2-b}(x).$$

Proof. We multiply either (5.1.4) or (5.1.5) by x^M and replace x with $1/x$ to give

$$x^M F_{2M+2-b}^r(1/x) = x^M \sum_{k=0}^{M} \binom{2M+1-b-k}{k} (x^{-1})^{M-k}$$

$$= \sum_{k=0}^{M} \binom{2M+1-b-k}{k} x^k = J_{2M+2-b}(x).$$

$\square$

LEMMA 5.2.2 (Jacobsthal-Lucas reciprocal)**.** *We have*

$$x^m L_{2m+b}^r(1/x) = j_{2m+b}(x).$$

Proof. We multiply either (5.1.8) or (5.1.9) by x^m and replace x with $1/x$ to obtain

$$x^m L^r_{2m+b}(1/x) = x^m \sum_{k=0}^{m} \frac{2m+b}{2m+b-k} \binom{2m+b-k}{k} (x^{-1})^{m-k}$$

$$= \sum_{k=0}^{m} \frac{2m+b}{2m+b-k} \binom{2m+b-k}{k} x^k = j_{2m+b}(x).$$

$\square$

These lemmas will be employed later in Chapter 8 concerning our work on establishing forms for the generating functions of our functions $\mathcal{L}_{s;abc}$.

5.3   The Chebyshev Polynomials $T_n(x)$, $U_n(x)$ $C_n(x)$, $S_n(x)$

Starting with the substitution $x = \cos\theta$, so that x is defined on the interval $[-1, 1]$, we define

$$T_n(\cos\theta) = \cos n\theta,$$

or putting $x = \cos\theta$,

$$T_n(x) = \cos\left(n\cos^{-1} x\right),$$

as the n-th Chebyshev polynomial of the first kind. It is a well establised result (see for example [5], [35] and [36]) that all n roots of this polynomial are real. By differentiation of $T_n(x)$ we obtain $n-1$ local extrema points. These points are given by

$$U_{n-1}(x) = \frac{1}{n}\frac{d}{dx}T_n(x) = \frac{\sin\left(n\cos^{-1} x\right)}{\sin\left(\cos^{-1} x\right)}, \tag{5.3.1}$$

where $U_n(x)$ is defined as the n-th Chebyshev polynomial of the second kind.

Three term recurrence relation

The polynomials themselves can be produced by a variety of methods. Perhaps the simplest being a three term recurrence relation that in terms of $T_n(x)$ is

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \tag{5.3.2}$$

where we have $T_0(x) = 1$ and $T_1(x) = x$.

We note that (5.3.2) follows immediately from the trigonometric relation

$$2\cos n\theta \cos m\theta = \cos(n+m)\theta + \cos(n-m)\theta \tag{5.3.3}$$

on putting $m = 1$.

The recurrence relation (5.3.2) holds equally for $U_n(x)$ with the minor modification that $U_0(x) = 1$ and $U_1(x) = 2x$.

5.3.1 Product of the roots

Given that the location of the roots of both $T_n(x)$ and $U_n(x)$ are known, (see for example, [18], [35] or [36]) their corresponding polynomials can be easily expressed as a product of linear factors. We find that

$$T_n(x) = 2^{n-1} \prod_{k=1}^{n} \left(x - \cos \frac{(2k-1)\pi}{2n} \right), \tag{5.3.4}$$

and

$$U_n(x) = 2^n \prod_{k=1}^{n} \left(x - \cos \frac{k\pi}{n+1} \right). \tag{5.3.5}$$

It is often desirable when working with these polynomials to express them in monic form. With $n \geq 1$, we then write (5.3.4) and (5.3.5) respectively as

$$C_n(x) = 2T_n(x/2) = \prod_{k=1}^{n} \left(x - 2\cos \frac{(2k-1)\pi}{2n} \right),$$

and

$$S_n(x) = U_n(x/2) = \prod_{k=1}^{n} \left(x - 2\cos \frac{k\pi}{n+1} \right).$$

It is these monic polynomials, $C_n(x)$ and $S_n(x)$, that have most relevance to us in this study.

5.3.2 Expression of the polynomial as a (binomial) sum

The expressions for these polynomials as sums of terms in the literature are written in different ways. We will start with a form given by Snyder [36] as

$$T_n(x) = \frac{1}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k 2^{n-2k} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k},$$

or in terms of the monic polynomial, and writing $n = 2m + b$ we obtain

$$C_{2m+b}(x) = \sum_{k=0}^{m} (-1)^k \frac{2m+b}{2m+b-k} \binom{2m+b-k}{k} x^{2m+b-2k}.$$

Then following the notation of Definition 5.0.2, and considering seperately the cases $b = 0$ and $b = 1$, we have

$$C_{2m}^r(x) = C_{2m}(\sqrt{x}) = \sum_{k=0}^{m} (-1)^k \frac{2m}{2m-k} \binom{2m-k}{k} x^{m-k},$$

and

$$C_{2m+1}^r(x) = (\sqrt{x})^{-1} C_{2m+1}(\sqrt{x}) = \sum_{k=0}^{m} (-1)^k \frac{2m+1}{2m+1-k} \binom{2m+1-k}{k} x^{m-k}.$$

Remark. We note that whereas each of the Fibonacci-type polnomials are composed natu-
rally of same sign terms, the Chebyshev polynomials are naturally of an alternating sign.

LEMMA 5.3.1 ($L_n - C_n$ association). *For $L_{2m+b}^r(x)$ and $C_{2m+b}^r(x)$ given as in Defini-
tion 5.0.2, where $L_n(x)$ represents the n-th Lucas polynomial and $C_n(x)$ the n-th monic
Chebyshev polynomial of the first kind, we have*

$$L_{2m+b}^r(-x) = (-1)^m C_{2m+b}^r(x).$$

Proof. We replace x with $-x$ in (5.1.7) to obtain

$$L_{2m+b}^r(-x) = \sum_{k=0}^{m} \frac{2m+b}{2m+b-k} \binom{2m+b-k}{k} (-x)^{m-k}$$

$$= (-1)^m \sum_{k=0}^{m} (-1)^k \frac{2m+b}{2m+b-k} \binom{2m+b-k}{k} (x)^{m-k} = (-1)^m C_{2m+b}^r(x).$$

$\square$

Remark. We associate the polynomials in [30] to our own polynomials as follows:

$$P_m(x) = L_{2m}^r(x) = (-1)^m C_{2m}^r(-x), \quad \text{and} \quad Q_m(x) = L_{2m+1}^r(x) = (-1)^m C_{2m+1}^r(-x).$$

Since for the purpose of our current investigation we wish to associate the (monic) Cheby-
shev polynomials of the second kind to the Fibonacci Polynomials, with $n = 2m + b$, it is
preferable to consider the polynomials, $U_{n-1}(x)$ and $S_{n-1}(x)$.

Using (5.3.1) we obtain the polynomials $U_{n-1}(x)$, by differentiation of $T_n(x)$.
With $n = 2m + b = 2M + (2 - b)$, where $M = m - (1 - b)$ we have

$$U_{2m+b-1}(x) = U_{2M+1-b}(x) = \sum_{k=0}^{M} (-1)^k \binom{2M+1-b-k}{k} (2x)^{2M+1-b-2k},$$

or in terms of the monic polynomials $S_{2M+1-b}(x)$,

$$S_{2M+1-b}(x) = \sum_{k=0}^{M} (-1)^k \binom{2M+1-b-k}{k} x^{2M+1-b-2k}. \tag{5.3.6}$$

Remark. We note that as with the Fibonacci polynomials using $M = m - (1-b)$ as opposed
to m in this manner preserves the upper summation value M.

The modified monic Chebyshev polynomials (of the second kind), $S_{2M+1-b}^r(x)$, for the
particular cases $b = 0$ and $b = 1$, are given respectively as

$$S_{2m-1}^r(x) = S_{2M+1}^r(x) = (\sqrt{x})^{-1} S_{2M+1}(\sqrt{x}) = \sum_{k=0}^{M} (-1)^k \binom{2M+1-k}{k} x^{M-k}, \tag{5.3.7}$$

and

$$S_{2m}^r(x) = S_{2M}^r(x) = S_{2M}(\sqrt{x}) = \sum_{k=0}^{M} (-1)^k \binom{2M-k}{k} x^{M-k}. \tag{5.3.8}$$

LEMMA 5.3.2 ($F_n^r - S_{n-1}^r$ association). *For $F_{2M+2-b}^r(x)$ and $S_{2M+b}^r(x)$ defined as in Definition 5.0.2, where $F_n(x)$ represents the Fibonacci polynomials and $S_{n-1}(x)$ the monic form of the Chebyshev polynomials of the second kind, we have that*

$$F_{2M+2-b}^r(-x) = (-1)^M S_{2M+1-b}^r(x).$$

Proof. We replace x with $-x$ in (5.1.3) to obtain

$$F_{2M+2-b}^r(-x) = \sum_{k=0}^{M} \binom{2M+1-b-k}{k}(-x)^{M-k}$$

$$= (-1)^M \sum_{k=0}^{M} (-1)^k \binom{2M+1-b-k}{k} x^{M-k} = (-1)^M S_{2M+1-b}^r(x).$$

$\square$

Remark. We associate the polynomials in [30] to our own polynomials as follows:

$$\mathcal{P}_m(x) = F_{2m+1}^r(x) = (-1)^m S_{2m}^r(-x), \quad \text{and} \quad \mathcal{Q}_m(x) = F_{2m}^r(x) = (-1)^m S_{2m-1}^r(-x).$$

Table 5.5: The monic Chebyshev polynomials, $S_{n-1}(x)$, and the modified monic Chebyshev polynomial, $S_{n-1}^r(x)$, for $1 \leq n \leq 9$, with $n-1 = 2m+b-1 = 2M+1-b$, where $M = m - (1-b)$ and b is the parity of n.

n	M	b	$S_{n-1}(x)$	$S_{n-1}^r(x)$
1	0	1	1	1
2	0	0	x	1
3	1	1	$x^2 - 1$	$x - 1$
4	1	0	$x^3 - 2x$	$x - 2$
5	2	1	$x^4 - 3x^2 + 1$	$x^2 - 3x + 1$
6	2	0	$x^5 - 4x^3 + 3x$	$x^2 - 4x + 3$
7	3	1	$x^6 - 5x^4 + 6x^2 - 1$	$x^3 - 5x^2 + 6x - 1$
8	3	0	$x^7 - 6x^5 + 10x^3 - 4x$	$x^3 - 6x^2 + 10x - 4$
9	4	1	$x^8 - 7x^6 + 15x^4 - 10x^2 + 1$	$x^4 - 7x^3 + 15x^2 - 10x^1 + 1$

5.4 Classification of the Fibonacci-type polynomials

To facilitate the expression of the linear recurrence polynomial $\mathcal{R}_{s;ab}(x, m)$ for the sum $\mathcal{L}_{s;abc}(r, t, q)$ we redefine Definitions 5.0.1 and 5.0.2 in terms of the parameters s, a and b and the variable m.

Table 5.6: The Chebyshev polynomials $C_n(x)$ and the modified Chebyshev polynomial, $C_n^r(x)$, for $0 \leq n \leq 8$, with $n = 2m + b$ where b is the parity of n.

n	m	b	$C_n(x)$	$C_n^r(x)$
0	0	0	2	2
1	0	1	x	1
2	1	0	$x^2 - 2$	$x - 2$
3	1	1	$x^3 - 3x$	$x - 3$
4	2	0	$x^4 - 4x^2 + 2$	$x^2 - 4x + 2$
5	2	1	$x^5 - 5x^3 + 5x$	$x^2 - 5x + 5$
6	3	0	$x^6 - 6x^4 + 9x^2 - 2$	$x^3 - 6x^2 + 9x - 2$
7	3	1	$x^7 - 7x^5 + 14x^3 - 7x$	$x^3 - 7x^2 + 14x - 7$
8	4	0	$x^8 - 8x^6 + 20x^4 - 16x^2 + 2$	$x^4 - 8x^3 + 20x^2 - 16x + 2$

Definition 5.4.1. *For (positive) integers $s \in \{0,1\}$; $a \in \{0,1\}$, $b \in \{0,1\}$, $q = 2m + b$ and $Q = q - (1-a)(1-s)$, we define the associated function $A_{s;ab}$ as*

$$A_{s;ab}(x, Q) = \prod_{d=1}^{2m+b+a-1} \left(x - 2\imath^s \cos \frac{(2d-a)\pi}{2(2m+b)} \right)$$

$$= \sum_{k=0}^{M} (-\gamma)^k \left(\frac{2M+B}{2M+B-k} \right)^a \binom{2M+B-k}{k} x^{2M+B-2k},$$

and the modified associated function $A_{s;ab}^r$ as

$$A_{s;ab}^r(x, Q) = (\sqrt{x})^{-\epsilon} \prod_{d=1}^{2m+b+a-1} \left(\sqrt{x} - 2\imath^s \cos \frac{(2d-a)\pi}{2(2m+b)} \right)$$

$$= \sum_{k=0}^{M} (-\gamma)^k \left(\frac{2M+B}{2M+B-k} \right)^a \binom{2M+B-k}{k} x^{M-k},$$

where $\gamma = (-1)^s$, $\epsilon = a(2b-1) + 1 - b$, $M = m - (1-a)(1-b)$ and $B = (1-a) + b(2a-1)$.

Connecting Definitions 5.0.2 and 5.4.1 we have

$$e = (1+s)(1-a) + b(2a-1), \qquad f = s(1-a),$$

and

$$\epsilon = e - f = a(2b-1) + 1 - b = \begin{cases} 0 & \text{if } a \neq b \\ 1 & \text{if } a = b. \end{cases} \tag{5.4.1}$$

Applying ϵ to the specific polynomials we obtain the relations given in Table 5.7.

Remark. For $A_{s;ab}$ and $A_{s;ab}^r$ defined as in Definition 5.4.1, we find that when $s = 0$, all the roots lie on the real axis and we are able to identify them as the roots of a monic Chebyshev polynomial of either the first or second kind, whereas when $s = 1$, these roots are spaced along the imaginary axis and are identified as the roots of either a Fibonacci or a Lucas polynomial.

Table 5.7: Relationship between ϵ and the Fibonacci type polynomials.

a	polynomial type	ϵ
0	$F_q^r(x), S_{q-1}^r(x)$	$1-b$
1	$L_q^r(x), C_q^r(x)$	b

We demonstrate the above remark with Theorem 5.4.1.

THEOREM 5.4.1 (expression of $A_{s;ab}(x,Q)$ as a Fibonacci type polynomial). *The polynomial $A_{s;ab}(x,Q)$ defined in Definition 5.4.1 is equated to a Fibonacci, Lucas or (monic) Chebyshev polynomial such that*

$$A_{s;ab}(x,Q) = \begin{cases} S_{2(m-1+b)+1-b}(x) & \text{if } s = 0,\ a = 0 \\ C_{2m+b}(x) & \text{if } s = 0,\ a = 1 \\ \\ F_{2(m-1+b)+2-b}(x) & \text{if } s = 1,\ a = 0 \\ L_{2m+b}(x) & \text{if } s = 1,\ a = 1, \end{cases}$$

where $Q = q - (1-a)(1-s)$, and $q = 2m+b$.

Proof. This follows on substitution of each value of each of the parameters s, a and b into the product and binomial and forms of $A_{s;ab}(x,Q)$ as given in Definition 5.4.1, and then compared with the corresponding (monic) Chebyshev, Fibonacci and Lucas polynomial forms. We detail the first case below and provide the full proof in Appendix B.1.

$$A_{0;00}(x, 2m-1) = A_{0;00}(x, 2(m-1)+1)$$
$$= \prod_{d=1}^{2m-1}\left(x - 2\cos\frac{\pi d}{2m}\right) = \sum_{k=0}^{m-1}(-1)^k\binom{2(m-1)+1-k}{k}x^{2(m-1)+1-2k}$$
$$= S_{2(m-1)+1}(x).$$

$\square$

From this theorem we similarly express the modified polynomials, $A_{s;ab}^r(x,Q)$, using the following corollary.

COROLLARY. *The modified polynomials $A_{s;ab}^r(x,Q)$ are related to a modified Chebyshev, Fibonacci or Lucas polynomial in the following manner:*

$$A_{s;ab}^r(x,Q) = \begin{cases} S_{2(m-1+b)+1-b}^r(x) & \text{if } s = 0,\ a = 0 \\ C_{2m+b}^r(x) & \text{if } s = 0,\ a = 1 \\ \\ F_{2(m-1+b)+2-b}^r(x) & \text{if } s = 1,\ a = 0 \\ L_{2m+b}^r(x) & \text{if } s = 1,\ a = 1. \end{cases}$$

Proof. From Definition 5.4.1 we have $A^r_{s;ab}(x,Q) = (\sqrt{x})^{-\epsilon} A_{s;ab}(\sqrt{x}, Q)$, where ϵ is given as in (5.4.1). Therefore, the result follows from Theorem 5.4.1, when we replace x with $\sqrt{x}$ and multiply by $(\sqrt{x})^{-\epsilon}$. $\qquad\qquad\square$

We similarly provide the first case here to illustrate the polynomials and detail all cases in Appendix $B.2$.

$$A^r_{0;00}(x, 2m-1) = A^r_{0;00}(x, 2(m-1)+1) = S^r_{2(m-1)+1}(x)$$

$$= \sum_{k=0}^{m-1} (-1)^k \binom{2(m-1)+1-k}{k} x^{m-1-k} = (\sqrt{x})^{-1} \prod_{d=1}^{2m-1} \left(\sqrt{x} - 2\cos\left(\frac{\pi d}{2m}\right) \right).$$

Except for the root at $x = 0$, (when it occurs), the roots of the polynomials $A_{s;ab}(x, Q)$ and $A^r_{s;ab}(x, Q)$ are symmetrically distributed about the origin. This enables a simplification of the product form given in Theorem 5.4.1 and the Corollary to it.

THEOREM 5.4.2 (simplification of the product form of $A_{s;ab}(x,Q)$). *For $A_{s;ab}(x,Q)$ defined as in Definition* 5.4.1 *we have*

$$A_{s;ab}(x, Q) = x^\epsilon \prod_{d=1}^{m-(1-a)(1-b)} \left(x^2 - 4\gamma \cos^2 \frac{(2d-a)\pi}{2(2m+b)} \right),$$

where $Q = q - (1-a)(1-s)$ and $\epsilon = a(2b-1)+1-b$.

Proof. From Theorem 5.4.1, we make a suitable "pairing" of terms. We illustrate with two cases and provide the full proof in Appendix $B.3$. Let us consider the cases for s. When $s = a = b = 0$, we have

$$A_{0;00}(x, 2m-1) = \prod_{d=1}^{2m-1} \left(x - 2\cos\frac{d\pi}{2m} \right)$$

$$= \left(x - 2\cos\frac{m\pi}{2m} \right) \prod_{d=1}^{m-1} \left(x - 2\cos\frac{d\pi}{2m} \right) \left(x - 2\cos\frac{(2m-k)\pi}{2m} \right)$$

$$= x \prod_{d=1}^{m-1} \left(x - 2\cos\frac{d\pi}{2m} \right) \left(x + 2\cos\frac{d\pi}{2m} \right)$$

$$= x \prod_{d=1}^{m-1} \left(x^2 - 4\cos^2\frac{d\pi}{2m} \right),$$

and when $s = 1$, $a = 0$, $b = 0$,

$$A_{1;00}(x, 2m) = \prod_{d=1}^{2m-1} \left(x - 2\imath \cos \frac{d\pi}{2m} \right)$$

$$= \left(x - 2\imath \cos \frac{m\pi}{2m} \right) \prod_{d=1}^{m-1} \left(x - 2\imath \cos \frac{d\pi}{2m} \right) \left(x - 2\imath \cos \frac{(2m-d)\pi}{2m} \right)$$

$$= x \prod_{d=1}^{m-1} \left(x - 2\imath \cos \frac{d\pi}{2m} \right) \left(x + 2\imath \cos \frac{d\pi}{2m} \right)$$

$$= x \prod_{d=1}^{m-1} \left(x^2 + 4 \cos^2 \frac{d\pi}{2m} \right).$$

$\square$

Using Theorem 5.4.2 we make a similar simplification of the modified polynomials, $A_{s;ab}^r(x, Q)$.

COROLLARY (simplified product form of $A_{s;ab}^r(x, Q)$)**.**

$$A_{s;ab}^r(x, Q) = \prod_{d=1}^{m-(1-a)(1-b)} \left(x - 4\gamma \cos^2 \frac{(2d - a)\pi}{2(2m + b)} \right).$$

Proof. We have from Definition 5.4.1 that $A_{s;ab}^r(x, Q) = (\sqrt{x})^{-\epsilon} A_{s;ab}(\sqrt{x}, Q))$, where $\epsilon = a(2b - 1) + 1 - b$. Therefore, if in each form of $A_{s;ab}(x, Q)$, derived in Theorem 5.4.2, we replace x with $\sqrt{x}$ and multiply by $(\sqrt{x})^{-\epsilon}$, we obtain the result. $\square$

We illustrate with two cases and provide all cases in Appendix $B.4$.

$$A_{0;00}^r(x, 2(m-1) + 1) = S_{2m-1}^r(x) = \prod_{d=1}^{m-1} \left(x - 4 \cos^2 \frac{d\pi}{2m} \right),$$

and

$$A_{0;11}^r(x, 2m + 1) = C_{2m+1}^r(x) = \prod_{d=1}^{m} \left(x - 4 \cos^2 \frac{(2d - 1)\pi}{2(2m + 1)} \right).$$

For the polynomials $A_{s;ab}(x, Q)$ or $A_{s;ab}^r(x, Q)$, the variable $M = m - (1 - a)(1 - b)$ was employed to indicate the order of the polynomial. In the forthcoming section we examine the linear recurrence polynomial, $\mathcal{R}_{s;ab}(x, m)$, satisfying the function $\mathcal{L}_{s;abc}$, and we denote the order of the polynomial by $M' = m + b(1 - a)$.

5.5 Recurrence polynomials for the family of functions $\mathcal{L}_{s;abc}$

To facilatate the development of a recurrence polynomial for each of the sequences $\mathcal{L}_{s;abc}(r, t, q)$, we wish to connect the roots of the polynomial to those of a corresponding power sum, as established in the Corollary of Theorem 4.4.2. We use Lemma 5.5.1 to provide this link.

LEMMA 5.5.1. *If the sequence $u_0, u_1, u_2, \ldots$ satisfies an order m linear recurrence of the form*

$$u_{n+m} + a_1 u_{n+m-1} + \cdots + a_m u_n = 0 \tag{5.5.1}$$

so that the characteristic polynomial is

$$P(x) = x^m + a_1 x^{m-1} + \cdots + a_m, \tag{5.5.2}$$

with distinct roots $\mu_1, \ldots, \mu_m$, then every solution of (5.5.1) can be written as a power-sum with coefficients:

$$u_n = b_1 \mu_1^n + \cdots + b_m \mu_m^n. \tag{5.5.3}$$

Proof. The sequence $u_0, u_1, u_2, \ldots$ satisfies a linear recurrence when (5.5.1) holds for all n. The values $u_0, \ldots, u_{m-1}$ determine all numbers in the sequence. If we fix coefficients $a_1, \ldots, a_m$, then the sequences satisfying (5.5.1) form a vector space of dimension m. A shift operator S acts on the space of sequences that satisfy (5.5.1) by

$$S \begin{pmatrix} u_0 \\ u_1 \\ \ldots \\ u_{m-2} \\ u_{m-1} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \ldots \\ u_{m-1} \\ u_m \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & \ldots & & \ldots & \\ 0 & 0 & 0 & \ldots & 1 \\ -a_m & -a_{m-1} & -a_{m-2} & \ldots & -a_1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \ldots \\ u_{m-2} \\ u_{m-1} \end{pmatrix} = A_{m \times m} \begin{pmatrix} u_0 \\ u_1 \\ \ldots \\ u_{m-2} \\ u_{m-1} \end{pmatrix}.$$

If the column operation $\mathbf{k}_1' = \mathbf{k}_1 + \mu \mathbf{k}_2 + \cdots + \mu^{m-1} \mathbf{k}_m$, followed by the consecutive row operations $\mathbf{r}_i' = \mathbf{r}_i + \mu \mathbf{r}_{i-1}$ $(2 \le i \le m)$, is applied to the characteristic polynomial, we obtain

$$|\mu I - A|_{m \times m} = \begin{vmatrix} \mu & -1 & 0 & \ldots & 0 \\ 0 & \mu & -1 & \ldots & 0 \\ & \ldots & & \ldots & \\ 0 & 0 & 0 & \ldots & -1 \\ a_m & a_{m-1} & a_{m-2} & \ldots & a_1 + \mu \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 & \ldots & 0 \\ 0 & 0 & -1 & \ldots & 0 \\ & \ldots & & \ldots & \\ 0 & 0 & 0 & \ldots & -1 \\ f(\mu) & a_{m-1} & a_{m-2} & \ldots & a_1 \end{vmatrix}, \tag{5.5.4}$$

where $f(x) = x^m + a_1 x^{m-1} + a_2 x^{m-2} + \cdots + a_m$, whose determinant is $f(\mu)$.

When the eigenvalues $\mu_1, \ldots, \mu_m$ are all distinct then the Vandemonde determinant is non-zero, and the m sequences $u_n = \mu_i^n$ are linearly independent (see Theorem 9.3 of [?]). These sequences form a basis for the vector space of sequences that satisfy (5.5.1). Therefore, every solution of (5.5.1) can be written as a power-sum with coefficients given by (5.5.3). $\qquad \square$

Remark. A similar proof to Lemma 5.5.1 is found in Lemma 3.5 of [9]. The above proof is provided due to its significance (for this work) and its brevity.

In the Corollary of Theorem 4.4.2 each of the sequence terms $\mathcal{L}_{s;abc}(r, t, q)$ are expressed as a power sum of the form given in (5.5.3), where for the cases $s = 0$ and $s = 1$, we have

$$x_d = 4 \cos^2 \left(\frac{\pi(2d-a)}{2q} \right), \quad \text{and} \quad x_d = -4 \sin^2 \left(\frac{\pi(2d-\epsilon)}{2q} \right). \tag{5.5.5}$$

Here we recall a, b and ϵ take the value 0 or 1; $\epsilon \equiv a + sb \pmod{2}$, and $a \le d \le m + b(1-a) + a - 1$.

We denote by $\mathcal{R}_{s;ab}(x,m)$, the recurrence polynomial that satisfy the sequences $\mathcal{L}_{s;abc}(r,t,q)$. From Lemma 5.5.1, the values, (5.5.5), comprise the roots of $\mathcal{R}_{s;ab}(x,m)$, and the order of the polynomial is m in all cases except when the parameters $a=0$ and $b=1$, (that is for the positive sum with odd modulus). In this latter case the order is $m+1$.

In the next theorem we simplify the two forms of (5.5.5) into the single form

$$x_D = -4\cos^2\left(\frac{\pi(2D-a)}{2q}\right),$$

where the set of the roots, x_D, are the same as those of x_d.

THEOREM 5.5.2 (product form of the recurrence polynomials, $\mathcal{R}_{s;ab}(x,m)$). *The recurrence polynomials $\mathcal{R}_{s;ab}(x,m)$ are, expressed as a product of their roots, given by*

$$\mathcal{R}_{s;ab}(x,m) = \prod_{d=a}^{m-(1-a)(1-b)}\left(x - 4\gamma\cos^2\left(\frac{\pi(2d-a)}{2q}\right)\right), \tag{5.5.6}$$

where $\gamma = (-1)^s$.

Proof. We consider each of the four cases for the parameters a and b, for both the cases $s=0$ and $s=1$. Here we detail the first case and give the proof in full in Appendix B.5.

Case 1: $a=0$, $b=0$.

From (5.5.5) the roots of the recurrence polynomial $\mathcal{R}_{0;00}(x,m)$ are $4\cos^2 d\pi/q$, where $0 \le d \le m-1$. On the other hand, the roots of $\mathcal{R}_{1;00}(x,m)$ are $-4\sin^2 d\pi/q$, where $1 \le d \le m$. Since $m/q = 1/2$ we have that

$$\sin\frac{(m-d)\pi}{q} = \cos\frac{d\pi}{q}, \tag{5.5.7}$$

so that

$$\prod_{d=1}^{m}\left(x + 4\sin^2\frac{d\pi}{q}\right) = \prod_{d=0}^{m-1}\left(x + 4\sin^2\frac{(m-d)\pi}{q}\right) = \prod_{d=0}^{m-1}\left(x + 4\cos^2\frac{d\pi}{q}\right).$$

$\square$

5.6 Association of the polynomials $\mathcal{R}_{s;ab}(x,m)$ to the modified polynomials $\mathcal{A}^r_{s;ab}(x,Q)$

We now associate the polynomials $\mathcal{R}_{s;ab}(x,m)$ to those of $\mathcal{A}^r_{s;ab}(x,Q)$ and consequently to those of either a monic Chebyshev, Fibonacci or Lucas polynomial.

THEOREM 5.6.1 (expression of $\mathcal{R}_{s;ab}(x,m)$ in terms of $A^r_{s;ab}(x,Q)$). *For $q = 2m+b$ and $Q = q - (1-a)(1-s)$, we have*

$$\mathcal{R}_{s;ab}(x,m) = (x - 4\gamma)^{1-a}\, A^r_{s;ab}(x,Q). \tag{5.6.1}$$

Proof. From the Corollary of Theorem 5.4.2 we have

$$A^r_{s;ab}(x, Q) = \prod_{d=1}^{m-(1-a)(1-b)} \left(x - 4\gamma \cos^2 \frac{(2d-a)\pi}{2(2m+b)} \right). \tag{5.6.2}$$

Multiplication of both sides of (5.6.2) by $(1 - 4\gamma x)^{1-a}$ produces

$$(x - 4\gamma)^{1-a} \prod_{d=1}^{m-(1-a)(1-b)} \left(x - 4\gamma \cos^2 \frac{(2d-a)\pi}{2(2m+b)} \right) = \prod_{d=a}^{m+b(1-a)+a-1} \left(x - 4\gamma \cos^2 \frac{(2d-a)\pi}{2(2m+b)} \right)$$

$$= \prod_{d=a}^{M'+a-1} \left(x - 4\gamma \cos^2 \frac{(2d-a)\pi}{2(2m+b)} \right) = \mathcal{R}_{s;ab}(x, m)$$

by Theorem 5.5.2. □

COROLLARY. *The linear recurrence polynomial $\mathcal{R}_{s;ab}(x, m)$ satisfied by the function $\mathcal{L}_{s;abc}$, can be expressed in terms of (modified) Fibonacci, Lucas or Chebyshev polynomial in the following manner:*

$$\mathcal{R}_{s;ab}(x, m) = \begin{cases} (x - 4)S^r_{2(m-1+b)+1-b}(x) & \text{if } s = 0, \ a = 0 \\ C^r_{2m+b}(x) & \text{if } s = 0, \ a = 1 \\ \\ (x + 4)F^r_{2(m-1+b)+2-b}(x) & \text{if } s = 1, \ a = 0 \\ L^r_{2m+b}(x) & \text{if } s = 1, \ a = 1. \end{cases} \tag{5.6.3}$$

Alternatively, with $q = 2m + b$ we can express (5.6.3) as

$$\mathcal{R}_{s;ab}(x, m) = \begin{cases} (x - 4)S^r_{q-1}(x) & \text{if } s = 0, \ a = 0 \\ C^r_q(x) & \text{if } s = 0, \ a = 1 \\ \\ (x + 4)F^r_q(x) & \text{if } s = 1, \ a = 0 \\ L^r_q(x) & \text{if } s = 1, \ a = 1. \end{cases} \tag{5.6.4}$$

Proof. The result follows from Theorem 5.6.1 and the Corollary to Theorem 5.4.1. □

5.7 Evaluation of the recurrence polynomials $\mathcal{R}_{s;ab}(x, q)$

A direct method of determining the coefficients of the polynomial, $\mathcal{R}_{s;ab}(x, m)$, is by expansion of (5.5.6) of Theorem 5.5.2.

For demonstrative purposes let us consider the case $s = 1$, so that $\gamma = -1$, and perform this expansion for either $m = 2$ or $m = 3$.

We have

$$x_d = 4\cos^2 \left(\frac{\pi(2d-a)}{2q} \right),$$

where $q = 2m + b$ and, $a \leq d \leq m + b - a - 1$.

So for $a = 0$, $b = 0$, and $m = 3$, then $d = 0, 1, 2$ and

$x_0 = 4\cos^2(0\pi/6) = 4(1) = 4,$

$x_1 = 4\cos^2(1\pi/6) = 4(\sqrt{3}/2)^2 = 3,$

$x_2 = 4\cos^2(2\pi/6) = 4(1/2)^2 = 1,$

$\mathcal{R}_{0;00}(x, 3) = (x + 4)(x + 3)(x + 1) = x^3 + 8x^2 + 19x + 12.$

For the case $a = 0$, $b = 1$, and $m = 2$, then $d = 0, 1, 2$ and

$x_0 = 4\cos^2(0\pi/5) = 4(1)^2 = 4,$

$x_1 = 4\cos^2(1\pi/5) = 4(\sqrt{5} + 1)^2/16 = (6 + 2\sqrt{5})/4,$

$x_2 = 4\cos^2(2\pi/5) = 4(\sqrt{5} - 1)^2/16 = (6 - 2\sqrt{5})/4,$

$\mathcal{R}_{0;01}(x, 2) = (x + 4)(x + (6 + 2\sqrt{5})/4)(x + (6 - 2\sqrt{5})/4) = x^3 + 7x^2 + 13x + 4.$

If $a = 1$, $b = 0$ and $m = 3$, then $d = 1, 2, 3$ and

$x_1 = 4\cos^2(1\pi/12) = 4\left(\left(\sqrt{2 + \sqrt{3}}\right)/2\right)^2 = 2 + \sqrt{3},$

$x_2 = 4\cos^2(3\pi/12) = 4(\sqrt{2}/2)^2 = 2,$

$x_3 = 4\cos^2(5\pi/12) = 4\left(\left(\sqrt{2 - \sqrt{3}}\right)/2\right)^2 = 2 - \sqrt{3},$

$\mathcal{R}_{0;10}(x, 3) = \left(x + (2 + \sqrt{3})\right)\left(x + (2 - \sqrt{3})\right)(x - 2) = x^3 + 6x^2 + 9x + 2.$

Finally when $a = 1$, $b = 1$, and $m = 2$, then $d = 1, 2$ and

$x_1 = 4\cos^2(1\pi/10) = 4\left(\sqrt{(5 + \sqrt{5})/8}\right)^2 = (5 + \sqrt{5})/2,$

$x_2 = 4\cos^2(3\pi/10) = 4\left(\sqrt{(5 - \sqrt{5})/8}\right)^2 = (5 - \sqrt{5})/2,$

$\mathcal{R}_{0;11}(x, 2) = (x + (5 + \sqrt{5})/2)(x + (5 - \sqrt{5})/2) = x^2 + 5x + 5.$

Remark. An immediate limitation of this method arises, the difficulty of obtaining explicit expressions for the roots when q has a prime factor ≥ 7.

In general a more practical approach is to employ Theorem 5.6.1. In the case of the parameter $a = 1$, we simply have

$$\mathcal{R}_{s;1b}(x, m) = A_{s;1b}^r(x, q) = \sum_{k=0}^{m}(-\gamma)^{m-k}\frac{2m + b}{m + k + b}\binom{m + k + b}{2k + b}x^k,$$

and when $a = 0$, we have

$$
\begin{aligned}
\mathcal{R}_{s;0b}(x, m) &= (x - 4\gamma) A^r_{s;0b}(x, q - s) \\
&= (x - 4\gamma) \sum_{k=0}^{m-1+b} (-\gamma)^{m-1+b-k} \binom{m+k}{2k+1-b} x^k \\
&= \sum_{k=0}^{m+b} (-\gamma)^{m+b-k} \left(\binom{m+k-1}{2k-1-b} + 4 \binom{m+k}{2k+1-b} \right) x^k.
\end{aligned}
$$

Here we just give one example of each type for the case when $s = 0$ and provide additional examples in Appendix $C.2$. We have

$$
\mathcal{R}_{0;00}(x, 3) = (x - 4) \sum_{k=0}^{2} (-1)^{2-k} \binom{3+k}{2k+1} x^k = x^3 - 8x^2 + 19x - 12,
$$

$$
\mathcal{R}_{0;01}(x, 3) = (x - 4) \sum_{k=0}^{3} (-1)^{3-k} \binom{3+k}{2k} x^k = x^4 - 9x^3 + 26x^2 - 25x + 4,
$$

$$
\mathcal{R}_{0;10}(x, 3) = \sum_{k=0}^{3} (-1)^{3-k} \frac{6}{3+k} \binom{3+k}{2k} x^k = x^3 - 6x^2 + 9x - 2,
$$

$$
\mathcal{R}_{0;11}(x, 3) = \sum_{k=0}^{3} (-1)^{3-k} \frac{7}{4+k} \binom{4+k}{2k+1} x^k = x^3 - 7x^2 + 14x - 7.
$$

Chapter 6

Differential Equations

This chapter examines the creation of a second order differential equation, a solution of which is one of the eight recurrence relation polynomials $\mathcal{R}_{s;ab}(x, m)$. With respect to this type of equation, we start in Section 6.1 by introducing the Jacobi polynomials, and a subclass of them, the Chebyshev polynomials of the first and second class. In Section 6.2 (with Theorem 6.2.2) and Section 6.3 (with Theorem 6.3.5), we establish a set of four second order differential equations, each one satisfied by one of the polynomials $\mathcal{R}_{s;1b}(x, m)$ and $\mathcal{R}_{s;0b}(x, m)$ respectively.

6.1 Jacobi poynomials

We begin with the generalised group of polynomials, the Jacobi polynomials $P_q^{(\alpha,\beta)}(x)$, satisfying the equation

$$(1 - x^2)y'' + \{\beta - \alpha - (\alpha + \beta + 2)x\}y' + q(q + \alpha + \beta + 1)y = 0. \qquad (6.1.1)$$

From our current perspective, particular cases of these polynomials are the Chebyshev polynomials of the first and second kind. For those of the first kind, $T_q(x)$, we have $\alpha = \beta = -1/2$, and for those of the second kind, $U_q(x)$, we have $\alpha = \beta = 1/2$. Substituting these values into (6.1.1), we obtain the respective second order differential equations

$$(1 - x^2)T_q''(x) - xT_q'(x) + q^2 T_q(x) = 0, \qquad (6.1.2)$$

and

$$(1 - x^2)U_q''(x) - 3xU_q'(x) + q(q + 2)U_q(x) = 0. \qquad (6.1.3)$$

In Theorem 5.6.1 we established, via the transformation $x = \sqrt{\gamma u}/2$ the close relationship between the polynomials $\mathcal{R}_{s;ab}(u, m)$ and those, when the parameter $s = 0$, of the Chebyshev $T_q(x)$ and $U_{q-1}(x)$ types, and those when $s = 1$, of the Fibonacci $F_q(x)$ and Lucas $L_q(x)$ types. Moreover, in Lemmas 5.3.1 and 5.3.2 we demonstrated a route between the interchange of the polynomials $L_q(x)$ and $T_q(x)$ and of $F_q(x)$ and $U_q(x)$ respectively. Given, therefore, the

well established (second order) equations for both the Chebyshev polynomials, (6.1.2) and (6.1.3), we can limit ourselves to these two equations for deriving the differential equations for each of the eight sequences $\{\mathcal{R}_{s;ab}(u,m)\}_{m=1}^{\infty}$.

6.2 The recurrence polynomials $\mathcal{R}_{s;1b}(u,m)$

We first consider the sequences with the alternating parameter $a = 1$, and start with a lemma.

LEMMA 6.2.1. *With $x = \sqrt{\gamma u}/2$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}x}T_q(x) = \gamma^m \imath^{sb} u^{\frac{b}{2}} \left(2u^{\frac{1}{2}}\mathcal{R}'_{s;1b}(u,m) + bu^{-\frac{1}{2}}\mathcal{R}_{s;1b}(u,m)\right),$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}T_q(x) = \gamma^m \imath^{sb} u^{\frac{b}{2}} \left(8\gamma u\mathcal{R}''_{s;1b}(u,m) + 4\gamma(2b+1)\mathcal{R}'_{s;1b}(u,m)\right).$$

Proof. We have from Theorem 5.6.1 that

$$T_q(x) = T_q(\sqrt{\gamma u}/2) = \frac{1}{2}C_q(\sqrt{\gamma u}) = \frac{(\sqrt{\gamma u})^b}{2}C_q^r(\gamma u) = \frac{\gamma^m(\sqrt{\gamma u})^b}{2}\mathcal{R}_{s;1b}(u,m). \tag{6.2.1}$$

On differentiating both sides of (6.2.1) with respect to x we obtain

$$T_q'(x) = \frac{\gamma^{m+b/2}}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(u^{\frac{b}{2}}\mathcal{R}_{s;1b}(u,m)\right),$$

the right hand side of which, via the chain rule, becomes

$$=\frac{\gamma^m \imath^{sb}}{2}\frac{\mathrm{d}}{\mathrm{d}u}\left(u^{\frac{b}{2}}\mathcal{R}_{s;1b}(u,m)\right)\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\gamma^m \imath^{sb}}{2}\left(u^{\frac{b}{2}}\mathcal{R}'_{s;1b}(u,m) + \frac{1}{2}bu^{\frac{b-2}{2}}\mathcal{R}_{s;1b}(u,m)\right)\frac{4u^{\frac{1}{2}}}{\imath^s}$$

$$=\gamma^m \imath^{s(b-1)}\left(2u^{\frac{b+1}{2}}\mathcal{R}'_{s;1b}(u,m) + bu^{\frac{b-1}{2}}\mathcal{R}_{s;1b}(u,m)\right)$$

$$=\gamma^m \imath^{s(b-1)}u^{\frac{b}{2}}\left(2u^{\frac{1}{2}}\mathcal{R}'_{s;1b}(u,m) + bu^{-\frac{1}{2}}\mathcal{R}_{s;1b}(u,m)\right).$$

Then on differentiating (6.2.1) a second time we have

$$T_q''(x) =\frac{\gamma^m \imath^{s(b-1)}}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(u^{\frac{b}{2}}\mathcal{R}_{s;1b}(u,m)\right) = \gamma^m \imath^{s(b-1)}\frac{\mathrm{d}}{\mathrm{d}x}\left(2u^{\frac{b+1}{2}}\mathcal{R}'_{s;1b}(u,m) + bu^{\frac{b-1}{2}}\mathcal{R}_{s;1b}(u,m)\right)$$

$$=\gamma^m \imath^{s(b-1)}\frac{\mathrm{d}}{\mathrm{d}u}\left(2u^{\frac{b+1}{2}}\mathcal{R}'_{s;1b}(u,m) + bu^{\frac{b-1}{2}}\mathcal{R}_{s;1b}(u,m)\right)\frac{4u^{\frac{1}{2}}}{\imath^s},$$

and following the second application of the chain rule we continue as

$$=\gamma^m \imath^{s(b-2)}\left(2u^{\frac{b+1}{2}}\mathcal{R}''_{s;1b}(u,m) + (b+1)u^{\frac{b-1}{2}}\mathcal{R}'_{s;1b}(u,m) + bu^{\frac{b-1}{2}}\mathcal{R}'_{s;1b}(u,m)\right.$$

$$\left.+\frac{1}{2}b(b-1)u^{\frac{b-3}{2}}\mathcal{R}_{s;1b}(u,m)\right)4u^{\frac{1}{2}}$$

$$=\gamma^{m+1}\imath^{sb}\left(8u^{\frac{3b}{2}}\mathcal{R}''_{s;1b}(u,m) + 4(b+1)u^{\frac{b}{2}}\mathcal{R}'_{s;1b}(u,m) + bu^{\frac{b}{2}}\mathcal{R}'_{s;1b}(u,m) + 0\right)$$

$$=\gamma^m \imath^{sb} u^{\frac{b}{2}}\left(8\gamma u\mathcal{R}''_{s;1b}(u,m) + 4\gamma(2b+1)\mathcal{R}'_{s;1b}(u,m)\right).$$

Thus the necessary results are obtained. $\qquad\square$

We are now in a position to state the following theorem.

THEOREM 6.2.2. *The polynomials $\mathcal{R}_{s;1b}(u,m)$ satisfy the second order differential equation*

$$4(u-4\gamma)u\mathcal{R}''_{s;1b}(u,m)+4\left((b+1)u-2\gamma(2b+1)\right)\mathcal{R}'_{s1b}(u,m)-(q^2-b)\mathcal{R}_{s;1b}(u,m)=0, \quad (6.2.2)$$

where $\gamma=(-1)^s$.

Proof. From (6.1.2) we have

$$(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}T_q(x)-x\frac{\mathrm{d}}{\mathrm{d}x}T_q(x)+q^2T_q(x)=0,$$

Let $x=\sqrt{\gamma u}/2=\imath^s\sqrt{u}/2$ and so,

$$\mathrm{d}x=\frac{\imath^s\mathrm{d}u}{4\sqrt{u}}, \qquad \text{or} \qquad \frac{\mathrm{d}u}{\mathrm{d}x}=\frac{4\sqrt{u}}{\imath^s}.$$

Substituting these forms into (6.1.2) we have

$$(1-\gamma u/4)\frac{\mathrm{d}^2}{\mathrm{d}x^2}T_q(\sqrt{\gamma u}/2)+\frac{\imath^s\sqrt{u}}{2}\frac{\mathrm{d}}{\mathrm{d}x}T_q(\sqrt{\gamma u}/2)+q^2T_q(\sqrt{\gamma u}/2)=0, \qquad (6.2.3)$$

and from Lemma 6.2.1 and equation (6.2.1) the left hand side can then be written as

$$\gamma^m\imath^{sb}u^{\frac{b}{2}}(1-\gamma u/4)\left(8\gamma u\mathcal{R}''_{s;ab}(u,m)+4\gamma(2b+1)\mathcal{R}'_{s;ab}(u,m)\right)$$
$$-\frac{1}{2}\gamma^m\imath^{sb}u^{\frac{b}{2}}\left(2u\mathcal{R}'_{s;1b}(u,m)+b\mathcal{R}_{s;1b}(u,m)\right)+\frac{1}{2}q^2\gamma^m\imath^{sb}u^{\frac{b}{2}}\mathcal{R}_{s;1b}(u,m).$$

Now after factoring out $-\frac{1}{2}\gamma^m\imath^{sb}u^{\frac{b}{2}}$ we are left with

$$-\gamma\frac{2}{4}(4-\gamma u)\left(8u\mathcal{R}''_{s;1b}(u,m)+4(2b+1)\mathcal{R}'_{s;1b}(u,m)\right)+\left(2u\mathcal{R}'_{s;1b}(u,m)+b\mathcal{R}_{s;1b}(u,m)\right)$$
$$-q^2\mathcal{R}_{s;1b}(u,m).$$

Whilst noting that $\gamma^2=1$, we further simplify as follows:

$$\frac{1}{2}(u-4\gamma)\left(8u\mathcal{R}''_{s;1b}(u,m)+4(2b+1)\mathcal{R}'_{s;1b}(u,m)\right)+2u\mathcal{R}'_{s;1b}(u,m)+(q^2-b)\mathcal{R}_{s;1b}(u,q)$$
$$=4(u-4\gamma)u\mathcal{R}''_{s;1b}(u,m)-(8\gamma(2b+1)+2(2b+2)u)\mathcal{R}'_{s;1b}(u,m)+(q^2-b)\mathcal{R}_{s;1b}(u,m)$$
$$=4(u-4\gamma)u\mathcal{R}''_{s;1b}(u,m)+4(2(2b+1)-\gamma(b+1)u)\mathcal{R}'_{s;1b}(u,m)+(q^2-b)\mathcal{R}_{s;1b}(u,m).$$
$$(6.2.4)$$

Therefore, the left hand side of (6.2.3) is (6.2.4) multiplied by the factor $-\frac{1}{2}\gamma^m\imath^{sb}u^{\frac{b}{2}}$ producing the equation

$$-\frac{1}{2}\gamma^m\imath^{sb}u^{\frac{b}{2}}\left(4(u-4\gamma)u\mathcal{R}''_{s;1b}(u,m)+4(2(2b+1)-\gamma(b+1)u)\mathcal{R}'_{s;1b}(u,m)\right.$$
$$\left.+(q^2-b)\mathcal{R}_{s;1b}(u,m)\right)=0, \qquad (6.2.5)$$

and this satisfies (6.2.2) as required. $\qquad\square$

Remark. (6.2.5) also has the solution $u = 0$ corresponding to the solution $x = 0$ in (6.1.2).

COROLLARY. *With consideration to each of the two parameters s and b we have on writing the polynomial $\mathcal{R}_{s;1b}(u, m)$ as $\mathcal{R}_{s;1b}$ the solutions*

$$(u - 4)u\mathcal{R}''_{0;10} + (u - 2)\mathcal{R}'_{0;10} - m^2\mathcal{R}_{0;10} = 0,$$

$$(u - 4)u\mathcal{R}''_{0;11} + 2(u - 3)\mathcal{R}'_{0;11} - m(m + 1)\mathcal{R}_{0;11} = 0,$$

$$(u + 4)u\mathcal{R}''_{1;10} + (u + 2)\mathcal{R}'_{1;10} - m^2\mathcal{R}_{1;10} = 0,$$

and

$$(u + 4)u\mathcal{R}''_{1;11} + 2(u + 3)\mathcal{R}'_{1;11} - m(m + 1)\mathcal{R}_{1;11} = 0.$$

Proof. The validity of each follows immediately from Theorem 6.2.2 on the appropriate substitution of each of the parameters s and b, replacing q with $2m + b$ and finally dividing out a common factor of 4. $\qquad\square$

6.3 The recurrence polynomials $\mathcal{R}_{s;0b}(u, m)$

We next turn to the non-alternating parameter $a = 0$. Again we first require some lemmas.

LEMMA 6.3.1 ($\sin\theta\sin q\theta$ solution). *The second order differential equation*

$$\sin^2\theta\frac{d^2y}{d\theta^2} - 2\sin\theta\cos\theta\frac{dy}{d\theta} + \left((q^2 + 1)\sin^2\theta + 2\cos^2\theta\right)y = 0, \tag{6.3.1}$$

has for a solution

$$y = \sin\theta\sin q\theta. \tag{6.3.2}$$

Proof. The first and second derivatives of y (with respect to θ are),

$$\frac{dy}{d\theta} = q\sin\theta\cos q\theta + \cos\theta\sin q\theta, \tag{6.3.3}$$

and

$$\frac{d^2y}{d\theta^2} = -q^2\sin\theta\sin q\theta + 2q\cos\theta\cos q\theta - \sin\theta\sin q\theta. \tag{6.3.4}$$

The necessary result is then obtained by multiplying (6.3.3) by $-2\sin\theta\cos\theta$, (6.3.4) by $\sin^2\theta$, (6.3.2) by $(q^2 + 1)\sin^2\theta + 2\cos^2\theta$ and then adding separately the left and right hand sides of each these scaled equations. $\qquad\square$

LEMMA 6.3.2. *The function $Y = \sin^2\theta U_{q-1}(\cos\theta)$ is a solution of (6.3.1).*

Proof. With $x = \cos\theta$, we have from (5.3.1) that

$$U_{q-1}(\cos\theta) = \frac{\sin q\theta}{\sin\theta}.$$

Multiplication by $\sin^2\theta$ gives us

$$\sin^2\theta U_{q-1}(\cos\theta) = \sin^2\theta\frac{\sin q\theta}{\sin\theta} = \sin\theta\sin q\theta.$$

Hence it follows from Lemma 6.3.1 that Y is a solution of (6.3.1). $\qquad\square$

LEMMA 6.3.3. *The function* $Y = \sin^2\theta U_{q-1}(\cos\theta)$ *is a solution of the differential equation*

$$(1-x^2)^2\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} + x(1-x^2)\frac{\mathrm{d}Y}{\mathrm{d}x} + \left((q^2+1)+(1-q)x^2\right)Y = 0. \tag{6.3.5}$$

Proof. Since we know Y is a solution to (6.3.1), we show that (6.3.1) is equivalent to (6.3.5) with the change of variable from θ to x. We make the appropriate changes using the substitution $x = \cos\theta$, and so $\mathrm{d}x = -\sin\theta\mathrm{d}\theta$. It is preferable to keep the variable θ for as long as possible, and so we initially express our derivatives with respect to x in the variable θ. Let us denote the derivative of Y with respect to x and θ as Y'_x and Y'_θ respectively. We have

$$Y'_x = Y'_\theta\frac{\mathrm{d}\theta}{\mathrm{d}x} - \sin\theta Y'_x = Y'_\theta, \tag{6.3.6}$$

and similarly for the second derivative

$$Y''_x = \frac{\mathrm{d}}{\mathrm{d}x}Y'_x = -\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{Y'_\theta}{\sin\theta}\right)\frac{\mathrm{d}\theta}{\mathrm{d}x} = -\left(\frac{\sin\theta Y''_\theta - \cos\theta Y'_\theta}{\sin^2\theta}\right)\frac{-1}{\sin\theta} = \left(\frac{\sin\theta Y''_\theta - \cos\theta Y'_\theta}{\sin^3\theta}\right).$$

Rearranging for Y''_θ in terms of Y'_x and Y''_x using (6.3.6), gives

$$Y''_\theta = \sin^2\theta Y''_x - \cos\theta Y'_x. \tag{6.3.7}$$

Substituting (6.3.6) and (6.3.7) into the left hand side of (6.3.1) we obtain

$$\sin^2\theta\left(\sin^2\theta Y''_x - \cos\theta Y'_x\right) - 2\sin\theta\cos\theta\left(-\sin\theta Y'_x\right) + \left((q^2+1)\sin^2\theta + 2\cos^2\theta\right)Y, \tag{6.3.8}$$

and on simplification this becomes

$$\sin^4\theta Y''_x + \cos\theta\sin^2\theta Y'_x + \left((q^2+1)\sin^2\theta + 2\cos^2\theta\right)Y. \tag{6.3.9}$$

With $x = \cos\theta$ and so $\sin\theta = (1-x^2)^{\frac{1}{2}}$, (6.3.9) transforms to

$$(1-x^2)^2\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} + x(1-x^2)\frac{\mathrm{d}Y}{\mathrm{d}x} + \left((q^2+1)(1-x^2)+2x^2\right)Y$$

which on manipulation of the Y coefficient yields the left hand side of (6.3.5) as desired. $\qquad\square$

LEMMA 6.3.4. *With* $x = \sqrt{\gamma u}/2$ *we have*

$$\frac{\mathrm{d}}{\mathrm{d}x}\{(1-x^2)U_{q-1}(x)\} = -\gamma^{M'}\imath^{-bs}\left(u^{\frac{2-b}{2}}\mathcal{R}'_{s;0b}(u,m) + \frac{1-b}{2}u^{\frac{-b}{2}}\mathcal{R}_{s;0b}(u,m)\right), \tag{6.3.10}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\{(1-x^2)U_{q-1}(x)\} = -4\gamma^{M'+1}\imath^{s(1-b)}u^{\frac{1-b}{2}}\left(u\mathcal{R}''_{s;0b}(u,m) + \frac{3-2b}{2}\mathcal{R}'_{s;0b}(u,m)\right). \tag{6.3.11}$$

Proof. We have from Theorem 5.6.1 and Lemma 5.3.2 with $q = 2m+b$, $M = m-1+b$ and $M' = m+b$ that

$$(1-x^2)U_{q-1}(x) = (1-\gamma u/4)U_{q-1}(\sqrt{\gamma u}/2) = -\frac{\gamma}{4}(u-4\gamma)S_{q-1}(\sqrt{\gamma u})$$

$$= -\frac{\gamma(\sqrt{\gamma u})^{1-b}}{4}(u-4\gamma)S_{q-1}^r(\gamma u) = -\frac{\gamma^{M'}(\sqrt{\gamma u})^{1-b}}{4}\mathcal{R}_{s;0b}(u,m). \quad (6.3.12)$$

On differentiating both sides of (6.3.12) with respect to x we obtain

$$Y_x' = -\frac{\gamma^{M'+(1-b)/2}}{4}\frac{\mathrm{d}}{\mathrm{d}x}\left(u^{\frac{1-b}{2}}\mathcal{R}_{s;0b}(u,m)\right), \quad (6.3.13)$$

the right hand side of which, via the chain rule, becomes

$$= -\frac{\gamma^{M'}\imath^{s(1-b)}}{4}\frac{\mathrm{d}}{\mathrm{d}u}\left(u^{\frac{1-b}{2}}\mathcal{R}_{s;0b}(u,m)\right)\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$= -\frac{\gamma^{M'}\imath^{s(1-b)}}{4}\left(u^{\frac{1-b}{2}}\mathcal{R}'_{s;0b}(u,m) + \frac{1-b}{2}u^{\frac{-1-b}{2}}\mathcal{R}_{s;0b}(u,m)\right)\frac{4u^{\frac{1}{2}}}{\imath^s}$$

$$= -\gamma^{M'}\imath^{-sb}\left(u^{\frac{2-b}{2}}\mathcal{R}'_{s;0b}(u,m) + \frac{1-b}{2}u^{\frac{-b}{2}}\mathcal{R}_{s;0b}(u,m)\right).$$

$$(6.3.14)$$

Then on differentiating (6.3.12) a second time we have

$$Y_x'' = -\frac{\gamma^{M'+(1-b)/2}}{4}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(u^{\frac{1-b}{2}}\mathcal{R}_{s;0b}(u,m)\right)$$

$$= -\gamma^{M'}\imath^{-sb}\frac{\mathrm{d}}{\mathrm{d}x}\left(u^{\frac{2-b}{2}}\mathcal{R}'_{s;0b}(u,m) + \frac{1-b}{2}u^{\frac{-b}{2}}\mathcal{R}_{s;0b}(u,m)\right)$$

$$= -\gamma^{M'}\imath^{-sb}\frac{\mathrm{d}}{\mathrm{d}u}\left(u^{\frac{2-b}{2}}\mathcal{R}'_{s;0b}(u,m) + \frac{1-b}{2}u^{\frac{-b}{2}}\mathcal{R}_{s;0b}(u,m)\right)\frac{4u^{\frac{1}{2}}}{\imath^s},$$

and following the second application of the chain rule we continue as

$$= -4\gamma^{M'}\imath^{s(1-b)}\imath^{-2s}u^{\frac{1}{2}}\left(u^{\frac{2-b}{2}}\mathcal{R}''_{s;0b}(u,m) + \frac{2-b}{2}u^{\frac{-b}{2}}\mathcal{R}'_{s;0b}(u,m) + \frac{1-b}{2}u^{\frac{-b}{2}}\mathcal{R}'_{s;0b}(u,m)\right.$$

$$\left. + \frac{1-b}{2}\left(\frac{-b}{2}\right)u^{\frac{-2-b}{2}}\mathcal{R}_{s;0b}(u,m)\right)$$

$$= -4\gamma^{M'}\gamma\imath^{s(1-b)}\left(u^{\frac{3-b}{2}}\mathcal{R}''_{s;0b}(u,m) + \frac{3-2b}{2}u^{\frac{1-b}{2}}\mathcal{R}'_{s;0b}(u,m)\right)$$

$$= -4\gamma^{M'+1}\imath^{s(1-b)}u^{\frac{1-b}{2}}\left(u\mathcal{R}''_{s;0b}(u,m) + \frac{3-2b}{2}\mathcal{R}'_{s;0b}(u,m)\right).$$

$$(6.3.15)$$

□

THEOREM 6.3.5. *The polynomials $\mathcal{R}_{s;0b}(u,m)$ satisfy the second order differential equation*

$$4u(u-4\gamma)^2\mathcal{R}''_{s;0b}(u,m) + 4(u-4\gamma)\left((1-b)u - 2\gamma(3-2b)\right)\mathcal{R}'_{s0b}(u,m)$$

$$-\left((q^2-b)u - 4\gamma(q^2+2-b)\right)\mathcal{R}_{s;0b}(u,m) = 0. \quad (6.3.16)$$

Proof. From Lemma 6.3.3 we have that the function $Y = \sin^2\theta U_{q-1}(\cos\theta)$ is a solution of

$$(1-x^2)^2 Y''_x + x(1-x^2)Y'_x + \left((q^2+1) + (1-q^2)x^2\right)Y = 0, \qquad (6.3.17)$$

and from Lemma 6.3.4 we have the relations

$$Y = (1-x^2)U_{q-1}(x) = -\frac{\gamma^{M'}\imath^{\frac{s(1-b)}{2}}u^{\frac{1-b}{2}}}{4}\mathcal{R}_{s;0b}(u,m),$$

$$Y'_x = -\gamma^{M'}\imath^{-bs}\left(u^{\frac{2-b}{2}}\mathcal{R}'_{s;0b}(u,m) + \frac{1-b}{2}u^{\frac{-b}{2}}\mathcal{R}_{s;0b}(u,m)\right),$$

and

$$Y''_x = -4\gamma^{M'}\imath^{-s(b+1)}u^{\frac{1-b}{2}}\left(u\mathcal{R}''_{s;0b}(u,m) + \frac{3-2b}{2}\mathcal{R}'_{s;0b}(u,m)\right).$$

Let $x = \sqrt{\gamma u}/2 = \imath^s\sqrt{u}/2$, and so $dx = \imath^s du/4\sqrt{u}$, or $du/dx = 4u^{\frac{1}{2}}/\imath^s$. Multiplying Y by $\left((q^2+1) + (1-q^2)x^2\right)$, Y'_x by $x(1-x^2)$ and Y''_x by $(1-x^2)^2$, then using the substitution $x = \imath^s u^{\frac{1}{2}}/2$ and factorising produces the expressions

$$-\frac{1}{16}\gamma^{M'}\imath^{\frac{s(1-b)}{2}}u^{\frac{1-b}{2}}\left(4(q^2+1) + \gamma(1-q^2)u\right)\mathcal{R}_{s;0b}(u,m),$$

$$\frac{1}{8}\gamma^{M'+1}\imath^{s(1-b)}u^{\frac{1}{2}}(u-4\gamma)\left(u^{\frac{2-b}{2}}\mathcal{R}'_{s;0b}(u,m) + \frac{1-b}{2}u^{\frac{-b}{2}}\mathcal{R}_{s;0b}(u,m)\right),$$

and

$$-\frac{1}{4}\gamma^{M'+1}\imath^{-s(b+1)}u^{\frac{1-b}{2}}(u-4\gamma)^2\left(u\mathcal{R}''_{s;0b}(u,m) + \frac{3-2b}{2}\mathcal{R}'_{s;0b}(u,m)\right).$$

Adding each of these expressions and then factorising by the term $-\gamma^{M'+1}\imath^{\frac{s(1-b)}{2}}u^{\frac{1-b}{2}}/16$ leaves the expression

$$4u(u-4\gamma)^2\mathcal{R}''_{s;0b}(u,m) + 2(u-4\gamma)\left((u-4\gamma)(3-2b) - u\right)\mathcal{R}'_{s;0b}(u,m)$$
$$+ \left(4\gamma(q^2+1) + (1-q^2)u - (u-4\gamma)(1-b)\right)\mathcal{R}_{s;0b}(u,m),$$

which on simplification produces the required result. $\qquad\square$

COROLLARY. *With consideration to each of the two parameters s and b, we have on writing the polynomial $\mathcal{R}_{s;0b}(u,m)$ as $\mathcal{R}_{s;0b}$ the solutions*

$$u(u-4)^2\mathcal{R}''_{0;00} + (u-4)(u-6)\mathcal{R}'_{0;00} - (m^2u - 2(2m^2+1))\mathcal{R}_{0;00} = 0,$$

$$u(u-4)^2\mathcal{R}''_{0;01} - 2(u-4)\mathcal{R}'_{0;01} - \left((m(m+1))u - 2(2m^2+2m+1)\right)\mathcal{R}_{0;01} = 0,$$

$$u(u+4)^2\mathcal{R}''_{1;00} + (u+4)(u+6)\mathcal{R}'_{1;00} - (m^2u + 2(2m^2+1))\mathcal{R}_{1;00} = 0,$$

and

$$u(u+4)^2\mathcal{R}''_{1;01} + 2(u+4)\mathcal{R}'_{1;01} - \left((m(m+1))u + 2(2m^2+2m+1)\right)\mathcal{R}_{1;01} = 0.$$

Proof. The validity of each follows immediately from Theorem 6.2.2 on the appropriate substitution of each of the parameters s and b, replacing q with $2m+b$ and finally dividing out a common factor of 4. $\qquad\square$

Chapter 7

Orthogonality

In Section 7.1, using the known orthogonality relations of the Chebyshev polynomials, we determine in Theorems 7.1.4 and 7.1.6, similar orthogonal relations for the polynomials $\mathcal{R}_{s;1b}(u,m)$ and $\mathcal{R}_{s;0b}(u,m)$ respectively. Then in Section 7.2 we exploit their orthogonality relations to establish some three term recurrence relations: in Theorem 7.2.4 we determine an intra sequence relation between consecutive values of the variable m and in Theorem 7.2.5 an inter sequence relation between consecutive values of the modulus $q = 2m + b$.

7.1 Orthogonal polynomial sequences

Let us begin this Section by introducing a definition adapted from Chihara [7].

Definition 7.1.1. *A polynomial sequence* $\{P_n(x)\}_{n=0}^{\infty}$ *is an orthogonal polynomial sequence with respect to a weight factor* $w(x)$ *on an interval* (a,b) *if* $P_n(x)$ *is a polynomial of degree* n *and*

$$\int_a^b P_n(x)P_m(x)w(x)\,\mathrm{d}x = \delta_{n,m}K_n,$$

where $\delta_{n,m}$ *is the Kronecker delta symbol and* K_n *is some* nonzero *constant. If additionally we have* $K_n = 1$ *then the polynomials* $P_n(x)$ *form an orthonormal polynomial sequence.*

We now consider the recurrence polynomials, $\mathcal{R}_{s;ab}(u,m)$ as forming (relative to each of three parameters) eight separate sequences of orthogonal polynomials of order $M' = m + b(1-a)$. These polynomials have been defined for $q \geq 1$, but to satisfy the properties of an orthogonal sequence we amend each sequence to $m \geq 1 + ab - (a+b)$, (or $q \geq 2(1-a) - (-1)^a b$). We summarise these conditions in Table 7.1.

As in Section 5 on recurrences we exploit the relationship between the polynomials, $\mathcal{R}_{s;ab}(u,m)$, and their associated Fibonacci, Lucas or Chebyshev polynomial representation. Since the orthogonality properties of the Chebyshev polynomials are well documented, (see for example [18]), we aim to establish the orthogonality of our polynomials in terms of them.

Table 7.1: Effect of the parameters a and b on the initial orthogonal sequence number m (and q).

a	b	m	q
0	0	1	2
0	1	0	1
1	0	0	0
1	1	0	1

7.1.1 The Chebyshev Polynomials

The orthogonality relations for Chebyshev polynomials (see for example [18] and [19]) follow easily from

$$T_n(x) = \cos n\theta = \cos\left(n \cos^{-1} x\right).$$

With the weight factor $\csc\theta = 1/\sqrt{(1 - x^2)}$, and the substitution $\mathrm{d}x = -\sin\theta\,\mathrm{d}\theta$, we have for $n \neq q$

$$\int_{-1}^{1} \frac{T_n(x)T_q(x)}{\sqrt{1 - x^2}}\,\mathrm{d}x = \int_{0}^{\pi} \cos n\theta \cos q\theta\,\mathrm{d}\theta = \frac{1}{2}\int_{0}^{\pi} \cos(n+q)\theta + \cos(n-q)\theta\,\mathrm{d}\theta \qquad (7.1.1)$$

$$= \frac{1}{2}\left[\frac{1}{n+q}\sin(n+q)\theta + \frac{1}{n-q}\sin(n-q)\theta\right]_{0}^{\pi} = 0,$$

If $n = q \neq 0$, (7.1.1) becomes

$$\int_{-1}^{1} \frac{T_n^2(x)}{\sqrt{1 - x^2}}\,\mathrm{d}x = \int_{0}^{\pi} \cos^2 n\theta\,\mathrm{d}\theta = \frac{1}{2}\int_{0}^{\pi} \cos 2n\theta + 1\,\mathrm{d}\theta = \frac{1}{2}\left[\frac{1}{2n}\sin 2n\theta + \theta\right]_{0}^{\pi} = \frac{\pi}{2}.$$

Finally when $n = q = 0$, (7.1.1) becomes simply

$$\int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}}\,\mathrm{d}x = \int_{0}^{\pi} \mathrm{d}\theta = \pi.$$

We shall deduce relations for the polynomials $\mathcal{R}_{s;ab}(u, m)$. Let us put $n = 0$, so that expression (7.1.1) reduces to

$$\int_{-1}^{1} \frac{T_q(x)}{\sqrt{1 - x^2}}\,\mathrm{d}x = \int_{0}^{\pi} \cos q\theta\,\mathrm{d}\theta = 0. \qquad (7.1.2)$$

In the generation of the polynomials $\mathcal{R}_{s;ab}(u, m)$, we have examined separately the cases of $T_q(x)$, when $q = 2m$ and $q = 2m + 1$. Moreover, with the change of variable $x = \sqrt{\gamma u}/2$, we find that we are only considering half of the interval $[-1, 1]$. If we start with (7.1.2), separate the integral into two halves and consider the particular cases of $q = 2m \geq 2$ and $q = 2m + 1 \geq 1$, we obtain

$$\int_{-1}^{0} \frac{T_{2m}(x)}{\sqrt{1 - x^2}}\,\mathrm{d}x = \int_{\pi/2}^{\pi} \cos 2m\theta\,\mathrm{d}\theta = 0, \qquad \int_{0}^{1} \frac{T_{2m}(x)}{\sqrt{1 - x^2}}\,\mathrm{d}x = \int_{0}^{\pi/2} \cos 2m\theta\,\mathrm{d}\theta = 0,$$

and

$$\int_{-1}^{0} \frac{T_{2m+1}(x)}{\sqrt{1-x^2}}\,dx = \int_{\pi/2}^{\pi} \cos{(2m+1)\theta}\,d\theta = \frac{(-1)^{m+1}}{2m+1}, \tag{7.1.3}$$

$$\int_{0}^{1} \frac{T_{2m+1}(x)}{\sqrt{1-x^2}}\,dx = \int_{0}^{\pi/2} \cos{(2m+1)\theta}\,d\theta = \frac{(-1)^{m}}{2m+1}. \tag{7.1.4}$$

So for $n = 0$, the even-numbered polynomials $T_{2m}(x)$ are orthogonal to constants on the half intervals $[-1, 0]$ and $[0, 1]$, whilst the odd-numbered $T_{2m+1}(x)$ are not so. However, for $n \geq 1$ and $n \neq q$, (7.1.3) and (7.1.4) are replaced by

$$-\int_{-1}^{0} \frac{T_{2m+1}(x)T_{2k+1}(x)}{\sqrt{1-x^2}}\,dx = \int_{0}^{1} \frac{T_{2m+1}(x)T_{2k+1}(x)}{\sqrt{1-x^2}}\,dx$$

$$= \frac{1}{2}\int_{0}^{\pi/2} \cos 2(m+k+1)\theta + \cos 2(m-k)\theta\,d\theta$$

$$= \frac{1}{2}\left[\frac{1}{2(m+k+1)}\sin{(2(m+k+1)\theta}) + \frac{1}{2(m-k)}\sin 2(m-k)\theta\right]_{0}^{\pi/2} = 0,$$

and this leads us to our next lemmas.

LEMMA 7.1.1 (different parity on half intervals). *For q odd and n even, we have*

$$\int_{0}^{\gamma 1} \frac{T_q(x)T_n(x)}{\sqrt{1-x^2}}\,dx = \frac{(-1)^{m+k}q}{(q+n)(q-n)}.$$

Proof. We consider separately the cases of the parameter s. When $s = 0$ we have

$$\int_{0}^{1} \frac{T_{2m+1}(x)T_{2k}(x)}{\sqrt{1-x^2}}\,dx = \frac{(-1)^2}{2}\int_{0}^{\pi/2}(\cos{(2(m+k)+1)\theta} + \cos{(2(m-k)+1)\theta})\,d\theta$$

$$= \frac{1}{2}\left[\frac{(-1)^{m+k}}{2(m+k)+1} + \frac{(-1)^{m-k}}{2(m-k)+1}\right] = \frac{1}{2}\left[\frac{(-1)^{m+k}2(2m+1)}{(2(m+k)+1)(2(m-k)+1)}\right]$$

$$= \frac{(-1)^{m+k}q}{(q+n)(q-n)}.$$

When $s = 1$ we have

$$\int_{0}^{-1} \frac{T_{2m+1}(x)T_{2k}(x)}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_{\pi/2}^{\pi}(\cos(2(m+k)+1)\theta + \cos{(2(m-k)+1)\theta})\,d\theta$$

$$= \frac{(-1)^2}{2}\left[\frac{(-1)^{m+k}}{2(m+k)+1} + \frac{(-1)^{m-k}}{2(m-k)+1}\right] = \frac{1}{2}\left[\frac{(-1)^{m+k}2(2m+1)}{(2(m+k)+1)(2(m-k)+1)}\right]$$

$$= \frac{(-1)^{m+k}q}{(q+n)(q-n)},$$

and on combining the two cases the result is obtained. $\qquad\square$

LEMMA 7.1.2 (same parity on half intervals). *For q and n of the same parity, we have*

$$\int_{0}^{\gamma 1} \frac{T_{2m+b}(x)T_{2k+b}(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 0 & \text{if } q \neq n \\ \gamma\pi/2 & \text{if } q = n = 0 \\ \gamma\pi/4 & \text{if } q = n \text{ and } q \geq 1, \end{cases}$$

where $\gamma = (-1)^s$.

Proof. We consider separately the cases of the parameter s. When $s = 0$ we have

$$\int_0^1 \frac{T_{2m+b}(x)T_{2k+b}(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{(-1)^2}{2}\int_0^{\pi/2}(\cos 2(m+k+b)\theta + \cos 2(m-k)\theta)\,\mathrm{d}\theta$$

$$= \frac{1}{2}\left[\epsilon\delta_{m,k}\theta + \delta_{m,k}\theta\right]_0^{\pi/2} = \frac{(\epsilon+1)\delta_{m,k}}{2}\left[\theta\right]_0^{\pi/2}$$

$$= \frac{(\epsilon+1)\pi\delta_{m,k}}{4} = \begin{cases} 0 & \text{if } q \neq n \\ \pi/2 & \text{if } q = n = 0 \\ \pi/4 & \text{if } q = n \text{ and } q \geq 1, \end{cases}$$

where

$$\epsilon = \begin{cases} 1 & \text{if } q = 0 \\ 0 & \text{if } q \geq 1, \end{cases}$$

and we note that ϵ takes on a different notation to that employed in Chapter 4.

When $s = 1$ we have

$$\int_0^{-1} \frac{T_{2m+b}(x)T_{2k+b}(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = -\frac{1}{2}\int_{\pi/2}^{\pi}(\cos 2(m+k+b)\theta + \cos 2(m-k)\theta)\,\mathrm{d}\theta$$

$$= -\frac{1}{2}\left[\epsilon\delta_{m,k}\theta + \delta_{m,k}\theta\right]_{\pi/2}^{\pi} = -\frac{(\epsilon+1)\delta_{m,k}}{2}\left[\theta\right]_{\pi/2}^{\pi}$$

$$= -\frac{(\epsilon+1)\pi\delta_{m,k}}{4} = \begin{cases} 0 & \text{if } q \neq n \\ -\pi/2 & \text{if } q = n = 0 \\ -\pi/4 & \text{if } q = n \text{ and } q \geq 1, \end{cases}$$

and so on combining the two cases the result is obtained. $\qquad\square$

Remark. Lemma 7.1.1 informs us that halving the integral length destroys the orthogonality relation between the sequence of polynomials $\{T_n(x)\}_{n=0}^{\infty}$. More specifically it demonstrates a destruction between the polynomials of different parity. On the other hand, Lemma 7.1.2 informs us that for the sequence $\{T_{2k}(x)\}_{k=0}^{\infty}$ the orthogonality property is preserved, whereas for the sequence $\{T_{2m+1}(x)\}_{m=0}^{\infty}$, not all of this property survives, (i.e. there is no constant term).

7.1.2 The recurrence polynomials $\mathcal{R}_{s;1b}(u,m)$

We observe that the polynomials $\mathcal{R}_{s;11}(u,m)$ have a "correction factor" of $u^{-1/2}$, so that $T_1(x) = x$ is mapped to $\mathcal{R}_{s;11}(u,0) = 1$, thus establishing the orthogonality of the polynomial sequences $\{\mathcal{R}_{s;11}(u,m)\}_{m=0}^{\infty}$, provided that the correction factor of both polynomials, u, is accounted for in the weight factor $w(u)$. After first introducing a lemma to help clarify one of the steps, we formulate these ideas as a theorem.

LEMMA 7.1.3. *We have*

$$\gamma\imath^{1+s} = \imath^{1-s} = \begin{cases} \imath & \text{if } s = 0 \\ 1 & \text{if } s = 1. \end{cases}$$

Proof. Recalling that $\gamma = (-1)^s$, the result follows on substitution of $s = 0$ and $s = 1$ into each of the terms. $\qquad\square$

THEOREM 7.1.4. *The polynomials $\mathcal{R}_{s;1b}(u, m)$ form a family of orthogonal polynomials with respect to the weight factor $(\gamma u)^b / \sqrt{\gamma u (u - 4\gamma)}$ on the interval $[0, 4\gamma]$, expressed by the integral*

$$\int_0^{4\gamma} \frac{(\gamma u)^b \mathcal{R}_{s;1b}(u, m) \mathcal{R}_{s;1b}(u, k)}{\sqrt{\gamma u}\sqrt{u - 4\gamma}} \, du = \begin{cases} 0 & \text{if } m \neq k \\ -4\pi\imath^{1-s} & \text{if } m = k \text{ and } q = 0 \\ -2\pi\imath^{1-s} & \text{if } m = k \text{ and } q \geq 1. \end{cases}$$

.

Proof. From Theorem 5.6.1 for the parameter $a = 1$,

$$\mathcal{R}_{s;1b}(u, m) = A^r_{s;1b}(u, 2m + b),$$

then from the Corollary to Theorem 5.4.1 and Lemma 5.3.1, the function $A^r_{s;1b}$ can be associated to the *modified* monic Chebyshev function C^r_q by the relations

$$A^r_{0;1b}(u, q) = C^r_q(u),$$

and

$$A^r_{1;1b}(u, q) = L^r_q(u) = (-1)^m L^r_q(-u) = (-1)^m C^r_q(-u).$$

Using these forms, the substitution $x = \sqrt{\gamma u/2}$ and Theorem 5.6.1 we can write, (for $q = 2m + b \geq 1$),

$$\mathcal{R}_{s;1b}(u, m) = A^r_{s;1b}(u, q) = \gamma^m C^r_q(\gamma u) = \gamma^m \frac{C_q(\sqrt{\gamma u})}{(\sqrt{\gamma u})^b} = \gamma^m \frac{2T_q(\sqrt{\gamma u}/2)}{(\sqrt{\gamma u})^b},$$

and when $q = 0$ we have $\mathcal{R}_{s;10}(u, 0) = 2T_0(x) = 2$. From (7.1.3) we find that on making the substitution $x = \sqrt{u}/2$ it is necessary to consider only half the original integral, let us select the interval $[0, 4\gamma]$. We have

$$\int_0^{4\gamma} \frac{(\gamma u)^b \mathcal{R}_{0;1b}(u, m) \mathcal{R}_{0;1b}(u, k)}{\sqrt{\gamma u}\sqrt{u - 4\gamma}} \, du$$

$$= \gamma^{m+k} \int_0^{4\gamma} \frac{(\gamma u)^b 2T_{2m+b}(\sqrt{\gamma u}/2) 2T_{2k+b}(\sqrt{\gamma u}/2)}{(\sqrt{\gamma u})^{2b} \sqrt{\gamma u}\sqrt{u - 4\gamma}} \, du$$

$$= 4\gamma^{m+k} \int_0^{4\gamma} \frac{T_{2m+b}(\sqrt{\gamma u}/2) T_{2k+b}(\sqrt{\gamma u}/2)}{\sqrt{u - 4\gamma}} \frac{du}{\sqrt{\gamma u}}$$

$$= 16\gamma^{m+k+1} \int_0^{4\gamma} \frac{T_{2m+b}(\sqrt{\gamma u}/2) T_{2k+b}(\sqrt{\gamma u}/2)}{\sqrt{-4\gamma}\sqrt{1 - \gamma u/4}} \frac{\gamma \, du}{4\sqrt{\gamma u}}$$

$$= -8\gamma^{m+k}\imath^{s+1} \int_0^{\gamma 1} \frac{T_{2m+b}(x) T_{2k+b}(x)}{\sqrt{1 - x^2}} \, dx.$$

The required result then follows from the application of Lemma 7.1.2 to the integral. $\qquad\square$

For demonstration we illustrate the case for $m = k$ with $q \geq 1$. So using Lemma 7.1.3 we have

$$-8\gamma^{m+k}\imath^{1+s}\frac{\gamma\pi}{4} = -2\gamma\pi\imath^{1+s} = -2\pi\imath^{1-s}.$$

7.1.3 The recurrence polynomials $\mathcal{R}_{s;0b}(u, m)$

In a similar manner we now examine and utilise the Chebyshev polynomials of the second kind, $U_n(x)$ that we recall from (5.3.1) are defined by

$$U_{n-1}(x) = \frac{\sin n\theta}{\sin \theta},$$

to enable us to extract the orthogonality of the sequences of polynomials $\mathcal{R}_{s;0b}(u, m)$. We find in these latter polynomials that the omnipresent $u - 4\gamma$ factor excludes the possibility in each sequence of the polynomial $P_0(u) = c$, (c a constant). Hence the creation of an orthogonal sequence for these polynomials requires the absorption of $u - 4\gamma$ into the weight factor. Alternatively, we select a weight factor that realigns the sequences $\{\mathcal{R}_{s;0b}(u, m)\}_{m=1-b}^{\infty}$ to those of $\{U_{2M+B}\}_{M=0}^{\infty}$, where $M = m - 1 + b$ and $B = 1 - b$. We start with a lemma.

LEMMA 7.1.5 (half interval integral). *For positive $Q = 2M + B$ and $N = 2K + B$ and constant $B \in \{0, 1\}$, we have*

$$\int_0^{\gamma 1} U_{2M+B}(x)U_{2K+B}(x)(\sqrt{1 - x^2})\,\mathrm{d}x = \begin{cases} 0 & \text{if } Q \neq N \\ \gamma\pi/4 & \text{if } Q = N \text{ and } Q \geq 1, \end{cases}$$

where $\gamma = (-1)^s$.

Proof. We consider separately the cases of the parameter s. When $s = 0$ we have

$$\int_0^1 U_{2M+B}(x)U_{2K+B}(x)\sqrt{1 - x^2}\,\mathrm{d}x$$

$$= (-1)^2 \int_0^{\pi/2} \sin(2m + b)\theta \sin(2k + b)\theta\,\mathrm{d}\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \cos 2(m - k)\theta - \cos 2(m + k + b)\theta\,\mathrm{d}\theta$$

$$= \frac{1}{2} [\delta_{m,k}\theta]_0^{\pi/2} = \frac{\delta_{m,k}\pi}{4} = \begin{cases} 0 & \text{if } Q \neq N \\ \pi/4 & \text{if } Q = N \text{ and } Q \geq 1. \end{cases}$$

When $s = 1$ we have

$$\int_0^{-1} U_{2M+B}(x)U_{2K+B}(x)\sqrt{1 - x^2}\,\mathrm{d}x$$

$$= -\frac{1}{2} \int_{\pi/2}^{\pi} \cos 2(m - k)\theta - \cos 2(m + k + b)\theta\,\mathrm{d}\theta$$

$$= -\frac{1}{2} [\delta_{m,k}\theta]_{\pi/2}^{\pi} = -\frac{\delta_{m,k}\pi}{4} = \begin{cases} 0 & \text{if } Q \neq N \\ -\pi/4 & \text{if } Q = N \text{ and } Q \geq 1, \end{cases}$$

and so on combining the two cases the result is obtained. $\square$

Now since we are using only using the Chebyshev $U_Q(x)$ polynomials, $M = m - (1 - b)$ and $B = 1 - b$.

THEOREM 7.1.6. *The polynomials $\mathcal{R}_{s;0b}(u, m)$ form an orthogonal polynomial sequence with respect to the weight factor $\sqrt{\gamma u}/((\gamma u)^b \sqrt{(u - 4\gamma)^3})$ on the interval $[0, 4\gamma]$, satisfying the integral*

$$\int_0^{4\gamma} \frac{\sqrt{\gamma u} \mathcal{R}_{s;0b}(u, m) \mathcal{R}_{s;0b}(u, k)}{(\gamma u)^b (\sqrt{u - 4\gamma})^3} \, du = \begin{cases} 0 & if \ m \neq k \\ 2\pi \imath^{s+1} & if \ m = k \ and \ q \geq 1. \end{cases}$$

Proof. From Theorem 5.6.1 with the parameter $a = 0$ we have

$$\mathcal{R}_{s;0b}(u, m) = (u - 4\gamma) A_{s;0b}^r(u, q),$$

then from the Corollary to Theorem 5.4.1 and Lemma 5.3.2 we rcall that the function $A_{s;0b}^r$ can be associated to the *modified* monic Chebyshev function S_{q-1}^r by the relation

$$A_{0;0b}^r(u, q - 1) = S_{q-1}^r(u), \quad and \quad A_{1;0b}^r(u, q) = F_q^r(u) = (-1)^m F_q^r(-u) = (-1)^m S_{q-1}^r(-u).$$

Using these forms, the substitution $x = \sqrt{\gamma u/2}$ and Theorem 5.6.1 we can write for $q \geq 1$, $(Q \geq 0)$,

$$\mathcal{R}_{s;0b}(u, m) = (u - 4\gamma) A_{s;0b}^r(u, q) = \gamma^M (u - 4\gamma) S_{q-1}^r(\gamma u)$$
$$= \gamma^M \frac{(u - 4\gamma) S_{q-1}(\sqrt{\gamma u})}{(\sqrt{\gamma u})^{1-b}} = \gamma^M \frac{(u - 4\gamma) U_{q-1}(\sqrt{\gamma u}/2)}{(\sqrt{\gamma u})^{1-b}}.$$

When $q = 1$ we have $\mathcal{R}_{s;01}(u, 1) = (u - 4\gamma)$, and when $q = 2$, we also find that $\mathcal{R}_{s;00}(u, 2) = (u - 4\gamma)$. From (7.1.3) we find that on making the substitution $x = \sqrt{u}/2$ it is necessary to consider only half the original integral, let us select the interval $[0, \gamma 1]$. We have

$$\int_0^{4\gamma} \frac{(\gamma u)^{1-b} \mathcal{R}_{0;1b}(u, m) \mathcal{R}_{0;1b}(u, k)}{\sqrt{\gamma u}(\sqrt{u - 4\gamma})^3} \, du$$
$$= \gamma^{M+K} \int_0^{4\gamma} \frac{(\gamma u)^{1-b}(u - 4\gamma) U_{q-1}(\sqrt{\gamma u}/2)(u - 4\gamma) U_{n-1}(\sqrt{\gamma u}/2)}{(\sqrt{\gamma u})^{2(1-b)} \sqrt{\gamma u}(\sqrt{u - 4\gamma})^3} \, du$$
$$= \gamma^{M+K} \int_0^{4\gamma} U_{2M+1-b}(\sqrt{\gamma u}/2) U_{2K+1-b}(\sqrt{\gamma u}/2)(\sqrt{u - 4\gamma}) \frac{du}{\sqrt{\gamma u}}$$
$$= 4\gamma^{M+K+1} \int_0^{4\gamma} U_{2M+1-b}(\sqrt{\gamma u}/2) U_{2K+1-b}(\sqrt{\gamma u}/2)(\sqrt{-4\gamma})(\sqrt{1 - \gamma u/4}) \frac{\gamma du}{4\sqrt{\gamma u}}$$
$$= 8\gamma^{M+K+1} \imath^{s+1} \int_0^{\gamma 1} U_{2M+1-b}(x) U_{2K+1-b}(x)(\sqrt{1 - x^2}) dx.$$

The desired from result then follows from application of Lemma 7.1.5 to the integral. □

We illustrate this for the case $m = k$ (and $q \geq 1$) and we obtain

$$8\gamma \imath^{1+s} \frac{\gamma \pi}{4} = 2\pi \gamma^2 \imath^{1+s} = 2\pi \imath^{1+s}.$$

We now combine Theorems 7.1.4 and 7.1.6.

THEOREM 7.1.7. *The polynomials $\mathcal{R}_{s;ab}(u, m)$ form an orthogonal polynomial sequence, with respect to the weight factor $w_{s;ab}(u)$ defined as*

$$w_{s;ab}(u) = \frac{(\sqrt{\gamma u})^{\lambda(1-2b)}}{(u - 4\gamma)^{2+\lambda}} \qquad where \qquad \lambda = (-1)^a,$$

on the interval $[0, 4\gamma]$ that satisfies the integral equation

$$\int_0^{4\gamma} \mathcal{R}_{s;ab}(u, m)\mathcal{R}_{s;ab}(u, k)w_{s;ab}(u)\, du = \begin{cases} 0 & \text{if } m \neq k \\ -4a\pi\imath^{1-s} & \text{if } m = k \text{ and } q = 0 \\ 2\pi\lambda\imath^{1+\lambda s} & \text{if } m = k \text{ and } q \geq 1. \end{cases} \qquad (7.1.5)$$

Proof. We consider the two cases of the parameter a. When $a = 1$, the weight factor becomes

$$w_{s;1b}(u) = \frac{(\sqrt{\gamma u})^{2b-1}}{(u - 4\gamma)},$$

and from Theorem 7.1.4, equation (7.1.5) is

$$\int_0^{4\gamma} \frac{(\sqrt{\gamma u})^{2b-1}\mathcal{R}_{s;1b}(u, m)\mathcal{R}_{s;1b}(u, k)}{(u - 4\gamma)}\, du = \begin{cases} 0 & \text{if } m \neq k \\ -4\pi\imath^{1-s} & \text{if } m = k \text{ and } q = 0 \\ -2\pi\imath^{1-s} & \text{if } m = k \text{ and } q \geq 1. \end{cases}$$

Then when $a = 0$, the weight factor becomes

$$w_{s;0b}(u) = \frac{(\sqrt{\gamma u})^{1-2b}}{(u - 4\gamma)^3},$$

and from Theorem 7.1.6, equation (7.1.5) is

$$\int_0^{4\gamma} \frac{(\sqrt{\gamma u})^{1-2b}\mathcal{R}_{s;0b}(u, m)\mathcal{R}_{s;0b}(u, k)}{(u - 4\gamma)^3}\, du = \begin{cases} 0 & \text{if } m \neq k \\ 0 & \text{if } m = k \text{ and } q = 0 \\ 2\pi\imath^{s+1} & \text{if } m = k \text{ and } q \geq 1, \end{cases}$$

therefore, establishing the result. $\qquad\qquad\square$

In isolating the individual sequences we obtain the following corollary.

COROLLARY. *For the (non)alternating parameter case $a = 0$, we have the orthogonal relations*

$$\int_0^4 \frac{\sqrt{u}\mathcal{R}_{0;00}(u, m)\mathcal{R}_{0;00}(u, k)}{(\sqrt{u} - 4)^3}\, du = \begin{cases} 0 & \text{if } m \neq k \\ 2\pi\imath & \text{if } m = k \text{ and } m \geq 1, \end{cases}$$

$$\int_0^4 \frac{\mathcal{R}_{0;01}(u, m)\mathcal{R}_{0;01}(u, k)}{\sqrt{u}\,(\sqrt{u} - 4)^3}\, du = \begin{cases} 0 & \text{if } m \neq k \\ 2\pi\imath & \text{if } m = k, \end{cases}$$

$$\int_{-4}^0 \frac{\sqrt{-u}\mathcal{R}_{1;00}(u, m)\mathcal{R}_{1;00}(u, k)}{(\sqrt{u} + 4)^3}\, du = \begin{cases} 0 & \text{if } m \neq k \\ 2\pi & \text{if } m = k \text{ and } m \geq 1, \end{cases}$$

and

$$\int_{-4}^{0} \frac{\mathcal{R}_{1;01}(u,m)\mathcal{R}_{1;01}(u,k)}{\sqrt{-u}\left(\sqrt{u+4}\right)^3}\, \mathrm{d}u = \begin{cases} 0 & \textit{if } m \neq k \\ 2\pi & \textit{if } m = k. \end{cases}$$

Also for the alternating parameter case $a = 1$, we have the orthogonal relations

$$\int_{0}^{4} \frac{\mathcal{R}_{0;10}(u,m)\mathcal{R}_{0;10}(u,k)}{\sqrt{u}\sqrt{u-4}}\, \mathrm{d}u = \begin{cases} 0 & \textit{if } m \neq k \\ -4\pi\imath & \textit{if } m = k \ \textit{and } m = 0 \\ -2\pi\imath & \textit{if } m = k \ \textit{and } m \geq 1, \end{cases}$$

$$\int_{0}^{4} \frac{\sqrt{u}\mathcal{R}_{0;11}(u,m)\mathcal{R}_{0;11}(u,k)}{\sqrt{u-4}}\, \mathrm{d}u = \begin{cases} 0 & \textit{if } m \neq k \\ -2\pi\imath & \textit{if } m = k, \end{cases}$$

$$\int_{-4}^{0} \frac{\mathcal{R}_{1;10}(u,m)\mathcal{R}_{1;10}(u,k)}{\sqrt{-u}\sqrt{u+4}}\, \mathrm{d}u = \begin{cases} 0 & \textit{if } m \neq k \\ 4\pi & \textit{if } m = k \ \textit{and } m = 0 \\ 2\pi & \textit{if } m = k \ \textit{and } m \geq 1, \end{cases}$$

and

$$\int_{-4}^{0} \frac{\sqrt{-u}\mathcal{R}_{1;11}(u,m)\mathcal{R}_{1;11}(u,k)}{\sqrt{u+4}}\, \mathrm{d}u = \begin{cases} 0 & \textit{if } m \neq k \\ 2\pi & \textit{if } m = k. \end{cases}$$

Proof. Each of these results follows directly upon substitution of each of the parameters s, a and b into (7.1.5) of Theorem 7.1.7 and **reversing the order of integration**, (and therefore changing the sign), when necessary. □

7.2 Three term order recurrences

In Section 5 we described an order $M' = m + b(1 - a)$ linear recurrence relation polynomial $\mathcal{R}_{s;ab}(u,m)$, that for fixed q and t, facilitates the calculation of the rth term, (for $r \geq M'$), of the sequence $\mathcal{L}_{s;abc}(r,t,q)$.

Alternatively, the word recurrence often refers to a relation between polynomials of consecutive orders. We will consider two types of such recurrences, those from the same sequence (that we refer to as an *intra* sequence) and those separated by the parameter b (that we refer to as an *inter* sequence).

7.2.1 Intra sequence recurrences

For fixed parameters s, a and b we consider the relation between the polynomial $\mathcal{R}_{s;ab}(u,m+2)$ in terms of the polynomials $\mathcal{R}_{s;ab}(u,m+1)$ and $\mathcal{R}_{s;ab}(u,m)$. To elucidate this relation we exploit the orthogonality of these polynomials shown in Section 7.1. We use a theorem to demonstrate a method described in [18].

THEOREM 7.2.1. *The orthogonal polynomial sequence $\{\phi_r(x)\}_{r=0}^{\infty}$ satisfies the three term relation*

$$\phi_{r+1}(x) = (\alpha_r x + \beta_r)\phi_r(x) + \gamma_{r-1}\phi_{r-1}(x). \tag{7.2.1}$$

Here the coefficients α_r, β_r and γ_{r-1} are given by

$$\alpha_r = \frac{A_{r+1}}{A_r}, \tag{7.2.2}$$

where A_i is the leading coefficient of the polynomial $\phi_i(x)$.

$$\beta_r = \frac{-\alpha_r}{k_r} \int_{-1}^{1} w(x) x \phi_r^2(x)\, dx \tag{7.2.3}$$

where $k_i \neq 0$ is defined by

$$k_i = \int_{-1}^{1} w(x)\phi_i^2(x)\, dx = \int_{-1}^{1} w(x)[A_i x^i + \phi_{i-1}(x)]\phi_i(x)\, dx = A_i \int_{-1}^{1} w(x)x^i \phi_i(x)\, dx, \tag{7.2.4}$$

and

$$\gamma_{r-1} = -\alpha_r \left(\frac{A_{r-1}}{A_r}\right) \left(\frac{k_r}{k_{r-1}}\right). \tag{7.2.5}$$

Proof. We choose α_r such that

$$\phi_{r+1}(x) - \alpha_r x \phi_r(x) \tag{7.2.6}$$

is a polynomial of degree r. Now due to the orthogonality of the sequence of polynomials $\{\phi_i(x)\}_{i=0}^{r}$, they are linearly independent and so span the space of the polynomials of degree r. We, therefore, select this sequence as a basis for the polynomials of degree r and write (7.2.6) as a linear combination of these polynomials. We have

$$\phi_{r+1}(x) - \alpha_r x \phi_r(x) = \beta_r \phi_r(x) + \gamma_{r-1}\phi_{r-1}(x) + \gamma_{r-2}\phi_{r-2}(x) + \ldots + \gamma_0\phi_0(x). \tag{7.2.7}$$

Multiplying (7.2.7) by $w(x)\phi_i(x)$, where $w(x)$ is the weight factor, and integrating (between -1 and 1), we find that for $0 \leq i \leq r - 1$,

$$\int_{-1}^{1} w(x)\phi_i(x)\{\phi_{r+1}(x) - \alpha_r x \phi_r(x)\}\, dx = \gamma_i \int_{-1}^{1} w(x)\phi_i^2(x)\, dx = \gamma_i k_i, \tag{7.2.8}$$

and when $i = r$ we have

$$\int_{-1}^{1} w(x)\phi_r(x)\{\phi_{r+1}(x) - \alpha_r x \phi_r(x)\}\, dx = \beta_r k_r. \tag{7.2.9}$$

Now since, for $0 \leq i \leq r - 2$, $\phi_{r+1}(x)$ is orthogonal to $\phi_i(x)$ and $\phi_r(x)$ is orthogonal to $x\phi_i(x)$, the left hand side of (7.2.8) disappears and so $\gamma_{r-2} = \gamma_{r-3} = \ldots = \gamma_0 = 0$.

However, when $i = r-1$, the terms $\phi_r(x)$ and $x\phi_{r-1}(x)$, both of degree r are no longer orthogonal, and similarly when $i = r$, the term $\alpha_r x \phi_r(x)$ is not orthogonal to $\phi_r(x)$). Consequently

in each case the left hand side of (7.2.8) and (7.2.9) respectively is nonzero. Therefore, (7.2.7) simplifies to

$$\phi_{r+1}(x) - \alpha_r x \phi_r(x) = \beta_r \phi_r(x) + \gamma_{r-1}\phi_{r-1}(x) \tag{7.2.10}$$

and on rearrangement of (7.2.10) we obtain (7.2.1).

To derive the coefficient α_r, we observe that its selection in (7.2.6) is such that on taking the coefficient of x^{r+1} we obtain

$$A_{r+1} - \alpha_r A_r = 0,$$

which on rearrangement gives (7.2.2).

For the coefficient β_r, we find that on simplifying (7.2.9) we obtain

$$-\alpha_r \int_{-1}^{1} w(x)x\phi_r^2(x)\,\mathrm{d}x = \beta_r k_r,$$

which on rearrangement produces (7.2.3). Finally for γ_{r-1}, we put $i = r-1$ into (7.2.8) so that on simplification (and use of 7.2.4) we have

$$-\alpha_r \int_{-1}^{1} w(x)x\phi_{r-1}(x)\phi_r(x)\mathrm{d}x = \gamma_{r-1}\int_{-1}^{1} w(x)\phi_{r-1}^2(x)\,\mathrm{d}x$$

$$-\alpha_r \int_{-1}^{1} w(x)[A_{r-1}x^r + x\phi_{r-2}(x)]\phi_r(x)\,\mathrm{d}x = \gamma_{r-1}k_{r-1}$$

$$-\alpha_r A_{r-1}\int_{-1}^{1} w(x)x^r \phi_r(x)\,\mathrm{d}x = \gamma_{r-1}k_{r-1}$$

$$-\alpha_r A_{r-1}\frac{k_r}{A_r} = \gamma_{r-1}k_{r-1}$$

$$-\alpha_r \frac{A_{r-1}}{A_r}\frac{k_r}{k_{r-1}} = \gamma_{r-1}.$$

$\square$

COROLLARY. *If $\{\phi_r(x)\}_{r=0}^{\infty}$ is a monic orthogonal polynomial sequence and there exists some i, such that for all $j \geq i$ we have $k_j = k_i$, where k_i is defined as in (7.2.4), then (7.2.1) simplifies to*

$$\phi_{j+2}(x) = (x + \beta_{j+1})\phi_{j+1}(x) - \phi_j(x), \tag{7.2.11}$$

where

$$\beta_{j+1} = \frac{-1}{k_i}\int_{-1}^{1} w(x)x\phi_{j+1}^2(x)\,\mathrm{d}x.$$

Proof. If $\phi_j(x)$ is monic, then we have that $A_{j+2} = A_{j+1} = 1$, therefore, $\alpha_{j+1} = 1$, and then since $k_i = k_{i+1}\ldots = k_{j+1}$, it follows from (7.2.5) that $\gamma_j = -1$. Then replacing r with $j+1$ in (7.2.3) and using the fact that $\alpha_{j+1} = 1$ and $k_{j+1} = k_i$, we obtain (7.2.11). $\square$

LEMMA 7.2.2. *For $q \geq 0$, we have*

$$\int_0^{4\gamma} (\mathcal{R}_{s;ab}(u,m))^2\, w_{s;ab}(u)\,\mathrm{d}u = \lambda 4\imath^{1+\lambda s}\int_0^{\pi/2} 1 - \lambda\cos 2(2m+b)\theta\,\mathrm{d}\theta, \tag{7.2.12}$$

where $\lambda = (-1)^a$ and $\gamma = (-1)^s$.

Proof. When $a = 1$, we have from Theorem 7.1.4 with $m = k$ and then Lemma 7.1.2 (that demonstrates the need for the additional factor γ when the variable changes from x to θ),

$$\int_0^{4\gamma} \frac{(\sqrt{\gamma u})^{2b-1} \left(\mathcal{R}_{s;1b}(u, m)\right)^2}{\sqrt{u - 4\gamma}} \, du$$

$$= -8\imath^{1+s} \int_0^{\gamma 1} \frac{(T_{2m+b}(x))^2}{\sqrt{1 - x^2}} \, dx$$

$$= -8\gamma\imath^{1+s} \int_0^{\pi/2} (\cos(2m + b)\theta)^2 \, d\theta$$

$$= -4\imath^{1-s} \int_0^{\pi/2} (1 + \cos 2(2m + b)\theta) \, d\theta.$$

and when $a = 0$, we have from Theorem 7.1.6 with $m = k$ and then Lemma 7.1.2 (also highlighting the additional γ factor when changing variable),

$$\int_0^{4\gamma} \frac{(\sqrt{\gamma u})^{1-2b} \left(\mathcal{R}_{s;0b}(u, m)\right)^2}{(\sqrt{u - 4\gamma})^3} \, du$$

$$= 8\gamma\imath^{1+s} \int_0^{\gamma 1} (U_{2m+b-1}(x))^2 \sqrt{1 - x^2} \, dx$$

$$= 8\gamma^2\imath^{1+s} \int_0^{\pi/2} (\sin(2m + b)\theta)^2 \, d\theta$$

$$= 4\imath^{1+s} \int_0^{\pi/2} (1 - \cos 2(2m + b)\theta) \, d\theta.$$

Combining these two results we obtain (7.2.12). $\qquad\square$

LEMMA 7.2.3 (first moment functional). *For $q \geq 0$, we have*

$$\int_0^{4\gamma} u \left(\mathcal{R}_{s;ab}(u, m)\right)^2 w_{s;ab}(u) \, du = \begin{cases} -8a\pi\imath^{1+s} & \text{if } q = 0 \\ \frac{6}{3^{1-a}} \pi\lambda\imath^{1-\lambda s} & \text{if } q = 1 \\ 4\pi\lambda\imath^{1-\lambda s} & \text{if } q \geq 2, \end{cases} \tag{7.2.13}$$

where $\lambda = (-1)^a$.

Proof. From Lemma 7.2.2 and recalling that $u = 4\gamma x^2 = 4\gamma \cos^2 \theta$, the left hand side of (7.2.13) can be written as

$$\lambda 4\imath^{1+\lambda s} \int_0^{\pi/2} 4\gamma \cos^2 \theta \, (1 - \lambda \cos 2(2m + b)\theta) \, d\theta. \tag{7.2.14}$$

If $q = 0$ then (7.2.14) becomes

$$4\lambda\imath^{1+\lambda s} \int_0^{\pi/2} \gamma(2a)4\cos^2 \theta \, d\theta = 16\lambda\gamma a\imath^{1+\lambda s} \int_0^{\pi/2} 1 + \cos 2\theta \, d\theta = -6\gamma a\imath^{1-s} \left[\frac{\pi}{2}\right] = -8a\pi\imath^{1+s}.$$

If $q \geq 1$ then on applying the identity $2\cos^2 \theta = \cos 2\theta + 1$ and multiplication, (7.2.14) becomes

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$$4\lambda\gamma\imath^{1+\lambda s}\int_0^{\pi/2}(2+2\cos 2\theta-\lambda\cos 2(2m+b+1)\theta-\lambda\cos 2(2m+b-1)\theta$$

$$-2\lambda\cos 2(2m+b)\theta)\,\mathrm{d}\theta. \tag{7.2.15}$$

Then if $q=1$, (7.2.15) simplifies to

$$4\lambda\gamma\imath^{1+\lambda s}\int_0^{\pi/2}(2+2\cos 2\theta-\lambda\cos 4\theta-\lambda-2\lambda\cos 2\theta)\,\mathrm{d}\theta$$

$$=4\lambda\imath^{1-\lambda s}\left[(2-\lambda)\theta\right]_0^{\pi/2}=2\lambda(2-\lambda)\pi\imath^{1-\lambda s}=\begin{cases}2\pi\lambda\imath^{1-\lambda s}&\text{if }a=0\\6\pi\lambda\imath^{1-\lambda s}&\text{if }a=1,\end{cases} \tag{7.2.16}$$

both cases being equivalent to the $q=1$ case in (7.2.13).

Finally, if $q\geq 2$, (7.2.15) becomes $4\lambda\gamma\imath^{1+\lambda s}\left[2\theta\right]_0^{\pi/2}=2\lambda\pi\imath^{1-\lambda s}$. $\qquad\square$

THEOREM 7.2.4 (intra sequence recurrence). *For fixed parameters s, a and b and $m\geq 1+ab-a-b$, with initial values given by*

$$\mathcal{R}_{s;00}(u,1)=u-4\gamma,\text{ and }\mathcal{R}_{s;00}(u,2)=u^2-6\gamma u+8,$$
$$\mathcal{R}_{s;01}(u,0)=u-4\gamma,\text{ and }\mathcal{R}_{s;01}(u,1)=u^2-5\gamma u+4,$$
$$\mathcal{R}_{s;10}(u,0)=2,\text{ and }\mathcal{R}_{s;10}(u,1)=u-2\gamma,$$
$$\mathcal{R}_{s;11}(u,0)=1,\text{ and }\mathcal{R}_{s;11}(u,1)=u-3\gamma,$$

we have

$$\mathcal{R}_{s;ab}(u,m+2)=(u-2\gamma)\mathcal{R}_{s;ab}(u,m+1)-\mathcal{R}_{s;ab}(u,m), \tag{7.2.17}$$

where $\gamma=(-1)^s$.

Proof. From Section 7.1 each of the polynomials $\mathcal{R}_{s;ab}(u,m)$ form an orthogonal monic polynomial sequence with respect to the weight factor $w_{s;ab}(u)$.

For $m\geq 1$ we have from Theorem 7.2.1 and Corollary 7.2.1, with $i=1$ and putting $j=m$,

$$\mathcal{R}_{s;ab}(u,m+2)=(u+g_{m+1})\mathcal{R}_{s;ab}(u,m+1)-\mathcal{R}_{s;ab}(u,m), \tag{7.2.18}$$

where

$$g_{m+1}=\frac{-1}{k_{m+1}}\int_0^{4\gamma}u\left(\mathcal{R}_{s;ab}(u,m+1)\right)^2 w_{s;ab}(u)\,\mathrm{d}u.$$

Here $k_{m+1}=k_1$ and as defined in (7.2.4) is from Theorem 7.1.7 found to be

$$k_{m+1}=2\lambda\pi\imath^{1+\lambda s},$$

and from Lemma 7.2.3 we have

$$\int_0^{4\gamma}u\left(\mathcal{R}_{s;ab}(u,m)\right)^2 w_{s;ab}(u)\mathrm{d}u=4\lambda\pi\imath^{1-\lambda s},$$

and so we find that

$$g_{m+1} = -\frac{4\lambda\pi\imath^{1-\lambda s}}{2\lambda\pi\imath^{1+\lambda s}} = -2\imath^{-2\lambda s} = -2((-1)^s)^{-\lambda} = -2\gamma,$$

and so (7.2.18) is obtained.

When $m = 0$ and $q = 0$, $\mathcal{R}_{s;00}(u, 0) = 0$, but $\mathcal{R}_{s;10}(u, 0) = 2$, (which is not monic) and we need to consider the implications to the corollary of Theorem 7.2.1. Inspection of the corollary with $j = 0$ reveals that $\alpha_1 = A_2/A_1$ is unaffected; then from Theorem 7.1.7 and Lemma 7.2.3,

$$\beta_1 = \frac{-1}{k_1} \int_0^{4\gamma} u \left(\mathcal{R}_{s;ab}(u, 1)\right)^2 w_{s;10}(u)\, \mathrm{d}u = -\frac{(-8)\pi\imath^{1-\lambda s}}{(-4)\pi\imath^{1+\lambda s}} = -2\gamma,$$

and the evaluation for γ_0 is

$$\gamma_0 = -\alpha_1 \frac{A_0}{A_1}\frac{k_1}{k_0} = -\frac{2}{1}\frac{(-4)\pi\imath^{1+\lambda s}}{(-8)\pi\imath^{1+\lambda s}} = -1,$$

showing that (7.2.18) remains valid.

When $m = 0$ and $q = 1$, $\mathcal{R}_{s;01}(u, 0) = u - 4\gamma$, but is equivalent to 1 after recalibration due to the amended weight factor. We also find (from Theorem 7.1.7 and Lemma 7.2.3) that $\beta_1 = -2\gamma$, (and since $k_0 = k_1$), $\gamma_0 = -1$, **where we note the different notations of** γ **and** γ_m.

Finally for $\mathcal{R}_{s;11}(u, 0) = 1$, applying the same theorem and lemma we also have $\beta_1 = -2\gamma$, and $\gamma_0 = -1$, and so we find that (7.2.18) holds for all $q \geq 1 - a$ as asserted. $\qquad\square$

7.2.2 Inter sequence recurrences

Consideration of the production of the polynomial $\mathcal{R}_{s;ab}(u, m+1)$ from those of $\mathcal{R}_{s;ab'}(u, m+b)$ and $\mathcal{R}_{s;ab}(u, m)$, where $b' = 1 - b$.

THEOREM 7.2.5 (inter sequence recurrence). *With $q = 2m + b \geq 1 - a$, such that the initial conditions are, if*

$$a = \begin{cases} 0 & then\ \mathcal{R}_{s;00}(u, 1) = u - 4\gamma,\ and\ \mathcal{R}_{s;01}(u, 0) = u - 4\gamma \\ 1 & then\ \mathcal{R}_{s;10}(u, 0) = 2, \qquad and\ \mathcal{R}_{s;11}(u, 0) = 1, \end{cases}$$

and $\gamma = (-1)^s$, then we have

$$\mathcal{R}_{s;ab}(u, m + 1) = u^{|b-a|}\mathcal{R}_{s;ab'}(u, m + b) - \gamma\mathcal{R}_{s;ab}(u, m), \tag{7.2.19}$$

where $|d|$ is the absolute value of d.

Proof. We need to consider the cases of the parameters a, b and s. Starting with case when $a = 1$ we have from (5.3.2)

$$T_{q+2}(x) = 2xT_{q+1}(x) - T_q(x),$$

and so with $x = \sqrt{u}/2$ we have,

$$T_{q+2}(\sqrt{u}/2) = \sqrt{u}T_{q+1}(\sqrt{u}/2) - T_q(\sqrt{u}/2). \tag{7.2.20}$$

From Theorem 5.6.1,

$$\mathcal{R}_{s;1b}(u, m) = \frac{\gamma^m 2 T_q(\sqrt{\gamma u}/2)}{(\sqrt{\gamma u})^b},$$

and when $q = 2m$ (7.2.20) becomes

$$\gamma^{-m-1}2^{-1}\mathcal{R}_{s;10}(u, m+1) = \gamma^{-m}2^{-1}\sqrt{\gamma u}\left(\sqrt{\gamma u}\mathcal{R}_{s;11}(u, m)\right) - \gamma^{-m}2^{-1}\mathcal{R}_{s;10}(u, m)$$

$$\gamma^{-1}\mathcal{R}_{s;10}(u, m+1) = \gamma u \mathcal{R}_{s;11}(u, m) - \mathcal{R}_{s;10}(u, m)$$

$$\mathcal{R}_{s;10}(u, m+1) = u\mathcal{R}_{s;11}(u, m) - \gamma\mathcal{R}_{s;10}(u, m),$$

where we have first divided through by $\gamma^{-m}2^{-1}$ and then multiplied by γ.

Conversely when $q = 2m + 1$, (7.2.20) becomes

$$\gamma^{-m-1}2^{-1}\sqrt{\gamma u}\mathcal{R}_{s;11}(u, m+1) = \gamma^{-m-1}2^{-1}\sqrt{\gamma u}\mathcal{R}_{s;10}(u, m+1) - \gamma^{-m}2^{-1}\sqrt{\gamma u}\mathcal{R}_{s;11}(u, m),$$

which on division by $\gamma^{-m-1}2^{-1}\sqrt{\gamma u}$ (taking $u \neq 0$ as fixed) gives

$$\mathcal{R}_{s;11}(u, m+1) = \mathcal{R}_{s;10}(u, m+1) - \gamma\mathcal{R}_{s;11}(u, m). \tag{7.2.21}$$

When $a = 0$ we have the additional factor of $(u - 4\gamma)$ to consider.

Once more starting with (5.3.2),

$$U_{q+1}(x) = 2xU_q(x) - U_{q-1}(x),$$

we put $x = \sqrt{\gamma u}/2$, before multiplying through by $u - 4\gamma$ to obtain

$$(u - 4\gamma)U_{q+1}(\sqrt{\gamma u}/2) = \sqrt{\gamma u}(u - 4\gamma)U_q(\sqrt{\gamma u}/2) - (u - 4\gamma)U_{q-1}(\sqrt{\gamma u}/2), \tag{7.2.22}$$

where the variable u is not to be confused with the polynomial $U_n(x)$, the Chebyshev polynomial of the second kind. Now from Theorem 5.6.1,

$$\mathcal{R}_{s;0b}(u, m) = \frac{\gamma^{m-1+b}(u - 4\gamma)U_{q-1}(\sqrt{\gamma u}/2)}{(\sqrt{\gamma u})^{1-b}},$$

and with $q = 2m + 1$, equation (7.2.22) is equivalent to

$$\gamma^{-m-1}\mathcal{R}_{s;01}(u, m+1) = \gamma^{-m}\sqrt{\gamma u}\left(\sqrt{\gamma u}\mathcal{R}_{s;00}(u, m+1)\right) - \gamma^{-m}\mathcal{R}_{s;01}(u, m)$$

$$\gamma^{-m}\mathcal{R}_{s;01}(u, m+1) = \gamma^{1-m}u\mathcal{R}_{s;00}(u, m+1) - \gamma^{-m}\mathcal{R}_{s;01}(u, m)$$

$$\mathcal{R}_{s;01}(u, m+1) = u\mathcal{R}_{s;00}(u, m+1) - \gamma\mathcal{R}_{s;01}(u, m).$$

For $q = 2m$ the case mirrors (7.2.21) and so division by $\gamma^{-m}\sqrt{\gamma u}$ (with $u \neq 0$) gives

$$\gamma^{-m}\sqrt{\gamma u}\mathcal{R}_{s;00}(u, m+1) = \gamma^{-m}\sqrt{\gamma u}\mathcal{R}_{s;01}(u, m) - \gamma^{1-m}\sqrt{\gamma u}\mathcal{R}_{s;00}(u, m)$$

$$\mathcal{R}_{s;00}(u, m+1) = \mathcal{R}_{s;01}(u, m) - \gamma\mathcal{R}_{s;00}(u, m).$$

and so we obtain (7.2.19) as required. $\qquad\square$

Chapter 8

Generating functions

We commence, in Section 8.1, with an overview of the generalised hypergeometric function, that play an important role in the establishment of key results in this chapter. In Section 8.2, we establish, in Lemma 8.2.1, a generalised generating function developed from the recurrence polynomial. Then in Section 8.3, we utilise this lemma and hypergeometric functions, to determine in Theorem 8.3.7 the generating function of the alternating sequences $\mathcal{L}_{s;1bc}(r,t,q)$. The non-alternating case $a = 0$, first requires some other prelimanary work, that we examine in Section 8.4, prior to Section 8.5, in which we culminate with Theorem 8.5.6, that determines the generating function of the sequences $\mathcal{L}_{s;0bc}(r,t,q)$.

8.1 The Generalised Hypergeometric Function (GHF)

For positive integers α and k, we denote the rising and falling factorials by

$$\alpha^{\overline{k}} = \alpha(\alpha+1)\ldots(\alpha+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}, \quad \text{and} \quad \alpha^{\underline{k}} = \alpha(\alpha-1)\ldots(\alpha-k+1) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-k)}.$$
$$(8.1.1)$$

Using the notation of (8.1.1) for the rising factorial, and citing [3] and [45], we have the following definition.

Definition 8.1.1 (generalised hypergeometric function). *A generalised hypergeometric series of the form*

$$\sum_{k\geq 0} T_k z^k = T_0 + T_1 z + T_2 z^2 + \ldots$$

is a power series in which the ratio of successive coefficients

$$\frac{T_{k+1}}{T_k} = \frac{(k+\alpha_1)\ldots(k+\alpha_m)}{(k+\beta_1)\ldots(k+\beta_n)(k+1)}, \tag{8.1.2}$$

indexed by k, is a rational function of k. The series, if convergent, defines a generalised hypergeometric function, denoted by

$$_mF_n\left(\begin{array}{c} \alpha_1,\alpha_2,\ldots\alpha_m \\ \beta_1,\beta_2,\ldots\beta_n \end{array} ; z\right) = \sum_{k=0}^{\infty} \frac{\alpha_1^{\overline{k}}\alpha_2^{\overline{k}}\ldots\alpha_m^{\overline{k}}x^k}{\beta_1^{\overline{k}}\beta_2^{\overline{k}}\ldots\beta_n^{\overline{k}}k!}. \tag{8.1.3}$$

where the parameters $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ are obtained directly from (8.1.2). When the series is not finite, its radius of convergence is given by

$$\rho = \begin{cases} \infty & \textit{if } m < n+1 \\ 1 & \textit{if } m = n+1 \\ 0 & \textit{if } m > n+1. \end{cases}$$

8.1.1 Overview to the application of generalised hypergeometric functions

The application of the generalised hypergeometric function (GHF) in this thesis involves the reduction of a sum of the product of two binomial coefficients to a single binomial coefficient. The general approach to achieve this is as follows:

1. Obtain the ratio T_{k+1}/T_k.

2. Identify the GHF by reading off the parameters from the ratio.

3. Multiply by the term T_0.

4. Associate the GHF to an established result.

5. Convert this result to a binomial coefficient (as the given result is usually expressed in terms of rising factorials.)

6. Check whether the GHF has the same number of terms as the given sum (and add / subtract terms if necessary).

8.1.2 Overview of applied properties and stated results

- If one (or more) of the parameters in the numerator is negative then the series is finite. Eg. the rising factorial of $(-n)^{\overline{n+1}} = 0$, so the series will vanish after n terms.

- Particularly for finite series we can replace z by a value, (such as 1 or -1), and the polynomial becomes a sum. To prevent a zero in the denominator of one of the terms, this statement may carry the caveat that the parameters satisfy some given criteria.

- There are many such GHFs whose sum has an established closed form. For example, if (8.1.3) satisfies (i) $m = n+1$ and (ii) $\alpha_1 + \ldots + \alpha_m + 1 = \beta_1 + \ldots + \beta_n$, and $z = 1$, the GHF is described as Saalchützian (or balanced).
  Two such results employed in this thesis relating to Saalchützian GHFs given in [37] are:

$$_3F_2 \left(\begin{array}{c} \alpha_1, \alpha_2, -m \\ \beta_1, \alpha_1 + \alpha_2 - m - \beta_1 + 1 \end{array} ; 1 \right) = \frac{(\beta_1 - \alpha_1)^{\overline{m}}(\beta_1 - \alpha_2)^{\overline{m}}}{\beta_1^{\overline{m}}(\beta_1 - \alpha_1 - \alpha_2)^{\overline{m}}}.$$

and (using the symbolism of Slater [37])

$$\frac{\Gamma(g)\Gamma(g-f-d)}{\Gamma(g-f)\Gamma(g-d)} \times {}_4F_3\left(\begin{array}{c} d, 1+f-g, f/2, f/2+1/2 \\ a, f/2+d/2-g/2, 1+f/2+d/2-g/2 \end{array}; 1\right)$$
$$= {}_3F_2\left(\begin{array}{c} f, 1+f-a, d \\ a, g \end{array}; -1\right).$$

In Appendix $D.1$ we provide a simple example to illustrate how hypergeometric functions are applied in this chapter.

8.2 Development of the generating function from the recurrence relation polynomial

Definition 8.2.1. *We have $\mathcal{L}_s(0), \mathcal{L}_s(1), \mathcal{L}_s(2), \ldots$ are two sequences of integers, where the parameter $s \in \{0, 1\}$, and are such that*

$$\mathcal{L}_1(r) = (-1)^{rs}\mathcal{L}_0(r) = \gamma^r \mathcal{L}_0(r),$$

where $\gamma = (-1)^s$.

Suppose a sequence $\mathcal{L}_s(0), \mathcal{L}_s(1), \mathcal{L}_s(2), \ldots$ satisfies a linear recurrence of order m, then recalling Lemma 5.5.1 we have

$$\mathcal{L}_s(r+m) + (-\gamma)a_1\mathcal{L}_s(r+m-1) + \ldots + (-\gamma)^m a_m \mathcal{L}_s(r) = 0. \tag{8.2.1}$$

Definition 8.2.2. *We denote the generating function of the sequence of terms $\mathcal{L}_s(r)$ by*

$$\mathcal{GL}_s(x) = \sum_{r=0}^{\infty} \mathcal{L}_s(r)x^r. \tag{8.2.2}$$

A method of determining the generating function equation from a three term linear recurrence relation is given by Koshy [29]. This is easily extended to an $m+1$ term relation. We express this method as a lemma.

LEMMA 8.2.1. *If the sequence of terms $\mathcal{L}_s(0), \mathcal{L}_s(1), \mathcal{L}_s(2), \ldots$ satisfy the $m+1$ term recurrence (8.2.1), then the generating function will have the form*

$$\mathcal{GL}_s(x) = \frac{\sum_{k=0}^{m-1}\sum_{j=0}^{k}(-\gamma)^j a_j \mathcal{L}_s(k-j)x^k}{\sum_{k=0}^{m}(-\gamma)^k a_k x^k}. \tag{8.2.3}$$

Proof. From (8.2.2) we have

$$\mathcal{GL}_s(x) = \sum_{r=0}^{\infty} \mathcal{L}_s(r)x^r = \mathcal{L}_s(0) + \mathcal{L}_s(1)x + \mathcal{L}_s(2)x^2 \ldots + \mathcal{L}_s(n)x^n + \ldots. \tag{8.2.4}$$

Multiplication of both sides of (8.2.4) by $(-\gamma)^k a_k x^k$ for $1 \leq k \leq m$, then produces the system of m additional equations

$$-\gamma a_1 x \mathcal{GL}_s(x) = -\gamma a_1 \mathcal{L}_s(0)x - \gamma a_1 \mathcal{L}_s(1)x^2 - \gamma a_1 \mathcal{L}_s(2)x^3 + \dots$$

$$a_2 x^2 \mathcal{GL}_s(x) = a_2 \mathcal{L}_s(0)x^2 + a_2 \mathcal{L}_s(1)x^3 + a_2 \mathcal{L}_s(2)x^4 + \dots$$

$$\vdots$$

$$(-\gamma)^{m-1} a_{m-1} x^{m-1} \mathcal{GL}_s(x) = (-\gamma)^{m-1} a_{m-1} \mathcal{L}_s(0)x^{m-1} + (-\gamma)^{m-1} a_{m-1} \mathcal{L}_s(1)x^m$$
$$+ (-\gamma)^{m-1} a_{m-1} \mathcal{L}_s(2)x^{m+1} + \dots$$

$$(-\gamma)^m a_m x^m \mathcal{GL}_s(x) = (-\gamma)^m a_m \mathcal{L}_s(0)x^m + (-\gamma)^m a_m \mathcal{L}_s(1)x^{m+1} + (-\gamma)^m a_m \mathcal{L}_s(2)x^{m+2} + \dots.$$

$$(8.2.5)$$

Summing the left and right hand sides of the $m + 1$ equations, collecting the terms for successive powers of x and then using the fact that $\sum_{k=0}^m (-\gamma)^{m-k} a_{m-k} \mathcal{L}_s(n+k) = 0$, we obtain

$$\sum_{k=0}^m (-\gamma)^k a_k x^k \mathcal{GL}_s(x)$$

$$= a_0 \mathcal{L}_s(0) + (a_0 \mathcal{L}_s(1) + (-\gamma)a_1 \mathcal{L}_s(0))\, x + (a_0 \mathcal{L}_s(2) + (-\gamma)a_1 \mathcal{L}_s(1) + a_2 \mathcal{L}_s(0))\, x^2$$

$$+ \dots + \left(\sum_{j=0}^{m-1} (-\gamma)^j a_j \mathcal{L}_s(m-1-j) \right) x^{m-1} + \left(\sum_{j=0}^m (-\gamma)^j a_j \mathcal{L}_s(m-j) \right) x^m$$

$$+ \dots + \left(\sum_{j=0}^m (-\gamma)^j a_j \mathcal{L}_s(n-j) \right) x^n + \dots$$

$$= \mathcal{L}_s(0) + (a_0 \mathcal{L}_s(1) - \gamma a_1 \mathcal{L}_s(0))\, x + (a_0 \mathcal{L}_s(2) - \gamma a_1 \mathcal{L}_s(1) + a_2 \mathcal{L}_s(0))\, x^2$$

$$+ \dots + \left(\sum_{j=0}^{m-1} (-\gamma)^j a_j \mathcal{L}_s(m-1-j) \right) x^{m-1} + 0 + \dots + 0 + \dots$$

$$= \sum_{k=0}^{m-1} \left(\sum_{j=0}^k (-\gamma)^j a_j \mathcal{L}_s(k-j) \right) x^k. \tag{8.2.6}$$

Division of both sides by $\sum_{k=0}^m (-\gamma)^k a_k x^k$ gives the result. $\qquad \square$

Remark. The generating function has "inverted" the coefficients of x^k in the sense that the coefficient a_k has been replaced by a_{m-k}, so that the recurrence polynomial in both the numerator and denominator of the generating functions are reciprocal polynomials.

The summation in (8.2.6) is not unique as we demonstrate in the Corollary.

COROLLARY. *For the sequence of terms $\mathcal{L}_s(0), \mathcal{L}_s(1), \mathcal{L}_s(2), \dots$, we have*

$$\mathcal{GL}_s(x) = \frac{\sum_{k=0}^{m-1} \sum_{j=0}^{m-k-1} (-\gamma)^j a_j \mathcal{L}_s(k)x^{j+k}}{\sum_{k=0}^m (-\gamma)^k a_k x^k}. \tag{8.2.7}$$

Proof. If, in Lemma 8.2.1, the remaining terms in (8.2.6) are collected for successive values of $\mathcal{L}_s(k)x^k$, then the result follows. $\qquad\square$

Remark. The terms a_j and $\mathcal{L}_s(k)$ in (8.2.7) are no longer inter-dependent in the summation, and this fact will be exploited in the generating function of the sequences $\mathcal{L}_{s;0bc}(r,t,q)$. A very similar equation to (8.2.7) is stated by Jordan ($p.27$, [28]).

We now wish to apply the result of Lemma 8.2.1 to determine the generating function of the sequences $\mathcal{L}_{s;abc}(r,t,q)$. The alternating case $a = 1$ is an easier proposition than the $a = 0$ case, so we shall consider this first.

8.3   The generating function of the sequences $\mathcal{L}_{s;1bc}(r,t,q)$

From the work in Section 8.2 we have Lemma 8.3.1.

LEMMA 8.3.1. *The generating function for the sequences $\mathcal{L}_{s;1bc}(r,t,q)$ has the form*

$$\mathcal{GL}_{s;1bc}(x,t,q) = \frac{\gamma^{t+1-c}\sum_{k=T}^{m-1}\sum_{j=0}^{k-T}(-1)^j\frac{2m+b}{2m+b-j}\binom{2m+b-j}{j}\binom{2k+2-c-2j}{k+t+1-c-j}(\gamma x)^k}{\sum_{k=0}^{m}(-\gamma)^k\frac{2m+b}{2m+b-k}\binom{2m+b-k}{k}x^k}, \tag{8.3.1}$$

where $0 \le t \le m$, and

$$T = \begin{cases} t & \text{if } t = 0 \\ t-1 & \text{if } t \ge 1. \end{cases} \tag{8.3.2}$$

When $b = c = 0$ and $t = m$, we have

$$\mathcal{GL}_{s;100}(x,m,2m) = 0. \tag{8.3.3}$$

Proof. From Lemma 8.2.1 we have

$$\mathcal{GL}_{s;1bc}(x,t,q) = \frac{\sum_{k=0}^{m-1}\sum_{j=0}^{k}(-\gamma)^j a_j \mathcal{L}_{s;1bc}(k-j,t,q)x^k}{\sum_{k=0}^{m}(-\gamma)^k a_k x^k}, \tag{8.3.4}$$

where we recall that the terms a_k are those of the corresponding recurrence polynomials, $\mathcal{R}_{s;1b}(x,m)$ (of order m), such that a_k is the coefficient of the term x^{m-k}. From Theorem 5.6.1 these are given by

$$\mathcal{R}_{s;1b}(x,m) = \sum_{k=0}^{m}(-\gamma)^{m-k}\frac{2m+b}{2m+b-k}\binom{2m+b-k}{k}x^{m-k}, \tag{8.3.5}$$

and we have

$$a_k = \frac{2m+b}{2m+b-k}\binom{2m+b-k}{k}. \tag{8.3.6}$$

The terms $\mathcal{L}_{s;1bc}(r,t,q)$ for $0 \le r \le m-1$, in the numerator of (8.3.4) are, with one exception, determined by the single binomial coefficients

$$\mathcal{L}_{s;1bc}(r,t,q) = \gamma^{r+t+1-c}\binom{2r+2-c}{r+t+1-c}. \tag{8.3.7}$$

This single exception occurs when the parameters $b = c = 0$ and the variables $r = m - 1$ and $t = m$. In this case (8.3.7) becomes

$$\mathcal{L}_{s;100}(m - 1, m, 2m) = \gamma^{2m} \left(\binom{2m}{0} - \binom{2m}{2m} \right) = 0. \tag{8.3.8}$$

Moreover, we also observe that when $t \geq 1$ and $r = t - 1$, the binomial coefficient in (8.3.7) is given by

$$\binom{2t - c}{2t - c},$$

and so if $t \geq 2$ and $0 \leq r \leq t - 2$, then (if we define the binomial coefficient $^nC_r = 0$ when $r > n$), $\mathcal{L}_{s;abc}(r, t, q) = 0$. Consequently this reduces the number of non-zero terms in the numerator of (8.3.4), and the upper limit of inner sum is reduced by T, as is the lower limit of outer sum raised to T, where T is given in (8.3.2).

Accordingly, substitution of each of the terms (8.3.6), (8.3.7) and (the discussed) placement of the variable T into (8.3.4) gives (8.3.1). Furthermore, in conjuction with (8.3.8), we have for $0 \leq r \leq m - 1$, that $\mathcal{L}_{s;100}(r, m, 2m) = 0$, and so we obtain (8.3.3). $\qquad \square$

Although we have established a form for the generating function of the sequences $\mathcal{L}_{s;1bc}(r, t, q)$, the numerator of each is a double sum and consequently rather unwieldy. We turn to generalised hypergeometric functions (introduced in Section 8.1) to reduce the inner (binomial) sum to a single term.

Let us first consider the particular case $t = 0$ (and $c = 0$) and denote the (inner sum) of the numerator of (8.3.1) as

$$\sum_{j=0}^{k} T_j = \sum_{j=0}^{k} (-1)^j \frac{2m + b}{2m + b - j} \binom{2m + b - j}{j} \binom{2k + 2 - 2j}{k + 1 - j}. \tag{8.3.9}$$

We require the following lemmas.

LEMMA 8.3.2. *For non-negative integers m, k and b with $0 \leq k \leq m - 1$ and $b \in \{0, 1\}$ we have*

$$\sum_{j=0}^{m} (-1)^j \frac{2m + b}{2m + b - j} \binom{2m + b - j}{j} \binom{2k + 2 - 2j}{k + 1 - j}$$

$$= \binom{2k + 2}{k + 1} {}_3F_2 \left(\begin{array}{c} -k - 1, 1/2 - b - m, -m \\ -k - 1/2, 1 - b - 2m \end{array} ; 1 \right).$$

Proof. Denote the left hand sum as

$$\sum_{j=0}^{m} T_j = \sum_{j=0}^{m} (-1)^j \frac{2m + b}{2m + b - j} \binom{2m + b - j}{j} \binom{2k + 2 - 2j}{k + 1 - j}, \tag{8.3.10}$$

then expressing it in terms of a hypergeometric function with

$$T_{j+1} = (-1)^{j+1} \frac{2m+b}{2m+b-j-1} \binom{2m+b-j-1}{j+1} \binom{2k-2j}{k-j}$$

$$= \frac{(-1)^{j+1}(2m+b)(2m+b-j-1)!(2k-2j)!}{(2m+b-j-1)(j+1)!(2m+b-2-2j)!(k-j)!(k-j)!},$$

and

$$T_j = (-1)^j \frac{2m+b}{2m+b-j} \binom{2m+b-j}{j} \binom{2k+2-2j}{k+1-j}$$

$$= \frac{(-1)^j(2m+b)(2m+b-j)!(2k+2-2j)!}{(2m+b-j)j!(2m+b-2j)!(k+1-j)!(k+1-j)!},$$

the ratio T_{j+1}/T_j is

$$= \frac{(-1)(2m+b)(2m+b-j-1)!(2k-2j)!(2m+b-j)j!}{(2m+b-j-1)(j+1)!(2m+b-2j-2)!(k-j)!(k-j)!}$$
$$\times \frac{(2m+b-2j)!(k+1-j)!(k+1-j)!}{(2m+b)(2m+b-j)!(2k+2-2j)!}$$
$$= \frac{(-1)(2m+b-2j)(2m+b-2j-1)(k+1-j)(k+1-j)}{(2m+b-j-1)(j+1)(2k+2-2j)(2k+1-2j)}$$
$$= \frac{4(-1)^5(j-m-b/2)(j-m+(1-b)/2)(j-k-1)(j-k-1)}{4(-1)^3(j+1-b-2m)(j-k-1)(j-k-1/2)(j+1)}$$
$$= \frac{(j-k-1)(j-m-b/2)(j-m+(1-b)/2)}{(j-k-1/2)(j+1-b-2m)(j+1)}. \tag{8.3.11}$$

We recall that b either takes the value 0 or 1, and so (8.3.11) can be equivalently written as

$$\frac{(j-k-1)(j+1/2-b-m)(j-m)}{(j-k-1/2)(j+1-b-2m)(j+1)}.$$

Also we have that

$$T_0 = \binom{2k+2}{k+1},$$

so that we can write

$$\sum_{j=0}^m T_j = \binom{2k+2}{k+1} {}_3F_2 \left(\begin{matrix} -k-1, 1/2-b-m, -m \\ -k-1/2, 1-b-2m \end{matrix} ; 1 \right)$$

as required. $\qquad\square$

LEMMA 8.3.3. *For non-negative integers m, k and b with $0 \le k \le m-1$ and $b \in \{0,1\}$ we have*

$$_3F_2 \left(\begin{matrix} -k-1, 1/2-b-m, -m \\ -k-1/2, 1-b-2m \end{matrix} ; 1 \right) = (-1)^{k+1} \binom{2m-k-2+b}{k+1}. \tag{8.3.12}$$

Proof. We observe that the sum of the numerator parameters is $-k - 1/2 - 2m$ and that this is one less than that of the denominator parameters. This fact enables us to employ Saalschütz's theorem given by

$$_3F_2 \left(\begin{array}{c} A, B, -m \\ C, A + B - m - C + 1 \end{array} ; 1 \right) = \frac{(C - A)^{\overline{m}}(C - B)^{\overline{m}}}{C^{\overline{m}}(C - A - B)^{\overline{m}}}. \tag{8.3.13}$$

We put $A = -k-1$, $B = 1/2-b-m$ and $C = -k-1/2$ (and so $A + B - m - C + 1 = 1 - b - 2m$), then using (8.3.13) and multiplying by T_0 we obtain

$$\binom{2k + 2}{k + 1} {}_3F_2 \left(\begin{array}{c} -k - 1, 1/2 - b - m, -m \\ -k - 1/2, 1 - b - 2m \end{array} ; 1 \right) = \binom{2k + 2}{k + 1} \frac{(1/2)^{\overline{m}}(m - k - 1 + b)^{\overline{m}}}{(-k - 1/2)^{\overline{m}}(m + b)^{\overline{m}}}. \tag{8.3.14}$$

Each of the rising factorials of (8.3.14) can be expressed in the following manner.

$$(1/2)^{\overline{m}} = (1/2)(3/2)\ldots((2m - 1)/2) = \frac{1 \times 2 \times \ldots (2m - 1)(2m)}{2^{2m}1 \times 2 \ldots m} = \frac{(2m)!}{2^{2m}m!},$$

$$(m - k - 1 + b)^{\overline{m}} = (m - k - 1 + b)(m - k + b)\ldots(m - k + m - 2 + b)$$
$$= (2m - k - 2 + b)\ldots(m - k - 1 + b)$$
$$= \frac{(2m - k - 2 + b)!}{(m - k - 2 + b)!},$$

$$(-k - 1/2)^{\overline{m}} = (-k - 1/2)(-k + 1/2)\ldots(-k + (2m - 3)/2)$$
$$= \frac{(-2k - 1)(-2k + 1)\ldots(-2k + 2m - 3)}{2^{2m}}$$
$$= \frac{(2m - 2k - 2)(2m - 2k - 3)\ldots 2.1.(-1).2.(-2k - 1)(2k + 2)}{2^{2m}(m - k - 1)\ldots 2.1.1.2.(k + 1)}$$
$$= \frac{(-1)^{k+1}(2m - 2k - 2)!(2k + 2)!}{2^{2m}(m - k - 1)!(k + 1)!},$$

and

$$(m + b)^{\overline{m}} = (m + b)(m + 1 + b)\ldots(m + m + b - 1) = (2m + b - 1)\ldots(m + b) = \frac{(2m + b - 1)!}{(m + b - 1)!}.$$

So we can write (8.3.14) as

$$\frac{(2k + 2)!}{(k + 1)!(k + 1)!} \times \frac{(-1)^{k+1}(2m)!(2m - k - 2 + b)!2^{2m}(m - k - 1)!(k + 1)!(m - 1 + b)!}{2^{2m}m!(m - k - 2 + b)!(2m - 2 - 2k)!(2k + 2)!(2m - 1 + b)!}, \tag{8.3.15}$$

and on recalling that b takes only the values 0 or 1 we can simplify (8.3.15) to

$$\frac{(-1)^{k+1}}{(k + 1)!} \frac{(2m)^{1-b}}{m^{1-b}} \frac{(m - k - 1)^{1-b}(2m - k - 2 + b)!}{(2m - 2 - 2k)!}$$
$$= \frac{(-1)^{k+1}(2m - k - 2 + b)!}{(k + 1)!(2m - 2k - 3 + b)!}$$
$$= (-1)^{k+1}\binom{2m - 2 + b - k}{k + 1}.$$

$\square$

Remark. We observe that the summation in Lemma 8.3.2 has $m + 1$ terms. However, in actual fact from either inspection of the product terms, or the hypergeometric numerator parameter $-(k + 1)$, only the first $k + 2$ terms are possibly non-zero.

We now demonstrate the following theorem.

THEOREM 8.3.4 (Lucas product closed form for $t = 0$ case). *For non-negative integers m, k and b with $0 \le k \le m - 1$ and $b \in \{0, 1\}$ we have*

$$\sum_{j=0}^{k} (-1)^j \frac{2m + b}{2m + b - j} \binom{2m + b - j}{j} \binom{2k + 2 - 2j}{k + 1 - j} = 2(-1)^k \binom{2m + b - 2 - k}{k}.$$

Proof. Let

$$T_j = (-1)^j \frac{2m + b}{2m + b - j} \binom{2m + b - j}{j} \binom{2k + 2 - 2j}{k + 1 - j}. \tag{8.3.16}$$

We write the left hand side of Lemma 8.3.2 as

$$\sum_{j=0}^{m} T_j = \sum_{j=0}^{k} T_j + \sum_{j=k+1}^{m} T_j, \tag{8.3.17}$$

and it is clear from (8.3.16) that $T_j = 0$ when $k + 2 \le j \le m$. Therefore, (8.3.17) simplifies to

$$\sum_{j=0}^{m} T_j = \sum_{j=0}^{k} T_j + T_{k+1}. \tag{8.3.18}$$

Now from Lemma 8.3.3, we have that

$$\sum_{j=0}^{m} T_j = (-1)^{k+1} \frac{(2m - 2 + b - k)!}{(k + 1)!(2m - 3 + b - 2k)!}$$

and the term T_{k+1} is given by

$$(-1)^{k+1} \frac{2m + b}{2m - 1 + b - k} \binom{2m - 1 + b - k}{k + 1} = (-1)^{k+1} \frac{(2m + b)(2m - 2 + b - k)!}{(k + 1)!(2m - 2 + b - 2k)!}.$$

So on rearranging (8.3.18) we have

$$\sum_{j=0}^{k} T_j = \sum_{j=0}^{M} T_j - T_{k+1}$$

$$= (-1)^{k+1} \frac{(2m - 2 + b - k)!}{(k + 1)!(2m - 3 + b - 2k)!} - (-1)^{k+1} \frac{(2m + b)(2m - 2 + b - k)!}{(k + 1)!(2m - 2 + b - 2k)!}$$

$$= (-1)^{k+1} \frac{(2m - 2 + b - k)!}{(k + 1)!(2m - 2 + b - 2k)!} ((2m - 2 + b - 2k) - (2m + b))$$

$$= 2(-1)^k \frac{(2m - 2 + b - k)!}{(k + 1)!(2m - 2 + b - 2k)!} (k + 1)$$

$$= 2(-1)^k \frac{(2m - 2 + b - k)!}{k!(2m - 2 + b - 2k)!} = 2(-1)^k \binom{2m - 2 + b - k}{k}.$$

□

We now turn to the other cases of (8.3.1) (i.e. when $t \neq 0$) and denote the (inner) sum of the numerator by

$$\sum_{j=0}^{k-t+1} T_j = \sum_{j=0}^{k-t+1} (-1)^j \frac{2m+b}{2m+b-j} \binom{2m+b-j}{j} \binom{2k+2-c-2j}{k+1-c+t-j},$$

(8.3.19)

and again we wish to reduce (8.3.19) to a single term. Before we embark on this, we use a lemma that will help clarify a piece of this task.

LEMMA 8.3.5 (negative gamma function). *For positve integers L, N and K such that $N - L \geq 1$, we have*

$$\frac{\Gamma(L+1-N)}{\Gamma(K+1)\Gamma(L+1-K-N)} = (-1)^K \binom{N+K-1-L}{K}.$$

Proof. Let $a = N - L$, then

$$\frac{\Gamma(L+1-N)}{\Gamma(L+1-K-N)} = \frac{(-a)(-(a+1))(-(a+2))\dots}{(-(a+K))(-(a+K+1))(-(a+K+2))\dots}$$

$$= (-a)(-(a+1))(-(a+2))\dots(-(a+K-1)) = (-1)^K (a+K-1)^{\underline{K}}$$

$$= (-1)^K K! \binom{a+K-1}{K} = (-1)^K K! \binom{N+K-1-L}{K}.$$

(8.3.20)

The result then follows on dividing (8.3.20) by $\Gamma(K+1)$. $\qquad\square$

We now extend the results of Theorem 8.3.4 to incorporate the cases when $1 \leq t \leq m$, and consequently we also have $b, c \in \{0, 1\}$.

THEOREM 8.3.6 (Lucas product closed form generalised case). *For non-negative integers m, k, b, c and t, with $0 \leq k \leq m-1$, $b, c \in \{0, 1\}$ and $1 \leq t \leq m$, we have*

$$\gamma^{t+1-c} \sum_{k=t-1}^{m-1} \sum_{j=0}^{k+1-t} (-1)^j \frac{2m+b}{2m+b-j} \binom{2m+b-j}{j} \binom{2k+2-c-2j}{k+1-c+t-j} (\gamma x)^k$$

$$= \sum_{k=0}^{m-t} (-\gamma)^k \binom{2(m-t)+b+c-1-k}{k} x^{k+t-1}.$$

Proof. We first consider the inner sum and establish that

$$\sum_{j=0}^{k+1-t} (-1)^j \frac{2m+b}{2m+b-j} \binom{2m+b-j}{j} \binom{2k+2-c-2j}{k+1-c+t-j}$$

$$(-1)^{k+1-t} \binom{2(m-t)+b+c-1-(k+1-t)}{k+1-t},$$

and then we determine that

$$\gamma^{t+1-c} \sum_{k=t-1}^{m-1} (-1)^{k+1-t} \binom{2(m-t)+b+c-1-(k+1-t)}{k+1-t} (\gamma x)^k$$

$$= \sum_{k=0}^{m-t} (-\gamma)^k \binom{2(m-t)+b+c-1-k}{k} x^{k+t-1}.$$

Let

$$T_j = (-1)^j \frac{2m+b}{2m+b-j} \binom{2m+b-j}{j} \binom{2k+2-c-2j}{k+t+1-c-j}$$

$$= \frac{(-1)^j!(2m+b)(2m+b-j)!(2k+2-c-2j)}{(2m+b-j)j!(2m+b+2-2j)!(k+t+1-c-j)!(k-t+1-j)!},$$

and

$$T_{j+1} = (-1)^{j+1} \frac{2m+b}{2m+b-j-1} \binom{2m+b-j-1}{j+1} \binom{2k-c-2j}{k+t-c-j}$$

$$= \frac{(-1)^{j+1}(2m+b)(2m+b-j-1)!(2k-c-2j)!}{(2m+b-j-1)j!(2m+b+2-2j)!(k+t-c-j)!(k-t-j)!}.$$

Then the ratio T_{j+1}/T_j is

$$= \frac{(-1)(2k-c-2j)!(2m+b)(2m+b-j-1)!(k+t+1-c-j)!}{(k+t-c-j)!(k-t-j)!(2m+b-j-1)(j+1)!(2m+b-2j-2)!}$$

$$\times \frac{(k-t+1-j)!(2m+b-j)j!(2m+b-2j)!}{(2k+2-c-2j)!(2m+b)(2m+b-j)!}$$

$$= \frac{(-1)(k+t+1-c-j)(k-t+1-j)(2m+b-2j)(2m+b-1-2j)}{(2k+2-c-2j)(2k+1-c-2j)(2m+b-1-j)(j+1)}$$

$$= \frac{4(-1)^5(j-k-t-1+c)(j-k+t-1)(j-m-b/2)(j-m+(1-b)/2}{4(-1)^3(j-k-1+c/2)(j-k-1/2+c/2)(j+1-b-2m)(j+1)}$$

$$= \frac{(j-k-t-1+c)(j-k+t-1)(j-m-b/2)(j-m+(1-b)/2)}{(j-k-1+c/2)(j-k-1/2+c/2)(j+1-b-2m)}.$$

Therefore, with $T_0 = \binom{2k+2-c}{k+1-c+t}$, we have

$$\sum_{j=0}^{k-t+1} T_j = \binom{2k+2-c}{k+1-c+t} \times {}_4F_3 \left(\begin{matrix} t-k-1, c-t-k-1, -b/2-m, 1/2-b/2-m \\ 1-b-2m, -k-1+c/2, -k+c/2-1/2 \end{matrix} ; 1 \right).$$

$$(8.3.21)$$

We note that in the hypergeometric function, (8.3.21), the sum of the denominator parameters is $c - 2m - b - 2k - 1/2$ and that this is one greater than that of the numerator parameters. Hence the function is Saalschützian.

As demonstrated in [37], (see $p.65$), Vandermonde's theorem can be employed to yield

$$\frac{\Gamma(g)\Gamma(g-f-d)}{\Gamma(g-f)\Gamma(g-d)} \times {}_4F_3 \left(\begin{matrix} d, 1+f-g, f/2, f/2+1/2 \\ a, f/2+d/2-g/2, 1+f/2+d/2-g/2 \end{matrix} ; 1 \right)$$

$$= {}_3F_2 \left(\begin{matrix} f, 1+f-a, d \\ a, g \end{matrix} ; -1 \right). \qquad (8.3.22)$$

We put $f = -b - 2m$, $d = t - k - 1$, $g = 2 + t + k - c - b - 2m$ and $a = 1 - b - 2m$ and we note that this gives $a = 1 + f$. Therefore, the right hand side of (8.3.22) simplifies to

$${}_3F_2 (f, 0, d; a, g; -1) = 1.$$

Then on rearrangement of (8.3.22) with the given substitutions of f, d, g and a we obtain

$$
{}_4F_3 \left(\begin{array}{c} t-k-1, c-t-k-1, -b/2-m, 1/2-b/2-m \\ 1-b-2m, -k-1+c/2, -k+c/2-1/2 \end{array} ; 1 \right)
$$
$$
= \frac{\Gamma(k+t+2-c)\Gamma(2k+3-c-b-2m)}{\Gamma(2k+3-c)\Gamma(k+t+2-c-b-2m)}.
$$

Therefore, we conclude that (8.3.21) simplifies to

$$
\sum_{j=0}^{k} T_j = \binom{2k+2-c}{k+1-c+t} \frac{\Gamma(k+t+2-c)\Gamma(2k+3-c-b-2m)}{\Gamma(2k+3-c)\Gamma(k+t+2-c-b-2m)}.
$$

We now use Lemma 8.3.5 with $L+1 = 2k+3-c-b-2m$, $K = k+1-t$ and $N = 2m$ and proceed as

$$
= (-1)^{k+1-t} \frac{(2k+2-c)!(k+t+1-c)!(2m-2-t+b+c-2k)^{\underline{k+1-t}}}{(k+1-c+t)!(k+1-t)!(2k+2-c)!}
$$
$$
= (-1)^{k+1-t} \binom{2(m-1)-t+b+c-k}{k+1-t}
$$
$$
= (-1)^{k+1-t} \binom{2(m-t)+b+c-1-(k+1-t)}{k+1-t},
$$

thus establishing the first part of the theorem. We now have

$$
\gamma^{t+1-c} \sum_{k=t-1}^{m-1} (-1)^{k+1-t} \binom{2(m-t)+b+c-1-(k+1-t)}{k+1-t} (\gamma x)^k,
$$

and this can be written

$$
\gamma^c \sum_{k=t-1}^{m-1} (-\gamma)^{k+1-t} \binom{2(m-t)+b+c-1-(k+1-t)}{k+1-t} x^k.
$$

The result follows on the rescaling of $k+1-t$ with k. $\qquad\square$

We now in a position to determine the generalised generating function for the function $\mathcal{L}_{s;1bc}$, that is the case when the alternating parameter $a = 1$.

THEOREM 8.3.7 (generating function of $\mathcal{L}_{s;1bc}$). *For $\mathcal{L}_{s;abc}(r,t,q)$ as given in 3.1.2 with $c \le t \le m$, we have when $a = 1$,*

$$
\mathcal{GL}_{s;1bc}(x,t,q) = \begin{cases} \dfrac{2\gamma J_{2(m-1)+1+b}(-\gamma x)}{j_{2m+b}(-\gamma x)} & \text{if } c = t = 0 \\[2ex] \dfrac{x^{t-1} J_{2(m-1-t)+2}(-\gamma x)}{j_{2m}(-\gamma x)} & \text{if } b = c = 0 \text{ and } t \ge 1 \\[2ex] \dfrac{\gamma^c x^{t-1} J_{2(m-t)+b+c}(-\gamma x)}{j_{2m+b}(-\gamma x)} & \text{otherwise.} \end{cases}
$$

Here, $J_N(x)$ is the Jacobsthal polynomial as defined in (5.2.2), $j_n(x)$ the Jacobsthal-Lucas polynomial as defined in (5.2.6) and $\gamma = (-1)^s$.

Proof. From Lemma 8.3.1 we have

$$\mathcal{GL}_{s;1bc}(x,t,q) = \frac{\gamma^{t+1-c}\sum_{k=T}^{m-1}\left(\sum_{j=0}^{k-T}(-1)^j \frac{2m+b}{2m+b-j}\binom{2m+b-j}{j}\binom{2k+2-c-2j}{k+t+1-c-j}\right)(\gamma x)^k}{\sum_{k=0}^{m}(-\gamma)^k \frac{2m+b}{2m+b-k}\binom{2m+b-k}{k}x^k},$$

where

$$T = \begin{cases} t & \text{if } t = 0 \\ t-1 & \text{if } t \geq 1. \end{cases}$$

When $t = 0$ (and $c = 0$) we have on substitution of the result of Theorem 8.3.4 into (8.3.1) that

$$\mathcal{GL}_{s;1b0}(x,0,q) = \frac{2\gamma\sum_{k=0}^{m-1}(-\gamma)^k\binom{2m-2+b-k}{k}x^k}{\sum_{k=0}^{m}(-\gamma)^k \frac{2m+b}{2m+b-k}\binom{2m+b-k}{k}x^k}, \tag{8.3.23}$$

and when $t \neq 0$ we have from Theorem 8.3.6,

$$\mathcal{GL}_{s;1bc}(x,t,q) = \frac{\gamma^{t+1-c}\sum_{k=t-1}^{m-1}(-1)^{k+1-t}\binom{2(m-t)+b+c-1-(k-t+1)}{k+1-t}(\gamma x)^k}{\sum_{k=0}^{m}(-\gamma)^k \frac{2m+b}{2m+b-k}\binom{2m+b-k}{k}x^k}$$

$$= \frac{\gamma^c\sum_{k=0}^{m-t-b'c'}(-\gamma)^k\binom{2(m-t)+b+c-1-k}{k}x^{k+t-1}}{\sum_{k=0}^{m}(-\gamma)^k \frac{2m+b}{2m+b-k}\binom{2m+b-k}{k}x^k}, \tag{8.3.24}$$

where $b' = 1-b$ and $c' = 1-c$.

So when $b = c = 0$, the numerator of (8.3.24) is

$$\sum_{k=0}^{m-1-t}(-\gamma)^k\binom{2(m-1-t)+1-k}{k}x^{k+t-1},$$

and for each of the other cases of b and c we have

$$\gamma^c\sum_{k=0}^{m-t}(-\gamma)^k\binom{2(m-t)+b+c-1-k}{k}x^{k+t-1}.$$

Now on applying (5.2.3) and (5.2.7) to express (8.3.23) and (8.3.24) in terms of Jacobsthal and Jacobsthal-Lucas polynomials the theorem follows. □

Remark. When $b = c = 0$ the summation of (8.3.24) only runs up to $k = m - 1 - t$ (as the term $k = m - t$ yields a negative binomial coefficient).

In the Corollary to Theorem 8.3.7 we now identify the generating function for each sequence when the parameter $a = 1$.

COROLLARY. *We have for $t = 0$,*

$$\mathcal{GL}_{0;11c}(x,0,1) = \frac{2^{1-c}J_0(-x)}{j_1(-x)} = 0,$$

$$\mathcal{GL}_{1;11c}(x,0,1) = \frac{(-2)^{1-c}J_0(x)}{j_1(x)} = 0,$$

$$\mathcal{GL}_{0;100}(x,0,2m) = \frac{2J_{2(m-1)+1}(-x)}{j_{2m}(-x)} = \frac{2\sum_{k=0}^{m-1}(-1)^k\binom{2(m-1)-k}{k}x^k}{\sum_{k=0}^{m}(-1)^k\frac{2m}{2m-k}\binom{2m-k}{k}x^k},$$

$$\mathcal{GL}_{0;110}(x,0,2m+1) = \frac{2J_{2(m-1)+2}(-x)}{j_{2m+1}(-x)} = \frac{2\sum_{k=0}^{m-1}(-1)^k\binom{2(m-1)+1-k}{k}x^k}{\sum_{k=0}^{m}(-1)^k\frac{2m+1}{2m+1-k}\binom{2m+1-k}{k}x^k},$$

$$\mathcal{GL}_{1;100}(x,0,2m) = \frac{-2J_{2(m-1)+1}(x)}{j_{2m}(x)} = \frac{-2\sum_{k=0}^{m-1}\binom{2(m-1)-k}{k}x^k}{\sum_{k=0}^{m}\frac{2m}{2m-k}\binom{2m-k}{k}x^k},$$

and

$$\mathcal{GL}_{1;110}(x,0,2m+1) = \frac{-2J_{2(m-1)+2}(x)}{j_{2m+1}(x)} = \frac{-2\sum_{k=0}^{m-1}\binom{2(m-1)+1-k}{k}x^k}{\sum_{k=0}^{m}\frac{2m+1}{2m+1-k}\binom{2m+1-k}{k}x^k}.$$

We have for $1 \le t \le m$,*

$$\mathcal{GL}_{0;100}(x,t,2m) = \frac{x^{t-1}J_{2(m-1-t)+2}(-x)}{j_{2m}(-x)} = \frac{\sum_{k=0}^{m-1-t}(-1)^k\binom{2(m-1-t)+1-k}{k}x^{k+t-1}}{\sum_{k=0}^{m}(-1)^k\frac{2m}{2m-k}\binom{2m-k}{k}x^k},$$

$$\mathcal{GL}_{0;101}(x,t,2m) = \frac{x^{t-1}J_{2(m-t)+1}(-x)}{j_{2m}(-x)} = \frac{\sum_{k=0}^{m-t}(-1)^k\binom{2(m-t)-k}{k}x^{k+t-1}}{\sum_{k=0}^{m}(-1)^k\frac{2m}{2m-k}\binom{2m-k}{k}x^k},$$

$$\mathcal{GL}_{0;110}(x,t,2m+1) = \frac{x^{t-1}J_{2(m-t)+1}(-x)}{j_{2m+1}(-x)} = \frac{\sum_{k=0}^{m-t}(-1)^k\binom{2(m-t)-k}{k}x^{k+t-1}}{\sum_{k=0}^{m}(-1)^k\frac{2m+1}{2m+1-k}\binom{2m+1-k}{k}x^k},$$

$$\mathcal{GL}_{0;111}(x,t,2m+1) = \frac{x^{t-1}J_{2(m-t)+2}(-x)}{j_{2m+1}(-x)} = \frac{\sum_{k=0}^{m-t}(-1)^k\binom{2(m-t)+1-k}{k}x^{k+t-1}}{\sum_{k=0}^{m}(-1)^k\frac{2m+1}{2m+1-k}\binom{2m+1-k}{k}x^k},$$

$$\mathcal{GL}_{1;100}(x,t,2m)x^r = \frac{x^{t-1}J_{2(m-1-t)+2}(x)}{j_{2m}(x)} = \frac{\sum_{k=0}^{m-1-t}\binom{2(m-1-t)+1-k}{k}x^{k+t-1}}{\sum_{k=0}^{m}\frac{2m}{2m-k}\binom{2m-k}{k}x^k},$$

$$\mathcal{GL}_{1;101}(x,t,2m) = \frac{-x^{t-1}J_{2(m-t)+1}(x)}{j_{2m}(x)} = \frac{(-1)\sum_{k=0}^{m-t}\binom{2(m-t)-k}{k}x^{k+t-1}}{\sum_{k=0}^{m}\frac{2m}{2m-k}\binom{2m-k}{k}x^k},$$

$$\mathcal{GL}_{1;110}(x,t,2m+1) = \frac{x^{t-1}J_{2(m-t)+1}(x)}{j_{2m+1}(x)} = \frac{\sum_{k=0}^{m-t}\binom{2(m-t)-k}{k}x^{k+t-1}}{\sum_{k=0}^{m}\frac{2m+1}{2m+1-k}\binom{2m+1-k}{k}x^k},$$

and

$$\mathcal{GL}_{1;111}(x,t,2m+1) = \frac{-x^{t-1}J_{2(m-t)+2}(x)}{j_{2m+1}(x)} = \frac{(-1)\sum_{k=0}^{m-t}\binom{2(m-t)+1-k}{k}x^{k+t-1}}{\sum_{k=0}^{m}\frac{2m+1}{2m+1-k}\binom{2m+1-k}{k}x^k}.$$

*When $b = c = 0$ and $t = m$, we have $\mathcal{GL}_{s;100}(x,m,2m) = 0$.

Proof. Each of the sequences follows immediately from Theorem 8.3.7 on substituting the appropriate values for each of the parameters s, b and c. For the case $q = 1$, we simplify the generating function of the sums $\mathcal{L}_{s;110}(r,0,q)$ and $\mathcal{L}_{s;111}(r,0,q) = -\mathcal{L}_{s;111}(r,1,q)$, and then we note that $J_0(x) = 0$ and $j_1(x) = 1$. The result for the sequences $\mathcal{L}_{s;100}(r,m,2m)$ is demonstrated in Lemma 8.3.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

8.4 Separation of the Lucas polynomial

The case $a = 0$, is complicated by the additional $x - 4\gamma$ factor in the recurrence polynomials, $\mathcal{R}_{s;0b}(x, m)$, so that coefficients, a_j, of these polynomials are no longer single binomial coefficients. Before examining this case, we consider an alternative way of considering the sums that we have been analysing in Lemmas 8.3.2 and 8.3.3. This work will then help overcome this difficulty. Let us employ some lemmas.

LEMMA 8.4.1 (Lucas product separation). *For non-negative integers m, k, b, c and t with $0 \le k \le m$, $0 \le t \le m$ and $b, c \in \{0, 1\}$ we have*

$$\sum_{j=0}^{k} (-1)^j \frac{2m+b}{2m+b-j} \binom{2m+b-j}{j} \binom{2k+2-c-2j}{k+1+t-c-j}$$

$$= \sum_{j=0}^{k} (-1)^j \binom{2m+b-j}{j} \binom{2k+2-c-2j}{k+1+t-c-j} - \sum_{j=0}^{k-1} (-1)^j \binom{2m-2+b-j}{j} \binom{2k-c-2j}{k+t-c-j}.$$

$$(8.4.1)$$

Proof. From the identity

$$\frac{X}{X-J} \binom{X}{J} = \binom{X}{J} + \binom{X-1}{J-1},$$

with $X = 2m + b$ and $J = j$ we can write the left hand side of (8.4.1) as

$$\sum_{j=0}^{k} (-1)^j \binom{2m+b-j}{j} \binom{2k+2-c-2j}{k+1+t-c-j} + \sum_{j=1}^{k} (-1)^j \binom{2m-1+b-j}{j-1} \binom{2k+2-c-2j}{k+1+t-c-j}$$

$$= \sum_{j=0}^{k} (-1)^j \binom{2m+b-j}{j} \binom{2k+2-c-2j}{k+1+t-c-j} - \sum_{j=0}^{k-1} (-1)^j \binom{2m-2+b-j}{j} \binom{2k-c-2j}{k+t-c-j}.$$

$\square$

In Theorem 8.3.4 using hypergeometric functions (with $t = c = 0$), we have derived a closed form expression for the left hand side of (8.4.1). We recall that this is

$$\sum_{j=0}^{k} (-1)^j \frac{2m+b}{2m+b-j} \binom{2m+b-j}{j} \binom{2k+2-2j}{k+1-j} = 2(-1)^k \binom{2m-2+b-k}{k}.$$

We observe from Lemma 8.4.1 that this summation can also be considered as the sum of two separate summations and it would be equally desirable to express these individual summations in closed form. Written in hypergeometric form (with $t = c = 0$), we have

$$\sum_{j=0}^{k} (-1)^j \binom{2m+b-j}{j} \binom{2k+2-2j}{k+1-j}$$

$$= \binom{2k+2}{k+1} {}_3F_2 \left(\begin{array}{c} -k-1, -m+1/2-b, -m \\ -k-1/2, -2m-b \end{array}; 1 \right) - (-1)^{k+1} \binom{2m-1+b-k}{k+1}, \quad (8.4.2)$$

and

$$\sum_{j=0}^{k-1}(-1)^j\binom{2m-2+b-j}{j}\binom{2k-2j}{k-j}$$
$$=\binom{2k}{k}{}_3F_2\left(\begin{array}{c}-k,-m+3/2-b,1-m\\1/2-k,2-2m-b\end{array};1\right)-(-1)^k\binom{2m-2+b-k}{k}. \tag{8.4.3}$$

Remark. Here we have subtracted the last term of the hypergeometric function as our required sum is one less than this series produces.

However, if we denote the numerator parameters of the hypergeometric functions by a, b and c, the denominator parameters by d and e and then consider the sum $\sigma = d + e - a - b - c$, we find that in both cases $\sigma = 0$. Therefore, we are unable to employ Saalschütz's theorem to derive a closed form as in our previous result.

Nevertheless, using Theorem 8.3.4 we will show that (8.4.2) and (8.4.3) respectively have the simpler forms

$$2(-1)^k\binom{2m-2+b-k}{k}{}_2F_1\left(\begin{array}{c}-k,1\\k+2-b-2m\end{array};-1\right), \tag{8.4.4}$$

and

$$2(-1)^{k-1}\binom{2m-4+b-(k-1)}{k-1}{}_2F_1\left(\begin{array}{c}-(k-1),1\\k+3-b-2m\end{array};-1\right). \tag{8.4.5}$$

In order to obtain these forms we first introduce a lemma.

LEMMA 8.4.2 (alternating binomial hypergeometric form). *For non-negative integers N and k with $0 \le k \le \lfloor N/2 \rfloor$ we have*

$$\sum_{j=0}^{k}(-1)^{k+j}\binom{N-k-j}{k-j}=(-1)^k\binom{N-k}{k}{}_2F_1\left(\begin{array}{c}-k,1\\k-N\end{array};-1\right).$$

Proof. Let

$$\sum_{j=0}^{k}T_j=\sum_{j=0}^{k}(-1)^{k+j}\binom{N-k-j}{k-j}.$$

We find that the ratio T_{j+1}/T_j is

$$\frac{(-1)^{k+j+1}\binom{N-k-j-1}{k-j-1}}{(-1)^{k+j}\binom{N-k-j}{k-j}}=\frac{(-1)(k-j)(j+1)}{(N-k-j)(j+1)}=\frac{(-1)^2(j-k)(j+1)}{(-1)(j+k-N)(j+1)},$$

which gives the hypergeometric function

$${}_2F_1\left(\begin{array}{c}-k,1\\k-N\end{array};-1\right).$$

Now on multiplication of the hypergeometric function by $T_0 = (-1)^k\binom{N-k}{k}$ we obtain the result. $\qquad\square$

The simpler forms stated in (8.4.4) and (8.4.5) are now established using the following theorem.

THEOREM 8.4.3 (simplified Fibonacci product). *For non-negative integers m and k, with $0 \leq k \leq m$, we have*

$$\sum_{j=0}^{k}(-1)^j \binom{2m+b-j}{j}\binom{2k+2-2j}{k+1-j} = 2\sum_{j=0}^{k}(-1)^{k+j}\binom{2m-2+b-k-j}{k-j}$$

$$=2(-1)^k\binom{2m-2+b-k}{k}{}_2F_1\left(\begin{array}{c}-k,1\\k+2-b-2m\end{array};-1\right).\tag{8.4.6}$$

Proof. We recall from Lemma 8.4.1 that

$$\sum_{j=0}^{k}(-1)^j \frac{2m+b}{2m+b-j}\binom{2m+b-j}{j}\binom{2k+2-2j}{k+1-j}$$

$$=\sum_{j=0}^{k}(-1)^j \binom{2m+b-j}{j}\binom{2k+2-2j}{k+1-j} - \sum_{j=0}^{k-1}(-1)^j\binom{2m-2+b-j}{j}\binom{2k-2j}{k-j}.\tag{8.4.7}$$

Also from Theorem 8.3.4 we have that

$$\sum_{j=0}^{k}(-1)^j \frac{2m+b}{2m+b-j}\binom{2m+b-j}{j}\binom{2k+2-2j}{k+1-j} = 2(-1)^k\binom{2m-2+b-k}{k},$$

and so (8.4.7) becomes

$$2(-1)^k\binom{2m-2+b-k}{k}$$

$$=\sum_{j=0}^{k}(-1)^j \binom{2m+b-j}{j}\binom{2k+2-2j}{k+1-j} - \sum_{j=0}^{k-1}(-1)^j\binom{2m-2+b-j}{j}\binom{2k-2j}{k-j}.$$

To demonstrate the first part of (8.4.6), we use induction on k. When $k = 0$ we have

$$(-1)^0\binom{2m+b}{0}\binom{2}{1} = 2(-1)^0\binom{2m-2+b}{0}\binom{2}{1} = 2.\tag{8.4.8}$$

Assuming the relation holds for all values up-to and including $k-1$, we find that for the value k we have

$$2(-1)^k\binom{2m-2+b-k}{k} = \sum_{j=0}^{k}(-1)^j\binom{2m+b-j}{j}\binom{2k+2-2j}{k+1-j}$$

$$-2\sum_{j=0}^{k-1}(-1)^{k-1+j}\binom{2(m-1)+b-2-(k-1)-j}{k-1-j}.\tag{8.4.9}$$

On rearranging (8.4.9) we have

$$\sum_{j=0}^{k}(-1)^j\binom{2m+b-j}{j}\binom{2k+2-2j}{k+1-j} = 2\sum_{j=0}^{k}(-1)^{k+j}\binom{2m-2+b-k-j}{k-j}.$$

Hence the simplified binomial form follows by induction. The hypergeometric form of (8.4.6) follows immediately from Lemma 8.4.2 with $N = 2m+b-2$. $\qquad\square$

As was illustrated in Theorem 8.3.4 on replacing k with $k-1$, and m with $m-1$, we similarly obtain

$$\sum_{j=0}^{k-1}(-1)^j \binom{2(m-1)+b-j}{j}\binom{2(k-1)+2-2j}{(k-1)+1-j}$$

$$= 2\sum_{j=0}^{k-1}(-1)^{k-1+j}\binom{2(m-1)+b-2-(k-1)-j}{k-1-j}$$

$$= 2(-1)^{k-1}\binom{2m-3+b-k}{k-1}{}_2F_1\left(\begin{array}{c}-(k-1),1\\k+3-b-2m\end{array};-1\right).$$

Remark. We see that the sum on the left hand side of (8.4.6) is equivalent to (twice) the alternating sign sum of a binomial coefficient that forms a diagonal on Pascal's triangle. Moreover, when the Lucas polynomial is split up we obtain the equivalent of two sums containing identical binomial coefficients except that the second sum commences with the second term (of the first sum). Since the sum is an alternating sign series, this has the consequence that all the terms will cancel except the first, thus obtaining the given result. This latter approach provides us with an additional insight not evident from the hypergeometric approach alone. However, predominantly it will provide us with a means of approaching the non-alternating case $a = 0$ that now follows.

8.5 The generating function of the sequences $\mathcal{L}_{s;0bc}(r,t,q)$

Our overall approach is similar to that of the case $a = 1$, and we begin with Lemma 8.5.1.

LEMMA 8.5.1. *The generating function for the sequences $\mathcal{L}_{s;0bc}(r,t,q)$ has the form*

$$\mathcal{GL}_{s;0bc}(x,t,q) = \frac{\gamma^{t+1-c}(1-4\gamma x)\sum_{k=T}^{M}\sum_{j=0}^{k-T}(-1)^j\binom{2M+1-b-j}{j}\binom{2k+2-c-2j}{k+t+1-c-j}(\gamma x)^k - Dx^{M+1}}{(1-4\gamma x)\sum_{k=0}^{M}(-\gamma)^k\binom{2M+1-b-k}{k}x^k},$$

(8.5.1)

where $D \in \mathbb{Z}$, $M = m+b-1$, $0 \le t \le m$, and

$$T = \begin{cases} t & \text{if } t = 0 \\ t-1 & \text{if } t \ge 1. \end{cases}$$

(8.5.2)

When $b = c = 0$ and $t = m$, we have

$$\mathcal{GL}_{s;000}(x,m,2m) = \frac{2}{(1-4\gamma x)\sum_{k=0}^{M}(-\gamma)^k\binom{2M+1-b-k}{k}x^k}.$$

(8.5.3)

Proof. From Lemma 8.2.1 we have

$$\mathcal{GL}_{s;0bc}(x,t,q) = \frac{\sum_{k=0}^{m+b-1}\sum_{j=0}^{k}(-\gamma)^j a_j \mathcal{L}_{s;0bc}(k-j,t,q)x^k}{\sum_{k=0}^{m+b}(-\gamma)^k a_k x^k},$$

(8.5.4)

where we recall that the terms a_k are those of the corresponding recurrence polynomials, $\mathcal{R}_{s;0b}(x, m)$ (of order $m + b$), such that a_k is the coefficient of the term x^{m+b-k}. From Theorem 5.6.1 with $M = m + b - 1$ these are given by

$$\mathcal{R}_{s;0b}(x, m) = (x - 4\gamma) \sum_{k=0}^{M} (-\gamma)^{M-k} \binom{2M + 1 - b - k}{k} x^{M-k}, \tag{8.5.5}$$

We recall that $\mathcal{R}_{s;0b}(x, m)$ is also composed of a Fibonacci polynomial of order $M = m+b-1$. Incorporating our notation from Definition 5.4.1, we have

$$A_{s;0b}^{r}(x) = \sum_{k=0}^{M} (-\gamma)^{M-k} b_k x^{M-k}, \tag{8.5.6}$$

where

$$b_k = \binom{2M + 1 - b - k}{k}. \tag{8.5.7}$$

Therefore, by comparing coefficients of x^i in (8.2.1) and (8.5.5), we have

$$a_i = b_i + 4b_{i-1}. \tag{8.5.8}$$

and from the Corollary of Lemma 8.2.1 we can express the numerator of (8.5.4) as

$$\gamma^{t+1-c} \sum_{k=0}^{M} \sum_{j=0}^{M-k} (-\gamma)^j a_j \mathcal{L}_s(k) x^{j+k} = \gamma^{t+1-c} \sum_{k=0}^{M} \sum_{j=0}^{M-k} (-\gamma)^j (b_j + 4b_{j-1}) \mathcal{L}_s(k) x^{j+k},$$

where $b_{-1} = 0$. Recombining the terms b_j in an alternative way we obtain

$$\gamma^{t+1-c} \sum_{k=0}^{M} \sum_{j=0}^{M-k} (-\gamma)^j (b_j x^j + 4x b_{j-1} x^{j-1}) \mathcal{L}_s(k) x^k$$

$$= \gamma^{t+1-c} (1 - 4\gamma x) \sum_{k=0}^{M} \sum_{j=0}^{M-k} (-\gamma)^j b_j \mathcal{L}_s(k) x^{j+k} - 4\gamma^{t+1-c} \sum_{j=0}^{M} (-\gamma)^{j+1} b_j \mathcal{L}_s(M - j) x^{M+1}$$

$$= \gamma^{t+1-c} (1 - 4\gamma x) \sum_{k=0}^{M} \sum_{j=0}^{k} (-\gamma)^j b_j \mathcal{L}_s(k - j) x^k - D x^{M+1}, \tag{8.5.9}$$

where

$$D = 4\gamma^{t+1-c} \sum_{j=0}^{M} (-\gamma)^{j+1} b_j \mathcal{L}_s(M - j). \tag{8.5.10}$$

As in Lemma 8.3.1 the terms $\mathcal{L}_{s;0bc}(r, t, q)$ for $0 \le r \le m - 1$, in the numerator of (8.5.4), are determined by the single binomial coefficients

$$\mathcal{L}_{s;0bc}(r, t, q) = \gamma^{r+t+1-c} \binom{2r + 2 - c}{r + t + 1 - c}. \tag{8.5.11}$$

However, when the parameters $b = c = 0$ and the variables $r = m - 1$ and $t = m$, (8.5.11) is replaced by

$$\mathcal{L}_{s;000}(m - 1, m, 2m) = \gamma^{2m} \left(\binom{2m}{0} + \binom{2m}{2m} \right) = 2. \tag{8.5.12}$$

Also as in Lemma 8.3.1 we have that $\mathcal{L}_{s;0bc}(r,t,q) = 0$ when $t \geq 2$ and $0 \leq r \leq t-2$, and we similarly adjust the limits of (8.5.9) by T given in (8.5.2).

On substitution of each of the terms (8.5.7), (8.5.11) and (the discussed) placement of the variable T into (8.5.9) gives (8.5.1). Furthermore, as in (8.3.8), we have for $0 \leq r \leq m-2$, that $\mathcal{L}_{s;000}(r,m,2m) = 0$; and from (8.5.12) that $\mathcal{L}_{s;000}(m-1,m,2m) = 2$, so we obtain (8.5.3). □

Remark. It will be observed that the "factor" $1 - 4\gamma x$ is not a complete factor of the numerator, due the remainder term Dx^{M+1} (required to cancel the additional term created when the $1 - 4\gamma x$ is "taken outside" of the summation). However, we consider the generating function of (8.5.1) as a transitional form as opposed to a practical one.

We now examine a means of simplifying the numerator of the generating function determined in Lemma 8.5.1. We start with the simpler case $t = 0$, with the aid of Lemma 8.5.2.

LEMMA 8.5.2 (binomial sum rearrangement). *For non-negative integers N and K, we have*

$$\binom{N-2}{K} + 4\binom{N-1}{K-1} = \binom{N}{K} + 2\binom{N-1}{K-1} + \binom{N-2}{K-2}.$$

We start with the expression on left hand side and manipulate as follows:

Proof.

$$\begin{aligned}
\binom{N-2}{K} + 4\binom{N-1}{K-1} &= \binom{N-2}{K} + \binom{N-1}{K-1} + \binom{N-1}{K-1} + 2\binom{N-1}{K-1} \\
&= \binom{N-2}{K} + \binom{N-2}{K-1} + \binom{N-2}{K-2} + \binom{N-1}{K-1} + 2\binom{N-1}{K-1} \\
&= \binom{N-1}{K} + \binom{N-1}{K-1} + 2\binom{N-1}{K-1} + \binom{N-2}{K-2} \\
&= \binom{N}{K} + 2\binom{N-1}{K-1} + \binom{N-2}{K-2}.
\end{aligned}$$

□

THEOREM 8.5.3. *For non-negative integers M, k and B, $M = m - (1-b)$ and $B = 1-b$, we have*

$$(1 - 4\gamma x) \sum_{k=0}^{M} \sum_{j=0}^{k} (-1)^j \binom{2M+B-j}{j} \binom{2k+2-2j}{k+1-j} (\gamma x)^k - Dx^{M+1}$$

$$= 2 \sum_{k=0}^{M} (-\gamma)^k \frac{2M+B}{2M+B-k} \binom{2M+B-k}{k} x^k. \tag{8.5.13}$$

Proof. With $0 \leq k \leq M$, we consider the coefficient of $(\gamma x)^k$ on the left hand side of (8.5.13). This gives

$$\sum_{j=0}^{k}(-1)^j \binom{2M+B-j}{j}\binom{2k+2-2j}{k+1-j} - 4\sum_{j=0}^{k-1}(-1)^j \binom{2M+B-j}{j}\binom{2(k-1)+2-2j}{(k-1)+1-j}$$

$$= \sum_{j=0}^{k}(-1)^j \binom{2M+B-j}{j}\binom{2k+2-2j}{k+1-j} - 4\sum_{j=0}^{k-1}(-1)^j \binom{2M+B-j}{j}\binom{2k-2j}{k-j}.$$

From Theorem 8.4.3 this simplifies to

$$2\left(\sum_{j=0}^{k}(-1)^{k+j}\binom{2M+B-2-k-j}{k-j} + 4\sum_{j=0}^{k-1}(-1)^{k+j}\binom{2M+B-1-k-j}{k-1-j}\right). \qquad (8.5.14)$$

We now use Lemma 8.5.2 with $N = 2M+B-k-j$ and $K = k-j$ to write the inner bracket of (8.5.14) as

$$\sum_{j=0}^{k}(-1)^{k+j}\binom{2M+B-k-j}{k-j} + 2\sum_{j=0}^{k-1}(-1)^{k+j}\binom{2M+B-1-k-j}{k-1-j}$$

$$+ \sum_{j=0}^{k-2}(-1)^{k+j}\binom{2M+B-2-k-j}{k-2-j},$$

or alternatively,

$$\sum_{j=0}^{k}(-1)^{k+j}\binom{2M+B-k-j}{k-j} + \sum_{j=0}^{k-1}(-1)^{k+j}\binom{2M+B-1-k-j}{k-1-j}$$

$$+ \sum_{j=0}^{k-1}(-1)^{k+j}\binom{2M+B-1-k-j}{k-1-j} + \sum_{j=0}^{k-2}(-1)^{k+j}\binom{2M+B-2-k-j}{k-2-j}. \qquad (8.5.15)$$

However, due to the cancellation of terms, (8.5.15) simplifies to

$$(-1)^k\left(\binom{2M+B-k}{k} + \binom{2M+B-1-k}{k-1}\right) = (-1)^k\frac{2M+B}{2M+B-k}\binom{2M+B-k}{k}. \qquad (8.5.16)$$

Substituting (8.5.16) into (8.5.14) obtains (8.5.13). Finally we choose D such that it cancels the coefficient of the term x^{M+1}. This is determined in (8.5.10) of Lemma 8.5.1. $\qquad \square$

We take a similar approach as in Theorem 8.5.3 to evaluate the general case when $t \neq 0$.

THEOREM 8.5.4 (simplified Fibonacci product sum general case). *For non-negative integers $M \geq 1$, $0 \leq k \leq M$ and $1 \leq t \leq M$, where $M = m - (1-b)$ and $B = 1-b$, we*

have

$$\sum_{j=0}^{k+1-t} (-1)^j \binom{2M+B-j}{j} \binom{2k+2-c-2j}{k+1+t-c-j}$$

$$= \sum_{j=0}^{k+1-t} (-1)^{k+1-t+j} \binom{2(M-t)+B+c-1-k-j}{k+1-t-j}$$

$$= (-1)^{k+1-t} \binom{2(M-t)+B+c-1-k}{k+1-t} {}_2F_1\left(\begin{array}{c} -k-1+t, 1 \\ k+1-2(M-t)-B-c \end{array}; -1\right). \quad (8.5.17)$$

Proof. We recall from Lemma 8.4.1 that we have

$$\sum_{j=0}^{k+1-t} (-1)^j \frac{2M+B}{2M+B-j} \binom{2M+B-j}{j} \binom{2k+2-c-2j}{k+1+t-c-j}$$

$$= \sum_{j=0}^{k+1-t} (-1)^j \binom{2M+B-j}{j} \binom{2k+2-c-2j}{k+1+t-c-j}$$

$$- \sum_{j=0}^{k-t} (-1)^j \binom{2M+B-2-j}{j} \binom{2k-c-2j}{k+t-c-j}. \quad (8.5.18)$$

Also from Theorem 8.3.6 we have that

$$\sum_{j=0}^{k+1-t} (-1)^j \frac{2M+B}{2M+B-j} \binom{2M+B-j}{j} \binom{2k+2-c-2j}{k+1+t-c-j}$$

$$= (-1)^{k+1-t} \binom{2(M-t)+B+c-1-(k+1-t)}{k+1-t},$$

and so (8.5.18) becomes

$$(-1)^{k+1-t} \binom{2(M-t)+B+c-1-(k+1-t)}{k+1-t}$$

$$= \sum_{j=0}^{k+1-t} (-1)^j \binom{2M+B-j}{j} \binom{2k+2-c-2j}{k+1+t-c-j} - \sum_{j=0}^{k-t} (-1)^j \binom{2M+B-2-j}{j} \binom{2k-c-2j}{k+t-c-j}.$$

$$(8.5.19)$$

As in Theorem 8.4.3, we use induction on the variable $K = k+1-t$, where $0 \le K \le M-t$. $K = k+1-t = 0$ implies $k+1 = t$ and this gives

$$(-1)^0 \binom{2M+B}{0} \binom{2t-c}{2t-c} = \binom{2(M-t)+B+c-1}{0} = 1.$$

Assuming the relation holds for all values up-to and including $K-1$ we have

$$- \sum_{j=0}^{k-t} (-1)^j \binom{2(M-1)+B-2-j}{j} \binom{2(k-1)+2-c-2j}{k+t-c-j}$$

$$= - \sum_{j=0}^{k-t} (-1)^{k-t+j} \binom{2(M-1-t)+B+c-1-(k-t)-j}{k-t-j}$$

$$= - \sum_{j=0}^{k-t} (-1)^{k-t+j} \binom{2(M-t)+B+c-2-(k+1-t)-j}{k-t-j}$$

$$= - \sum_{j=1}^{k+1-t} (-1)^{k+1-t+j} \binom{2(M-t)+B+c-1-(k+1-t)-j}{k+1-t-j}. \tag{8.5.20}$$

Putting (8.5.20) into (8.5.19) and rearranging the latter equation we have

$$\sum_{j=0}^{k+1-t} (-1)^j \binom{2M+B-j}{j} \binom{2k+2-c-2j}{k+1+t-c-j}$$

$$= \sum_{j=0}^{k+1-t} (-1)^{k+1-t+j} \binom{2(M-t)+B+c-1-(k+1-t)-j}{k+1-t-j}.$$

Hence the simplified binomial form then follows by induction. The hypergeometric form of (8.5.17) follows immediately from Lemma 8.4.2 with $N = 2M - 3t + B + c$ and replacing k with $k + 1 - t$. $\qquad \Box$

THEOREM 8.5.5. *For non-negative integers $M = m + b - 1$, $0 \le k \le M$, $B = 1 - b$ and D as given in (8.5.10) of Lemma 8.5.1, we have*

$$\gamma^{t+1-c}(1-4\gamma x) \sum_{k=t-1}^{M} \left(\sum_{j=0}^{k-t+1} (-1)^j \binom{2M+B-j}{j} \binom{2k+2-c-2j}{k+1+t-c-j} \right) (\gamma x)^k - Dx^{M+1}$$

$$= \gamma^c \sum_{k=0}^{m+bc-t} (-1)^k \frac{2(m-t)+b+c}{2(m-t)+b+c-k} \binom{2(m-t)+b+c-k)}{k} (\gamma x)^{k+t-1}. \tag{8.5.21}$$

Proof. From Theorem 8.5.4 we have

$$\sum_{j=0}^{k+1-t} (-1)^j \binom{2M+B-j}{j} \binom{2k+2-c-2j}{k+1+t-c-j}$$

$$= \sum_{j=0}^{k+1-t} (-1)^{k+1-t+j} \binom{2(M-t)+B+c-1-(k+1-t)-j}{k+1-t-j},$$

and the left hand side of (8.5.21) becomes

$$(1-4\gamma x) \sum_{k=t-1}^{M} \left(\sum_{j=0}^{k+1-t} (-1)^{k+1-t+j} \binom{2(M-t)+B+c-1-(k+1-t)-j}{k+1-t-j} \right) (\gamma x)^k. \tag{8.5.22}$$

For $0 \le k \le M$ we consider the coefficient of $(\gamma x)^k$ in (8.5.22) to obtain

$$\sum_{j=0}^{k+1-t} (-1)^{k+1-t+j} \binom{2(M-t)+B+c-1-(k+1-t)-j}{k+1-t-j}$$

$$-4\sum_{j=0}^{k-t}(-1)^{k-t+j}\binom{2(M-t)+B+c-1-(k-t)-j}{k-t-j}. \tag{8.5.23}$$

Now we use Lemma 8.5.2 with $N = 2(M-t)+B+c+1-(k+1-t)$ and $K = k+1-t$ (and also using this notation in order to condense the binomial coefficients) we write (8.5.23) as

$$\sum_{j=0}^{K}(-1)^{K+j}\binom{N-j}{K-j} + 2\sum_{j=0}^{K-1}(-1)^{K+j}\binom{N-1-j}{K-1-j} + \sum_{j=0}^{K-2}(-1)^{K+j}\binom{N-2-j}{K-2-j},$$

or alternatively,

$$\sum_{j=0}^{K}(-1)^{K+j}\binom{N-j}{K-j} + \sum_{j=0}^{K-1}(-1)^{K+j}\binom{N-1-j}{K-1-j}$$

$$+ \sum_{j=0}^{K-1}(-1)^{K+j}\binom{N-1-j}{K-j} + \sum_{j=0}^{K-2}(-1)^{K+j}\binom{N-2-j}{K-2-j}. \tag{8.5.24}$$

However, (8.5.24) simplifies to

$$(-1)^K\left(\binom{N}{K} + \binom{N-1}{K-1}\right) = (-1)^K\frac{N+K}{N}\binom{N}{K},$$

so that on replacing N and K with their original values and on recalling that $M = m - (1-b)$ and $B = 1 - b$ we continue as

$$=(-1)^{k+1-t}\frac{2(M-t)+B+c+1}{2(M-t)+B+c+1-(k+1-t)}\binom{2(M-t)+B+c+1-(k+1-t)}{k+1-t}$$

$$=(-1)^{k+1-t}\frac{2(m-t)+b+c}{2(m-t)+b+c-(k+1-t)}\binom{2(m-t)+b+c-(k+1-t)}{k+1-t}.$$

We now have

$$\gamma^{t+1-c}\sum_{k=t-1}^{M=m+b-1}(-1)^{k+1-t}\frac{2(m-t)+b+c}{2(m-t)+b+c-(k+1-t)}\binom{2(m-t)+b+c-(k+1-t)}{k+1-t}(\gamma x)^k$$

and this is equivalent to

$$\gamma^c\sum_{k=t-1}^{m+b-1}(-\gamma)^{k+1-t}\frac{2(m-t)+b+c}{2(m-t)+b+c-(k+1-t)}\binom{2(m-t)+b+c-(k+1-t)}{k+1-t}x^k$$

and on the rescaling of $k + 1 - t$ with k we obtain

$$\gamma^c\sum_{k=0}^{m+b-t}(-\gamma)^k\frac{2(m-t)+b+c}{2(m-t)+b+c-k}\binom{2(m-t)+b+c-k}{k}x^{k+t-1}. \tag{8.5.25}$$

However, we observe that the binomial coefficient in (8.5.25) when $k = m + b - t$ and the parameters $b = 1$ and $c = 0$ becomes $\binom{m-t}{m+1-t} = 0$. Consequently we can write (8.5.25) as (8.5.21). Finally as in Theorem 8.5.3 we choose D such that it cancels the coefficient of the term x^{M+1}, and we recall that this value is determined in (8.5.10) of Lemma 8.5.1. $\qquad\square$

This then motivates the following theorem.

THEOREM 8.5.6 (generating function of $\mathcal{L}_{s;0bc}$). *For $\mathcal{L}_{s;abc}(r,t,q)$ as given in Definition 3.1.2 with $c \le t \le M$ and $M = m - 1 + b$, we have with $a = 0$,*

$$\mathcal{GL}_{s;0bc}(x,t,q) = \begin{cases} \dfrac{2\gamma j_{2M+1-b}(-\gamma x)}{(1-4\gamma x)J_{2M+2-b}(-\gamma x)} & \text{if } c = t = 0 \\[3mm] \dfrac{\gamma^c x^{t-1} j_{2(m+bc-t)+b+c-2bc}(-\gamma x)}{(1-4\gamma x)J_{2M+2-b}(-\gamma x)} & \text{otherwise.} \end{cases}$$

Here, $J_N(x)$ is the Jacobsthal polynomial as defined in (5.2.2), $j_n(x)$ the Jacobsthal-Lucas polynomial as defined in (5.2.6) and $\gamma = (-1)^s$.

Proof. From Lemma 8.5.1 we have

$$\mathcal{GL}_{s;0bc}(x,t,q) = \frac{\gamma^{t+1-c}(1-4\gamma x)\sum_{k=T}^{M}\sum_{j=0}^{k-T}(-1)^j\binom{2M+1-b-j}{j}\binom{2k+2-c-2j}{k+t+1-c-j}(\gamma x)^k - Dx^{M+1}}{(1-4\gamma x)\sum_{k=0}^{M}(-\gamma)^k\binom{2M+1-b-k}{k}x^k},$$

$$(8.5.26)$$

where

$$T = \begin{cases} t & \text{if } t = 0 \\ t-1 & \text{if } t \ge 1. \end{cases}$$

Then when $t = 0$ (and $c = 0$) we have from Theorem 8.5.3 that (8.5.26) simplifies to,

$$\mathcal{GL}_{s;0b0}(x,0,q) = \frac{2\gamma\sum_{k=0}^{m-1+b}(-\gamma)^k\frac{2m-1+b}{2m-1+b-k}\binom{2(m-1+b)+1-b-k}{k}x^k}{(1-4\gamma x)\sum_{k=0}^{m-1+b}(-\gamma)^k\binom{2(m-1+b)+1-b-k}{k}x^k}, \qquad (8.5.27)$$

and when $t \ge 1$ we have from Theorem 8.5.5 that

$$\mathcal{GL}_{s;0bc}(x,t,q) = \frac{\gamma^c\sum_{k=0}^{m-t+bc}(-\gamma)^k\frac{2(m-t+bc)+b+c-2bc}{2(m-t+bc)+b+c-2bc-k}\binom{2(m-t+bc)+b+c-2bc-k}{k}x^{k+t-1}}{(1-4\gamma x)\sum_{k=0}^{m-1+b}(-\gamma)^k\binom{2(m-1+b)+1-b-k}{k}x^k}. \qquad (8.5.28)$$

Now on applying (5.2.3) and (5.2.7) to express (8.3.23) and (8.3.24) in terms of Jacobsthal and Jacobsthal-Lucas polynomials the theorem follows. $\qquad\square$

From Theorem 8.5.6 we identify the generating function for each of the sequences when $a = 0$.

COROLLARY. *We have for $t = 0$,*

$$\mathcal{GL}_{0;01c}(x,0,1) = \frac{2^{1-c}j_0(-x)}{(1-4x)J_1(-x)} = \frac{4}{2^c(1-4x)},$$

$$\mathcal{GL}_{1;01c}(x,0,1) = \frac{(-2)^{1-c}j_0(x)}{(1+4x)J_1(x)} = \frac{(-2)^{1-c}2}{(1+4x)},$$

$$\mathcal{GL}_{0;000}(x,0,2m) = \frac{2j_{2(m-1)+1}(-x)}{(1-4x)J_{2(m-1)+2}(-x)} = \frac{2\sum_{k=0}^{m-1}(-1)^k \frac{2m-1}{2m-1-k}\binom{2(m-1)+1-k}{k}x^k}{(1-4x)\sum_{k=0}^{m-1}(-1)^k\binom{2(m-1)+1-k}{k}x^k},$$

$$\mathcal{GL}_{0;010}(x,0,2m+1) = \frac{2j_{2m}(-x)}{(1-4x)J_{2m+1}(-x)} = \frac{2\sum_{k=0}^{m}(-1)^k \frac{2m}{2m-k}\binom{2m-k}{k}x^k}{(1-4x)\sum_{k=0}^{m}(-1)^k\binom{2m-k}{k}x^k},$$

$$\mathcal{GL}_{1;000}(x,0,2m) = \frac{-2j_{2(m-1)+1}(x)}{(1+4x)J_{2(m-1)+2}(x)} = \frac{-2\sum_{k=0}^{m-1}\frac{2m-1}{2m-1-k}\binom{2(m-1)+1-k}{k}x^k}{(1+4x)\sum_{k=0}^{m-1}\binom{2(m-1)+1-k}{k}x^k},$$

and

$$\mathcal{GL}_{1;010}(x,0,2m+1) = \frac{-2j_{2m}(x)}{(1+4x)J_{2m+1}(x)} = \frac{-2\sum_{k=0}^{m}\frac{2m}{2m-k}\binom{2m-k}{k}x^k}{(1+4x)\sum_{k=0}^{m}\binom{2m-k}{k}x^k}.$$

We have for $1 \le t \le m$,

$$\mathcal{GL}_{0;000}(x,t,2m) = \frac{x^{t-1}j_{2(m-t)}(-x)}{(1-4x)J_{2(m-1)+2}(-x)} = \frac{\sum_{k=0}^{m-t}(-1)^k \frac{2(m-t)}{2(m-t)-k}\binom{2(m-t)-k}{k}x^{k+t-1}}{(1-4x)\sum_{k=0}^{m-1}(-1)^k\binom{2(m-1)+1-k}{k}x^k},$$

(when we define the limit of the numerator when $t = m$ as 2),

$$\mathcal{GL}_{0;001}(x,t,2m) = \frac{x^{t-1}j_{2(m-t)+1}(-x)}{(1-4x)J_{2(m-1)+2}(-x)} = \frac{\sum_{k=0}^{m-t}(-1)^k \frac{2(m-t)+1}{2(m-t)+1-k}\binom{2(m-t)+1-k}{k}x^{k+t-1}}{(1-4x)\sum_{k=0}^{m-1}(-1)^k\binom{2(m-1)+1-k}{k}x^k},$$

$$\mathcal{GL}_{0;010}(x,t,2m+1) = \frac{x^{t-1}j_{2(m-t)+1}(-x)}{(1-4x)J_{2m+1}(-x)} = \frac{\sum_{k=0}^{m-t}(-1)^k \frac{2(m-t)+1}{2(m-t)+1-k}\binom{2(m-t)+1-k}{k}x^{k+t-1}}{(1-4x)\sum_{k=0}^{m}(-1)^k\binom{2m-k}{k}x^k},$$

$$\mathcal{GL}_{0;011}(x,t,2m+1) = \frac{x^{t-1}j_{2(m+1-t)}(-x)}{(1-4x)J_{2m+1}(-x)} = \frac{\sum_{k=0}^{m+1-t}(-1)^k \frac{2(m+1-t)}{2(m+1-t)-k}\binom{2(m+1-t)-k}{k}x^{k+t-1}}{(1-4x)\sum_{k=0}^{m}(-1)^k\binom{2m-k}{k}x^k},$$

$$\mathcal{GL}_{1;000}(x,t,2m) = \frac{x^{t-1}j_{2(m-t)}(x)}{(1+4x)J_{2(m-1)+2}(x)} = \frac{\sum_{k=0}^{m-t} \frac{2(m-t)}{2(m-t)-k}\binom{2(m-t)-k}{k}x^{k+t-1}}{(1+4x)\sum_{k=0}^{m-1}\binom{2(m-1)+1-k}{k}x^k},$$

$$\mathcal{GL}_{1;001}(x,t,2m) = \frac{-x^{t-1}j_{2(m-t)+1}(x)}{(1+4x)J_{2(m-1)+2}(x)} = \frac{(-1)\sum_{k=0}^{m-t} \frac{2(m-t)+1}{2(m-t)+1-k}\binom{2(m-t)+1-k}{k}x^{k+t-1}}{(1+4x)\sum_{k=0}^{m-1}\binom{2(m-1)+1-k}{k}x^k},$$

$$\mathcal{GL}_{1;010}(x,t,2m+1) = \frac{x^{t-1}j_{2(m-t)+1}(x)}{(1+4x)J_{2m+1}(x)} = \frac{\sum_{k=0}^{m-t} \frac{2(m-t)+1}{2(m-t)+1-k}\binom{2(m-t)+1-k}{k}x^{k+t-1}}{(1+4x)\sum_{k=0}^{m}\binom{2m-k}{k}x^k},$$

and

$$\mathcal{GL}_{1;011}(x,t,2m+1) = \frac{-x^{t-1}j_{2(m+1-t)}(x)}{(1+4x)J_{2m+1}(x)} = \frac{(-1)\sum_{k=0}^{m+1-t} \frac{2(m+1-t)}{2(m+1-t)-k}\binom{2(m+1-t)-k}{k}x^{k+t-1}}{(1+4x)\sum_{k=0}^{m}\binom{2m-k}{k}x^k}.$$

Proof. Each of the sequences follows immediately from Theorem 8.5.6 on substituting the appropriate values for each of the parameters s, b and c. For the case $q = 1$, we simplify the generating function of the sums $\mathcal{L}_{s;010}(r,0,q)$ and $\mathcal{L}_{s;011}(r,0,q) = -\mathcal{L}_{s;011}(r,1,q)$, and then we note that $j_0(x) = 2$ and $J_1(x) = 1$. The result for the sequences $\mathcal{L}_{s;000}(r,m,2m)$ is demonstrated in Lemma 8.5.1 which introduces an additional factor of 2 that is not displayed in the equation. $\qquad\square$

Chapter 9

Minor Corner Layered (MCL) Determinants

In this chapter we wish to establish the relationship between the generating function, the recurrence relation polynomial and MCL determinants of type 1 and 2 (as defined in [31]).

9.1 Minor Corner Layered (MCL) Determinants

In the works of Lettington [30], [31] and Coffey, Hindmarsh, Lettington and Pryce [9], three types of determinants were described, two of which defined below are necessary for our calculations.

Definition 9.1.1 (MCL (and half-weighted) Determinant)**.** *Let* $\Delta_r(\vec{\mathbf{h}})$ *be an* $r \times r$ *minor corner layered (MCL) determinant, where* $\vec{\alpha}$ *means entry index* i *in* α_i *increasing left to right, be defined such that* $\Delta_0(\vec{\mathbf{h}}) = 1$*, and for* $r \geq 1$ *we have*

$$\Delta_r(\vec{\mathbf{h}}) = (-1)^r \begin{vmatrix} h_1 & 1 & 0 & 0 & \dots & 0 \\ h_2 & h_1 & 1 & 0 & \dots & 0 \\ h_3 & h_2 & h_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{r-1} & h_{r-2} & h_{r-3} & h_{r-4} & \dots & 1 \\ h_r & h_{r-1} & h_{r-2} & h_{r-3} & \dots & h_1 \end{vmatrix}, \tag{9.1.1}$$

where the vector $\vec{\mathbf{h}} = (h_1, h_2, h_3, \dots)$ *is an infinite dimensional vector.*

Similarly, let $\Psi_r(\vec{\mathbf{h}}, \vec{\mathbf{H}})$ *be a half-weighted* $r \times r$ *MCL determinant defined such that we have*

$$\Psi_r(\vec{\mathbf{h}}, \vec{\mathbf{H}}) = (-1)^r \begin{vmatrix} H_1 & 1 & 0 & 0 & \dots & 0 \\ H_2 & h_1 & 1 & 0 & \dots & 0 \\ H_3 & h_2 & h_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{r-1} & h_{r-2} & h_{r-3} & h_{r-4} & \dots & 1 \\ H_r & h_{r-1} & h_{r-2} & h_{r-3} & \dots & h_1 \end{vmatrix}, \tag{9.1.2}$$

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where $\vec{\mathbf{h}} = (h_1, h_2, h_3, \ldots)$ *and* $\vec{\mathbf{H}} = (H_1, H_2, H_3, \ldots)$ *correspond to two infinite dimensional vectors.*

Furthermore, the works of [30] and [31] have established the following lemmas associating an MCL determinant with a recurrence relation polynomial.

LEMMA 9.1.1. *Let* g_r *be a function satisfying an* $(r + 1)$ *term recurrence relation (or* r-*th order relation) such that*

$$g_r = -\sum_{k=0}^{r-1} h_{r-k} g_k.$$

Then for $r \geq 1$ *we have*

$$g_r = \Delta_r(\vec{\mathbf{h}}),$$

where $\Delta_r(\vec{\mathbf{h}})$ *is defined as in Definition 9.1.1.*

Proof. See the Corollary of Lemma 6.1 of [30]. $\square$

LEMMA 9.1.2. *We have for* $r \geq 1$ *that*

$$\Psi_r(\vec{\mathbf{h}}, \vec{\mathbf{H}}) = -\sum_{k=0}^{r-1} H_{r-k} \Delta_k(\vec{\mathbf{h}}), \tag{9.1.3}$$

where $\Delta_r(\vec{\mathbf{h}})$ *and* $\Psi_r(\vec{\mathbf{h}}, \vec{\mathbf{H}})$ *are defined as in Definition 9.1.1.*

Proof. See display (3.2) of Lemma 3.1 in [31]. $\square$

COROLLARY. *If the vector* $\vec{\mathbf{H}} = (1, 0, 0, 0, \ldots)$, *then we have*

$$\Psi_r(\vec{\mathbf{h}}, \vec{\mathbf{H}}) = -\Delta_{r-1}(\vec{\mathbf{h}}).$$

Proof. We put $H_1 = 1$, and for $r \geq 2$ we put $H_r = 0$, into (9.1.3). $\square$

In preparation of the work that we introduce in Section 9.2 it is necessary to elaborate on Definition 9.1.1 in the following manner.

Definition 9.1.2 (signed MCL (and half-weighted) Determinant)**.** *Let* $\Delta_r^\rho(\vec{\mathbf{a}}_\mathbf{n})$ *be an* $r \times r$ *MCL determinant with* $\rho \in \{0, 1\}$ *and defined such that we have* [1]

$$\Delta_r^\rho(\vec{\mathbf{a}}_\mathbf{n}) = (-1)^r \times \begin{vmatrix} a_{n,1}^\rho & 1 & 0 & 0 & \ldots & 0 \\ a_{n,2} & a_{n,1}^\rho & 1 & 0 & \ldots & 0 \\ a_{n,3}^\rho & a_{n,2} & a_{n,1}^\rho & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,r-1}^\rho & a_{n,r-2}^\rho & a_{n,r-3}^\rho & a_{n,r-4}^\rho & \ldots & 1 \\ a_{n,r}^\rho & a_{n,r-1}^\rho & a_{n,r-2}^\rho & a_{n,r-3}^\rho & \ldots & a_{n,1}^\rho \end{vmatrix},$$

---

[1]Throughout, we use $a_{n,k}^\rho = (-1)^{\rho k} a_{n,k}$ and $A_{N,k}^\rho = (-1)^{\rho k} A_{N,k}$ for layout considerations.

where the vector $\vec{\mathbf{a}}_{\mathbf{n}} = ((-1)^{\rho}a_{n,1}, a_{n,2}, (-1)^{\rho}a_{n,3}, \dots,$ and $a_k = 0$ when $k > n$.
Similarly, let $\Psi_r^{\rho}(\vec{\mathbf{a}}_{\mathbf{n}}, \vec{\mathbf{A}}_{\mathbf{N},\mathbf{T}})$ be a half-weighted $(r+1) \times (r+1)$ MCL determinant defined such that (when $T = 0$),

$$
\Psi_r^{\rho}(\vec{\mathbf{a}}_{\mathbf{n}}, \vec{\mathbf{A}}_{\mathbf{N},\mathbf{0}}) = (-1)^r \times
\begin{vmatrix}
A_{N,0} & 1 & 0 & 0 & \dots & 0 \\
A_{N,1}^{\rho} & a_{n,1}^{\rho} & 1 & 0 & \dots & 0 \\
A_{N,2} & a_{n,2} & a_{n,1}^{\rho} & 1 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A_{N,r-1}^{\rho} & a_{n,r-1}^{\rho} & a_{n,r-2}^{\rho} & a_{n,r-3}^{\rho} & \dots & 1 \\
A_{N,r}^{\rho} & a_{n,r}^{\rho} & a_{n,r-1}^{\rho} & a_{n,r-2}^{\rho} & \dots & a_{n,1}^{\rho}
\end{vmatrix},
$$

where the vectors $\vec{\mathbf{a}}_{\mathbf{n}} = ((-1)^{\rho}a_{n,1}, a_{n,2}, (-1)^{\rho}a_{n,3}, \dots,$
and $\vec{\mathbf{A}}_{\mathbf{N},\mathbf{0}} = ((-1)^{\rho}A_{N,1}, A_{N,2}, (-1)^{\rho}A_{N,3}, \dots)$; $a_k = 0$ when $k > n$ and $A_k = 0$ when $k > N$.
The positive integer T corresponds to a downwards displacement from the top of each of the elements $A_{N,k}$, $(0 \leq k \leq N)$, such that the element $A_{N,k}$ is shifted downwards from row $k+1$ to row $k+T+1$. (When $T = 0$ in the set of determinants under consideration, the subscript T may be dropped for clarity, so that $\vec{\mathbf{A}}_{\mathbf{N},\mathbf{0}} = \vec{\mathbf{A}}_{\mathbf{N}}$.)

Altering the labelling of H_k to $A_{N,k-1}$ has an impact on Lemma 9.1.2 that we observe in Lemma 9.1.3.

LEMMA 9.1.3. *We have for $r \geq 0$ that*

$$
\Psi_r^0(\vec{\mathbf{a}}_{\mathbf{n}}, \vec{\mathbf{A}}_{\mathbf{N}}) = \sum_{k=0}^{r} A_{N,r-k}\Delta_k^0(\vec{\mathbf{a}}_{\mathbf{n}}), \tag{9.1.4}
$$

where $\Delta_r^0(\vec{\mathbf{a}}_{\mathbf{n}})$ and $\Psi_r^0(\vec{\mathbf{a}}_{\mathbf{n}}, \vec{\mathbf{A}}_{\mathbf{N}})$ are defined as in Definition 9.1.2.

Proof. Using Lemma 9.1.2 we replace H_k with $A_{N,k-1}$ and define
$\Psi_0^{\rho}(\vec{\mathbf{a}}_{\mathbf{n}}, \vec{\mathbf{A}}_{\mathbf{N}}) = A_{N,0}$ (as opposed to $\Psi_1(\vec{\mathbf{h}}, \vec{\mathbf{H}}) = -H_1$). $\qquad\square$

COROLLARY 1. *If the vector $\vec{\mathbf{A}}_{\mathbf{N}} = (1, 0, 0, 0, \dots)$, then we have*

$$
\Psi_r^0(\vec{\mathbf{a}}_{\mathbf{n}}, \vec{\mathbf{A}}_{\mathbf{N}}) = \Delta_r^0(\vec{\mathbf{a}}_{\mathbf{n}}).
$$

Proof. We put $A_{N,0} = 1$, and for $r \geq 2$ we put $A_{N,r} = 0$, into (9.1.4). $\qquad\square$

Remark. We note that in the particular case $\rho = 0$, $n = N = r$ and $T = 0$, we have

$$
\Psi_r^0(\vec{\mathbf{a}}_{\mathbf{r}}, \vec{\mathbf{A}}_{\mathbf{r}}) = -\Psi_{r+1}(\vec{\mathbf{h}}, \vec{\mathbf{H}}).
$$

9.2 Relationship between the generating function and the recurrence polynomial.

We start by examining the relationship between the generating function of a function P^{ρ} (as defined in Definition 9.2.1) and its recurrence relation polynomial.

Definition 9.2.1. *We denote by P^ρ, a function that takes the values $P^\rho(r,T,N,n)$, where r, T, N and n are non-negative integers, and it has generating function $\mathcal{G}P^\rho$ given by*

$$\mathcal{G}P^\rho(x,T,N,n) = \frac{\sum_{k=0}^N (-1)^{\rho k} A_{N,k} x^{k+T}}{\sum_{k=0}^n (-1)^{\rho k} a_{n,k} x^k} = \sum_{k=0}^\infty P^\rho(k,T,N,n) x^k.$$

Here the parameter $\rho \in \{0,1\}$ is a sign indicator, whilst the variable r is the term number, T is the "shift" value, n is the order of the denominator polynomial and N the order of the numerator. Moreover, the coefficients $a_{n,k} \neq 0$ $(0 \le k \le n)$ and $A_{N,k} \neq 0$ $(0 \le k \le N)$ are determined by n and N respectively.

From this generating function, we now associate the function P^ρ to a recurrence relation polynomial using Theorems 9.2.1 and 9.2.2.

THEOREM 9.2.1. *If P^ρ is a function given as in Definition 9.2.1 with $P^\rho(0,0,0,n) = 1$ and*

$$\mathcal{G}P^\rho(x,0,0,n) = \frac{1}{\sum_{k=0}^n (-1)^{\rho k} a_{n,k} x^k}, \tag{9.2.1}$$

then for $r \ge 1$, P^ρ obeys the recurrence relation

$$P^\rho(r,0,0,n) = -\sum_{k=0}^{r-1} (-1)^{\rho(r-k)} a_{n,r-k} P^\rho(k,0,0,n), \tag{9.2.2}$$

where for $r > n$ we have $a_{n,r} = 0$.
Conversely, if P^ρ satisfies the recurrence relation given in (9.2.2), then the generating function is given by (9.2.1).

Proof. Since for the theorem we have that N, n and T are constants, for clarity we write $P^\rho(r) = P^\rho(r,0,0,n)$. Then

$$\mathcal{G}P^\rho(x,0,0,n) = \frac{1}{\sum_{k=0}^n (-1)^{\rho k} a_{n,k} x^k} = P^\rho(0) + P^\rho(1)x + P^\rho(2)x^2 + \dots,$$

and on multiplication of both sides by $\sum_{k=0}^n (-1)^{\rho k} a_{n,k} x^k$, and comparing the coefficients of x^r we obtain the system of equations

$$1 = a_{n,0} P^\rho(0)$$
$$0 = (-1)^\rho a_{n,1} P^\rho(0) + a_{n,0} P^\rho(1)$$
$$0 = a_{n,2} P^\rho(0) + (-1)^\rho a_{n,1} P^\rho(1) + a_{n,0} P^\rho(2)$$
$$\vdots$$
$$0 = (-1)^{\rho r} a_{n,r} P^\rho(0) + (-1)^{\rho(r-1)} a_{n,r-1} P^\rho(1) + \dots + a_{n,0} P^\rho(r). \tag{9.2.3}$$

On rearrangement of (9.2.3) and putting $a_{n,0} = 1$ we have for $1 \le r \le n$,

$$P^\rho(r) = -\sum_{k=0}^{r-1} (-1)^{\rho(r-k)} a_{n,r-k} P^\rho(k). \tag{9.2.4}$$

If $r > n$, on letting $r = n + h$, (9.2.3) becomes

$$0 = 0P^\rho(0) + 0P^\rho(1) + \ldots + (-1)^{\rho n}a_{n,n}P^\rho(h) + (-1)^{\rho(n-1)}a_{n,n-1}P^\rho(h+1)$$
$$+ \ldots + a_{n,0}P^\rho(h+n), \qquad (9.2.5)$$

and on rearrangement of (9.2.5) we once more obtain (9.2.4), where $a_{n,r-k} = 0$ when $r-k > n$.

Conversely, suppose that the function P^ρ satisfies (9.2.2) for all $r \geq 1$. Then on rearrangement of (9.2.4) we obtain the system of $r + 1$ equations (where $r \to \infty$) as in (9.2.3). Multiplication of the first equation by x^0, the second by x^1 and the $(r + 1)$-th equation by x^r produces

$$1 = a_{n,0}P^\rho(0)$$
$$0 = (-1)^\rho a_{n,1}P^\rho(0)x + a_{n,0}P^\rho(1)x$$
$$0 = a_{n,2}P^\rho(0)x^2 + (-1)^\rho a_{n,1}P^\rho(1)x^2 + a_{n,0}P^\rho(2)x^2$$
$$\vdots$$
$$0 = (-1)^{\rho n}a_{n,n}P^\rho(0)x^n + (-1)^{\rho(n-1)}a_{n,n-1}P^\rho(1)x^n + \ldots + a_{n,0}P^\rho(n)x^n$$
$$\vdots$$
$$0 = (-1)^{\rho r}a_{n,r}P^\rho(0)x^r + (-1)^{\rho(r-1)}a_{n,r-1}P^\rho(1)x^r + \ldots + (-1)^{\rho n}a_{n,n}P^\rho(r-n)x^r$$
$$+ \ldots + a_{n,0}P^\rho(r)x^r.$$

Then summing both sides with the right hand side being summed with respect to the second parameter of the constant $a_{n,k}$ for $0 \leq k \leq n$ across the (top left to bottom right) diagonals we have

$$1 = a_{n,0}\left(P^\rho(0) + P^\rho(1)x + P^\rho(2)x^2 + \ldots\right)$$
$$+ (-1)^\rho a_{n,1}x\left(P^\rho(0) + P^\rho(1)x + P^\rho(2)x^2 + \ldots\right)$$
$$+ a_{n,2}x^2\left(P^\rho(0) + P^\rho(1)x + P^\rho(2)x^2 + \ldots\right)$$
$$\vdots$$
$$+ (-1)^{\rho n}a_{n,n}x^n\left(P^\rho(0) + P^\rho(1)x + P^\rho(2)x^2 + \ldots\right),$$
$$(9.2.6)$$

and on factorising this gives us

$$1 = \left(a_{n,0} + (-1)^\rho a_{n,1}x + a_{n,2}x^2 + \ldots + (-1)^{\rho n}a_{n,n}x^n\right) \times$$
$$\left(P^\rho(0) + P^\rho(1)x + P^\rho(2)x^2 + \ldots\right), \qquad (9.2.7)$$

so that on division by $\sum_{k=0}^{n}(-1)^{\rho k}a_{n,k}x^k$ on both sides of (9.2.7) we obtain

$$\frac{1}{\sum_{k=0}^{n}(-1)^{\rho k}a_{n,k}x^k} = P^\rho(0) + P^\rho(1)x + P^\rho(2)x^2 + \ldots.$$

This forces each coefficient of x^r, $(r \geq 1)$ in the numerator to be 0, and hence the result. $\square$

COROLLARY 1. *Let P^ρ be a function defined as in Definition 9.2.1 with $P^\rho(j, j, 0, n) = 1$ and $P^\rho(i, j, 0, n) = 0$ if $i < j$, and so $A_{0,0} = 1$. Then if*

$$\mathcal{G}P^\rho(x, j, 0, n) = \frac{x^j}{\sum_{k=0}^{n}(-1)^{\rho k}a_{n,k}x^k},$$ (9.2.8)

then for $r \geq j + 1$, P^ρ obeys the recurrence relation

$$P^\rho(r, j, 0, n) = -\sum_{k=j}^{r-1}(-1)^{r-k}a_{n,r-k}P^\rho(k, j, 0, n),$$ (9.2.9)

where $a_{n,R} = 0$ when $R > n$. Conversely, if P^ρ satisfies the recurrence relation given in (9.2.9), then the generating function is given by (9.2.8).

Proof. We have

$$x^j\frac{1}{\sum_{k=0}^{n}(-1)^{\rho k}a_{n,k}x^k} = x^j\mathcal{G}P^\rho(x, 0, 0, n)$$

$$= x^j\left(P^\rho(0, 0, 0, n) + P^\rho(1, 0, 0, n)x^1 + P^\rho(2, 0, 0, n)x^2 + \ldots\right)$$

$$= P^\rho(j, j, 0, n)x^j + P^\rho(j + 1, j, 0, n)x^{j+1} + P^\rho(j + 2, j, 0, n)x^{j+2} + \ldots.$$

The recurrence relation (9.2.9) then follows from Theorem 9.2.1 on comparsion of the co-efficient of x^{r+j} (as opposed to x^r). Similarly the converse follows from Theorem 9.2.1 on respectively replacing "multiplication by x^0, x^1, x^2, ..." by "multiplication by x^j, x^{j+1}, x^{j+2}, ...". □

Remark. We could still begin with $r = 1, 2, 3, \ldots$, rather than j and obtain the same result, as the extra terms simply correspond to coefficients 0 in the recurrence.

COROLLARY 2. *We have for $\mathbf{r \geq N + 1}$,*

$$P^\rho(r, 0, N, n) = -\sum_{k=0}^{r-1}(-1)^{\rho(r-k)}a_{n,r-k}P^\rho(k, 0, N, n),$$ (9.2.10)

where $a_{n,r-k} = 0$ if $r - k > n$.

Proof. For brevity we write $P_N^\rho(r) = P^\rho(r, 0, N, n)$, (using the subscript N to avoid confusion with $P^\rho(r) = P^\rho(r, 0, 0, n)$ used in Theorem 9.2.1.
When $N \geq 1$, (9.2.3) becomes

$$A_{N,0} = a_{n,0}P_N^\rho(0)$$

$$(-1)^\rho A_{N,1} = (-1)^\rho a_{n,1}P_N^\rho(0) + a_{n,0}P_N^\rho(1)$$

$$\vdots$$

$$(-1)^{\rho N}A_{N,N} = (-1)^{\rho N}a_{n,N}P_N^\rho(0) + (-1)^{\rho(N-1)}a_{n,N-1}P_N^\rho(1) + \ldots + a_{n,0}P_N^\rho(N)$$

$$0 = (-1)^{\rho(N+1)}a_{n,N+1}P_N^\rho(0) + (-1)^{\rho N}a_{n,N}P_N^\rho(1) + \ldots + a_{n,0}P_N^\rho(N+1)$$

$$\vdots$$

$$0 = (-1)^{\rho r}a_{n,r}P_N^\rho(0) + (-1)^{\rho(r-1)}a_{n,r-1}P_N^\rho(1) + \ldots + a_{n,0}P_N^\rho(r).$$

And so we have that

$$P_N^\rho(0) = A_0$$
$$P_N^\rho(1) = (-1)^\rho A_{N,1} - (-1)^\rho a_{n,1} P_N^\rho(0)$$
$$P_N^\rho(2) = A_{N,2} - a_{n,2} P_N^\rho(0) - (-1)^\rho a_{n,1} P_N^\rho(1)$$
$$\vdots$$
$$P_N^\rho(r) = (-1)^{\rho r} A_{N,r} - \sum_{k=0}^{r-1} (-1)^{\rho(r-k)} a_{n,r-k} P_N^\rho(k). \qquad (9.2.11)$$

That is (9.2.10) will only be satisfied when $r \geq \mathbf{N}+\mathbf{1}$ (when $A_{N,r} = 0$). $\qquad \square$

Remark. Corollary 2 could be alternatively stated as:

The function P^ρ with $P^\rho(0,0,N,n) = A_{N,0}$, (and so $a_{n,0} = 1$) will satisfy (9.2.10) only when $r \geq N+1$.

COROLLARY 3. *We have for $r \leq N$,*

$$P_N^\rho(r,0,N,n) = (-1)^{\rho r} A_{N,r} - \sum_{k=0}^{r-1} (-1)^{\rho(r-k)} a_{n,r-k} P_N^\rho(k,0,N,n).$$

Proof. This follows from Corollary 2, in which $P_N^\rho(r,0,N,n)$ is given by (9.2.11). $\qquad \square$

THEOREM 9.2.2. *We have*

$$P^\rho(r,0,N,n) = \sum_{k=0}^{r} (-1)^{\rho(r-k)} A_{N,r-k} P^\rho(k,0,0,n).$$

Here P^ρ is given as in Definition 9.2.1 via the generating function

$$\mathcal{G}P^\rho(x,0,N,n) = \frac{\sum_{k=0}^{N} (-1)^{\rho k} A_{N,k} x^k}{\sum_{k=0}^{n} (-1)^{\rho k} a_{n,k} x^k}, \quad (n \geq N),$$

where n is the order of the denominator and N is the order of the numerator.

Proof. From the generating function of P^ρ when $N \geq 1$ we have that

$$\mathcal{G}P^\rho(x,0,N,n) = \frac{\sum_{k=0}^{N} (-1)^{\rho k} A_{N,k} x^k}{\sum_{k=0}^{n} (-1)^{\rho k} a_{n,k} x^k}$$
$$= \sum_{k=0}^{N} (-1)^{\rho k} A_{N,k} x^k \mathcal{G}P^\rho(x,0,0,n)$$
$$= \sum_{k=0}^{N} (-1)^{\rho k} A_{N,k} x^k \left(P^\rho(0,0,0,n) + P^\rho(1,0,0,n)x + P^\rho(2,0,0,n)x^2 + \ldots \right).$$

If $r \leq N \leq n$, then we find that on equating the coefficient of x^r we have

$$P^\rho(r,0,N,n) = A_{N,0}P^\rho(r,0,0,n) + (-1)^\rho A_{N,1}P^\rho(r-1,0,0,n) + \ldots + (-1)^{\rho r}A_{N,r}P^\rho(0,0,0,n)$$

$$= \sum_{k=0}^{r}(-1)^{\rho(r-k)}A_{N,r-k}P^\rho(k,0,0,n).$$

Furthermore, if $r > N$, then the equation becomes

$$P^\rho(r,0,N,n) = A_{N,0}P^\rho(r,0,0,n) + (-1)^\rho A_{N,1}P^\rho(r-1,0,0,n) + \ldots$$

$$+ (-1)^{\rho N}A_{N,N}P^\rho(r-N,0,0,n) + 0P^\rho(r-N-1,0,0,n) + \ldots + 0P^\rho(0,0,0,n)$$

$$= \sum_{k=0}^{r}(-1)^{\rho(r-k)}A_{N,r-k}P^\rho(k,0,0,n). \tag{9.2.12}$$

where we recall that $A_{N,r-k} = 0$ when $r - k > N$. $\qquad\square$

Remark. We note that (9.2.12) can be expressed more succinctly as

$$P^\rho(r,0,N,n) = \sum_{k=0}^{N}(-1)^{\rho k}A_{N,k}P^\rho(r-k,0,0,n),$$

and if we put $N = 0$, we observe that the sum $P^\rho(r,0,N,n)$ simplifies to $P^\rho(r,0,0,n)$.

9.2.1 Relationship between $\rho = 0$ and $\rho = 1$ cases.

We relate the sum $P^1(r,0,0,n)$ to $P^0(r,0,0,n)$, (as given by Definition 9.2.1) by Lemma 9.2.3.

LEMMA 9.2.3. *With $P^1(0,0,0,n) = P^0(0,0,0,n) = 1$, and thereafter for all $r \geq 1$, we have*

$$P^1(r,0,0,n) = (-1)^r P^0(r,0,0,n).$$

Proof. The identity is true for $k = 0$, Then assuming the relationship holds true for $k \leq r$, we use Theorem 9.2.1 to obtain

$$P^1(r+1,0,0,n) = -\sum_{k=0}^{r}(-1)^{r+1-k}a_{n,r+1-k}P^1(k,0,0,n)$$

$$= -\sum_{k=0}^{r}(-1)^{r+1-k}a_{n,r+1-k}(-1)^k P^0(k,0,0,n)$$

$$= (-1)^{r+1}\left(-\sum_{k=0}^{r}a_{n,r+1-k}P^0(k,0,0,n)\right)$$

$$= (-1)^{r+1}P^0(r+1,0,0,n).$$

$\qquad\square$

Similarly we relate the sum $P^1(r,0,N,n)$ to $P^0(r,0,N,n)$, (as given by Definition 9.2.1).

LEMMA 9.2.4. *For $r \geq 0$ we have*

$$P^1(r, 0, N, n) = (-1)^r P^0(r, 0, N, n).$$

Proof. From Theorem 9.2.2 and Lemma 9.2.3, for all non-negative r we have

$$
\begin{aligned}
P^1(r, 0, N, n) &= \sum_{k=0}^{r} (-1)^{r-k} A_{N,r-k} P^1(k, 0, 0, n) \\
&= \sum_{k=0}^{r} (-1)^{r-k} A_{N,r-k} (-1)^k P^0(k, 0, 0, n) \\
&= (-1)^r \sum_{k=0}^{r} A_{N,r-k} P^0(k, 0, 0, n) \\
&= (-1)^r P^0(r, 0, N, n).
\end{aligned}
$$

$\square$

COROLLARY (to Lemmas 9.2.3 and 9.2.4). *These lemmas demontrate that by alternating the sign of the terms as in Definition 9.1.2, the absolute value of the sequence term r is unaffected, but when the parity of r is 1, the value of the term is multiplied by -1.*

9.2.2   Association of a generating function with an MCL determinant

THEOREM 9.2.5. *We have $P^\rho(0, 0, 0, n) = \Delta_0^\rho(\vec{\mathbf{a}}_\mathbf{n}) = 1$ and for $r \geq 1$ that*

$$P^\rho(r, 0, 0, n) = \Delta_r^\rho(\vec{\mathbf{a}}_\mathbf{n}).$$

Here P^ρ is given as in Definition 9.2.1 via the generating function

$$\mathcal{G}P^\rho(x, 0, 0, n) = \frac{1}{\sum_{k=0}^{n} (-1)^{\rho k} a_{n,k} x^k}, \quad (n \geq N),$$

so that for $r \leq n$,

$$
\Delta_r^\rho(\vec{\mathbf{a}}_\mathbf{n}) = (-1)^r
\begin{vmatrix}
a_{n,1}^\rho & 1 & 0 & 0 & \ldots & 0 \\
a_{n,2} & a_{n,1}^\rho & 1 & 0 & \ldots & 0 \\
a_{n,3}^\rho & a_{n,2} & a_{n,1}^\rho & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n,r-1}^\rho & a_{n,r-2}^\rho & a_{n,r-3}^\rho & a_{n,r-4}^\rho & \ldots & 1 \\
a_{n,r}^\rho & a_{n,r-1}^\rho & a_{n,r-2}^\rho & a_{n,r-3}^\rho & \ldots & a_{n,1}^\rho
\end{vmatrix},
$$

and for $r > n$,

$$\Delta_r^\rho(\vec{\mathbf{a}}_\mathbf{n}) = (-1)^r \begin{vmatrix} a_{n,1}^\rho & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_{n,2} & a_{n,1}^\rho & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_{n,3}^\rho & a_{n,2} & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,n-1}^\rho & a_{n,n-2}^\rho & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 \\ a_{n,n}^\rho & a_{n,n-1}^\rho & \cdots & a_{n,1}^\rho & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & a_{n,2} & a_{n,1}^\rho & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & a_{n,n}^\rho & a_{n,n-1}^\rho & a_{n,n-2}^\rho & \cdots & a_{n,1}^\rho & 1 & 0 \\ 0 & \cdots & 0 & a_{n,n}^\rho & a_{n,n-1}^\rho & \cdots & a_{n,2} & a_{n,1}^\rho & 1 \\ 0 & \cdots & 0 & 0 & a_{n,n}^\rho & \cdots & a_{n,3}^\rho & a_{n,2} & a_{n,1}^\rho \end{vmatrix},$$

where $(\vec{\mathbf{a}}_\mathbf{n} = ((-1)^\rho a_{n,1}, a_{n,2}, \ldots, (-1)^{\rho n} a_{n,n}, 0, 0, 0, \ldots))$.

Proof. From Theorem 9.2.1 we have

$$P^\rho(r, 0, 0, n) = -\sum_{k=0}^{r-1} (-1)^{\rho(r-k)} a_{n,r-k} P^\rho(k, 0, 0, n),$$

and from Lemma 9.1.1,

$$g_r = -\sum_{k=0}^{r-1} h_{r-k} g_k = \Delta_r(\vec{\mathbf{h}}).$$

The theorem is then obtained (from also using Lemma 9.2.3) on putting $g_k = P^\rho(k, 0, 0, n)$, $h_k = (-1)^{\rho k} a_k$ and $\Delta_r(\vec{\mathbf{h}}) = \Delta_r^\rho(\vec{\mathbf{a}}_\mathbf{n})$. $\qquad\square$

THEOREM 9.2.6. *We have $P^\rho(0, 0, N, n) = \Psi_0^\rho(\vec{\mathbf{a}}_\mathbf{n}, \vec{\mathbf{A}}_{\mathbf{N,0}}) = A_0$, and for $r \geq 1$ that*

$$P^\rho(r, 0, N, n) = \Psi_r^\rho(\vec{\mathbf{a}}_\mathbf{n}, \vec{\mathbf{A}}_{\mathbf{N,0}}).$$

Here P^ρ is given as in Definition 9.2.1 via the generating function

$$\mathcal{G}P^\rho(x, 0, N, n) = \frac{\sum_{k=0}^{N}(-1)^{\rho k} A_{N,k} x^k}{\sum_{k=0}^{n}(-1)^{\rho k} a_{n,k} x^k}, \quad (n \geq N), \tag{9.2.13}$$

so that for $r \leq N \leq n$,

$$\Psi_r^\rho(\vec{\mathbf{a}}_\mathbf{n}, \vec{\mathbf{A}}_{\mathbf{N,0}}) = (-1)^r \begin{vmatrix} A_{N,0} & 1 & 0 & 0 & \cdots & 0 \\ A_{N,1}^\rho & a_{n,1}^\rho & 1 & 0 & \cdots & 0 \\ A_{N,2} & a_{n,2} & a_{n,1}^\rho & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{N,r-1}^\rho & a_{n,r-1}^\rho & a_{n,r-2}^\rho & a_{n,r-3}^\rho & \cdots & 1 \\ A_{N,r}^\rho & a_{n,r}^\rho & a_{n,r-1}^\rho & a_{n,r-2}^\rho & \cdots & a_{n,1}^\rho \end{vmatrix},$$

for $N < r \leq n$,

$$\Psi_r^\rho(\vec{\mathbf{a}}_\mathbf{n}, \vec{\mathbf{A}}_{\mathbf{N,0}}) = (-1)^r \begin{vmatrix} A_{N,0} & 1 & 0 & 0 & \dots & 0 \\ A_{N,1}^\rho & a_{n,1}^\rho & 1 & 0 & \dots & 0 \\ A_{N,2} & a_{n,2} & a_{n,1}^\rho & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{N,N}^\rho & a_{n,N}^\rho & a_{n,N-1}^\rho & a_{n,N-2}^\rho & \dots & 0 \\ 0 & a_{n,N+1}^\rho & a_{n,N}^\rho & a_{n,N-1}^\rho & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n,r}^\rho & a_{n,r-1}^\rho & a_{n,r-2}^\rho & \dots & a_{n,1}^\rho \end{vmatrix},$$

and for $r > n$, $\Psi_r^\rho(\vec{\mathbf{a}}_\mathbf{n}, \vec{\mathbf{A}}_{\mathbf{N,0}}) =$

$$(-1)^r \begin{vmatrix} A_{N,0} & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ A_{N,1}^\rho & a_{n,1}^\rho & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ A_{N,2} & a_{n,2} & a_{n,1}^\rho & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ A_{N,3}^\rho & a_{n,3}^\rho & a_{n,2} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ A_{N,N}^\rho & a_{n,N}^\rho & a_{n,N-1}^\rho & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & a_{n,N+1}^\rho & a_{n,N}^\rho & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & a_{n,n}^\rho & a_{n,n-1}^\rho & \dots & a_{n,1}^\rho & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & a_{n,2} & a_{n,1}^\rho & 1 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_{n,n}^\rho & a_{n,n-1}^\rho & a_{n,n-2}^\rho & \dots & \dots & a_{n,1}^\rho & 1 & 0 \\ 0 & 0 & \dots & 0 & a_{n,n}^\rho & a_{n,n-1}^\rho & \dots & \dots & a_{n,2} & a_{n,1}^\rho & 1 \\ 0 & 0 & \dots & 0 & 0 & a_{n,n}^\rho & \dots & \dots & a_{n,3}^\rho & a_{n,2} & a_{n,1}^\rho \end{vmatrix}.$$

Proof. From Theorem 9.2.2 we have

$$P^\rho(r,0,N,n) = \sum_{k=0}^r (-1)^{\rho(r-k)} A_{N,r-k} P^\rho(k,0,0,n),$$

and from Theorem 9.2.5 that

$$P^\rho(r,0,0,n) = \Delta_r^\rho(\vec{\mathbf{a}}_\mathbf{n}), \quad (r \geq 0). \tag{9.2.14}$$

Then using Lemmas 9.1.3 and 9.2.4 we obtain

$$\Psi_r^\rho(\vec{\mathbf{a}}_\mathbf{n}, \vec{\mathbf{A}}_\mathbf{N}) = \sum_{k=0}^r (-1)^{\rho(r-k)} A_{N,r-k} \Delta_k^\rho(\vec{\mathbf{a}}_\mathbf{n}). \tag{9.2.15}$$

On substitution of (9.2.14) into (9.2.15) we conclude that $P^\rho(r,0,N,n) = \Psi_r^\rho(\vec{\mathbf{a}}_\mathbf{n}, \vec{\mathbf{A}}_\mathbf{N})$. $\quad\square$

An important corollary to Theorem 9.2.6 concerns the effect of the "shift" variable T on the half weighted MCL determinant.

COROLLARY 1. *We have*

$$P^\rho(r, T, N - T, n) = \Psi_r^\rho(\vec{\mathbf{a}}_\mathbf{n}, \vec{\mathbf{A}}_{\mathbf{N-T},\mathbf{T}}),$$

with P^ρ defined as in Definition 9.2.1 via the generating function

$$\mathcal{G}P^\rho(x, T, N - T, n) = \frac{\sum_{k=0}^{N-T}(-1)^{\rho k}A_{N-T,k}x^{k+T}}{\sum_{k=0}^{n}(-1)^{\rho k}a_{n,k}x^k}, \ (n \geq N),$$

and the half weighted $(r + 1) \times (r + 1)$ MCL determinant
$\Psi_r^\rho(\vec{\mathbf{a}}_\mathbf{n}, \vec{\mathbf{A}}_{\mathbf{N-T},\mathbf{T}}) = (-1)^r \times$

$$\begin{vmatrix}
0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & a_{n,1}^\rho & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
A_{N-T,0}^\rho & a_{n,T}^\rho & a_{n,T-1}^\rho & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
A_{N-T,N-T}^\rho & a_{n,N}^\rho & a_{n,N-1}^\rho & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & a_{n,N+1}^\rho & a_{n,N}^\rho & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & a_{n,n}^\rho & a_{n,n-1}^\rho & \cdots & a_{n,1}^\rho & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & a_{n,2} & a_{n,1}^\rho & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{n,n}^\rho & a_{n,n-1}^\rho & a_{n,n-2}^\rho & \cdots & \cdots & a_{n,1}^\rho & 1 & 0 \\
0 & 0 & \cdots & 0 & a_{n,n}^\rho & a_{n,n-1}^\rho & \cdots & \cdots & a_{n,2} & a_{n,1}^\rho & 1 \\
0 & 0 & \cdots & 0 & 0 & a_{n,n}^\rho & \cdots & \cdots & a_{n,3}^\rho & a_{n,2} & a_{n,1}^\rho
\end{vmatrix},$$

where $\vec{\mathbf{a}}_\mathbf{n} = ((-1)^\rho a_{n,1}, a_{n,2}, \ldots, (-1)^{\rho n}a_{n,n}, 0, 0, \ldots)$,
and $\vec{\mathbf{A}}_{\mathbf{N-T},\mathbf{T}} = (\underbrace{0, 0, \ldots, 0}_{T \ times}, A_{N-T,0}, (-1)^\rho A_{N-T,1}, \ldots, (-1)^{\rho(N-T)}A_{N-T,N-T}, 0, \ldots).$

Proof. Using (9.2.13) of Theorem 9.2.6 we put $A_{N,k} = 0$ for $k \leq T - 1$. The numerator then becomes

$$\sum_{k=T}^{N}(-1)^{\rho k}A_{N,k}x^k = \sum_{k=0}^{N-T}(-1)^{\rho k}A_{N,k+T}x^{k+T}. \tag{9.2.16}$$

So on renumbering the coefficients $A_{N,k+T}$ of (9.2.16) we then obtain

$$\frac{\sum_{k=0}^{N-T}(-1)^{\rho k}A_{N-T,k}x^{k+T}}{\sum_{k=0}^{n}(-1)^{\rho k}a_{n,k}x^k} = \mathcal{G}P^\rho(x, T, N - T, n).$$

The result then follows from Theorem 9.2.6. $\qquad\square$

Of particular interest to Corollary 1 is the case $T = N$. We then have

$$P^\rho(r, T, 0, n) = \Psi_r^\rho(\vec{\mathbf{a}}_\mathbf{n}, \vec{\mathbf{A}}_{\mathbf{0},\mathbf{T}})$$

$$= (-1)^r \begin{vmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & a_{n,1}^\rho & 1 & 0 & \ldots & 0 \\ 0 & a_{n,2} & a_{n,1}^\rho & 1 & \ldots & 0 \\ 0 & a_{n,3}^\rho & a_{n,2} & a_{n,1}^\rho & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{n,T}^\rho & a_{n,T-1}^\rho & a_{n,T-2}^\rho & a_{n,T-3}^\rho & \cdots \\ 0 & a_{n,T+1}^\rho & a_{n,T}^\rho & a_{n,T-1}^\rho & a_{n,T-2}^\rho & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{vmatrix},$$

where $\vec{\mathbf{a}}_{\mathbf{n}} = ((-1)^\rho a_{n,1}, a_{n,2}, \ldots, (-1)^{\rho n} a_{n,n}, 0, 0, 0, \ldots)$, and $\vec{\mathbf{A}}_{\mathbf{0,T}} = (\underbrace{0, 0, \ldots, 0}_{\text{T times}}, 1, 0, 0, 0, \ldots)$.

9.3 $\quad \mathcal{L}_{s;abc}(r, t, q)$ as a half weighted MCL determinant

Lemmas 9.1.1 and 9.1.3 were employed in [30] to express, as a type 2 MCL determinant, the specific functions (that we denote as) $\mathcal{L}_{1;11c}$, (where $c \in \{0, 1\}$).

From the results obtained in this chapter, we now express each of the sixteen forms of the function $\mathcal{L}_{s;abc}$ as a type 2 MCL determinant.

In Chapter 5 we determined an order $M' = m + b(1 - a)$ recurrence relation for the function $\mathcal{L}_{s;abc}$ of the form

$$x^{M'} - \gamma a_1 x^{M'-1} + \ldots + (-\gamma)^{M'-1} a_{M'-1} x + (-\gamma)^{M'} a_{M'} = 0,$$

where we recall $\gamma = (-1)^s$ or equivalently $-\gamma = (-1)^{1-s}$.

Let us put $a_{n,k} = a_k$ and $A_{N,k} = A_k$. In Chapter 8 on application of Theorem 8.3.7 and Theorem 8.5.6, we were able to express the generating function of $\mathcal{L}_{s;abc}$ in the form

$$\mathcal{GL}_{s;ab0}(x, 0, q) = \frac{2\gamma \sum_{k=0}^{N} (-\gamma)^k A_{N,k} x^k}{\sum_{k=0}^{n} (-\gamma)^k a_{n,k} x^k}, \qquad t = 0, \ N = M' - 1, \qquad (9.3.1)$$

or

$$\mathcal{GL}_{s;abc}(x, t, q) = \frac{x^{t-1} \sum_{k=0}^{N} (-\gamma)^k A_{N,k} x^k}{\sum_{k=0}^{n} (-\gamma)^k a_{n,k} x^k}, \qquad t \geq 1, \ N = M' - t. \qquad (9.3.2)$$

We recall that the polynomials in both numerator and denominator are reciprocal polynomials so that the coefficient of x^k ($0 \leq k \leq n$) becomes $a_{n,k}$.

Now we associate the function $\mathcal{L}_{s;abc}$ with a function L_{abc}^ρ as given in Definition 9.3.1, (see also Definition 9.2.1), where the parameters a, b and c determine the coefficients $a_{n,k}$ and $A_{N,k}$.

On separation of the cases for the parameter $a = 1$ and $a = 0$, we then express each of the functions $\mathcal{L}_{s;abc}$ as a type 2 MCL determinant via the corresponding function L_{abc}^{1-s} using Theorems 9.3.1 and 9.3.2 respectively.

Definition 9.3.1. *Let L_{abc}^{ρ} be a function taking the values $L_{abc}^{\rho}(r, T, N-T, n)$ with generating function*

$$\mathcal{G}L_{abc}^{\rho}(x, T, N-T, n) = \frac{\sum_{k=0}^{N-T}(-1)^{\rho k}A_{N-T,k}x^{k+T}}{\sum_{k=0}^{n}(-1)^{\rho k}a_{n,k}x^{k}} = \sum_{k=0}^{\infty}L_{abc}^{\rho}(k, T, N-T, n)x^{k}.$$

where the coefficients $a_{n,k}$ and $A_{N-T,k}$ are determined by the parameter a according to the criteria:

When $a = 0$, let $\qquad a_{n,k} = J_{n-1,k} + 4J_{n-1,k-1}, \qquad$ and $\qquad A_{N-T,k} = j_{N-T,k},$

($J_{n-1,k-1} = 0$, if $k = 0$), and

when $a = 1$, let $\qquad a_{n,k} = j_{n,k}, \qquad$ and $\qquad A_{N-T,k} = J_{N-T,k}.$

Here $j_{n,k}$ is the k-th coefficient of the Jacobsthal-Lucas polynomial of order n and $J_{N-T,k}$ is the k-th coefficient of the Jacobsthal polynomial of order $N-T$, (see Section 5.2). Furthermore, the variables n and N are determined in part by the parameters b and c.

THEOREM 9.3.1. *We have that*

$$\mathcal{L}_{s;1bc}(r, t, q) = L_{1bc}^{1-s}(r, T, M_1 - T, m) = K\Psi_r^{1-s}(\vec{\mathbf{a}}_{\mathbf{m}}, \vec{\mathbf{A}}_{\mathbf{M_1-T,T}}).$$

Here, $\Psi_r^{1-s}(\vec{\mathbf{a}}_{\mathbf{m}}, \vec{\mathbf{A}}_{\mathbf{M_1-T,T}}) = (-1)^r \times$

$$\begin{vmatrix}
0 & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & a_{m,1}^{1-s} & 1 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ldots & 0 & 0 & \ldots & 0 & 0 \\
A_{M_1-T,0}^{1-s} & a_{m,T}^{1-s} & a_{m,T-1}^{1-s} & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
A_{M_1-T,M_1-T}^{1-s} & a_{m,M_1}^{1-s} & a_{m,M_1-1}^{1-s} & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & a_{m,M_1+1}^{1-s} & a_{m,M_1}^{1-s} & \ldots & 1 & 0 & \ldots & 0 & 0 \\
0 & a_{m,m}^{1-s} & a_{m,m-1}^{1-s} & \ldots & a_{m,1}^{1-s} & 1 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & a_{m,2} & a_{m,1}^{1-s} & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \ldots & a_{m,m-1}^{1-s} & a_{m,m-2}^{1-s} & \ldots & 1 & 0 \\
0 & 0 & \ldots & \ldots & a_{m,m}^{1-s} & a_{m,m-1}^{1-s} & \ldots & a_{m,1}^{1-s} & 1 \\
0 & 0 & \ldots & \ldots & 0 & a_{m,m}^{1-s} & \ldots & a_{m,2} & a_{m,1}^{1-s}
\end{vmatrix},$$

where $\vec{\mathbf{a}}_{\mathbf{m}} = (-\gamma a_{m,1}, a_{m,2}, \ldots, (-\gamma)^m a_{m,m}, 0, 0, 0, \ldots)$,
and $\vec{\mathbf{A}}_{\mathbf{M_1-T,T}} = (\underbrace{0, 0, \ldots, 0}_{T \text{ times}}, A_{M-1,0}, -\gamma A_{M_1-T,1}, \ldots, (-\gamma)^{M_1-T}A_{M_1-T,M_1-T}, 0, 0, 0, \ldots)$.

Furthermore,

$$K = \begin{cases} 2\gamma & \text{if } t = 0 \\ \gamma^c & \text{if } t \geq 1; \end{cases} \qquad T = \begin{cases} t & \text{if } t = 0 \\ t-1 & \text{if } t \geq 1, \end{cases} \qquad \text{and} \qquad M_1 = \begin{cases} m-1 & \text{if } t = 0 \\ m-1-b'c' & \text{if } t \geq 1, \end{cases}$$

where $b' = 1 - b$ and $c' = 1 - c$.

Moreover, the term $a_{m,k}$ represents the coefficient of the term x^k of the $(2m+b)$-th Jacobsthal-Lucas polynomial, and similarly the term $A_{M_1-T,k}$, represents the coefficient of the term x^k of a Jacobsthal polynomial according to

$$\sum_{k=0}^{M_1-T} (-\gamma)^k A_{M_1-T,k} = \begin{cases} 2\gamma J_{2(m-1)+1+b}(-\gamma x) & \text{if } c = t = 0 \\ J_{2(m-1-t)+2}(-\gamma x) & \text{if } b = c = 0 \text{ and } t \geq 1 \\ \gamma^c J_{2(m-t)+b+c}(-\gamma x) & \text{otherwise.} \end{cases}$$

Proof. We first isolate the particular sum $\mathcal{L}_{s;1b0}(r,0,q)$ before considering the other cases (when $t \geq 1$).

For the sum $\mathcal{L}_{s;1b0}(r,0,q)$ we recall from Theorem 8.3.7 that

$$\begin{aligned}
\mathcal{GL}_{s;1b0}(r,0,q) &= \frac{2\gamma J_{2m+b-1}(-\gamma x)}{j_{2m+b}(-\gamma x)} \\
&= \frac{2\gamma \sum_{k=0}^{m-1}(-\gamma)^k \binom{2(m-1)+b-k}{k} x^k}{\sum_{k=0}^m (-\gamma)^k \frac{2m+b}{2m+b-k}\binom{2m+b-k}{k} x^k} \\
&= \frac{2\gamma \sum_{k=0}^{m-1}(-\gamma)^k A_{m-1,k} x^k}{\sum_{k=0}^m (-\gamma)^k a_{m,k} x^k}.
\end{aligned}$$

For each of the other sums, we have from Theorem 8.3.7 that

$$\begin{aligned}
\mathcal{GL}_{s;1bc}(r,t,q) &= \frac{\gamma^c x^{t-1} J_{2(m-t)+b+c}(-\gamma x)}{j_{2m+b}(-\gamma x)} \\
&= \frac{\gamma^c \sum_{k=0}^{m-t-b'c'}(-\gamma)^k \binom{2(m-t)+b+c-1-k}{k} x^{k+t-1}}{\sum_{k=0}^m (-\gamma)^k \frac{2m+b}{2m+b-k}\binom{2m+b-k}{k} x^k} \\
&= \frac{\gamma^c \sum_{k=0}^{m-t-b'c'}(-\gamma)^k A_{m-t-b'c',k} x^{k+t-1}}{\sum_{k=0}^m (-\gamma)^k a_{m,k} x^k}.
\end{aligned}$$

From Definition 9.3.1 we have

$$a_{m,k} = \frac{2m+b}{2m+b-k}\binom{2m+b-k}{k}, \quad \text{and} \quad A_{m-1,k} = \begin{cases} \binom{2(m-1)+b-k}{k} & \text{if } t = 0 \\ \binom{2(m-t)+b+c-1-k}{k} & \text{if } t \geq 1. \end{cases}$$

So with $t = 0$, we have $T = 0$, and $M_1 - T = m - 1$, and from Theorem 9.2.6

$$\mathcal{L}_{s;1b0}(r,0,q) = 2\gamma L_{1bc}^{1-s}(r,0,m-1,m) = 2\gamma \Psi_k^{1-s}(\vec{\mathbf{a}}_m, \vec{\mathbf{A}}_{m-1,0}).$$

On putting $K = 2\gamma$ the identity (for $t = 0$) is established.

When $t \geq 1$, we have $T = t - 1$, and $M_1 - T = m - 1 - b'c' - (t-1) = m - t - b'c'$. So that from Theorem 9.2.6 and Corollary 1 to this theorem we have when $b = c = 0$,

$$\mathcal{L}_{s;100}(r,t,q) = L_{1bc}^{1-s}(r,t-1,m-t-1,m) = \Psi_r^{1-s}(\vec{\mathbf{a}}_m, \vec{\mathbf{A}}_{m-t-1,t-1}),$$

and otherwise,

$$\mathcal{L}_{s;1bc}(r,t,q) = \gamma^c L_{1bc}^{1-s}(r,t-1,m-t,m) = \gamma^c \Psi_r^{1-s}(\vec{\mathbf{a}}_m, \vec{\mathbf{A}}_{m-t,t-1}).$$

On putting $K = \gamma^c$ the identity (for $t \geq 1$) is established.

$\square$

THEOREM 9.3.2. *We have that*

$$\mathcal{L}_{s;0bc}(r,t,q) = L_{0bc}^{1-s}(r,T,M_0-T,m+b) = K\Psi_r^{1-s}(\vec{\mathbf{a}}_{m+b}, \vec{\mathbf{A}}_{M_0-T,T}).$$

Here, $\Psi_r^{1-s}(\vec{\mathbf{a}}_{m+b}, \vec{\mathbf{A}}_{M_0-T,T}) = (-1)^r \times$

$$\begin{vmatrix}
0 & 1 & 0 & \dots & 0 & \dots & 0 & 0 \\
0 & a_{m+b,1}^{1-s} & 1 & \dots & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \dots & 0 & \dots & 0 & 0 \\
A_{M_0-T,0} & a_{m+b,T}^{1-s} & a_{m+b,T-1}^{1-s} & \dots & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
A_{M_0-T,M_0-T}^{1-s} & a_{m+b,M_0}^{1-s} & a_{m+b,M_0-1}^{1-s} & \dots & 0 & \dots & 0 & 0 \\
0 & a_{m+b,M_0+1}^{1-s} & a_{m+b,M_0}^{1-s} & \dots & 0 & \dots & 0 & 0 \\
0 & a_{m+b,m+b}^{1-s} & a_{m+b,m+b-1}^{1-s} & \dots & 1 & \dots & 0 & 0 \\
0 & 0 & a_{m+b,m+b}^{1-s} & \dots & a_{m+b,1}^{1-s} & \dots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & \dots & a_{m+b,m+b-2}^{1-s} & \dots & 1 & 0 \\
0 & 0 & \dots & \dots & a_{m+b,m+b-1}^{1-s} & \dots & a_{m+b,1}^{1-s} & 1 \\
0 & 0 & \dots & \dots & a_{m+b,m+b}^{1-s} & \dots & a_{m+b,2} & a_{m+b,1}^{1-s}
\end{vmatrix},$$

where $\vec{\mathbf{a}}_{m+b} = (-\gamma a_{m+b,1}, a_{m+b,2}, \dots, (-\gamma)^{m+b} a_{m+b,m+b}, 0, 0, 0, \dots),$
and $\vec{\mathbf{A}}_{M_0-T,T} = (\underbrace{0,0,\dots,0}_{T \ times}, A_{M_0-1,0}, -\gamma A_{M_0-T,1}, \dots, (-\gamma)^{M_0-T} A_{M_0-T,M_0-T}, 0,0,0,\dots).$
Furthermore,

$$K = \begin{cases} 2\gamma & \text{if } t = 0 \\ \gamma^c & \text{if } t \ge 1; \end{cases} \qquad T = \begin{cases} t & \text{if } t = 0 \\ t-1 & \text{if } t \ge 1, \end{cases} \qquad \text{and} \qquad M_0 = \begin{cases} m-1+b & \text{if } t = 0 \\ m-1+bc & \text{if } t \ge 1. \end{cases}$$

Moreover, the term $a_{m+b,k}$ *represents the coefficient of the term* x^k *created from the product of the* $(2m+b)$*-th Jacobsthal polynomial with the factor* $(1-4\gamma x)$*. Similarly the term* $A_{M_0-T,k}$*, represents the coefficient of the term* x^k *of a Jacobsthal-Lucas polynomial according to*

$$\sum_{k=0}^{M_0-T} (-\gamma)^k A_{M_0-T,k} = \begin{cases} 2\gamma j_{2(m-1+b)+1-b}(-\gamma x) & \text{if } c = t = 0 \\ \gamma j_{2(m-t+1)}(-\gamma x) & \text{if } b = c = 1 \text{ and } t \ge 1 \\ \gamma^c j_{2(m-t)+b+c}(-\gamma x) & \text{otherwise.} \end{cases}$$

Proof. We first isolate the particular sum $\mathcal{L}_{s;0b0}(r,0,q)$ before considering the other cases (when $t \ge 1$).
For the sum $\mathcal{L}_{s;0b0}(r,0,q)$ we recall from Theorem 8.5.6 that

$$\begin{aligned}
\mathcal{GL}_{s;0b0}(r,0,q) &= \frac{2\gamma j_{2(m-1+b)+1-b}(-\gamma x)}{(1-4\gamma x) J_{2(m-1+b)+2-b}(-\gamma x)} \\
&= \frac{2\gamma \sum_{k=0}^{m-1+b}(-\gamma)^k \frac{2m-1+b}{2m-1+b-k}\binom{2(m-1+b)+1-b-k}{k} x^k}{(1-4\gamma x) \sum_{k=0}^{m-1+b}(-\gamma)^k \binom{2(m-1+b)+1-b-k}{k} x^k} \\
&= \frac{2\gamma \sum_{k=0}^{m-1+b}(-\gamma)^k A_{m-1+b,k} x^k}{\sum_{k=0}^{m+b}(-\gamma)^k a_{m+b,k} x^k}.
\end{aligned}$$

For each of the other sums, we have from Theorem 8.5.6 that

$$\mathcal{GL}_{s;0bc}(r,t,q) = \frac{\gamma^c j_{2(m-t+bc)+b+c-2bc}(-\gamma x)}{(1-4\gamma x)J_{2(m-1+b)+2-b}(-\gamma x)}$$

$$= \frac{\gamma^c \sum_{k=0}^{m-t+bc}(-\gamma)^k \frac{2(m-t)+b+c}{2(m-t)+b+c-k}\binom{2(m-t+bc)+b+c-2bc-k}{k}x^k}{(1-4\gamma x)\sum_{k=0}^{m-1+b}(-\gamma)^k \binom{2(m-1+b)+1-b-k}{k}x^k}$$

$$= \frac{\gamma^c \sum_{k=0}^{m-t+bc}(-\gamma)^k A_{m-t+bc,k}x^k}{\sum_{k=0}^{m+b}(-\gamma)^k a_{m+b,k}x^k}.$$

From Definition 9.3.1 we have

$$a_{m+b,k} = \binom{2m+b-1-k}{k} + 4\binom{2m+b-k}{k-1}, \tag{9.3.3}$$

and

$$A_{m-1,k} = \begin{cases} \frac{2m-1+b}{2m-1+b-k}\binom{2m-1+b-k}{k} & \text{if } t = 0 \\ \frac{2(m-t)+b+c}{2(m-t)+b+c-k}\binom{2(m-t)+b+c-k}{k} & \text{if } t \geq 1. \end{cases}$$

So with $t = 0$, we have $T = 0$, and $M_0 - T = m - 1 + b$. and from Theorem 9.2.6 we have that

$$\mathcal{L}_{s;0b0}(r,0,q) = 2\gamma L_{abc}^{1-s}(r,0,m-1+b,m+b) = 2\gamma\Psi_r^{1-s}(\vec{\mathbf{a}}_{m+b}, \vec{\mathbf{A}}_{m-1+b,0}).$$

On putting $K = 2\gamma$ (when $t = 0$) the identity is established.

When $t \geq 1$, we have $T = t - 1$, and $M_0 - T = m - 1 + bc - (t-1) = m - t + bc$.
So that from Theorem 9.2.6 and Corollary 1 to this theorem we have when $b = c = 1$,

$$\mathcal{L}_{s;011}(r,t,q) = \gamma L_{abc}^{1-s}(r,t-1,m-t+1,m+1) = \gamma\Psi_r^{1-s}(\vec{\mathbf{a}}_{m+1}, \vec{\mathbf{A}}_{m-t+1,t-1}),$$

and otherwise,

$$\mathcal{L}_{s;0bc}(r,t,q) = \gamma^c L_{abc}^{1-s}(r,t-1,m-t,m+b) = \gamma^c\Psi_r^{1-s}(\vec{\mathbf{a}}_{m+b}, \vec{\mathbf{A}}_{m-t,t-1}).$$

On putting $K = \gamma^c$ (when $t \geq 1$) the identity is established. $\qquad\square$

We illustrate Theorem 9.3.1 and Theorem 9.3.2 with two examples.

Example 1.

$$\mathcal{L}_{1;110}(6,0,7) = (-1)^{6+1}\begin{vmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 8 & 7 & 1 & 0 & 0 & 0 & 0 \\ 6 & 14 & 7 & 1 & 0 & 0 & 0 \\ 0 & 7 & 14 & 7 & 1 & 0 & 0 \\ 0 & 0 & 7 & 14 & 7 & 1 & 0 \\ 0 & 0 & 0 & 7 & 14 & 7 & 1 \\ 0 & 0 & 0 & 0 & 7 & 14 & 7 \end{vmatrix} = -3430,$$

$$\mathcal{L}_{1;110}(6,1,7) = (-1)^6 \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 7 & 1 & 0 & 0 & 0 & 0 \\ 1 & 14 & 7 & 1 & 0 & 0 & 0 \\ 0 & 7 & 14 & 7 & 1 & 0 & 0 \\ 0 & 0 & 7 & 14 & 7 & 1 & 0 \\ 0 & 0 & 0 & 7 & 14 & 7 & 1 \\ 0 & 0 & 0 & 0 & 7 & 14 & 7 \end{vmatrix} = 2989,$$

$$\mathcal{L}_{1;110}(6,2,7) = (-1)^6 \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 7 & 1 & 0 & 0 & 0 & 0 \\ 1 & 14 & 7 & 1 & 0 & 0 & 0 \\ 0 & 7 & 14 & 7 & 1 & 0 & 0 \\ 0 & 0 & 7 & 14 & 7 & 1 & 0 \\ 0 & 0 & 0 & 7 & 14 & 7 & 1 \\ 0 & 0 & 0 & 0 & 7 & 14 & 7 \end{vmatrix} = -1911,$$

$$\mathcal{L}_{1;110}(6,3,7) = (-1)^6 \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 & 0 & 0 \\ 1 & 14 & 7 & 1 & 0 & 0 & 0 \\ 0 & 7 & 14 & 7 & 1 & 0 & 0 \\ 0 & 0 & 7 & 14 & 7 & 1 & 0 \\ 0 & 0 & 0 & 7 & 14 & 7 & 1 \\ 0 & 0 & 0 & 0 & 7 & 14 & 7 \end{vmatrix} = 637.$$

Example 2.

$$\mathcal{L}_{0;001}(6,1,6) = (-1)^6 \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -5 & -8 & 1 & 0 & 0 & 0 & 0 \\ 5 & 19 & -8 & 1 & 0 & 0 & 0 \\ 0 & -12 & 19 & -8 & 1 & 0 & 0 \\ 0 & 0 & -12 & 19 & -8 & 1 & 0 \\ 0 & 0 & 0 & -12 & 19 & 7 & 1 \\ 0 & 0 & 0 & 0 & -12 & 19 & -8 \end{vmatrix} = 1730,$$

$$\mathcal{L}_{0;001}(6,2,6) = (-1)^6 \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -8 & 1 & 0 & 0 & 0 & 0 \\ -3 & 19 & -8 & 1 & 0 & 0 & 0 \\ 0 & -12 & 19 & -8 & 1 & 0 & 0 \\ 0 & 0 & -12 & 19 & -8 & 1 & 0 \\ 0 & 0 & 0 & -12 & 19 & 7 & 1 \\ 0 & 0 & 0 & 0 & -12 & 19 & -8 \end{vmatrix} = 1365,$$

$$\mathcal{L}_{0;001}(6,3,6) = (-1)^6 \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -8 & 1 & 0 & 0 & 0 & 0 \\ 1 & 19 & -8 & 1 & 0 & 0 & 0 \\ 0 & -12 & 19 & -8 & 1 & 0 & 0 \\ 0 & 0 & -12 & 19 & -8 & 1 & 0 \\ 0 & 0 & 0 & -12 & 19 & 7 & 1 \\ 0 & 0 & 0 & 0 & -12 & 19 & -8 \end{vmatrix} = 1001.$$

9.4 Expression of the sums $\mathcal{L}_{s;abc}(r, t, 2m + b)$ for generalised m

In Section 3.1, (and also see Appendix A), each of the sums $\mathcal{L}_{s;abc}(r, t, q)$ were expressed in terms of sums of binomial coefficients. When

$$r + t + 1 - c < q = 2m + b, \tag{9.4.1}$$

the sum is composed of the single binomial coefficient

$$\binom{2r + 2 - c}{r + t + 1 - c}, \tag{9.4.2}$$

and is, therefore, independent of the modulus q (and the parameters a and b).

In Theorems 9.3.2 and 9.3.2 we expressed the values of $\mathcal{L}_{s;abc}(r, t, q)$ using an MCL determinant. If the value of m remains unspecified, then each of the nonzero entries ($\neq 1$) becomes a polynomial in m, and we might expect that the determinant also yields a polynomial in m. However, we have the following lemma.

LEMMA 9.4.1. *The MCL determinant of the terms $\mathcal{L}_{s;abc}(r, t, q)$ when the variable m remains unspecified is an integer value.*

Proof. When m is not specified we find that the condition is equivalent to (9.4.1) and so (9.4.2) is also obtained. Consequently the result of determining the MCL determinant in m is the numerical value (9.4.2): all terms of the final polynomial cancelling except for the constant terms. □

We demonstrate Lemma 9.4.1 with the following example.

Example.

$\mathcal{L}_{1;110}(4, 1, q) =$

$$(-1)^4 \times \begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 2m - 3 & 2m + 1 & 1 & 0 & 0 \\ \frac{1}{2!}(2m-4)^{\underline{2}} & \frac{(2m+1)}{2!}(2m-2) & 2m+1 & 1 & 0 \\ \frac{1}{3!}(2m-5)^{\underline{3}} & \frac{(2m+1)}{3!}(2m-3)^{\underline{2}} & \frac{(2m+1)}{2!}(2m-2) & 2m+1 & 1 \\ \frac{1}{4!}(2m-6)^{\underline{4}} & \frac{(2m+1)}{4!}(2m-4)^{\underline{3}} & \frac{(2m+1)}{3!}(2m-3)^{\underline{2}} & \frac{(2m+1)}{2!}(2m-2) & 2m+1 \end{vmatrix}$$

$$= \binom{2 \times 4 + 2}{4 + 1 + 1} = \binom{10}{6} = 210.$$

9.4.1 The functions $\mathcal{L}_{s;abc}$ for negative r

Contrariwise, if we consider the sums $\mathcal{L}_{s;abc}(-r, t, q)$, that is if we run the sequences backwards, their evaluations using sums of binomial coefficients no longer make sense and a term independent of m such as (9.4.2) does not exist. Instead we turn to methods such as the roots

of unity and trigonometric sums employed in Chapter 4. In consequence to this dependence on the variable m, we find that when determining the values of $\mathcal{L}_{s;abc}(-r, t, q)$ for unspecified m, there is no wholesale cancellation of the polynomial in m.

In a study by Lettington [30] these polynomials were investigated for each of the sequences of the sums $\mathcal{L}_{1;11c}(-r, t, 2m+1)$, $(c \in \{0, 1\})$. Interesting results were discovered for the leading coefficient of the particular cases of the variable $t = 1$ and $t = m$. More specifically when $t = 1$ (and with m replaced with π) this leading coefficient of the term x^k was identified to be equal to the even zeta function, $\zeta(2k)$, when $c = 1$, and $\zeta(2k)/2$ when $c = 0$.

We now broaden these findings to include the sixteen cases produced by varying the parameters a, b, c and s, when $t = m$ and $t = 1$. We determine the generating function of the terms $\mathcal{L}_{s;abc}(-r, t, q)$ and from them obtain the required polynomial from the MCL determinant. Since it is the leading coefficient of this polynomial that is of particular interest, we construct an amended function that produces precisely this, and in doing so immensely simplifies each of the individual polynomial entries of this determinant.

Let us introduce some definitions.

Definition 9.4.1. *We denote by $\mathcal{L}_{s;abc}^{-}$ the function such that for integer $r \geq 0$ it produces the values*

$$\mathcal{L}_{s;abc}^{-}(r, t, q) = \mathcal{L}_{s;abc}(-r, t, q).$$

Definition 9.4.2. *We denote by $\mathcal{L}_{s;abc}^{T-}$ the function such that*

$$\mathcal{L}_{s;abc}^{T-}(r, t, q) = \text{leading coefficient of } \mathcal{L}_{s;abc}^{-}(r, t, q).$$

Definition 9.4.3. *Let the polynomial*

$$p_n(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x + a_0,$$

and

$$x^T p_n(x) = a_n x^{n+T} + a_{n-1} x^{n+T-1} + a_{n-2} x^{n+T-2} + \ldots + a_1 x^{T+1} + a_0 x^T. \tag{9.4.3}$$

Then let $N = n + T$ and relabel the coefficients such that

$$A_N = \begin{cases} a_n & \text{if } n \geq 0 \\ 0 & \text{if } N \leq T - 1, \end{cases}$$

so that (9.4.3) becomes

$$P_N(x) = x^T p_n(x) = A_N x^N + A_{N-1} x^{N-1} + \ldots + A_{T+1} x^{T+1} + A_T x^T + 0 x^{T-1} + \ldots + 0x + 0.$$

LEMMA 9.4.2. *The reciprocal polynomial $p_n^*(x)$ of the polynomial $x^T p_n(x)$ as defined in Definition 9.4.3 is given by*

$$p_n^*(x) = x^n p_n(x^{-1}) = a_n + a_{n-1} x + a_{n-2} x^2 + \ldots + a_1 x^{n-1} + a_0 x^n.$$

Proof. From Definition 9.4.3 we have

$$P_N(x) = x^T p_n(x) = A_N x^N + A_{N-1} x^{N-1} + \ldots + A_{T+1} x^{T+1} + A_T x^T + 0 x^{T-1} + \ldots + 0x + 0.$$

The reciprocal polynomial $P_N^*(x)$ of the polynomial $P_N(x)$ is given by

$$
\begin{aligned}
P_N^*(x) &= x^N P_N(x^{-1}) \\
&= A_N + A_{N-1} x + \ldots + A_T x^{N-T} + 0 x^{N-T+1} + \ldots + 0 x^{N-1} + 0 x^N \\
&= a_n + a_{n-1} x + a_{n-2} x^2 + \ldots + a_1 x^{n-1} + a_0 x^n \\
&= p_n^*(x)
\end{aligned}
$$

as required. $\square$

Remark. When the reciprocal of a polynomial $x^T p_n(x)$ is determined the power of x^T is effectively removed.

We now consider separately the generating functions of the terms $\mathcal{L}_{s;1bc}^-$ and $\mathcal{L}_{s;0bc}^-$.

THEOREM 9.4.3. *The generating function of the function $\mathcal{L}_{s;1bc}^-$, as defined in Definition 9.4.1, is given by*

$$
\mathcal{GL}_{s;1bc}^-(x,t,q) =
\begin{cases}
\dfrac{-2 F_{2(m-1)+1+b}(\sqrt{-\gamma x})}{L_{2m+b}(\sqrt{-\gamma x})} & \text{if } t = 0 \\[2ex]
\dfrac{(-\gamma)^{t+1} F_{2(m-1-t)+2}(\sqrt{-\gamma x})}{\sqrt{-\gamma x} L_{2m}\sqrt{-\gamma x}} & \text{if } b = c = 0 \text{ and } t \geq 1 \\[2ex]
\dfrac{\gamma^c (-\gamma)^t (\sqrt{-\gamma x})^{bc} F_{2(m-t)+b+c}(\sqrt{-\gamma x})}{L_{2m+b}(\sqrt{-\gamma x})} & \text{otherwise}
\end{cases}
$$

$$
=
\begin{cases}
\dfrac{-2 \sum_{k=0}^{m-1} \binom{(m-1)+b+k}{2k+b}(-\gamma x)^k}{\sum_{k=0}^{m} \frac{2m}{m+b+k} \binom{m+b+k}{2k+b}(-\gamma x)^k} & \text{if } t = 0 \\[3ex]
\dfrac{(-\gamma)^{t+1} \sum_{k=0}^{m-1-t} \binom{(m-1-t)+1+k}{2k+1}(-\gamma x)^k}{\sum_{k=0}^{m} \frac{2m}{m+k} \binom{m+k}{2k}(-\gamma x)^k} & \text{if } b = c = 0 \text{ and } t \geq 1 \qquad (9.4.4) \\[3ex]
\dfrac{\gamma^c (-\gamma)^t \sum_{k=0}^{m-t} \binom{(m-t)+b+c-1+k}{2k+bc}(-\gamma x)^k}{\sum_{k=0}^{m} \frac{2m+b}{m+b+k} \binom{m+b+k}{2k+b}(-\gamma x)^k} & \text{otherwise.}
\end{cases}
$$

Proof. From Theorem 8.3.7 we have

$$
\mathcal{GL}_{s;1bc}(x,t,q) =
\begin{cases}
\dfrac{2\gamma J_{2(m-1)+1+b}(-\gamma x)}{j_{2m+b}(-\gamma x)} & \text{if } t = 0 \\[2ex]
\dfrac{x^{t-1} J_{2(m-1-t)+2}(-\gamma x)}{j_{2m}(-\gamma x)} & \text{if } b = c = 0 \text{ and } t \geq 1 \qquad (9.4.5) \\[2ex]
\dfrac{\gamma^c x^{t-1} J_{2(m-t)+b+c}(-\gamma x)}{j_{2m+b}(-\gamma x)} & \text{otherwise.}
\end{cases}
$$

Let us generalise the generating function of (9.4.5) as

$$
\begin{aligned}
\mathcal{GL}_{s;1bc}(x,t,q) &= \frac{K x^T J_{2n+e}(-\gamma x)}{j_{2m+b}(-\gamma x)} \\
&= \frac{K x^T \sum_{k=0}^{n}(-\gamma)^k \binom{2n+e-1-k}{k} x^k}{\sum_{k=0}^{m}(-\gamma)^k \frac{2m+b}{2m+b-k} \binom{2m+b-k}{k} x^k},
\end{aligned}
$$

where K is a constant and the parameter $e \in \{1, 2\}$.

Now since

$$\mathcal{L}_{s;abc}^-(r, t, q) = \mathcal{L}_{s;abc}(-r, t, q), \qquad (r \geq 0),$$

we determine the reciprocal polynomial of the numerator and denominator of (9.4.5).

From Lemma 9.4.2 we recall that the reciprocal polynomial $p_n^*(x)$ of $x^T p_n(x)$ is

$$p_n^*(x) = np_n(x^{-1}) = a_n + a_{n-1}x + a_{n-2}x^2 + \ldots + a_1 x^{n-1} + a_0 x^n.$$

We then have that $\mathcal{GL}_{s;1bc}^-$ is given as

$$
\begin{aligned}
\mathcal{GL}_{s;1bc}^-(x, t, q) &= \frac{K \sum_{k=0}^n (-\gamma)^k \binom{2n+e-1-k}{k+e-1} x^{n-k}}{\sum_{k=0}^m (-\gamma)^k \frac{2m+b}{2m+b-k} \binom{2m+b-k}{k} x^{m-k}} \\
&= \frac{K \sum_{k=0}^n (-\gamma)^{n-k} \binom{n+e-1+k}{2k+e-1} x^k}{\sum_{k=0}^m (-\gamma)^{m-k} \frac{2m+b}{m+b+k} \binom{m+b+k}{2k+b} x^k} \\
&= \frac{(-\gamma)^n K \sum_{k=0}^n (-\gamma)^k \binom{n+e-1+k}{2k+e-1} x^k}{(-\gamma)^m \sum_{k=0}^m (-\gamma)^{m-k} \frac{2m+b}{m+b+k} \binom{m+b+k}{2k+b} x^k} \\
&= (-\gamma)^{n-m} \frac{K(\sqrt{-\gamma x})^d F_{2n+e}\sqrt{-\gamma x}}{L_{2m+b}\sqrt{-\gamma x}}, \qquad d \in \{-1, 0, 1\}.
\end{aligned}
$$

So if $t = 0$, we put $K = 2\gamma$, $n = m - 1$, $e = 1 + b$ and $d = 0$;

whilst if $t \geq 1$, and $b = c = 0$, we put $K = 1$, $n = m - 1 - t$, $e = 2$ and $d = -1$,

and in all other cases we put $K = \gamma^c$, $n = m - t$, $e = b + c$ and $d = bc$, we then obtain (9.4.4) and the theorem follows. □

Finally, we have two corollaries to give the explicit forms for each of the parameters b and c, when $a = 1$ and the variables $t = m$ and $t = 1$.

COROLLARY 1. *When the variable $t = m$, we have*

$$\mathcal{GL}_{s;100}^-(x, m, q) = \frac{F_0(\sqrt{-\gamma x})}{\sqrt{-\gamma x}L_{2m}(\sqrt{-\gamma x})} = 0,$$

$$\mathcal{GL}_{s;101}^-(x, m, q) = \frac{\gamma(-\gamma)^m F_1(\sqrt{-\gamma x})}{L_{2m}(\sqrt{-\gamma x})} = \frac{\gamma(-\gamma)^m}{\sum_{k=0}^m \frac{2m}{m+k}\binom{m+k}{2k}(-\gamma x)^k},$$

$$\mathcal{GL}_{s;110}^-(x, m, q) = \frac{(-\gamma)^m \sqrt{-\gamma x} F_1(\sqrt{-\gamma x})}{L_{2m+1}(\sqrt{-\gamma x})} = \frac{(-\gamma)^m}{\sum_{k=0}^m \frac{2m+1}{m+k+1}\binom{m+k+1}{2k+1}(-\gamma x)^k},$$

and

$$\mathcal{GL}_{s;111}^-(x, m, q) = \frac{\gamma(-\gamma)^m F_2(\sqrt{-\gamma x})}{L_{2m+1}(\sqrt{-\gamma x})} = \frac{\gamma(-\gamma)^m}{\sum_{k=0}^m \frac{2m+1}{m+k+1}\binom{m+k+1}{2k+1}(-\gamma x)^k}.$$

Proof. Each of the results follow from substitution of the parameters b and c and the variable $t = m$ into Theorem 9.4.3. □

COROLLARY 2. *When the variable $t = 1$, we have*

$$\mathcal{GL}_{s;100}^{-}(x,1,q) = \frac{F_{2(m-2)+2}(\sqrt{-\gamma x})}{\sqrt{-\gamma x}L_{2m}(\sqrt{-\gamma x})} = \frac{\sum_{k=0}^{m-2}\binom{(m-2)+1+k}{2k+1}(-\gamma x)^k}{\sum_{k=0}^{m}\frac{2m}{m+k}\binom{m+k}{2k}(-\gamma x)^k},$$

$$\mathcal{GL}_{s;101}^{-}(x,1,q) = \frac{-F_{2(m-1)+1}(\sqrt{-\gamma x})}{L_{2m}(\sqrt{-\gamma x})} = \frac{-\sum_{k=0}^{m-1}\binom{m-1+k}{2k}(-\gamma x)^k}{\sum_{k=0}^{m}\frac{2m}{m+k}\binom{m+k}{2k}(-\gamma x)^k},$$

$$\mathcal{GL}_{s;110}^{-}(x,1,q) = \frac{-\gamma\sqrt{-\gamma x}F_{2(m-1)+1}(\sqrt{-\gamma x})}{L_{2m+1}(\sqrt{-\gamma x})} = \frac{-\gamma\sum_{k=0}^{m-1}\binom{m-1+k}{2k}(-\gamma x)^k}{\sum_{k=0}^{m}\frac{2m+1}{m+k+1}\binom{m+k+1}{2k+1}(-\gamma x)^k},$$

and

$$\mathcal{GL}_{s;111}^{-}(x,1,q) = \frac{-F_{2(m-1)+2}(\sqrt{-\gamma x})}{L_{2m+1}(\sqrt{-\gamma x})} = \frac{-\sum_{k=0}^{m-1}\binom{(m-1)+1+k}{2k+1}(-\gamma x)^k}{\sum_{k=0}^{m}\frac{2m+1}{m+k+1}\binom{m+k+1}{2k+1}(-\gamma x)^k}.$$

Proof. Each of the results follow from substitution of the parameters b and c and the variable $t = 1$ into Theorem 9.4.3. $\qquad\square$

THEOREM 9.4.4. *The generating function of the function $\mathcal{L}_{s;0bc}^{-}$ as defined as in Definition 9.4.1 is given by*

$$\mathcal{GL}_{s;0bc}^{-}(x,t,q) = \begin{cases} \frac{2\gamma L_{2(m-1+b)+1-b}(\sqrt{-\gamma x})}{(x-4\gamma)F_{2(m-1+b)+2-b}(\sqrt{-\gamma x})} & \text{if } t = 0 \\ \frac{\gamma(-\gamma)^{t-1}L_{2(m+1-t)}(\sqrt{-\gamma x})}{(x-4\gamma)F_{2m+1}\sqrt{-\gamma x})} & \text{if } b = c = 1 \text{ and } t \geq 1 \\ \frac{\gamma^c(-\gamma)^{t+1-b}(\sqrt{-\gamma x})^{1-2b-c}L_{2(m-t)+b+c}(\sqrt{-\gamma x})}{(x-4\gamma)F_{2(m-1+b)+2-b}(\sqrt{-\gamma x})} & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{2\gamma\sum_{k=0}^{m-1+b}\frac{2(m-1+b)+1-b}{(m-1+b)+1-b+k}\binom{(m-1+b)+1-b+k}{2k+1-b}(-\gamma x)^k}{(x-4\gamma)\sum_{k=0}^{m-1+b}\binom{(m-1+b)+1-b+k}{2k+1-b}(-\gamma x)^k} & \text{if } t = 0 \\[2em] \frac{\gamma(-\gamma)^{t-1}\sum_{k=0}^{m+1-t}\frac{2(m+1-t)}{(m+1-t)+k}\binom{(m+1-t)+k}{2k}(-\gamma x)^k}{(x-4\gamma)\sum_{k=0}^{m}\binom{(m+k)}{2k}(-\gamma x)^k} & \text{if } b = c = 1 \text{ and } t \geq 1 \\[2em] \frac{\gamma^c(-\gamma)^{t+1-b}\sum_{k=0}^{m-t}\frac{2(m-t)+b+c}{(m-t)+b+c+k}\binom{(m-t)+b+c+k}{2k+b+c}(-\gamma x)^k}{(x-4\gamma)\sum_{k=0}^{m-1+b}\binom{(m-1+b)+1-b+k}{2k+1-b}(-\gamma x)^k} & \text{otherwise.} \end{cases}$$

$$(9.4.6)$$

Proof. From Theorem 8.3.7 we have

$$\mathcal{GL}_{s;0bc}(x,t,q) = \begin{cases} \frac{2\gamma j_{2(m-1+b)+1-b}(-\gamma x)}{(1-4\gamma x)J_{2(m-1+b)+2-b}(-\gamma x)} & \text{if } t = 0 \\ \frac{\gamma x^{t-1}j_{2(m+1-t)}(-\gamma x)}{(1-4\gamma x)J_{2m+1}(-\gamma x)} & \text{if } t \geq 1 \text{ and } b = c = 1 \\ \frac{\gamma^c x^{t-1}j_{2(m-t)+b+c}(-\gamma x)}{(1-4\gamma x)J_{2(m-1+b)+2-b}(-\gamma x)} & \text{otherwise.} \end{cases} \qquad (9.4.7)$$

Let us generalise the generating function of (9.4.7) as

$$\mathcal{GL}_{s;0bc}(x,t,q) = \frac{Kx^T j_{2n+f}(-\gamma x)}{(1-4\gamma x)J_{2M+2-b}(-\gamma x)}$$

$$= \frac{Kx^T\sum_{k=0}^{N}(-\gamma)^k\frac{2n+f}{2n+f-k}\binom{2n+f-k}{k}x^k}{(1-4\gamma x)\sum_{k=0}^{M}(-\gamma)^k\binom{2M+1-b-k}{k}x^k},$$

where K is a constant and the parameter $f \in \{0, 1\}$.

Now since

$$\mathcal{L}_{s;abc}^{-}(r, t, q) = \mathcal{L}_{s;abc}(-r, t, q), \qquad (r \geq 0),$$

we determine the reciprocal polynomial of the numerator and denominator of (9.4.7).

From Lemma 9.4.2 we recall that the reciprocal polynomial $p_n^*(x)$ of $x^T p_n(x)$ is

$$p_n^*(x) = x^n p_n(x^{-1}) = a_n + a_{n-1}x + a_{n-2}x^2 + \ldots + a_1 x^{n-1} + a_0 x^n.$$

We then have that $\mathcal{GL}_{s;0bc}^{-}$ is given as

$$
\begin{aligned}
\mathcal{GL}_{s;0bc}^{-}(x, t, q) &= \frac{K \sum_{k=0}^{n}(-\gamma)^k \frac{2n+f}{2n+f-k}\binom{2n+f-k}{k}x^{n-k}}{(x - 4\gamma)\sum_{k=0}^{M}(-\gamma)^k \binom{2M+1-b-k}{k}x^{M-k}} \\
&= \frac{K \sum_{k=0}^{n}(-\gamma)^{n-k} \frac{2n+f}{n+f+k}\binom{n+f+k}{2k+f}x^k}{(x - 4\gamma)\sum_{k=0}^{M}(-\gamma)^{M-k}\binom{M+1-b+k}{2k+1-b}x^k} \\
&= \frac{(-\gamma)^n K \sum_{k=0}^{n}(-\gamma)^k \frac{2n+f}{n+f+k}\binom{n+f+k}{2k+f}x^k}{(-\gamma)^M (x - 4\gamma)\sum_{k=0}^{M}(-\gamma)^k\binom{M+1-b+k}{2k+1-b}x^k} \\
&= (-\gamma)^{n-M} \frac{K(\sqrt{-\gamma x})^d L_{2n+f}(\sqrt{-\gamma x})}{(x - 4\gamma)F_{2M+2-b}(\sqrt{-\gamma x})}, \qquad d \in \{-1, 0, 1\}.
\end{aligned}
$$

So with $M = m - 1 + b$ and if $t = 0$, we put $K = 2\gamma$, $n = m - 1$, $f = 1 - b$ and $d = 0$; whilst if $t \geq 1$, and $b = c = 1$, we put $K = \gamma$, $n = m - 1 - t$, $f = 0$ and $d = 0$, and in all other cases we put $K = \gamma^c$, $n = m - t$, $f = b + c$ and $d = 1 - 2b - c$, we then obtain (9.4.6) and the result follows. $\qquad\square$

Finally, we have two corollaries to give the explicit forms for each of the parameters b and c when $a = 0$ and the variables $t = m + bc$ and $t = 1$.

COROLLARY 1. *When the variable $t = m$, we have*

$$\mathcal{GL}_{s;000}^{-}(x, m, q) = \frac{(-\gamma)^{m-1}\sqrt{-\gamma x}L_0(\sqrt{-\gamma x})}{((x - 4\gamma)F_{2(m-1)+2}(\sqrt{-\gamma x})} = \frac{2(-\gamma)^{m-1}}{(x - 4\gamma)\sum_{k=0}^{m-1}\binom{(m-1)+1+k}{2k+1}(-\gamma x)^k},$$

$$\mathcal{GL}_{s;001}^{-}(x, m, q) = \frac{\gamma(-\gamma)^{m-1}L_1(\sqrt{-\gamma x})}{(x - 4\gamma)F_{2(m-1)+2}(\sqrt{-\gamma x})} = \frac{\gamma(-\gamma)^{m-1}}{(x - 4\gamma)\sum_{k=0}^{m-1}\binom{(m-1)+1+k}{2k+1}(-\gamma x)^k},$$

$$\mathcal{GL}_{s;010}^{-}(x, m, q) = \frac{(-\gamma)^m L_1(\sqrt{-\gamma x})}{\sqrt{-\gamma x}(x - 4\gamma)F_{2m+1}(\sqrt{-\gamma x})} = \frac{(-\gamma)^m}{(x - 4\gamma)\sum_{k=0}^{m}\binom{m+k}{2k}(-\gamma x)^k},$$

and

$$\mathcal{GL}_{s;011}^{-}(x, m + 1, q) = \frac{\gamma(-\gamma)^m L_0(\sqrt{-\gamma x})}{(x - 4\gamma)F_{2m+1}(\sqrt{-\gamma x})} = \frac{2\gamma(-\gamma)^m}{(x - 4\gamma)\sum_{k=0}^{m}\binom{m+k}{2k}(-\gamma x)^k}.$$

Proof. Each of the results follow from substitutiom of the parameters b and c and the variable $t = m + bc$ into Theorem 9.4.4. $\qquad\square$

COROLLARY 2. *When the variable $t = 1$, we have*

$$\mathcal{GL}_{s;000}^{-}(x,1,q) = \frac{\sqrt{-\gamma x}L_{2(m-1)}(\sqrt{-\gamma x})}{(x-4\gamma)F_{2(m-1)+2}(\sqrt{-\gamma x})} = \frac{\sum_{k=0}^{m-1}\frac{2(m-1)}{m-1+k}\binom{m-1+k}{2k}(-\gamma x)^k}{(x-4\gamma)\sum_{k=0}^{m-1}\binom{(m-1)+1+k}{2k+1}(-\gamma x)^k},$$

$$\mathcal{GL}_{s;001}^{-}(x,1,q) = \frac{\gamma L_{2(m-1)+1}(\sqrt{-\gamma x})}{(x-4\gamma)F_{2(m-1)+2}(\sqrt{-\gamma x})} = \frac{\gamma\sum_{k=0}^{m-1}\frac{2(m-1)+1}{(m-1)+1+k}\binom{(m-1)+1+k}{2k+1}(-\gamma x)^k}{(x-4\gamma)\sum_{k=0}^{m-1}\binom{(m-1)+1+k}{2k+1}(-\gamma x)^k},$$

$$\mathcal{GL}_{s;010}^{-}(x,1,q) = \frac{-\gamma L_{2(m-1)+1}(\sqrt{-\gamma x})}{\sqrt{-\gamma x}(x-4\gamma)F_{2m+1}(\sqrt{-\gamma x})}$$

$$= \frac{-\gamma\sum_{k=0}^{m-1}\frac{2(m-1)+1}{(m-1)+1+k}\binom{(m-1)+1+k}{2k+1}(-\gamma x)^k}{(x-4\gamma)\sum_{k=0}^{m}\binom{m+k}{2k}(-\gamma x)^k},$$

and

$$\mathcal{GL}_{s;011}^{-}(x,1,q) = \frac{\gamma L_{2m}(\sqrt{-\gamma x})}{(x-4\gamma)F_{2m+1}(\sqrt{-\gamma x})} = \frac{\gamma\sum_{k=0}^{m}\frac{2m}{m+k}\binom{m+k}{2k}(-\gamma x)^k}{(x-4\gamma)\sum_{k=0}^{m}\binom{m+k}{2k}(-\gamma x)^k}.$$

Proof. Each of the results follow from substitutiom of the parameters b and c and the variable $t = 1$ into Theorem 9.4.4. $\qquad\square$

9.4.2 Determination of the leading coefficient of the polynomial $\mathcal{L}_{s;abc}^{-}(x,1,q)$

We now truncate the polynomials created in the Corollary to Theorems 9.4.3 and 9.4.4 to obtain the function $\mathcal{L}_{s;abc}^{T-}$ as defined in Definition 9.4.2. To achieve this we identify the leading coefficient(s) of the polynomial in m obtained from the term x^k of both the numerator and denominator of the generating function of $\mathcal{L}_{s;abc}^{-}$. These polynomials derive from the binomial coefficients, in the variables m and k, of either the Fibonacci or Lucas polynomials. We first look at the binomial coefficients from the Fibonacci polynomial and consider the sum

$$\sum_{k=0}^{n}\binom{N+k}{2k+b}x^k.$$

We have the following lemmas.

LEMMA 9.4.5 (leading coefficients 1). *We have*

$$\sum_{k=0}^{n}\binom{N+k}{2k}x^k = \sum_{k=0}^{n}\frac{N^{2k}+kN^{2k-1}+\text{lower degree terms}}{(2k)!}x^k,$$

and

$$\sum_{k=0}^{n}\binom{N+k}{2k+1}x^k = \sum_{k=0}^{n}\frac{N^{2k+1}+0N^{2k}+\text{lower degree terms}}{(2k+1)!}x^k.$$

Proof. For the case $b = 0$,

$$\sum_{k=0}^{n}\binom{N+k}{2k}x^k = 1 + \frac{(N+1)^{\underline{2}}}{2!}x + \frac{(N+2)^{\underline{4}}}{4!}x^2 + \frac{(N+3)^{\underline{6}}}{6!}x^3 + \dots$$

$$= 1 + \frac{(N+1)N}{2!}x + \frac{(N+2)(N+1)N(N-1)}{4!}x^2$$

$$+ \frac{(N+3)(N+2)(N+1)N(N-1)(N-2)}{6!}x^3 + \dots \quad (9.4.8)$$

Truncating (9.4.8) to the first two coefficients we have

$$1 + \frac{N^2+N}{2!}x + \frac{N^4+2N^3}{4!}x^2 + \dots + \frac{N^{2n}+nN^{2n-1}}{(2n)!}x^n = \sum_{k=0}^{n}\frac{N^{2k}+kN^{2k-1}}{(2k)!}x^k.$$

Then for $b = 1$,

$$\sum_{k=0}^{n}\binom{N+k}{2k+1}x^k = N + \frac{(N+1)^{\underline{3}}}{3!}x + \frac{(N+2)^{\underline{5}}}{5!}x^2 + \frac{(N+3)^{\underline{7}}}{7!}x^3 + \dots$$

$$= N + \frac{(N+1)N(N-1)}{3!}x + \frac{(N+2)(N+1)N(N-1)(N-2)}{5!}x^2$$

$$+ \frac{(N+3)(N+2)(N+1)N(N-1)(N-2)(N-3)}{7!}x^3 + \dots \quad (9.4.9)$$

Truncating (9.4.9) to the first two coefficients we have

$$N + \frac{N^3+0N^2}{3!}x + \frac{N^5+0N^4}{5!}x^2 + \dots + \frac{N^{2n+1}+0N^{2n}}{(2n+1)!}x^n + \dots = \sum_{k=0}^{n}\frac{N^{2k+1}}{(2k+1)!}x^k.$$

$\square$

LEMMA 9.4.6 (leading coefficients 2)**.** *We have*

$$(x+4)\sum_{k=0}^{n}\binom{N+k}{2k}x^k = 4\sum_{k=0}^{n}\frac{N^{2k}+kN^{2k-1}+ \textit{lower degree terms}}{(2k)!}x^k, \quad (9.4.10)$$

and

$$(x+4)\sum_{k=0}^{n}\binom{N+k}{2k+1}x^k = 4\sum_{k=0}^{n}\frac{N^{2k+1}+0N^{2k}+ \textit{lower degree terms}}{(2k+1)!}x^k. \quad (9.4.11)$$

Proof. The coefficient of x^k on the left hand side of (9.4.10) and (9.4.11) is given by

$$\binom{N+k-1}{2(k-1)+b} + 4\binom{N+k}{2k+b}. \quad (9.4.12)$$

The leading two coefficients comes solely from the second term of (9.4.12) and so on application of Lemma 9.4.5 we obtain desired result. $\square$

We also require the leading coefficents for the Lucas polynomial and consider the sum

$$\sum_{k=0}^{n} \frac{2M+b}{M+b+k} \binom{M+b+k}{2k+b} x^k,$$

and use the following lemma.

LEMMA 9.4.7 (leading coefficients 3). *We have*

$$\sum_{k=0}^{n} \frac{2M}{M+k} \binom{M+k}{2k} x^k = 2 \sum_{k=0}^{n} \frac{M^{2k} + 0 M^{2k-1}}{(2k)!} x^k,$$

and

$$\sum_{k=0}^{n} \frac{2M+1}{M+1+k} \binom{M+1+k}{2k+1} x^k = (2M+1) \sum_{k=0}^{n} \frac{M^{2k} + k M^{2k-1}}{(2k+1)!} x^k.$$

Proof. When $b = 0$,

$$\frac{2M}{M+k} \sum_{k=0}^{n} \binom{M+k}{2k} x^k = 2 + 2M \frac{M}{2!} x + 2M \frac{(M+1)^{\underline{3}}}{4!} x^2 + 2M \frac{(M+2)^{\underline{5}}}{6!} x^3 + \dots$$

$$= 2 + 2M \frac{M}{2!} x + 2M \frac{(M+1)M(M-1)}{4!} x^2 + 2M \frac{(M+2)(M+1)M(M-1)(M-2)}{6!} x^3 + \dots$$

$$= 2 \left(1 + \frac{M^2}{2!} x + \frac{(M+1)M^2(M-1)}{4!} x^2 + \frac{(M+2)(M+1)M^2(M-1)(M-2)}{6!} x^3 + \dots \right). \tag{9.4.13}$$

Truncating (9.4.13) to the first two coefficients we have

$$2 \left(1 + \frac{M^2}{2!} x + \frac{M^4 + 0 M^3}{4!} x^2 + \dots + \frac{M^{2n} + 0 M^{2n-1}}{(2n)!} x^n \right) = 2 \sum_{k=0}^{n} \frac{M^{2k} x^k}{(2k)!}.$$

Then when $b = 1$,

$$\sum_{k=0}^{n} \frac{2M+1}{M+k+1} \binom{M+k+1}{2k+1}$$

$$= (2M+1) \left(1 + \frac{(M+1)^{\underline{2}}}{3!} x + \frac{(M+2)^{\underline{4}}}{5!} x^2 + \frac{(M+3)^{\underline{6}}}{7!} x^3 + \dots \right)$$

$$= (2M+1) \left(1 + \frac{(M+1)M}{3!} x + \frac{(M+2)(M+1)M(M-1)}{5!} x^2 \right.$$

$$\left. + \frac{(M+3)(M+2)(M+1)M(M-1)(M-2)}{7!} x^3 + \dots \right). \tag{9.4.14}$$

Truncating (9.4.14) to the first two coefficients we have

$$(2M+1) \left(1 + \frac{M^2 + M}{3!} x + \frac{M^4 + 2M^3}{5!} x^2 + \frac{M^6 + 3M^6}{7!} x^3 + \dots + \frac{M^{2n} + n M^{2n-1}}{(2n+1)!} x^n \right)$$

$$= (2M+1) \sum_{k=0}^{n} \frac{M^{2k} + k M^{2k-1}}{(2k+1)!} x^k.$$

$\square$

We now employ lemmas 9.4.5 and 9.4.7 (in Theorem 9.4.8) to determine the leading coefficient of the polynomial in the variable m of the r-th term of the alternating sequences $a = 1$.

THEOREM 9.4.8 (leading coefficient of the alternating $a = 1$ sequences). *The generating function of the sequences,* $\mathcal{L}^{T-}_{s;1bc}(r, t, m)$, *derived from the leading coefficient of the polynomial sequences,* $\mathcal{L}^{-}_{s;1bc}(r, t, m)$, *when $t = m$, and $r \geq 0$, are given by*

$$\mathcal{GL}^{T-}_{s;100}(x, m, q) = 0, \qquad \mathcal{GL}^{T-}_{s;101}(x, m, q) = \frac{\gamma(-\gamma)^m}{2 \sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k)!} x^k},$$

$$\mathcal{GL}^{T-}_{s;11c}(x, m+c, q) = \frac{\gamma^c(-\gamma)^m}{(2m+1) \sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k+1)!} x^k},$$

and when $t = 1$, by

$$\mathcal{GL}^{T-}_{s;100}(x, 1, q) = \frac{\sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k+1}}{(2k+1)!} x^k}{2 \sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k)!} x^k}, \tag{9.4.15}$$

$$\mathcal{GL}^{T-}_{s;101}(x, 1, q) = \frac{-1}{2} \left(1 - \frac{\sum_{k=1}^{r} \frac{(-\gamma)^k k m^{2k-1}}{(2k)!} x^k}{\sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k)!} x^k} \right), \tag{9.4.16}$$

$$\mathcal{GL}^{T-}_{s;110}(x, 1, q) = \frac{-\gamma}{2m+1} \frac{\sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k)!} x^k}{\sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k+1)!} x^k}, \tag{9.4.17}$$

and

$$\mathcal{GL}^{T-}_{s;111}(x, 1, q) = \frac{-m}{2m+1} \left(1 - \frac{\sum_{k=1}^{r} \frac{(-\gamma)^k k m^{2k-1}}{(2k+1)!} x^k}{\sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k+1)!} x^k} \right). \tag{9.4.18}$$

Proof. We employ the generating function of the function $\mathcal{L}^{-}_{s;1bc}$ and choose $r \leq m - \epsilon$, where $\epsilon \in \{0, 1, 2\}$. Then in the case $t = m$, we use Lemma 9.4.7 to identify the leading coefficient, whilst in the case $t = 1$, we use Lemmas 9.4.5 and 9.4.7, to identify the coefficients of the leading two terms of both numerator and denominator. When the factorials of top and bottom are of different parity there is no cancellation of terms and consequently the leading coefficient of both is sufficient to determine $\mathcal{L}^{T-}_{s;1bc}$. Conversely when they are of the same parity, consideration will required to be given to a second coefficient.

For the polynomial sequences of the type $\mathcal{L}^{-}_{s;1bc}(r, m, q)$, $(r \geq 0)$, we have

$$\mathcal{GL}^{-}_{s;100}(x, m, q) = 0, \qquad \mathcal{GL}^{-}_{s;101}(x, m, q) = \frac{\gamma^c(-\gamma)^m}{\sum_{k=0}^{m} \frac{2m}{m+k} \binom{m+k}{2k} (-\gamma x)^k},$$

and

$$\mathcal{GL}^{-}_{s;11c}(x, m, q) = \frac{\gamma^c(-\gamma)^m}{\sum_{k=0}^{m} \frac{2m}{m+k+1} \binom{m+k+1}{2k+1} (-\gamma x)^k}.$$

The first result is complete, and the other two follow from application of Lemma 9.4.7 and replacing x with $-\gamma x$.

For the polynomial sequence, $\mathcal{L}_{s;100}^{-}(r,1,q)$ we have

$$\mathcal{GL}_{s;100}^{-}(x,1,q) = \frac{\sum_{k=0}^{m-2}\left(\binom{m-1+k}{2k+1}\right)(-\gamma x)^k}{\sum_{k=0}^{m}\frac{2m}{m+k}\binom{m+k}{2k}(-\gamma x)^k}.$$

With $0 \le r \le m-2$, and $N = m-1$, by applying Lemmas 9.4.5 and 9.4.7, we select the leading coefficient of both numerator and denominator. The result is (9.4.15).

For $\mathcal{L}_{s;101}^{-}(r,1,q)$ we have

$$\mathcal{GL}_{s;101}^{-}(x,1,q) = \frac{\gamma\sum_{k=0}^{m-1}\binom{m-1+k}{2k}(-\gamma x)^k}{\sum_{k=0}^{m}\frac{2m}{m+k}\binom{m+k}{2k}(-\gamma x)^k}. \tag{9.4.19}$$

With $r \le m-1$, and $N = m-1$, we have from Lemma 9.4.5, that the numerator of (9.4.19) is

$$\sum_{k=0}^{r}\frac{(m-1)^{2k} + k(m-1)^{2k-1} + \text{lower degree terms}}{(2k)!}(-\gamma x)^k$$

$$= \sum_{k=0}^{r}\frac{m^{2k} - 2km^{2k-1} + km^{2k-1} + \text{lower degree terms}}{(2k)!}(-\gamma x)^k$$

$$= \sum_{k=0}^{r}\frac{m^{2k} - km^{2k-1} + \text{lower degree terms}}{(2k)!}(-\gamma x)^k. \tag{9.4.20}$$

From (9.4.20) and Lemma 9.4.7 we then obtain

$$\mathcal{GL}_{s;101}^{T-}(x,1,q) = \frac{\gamma\sum_{k=0}^{r}\frac{(-\gamma)^k\left(m^{2k}-km^{2k-1}\right)}{(2k)!}x^k}{2\sum_{k=0}^{r}\frac{(-\gamma)^k m^{2k}}{(2k)!}x^k} = \frac{\gamma}{2}\left(1 - \frac{\sum_{k=1}^{r}\frac{(-\gamma)^k km^{2k-1}}{(2k)!}x^k}{\sum_{k=0}^{r}\frac{(-\gamma)^k m^{2k}}{(2k)!}x^k}\right)$$

as required.

For $\mathcal{L}_{s;110}^{-}(r,1,q)$ we have

$$\mathcal{GL}_{s;110}^{-}(x,1,q) = \frac{\sum_{k=0}^{m-1}\binom{m-1+k}{2k}(-\gamma x)^k}{\sum_{k=0}^{m}\frac{2m+1}{m+k+1}\binom{m+k+1}{2k+1}(-\gamma x)^k}.$$

With $r \le m-1$, and $N = m-1$, on application of Lemmas 9.4.5 and 9.4.7 we obtain (9.4.17).

Finally, for $\mathcal{L}_{s;111}^{-}(r,1,q)$ we have

$$\mathcal{GL}_{s;111}^{-}(x,1,q) = \frac{\gamma\sum_{k=0}^{m-1}\binom{m-k}{2k+1}(-\gamma x)^k}{\sum_{k=0}^{m}\frac{2m+1}{m+k+1}\binom{m+k+1}{2k+1}(-\gamma x)^k}.$$

With $r \leq m-1$, we have from Lemmas 9.4.5 and 9.4.7,

$$\frac{\gamma \sum_{k=0}^{r} \frac{m^{2k+1}}{(2k+1)!}(-\gamma x)^k}{2m+1 \sum_{k=0}^{r} \frac{m^{2k}+km^{2k-1}}{(2k+1)!}(-\gamma x)^k}$$

$$=\frac{\gamma m}{2m+1} \frac{\sum_{k=0}^{r} \frac{m^{2k}+km^{2k-1}-km^{2k-1}}{(2k+1)!}(-\gamma x)^k}{\sum_{k=0}^{r} \frac{m^{2k}+km^{2k-1}}{(2k+1)!}(-\gamma x)^k}$$

$$=\frac{\gamma m}{2m+1}\left(1-\frac{\sum_{k=0}^{r} \frac{km^{2k-1}}{(2k+1)!}(-\gamma x)^k}{\sum_{k=0}^{r} \frac{m^{2k}+km^{2k-1}}{(2k+1)!}(-\gamma x)^k}\right)$$

$$=\frac{\gamma m}{2m+1}\left(1-\frac{\sum_{k=0}^{r} \frac{km^{2k-1}}{(2k+1)!}(-\gamma x)^k}{\sum_{k=0}^{r} \frac{m^{2k}}{(2k+1)!}(-\gamma x)^k}\right)$$

as required. $\qquad\square$

Next we utilise Lemmas 9.4.6 and 9.4.7 in Theorem 9.4.9 to determine the leading coefficient of the polynomial (in the variable m) of the r-th term of the sequences of the parameter $a = 0$.

THEOREM 9.4.9 (leading coefficient of the nonalternating $a = 0$ sequences). *The generating function of the sequences, $\mathcal{L}_{s;0bc}^{T-}(r, t, m)$, derived from the leading coefficient of the polynomial sequences, $\mathcal{L}_{s;0bc}^{-}(r, t, q)$, when $t = m + bc$, are given by*

$$\mathcal{GL}_{s;00c}^{T-}(x, m, q) = \frac{2^{1-c}\gamma^c(-\gamma)^m}{4m \sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k+1)!} x^k}, \quad and \quad \mathcal{GL}_{s;01c}^{T-}(x, m+c, q) = \frac{2^c \gamma^c(-\gamma)^{m+1}}{4 \sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k)!} x^k},$$

and when $t = 1$, by

$$\mathcal{GL}_{s;000}^{T-}(x, 1, q) = \frac{-\gamma}{2m} \frac{\sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k)!} x^k}{\sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k+1)!} x^k},$$

$$\mathcal{GL}_{s;001}^{T-}(x, 1, q) = \frac{-(2m-1)}{4m}\left(1 - \frac{\sum_{k=1}^{r} \frac{(-\gamma)^k k m^{2k-1}}{(2k+1)!} x^k}{\sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k+1)!} x^k}\right),$$

$$\mathcal{GL}_{s;010}^{T-}(x, 1, q) = \frac{(2m-1)}{4} \frac{\sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k+1)!} x^k}{\sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k)!} x^k},$$

and

$$\mathcal{GL}_{s;011}^{T-}(x, 1, q) = \frac{\gamma}{2}\left(1 - \frac{\sum_{k=1}^{r} \frac{(-\gamma)^k k m^{2k-1}}{(2k)!} x^k}{\sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k)!} x^k}\right).$$

Proof. We use the same approach as in Theorem 9.4.8, but also require Lemma 9.4.6. For the polynomial sequences of the type $\mathcal{L}_{s;0bc}^{-}(r, m+bc, q)$ we have

$$\mathcal{GL}_{s;00c}^{-}(x, m, q) = \frac{2^{1-c}\gamma^c(-\gamma)^{m-1}}{(x-4\gamma) \sum_{k=0}^{m-1} \binom{(m-1)+1+k}{2k+1}(-\gamma x)^k},$$

and

$$\mathcal{GL}^-_{s;01c}(x, m, q) = \frac{2^c \gamma^c (-\gamma)^m}{(x - 4\gamma) \sum_{k=0}^{m-1} \binom{(m-1)+1+k}{2k+1} (-\gamma x)^k}.$$

Both results follow from application of Lemma 9.4.6 on replacing 4 with -4γ, and within the summation replacing x with $-\gamma x$.

For the polynomial sequences type $\mathcal{L}^-_{s;000}(r, 1, q)$ we have

$$\mathcal{GL}^-_{s;000}(x, 1, q) = \frac{\sum_{k=0}^{m-1} \frac{2(m-1)}{m-1+k} \binom{m-1+k}{2k} (-\gamma x)^k}{(x - 4\gamma) \sum_{k=0}^{m-1} \binom{(m-1)+1+k}{2k+1} (-\gamma x)^k}.$$

With $0 \leq r \leq m - 1$ and $M = m - 1$, on application of Lemmas 9.4.6 and 9.4.7 we obtain

$$\mathcal{GL}^{T-}_{s;000}(x, 1, q) = \frac{2 \sum_{k=0}^{r} \frac{(-\gamma)^k (m-1)^{2k}}{(2k)!} x^k}{-4\gamma \sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k+1}}{(2k+1)!} x^k} = \frac{-\gamma \sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k)!} x^k}{2 \sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k+1}}{(2k+1)!} x^k}.$$

For $\mathcal{L}^-_{s;001}(r, 1, q)$ we have

$$\mathcal{GL}^-_{s;001}(x, 1, q) = \frac{\gamma \sum_{k=0}^{m-1} \frac{2(m-1)+1}{(m-1)+1+k} \binom{(m-1)+1+k}{2k+1} (-\gamma x)^k}{(x - 4\gamma) \sum_{k=0}^{m-1} \binom{(m-1)+1+k}{2k+1} (-\gamma x)^k},$$

and so

$$\begin{aligned}
\mathcal{GL}^{T-}_{s;001}(x, 1, q) &= \frac{\gamma(2m-1) \sum_{k=0}^{r} \frac{(m-1)^{2k} + k(m-1)^{2k-1}}{(2k+1)!} (-\gamma x)^k}{-4\gamma \sum_{k=0}^{r} \frac{m^{2k+1}}{(2k+1)!} (-\gamma x)^k} \\
&= \frac{\gamma(2m-1) \sum_{k=0}^{r} \frac{m^{2k} - km^{2k-1}}{(2k+1)!} (-\gamma x)^k}{-4\gamma m \sum_{k=0}^{r} \frac{m^{2k}}{(2k+1)!} (-\gamma x)^k} \\
&= \frac{-(2m-1)}{4m} \left(1 - \frac{\sum_{k=0}^{r} \frac{(-\gamma)^k k m^{2k-1}}{(2k+1)!} x^k}{\sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k+1)!} x^k} \right).
\end{aligned} \qquad (9.4.21)$$

For $\mathcal{L}^-_{s;010}(r, 1, q)$ we have

$$\mathcal{GL}^-_{s;010}(x, 1, q) = \frac{-\gamma \sum_{k=0}^{m-1} \frac{2(m-1)+1}{(m-1)+1+k} \binom{(m-1)+1+k}{2k+1} (-\gamma x)^k}{(x - 4\gamma) \sum_{k=0}^{m} \binom{m+k}{2k} (-\gamma x)^k},$$

and so

$$\mathcal{GL}^{T-}_{s;010}(x, 1, q) = \frac{-\gamma(2m-1) \sum_{k=0}^{r} \frac{(-\gamma)^k (m-1)^{2k}}{(2k+1)!} x^k}{-4\gamma \sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k)!} x^k} = \frac{(2m-1)}{4} \frac{\sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k+1)!} x^k}{\sum_{k=0}^{r} \frac{(-\gamma)^k m^{2k}}{(2k)!} x^k}.$$

Finally for $\mathcal{L}^-_{s;011}(r, 1, q)$ we have

$$\mathcal{GL}^-_{s;011}(x, 1, q) = \frac{\gamma \sum_{k=0}^{m} \frac{2m}{m+k} \binom{m+k}{2k} (-\gamma x)^k}{(x - 4\gamma) \sum_{k=0}^{m} \binom{m+k}{2k} (-\gamma x)^k},$$

and then

$$\mathcal{GL}_{s;011}^{T-}(x,1,q) = \frac{2\gamma \sum_{k=1}^{r} \frac{(-\gamma)^k m^{2k}}{(2k)!} x^k}{-4\gamma \sum_{k=0}^{r} \frac{m^{2k}+km^{2k-1}}{(2k)!}(-\gamma x)^k} = \frac{2\gamma \sum_{k=1}^{r} \frac{(-\gamma)^k m^{2k}}{(2k)!} x^k}{-4\gamma \sum_{k=0}^{r} \frac{m^{2k}+km^{2k-1}}{(2k)!}(-\gamma x)^k}$$

$$= \frac{-1}{2}\left(1 - \frac{\sum_{k=1}^{r} \frac{(-\gamma)^k km^{2k-1}}{(2k)!} x^k}{\sum_{k=0}^{r} \frac{m^{2k}+km^{2k-1}}{(2k)!}(-\gamma x)^k}\right) = \frac{-1}{2}\left(1 - \frac{\sum_{k=1}^{r} \frac{(-\gamma)^k km^{2k-1}}{(2k)!} x^k}{\sum_{k=0}^{r} \frac{m^{2k}}{(2k)!}(-\gamma x)^k}\right).$$

The result follows on applying the same manipulation as employed in (9.4.21). □

9.5 The polynomials $D_e^\rho(r,0,0,n)$ and $D_{de}^\rho(r,T,N,n)$

In the previous section we established the generating function $\mathcal{GL}_{s;abc}^{T-}(x,t,q)$ that we employed to determine, (when q is not specified), the leading coefficient of the terms $\mathcal{L}_{s;abc}^{-}(r,1,q)$ and $\mathcal{L}_{s;abc}^{-}(r,m,q)$. Extending the work of [30] we now wish to relate these coefficients to a known Dirichlet series. We find this easier to achieve by first relating them to an "intermediary polynomial" and then relating the latter to the Dirichlet series. In this manner and replicating the format of Definition 9.2.1 we employ the following definitions.

Definition 9.5.1. *Let us denote by D_e^ρ a function, that for non-negative integers r and n, takes the values $D_e^\rho(r,0,0,n)$, and has generating function*

$$\mathcal{GD}_e^\rho(x,0,0,n) = \frac{1}{\sum_{k=0}^{n} \frac{(-1)^{\rho k}\pi^{2k}}{(2k+e)!} x^k}.$$

Here we have $a_{n,k} = \pi^{2k}/(2k+e)!$ and the parameter e represents the parity of the factorial in the denominator.

Similarly, we denote by D_{de}^ρ, a function that for non-negative integers r, T, N and n takes the values $D_{de}^\rho(r,T,N,n)$, and as has generating function

$$\mathcal{GD}_{de}^\rho(x,\delta,n-\delta,n) = (-1)^\delta \frac{\sum_{k=0}^{n-\delta} \frac{(-1)^{\rho k}k^\delta \pi^{2k}}{(2k+d)!} x^k}{\sum_{k=0}^{n} \frac{(-1)^{\rho k}\pi^{2k}}{(2k+e)!} x^k}. \tag{9.5.1}$$

Here we have $a_{n,k} = \pi^{2k}/(2k+e)!$ and $A_{N,k} = k^\delta \pi^{2k}/(2k+d)!$; the parameters d and e represents the parity of the factorial in the numerator and denominator respectively; $\delta = \delta_{d,e}$, is the Kronecker delta function and $\rho \in \{0,1\}$.

9.5.1 Expression of the function D_e^ρ

We first turn our attention to the function D_e^ρ, and employ two lemmas.

LEMMA 9.5.1 (even parity)**.** *We have*

$$\lim_{n\to\infty} \mathcal{GD}_0^\rho(x,0,0,n) = \begin{cases} \operatorname{sech}(\pi\sqrt{x}) & \text{if } \rho = 0 \\ \sec(\pi\sqrt{x}) & \text{if } \rho = 1. \end{cases} \tag{9.5.2}$$

Proof. We have when $\rho = 0$,

$$\lim_{n \to \infty} \mathcal{G}D_0^0(x, 0, 0, n) = \lim_{n \to \infty} \frac{1}{\sum_{k=0}^{n} \frac{\pi^{2k}}{(2k)!} x^k} = \frac{1}{\sum_{k=0}^{\infty} \frac{\pi^{2k}}{(2k)!} x^k} = \frac{1}{\cosh(\pi\sqrt{x})} = \operatorname{sech}(\pi\sqrt{x}),$$

and $\rho = 1$,

$$\lim_{n \to \infty} \mathcal{G}D_0^1(x, 0, 0, n) = \lim_{n \to \infty} \frac{1}{\sum_{k=0}^{n} \frac{(-1)^k \pi^{2k}}{(2k)!} x^k} = \frac{1}{\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k}}{(2k)!} x^k} = \frac{1}{\cos(\pi\sqrt{x})} = \sec(\pi\sqrt{x}).$$

$\square$

We note that when $\rho = 0$, the expansion of (9.5.2) gives

$$\left(1 - \frac{\pi^2}{2} x + \frac{5\pi^4}{24} x^2 - \frac{61\pi^6}{720} x^3 + \frac{277\pi^8}{8064} x^4 - \frac{50521\pi^{10}}{3628800} x^5 + \cdots \right),$$

and when $\rho = 1$, we have

$$\left(1 + \frac{\pi^2}{2} x + \frac{5\pi^4}{24} x^2 + \frac{61\pi^6}{720} x^3 + \frac{277\pi^8}{8064} x^4 + \frac{50521\pi^{10}}{3628800} x^5 + \cdots \right).$$

LEMMA 9.5.2 (odd parity). *We have*

$$\lim_{n \to \infty} \mathcal{G}D_1^\rho(x, 0, 0, n) = \begin{cases} \pi\sqrt{x}\operatorname{csch}(\pi\sqrt{x}) & \text{if } \rho = 0 \\ \pi\sqrt{x}\csc(\pi\sqrt{x}) & \text{if } \rho = 1. \end{cases} \tag{9.5.3}$$

Proof. We have

$$\lim_{n \to \infty} \mathcal{G}D_1^\rho(x, 0, 0, n) = \lim_{n \to \infty} \frac{1}{\sum_{k=0}^{n} \frac{(-1)^{\rho k} \pi^{2k}}{(2k+1)!} x^k} = \frac{1}{\sum_{k=0}^{\infty} \frac{(-1)^{\rho k} \pi^{2k}}{(2k+1)!} x^k}.$$

On expansion of the denominator when $\rho = 0$, we have

$$1 + \frac{\pi^2}{3!} x + \frac{\pi^4}{5!} x^2 + \frac{\pi^6}{7!} x^3 + \frac{\pi^8}{9!} x^4 + \frac{\pi^{10}}{11!} x^5 + \cdots$$

$$= \frac{1}{\pi\sqrt{x}} \left(\pi\sqrt{x} + \frac{\pi^3}{3!} x^{3/2} + \frac{\pi^5}{5!} x^{5/2} + \frac{\pi^7}{7!} x^{7/2} + \frac{\pi^9}{9!} x^{9/2} + \cdots \right)$$

$$= \frac{1}{\pi\sqrt{x}} \sinh(\pi\sqrt{x}), \tag{9.5.4}$$

and when $\rho = 1$, we have

$$\frac{1}{\pi\sqrt{x}} \left(\pi\sqrt{x} - \frac{\pi^3}{3!} x^{3/2} + \frac{\pi^5}{5!} x^{5/2} - \frac{\pi^7}{7!} x^{7/2} + \frac{\pi^9}{9!} x^{9/2} - \cdots \right) = \frac{1}{\pi\sqrt{x}} \sin(\pi\sqrt{x}). \tag{9.5.5}$$

Therefore, on taking separately the reciprocal of (9.5.4) and (9.5.5) we obtain the result. $\square$

We note that when $\rho = 0$, the expansion of (9.5.3) produces the terms

$$1 - \frac{\pi^2}{6} x + \frac{7\pi^4}{360} x^2 - \frac{31\pi^6}{15120} x^3 + \frac{127\pi^8}{604,800} x^4 - \frac{73\pi^{10}}{3421440} x^5 + \cdots,$$

and when $\rho = 1$,

$$1 + \frac{\pi^2}{6} x + \frac{7\pi^4}{360} x^2 + \frac{31\pi^6}{15120} x^3 + \frac{127\pi^8}{604,800} x^4 + \frac{73\pi^{10}}{3421440} x^5 + \cdots.$$

Employing these two lemmas we now state Theorem 9.5.3.

THEOREM 9.5.3. *The terms $D_e^\rho(r,0,0,n)$ as defined in Definition 9.5.1 and with $r \le n$ are determined by:*

(1) The generating function.

We have

$$\mathcal{G}D_0^\rho(x,0,0,n) = \frac{1}{\sum_{k=0}^n \frac{(-1)^{\rho k}\pi^{2k}}{(2k)!}x^k} = \begin{cases} \mathrm{sech}(\pi\sqrt{x}) & \text{if } \rho = 0 \\ \sec(\pi\sqrt{x}) & \text{if } \rho = 1, \end{cases}$$

and

$$\mathcal{G}D_1^\rho(x,0,0,n) = \frac{1}{\sum_{k=0}^n \frac{(-1)^{\rho k}\pi^{2k}}{(2k+1)!}x^k} = \begin{cases} \pi\sqrt{x}\,\mathrm{csch}(\pi\sqrt{x}) & \text{if } \rho = 0 \\ \pi\sqrt{x}\,\csc(\pi\sqrt{x}) & \text{if } \rho = 1. \end{cases}$$

(2) The recurrence polynomial.

With $D_e^\rho(0,0,0,n) = 1$, we have

$$D_0^\rho(r,0,0,n) = \begin{cases} -\sum_{k=0}^{r-1} \frac{\pi^{2(r-k)}}{(2(r-k))!}D_0^0(k,0,0,n) & \text{if } \rho = 0 \\ -\sum_{k=0}^{r-1} \frac{(-1)^{r-k}\pi^{2(r-k)}}{(2(r-k))!}D_0^1(k,0,0,n) & \text{if } \rho = 1, \end{cases} \tag{9.5.6}$$

and

$$D_1^\rho(r,0,0,n) = \begin{cases} -\sum_{k=0}^{r-1} \frac{\pi^{2(r-k)}}{(2(r-k)+1)!}D_1^0(k,0,0,n) & \text{if } \rho = 0 \\ -\sum_{k=0}^{r-1} \frac{(-1)^{r-k}\pi^{2(r-k)}}{(2(r-k)+1)!}D_1^1(k,0,0,n) & \text{if } \rho = 1. \end{cases} \tag{9.5.7}$$

(3) The MCL (type 1) determinant.

We have $D_0^\rho(r,0,0,n) = \Delta_r(\vec{\mathbf{a}}_n^{(0)}) =$

$$(-1)^r \times \begin{vmatrix} \frac{(-1)^\rho \pi^2}{2!} & 1 & 0 & 0 & \cdots & 0 \\ \frac{\pi^4}{4!} & \frac{(-1)^\rho \pi^2}{2!} & 1 & 0 & \cdots & 0 \\ \frac{(-1)^\rho \pi^6}{6!} & \frac{\pi^4}{4!} & \frac{(-1)^\rho \pi^2}{2!} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^{\rho(r-1)}\pi^{2r-2}}{(2r-2)!} & \frac{(-1)^{\rho(r-2)}\pi^{2r-4}}{(2r-4)!} & \frac{(-1)^{\rho(r-3)}\pi^{2r-6}}{(2r-6)!} & \frac{(-1)^{\rho(r-4)}\pi^{2r-8}}{(2r-8)!} & \cdots & 1 \\ \frac{(-1)^{\rho r}\pi^{2r}}{(2r)!} & \frac{(-1)^{\rho(r-1)}\pi^{2r-2}}{(2r-2)!} & \frac{(-1)^{\rho(r-2)}\pi^{2r-4}}{(2r-4)!} & \frac{(-1)^{\rho(r-3)}\pi^{2r-6}}{(2r-6)!} & \cdots & \frac{(-1)^\rho \pi^2}{2!} \end{vmatrix},$$

and $D_1^\rho(r,0,0,n) = \Delta_r(\vec{\mathbf{a}}_n^{(1)}) =$

$$(-1)^r \times \begin{vmatrix} \frac{(-1)^\rho \pi^2}{3!} & 1 & 0 & 0 & \cdots & 0 \\ \frac{\pi^4}{5!} & \frac{(-1)^\rho \pi^2}{3!} & 1 & 0 & \cdots & 0 \\ \frac{(-1)^\rho \pi^6}{7!} & \frac{\pi^4}{5!} & \frac{(-1)^\rho \pi^2}{3!} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^{\rho(r-1)}\pi^{2r-2}}{(2r-1)!} & \frac{(-1)^{\rho(r-2)}\pi^{2r-4}}{(2r-3)!} & \frac{(-1)^{\rho(r-3)}\pi^{2r-6}}{(2r-5)!} & \frac{(-1)^{\rho(r-4)}\pi^{2r-8}}{(2r-7)!} & \cdots & 1 \\ \frac{(-1)^{\rho r}\pi^{2r}}{(2r+1)!} & \frac{(-1)^{\rho(r-1)}\pi^{2r-2}}{(2r-1)!} & \frac{(-1)^{\rho(r-2)}\pi^{2r-4}}{(2r-3)!} & \frac{(-1)^{\rho(r-3)}\pi^{2r-6}}{(2r-5)!} & \cdots & \frac{(-1)^\rho \pi^2}{3!} \end{vmatrix}$$

Proof. We have that

(1) follows from Lemmata 9.5.1 and 9.5.2 respectively,

(2) follows from Theorem 9.2.1, and

(3) follows from Theorem 9.2.5. $\qquad\square$

We now relate the terms $\mathcal{L}^{T-}_{s,abc}(r, m + a'bc, q)$ to those of $D^{\rho}_e(r, 0, 0, n)$.

COROLLARY. *We have for all positive integers $r \leq n$,*

$$\mathcal{L}^{T-}_{s,00c}(r, m, 2\pi) = \frac{(-\gamma)^m \gamma^c 2^{1-c}}{4\pi} D^{1-s}_1(r, 0, 0, n),$$

$$\mathcal{L}^{T-}_{s,01c}(r, m + c, 2\pi + 1) = \frac{(-\gamma)^{m+1} \gamma^c 2^c}{4} D^{1-s}_0(r, 0, 0, n),$$

$$\mathcal{L}^{T-}_{s,101}(r, m, 2\pi) = \frac{(-\gamma)^m \gamma}{2} D^{1-s}_0(r, 0, 0, n),$$

and

$$\mathcal{L}^{T-}_{s,11c}(r, m, 2\pi + 1) = \frac{(-\gamma)^m \gamma^c}{(2\pi + 1)} D^{1-s}_1(r, 0, 0, n).$$

Proof. The creation of the terms $\mathcal{L}^{T-}_{s,abc}(r, m, 2\pi + b)$ by the truncation of the polynomials $\mathcal{L}^{-}_{s,abc}(r, m, 2\pi + b)$ removes the dependence of r on m (i.e. $r \leq m$), therefore, we are at liberty to make the substitution $m = \pi$. We also put $\rho = 1 - s$ and the result now follows from comparison of the expressions in Theorems 9.4.8 and 9.4.9 with that of Theorem 9.5.3. $\qquad\square$

9.5.2 Expression of the function D^{ρ}_{de}

When the order of the numerator of the generating function of D^{ρ}_{de} is non-zero, we observe from (9.5.1) of Definition 9.5.1 that there are four types, which we consider in pairs, dependent upon the parity of the factorials in the denominator.

We first address the pair of the form, D^{ρ}_{d1}, and require a few lemmas to associate their generating function with a trigonometric expression.

LEMMA 9.5.4. *We have*

$$\lim_{n\to\infty} \mathcal{G}D^{\rho}_{01}(x, 0, n, n) = \frac{\sum_{k=0}^{\infty} \frac{(-1)^{\rho k} \pi^{2k} x^k}{(2k)!}}{\sum_{k=0}^{\infty} \frac{(-1)^{\rho k} \pi^{2k} x^k}{(2k+1)!}} = \begin{cases} \pi\sqrt{x} \coth(\pi\sqrt{x}) & \text{if } \rho = 0 \\ \pi\sqrt{x} \cot(\pi\sqrt{x}) & \text{if } \rho = 1. \end{cases} \tag{9.5.8}$$

Proof. For the case $\rho = 1$, we have on expanding $\sin(\pi\sqrt{x})$ and $\cos(\pi\sqrt{x})$ about $x = 0$,

$$\sin(\pi\sqrt{x}) = \pi\sqrt{x} - \frac{\pi^3 x^{3/2}}{3!} + \frac{\pi^5 x^{5/2}}{5!} - \frac{\pi^7 x^{7/2}}{7!} + \frac{\pi^9 x^{9/2}}{9!} - \dots, \tag{9.5.9}$$

and

$$\cos(\pi\sqrt{x}) = 1 - \frac{\pi^2 x}{2!} + \frac{\pi^4 x^2}{4!} - \frac{\pi^6 x^3}{6!} + \frac{\pi^8 x^4}{8!} - \dots. \tag{9.5.10}$$

From (9.5.9) and (9.5.10) we write the second member of (9.5.8) as

$$\frac{\cos\left(\pi\sqrt{x}\right)}{\frac{1}{\pi\sqrt{x}}\sin\left(\pi\sqrt{x}\right)} = \pi\sqrt{x}\cot\left(\pi\sqrt{x}\right)$$

as required.

When $\rho = 0$, we simply replace the expansions (9.5.9) and (9.5.10) with $\sinh\left(\pi\sqrt{x}\right)$ and $\cosh\left(\pi\sqrt{x}\right)$ respectively. $\qquad\square$

LEMMA 9.5.5. *We have*

$$-\sum_{k=0}^{\infty}\frac{(-1)^{\rho k}\pi^{2k}kx^k}{(2k+1)!} = \begin{cases}\frac{1}{2\pi\sqrt{x}}\left(\sinh\left(\pi\sqrt{x}\right) - \pi\sqrt{x}\cosh\left(\pi\sqrt{x}\right)\right) & \text{if } \rho = 0 \\ \frac{1}{2\pi\sqrt{x}}\left(\sin\left(\pi\sqrt{x}\right) - \pi\sqrt{x}\cos\left(\pi\sqrt{x}\right)\right) & \text{if } \rho = 1.\end{cases}$$

Proof. We demonstate the case when $\rho = 1$. For the case $\rho = 0$, we replace sin with sinh and cos with cosh.

From the expansion of $\sin\left(\pi\sqrt{x}\right)$ in (9.5.9), and $\cos\left(\pi\sqrt{x}\right)$ in (9.5.10), we multiply (9.5.10) by $\pi\sqrt{x}$ and subtract the result from (9.5.9). We then obtain

$$\sin\left(\pi\sqrt{x}\right) - \pi\sqrt{x}\cos\left(\pi\sqrt{x}\right)$$

$$= \pi\sqrt{x} - \frac{\pi^3 x^{3/2}}{3!} + \frac{\pi^5 x^{5/2}}{5!} - \frac{\pi^7 x^{7/2}}{7!} + \frac{\pi^9 x^{9/2}}{9!} - \cdots$$

$$- \pi\sqrt{x} + \frac{\pi^3 x^{3/2}}{2!} - \frac{\pi^5 x^{5/2}}{4!} + \frac{\pi^7 x^{7/2}}{6!} - \frac{\pi^9 x^{9/2}}{8!} - \cdots$$

$$= +\frac{3-1}{3!}\pi^3 x^{3/2} - \frac{5-1}{5!}\pi^5 x^{5/2} + \frac{7-1}{7!}\pi^7 x^{7/2} - \frac{9-1}{9!}\pi^9 x^{9/2} - \cdots$$

$$= +\frac{2}{3!}\pi^3 x^{3/2} - \frac{4}{5!}\pi^5 x^{5/2} + \frac{6}{7!}\pi^7 x^{7/2} - \frac{8}{9!}\pi^9 x^{9/2} + \cdots$$

$$= 2\pi\sqrt{x}\left(\frac{\pi^2 x}{3!} - \frac{2\pi^4 x^2}{5!} + \frac{3\pi^6 x^3}{7!} - \frac{4\pi^8 x^4}{9!} + \cdots\right)$$

$$= 2\pi\sqrt{x}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\pi^{2k}kx^k}{(2k+1)!}.$$

Division by $2\pi\sqrt{x}$ then produces the result. $\qquad\square$

LEMMA 9.5.6. *We have*

$$\lim_{n\to\infty}\mathcal{G}D_{11}^1(x,1,n-1,n) = -\frac{\sum_{k=0}^{\infty}\frac{(-1)^{\rho k}\pi^{2k}kx^k}{(2k+1)!}}{\sum_{k=0}^{\infty}\frac{(-1)^{\rho k}\pi^{2k}x^k}{(2k+1)!}} = \begin{cases}\frac{1-\pi\sqrt{x}\coth\left(\pi\sqrt{x}\right)}{2} & \text{if } \rho = 0 \\ \frac{1-\pi\sqrt{x}\cot\left(\pi\sqrt{x}\right)}{2} & \text{if } \rho = 1.\end{cases} \qquad (9.5.11)$$

Proof. For the case $\rho = 1$, we have from Lemma 9.5.5, that the numerator of the expression in (9.5.11) is given as

$$-\sum_{k=0}^{\infty}\frac{(-1)^k\pi^{2k}kx^k}{(2k+1)!} = \frac{1}{2\pi\sqrt{x}}\left(\sin\left(\pi\sqrt{x}\right) - \pi\sqrt{x}\cos\left(\pi\sqrt{x}\right)\right) \qquad (9.5.12)$$

and for the denominator we have

$$\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} x^k}{(2k+1)!} = 1 - \frac{\pi^2 x}{3!} + \frac{\pi^4 x^2}{5!} - \frac{\pi^6 x^3}{7!} + \frac{\pi^8 x^4}{9!} - \cdots$$

$$= \frac{1}{\pi\sqrt{x}} \left(\pi\sqrt{x} - \frac{\pi^3 x^{3/2}}{3!} + \frac{\pi^5 x^{5/2}}{5!} - \frac{\pi^7 x^{7/2}}{7!} + \frac{\pi^9 x^{9/2}}{9!} - \cdots \right)$$

$$= \frac{1}{\pi\sqrt{x}} \sin\left(\pi\sqrt{x}\right). \tag{9.5.13}$$

Then on division of (9.5.12) by (9.5.13) we obtain the result.

When $\rho = 0$, for the numerator we replace (9.5.12) with

$$-\sum_{k=0}^{\infty} \frac{\pi^{2k} k x^k}{(2k+1)!} = \frac{1}{2\pi\sqrt{x}} \left(\sinh\left(\pi\sqrt{x}\right) - \pi\sqrt{x} \cosh\left(\pi\sqrt{x}\right) \right), \tag{9.5.14}$$

and for denominator we have

$$\sum_{k=0}^{\infty} \frac{\pi^{2k} x^k}{(2k+1)!} = 1 + \frac{\pi^2 x}{3!} + \frac{\pi^4 x^2}{5!} + \frac{\pi^6 x^3}{7!} + \frac{\pi^8 x^4}{9!} + \cdots$$

$$= \frac{1}{\pi\sqrt{x}} \left(\pi\sqrt{x} + \frac{\pi^3 x^{3/2}}{3!} + \frac{\pi^5 x^{5/2}}{5!} + \frac{\pi^7 x^{7/2}}{7!} + \frac{\pi^9 x^{9/2}}{9!} + \cdots \right)$$

$$= \frac{1}{\pi\sqrt{x}} \sinh\left(\pi\sqrt{x}\right). \tag{9.5.15}$$

Then on division of (9.5.14) by (9.5.15) we obtain the lemma.

$$\square$$

Employing these lemmas, we now state Theorem 9.5.7.

THEOREM 9.5.7. *The terms* $D_{d1}^{\rho}(r, \delta, n - \delta, n)$ *with* $r \le n$, *are given by:*

(1) The generating function.

$$\mathcal{G}D_{01}^{\rho}(x, 0, n, n) = \frac{\sum_{k=0}^{n} \frac{(-1)^{\rho k} \pi^{2k}}{(2k)!} x^k}{\sum_{k=0}^{n} \frac{(-1)^{\rho k} \pi^{2k}}{(2k+1)!} x^k} = \begin{cases} \pi\sqrt{x} \coth\left(\pi\sqrt{x}\right) & \text{if } \rho = 0 \\ \pi\sqrt{x} \cot\left(\pi\sqrt{x}\right) & \text{if } \rho = 1, \end{cases}$$

and

$$\mathcal{G}D_{11}^{\rho}(x, 1, n-1, n) = -\frac{\sum_{k=0}^{n} \frac{(-1)^{\rho k} k \pi^{2k}}{(2k+1)!} x^k}{\sum_{k=0}^{n} \frac{(-1)^{\rho k} \pi^{2k}}{(2k+1)!} x^k} = \begin{cases} \frac{1 - \pi\sqrt{x} \coth\left(\pi\sqrt{x}\right)}{2} & \text{if } \rho = 0 \\ \frac{1 - \pi\sqrt{x} \cot\left(\pi\sqrt{x}\right)}{2} & \text{if } \rho = 1. \end{cases}$$

(2) The recurrence polynomial.
(i) In terms of the function D_1^{ρ}.

$$D_{01}^{\rho}(r, 0, n, n) = \sum_{k=0}^{r} \frac{(-1)^{\rho(r-k)} \pi^{2(r-k)}}{(2r - 2k)!} D_1^{\rho}(k, 0, 0, n), \tag{9.5.16}$$

and

$$D_{11}^{\rho}(r, 1, n-1, n) = -\sum_{k=0}^{r-1} \frac{(-1)^{\rho(r-k)} \pi^{2(r-k)} (r-k)}{(2r - 2k + 1)!} D_1^{\rho}(k, 0, 0, n). \tag{9.5.17}$$

(ii) In terms of themselves.

$$D_{01}^{\rho}(r,0,n,n) = \frac{(-1)^{\rho r}\pi^{2r}}{(2r)!} - \sum_{k=0}^{r-1} \frac{(-1)^{\rho(r-k)}\pi^{2(r-k)}}{(2r-2k+1)!} D_{01}^{\rho}(k,0,n,n), \qquad (9.5.18)$$

and

$$D_{11}^{\rho}(r,1,n-1,n) = \frac{(-1)^{\rho(r+1)}r\pi^{2r}}{(2r+1)!} - \sum_{k=1}^{r-1} \frac{(-1)^{\rho(r-k)}\pi^{2(r-k)}}{(2r-2k+1)!} D_{11}^{\rho}(k,1,n-1,n). \qquad (9.5.19)$$

(3) The MCL (type 2) determinant.

$$D_{01}^{\rho}(r,0,n,n) = \Psi_r(\vec{\mathbf{a}}_n^{(1)}, \vec{\mathbf{A}}_n^{(0)}) =$$

$$(-1)^r \times \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ \frac{(-1)^{\rho}\pi^2}{2!} & \frac{(-1)^{\rho}\pi^2}{3!} & 1 & 0 & \cdots & 0 \\ \frac{\pi^4}{4!} & \frac{\pi^4}{5!} & \frac{(-1)^{\rho}\pi^2}{3!} & 1 & \cdots & 0 \\ \frac{(-1)^{\rho}\pi^6}{6!} & \frac{(-1)^{\rho}\pi^6}{7!} & \frac{\pi^4}{5!} & \frac{(-1)^{\rho}\pi^2}{3!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^{\rho(r-1)}\pi^{2r-2}}{(2r-2)!} & \frac{(-1)^{\rho(r-1)}\pi^{2r-2}}{(2r-1)!} & \frac{(-1)^{\rho(r-2)}\pi^{2r-4}}{(2r-3)!} & \frac{(-1)^{\rho(r-3)}\pi^{2r-6}}{(2r-5)!} & \cdots & 1 \\ \frac{(-1)^{\rho r}\pi^{2r}}{(2r)!} & \frac{(-1)^{\rho r}\pi^{2r}}{(2r+1)!} & \frac{(-1)^{\rho(r-1)}\pi^{2r-2}}{(2r-1)!} & \frac{(-1)^{\rho(r-2)}\pi^{2r-4}}{(2r-3)!} & \cdots & \frac{(-1)^{\rho}\pi^2}{3!} \end{vmatrix},$$

and $D_{11}^{\rho}(r,1,n-1,n) = \Psi_r(\vec{\mathbf{a}}_n^{(1)}, \vec{\mathbf{A}}_{n-1}^{(11)}) =$

$$(-1)^r \times \begin{vmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ \frac{\pi^2}{3!} & \frac{(-1)^{\rho}\pi^2}{3!} & 1 & 0 & \cdots & 0 \\ \frac{(-1)^{\rho}2\pi^4}{5!} & \frac{\pi^4}{5!} & \frac{(-1)^{\rho}\pi^2}{3!} & 1 & \cdots & 0 \\ \frac{3\pi^6}{7!} & \frac{(-1)^{\rho}\pi^6}{7!} & \frac{\pi^4}{5!} & \frac{(-1)^{\rho}\pi^2}{3!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^{\rho(r-2)}(r-1)\pi^{2r-2}}{(2r-1)!} & \frac{(-1)^{\rho(r-1)}\pi^{2r-2}}{(2r-1)!} & \frac{(-1)^{\rho(r-2)}\pi^{2r-4}}{(2r-3)!} & \frac{(-1)^{\rho(r-3)}\pi^{2r-6}}{(2r-5)!} & \cdots & 1 \\ \frac{(-1)^{\rho(r-1)}r\pi^{2r}}{(2r+1)!} & \frac{(-1)^{\rho r}\pi^{2r}}{(2r+1)!} & \frac{(-1)^{\rho(r-1)}\pi^{2r-2}}{(2r-1)!} & \frac{(-1)^{\rho(r-2)}\pi^{2r-4}}{(2r-3)!} & \cdots & \frac{(-1)^{\rho}\pi^2}{3!} \end{vmatrix}.$$

Proof. We have that

(1) follows from Lemmas 9.5.4 and 9.5.6;

(2) follows from Theorem 9.2.2, and (2ii) from Corollary 3 of Theorem 9.2.1, and

(3) from Theorem 9.2.6. $\qquad\square$

COROLLARY. *We have for positive integer* $r \le n$,

$$\mathcal{L}_{s;000}^{T-}(r,1,2\pi) = \frac{-\gamma}{2\pi} D_{01}^{1-s}(r,0,n,n), \qquad \mathcal{L}_{s;001}^{T-}(r,1,2\pi) = \frac{-(2\pi-1)}{4\pi^2} D_{11}^{1-s}(r,1,n-1,n),$$

$$\mathcal{L}_{s;110}^{T-}(r,1,2\pi+1) = \frac{-\gamma}{(2\pi+1)} D_{01}^{1-s}(r,0,n,n),$$

and

$$\mathcal{L}_{s;111}^{T-}(r,1,2\pi+1) = \frac{-1}{(2\pi+1)} D_{11}^{1-s}(r,1,n-1,n).$$

Proof. As (reasoned) in the Corollary to Theorem 9.5.3, we put $m = \pi$, and also put $\rho = 1-s$. The Corollary then similarly follows from comparison of the expressions in Theorems 9.4.8 and 9.4.9 with that of Theorem 9.5.7. $\qquad\square$

We now consider the pair of functions of the form D_{d0}^{ρ}, and similarly commence with some lemmas that associate each of the generating functions with a trigonometric function.

LEMMA 9.5.8. *We have*

$$\lim_{n\to\infty} \mathcal{G}D_{00}^{\rho}(x,0,n,n) = \frac{-\sum_{k=0}^{\infty} \frac{(-1)^{\rho k}\pi^{2k}kx^k}{(2k)!}}{\sum_{k=0}^{\infty} \frac{(-1)^{\rho k}\pi^{2k}x^k}{(2k)!}} = \begin{cases} \frac{\pi\sqrt{x}\tanh(\pi\sqrt{x})}{2} & \text{if } \rho = 0 \\ \frac{\pi\sqrt{x}\tan(\pi\sqrt{x})}{2} & \text{if } \rho = 1. \end{cases}$$

Proof. When $\rho = 1$, we first observe that

$$\begin{aligned}
\frac{\pi\sqrt{x}}{2}\sin(\pi\sqrt{x}) &= \frac{\pi^2 x}{2} - \frac{\pi^4 x^2}{2.3!} + \frac{\pi^6 x^3}{2.5!} - \frac{\pi^8 x^4}{2.7!} + \frac{\pi^{10} x^5}{2.9!} - \cdots \\
&= \frac{\pi^2 x}{2!} - \frac{2\pi^4 x^2}{4!} + \frac{3\pi^6 x^3}{6!} - \frac{4\pi^8 x^4}{8!} + \frac{5\pi^{10} x^5}{10!} - \cdots \\
&\quad - \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} k x^k}{(2k)!}.
\end{aligned}$$

So we have that

$$\frac{-\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} k x^k}{(2k)!}}{\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} x^k}{(2k)!}} = \frac{\pi\sqrt{x}\sin(\pi\sqrt{x})}{2\cos(\pi\sqrt{x})} = \frac{\pi\sqrt{x}\tan(\pi\sqrt{x})}{2}.$$

For the case $\rho = 0$ the proof is identical, except that we replace each of the trigonometric functions with their hyperbolic equivalent. $\qquad\square$

Finally for the expression (9.5.21) of Type (10) we use Lemma 9.5.9.

LEMMA 9.5.9. *We have*

$$\lim_{n\to\infty} \mathcal{G}D_{10}^{\rho}(x,0,n,n) = \frac{\sum_{k=0}^{\infty} \frac{(-1)^{\rho k}\pi^{2k}x^k}{(2k+1)!}}{\sum_{k=0}^{\infty} \frac{(-1)^{\rho k}\pi^{2k}x^k}{(2k)!}} = \begin{cases} \frac{\tanh(\pi\sqrt{x})}{\pi\sqrt{x}} & \text{if } \rho = 0 \\ \frac{\tan(\pi\sqrt{x})}{\pi\sqrt{x}} & \text{if } \rho = 1. \end{cases}$$

Proof. For the case $\rho = 1$ we have from (9.5.9) and (9.5.10),

$$\frac{\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} x^k}{(2k+1)!}}{\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} x^k}{(2k)!}} = \frac{\frac{1}{\pi\sqrt{x}}\sin(\pi\sqrt{x})}{\cos(\pi\sqrt{x})} = \frac{\tan(\pi\sqrt{x})}{\pi\sqrt{x}}.$$

Similarly for $\rho = 0$, we replace the trigonometric function with its hyperbolic equivalent. $\qquad\square$

Using Lemmas 9.5.8 and 9.5.9 we have Theorem 9.5.10.

THEOREM 9.5.10. *The terms $D_{d0}^\rho(r,0,0,n)$ as defined in Definition 9.5.1, and with $r \le n$, are determined by:*

(1) The generating function.

We have

$$\mathcal{G}D_{00}^\rho(x,1,n-1,n) = -\frac{\sum_{k=1}^n \frac{(-1)^{\rho k} k \pi^{2k}}{(2k)!} x^k}{\sum_{k=0}^n \frac{(-1)^{\rho k} \pi^{2k}}{(2k)!} x^k} = \begin{cases} \frac{\pi\sqrt{x}\tanh(\pi\sqrt{x})}{2} & \text{if } \rho = 0 \\ \frac{\pi\sqrt{x}\tan(\pi\sqrt{x})}{2} & \text{if } \rho = 1, \end{cases} \qquad (9.5.20)$$

and

$$\mathcal{G}D_{10}^\rho(x,0,n,n) = \frac{\sum_{k=0}^n \frac{(-1)^{\rho k}\pi^{2k}}{(2k+1)!} x^k}{\sum_{k=0}^n \frac{(-1)^{\rho k}\pi^{2k}}{(2k)!} x^k} = \begin{cases} \frac{\tanh(\pi\sqrt{x})}{\pi\sqrt{x}} & \text{if } \rho = 0 \\ \frac{\tan(\pi\sqrt{x})}{\pi\sqrt{x}} & \text{if } \rho = 1. \end{cases} \qquad (9.5.21)$$

(2) The recurrence polynomial.

(i) In terms of the function D_0^ρ.

$$D_{10}^\rho(r,0,n,n) = \sum_{k=0}^r \frac{(-1)^{\rho(r-k)}\pi^{2(r-k)}}{(2r-2k+1)!} D_0^\rho(k,0,0,n),$$

and

$$D_{00}^\rho(r,1,n-1,n) = -\sum_{k=0}^{r-1} \frac{(-1)^{\rho(r-k)}\pi^{2(r-k)}(r-k)}{(2r-2k)!} D_0^\rho(k,0,0,n).$$

(ii) In terms of themselves.

$$D_{10}^\rho(r,0,n,n) = \frac{(-1)^{\rho r}\pi^{2r}}{(2r+1)!} - \sum_{k=0}^{r-1} \frac{(-1)^{\rho(r-k)}\pi^{2(r-k)}}{(2r-2k)!} D_{10}^\rho(k,0,n,n), \qquad (9.5.22)$$

and

$$D_{00}^\rho(r,1,n-1,n) = \frac{(-1)^{\rho(r+1)}r\pi^{2r}}{(2r)!} - \sum_{k=0}^{r-1} \frac{(-1)^{\rho(r-k)}\pi^{2(r-k)}}{(2r-2k)!} D_{00}^\rho(k,1,n-1,n). \qquad (9.5.23)$$

(3) The MCL (type 2) determinant.
$D_{00}^\rho(r,1,n-1,n) = \Psi_r(\vec{\mathbf{a}}_n^{(0)}, \vec{\mathbf{A}}_{n-1}^{(00)}) =$

$$(-1)^r \times \begin{vmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ \frac{\pi^2}{2!} & \frac{(-1)^\rho \pi^2}{2!} & 1 & 0 & \dots & 0 \\ \frac{(-1)^\rho 2\pi^4}{4!} & \frac{\pi^4}{4!} & \frac{(-1)^\rho \pi^2}{2!} & 1 & \dots & 0 \\ \frac{3\pi^6}{6!} & \frac{-\pi^6}{6!} & \frac{\pi^4}{4!} & \frac{(-1)^\rho \pi^2}{2!} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^{\rho(r-2)}(r-1)\pi^{2r-2}}{(2r-2)!} & \frac{(-1)^{\rho(r-1)}\pi^{2r-2}}{(2r-2)!} & \frac{(-1)^{\rho(r-2)}\pi^{2r-4}}{(2r-4)!} & \frac{(-1)^{\rho(r-3)}\pi^{2r-6}}{(2r-6)!} & \dots & 1 \\ \frac{(-1)^{\rho(r-1)}r\pi^{2r}}{(2r)!} & \frac{(-1)^{\rho r}\pi^{2r}}{(2r)!} & \frac{(-1)^{\rho(r-1)}\pi^{2r-2}}{(2r-2)!} & \frac{(-1)^{\rho(r-2)}\pi^{2r-4}}{(2r-4)!} & \dots & \frac{(-1)^\rho \pi^2}{3!} \end{vmatrix},$$

and

$$D_{10}^{\rho}(r, 0, n, n) = \Psi_r(\vec{\mathbf{a}}_n^{(0)}, \vec{\mathbf{A}}_n^{(1)}) =$$

$$(-1)^r \times \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ \frac{(-1)^{\rho}\pi^2}{3!} & \frac{(-1)^{\rho}\pi^2}{2!} & 1 & 0 & \cdots & 0 \\ \frac{\pi^4}{5!} & \frac{\pi^4}{4!} & \frac{(-1)^{\rho}\pi^2}{2!} & 1 & \cdots & 0 \\ \frac{(-1)^{\rho}\pi^6}{7!} & \frac{(-1)^{\rho}\pi^6}{6!} & \frac{\pi^4}{4!} & \frac{(-1)^{\rho}\pi^2}{2!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^{\rho(r-1)}\pi^2(r-1)}{(2r-1)!} & \frac{(-1)^{\rho(r-1)}\pi^2(r-1)}{(2r-2)!} & \frac{(-1)^{\rho(r-2)}\pi^2(r-2)}{(2r-4)!} & \frac{(-1)^{\rho(r-3)}\pi^2(r-3)}{(2r-6)!} & \cdots & 1 \\ \frac{(-1)^{\rho r}\pi^2 r}{(2r+1)!} & \frac{(-1)^{\rho r}\pi^2 r}{(2r)!} & \frac{(-1)^{\rho(r-1)}\pi^2(r-1)}{(2r-2)!} & \frac{(-1)^{\rho(r-2)}\pi^2(r-2)}{(2r-4)!} & \cdots & \frac{(-1)^{\rho}\pi^2}{3!} \end{vmatrix}.$$

Proof. We have that

(1) follows from Lemmas 9.5.8 and 9.5.9;

(2i) follows from Theorem 9.2.2 and (2ii) from Corollary 3 of Theorem 9.2.1, and

(3) from Theorem 9.2.6. $\qquad \square$

COROLLARY. *We have*

$$\mathcal{L}_{s;010}^{T-}(r, 1, 2\pi + 1) = \frac{2\pi - 1}{4} D_{10}^{1-s}(r, 0, n, n), \qquad \mathcal{L}_{s;011}^{T-}(r, 1, 2\pi + 1) = \frac{\gamma}{2} D_{00}^{1-s}(r, 1, n - 1, n),$$

$$\mathcal{L}_{s;100}^{T-}(r, 1, 2\pi) = \frac{1}{2} D_{10}^{1-s}(r, 0, n, n), \quad and \quad \mathcal{L}_{s;101}^{T-}(r, 1, 2\pi) = \frac{-1}{2} D_{00}^{1-s}(r, 1, n - 1, n).$$

Proof. As reasoned in the Corollary to Theorem 9.5.3, we put $m = \pi$, and also put $\rho = 1 - s$, and then the corollary similarly follows from comparison of the expressions in Theorem 9.4.8 and Theorem 9.4.9 with that of Theorem 9.5.7. $\qquad \square$

9.5.3 Connecting the functions D_e^1 and D_{de}^1 to Dirichlet functions

We now connect the functions D_e^1 and D_{de}^1 to familiar Dirichlet functions, and to the Bernoulli numbers, B_n, (which we shall discuss in a subsequent chapter), and Euler numbers, E_n.

Definition 9.5.2. *convenient notation. We write*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ so that } \zeta(2r) = \frac{(-1)^{r+1}(2\pi)^{2r}B_{2r}}{2(2r)!},$$

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \text{ so that } \eta(2r) = \frac{(-1)^{r+1}(2^{2r-1} - 1)\pi^{2r}B_{2r}}{(2r)!},$$

$$\lambda(s) = \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^s} \text{ so that } \lambda(2r) = \frac{(-1)^{r+1}(2^{2r} - 1)\pi^{2r}B_{2r}}{2(2r)!},$$

and with $\chi(n)$ denoting the non-trivial Dirichlet character modulo 4, and E_n denoting Euler numbers

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^s} \text{ so that } \beta(2r + 1) = \frac{(-1)^r \pi^{2r+1} E_{2r}}{4^{r+1}(2r)!};$$

the explicit values are at odd, not even, integers since $\chi(n)$ is an odd Dirichlet character with $\chi(-1) = -1$.

We also have the following lemmas.

LEMMA 9.5.11 (even eta relation). *We have for positive integers $r \leq n$,*

$$\frac{D_1^1(r, 0, 0, n)}{2} = \eta(2r).$$

Proof. From Theorem 9.5.3 we have, for $r \leq n$, that

$$\mathcal{G}D_1^1(x, 0, 0, n) = \frac{1}{\sum_{k=0}^{n} \frac{(-1)^k \pi^{2k}}{(2k+1)!} x^k} = \pi\sqrt{x} \csc\left(\pi\sqrt{x}\right).$$

For $0 \leq r \leq n$, we consider the coefficient of x^r in the expansion of $z \csc z/2$ with $z = \pi\sqrt{x}$, that is

$$\frac{(-1)^{r+1} 2\left(2^{2r-1} - 1\right) \pi^{2r} B_{2r}}{2(2r)!} = \frac{(-1)^{r+1}\left(2^{2r-1} - 1\right) \pi^{2r} B_{2r}}{(2r)!} = \eta(2r).$$

$\square$

LEMMA 9.5.12 (even zeta relation). *We have for positive integers $r \leq n$,*

$$D_{11}^1(r, 1, n-1, n) = -\frac{1}{2}D_{01}^1(r, 0, n, n) = \zeta(2r). \tag{9.5.24}$$

Proof. From Theorem 9.5.7 we have, for $1 \leq r \leq n$, that

$$\mathcal{G}D_{11}^1(x, 1, n-1, n) = -\frac{1}{2}\mathcal{G}D_{01}^1(x, 0, n, n) = \frac{-\pi\sqrt{x}\cot\left(\pi\sqrt{x}\right)}{2}.$$

For $1 \leq r \leq n$, we consider the coefficient of x^r in the expansion of $-z \cot z/2$, with $z = \pi\sqrt{x}$, that gives

$$\frac{-(-1)^r 2^{2r} \pi^{2r} B_{2r}}{2(2r)!} = \frac{(-1)^{r+1}(2\pi)^{2r} B_{2r}}{2(2r)!} = \zeta(2r).$$

$\square$

LEMMA 9.5.13 (even lambda relation). *We have for positive integers $r \leq n$,*

$$\frac{1}{2^{2r}}D_{00}^1(r, 1, n-1, n) = \frac{1}{2^{2r+1}}D_{10}^1(r-1, 0, n, n) = \lambda(2r).$$

Proof. From Theorem 9.5.10 we have, for $1 \leq r \leq n$, that

$$\mathcal{G}D_{00}^1(x, 1, n-1, n) = \frac{\pi^2 x}{2}\mathcal{G}D_{10}^1(x, 0, n, n) = \frac{\pi\sqrt{x}\tan\left(\pi\sqrt{x}\right)}{2}.$$

For $1 \leq r \leq n$ we consider the coefficient of x^r in the expansion of $z \tan z/2$, with $z = \pi\sqrt{x}$. We then have

$$\frac{(-1)^{r+1} 2^{2r}\left(2^{2r} - 1\right) \pi^{2r} B_{2r}}{2(2r)!} = 2^{2r}\frac{(-1)^{r+1}\left(2^{2r} - 1\right) \pi^{2r} B_{2r}}{2(2r)!} = 2^{2r}\lambda(2r).$$

Consequently we have

$$D_{00}^1(r, 1, n-1, n) = \frac{\pi^2}{2} D_{10}^1(r-1, 0, n, n) = 2^{2r} \lambda(2r),$$

and on division by 2^{2r} we obtain the lemma. $\qquad\square$

LEMMA 9.5.14 (odd beta relation). *We have for non-negative integers $r \leq n$,*

$$\frac{\pi D_0^1(r, 0, 0, n)}{4^{r+1}} = \beta(2r+1).$$

Proof. From Theorem 9.5.3 we have, for $r \leq n$, that

$$\mathcal{G}D_0^1(x, 0, 0, n) = \frac{1}{\sum_{k=0}^{n} \frac{(-1)^k \pi^{2k}}{(2k)!} x^k} = \sec\left(\pi\sqrt{x}\right).$$

For, $0 \leq r \leq n$, we consider the coefficient of x^r in the expansion of $\pi \sec z/4^{r+1}$, with $z = \pi\sqrt{x}$. We then proceed as

$$\frac{(-1)^r \pi^{2r+1} E_{2r}}{4^{r+1}(2r)!} = \beta(2r+1).$$

$\qquad\square$

From the results of the Corollaries to Theorem 9.5.7 and to Theorem 9.5.10 and Lemmas 9.5.11 - 9.5.14, we can associate each of the sequences $\mathcal{L}_{0;abc}^{T-}(r, m, 2m+b)$ and $\mathcal{L}_{0;abc}^{T-}(r, 1, 2m+b)$ to a particular Dirichlet function. We summarise these connections in Tables 9.1 and 9.2.

Table 9.1: Relationship between the terms of the sequences $\mathcal{L}_{0;abc}^{T-}(r, m, 2m+b)$, $D_e^1(r, 0, 0, n)$ and a Dirichlet series.

$\mathcal{L}_{0;abc}^{T-}$	D_e^1	Dirichlet series
a+b	e	type
even	1	$\eta(2r)$
odd	0	$\beta(2r+1)$

Table 9.2: Relationship between the terms of the sequences $\mathcal{L}_{0;abc}^{T-}(r, 1, 2m+b)$, $D_{de}^1(r, \delta_{d,e}, n-\delta_{d,e}, n)$ and a Dirichlet series.

$\mathcal{L}_{0;abc}^{T-}$	D_{de}^1	Dirichlet series
a+b	e	type
even	1	$\zeta(2r)$
odd	0	$\lambda(2r)$

In the final part of this chapter we consider (in Theorem 9.5.15) a collection of linear recurrence relations involving the Dirichlet η and ζ functions, that occur naturally from Theorem 9.5.3 and Theorem 9.5.7 and Lemmas 9.5.11 and 9.5.12. Similarly (in Theorem 9.5.16) we have a set of recurrence relations involving the Dirichlet β and λ functions, that occur naturally from Theorem 9.5.3 and Theorem 9.5.10, and Lemmas 9.5.13 and 9.5.14.

THEOREM 9.5.15. *With $\eta(0) = 1/2$, $\zeta(0) = -1/2$ and r a positive integer, we have the following linear recurrence relations:*

$$\eta(2r) = -\sum_{k=0}^{r-1} \frac{(-1)^{r-k}\pi^{2(r-k)}}{(2r-2k+1)!}\eta(2k), \tag{9.5.25}$$

$$\zeta(2r) = -\sum_{k=0}^{r} \frac{(-1)^{r-k}\pi^{2(r-k)}}{(2r-2k)!}\eta(2k), \tag{9.5.26}$$

$$\zeta(2r) = -2\sum_{k=0}^{r-1} \frac{(-1)^{r-k}\pi^{2(r-k)}(r-k)}{(2r-2k+1)!}\eta(2k), \tag{9.5.27}$$

and

$$\zeta(2r) = \frac{(-1)^{r+1}\pi^{2r}}{2(2r)!} - \sum_{k=0}^{r-1} \frac{(-1)^{r-k}\pi^{2(r-k)}}{(2r-2k+1)!}\zeta(2k). \tag{9.5.28}$$

Proof. For the first relation we have from Lemma 9.5.11, that for $0 \le r \le n$,

$$D_1^1(k, 0, 0, n) = 2\eta(2k). \tag{9.5.29}$$

We substitute (9.5.29) into (9.5.7) (of Theorem 9.5.3), and (9.5.25) follows on cancellation of the factor of 2.

For (9.5.26) and (9.5.27), we have from Lemma 9.5.12, that for $1 \le r \le n$,

$$D_{11}^1(r, 1, n-1, n) = -\frac{1}{2}D_{01}^1(r, 0, n, n) = \zeta(2r). \tag{9.5.30}$$

We then substitute (9.5.30) and (9.5.29) into (9.5.16) and (9.5.17) of (2i) of Theorem 9.5.7 and results follow (on cancellation).

For the last relation we use either (9.5.18) or (9.5.19) of (2ii) of Theorem 9.5.7, and (as in the previous two relations) use (9.5.30) to convert to a relation involving $\zeta(2r)$. $\qquad\square$

THEOREM 9.5.16. *With $\beta(1) = \pi/4$, $\lambda(0) = 0$ and r a positive integer, we have the following linear recurrence relations:*

$$\beta(2r+1) = -\sum_{k=0}^{r-1} \frac{(-1)^{r-k}\pi^{2(r-k)}}{4^{r-k}(2r-2k)!}\beta(2k+1), \tag{9.5.31}$$

$$\lambda(2r) = \frac{\pi}{2}\sum_{k=0}^{r} \frac{(-1)^{r-k}\pi^{2(r-k)}}{4^{r-k}(2r-2k+1)!}\beta(2k+1), \tag{9.5.32}$$

$$\lambda(2r) = -\frac{1}{\pi} \sum_{k=0}^{r-1} \frac{(-1)^{r-k}(r-k)\pi^{2(r-k)}}{4^{r-k-1}(2r-2k)!} \beta(2k+1), \tag{9.5.33}$$

and

$$\lambda(2r) = \frac{(-1)^{r+1}\pi^{2r}r}{4^r(2r)!} - \sum_{k=1}^{r-1} \frac{(-1)^{r-k}\pi^{2(r-k)}}{4^{r-k}(2r-2k)!} \lambda(2k). \tag{9.5.34}$$

Proof. For the first relation we have from Lemma 9.5.14, that for $0 \le r \le n$,

$$\frac{\pi D_0^1(r,0,0,n)}{4^{r+1}} = \beta(2r+1). \tag{9.5.35}$$

We substitute (9.5.35) into (9.5.6) of Theorem 9.5.3, and (9.5.31) follows on division of the factor 4^{r+1}. For (9.5.32) and (9.5.33), we have from Lemma 9.5.13, that for $1 \le r \le n$,

$$\frac{1}{2^{2r}} D_{00}^1(r,1,n-1,n) = \frac{1}{2^{2r+1}} D_{10}^1(r-1,0,n,n) = \lambda(2r). \tag{9.5.36}$$

We then substitute (9.5.36) and (9.5.35) into (9.5.16) and (9.5.17) of ($2i$) of Theorem 9.5.10 and results follow on cancellation.

For the last relation we use either (9.5.22) or (9.5.23) of ($2ii$) of Theorem 9.5.10, and (as in the previous two relations) use (9.5.36) to convert to a relation involving $\lambda(2r)$. $\square$

These relations establish that the recurrences for the η, ζ and λ functions at even integer arguments, and the β function at odd integer arguments are embedded in the negative continuation of the sequences $\mathcal{L}_{0;abc}(r,t,q)$.

Chapter 10

Bernoulli numbers of the first and second kind

In Section 10.1 we introduce modified Bernoulli numbers of the first kind and the Bernoulli numbers of the second kind, outlining their determination from the generating function and recurrence polynomial. This is followed in Section 10.2 by their construction as a MCL determinant, and then finally in Section 10.3, with Theorems 10.3.3 and 10.3.9, we investigate the uncancelled denonominator of these two kinds of numbers, that culminates in a corollary on the uncancelled denominator of the even zeta function. Here we establish that the exponent of each prime p occurring in the product of the n-th uncancelled Bernoulli number (of the first kind) is that of the Fleck quotient.

10.1 Modified Bernoulli numbers of the first kind

Let B_r represent the r-th Bernoulli number of the first kind and $\mathcal{B}_r = B_r/r!$ represent the r-th "modified" Bernoulli number. The generating function of these numbers is [28]

$$
\begin{aligned}
\frac{x}{e^x - 1} &= \frac{1}{1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots} \\
&= 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \frac{1}{30,240}x^6 - \frac{1}{1,209,600}x^8 + \dots \\
&= 1 + \frac{-1}{2}x + \frac{1}{6}\frac{x^2}{2!} + \frac{-1}{30}\frac{x^4}{4!} + \frac{1}{42}\frac{x^6}{6!} + \frac{-1}{30}\frac{x^8}{8!} + \dots \\
&= \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \sum_{k=0}^{\infty} \mathcal{B}_k x^k,
\end{aligned}
\tag{10.1.1}
$$

where $\mathcal{B}_r$ denotes the (usual) sequence of modified Bernoulli numbers that has $\mathcal{B}_1 = -1/2$. Alternatively, with an alteration to the denominator we obtain

$$
\begin{aligned}
\frac{x}{1 - e^{-x}} &= \frac{1}{1 - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \dots} \\
&= 1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \frac{1}{30,240}x^6 - \frac{1}{1,209,600}x^8 + \dots \\
&= \sum_{k=0}^{\infty} \mathcal{B}_k^+ x^k, \quad\quad\quad\quad\quad\quad\quad (10.1.2)
\end{aligned}
$$

where we let $\mathcal{B}_r^+$ denote the sequence of (modified) Bernoulli numbers that has $\mathcal{B}_1^+ = B_1^+ = 1/2$.

To express the Bernoulli numbers, B_r, in terms of a recurrence polynomial we have [28],

$$
\binom{r+1}{0}B_0 + \binom{r+1}{1}B_1 + \binom{r+1}{2}B_2 + \dots + \binom{r+1}{r}B_r = 0,
$$

which on rearrangement and division by $(r+1)!$ gives us a corresponding recurrence for the modified Bernoulli numbers of the form

$$
\mathcal{B}_r = -\frac{1}{(r+1)!}\mathcal{B}_0 - \frac{1}{r!}\mathcal{B}_1 - \frac{1}{(r-1)!}\mathcal{B}_2 - \dots - \frac{1}{2!}\mathcal{B}_{r-1}. \quad\quad (10.1.3)
$$

10.1.1 Expression of the modified Bernoulli numbers in terms of $\mathcal{B}^\rho$

Definition 10.1.1. *Let us denote by $\mathcal{B}^\rho$, a function that for non-negative integers, r and n takes the values $\mathcal{B}^\rho(r, 0, 0, n)$, and as in Definition 9.2.1 has generating function $\mathcal{GB}^\rho$ given by*

$$
\mathcal{GB}^\rho(x, 0, 0, n) = \frac{1}{\sum_{k=0}^{n} \frac{(-1)^{\rho k}}{(k+1)!} x^k} = \sum_{k=0}^{\infty} \mathcal{B}^\rho(k, 0, 0, n) x^k. \quad\quad (10.1.4)
$$

LEMMA 10.1.1. *We have*

$$
\lim_{n \to \infty} \mathcal{B}^\rho(r, 0, 0, n) = \begin{cases} \mathcal{B}_r & \text{if } \rho = 0 \\ \mathcal{B}_r^+ & \text{if } \rho = 1. \end{cases}
$$

Proof. We have on comparison of the generating function (10.1.4) with (10.1.1) and (10.1.2),

$$
\lim_{n \to \infty} \mathcal{GB}^0(x, 0, 0, n) = \frac{1}{\sum_{k=0}^{\infty} \frac{1}{(k+1)!} x^k} = \frac{x}{e^x - 1} = \mathcal{GB}_r,
$$

and

$$
\lim_{n \to \infty} \mathcal{GB}^1(x, 0, 0, n) = \frac{1}{\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} x^k} = \frac{x}{1 - e^{-x}} = \mathcal{GB}_r^+.
$$

$\square$

THEOREM 10.1.2. *The terms $\mathcal{B}^\rho(r,0,0,n)$, for $r \leq n$, are given by:*

(1) The generating function.

$$\mathcal{B}^\rho(r,0,0,n) = \begin{cases} \frac{x}{e^x - 1} = \mathcal{GB}_r & \text{if } \rho = 0 \\ \frac{x}{1 - e^{-x}} = \mathcal{GB}_r^+ & \text{if } \rho = 1. \end{cases}$$

(2) The recurrence polynomial. With $\mathcal{B}^\rho(0,0,0,n) = 1$, we have

$$\mathcal{B}^\rho(r,0,0,n) = -\sum_{k=0}^{r-1} \frac{(-1)^{\rho(r-k)}}{(r+1-k)!} \mathcal{B}^\rho(k,0,0,n).$$

(3) The MCL (type 1) determinant. We have for $1 \leq r \leq n$,

$$\mathcal{B}^\rho(r,0,0,n) = \Delta_r^\rho(\vec{\mathbf{a}}_r)$$

$$= (-1)^r \begin{vmatrix} \frac{(-1)^\rho}{2!} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{3!} & \frac{(-1)^\rho}{2!} & 1 & 0 & \cdots & 0 \\ \frac{(-1)^\rho}{4!} & \frac{1}{3!} & \frac{(-1)^\rho}{2!} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^{\rho(r-1)}}{r!} & \frac{(-1)^{\rho(r-2)}}{(r-1)!} & \frac{(-1)^{\rho(r-3)}}{(r-2)!} & \frac{(-1)^{\rho(r-4)}}{(r-3)!} & \cdots & 1 \\ \frac{(-1)^{\rho r}}{(r+1)!} & \frac{(-1)^{r-1}}{r!} & \frac{(-1)^{r-2}}{(r-1)!} & \frac{(-1)^{r-3}}{(r-2)!} & \cdots & \frac{(-1)^\rho}{2!} \end{vmatrix},$$

where $\Delta_r^\rho(\vec{\mathbf{a}}_r)$ is a MCL determinant with $\vec{\mathbf{a}}_r = (\frac{(-1)^\rho}{2!}, \frac{1}{3!}, \ldots, \frac{(-1)^{\rho(r-1)}}{r!}, \frac{(-1)^{\rho r}}{(r+1)!})$.

Proof. We have that:

(1) follows from Lemma 10.1.1;

(2) follows from Theorem 9.2.1, and

(3) follows from Theorem 9.2.5. $\qquad\square$

COROLLARY. *We have*

$$r! \Delta_r^0(\vec{\mathbf{a}}_r) = B_r.$$

Proof. This follows directly from Theorem 10.1.2 on recalling that $\mathcal{B}^0(r,0,0,n) = \mathcal{B}_r = B_r/r!$. $\qquad\square$

Remark. To generate any B_r value, we can truncate the infinite summation of the denominator of (10.1.1) to the finite summation, and then consider a function $\mathcal{B}^0$.

10.2 Bernoulli numbers of the second kind

The Bernoulli numbers of the second kind, b_r, are determined by the generating function [28], (p259),

$$\mathcal{G}b_r = \frac{x}{\log 1 + x} = \frac{1}{1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots}$$

$$= 1 + \frac{1}{2}x - \frac{1}{12}x^2 + \frac{1}{24}x^3 - \frac{19}{720}x^4 + \ldots = \sum_{k=0}^{\infty} b_k x^k. \tag{10.2.1}$$

If we replace x with $-x$ we obtain

$$\mathcal{G}b_r^- = \frac{-x}{\log 1 - x} = \frac{1}{1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \dots}$$

$$= 1 - \frac{1}{2}x - \frac{1}{12}x^2 - \frac{1}{24}x^3 - \frac{19}{720}x^4 - \dots = \sum_{k=0}^{\infty} b_k^- x^k, \qquad (10.2.2)$$

where we let b_r^- denote an alternative sequence of Bernoulli numbers of the second kind such that for $r \geq 1$, we have $b_r < 0$.

10.2.1 Expression of the Bernoulli number in terms of a function b^ρ

Definition 10.2.1. *Let us denote by b^ρ, a function that for non-negative integers, r and n takes the values $b^\rho(r, 0, 0, n)$, and as in Definition 9.2.1 has generating function $\mathcal{G}b^\rho$ given by*

$$\mathcal{G}b^\rho(x, 0, 0, n) = \frac{1}{\sum_{k=0}^{n} \frac{(-1)^{\rho k}}{k+1} x^k} = \sum_{k=0}^{\infty} b^\rho(k, 0, 0, n) x^k.$$

LEMMA 10.2.1. *We have*

$$\lim_{n \to \infty} b^\rho(r, 0, 0, n) = \begin{cases} b_r^- & \text{if } \rho = 0 \\ b_r & \text{if } \rho = 1. \end{cases}$$

Proof. We have from the generating functions (10.2.1) and (10.2.2),

$$\lim_{n \to \infty} \mathcal{G}b^0(x, 0, 0, n) = \frac{1}{\sum_{k=0}^{\infty} \frac{1}{k+1} x^k} = \frac{-x}{\log 1 - x} = \mathcal{G}b_r^-,$$

and

$$\lim_{n \to \infty} \mathcal{G}b^1(x, 0, 0, n) = \frac{1}{\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^k} = \frac{x}{\log 1 + x} = \mathcal{G}b_r.$$

$\square$

THEOREM 10.2.2. *The terms $b^\rho(r, 0, 0, n)$, for $r \leq n$, are given by:*
(1) The generating function.
We have

$$\mathcal{G}b^\rho(r, 0, 0, n) = \begin{cases} \frac{-x}{\log 1 - x} = \mathcal{G}b_r^- & \text{if } \rho = 0 \\ \frac{x}{\log 1 + x} = \mathcal{G}b_r & \text{if } \rho = 1. \end{cases}$$

(2) The recurrence polynomial.
With $b^\rho(0, 0, 0, n) = 1$, we have

$$b^\rho(r, 0, 0, n) = -\sum_{k=0}^{r-1} \frac{(-1)^{\rho(r-k)}}{r + 1 - k} b^\rho(k, 0, 0, n).$$

(3) The MCL (type 1) determinant.
We have for $1 \leq r \leq n$,

$$b^\rho(r, 0, 0, n) = \Delta_r^\rho(\vec{\mathbf{a}}_r)$$

$$= (-1)^r \begin{vmatrix} \frac{(-1)^\rho}{2} & 1 & 0 & 0 & \ldots & 0 \\ \frac{1}{3} & \frac{(-1)^\rho}{2} & 1 & 0 & \ldots & 0 \\ \frac{(-1)^\rho}{4} & \frac{1}{3} & \frac{(-1)^\rho}{2} & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^{\rho(r-1)}}{r} & \frac{(-1)^{\rho(r-2)}}{r-1} & \frac{(-1)^{\rho(r-3)}}{r-2} & \frac{(-1)^{\rho(r-4)}}{r-3} & \ldots & 1 \\ \frac{(-1)^{\rho r}}{r+1} & \frac{(-1)^{r-1}}{r} & \frac{(-1)^{r-2}}{r-1} & \frac{(-1)^{r-3}}{r-2} & \ldots & \frac{(-1)^\rho}{2} \end{vmatrix},$$

where $\Delta_r^\rho(\vec{\mathbf{a}}_r)$ is a MCL determinant with $\vec{\mathbf{a}}_r = (\frac{(-1)^\rho}{2}, \frac{1}{3}, \ldots, \frac{(-1)^{\rho(r-1)}}{r}, \frac{(-1)^{\rho r}}{r+1})$.

Proof. We have that:

(1) follows from Lemma 10.2.1;

(2) follows from Theorem 9.2.1, and

(3) follows from Theorem 9.2.5. □

10.3 Bernoulli numbers and their (uncancelled) denonimators

We examine the natural (uncancelled) denominator of the Bernoulli numbers of the second kind and the (modified) Bernoulli numbers of the first kind, and find that for both, the power of each prime p, $(p - 1 \leq n)$, satisfy the Fleck congruence $\lfloor n/(p - 1) \rfloor$.

We begin by examining the Bernoulli Numbers of the second kind.

10.3.1 Denominator theorem: Bernoulli numbers of the second Kind

Definition 10.3.1. *We have $b_0 = 1$, and*

$$b_n = \frac{1}{2}b_{n-1} - \frac{1}{3}b_{n-2} + \ldots + (-1)^m \frac{1}{m}b_{n-m+1} + \ldots + (-1)^n \frac{1}{n}b_1 + (-1)^{n+1}\frac{1}{n+1}b_0. \quad (10.3.1)$$

We denote by e_n the lowest common multiple of the denominators in (10.3.1) before any cancellation has occured, so that

$$e_n = LCM\left[(n+1)e_0, ne_1, (n-1)e_2, \ldots, me_{n-m+1}, \ldots, 2e_{n-1}\right], \quad (10.3.2)$$

and

$$e_n = \prod_{p \leq n+1} p^{\gamma(n,p)}.$$

Then let $b_n = f_n/e_n$, where $f_n \in \mathbb{Z}$ and $b_0 = e_0 = 1$. Furthermore, let

$$E_n = \prod_{p \leq n+1} p^{\delta(n,p)}, \quad where \quad \delta(n,p) = \left\lfloor \frac{n}{p-1} \right\rfloor. \quad (10.3.3)$$

It is observed that for each prime $p \leq n + 1$ its first occurence in the term e_M is when $M = n - m + 1 = p - 1$ and thereafter, the exponent of the prime p increases by 1 after each interval of $p - 1$ terms. We propose that $\gamma(n,p) = \delta(n,p)$.

However, in order to prove this result it will be helpful to introduce some lemmas.

LEMMA 10.3.1. *For $\delta(n,p)$ defined as in* (10.3.3) *we have*

$$\delta(n,p) = \delta(n-p+1,p) + 1$$

Proof. From (10.3.3) we have

$$\delta(n-p+1,p) + 1 = \left\lfloor \frac{n-p+1}{p-1} \right\rfloor + \frac{p-1}{p-1} = \left\lfloor \frac{n}{p-1} \right\rfloor = \delta(n,p).$$

$\square$

LEMMA 10.3.2. *With $2 \le m \le n+1$ and r an integer we have*

$$\delta(n,p) \ge \delta(n-m+1,p) + r,$$

whenever p^r divides m.

Proof. We use $\lfloor u \rfloor + \lfloor v \rfloor \le \lfloor u+v \rfloor$.

Take $u = (n+1-m)/(p-1)$, and, $v = (m-1)/(p-1)$, to obtain

$$\left\lfloor \frac{n+1-m}{p-1} \right\rfloor + \left\lfloor \frac{m-1}{p-1} \right\rfloor \le \left\lfloor \frac{n}{p-1} \right\rfloor,$$

which implies

$$\delta(n-m+1,p) + \delta(m-1,p) \le \delta(n,p).$$

Then rearranging and putting $m = jp^r$, where $j \nmid p$, we have

$$\delta(n,p) - \delta(n-m+1,p) \ge \left\lfloor \frac{m-1}{p-1} \right\rfloor \ge \left\lfloor \frac{jp^r - 1}{p-1} \right\rfloor \ge \left\lfloor \frac{p^r - 1}{p-1} \right\rfloor,$$

and

$$\left\lfloor \frac{p^r - 1}{p-1} \right\rfloor = \left\lfloor 1 + p + p^2 + \ldots + p^{r-1} \right\rfloor \ge r,$$

so that $\delta(n,p) \ge \delta(n-m+1,p) + r$ as required. $\square$

Remark. If $p \nmid m$, then $r = 0$ and $m = j$, and Lemma 10.3.2 still holds.

We are now in a position to state

THEOREM 10.3.3 (denominator theorem: Bernoulli numbers second kind). *With e_n, E_n, $\gamma(n,p)$ and $\delta(n,p)$ given as in Definition* 10.3.1 *we have*

$$\gamma(n,p) = \delta(n,p). \tag{10.3.4}$$

Proof. We show the equality of (10.3.4) using induction on N, ($N \geq 0$), and for $p \geq 2$. For $N = 0$, we have $\delta(0, p) = \lfloor 0/(p-1) \rfloor = 0$, and also $\gamma(0, p) = 0$ follows from $e_0 = 1$. So it is true for $N = 0$.

When $N = M$, let us assume $\gamma(M, p) = \delta(M, p)$, for $M = n - 1, n - 2, \ldots$ (as many as required). We wish to demonstrate the case $N = n$.

Let us put $M = n - p + 1$ (and $m = p$). Now from (10.3.2),

$$pe_{n-p+1} \mid e_n$$

and from the induction hypothesis

$$\gamma(n - p + 1, p) = \delta(n - p + 1, p) = \left\lfloor \frac{n - p}{n - 1} \right\rfloor$$

so that from Lemma 10.3.1 the exponent of p in the term pe_{n-p+1} is

$$\gamma(n - p + 1, p) + 1 = \delta(n - p + 1) + 1 = \delta(n, p).$$

Therefore, when $N = n$, we have that $\delta(n, p)$ is a factor of $\gamma(n, p)$. Is it the greatest factor? Any such term of (10.3.2) with a greater exponent of p will require $m = p^r$. We have

$$p^r e_{n-p^r+1} = \gamma(n - p^r + 1, p) + r = \delta(n - p^r + 1, p) + r \leq \delta(n, p)$$

from Lemma 10.3.2. That is no there is no term me_{n-m+1} with a greater exponent than pe_{n-p+1}. Therefore, we conclude that

$$\gamma(n, p) = \delta(n, p) = \left\lfloor \frac{n}{p - 1} \right\rfloor$$

for each $p \leq n + 1$, and the theorem follows. $\qquad \square$

10.3.2 Denominator theorem: (Modified) Bernoulli numbers

We consider the numbers $\mathcal{B}_n = B_n/n!$, where B_n is a Bernoulli number of the first kind, so that, for example, we have $\mathcal{B}_0 = 1$, $\mathcal{B}_1 = 1/2$, $\mathcal{B}_2 = 1/12$, $\mathcal{B}_3 = 0$, $\mathcal{B}_4 = -1/720$, …. Then since for $r \geq 1$, $\mathcal{B}_{2r+1} = 0$, for $n \geq 2$ we write $n = 2r$. Analogous to Definition 10.3.1 we define

Definition 10.3.2. *Let $\mathcal{B}_0 = 1$, and (from (10.1.3)) we have*

$$\mathcal{B}_n = \frac{(-1)^{n+1}}{(n+1)!} \mathcal{B}_0 + \frac{(-1)^n}{n!} \mathcal{B}_1 + \frac{(-1)^{n-1}}{(n-1)!} \mathcal{B}_2 + \ldots + \frac{1}{2!} \mathcal{B}_{n-1}. \qquad (10.3.5)$$

Denote by e_n the lowest common multiple of the denominators in (10.3.5) before any cancellation has occurred, so that

$$e_n = LCM\left[(n+1)! e_0, n! e_1, (n-1)! e_2, \ldots, m! e_{n-m+1}, \ldots, 2! e_{n-1} \right],$$

whilst for $n = 2r \geq 2$,

$$e_{2r} = LCM\left[(2r+1)!e_0, (2r)!e_1, (2r-1)!e_2, \ldots, (2s+1)!e_{2(r-s)}, \ldots, 3!e_{2r-2}\right], \quad (10.3.6)$$

and

$$e_n = \prod_{p \leq n+1} p^{\gamma(n,p)}.$$

Then let $\mathcal{B}_n = f_n/e_n$, where $f_n \in \mathbb{Z}$ and $\mathcal{B}_0 = e_0 = 1$. We also write $m = 2s + b$, where $b \in \{0, 1\}$ is the parity of m.

Furthermore, let

$$E_n = \frac{1}{2^{D-1}} \prod_{p \leq n+1} p^{\delta(n,p)}, \quad (10.3.7)$$

where $\delta(n, p) = \lfloor n/(p-1) \rfloor$ and D is the sum of the digits of n expressed in binary form.

By definition (of the lowest common multiple) it is evident that for each such p, its (greatest) exponent in e_n must equal its (greatest) exponent in at least one of the terms $m!e_{n-m+1}$. As in Theorem 10.3.3 we put $m = p$ and note that for $p = 2s + 1 \geq 3$,

$$n - p + 1 = 2r - (2s+1) + 1 = 2(r-s),$$

so that $e_{n-p+1} \neq 0$. Conversely when $p = 2$ and $n \geq 4$, we have $e_{n-2+1} = 0$.

It is observed that for each prime $p \leq n+1$, its first occurence in the term e_n is when $n = p-1$, and thereafter, (for $p \geq 3$), the exponent of the prime p increases by 1 after each interval of $(p-1)$ terms. For $p = 2$, this pattern is affected by the fact that for $n \geq 1$, $\mathcal{B}_{2r+1} = 0$. Once again, (perhaps surprisingly), we are motivated to conjecture that for $n \geq 1$,

$$\gamma(n, 2) = \delta(n, 2) + 1 - D, \qquad \text{and that for } p \geq 3, \qquad \gamma(n, p) = \delta(n, p). \qquad (10.3.8)$$

The demonstration of (10.3.8) requires accounting for the fact that each of the terms comprising the lowest common multiple now has the form $m!e_{n-m+1}$ (as opposed to me_{n-m+1}), and in the case of the prime $p = 2$, the determination of the offset, $1 - D$. The following set of lemmas enable us to establish these amendments.

First we require a definition.

Definition 10.3.3. *Let k be a positive integer expressed in the scale of $b \geq 2$, whose digits are $d_j(k)$, $d_{j-1}(k)$, $\ldots$, $d_2(k)$, $d_1(k)$, $d_0(k)$, where for $0 \leq i \leq j$, $0 \leq d_i(k) \leq b - 1$. We express the sum of the digits of k in the scale of (base) b by*

$$D(k) = \sum_{i=0}^{j} d_i(k). \qquad (10.3.9)$$

Also let

$$R(k) = \sum_{i=0}^{j} (b^i - 1) d_i(k), \qquad (10.3.10)$$

then we have

$$k = D(k) + R(k).$$

If k is evident, then we simplify (10.3.9) *and* (10.3.10) *to*

$$D = \sum_{i=0}^{j} d_i, \quad and \quad R = \sum_{i=0}^{j} (b^i - 1) d_i$$

respectively.

LEMMA 10.3.4 (factorial factor). *With E_n defined as in* (10.3.7) *for some integer k we have*

$$E_n = k(n+1)!.$$

Proof. The power of p in $(n+1)!$ is given by

$$t = \lfloor (n+1)/p \rfloor + \lfloor (n+1)/p^2 \rfloor + \lfloor (n+1)/p^3 \rfloor + \dots$$
$$< (n+1)/p + (n+1)/p^2 + (n+1)/p^3 + \dots. \tag{10.3.11}$$

This gives $t < (n+1)/(p-1)$ which (since t is a positive integer) implies that $t \leq \lfloor n/(p-1) \rfloor$. Therefore, we have

$$\prod_{p \leq n+1} p^{\left\lfloor \frac{n}{p-1} \right\rfloor} = k(n+1)!$$

for some integer $k \geq 1$. $\qquad \square$

Remark. There cannot be equality in (10.3.11), because $[(n+1)/p^r]$ is 0 when r is large.

LEMMA 10.3.5 (power of prime p in factorial n). *We have that the power, e, of a prime p in factorial n is given by*

$$e = \sum_{k=1}^{\infty} \left[\frac{n}{p^k} \right].$$

Proof. There are $[n/p]$ multiples of p in $1, 2, 3, \dots, n$. Each of these contributes at least one factor p to $n!$.

The multiples of p^2 contribute an extra power of p. There are $[n/p^2]$ of them in $1, 2, 3, \dots, n$. The multples of p^3 contribute an extra power of p. There are $[n/p^3]$ of them in $1, 2, 3, \dots, n$. And so on. This sequence of steps stops after $[\log n / \log p]$ steps.

So the prime p is raised to the power

$$e = \sum_{k=1}^{\infty} \left[\frac{n}{p^k} \right].$$

$\qquad \square$

We improve on the result of Lemma 10.3.5 by presenting a more applicable formula for e.

LEMMA 10.3.6 (power of prime p in factorial n: Exact Formula). *We have that the power, e, of a prime p in factorial n is given by*

$$\frac{1}{p-1}\left(n - \sum d_s\right).$$

Proof. The positive integer n can be written in the scale of p (with digits $0, 1, \ldots, p-1$) as $d_r d_{r-1} \ldots d_1 d_0$.

This is formally

$$\sum_{s=1}^{\infty} d_s p^s,$$

where $d_s = 0$ for $s > r$. So we have the double sum

$$e = \sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right] = \sum_{k=1}^{\infty} \left(\frac{n}{p^k} - \frac{d_{k-1}p^{k-1} + \cdots + d_1 p + d_0}{p^k}\right) = \sum_{k=1}^{\infty} \left(\frac{n}{p^k} - \sum_{s=0}^{k-1} \frac{d_s}{p^{k-s}}\right)$$

$$= \sum_{k=1}^{\infty} \frac{n}{p^k} - \sum_{s=0}^{\infty} d_s \sum_{k=s+1}^{\infty} \frac{1}{p^{k-s}} = \sum_{\ell=1}^{\infty} \frac{1}{p^\ell}\left(n - \sum_{s=0}^{\infty} d_s\right) = \frac{1}{p-1}\left(n - \sum d_s\right).$$

$\square$

LEMMA 10.3.7. *Let $D(n)$ and $R(n)$ be defined as in Definition 10.3.3. Then for a positive integer n expressed in the scale of $b \geq 2$, whose sum of digits is given by $D(n)$, we have that for $0 \leq m \leq n$,*

$$D(n) + (b-1)c = D(m) + D(n-m), \tag{10.3.12}$$

or equivalently

$$R(n) - (R(m) + R(n-m)) = (b-1)c. \tag{10.3.13}$$

Here $c \geq 0$ denotes the number of times b is carried over in the sum $D(m) + D(n-m)$.

Proof. Let us consider the sum of the right hand side of (10.3.12). From Definition 10.3.3 we have

$$D(k) = \sum_{i=0}^{j} d_i(k),$$

so that

$$D(m) + D(n-m) = \sum_{i=0}^{j} d_i(m) + \sum_{i=0}^{j} d_i(n-m)$$

$$= \sum_{i=0}^{j} d_i(m) + d_i(n-m). \tag{10.3.14}$$

When $i = 0$, we have one of the conditions

$$d_0(m) + d_0(n-m) = d_0(n), \qquad (1)$$

$$d_0(m) + d_0(n-m) = d_0(n) + b. \qquad (2)$$

When $1 \leq i \leq j - 1$, one of the conditions

$$d_i(m) + d_i(n - m) = d_i(n), \qquad (1)$$
$$d_i(m) + d_i(n - m) = d_i(n) + b, \qquad (2)$$
$$1 + d_i(m) + d_i(n - m) = d_i(n), \qquad (3)$$
$$1 + d_i(m) + d_i(n - m) = d_i(n) + b. \qquad (4)$$

Finally when $i = j$, it is one of the conditions

$$d_j(m) + d_j(n - m) = d_j(n), \qquad (1)$$
$$1 + d_i(m) + d_i(n - m) = d_i(n). \qquad (3)$$

If for every i, $(0 \leq i \leq j)$, condition (1) is satisfied then the number of carries $c = 0$, and we have

$$D(m) + D(n - m) = D(n).$$

If the number of carries $c \geq 1$, then each time condition (2) or (4) is satisfied in the i-th sum, $(0 \leq i \leq j - 1)$, i.e., having b on the right hand side, we obtain a 1 on the left hand side of the $(i + 1)$-th sum. To the total sum (10.3.14), the former adds b whilst the latter subtracts 1. Let c denote the number of times condition (2) or (4) occurs in this sum, then (10.3.14) becomes

$$D(m) + D(n - m) = \sum_{i=0}^{j} d_i(m) + d_i(n - m) = \sum_{i=0}^{j} d_i(n) + (b - 1)c = D(n) + (b - 1)c,$$

and so we obtain (10.3.12).
Also from Definition 10.3.3 we have

$$n = m + n - m$$
$$D(n) + R(n) = D(m) + R(m) + D(n - m) + R(n - m)$$
$$D(n) + R(n) - (R(m) + R(n - m)) = D(m) + D(n - m),$$

and so we obtain (10.3.13). $\qquad \square$

COROLLARY. *For a positive integer n expressed in the scale of $b \geq 2$, whose sum of digits is $D(n)$, we have that for $0 \leq m \leq n$,*

$$D(n) \leq D(m) + D(n - m).$$

Proof. This follows directly from (10.3.12) of Lemma 10.3.7 on noting that since $b \geq 2$ and $c \geq 0$, then $(b - 1)c \geq 0$. $\qquad \square$

LEMMA 10.3.8. *The exponent of the prime factor* 2 *in the uncancelled denominator of* $\mathcal{B}_0$ *is* 0, *that of* $\mathcal{B}_1$ *is* 1 *and thereafter, for* $n = 2r$, *is given by*

$$\gamma(2r, 2) = 2r + 1 - D(2r), \qquad (10.3.15)$$

where $D(2r)$ *is the sum of the digits of* $2r$ *expressed in the scale of* 2.

Proof. We have $\mathcal{B}_0 = 1$ and so $\gamma(0, 2) = 0$. Then for $n \geq 1$, we use induction on n. When $n = 1$, we have $\gamma(1, 2) = 1 = 1 + 1 - 1$. Let us assume that (10.3.15) holds for $N \leq 2r - 2$, then for $N = n = 2r$ we have from (10.3.6)

$$e_{2r} = LCM\left[(2r+1)!e_0, (2r)!e_1, (2r-1)!e_2, \ldots, (2s+1)!e_{2(r-s)}, \ldots, 3!e_{2r-2}\right]. \qquad (10.3.16)$$

From Lemma 10.3.6, and the induction hypothesis, the power of the prime factor 2 in (10.3.16) is

$$\gamma(2r, 2) = \max[2r + 1 - D(2r+1), 2r - D(2r) + \gamma(1, 2), 2r - 1 - D(2r-1) + \gamma(2, 2),$$
$$\ldots, 2s + 1 - D(2s+1) + \gamma(2(r-s), 2), \ldots, 3 - D(3) + \gamma(2r-2, 2)]$$
$$= \max[2r + 1 - D(2r+1), 2r - D(2r) + 2 - D(1), 2r - 1 - D(2r-1) + 3 - D(2),$$
$$\ldots, 2s + 1 - D(2s+1) + 2(r - s) + 1 - D(2r - 2s), \ldots, 3 - D(3) + 2r - 1 - D(2r-2)]$$
$$= \max[2r + 1 - D(2r+1), 2r + 2 - (D(2r) + D(1)), 2r + 2 - (D(2r-1) + D(2)),$$
$$\ldots, 2r + 2 - (D(2s+1) + D(2r - 2s)), \ldots, 2r + 2 - (D(3) + D(2r-2))]. \qquad (10.3.17)$$

Now since $\gamma(1, 2) = 1$, the second term can be written

$$2r + 1 - D(2r) > 2r + 1 - D(2r + 1),$$

therefore, demonstrating that its exponent of 2 exceeds (by 1) that of the first term. Then from the Corollary of Lemma 10.3.7 we have

$$D(2r + 1) = D(2r) + D(1) \leq D(2r + 1 - m) + D(m),$$

and so from (10.3.17) no other term can have an exponent of 2 that exceeds the term $(2r)!e_1$ and this exponent is indeed given by

$$\gamma(2r, 2) = 2r + 1 - D(2r).$$

$\square$

This now leads us to state

THEOREM 10.3.9 (denominator theorem). *With* e_n, E_n, $\gamma(n, p)$, $\delta(n, p)$ *and* D *given as in Definition* 10.3.2, *we have for* $n = 0$,

$$\gamma(0, p) = \delta(0, p) = 0,$$

and for $n = 1$, *and thereafter, for* $n = 2r$, *that*

$$\gamma(n, 2) = \delta(n, 2) + 1 - D, \qquad \text{and for } p \geq 3, \qquad \gamma(n, p) = \delta(n, p). \qquad (10.3.18)$$

Proof. Our method of determination is similar in approach to that of Theorem 10.3.3. We show the equality of (10.3.18) using induction on N, for all non-negative values of N and for all prime $p \leq n + 1$. We consider separately the cases $p = 2$ and $p \geq 3$.

For the case $p = 2$, we have from Lemma 10.3.8 that for $n = 0$,

$$\gamma(0, 2) = \delta(0, 2) = 0,$$

and for $n \geq 1$,

$$\gamma(n, 2) = n + 1 - D = \left\lfloor \frac{n}{2 - 1} \right\rfloor + 1 - D = \delta(n, 2) + 1 - D,$$

where since n, and $p = 2$, are constant we write D for $D(n)$.

We now proceed with the case $p \geq 3$. For $N = 0$, we have $\delta(0, p) = \lfloor 0/(p - 1) \rfloor = 0$, and also $\gamma(0, p) = 0$ follows from $e_0 = 1$. So it is true for $N = 0$.

When $N = M$, let us assume $\gamma(M, p) = \delta(M, p)$, for $M = n - 1, n - 2, \ldots$ (as many as required). We wish to demonstrate the case $N = n$. Let us put $M = n - p + 1$ (and $m = p$), and first note that in consideration of the LCM of the term $a_n = a_{2r}$, the term $p! e_{n-p+1} \neq 0$. Now from (10.3.2),

$$p! e_{n-p+1} \mid e_n$$

and from the induction hypothesis

$$\gamma(n - p + 1, p) = \delta(n - p + 1, p) = \left\lfloor \frac{n - p + 1}{p - 1} \right\rfloor$$

so that from Lemma 10.3.1 the exponent of prime p in the term $p! e_{n-p+1}$ is

$$\gamma(n - p + 1, p) + 1 = \delta(n - p + 1, p) + 1 = \delta(n, p).$$

Therefore, when $N = n$, $\delta(n, p)$ is a factor of $\gamma(n, p)$. Is it the greatest factor? We need to determine the exponent of p in $m!$. Now from Lemma 10.3.4 this exponent is bounded by $\delta(m - 1, p)$ and so (assuming the induction hypothesis) we have that the exponent of p in the term $m! e_{n-m+1}$ is bounded by

$$\delta(m - 1, p) + \gamma(n - m + 1, p)$$
$$= \delta(m - 1, p) + \delta(n - m + 1, p)$$
$$= \left\lfloor \frac{m - 1}{p - 1} \right\rfloor + \left\lfloor \frac{n - m + 1}{p - 1} \right\rfloor$$
$$\leq \left\lfloor \frac{n}{p - 1} \right\rfloor = \delta(n, p).$$

That is, no term $m! e_{n-m+1}$ has a greater exponent of p than $p! e_{n-p+1}$. Therefore, we conclude that

$$\gamma(n, p) = \delta(n, p) = \left\lfloor \frac{n}{p - 1} \right\rfloor$$

for each $p \leq n + 1$ and the theorem follows. □

Table 10.1: Bernoulli values of the first kind for $0 \le n \le 16$, in its usual (cancelled) form B_n; its modified (and uncancelled) form $\mathcal{B}_n = B_n/n!$, and the uncancelled denonimator E_n of $\mathcal{B}_n$.

n	0	1	2	4	6	8	10
B_n	1	$\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$
$\mathcal{B}_n$	1 $= \frac{1}{E_0}$	$\frac{1}{2}$ $= \frac{1}{E_1}$	$\frac{1}{12}$ $= \frac{1}{E_2}$	$-\frac{1}{720}$ $= -\frac{1}{E_4}$	$\frac{1}{30240}$ $= \frac{1}{E_6}$	$-\frac{3}{3628800}$ $= -\frac{3}{E_8}$	$\frac{5}{239500800}$ $= \frac{5}{E_{10}}$
E_n	1	2	$2^2.3$	$2^4.3^2.5$	$2^5.3^3.5.7$	$2^8.3^4.5^2.7$	$2^9.3^5.5^2.7.11$

n	12	14	16
B_n	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$
$\mathcal{B}_n$	$-\frac{691}{1307674368000}$ $= -\frac{691}{E_{12}}$	$\frac{105}{7846046208000}$ $= \frac{3.5.7}{E_{14}}$	$-\frac{10851}{32011868528640000}$ $= -\frac{3.3617}{E_{16}}$
E_n	$2^{11}.3^6.5^3.7^2.11.13$	$2^{12}.3^7.5^3.7^2.11.13$	$2^{16}.3^8.5^4.7^2.11.13.17$

COROLLARY. *We have*

$$\zeta(2r) = \frac{(2\pi)^{2r}}{2(2r)!}|B_{2r}|, \qquad (10.3.19)$$

and the natural denominator (before cancelling) of the value of (10.3.19) *is given by*

$$F_{2r} = 2^{2-D} \prod_{3 \le p \le 2r+1} p^{\left\lfloor \frac{2r}{p-1} \right\rfloor},$$

where D is the sum of the digits of $2r$ expressed in the scale of 2.

Proof. We have ([28]) that

$$\zeta(2r) = \frac{(2\pi)^{2r}}{2}|\mathcal{B}_{2r}|,$$

where we recall $\mathcal{B}_n = B_n/n!$, and B_n is a Bernoulli number of the first kind, so that on application of Theorem 10.3.9 and cancellation of the factor 2^{2r-1} the result is obtained. $\square$

Table 10.2: Positive Zeta values in its cancelled, $\zeta(2r)$, and uncancelled, $\zeta(2r)^*$, form; and its uncancelled denominator F_{2r}.

r	1	2	3	4	5	6	7
$\zeta(2r)$	$\frac{\pi^2}{6}$	$\frac{\pi^4}{90}$	$\frac{\pi^6}{945}$	$\frac{\pi^8}{9450}$	$\frac{\pi^{10}}{93555}$	$\frac{691\pi^{12}}{638512875}$	$\frac{2\pi^{14}}{18243225}$
$\zeta(2r)^*$	$\frac{\pi^2}{6}$ $= \frac{\pi^2}{F_2}$	$\frac{\pi^4}{90}$ $= \frac{\pi^4}{F_4}$	$\frac{\pi^6}{945}$ $= \frac{\pi^6}{F_6}$	$\frac{3\pi^8}{28350}$ $= \frac{3\pi^8}{F_8}$	$\frac{5\pi^{10}}{467775}$ $= \frac{5\pi^{10}}{F_{10}}$	$\frac{691\pi^{12}}{638512875}$ $= \frac{691\pi^{12}}{F_{12}}$	$\frac{2.105\pi^{14}}{1915538625}$ $= \frac{3.5.7\pi^{14}}{F_{14}}$
F_{2r}	2.3	$2.3^2.5$	$3^3.5.7$	$2.3^4.5^2.7$	$3^5.5^2.7.11$	$3^6.5^3.7^2.11.13$	$2^{-1}.3^7.5^3.7^2.11.13$

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Appendix A

Some expressions for $\mathcal{L}_{s;abc}(r,t,q)$

A.1 Closed binomial forms for $\mathcal{L}_{s;abc}(r,t,q)$

$$
\begin{aligned}
\mathcal{L}_{s;abc}(r,t,q) &= \mathcal{F}_{s;ab}(R,T,q) \\
&= \sum_{k\equiv T \pmod{2}q} \gamma^k \binom{R}{k} + \lambda\gamma^b \sum_{k\equiv T+q \pmod{2}q} \gamma^k \binom{R}{k} \\
&= \gamma^T \lambda^{\lfloor T/q \rfloor} \sum_{d=0}^{\lfloor (R-T_q)/q \rfloor} \lambda^d \binom{R}{T_q + dq}.
\end{aligned}
$$

When $a = c = 0$ we have

$$
\begin{aligned}
\mathcal{L}_{s;0b0}(r,t,2m+b) &= \gamma^{r+t+1} \mathcal{F}_{s;0b}(2r+2, r+t+1, 2m+b) \\
&= \sum_{k\equiv T \pmod{2}q} \gamma^k \binom{2r+2}{k} + \sum_{k\equiv T+q \pmod{2}q} \gamma^k \binom{2r+2}{k} \\
&= \gamma^{r+t+1} \sum_{d=0}^{\lfloor (R-T_q)/q \rfloor} \binom{2r+2}{T_q + dq}.
\end{aligned}
$$

When $a = 0$ and $c = 1$ we have

$$
\begin{aligned}
\mathcal{L}_{s;0b0}(r,t,2m+b) &= \gamma^{r+t} \mathcal{F}_{s;0b}(2r+1, r+t, 2m+b) \\
&= \sum_{k\equiv T \pmod{2}q} \gamma^k \binom{2r+1}{k} + \sum_{k\equiv T+q \pmod{2}q} \gamma^k \binom{2r+1}{k} \\
&= \gamma^{r+t} \sum_{d=0}^{\lfloor (R-T_q)/q \rfloor} \binom{2r+1}{T_q + dq}.
\end{aligned}
$$

When $a = 1$, $b = 0$ and $c = 0$ we have

$$\mathcal{L}_{s;100}(r, t, 2m) = \mathcal{F}_{s;10}(R, T, 2m)$$

$$= \sum_{k \equiv T \pmod{2)q}} \gamma^k \binom{2r+2}{k} - \sum_{k \equiv T+q \pmod{2)q}} \gamma^k \binom{2r+2}{k}$$

$$= \gamma^{r+t+1}(-1)^{\lfloor (r+t+1)/q \rfloor} \sum_{d=0}^{\lfloor (R-T_q)/q \rfloor} (-1)^d \binom{2r+2}{T_q + dq}.$$

When $a = 1$, $b = 0$ and $c = 1$ we have

$$\mathcal{L}_{s;101}(r, t, 2m) = \mathcal{F}_{s;10}(R, T, 2m)$$

$$= \sum_{k \equiv T \pmod{2)q}} \gamma^k \binom{2r+1}{k} - \sum_{k \equiv T+q \pmod{2)q}} \gamma^k \binom{2r+1}{k}$$

$$= \gamma^{r+t}(-1)^{\lfloor (r+t)/q \rfloor} \sum_{d=0}^{\lfloor (R-T_q)/q \rfloor} (-1)^d \binom{2r+1}{T_q + dq}.$$

When $a = b = 1$ and $c = 0$ we have

$$\mathcal{L}_{s;110}(r, t, 2m+1) = \mathcal{F}_{s;11}(R, T, 2m+1)$$

$$= \sum_{k \equiv T \pmod{2)q}} \gamma^k \binom{2r+2}{k} - \gamma \sum_{k \equiv T+q \pmod{2)q}} \gamma^k \binom{2r+2}{k}$$

$$= \gamma^{r+t+1}(-1)^{\lfloor (r+t+1)/q \rfloor} \sum_{d=0}^{\lfloor (R-T_q)/q \rfloor} (-1)^d \binom{2r+2}{T_q + dq}.$$

When $a = b = c = 1$ we have

$$\mathcal{L}_{s;111}(r, t, 2m+1) = \mathcal{F}_{s;11}(R, T, 2m+1)$$

$$= \sum_{k \equiv T \pmod{2)q}} \gamma^k \binom{2r+1}{k} - \gamma \sum_{k \equiv T+q \pmod{2)q}} \gamma^k \binom{2r+1}{k}$$

$$= \gamma^{r+t}(-1)^{\lfloor (r+t)/q \rfloor} \sum_{d=0}^{\lfloor (R-T_q)/q \rfloor} (-1)^d \binom{2r+1}{T_q + dq}.$$

A.2 Expression as a sum of $(r+1)$-th powers

To express the term $\mathcal{L}_{s;abc}(r, t, q)$ as a sum of r-th powers we recall Theorem 4.4 that states

$$\mathcal{L}_{s;abc}(r, t, q) = \frac{\gamma^{r+1-c}2^{2r+3-c}}{q} \times$$

$$\left(\frac{(a-1)^{st}}{2} + \sum_{d=1}^{\lfloor (q+a-1)/2 \rfloor} \cos\left(\frac{\pi(c-2t)(2d - \epsilon - scq)}{2q} \right) \left(\cos\left(\frac{\pi(2d - \epsilon - sq)}{2q} \right) \right)^{2r+2-c} \right),$$

where

$$\epsilon \text{ is } \begin{cases} 0 & \text{if } a = sb, \\ 1 & \text{if } a \neq sb. \end{cases}$$

If we let $x_{m,0} = 4\gamma$, and for $d \geq 1$,

$$x_{m,d} = 4\gamma \cos^2\left(\frac{\pi(2(d - s(m + a'b)) - a)}{2q}\right) = 4\gamma \cos^2\left(\frac{\pi(2D - a)}{2q}\right),$$

where $D = d - s(m - a'b)$, $a' = 1 - a$ and $q = 2m + b$, then for all non-negative integers r, the function $\mathcal{L}_{s;abc}$ takes the form

$$q\mathcal{L}_{s;abc}(r, t, q) = \alpha_{m,t,0}x_{m,0}^r + \alpha_{m,t,1}x_{m,1}^r + \alpha_{m,t,2}x_{m,2}^r + \ldots + \alpha_{m,t,M}x_{m,M}^r.$$

Here we define

$$\alpha_{m,t,0} = \begin{cases} \frac{4}{2^c}\gamma^{t+1-c} & \text{if } a = 0 \\ 0 & \text{if } a = 1, \end{cases}$$

and more generally for $1 \leq d \leq M = m + (b - 1)(1 - a)$,

$$\alpha_{m,t,d} = \frac{\gamma 8}{2^c} \cos\frac{\pi(c - 2t)(2D - a)}{2q} \cos^{2-c}\frac{\pi(2D - a)}{2q}.$$

We also note that each $\alpha_{m,t,d}$ is real.

Remark. We note that upon fixing the parameters a, b and c, the variables affecting the x terms are m and d and that of the α terms are m, t and d. Furthermore, whereas the sums produced (for positive r) are integers, the scalars $\alpha_{m,t,d}$, are in general not integers.

However, in respect to the above remark it is apparent that certain values of t produce "nice" values for $\alpha_{m,t,d}$. In particular, we find that for the function $\mathcal{L}_{s;ab1}$, with $t = 1$, that (for $d \geq 1$) $\alpha_{m,t,d} = x_{m,d}$, and so we obtain

$$q\mathcal{L}_{s;ab1}(r, 1, q) = \gamma^{r+1}2^{2r+2-c} + x_{m,1}^{r+1} + x_{m,2}^{r+1} + \ldots + x_{m,M}^{r+1}.$$

So for example, when $t = 1$, $q = 6$, $(b = 0)$, $a = 1$ and $s = 0$, with $1 \leq d \leq 3$ we have

$$6l_{101}(r, 1, 6) = 2^{r+1}\left(\left(1 + \frac{\sqrt{3}}{2}\right)^{r+1} + 1 + \left(1 - \frac{\sqrt{3}}{2}\right)^{r+1}\right),$$

and when $t = 1$, $q = 6$, $(b = 0)$, $a = 0$ and $s = 1$, with $0 \leq d \leq 2$ the expression becomes

$$6L_{001}(r, 1, 6) = (-1)^{r+1}\left(\frac{4^{r+1}}{2} + 1^{r+1} + 3^{r+1}\right).$$

Moreover, it is evident that when the parameter $c = 1$, we have that $\alpha_{m,0,d} = \alpha_{m,1,d}$, so that the sequences for $t = 0$ and $t = 1$ are identical.

Appendix B

The polynomials $A_{s;ab}(x, Q)$

B.1 Expression as Fibonacci type polynomials

THEOREM B.1.1 (Theorem 5.4.1). *The polynomial $A_{s;ab}(x, Q)$ defined in Definition 5.4.1 is equated to a Fibonacci, Lucas or (monic) Chebyshev polynomial such that*

$$A_{s;ab}(x, Q) = \begin{cases} S_{2(m-1+b)+1-b}(x) & \text{if } s = 0,\ a = 0 \\ C_{2m+b}(x) & \text{if } s = 0,\ a = 1 \\ \\ F_{2(m-1+b)+2-b}(x) & \text{if } s = 1,\ a = 0 \\ L_{2m+b}(x) & \text{if } s = 1,\ a = 1, \end{cases}$$

where $Q = q - (1-a)(1-s)$ and $q = 2m + b$.

Proof. This follows on substitution of each value of each of the parameters s, a and b into the product and binomial and forms of $A_{s;ab}(x, Q)$ as given in Definition 5.4.1, and then compared with the corresponding (monic) Chebyshev, Fibonacci and Lucas polynomial forms.

$$\begin{aligned} A_{0;00}(x, 2m-1) &= A_{0;00}(x, 2(m-1)+1) \\ &= \prod_{d=1}^{2m-1} \left(x - 2\cos\frac{\pi d}{2m} \right) = \sum_{k=0}^{m-1} (-1)^k \binom{2(m-1)+1-k}{k} x^{2(m-1)+1-2k} \\ &= S_{2(m-1)+1}(x), \end{aligned}$$

$$A_{0;01}(x, 2m) = \prod_{d=1}^{2m} \left(x - 2\cos\frac{\pi d}{2m+1} \right) = \sum_{k=0}^{m} (-1)^k \binom{2m-k}{k} x^{2m-2k} = S_{2m}(x),$$

$$\begin{aligned} A_{1;00}(x, 2m) &= A_{2(m-1)+2}(x) \\ &= \prod_{d=1}^{2m-1} \left(x - 2\imath\cos\frac{\pi d}{2m} \right) = \sum_{k=0}^{m-1} \binom{2(m-1)+1-k}{k} x^{2(m-1)+1-2k} \\ &= F_{2(m-1)+2}(x), \end{aligned}$$

$$A_{1;01}(x, 2m+1) = \prod_{d=1}^{2m} \left(x - 2\imath \cos \frac{\pi d}{2m+1} \right) = \sum_{k=0}^{m} \binom{2m-k}{k} x^{2m-2k} = F_{2m+1}(x),$$

$$A_{0;10}(x, 2m) = \prod_{d=1}^{2m} \left(x - 2 \cos \frac{(2d-1)\pi}{4m} \right) = \sum_{k=0}^{m} (-1)^k \frac{2m}{2m-k} \binom{2m-k}{k} x^{2m-2k} = C_{2m}(x),$$

$$A_{0;11}(x, 2m+1) = \prod_{d=1}^{2m+1} \left(x - 2 \cos \frac{(2d-1)\pi}{2(2m+1)} \right) = \sum_{k=0}^{m} (-1)^k \frac{2m+1}{2m+1-k} \binom{2m+1-k}{k} x^{2m+1-2k}$$
$$= C_{2m+1}(x),$$

$$A_{1;10}(x, 2m) = \prod_{d=1}^{2m} \left(x - 2\imath \cos \frac{(2d-1)\pi}{4m} \right) = \sum_{k=0}^{m} \frac{2m}{2m-k} \binom{2m-k}{k} x^{2m-2k} = L_{2m}(x),$$

and

$$A_{1;11}(x, 2m+1) = \prod_{d=1}^{2m+1} \left(x - 2\imath \cos \frac{(2d-1)\pi}{2(2m+1)} \right) = \sum_{k=0}^{m} \frac{2m+1}{2m+1-k} \binom{2m+1-k}{k} x^{2m+1-2k}$$
$$= L_{2m+1}(x).$$

$$\square$$

B.2 Expression as modified Fibonacci type polynomials

Expressing each of the modified polynomials that follows from the Corollary to Theorem 5.4.1.

$$A_{0;00}^{r}(x, 2m-1) = S_{2(m-1)+1}^{r}(x)$$
$$= (\sqrt{x})^{-1} \prod_{d=1}^{2m-1} \left(\sqrt{x} - 2 \cos \frac{\pi d}{2m} \right) = \sum_{k=0}^{m-1} (-1)^k \binom{2(m-1)+1-k}{k} x^{m-1-k},$$

$$A_{0;01}^{r}(x, 2m) = S_{2m}^{r}(x)$$
$$= \prod_{d=1}^{2m} \left(\sqrt{x} - 2 \cos \frac{\pi d}{2m+1} \right) = \sum_{k=0}^{m} (-1)^k \binom{2m-k}{k} x^{m-k},$$

$$A_{1;00}^{r}(x, 2m) = F_{2(m-1)+2}^{r}(x)$$
$$= (\sqrt{x})^{-1} \prod_{d=1}^{2m-1} \left(\sqrt{x} - 2\imath \cos \frac{\pi d}{2m} \right) = \sum_{k=0}^{m-1} \binom{2(m-1)+1-k}{k} x^{m-1-k},$$

$$A^r_{1;01}(x, 2m+1) = F^r_{2m+1}(x)$$

$$= \prod_{d=1}^{2m} \left(\sqrt{x} - 2\imath \cos \frac{\pi d}{2m+1} \right) = \sum_{k=0}^{m} \binom{2m-k}{k} x^{m-k},$$

$$A^r_{0;10}(x, 2m) = C^r_{2m}(x)$$

$$= \prod_{d=1}^{2m} \left(\sqrt{x} - 2 \cos \frac{(2d-1)\pi}{4m} \right) = \sum_{k=0}^{m} (-1)^k \frac{2m}{2m-k} \binom{2m-k}{k} x^{m-k},$$

$$A^r_{0;11}(x, 2m+1) = C^r_{2m+1}(x)$$

$$= (\sqrt{x})^{-1} \prod_{d=1}^{2m+1} \left(\sqrt{x} - 2 \cos \frac{(2d-1)\pi}{2(2m+1)} \right) = \sum_{k=0}^{m} (-1)^k \frac{2m+1}{2m+1-k} \binom{2m+1-k}{k} x^{m-k},$$

$$A^r_{1;10}(x, 2m) = L^r_{2m}(x)$$

$$= \prod_{d=1}^{2m} \left(\sqrt{x} - 2\imath \cos \frac{(2d-1)\pi}{4m} \right) = \sum_{k=0}^{m} \frac{2m}{2m-k} \binom{2m-k}{k} x^{m-k},$$

and

$$A^r_{1;11}(x, 2m+1) = L^r_{2m+1}(x)$$

$$= (\sqrt{x})^{-1} \prod_{d=1}^{2m+1} \left(\sqrt{x} - 2\imath \cos \frac{(2d-1)\pi}{2(2m+1)} \right) = \sum_{k=0}^{m} \frac{2m+1}{2m+1-k} \binom{2m+1-k}{k} x^{m-k}.$$

B.3 Simplification of expression as Fibonacci type polynomials

THEOREM B.3.1 (Theorem 5.4.2). *For $A_{s;ab}(x, Q)$ defined as in Definition 5.4.1 we have*

$$A_{s;ab}(x, Q) = x^\epsilon \prod_{d=1}^{m-(1-a)(1-b)} \left(x^2 - 4\gamma \cos^2 \frac{(2d-a)\pi}{2(2m+b)} \right),$$

where $Q = q - (1-a)(1-s)$ and $\epsilon = a(2b-1) + 1 - b$.

Proof. The proof is a demonstration by subsitution of each value of each of the parameters s, a and b into Definition 5.4.1 and suitable "pairing" of terms. Let us first consider the cases for s.

When $s = 0$, we have

$$A_{0;ab}(x, 2m + b - (1-a)) = \prod_{d=1}^{2m+b+a-1} \left(x - 2 \cos \frac{(2d-a)\pi}{2(2m+b)} \right),$$

on the other hand when $s = 1$,

$$A_{1;ab}(x, 2m+b) = \prod_{d=1}^{2m+b+a-1} \left(x - 2\imath \cos\frac{(2d-a)\pi}{2(2m+b)} \right).$$

Now with $a = 0$ we obtain

$$A_{s;0b}(x, 2m+b-(1-s)) = \prod_{d=1}^{q-1} \left(x - 2\imath^s \cos\frac{2d\pi}{2q} \right) = \prod_{d=1}^{2m+b-1} \left(x - 2\imath^s \cos\frac{d\pi}{(2m+b)} \right).$$

More specifically, when $a = b = 0$, we have

$$A_{0;00}(x, 2m-1) = \prod_{d=1}^{2m-1} \left(x - 2\cos\frac{d\pi}{2m} \right)$$

$$= \left(x - 2\cos\frac{m\pi}{2m} \right) \prod_{d=1}^{m-1} \left(x - 2\cos\frac{d\pi}{2m} \right) \left(x - 2\cos\frac{(2m-k)\pi}{2m} \right)$$

$$= x \prod_{d=1}^{m-1} \left(x - 2\cos\frac{d\pi}{2m} \right) \left(x + 2\cos\frac{d\pi}{2m} \right)$$

$$= x \prod_{d=1}^{m-1} \left(x^2 - 4\cos^2\frac{d\pi}{2m} \right),$$

and

$$A_{1;00}(x, 2m) = \prod_{d=1}^{2m-1} \left(x - 2\imath \cos\frac{d\pi}{2m} \right)$$

$$= \left(x - 2\imath \cos\frac{m\pi}{2m} \right) \prod_{d=1}^{m-1} \left(x - 2\imath \cos\frac{d\pi}{2m} \right) \left(x - 2\imath \cos\frac{(2m-d)\pi}{2m} \right)$$

$$= x \prod_{d=1}^{m-1} \left(x - 2\imath \cos\frac{d\pi}{2m} \right) \left(x + 2\imath \cos\frac{d\pi}{2m} \right)$$

$$= x \prod_{d=1}^{m-1} \left(x^2 + 4\cos^2\frac{d\pi}{2m} \right).$$

Now with $a = 0$ and $b = 1$, we have

$$A_{0;01}(x, 2m) = \prod_{d=1}^{2m} \left(x - 2\cos\frac{d\pi}{2m+1} \right)$$

$$= \prod_{d=1}^{m} \left(x - 2\cos\frac{d\pi}{2m+1} \right) \left(x - 2\cos\frac{(2m+1-d)\pi}{2m+1} \right)$$

$$= \prod_{d=1}^{m} \left(x - 2\cos\frac{d\pi}{2m+1} \right) \left(x + 2\cos\frac{d\pi}{2m+1} \right)$$

$$= \prod_{d=1}^{m} \left(x^2 - 4\cos^2\frac{d\pi}{2m+1} \right),$$

and

$$A_{1;01}(x, 2m+1) = \prod_{d=1}^{2m} \left(x - 2\imath \cos \frac{k\pi}{2m+1} \right)$$

$$= \prod_{d=1}^{m} \left(x - 2\imath \cos \frac{d\pi}{2m+1} \right) \left(x - 2\imath \cos \frac{(2m+1-d)\pi}{2m+1} \right)$$

$$= \prod_{d=1}^{m} \left(x - 2\imath \cos \frac{d\pi}{2m+1} \right) \left(x + 2\imath \cos \frac{d\pi}{2m+1} \right)$$

$$= \prod_{d=1}^{m} \left(x^2 + 4 \cos^2 \frac{d\pi}{2m+1} \right).$$

Next we look at the cases when $a = 1$ and in general obtain

$$A_{s;1b}(x, 2m+b) = \prod_{d=1}^{2m+b} \left(x - 2\imath^s \cos \frac{(2d-1)\pi}{2(2m+b)} \right)$$

$$= \prod_{d=1}^{q} \left(x - 2\imath^s \cos \frac{(2d-1)\pi}{2q} \right).$$

Repeating the first four cases for $a = 1$, we have

$$A_{0;10}(x, 2m) = \prod_{d=1}^{2m} \left(x - 2 \cos \frac{(2d-1)\pi}{4m} \right)$$

$$= \prod_{d=1}^{m} \left(x - 2 \cos \frac{(2d-1)\pi}{4m} \right) \left(x + 2 \cos \frac{(2d-1)\pi}{4m} \right)$$

$$= \prod_{d=1}^{m} \left(x^2 - 4 \cos^2 \frac{(2d-1)\pi}{4m} \right),$$

and

$$A_{1;10}(x, 2m) = \prod_{d=1}^{2m} \left(x - 2\imath \cos \frac{(2d-1)\pi}{4m} \right)$$

$$= \prod_{d=1}^{m} \left(x - 2\imath \cos \frac{(2d-1)\pi}{4m} \right) \left(x - 2\imath \cos \frac{(2m-(2d-1))\pi}{4m} \right)$$

$$= \prod_{d=1}^{m} \left(x - 2\imath \cos \frac{(2d-1)\pi}{4m} \right) \left(x + 2\imath \cos \frac{(2d-1)\pi}{4m} \right)$$

$$= \prod_{d=1}^{m} \left(x^2 + 4 \cos^2 \frac{(2d-1)\pi}{4m} \right).$$

Finally when $a = b = 1$, we have

$$A_{0;11}(x, 2m + 1) = \prod_{d=1}^{2m+1} \left(x - 2\cos\frac{(2d-1)\pi}{2(2m+1)} \right)$$

$$= \left(x - 2\cos\frac{(2m+1)\pi}{2(2m+1)} \right) \prod_{d=1}^{m} \left(x - 2\cos\frac{(2d-1)\pi}{2(2m+1)} \right) \left(x - 2\cos\frac{(2m+1-(2d-1)))\pi}{2(2m+1)} \right)$$

$$= x \prod_{d=1}^{m} \left(x - 2\cos\frac{(2d-1)\pi}{2(2m+1)} \right) \left(x + 2\cos\frac{(2d-1)\pi}{2(2m+1)} \right)$$

$$= x \prod_{d=1}^{m} \left(x^2 - 4\cos^2\frac{(2d-1)\pi}{2(2m+1)} \right),$$

and

$$A_{1;11}(x, 2m + 1) = \prod_{d=1}^{2m+1} \left(x - 2\imath\cos\frac{(2d-1)\pi}{2(2m+1)} \right)$$

$$= \left(x - 2\imath\cos\frac{\pi}{2} \right) \prod_{d=1}^{m} \left(x - 2\imath\cos\frac{(2d-1)\pi}{2(2m+1)} \right) \left(x - 2\imath\cos\frac{(2m+1-(2d-1)))\pi}{2(2m+1)} \right)$$

$$= x \prod_{d=1}^{m} \left(x - 2\imath\cos\frac{(2d-1)\pi}{2(2m+1)} \right) \left(x + 2\imath\cos\frac{(2d-1)\pi}{2(2m+1)} \right)$$

$$= x \prod_{d=1}^{m} \left(x^2 + 4\cos^2\frac{(2d-1)\pi}{2(2m+1)} \right)$$

$$= x \prod_{d=1}^{m} \left(x^2 - 4\gamma\cos^2\frac{(2d-1)\pi}{2(2m+1)} \right).$$

$$\square$$

B.4 Simplification of expression as modified Fibonacci type polynomials

Simplifying the product form of $A^r_{s;ab}(x, Q)$ that follows from the Corollary to Theorem 5.4.2 we have

$$A^r_{0;00}(x, 2(m-1) + 1) = S^r_{2m-1}(x) = \prod_{d=1}^{m-1} \left(x - 4\cos^2\frac{d\pi}{2m} \right),$$

$$A^r_{0;01}(x, 2m) = S^r_{2m}(x) = \prod_{d=1}^{m} \left(x - 4\cos^2\frac{d\pi}{2m+1} \right),$$

$$A^r_{1;00}(x, 2(m-1) + 2) = F^r_{2m}(x) = \prod_{d=1}^{m-1} \left(x + 4\cos^2\frac{d\pi}{2m} \right),$$

$$A^r_{1;01}(x, 2m + 1) = F^r_{2m+1}(x) = \prod_{d=1}^{m} \left(x + 4\cos^2\frac{d\pi}{2m+1} \right),$$

$$A_{0;10}^r(x, 2m) = C_{2m}^r(x) = \prod_{d=1}^{m} \left(x - 4\cos^2 \frac{(2d-1)\pi}{4m} \right),$$

$$A_{0;11}^r(x, 2m+1) = C_{2m+1}^r(x) = \prod_{d=1}^{m} \left(x - 4\cos^2 \frac{(2d-1)\pi}{2(2m+1)} \right),$$

$$A_{1;10}^r(x, 2m) = L_{2m}^r(x) = \prod_{d=1}^{m} \left(x + 4\cos^2 \frac{(2d-1)\pi}{4m} \right),$$

and

$$A_{1;11}^r(x, 2m+1) = L_{2m+1}^r(x) = \prod_{d=1}^{m} \left(x + 4\cos^2 \frac{(2d-1)\pi}{2(2m+1)} \right).$$

B.5 The recurrence polynomial, $\mathcal{R}_{s;ab}(x, m)$

THEOREM B.5.1 (Theorem 5.5.2). *The recurrence polynomials $\mathcal{R}_{s;ab}(x, m)$ are, expressed as a product of their roots, given by*

$$\mathcal{R}_{s;ab}(x, m) = \prod_{d=a}^{m-(1-a)(1-b)} \left(x - 4\gamma \cos^2 \left(\frac{\pi(2d-a)}{2q} \right) \right),$$

where $\gamma = (-1)^s$.

Proof. We consider each of the four cases for the parameters a and b for both the cases $s = 0$ and $s = 1$.

Case 1: $a = 0$, $b = 0$.
From (5.5.5) the roots of the recurrence polynomial $\mathcal{R}_{0;00}(x, m)$ are $4\cos^2 d\pi/q$, where $0 \le d \le m-1$. On the other hand, the roots of $\mathcal{R}_{1;00}(x, m)$ are $-4\sin^2 d\pi/q$, where $1 \le d \le m$. Since $m/q = 1/2$ we have that

$$\sin \frac{(m-d)\pi}{q} = \cos \frac{d\pi}{q},$$

so that

$$\prod_{d=1}^{m} \left(x + 4\sin^2 \frac{d\pi}{q} \right) = \prod_{d=0}^{m-1} \left(x + 4\sin^2 \frac{(m-d)\pi}{q} \right) = \prod_{d=0}^{m-1} \left(x + 4\cos^2 \frac{d\pi}{q} \right).$$

Case 2: $a = 0$, $b = 1$.
From (5.5.5) the roots of the recurrence polynomial $\mathcal{R}_{0;01}(x, m)$ are still of the form $4\cos^2 d\pi/q$, but now $0 \le d \le m$. However, we see that those of $\mathcal{R}_{1;01}(x, m)$ are given by $-4\sin^2 (2d-1)\pi/2q$, where $1 \le d \le m+1$.
Noting that

$$\sin \frac{(q-2d)\pi}{2q} = \cos \left(-\frac{2d\pi}{2q} \right) = \cos \frac{d\pi}{q},$$

we then have

$$\prod_{d=1}^{m+1}\left(x+4\sin^2\frac{(2d-1)\pi}{2q}\right)=\prod_{d=0}^{m}\left(x+4\sin^2\frac{(2m+1-2d)\pi}{2q}\right)$$
$$=\prod_{d=0}^{m}\left(x+4\sin^2\frac{(q-2d)\pi}{2q}\right)=\prod_{d=0}^{m}\left(x+4\cos^2\frac{d\pi}{q}\right).$$

Case 3: $a=1$, $b=0$.

When the sum alternates in sign, we observe from (5.5.5), that the roots of $\mathcal{R}_{0;10}(x,m)$ and $\mathcal{R}_{1;10}(x,m)$ are respectively $4\cos^2(2d-1)\pi/2q$ and $-4\sin^2(2d-1)\pi/2q$, both with $1\leq d\leq m$. We have

$$\sin\frac{(q-(2d-1))\pi}{2q}=\cos\left(-\frac{(2d-1)\pi}{2q}\right)=\cos\frac{(2d-1)\pi}{2q},$$

so that

$$\prod_{d=1}^{m}\left(x+4\sin^2\frac{(2d-1)\pi}{2q}\right)=\prod_{d=1}^{m}\left(x+4\sin^2\frac{(2m+1-2d)\pi}{2q}\right)$$
$$=\prod_{d=1}^{m}\left(x+4\sin^2\frac{(q-(2d-1))\pi}{2q}\right)=\prod_{d=1}^{m}\left(x+4\cos^2\frac{(2d-1)\pi}{2q}\right).$$

Case 4: $a=1$, $b=1$.

From (5.5.5) the parameter b has no effect on the recurrence polynomial $\mathcal{R}_{0;11}(x,m)$ and so we have that

$$\mathcal{R}_{0;10}(x,m)=\mathcal{R}_{0;11}(x,m)=\prod_{d=1}^{m}\left(x-4\cos^2\frac{(2d-1)\pi}{2q}\right).$$

Conversely, for $\mathcal{R}_{1;11}(x,m)$, we note that $\epsilon\equiv 0\pmod 2$, and consequently we have

$$\prod_{d=1}^{m}\left(x+4\sin^2\frac{d\pi}{q}\right)=\prod_{d=1}^{m}\left(x+4\sin^2\frac{(m+1-d)\pi}{q}\right)$$
$$=\prod_{d=1}^{m}\left(x+4\sin^2\frac{(2m+2-2d)\pi}{2q}\right)=\prod_{d=1}^{m}\left(x+4\sin^2\frac{(2m+1-(2d-1))\pi}{2q}\right)$$
$$=\prod_{d=1}^{m}\left(x+4\sin^2\frac{(q-(2d-1))\pi}{2q}\right)=\prod_{d=1}^{m}\left(x+4\cos^2\frac{(2d-1)\pi}{2q}\right).$$

$\square$

Appendix C

Some calculations of the recurrence polynomial $\mathcal{R}_{s;ab}(x, m)$

C.1 Evaluation of the coefficients from the roots

We commence with the (non)alternating parameter case $a = 0$. From (5.5.5), the dth root of the function $l_{0bc}(r, t, q)$ is given by

$$x_d = 4\cos^2\left(\frac{\pi d}{2m + b}\right),$$

where $0 \leq d \leq m + b - 1$.

So for the even base parameter case $b = 0$, when $q = 2m = 2$, $d = 0$ and
$x_0 = 4\cos^2(0\pi/2) = 4(1)^2 = 4$,
$\mathcal{R}_{0;00}(x, 1) = (x - 4)$.

When $q = 4$, then $d = 0, 1$ and
$x_0 = 4\cos^2(0\pi/4) = 4(1)^2 = 4$,
$x_1 = 4\cos^2(\pi/4) = 4(\sqrt{2}/2)^2 = 2$,
$\mathcal{R}_{0;00}(x, 2) = (x - 4)(x - 2) = x^2 - 6x + 8$.

When $q = 6$, then $d = 0, 1, 2$
$x_0 = 4\cos^2(0\pi/6) = 4(1) = 4$,
$x_1 = 4\cos^2(\pi/6) = 4(\sqrt{3}/2)^2 = 3$,
$x_2 = 4\cos^2(2\pi/6) = 4(1/2)^2 = 1$,
$\mathcal{R}_{0;00}(x, 3) = (x - 4)(x - 3)(x - 1) = x^3 - 8x^2 + 19x - 12$.

When $q = 8$, then $d = 0, 1, 2, 3$ and
$x_0 = 4\cos^2(0\pi/8) = 4(1)2 = 4$,
$x_1 = 4\cos^2(\pi/8) = 4(\sqrt{2 + \sqrt{2}}/2)^2 = 2 + \sqrt{2}$,

$x_2 = 4\cos^2(2\pi/8) = 4(\sqrt{2}/2)^2 = 2,$

$x_3 = 4\cos^2(3\pi/8) = 4(\sqrt{2-\sqrt{2}}/2)^2 = 2 - \sqrt{2},$

$\mathcal{R}_{0;00}(x,4) = (x-4)(x-2)(x-2-\sqrt{2})(x-2+\sqrt{2}) = x^4 - 10x^3 + 34x^2 - 44x + 16.$

For the case $a = 0$ with odd base $b = 1$, the form (5.5.5) remains unaltered but from Theorem 4.4.2 there are now $m + 1$ roots. So when $q = 2m + 1 = 3$, then $d = 0, 1$ and we have

$x_0 = 4\cos^2(0\pi/3) = 4(1)^2 = 4,$

$x_1 = 4\cos^2(\pi/3) = 4(1/2)^2 = 1,$

$\mathcal{R}_{0;01}(x,1) = (x-4)(x-1) = x^2 - 5x + 4.$

And when $q = 5$, then $d = 0, 1, 2$ and

$x_0 = 4\cos^2(0\pi/5) = 4(1)^2 = 4,$

$x_1 = 4\cos^2(\pi/5) = 4(\sqrt{5}+1)^2/16 = (6 + 2\sqrt{5})/4,$

$x_2 = 4\cos^2(2\pi/5) = 4(\sqrt{5}-1)^2/16 = (6 - 2\sqrt{5})/4,$

$\mathcal{R}_{0;01}(x,2) = (x-4)(x-(6+2\sqrt{5})/4)(x-(6-2\sqrt{5})/4) = x^3 - 7x^2 + 13x - 4.$

If, on the other hand, the alternating parameter case is $a = 1$, we simplify (5.5.5) to

$$x_d = 4\cos^2(\pi(2d-1)/2(2m+b)), \qquad \text{where } 1 \le d \le m.$$

So for even base we have $q = 2m = 2$, $d = 1$ and

$x_1 = 4\cos^2(\pi/4) = 4(1/2) = 2,$

$\mathcal{R}_{0;10}(x,1) = (x-2).$

When $q = 4$, then $d = 1, 2$ and

$x_1 = 4\cos^2(\pi/8) = 4(\sqrt{2+\sqrt{2}}/2)^2 = 2 + \sqrt{2},$

$x_2 = 4\cos^2(3\pi/8) = 4(\sqrt{2-\sqrt{2}}/2)^2 = 2 - \sqrt{2},$

$\mathcal{R}_{0;10}(x,2) = x^2 - 4x + 2.$

When $q = 6$, then $d = 1, 2, 3$ and

$x_1 = 4\cos^2(\pi/12) = 4(\sqrt{2+\sqrt{3}}/2)^2 = 2 + \sqrt{3},$

$x_2 = 4\cos^2(3\pi/12) = 4(\sqrt{2}/2)^2 = 2,$

$x_3 = 4\cos^2(5\pi/12) = 4(\sqrt{2-\sqrt{3}}/2)^2 = 2 - \sqrt{3},$

$\mathcal{R}_{0;10}(x,3) = \left(x - (2+\sqrt{3})\right)\left(x - (2-\sqrt{3})\right)(x-2) = x^3 - 6x^2 + 9x - 2.$

When $q = 8$, then $d = 1, 2, 3, 4$ and

$x_1 = 4\cos^2(\pi/16) = 4(\sqrt{2+\sqrt{2+\sqrt{2}}}/2)^2 = 2 + \sqrt{2+\sqrt{2}},$

$x_2 = 4\cos^2(3\pi/16) = 4(\sqrt{2+\sqrt{2-\sqrt{2}}}/2)^2 = 2 + \sqrt{2-\sqrt{2}},$

$$x_3 = 4\cos^2\left(5\pi/16\right) = 4(\sqrt{2 - \sqrt{2 - \sqrt{2}}}/2)^2 = 2 - \sqrt{2 - \sqrt{2}},$$

$$x_4 = 4\cos^2\left(7\pi/16\right) = 4(\sqrt{2 - \sqrt{2 + \sqrt{2}}}/2)^2 = 2 - \sqrt{2 + \sqrt{2}},$$

$$\mathcal{R}_{0;10}(x, 4) = \left(x - (2 + \sqrt{2 + \sqrt{2}})\right)\left(x - (2 - \sqrt{2 + \sqrt{2}})\right)$$
$$\times \left(x - (2 - \sqrt{2 - \sqrt{2}})\right)\left(x - (2 - \sqrt{2 + \sqrt{2}})\right)$$
$$= x^4 - 8x^3 + 20x^2 - 16x + 2.$$

Finally for the case $a = 1$ with odd base $b = 1$, when $q = 3$, $d = 1$ and

$$x_1 = 4\cos^2\left(\pi/6\right) = 4(\sqrt{3}/2)^2 = 3,$$

$$\mathcal{R}_{0;11}(x, 1) = (x - 3).$$

When $q = 2m + 1 = 5$, $d = 1, 2$ and

$$x_1 = 4\cos^2\left(\pi/10\right) = 4\left(\sqrt{(5 + \sqrt{5})/8}\right)^2 = (5 + \sqrt{5})/2,$$

$$x_2 = 4\cos^2\left(3\pi/10\right) = 4\left(\sqrt{(5 - \sqrt{5})/8}\right)^2 = (5 - \sqrt{5})/2,$$

$$\mathcal{R}_{0;11}(x, 2) = (x - (5 + \sqrt{5})/2)(x - (5 - \sqrt{5})/2) = x^2 - 5x + 5.$$

C.2 Evaluation of the coefficients using Theorem 5.6.1

When $a = 0$ we have

$$\mathcal{R}_{0;0b}(x, m) = (x - 4)(\sqrt{x})^{b-1}S_{2m+b-1}(\sqrt{x}) = (x - 4)S^1_{2m+b-1}(x).$$

In the case $b = 0$, the roots are given by $(\sqrt{x})^{-1}S_{2m-1}(\sqrt{x})$ and so we have

$$\mathcal{R}_{0;00}(x, m) = (x - 4)(\sqrt{x})^{-1}S_{2m-1}(\sqrt{x}) = (x - 4)S^1_{2m+b-1}(x).$$

The first few polynomials are given by

$$\mathcal{R}_{0;00}(x,1) = (x-4)(\sqrt{x})^{-1}S_1(\sqrt{x}) = (x-4)(\sqrt{x})^{-1}(\sqrt{x})$$
$$= x - 4,$$
$$\mathcal{R}_{0;00}(x,2) = (x-4)(\sqrt{x})^{-1}S_3(\sqrt{x}) = (x-4)(\sqrt{x})^{-1}\left((\sqrt{x})^3 - 2\sqrt{x}\right)$$
$$= (x-4)(x-2)$$
$$= x^2 - 6x + 8,$$
$$\mathcal{R}_{0;00}(x,3) = (x-4)(\sqrt{x})^{-1}S_5(\sqrt{x}) = (x-4)(\sqrt{x})^{-1}\left((\sqrt{x})^5 - 4(\sqrt{x})^3 + 3\sqrt{x}\right)$$
$$= (x-4)(x^2 - 4x + 3)$$
$$= x^3 - 8x^2 + 19x - 12,$$
$$\mathcal{R}_{0;00}(x,4) = (x-4)(\sqrt{x})^{-1}S_7(\sqrt{x}) = (x-4)(\sqrt{x})^{-1}\left((\sqrt{x})^7 - 6(\sqrt{x})^5 + 10(\sqrt{x})^3 - 4\sqrt{x}\right)$$
$$= (x-4)(x^3 - 6x^2 + 10x - 4)$$
$$= x^4 - 10x^3 + 34x^2 - 44x + 16,$$
$$\mathcal{R}_{0;00}(x,5) = (x-4)(\sqrt{x})^{-1}S_9(\sqrt{x}) = (x-4)(\sqrt{x})^{-1}\left((\sqrt{x})^9 - 8(\sqrt{x})^7 + 21(\sqrt{x})^5 - 20(\sqrt{x})^3 + 5\sqrt{x}\right)$$
$$= (x-4)(x^4 - 8x^3 + 21x^2 - 20x + 5$$
$$= x^5 - 12x^4 + 53x^3 - 104x^2 + 85x - 20).$$

In the case $b = 1$ the roots are determined by

$$\mathcal{R}_{0;01}(x,m) = (x-4)S_{2m}(\sqrt{x}) \qquad \text{of order } m+1,$$

so that,

$$\mathcal{R}_{0;01}(x,1) = (x-4)S_2(\sqrt{x}) = (x-4)\left((\sqrt{x})^2 - 1\right)$$
$$= (x-4)(x-1)$$
$$= x^2 - 5x + 4,$$
$$\mathcal{R}_{0;01}(x,2) = (x-4)S_4(\sqrt{x}) = (x-4)\left((\sqrt{x})^4 - 3(\sqrt{x})^2 + 1\right)$$
$$= (x-4)(x^2 - 3x + 1)$$
$$= x^3 - 7x^2 + 13x - 4,$$
$$\mathcal{R}_{0;01}(x,3) = (x-4)S_6(\sqrt{x}) = (x-4)\left((\sqrt{x})^6 - 5(\sqrt{x})^4 + 6(\sqrt{x})^4 - 1\right)$$
$$= (x-4)(x^3 - 5x^2 + 6x - 1)$$
$$= x^4 - 9x^3 + 26x^2 - 25x + 4,$$
$$\mathcal{R}_{0;01}(x,4) = (x-4)S_8(\sqrt{x}) = (x-4)\left((\sqrt{x})^8 - 7(\sqrt{x})^6 + 15(\sqrt{x})^4 - 10(\sqrt{x})^2 + 1\right)$$
$$= (x-4)(x^4 - 7x^3 + 15x^2 - 10x + 1)$$
$$= x^5 - 11x^4 + 43x^3 - 70x^2 + 41x - 4.$$

When $a = 1$ with $b = 0$ we identify with the polynomial $C_{2m}(\sqrt{x})$ so that the first few polynomials are given by

$$\mathcal{R}_{0;10}(x, 1) = C_2(\sqrt{x}) = (\sqrt{x})^2 - 2$$
$$= x - 2,$$
$$\mathcal{R}_{0;10}(x, 2) = C_4(\sqrt{x}) = (\sqrt{x})^4 - 4(\sqrt{x})^2 + 2$$
$$= x^2 - 4x + 2,$$
$$\mathcal{R}_{0;10}(x, 3) = C_6(\sqrt{x}) = (\sqrt{x})^6 - 6(\sqrt{x})^4 + 9(\sqrt{x})^4 - 2$$
$$= x^3 - 6x^2 + 9x - 2,$$
$$\mathcal{R}_{0;10}(x, 4) = C_8(\sqrt{x}) = (\sqrt{x})^8 - 8(\sqrt{x})^6 + 20(\sqrt{x})^4 - 16(\sqrt{x})^2 + 2$$
$$= x^4 - 8x^3 + 20x^2 - 16x + 2.$$

and when $b = 1$ we have $\mathcal{R}_{0;11}(x, 2m + 1)$ so that

$$\mathcal{R}_{0;11}(x, 1) = (\sqrt{x})^{-1}C_3(\sqrt{x}) = (\sqrt{x})^{-1}\left((\sqrt{x})^3 - 3\sqrt{x}\right)$$
$$= x - 3,$$
$$\mathcal{R}_{0;11}(x, 2) = (\sqrt{x})^{-1}C_5(\sqrt{x}) = (\sqrt{x})^{-1}\left((\sqrt{x})^5 - 5(\sqrt{x})^3 + 5\sqrt{x}\right)$$
$$= x^2 - 5x + 5,$$
$$\mathcal{R}_{0;11}(x, 3) = (\sqrt{x})^{-1}C_7(\sqrt{x}) = (\sqrt{x})^{-1}\left((\sqrt{x})^7 - 7(\sqrt{x})^5 + 14(\sqrt{x})^3 - 7\sqrt{x}\right)$$
$$= x^3 - 7x^2 + 14x - 7,$$
$$\mathcal{R}_{0;11}(x, 4) = (\sqrt{x})^{-1}C_9(\sqrt{x}) = (\sqrt{x})^{-1}\left((\sqrt{x})^9 - 9(\sqrt{x})^7 + 27(\sqrt{x})^5 - 30(\sqrt{x})^3 + 9\sqrt{x}\right)$$
$$= x^4 - 9x^3 + 27x^2 - 30x + 9.$$

Appendix D

The hypergeometric function

For demonstration purposes, we simplify the generalised hypergeometric function given in Definition 8.1.1 to the Gauss hypergeometric function

$$_2F_1\left(\begin{array}{c} \alpha_1, \alpha_2 \\ \beta \end{array}; 1\right) = \sum_{k=0}^{\infty} \frac{\alpha_1^{\overline{k}} \alpha_2^{\overline{k}}}{\beta^{\overline{k}}} \frac{1}{k!}.$$

Here, if all the parameters are positive, we require $\Re[\beta - \alpha_1 - \alpha_2] > 0$.

Now to help illustrate how we have applied the hypergeometric function in Chapter 8, let us consider the following example which involves the simplificaton of a sum product to a single binomial coefficent.

D.1 A worked example

Example. For positive integers n, x and r we have

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{x+k}{r} = (-1)^n \binom{x}{r-n}, \tag{D.1.1}$$

where we define the binomial coefficient $\binom{x}{m} = 0$, if either $m < 0$ or $m > x$.

Proof. We denote the sum of the first member of $(D.1.1)$ as

$$\sum_{k=0}^{n} T_k$$

and find the ratio T_{k+1}/T_k.

$$T_{k+1} = \frac{(-1)^{k+1}n!(x+k+1)!}{(k+1)!(n-k-1)!r!(x+k+1-r)!}, \quad \text{and } T_k = \frac{(-1)^k n!(x+k)!}{k!(n-k)!r!(x+k-r)!},$$

so that

$$\frac{T_{k+1}}{T_k} = \frac{(-1)^{k+1}n!(x+k+1)!k!(n-k)!r!(x+k-r)!}{(-1)^k n!(x+k)!(k+1)!(n-k-1)!r!(x+k+1-r)!}$$

$$= \frac{(-1)(x+k+1)(n-k)}{(x+k+1-r)(k+1)} = \frac{(-1)^2(k-n)(k+x+1)}{(k+x+1-r)(k+1)}.$$

We will then have

$$\sum_{k=0}^{n} T_k = T_0 \times {}_2F_1 \left(\begin{array}{c} -n, x+1 \\ x+1-r \end{array} ; 1 \right) = \binom{x}{r} {}_2F_1 \left(\begin{array}{c} -n, x+1 \\ x+1-r \end{array} ; 1 \right), \qquad (D.1.2)$$

where we observe that in consequence of the parameter $-n$, this hypergeometric function sum will be finite and terminate after the term that includes $(-n)^{\overline{n}}$. This also removes the restriction $\Re[\beta - (-n) - \alpha] = n - r > 0$.

Therefore, to evaluate the second member of $(D.1.2)$, we apply the result of Vandermonde's summation formula

$$_2F_1 \left(\begin{array}{c} -n, \alpha \\ \beta \end{array} ; 1 \right) = \frac{(\beta - \alpha)^{\overline{n}}}{\beta^{\overline{n}}}.$$

We then evaluate

$$(\beta-\alpha)^{\overline{n}} = (-r)^{\overline{n}} = (-r)(-r+1)\ldots(-r-1+n) = (-1)^n r(r-1)\ldots(r+1-n) = (-1)^n \frac{r!}{(r-n)!},$$
$$(D.1.3)$$

and

$$\beta^{\overline{n}} = (x+1-r)^{\overline{n}} = (x+1-r)(x+2-r)\ldots(x+n-r) = \frac{(x+n-r)!}{(x-r)!}. \qquad (D.1.4)$$

Then substituting $(D.1.3)$ and $(D.1.4)$ into the second member of $(D.1.2)$ we have

$$\binom{x}{r} {}_2F_1 \left(\begin{array}{c} -n, x+1 \\ x+1-r \end{array} ; 1 \right) = (-1)^n \frac{x!}{r!(x-r)!} \frac{r!}{(r-n)!} \frac{(x-r)!}{(x+n-r)!} = (-1)^n \binom{x}{r-n}.$$

$$\square$$

Appendix E

E.1 Tables of values of $\mathcal{F}_{s;ab}(r,t,q)$ for $q=6$ and $q=7$

Table E.1: $f_{00}(r,t,6)$, $0 \le r \le 14$, $0 \le t \le 11$.

t\r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	1	1	1	1	1	2	8	29	85	211	463	926	1730	3095
1	0	1	2	3	4	5	6	8	16	45	130	341	804	1730	3460
2	0	0	1	3	6	10	15	21	29	45	90	220	561	1365	3095
3	0	0	0	1	4	10	20	35	56	85	130	220	440	1001	2366
4	0	0	0	0	1	5	15	35	70	126	211	341	561	1001	2002
5	0	0	0	0	0	1	6	21	56	126	252	463	804	1365	2366
6	1	1	1	1	1	1	2	8	29	85	211	463	926	1730	3095
7	0	1	2	3	4	5	6	8	16	45	130	341	804	1730	3460
8	0	0	1	3	6	10	15	21	29	45	90	220	561	1365	3095
9	0	0	0	1	4	10	20	35	56	85	130	220	440	1001	2366
10	0	0	0	0	1	5	15	35	70	126	211	341	561	1001	2002
11	0	0	0	0	0	1	6	21	56	126	252	463	804	1365	2366

Table E.2: $f_{01}(r,t,7)$, $0 \le r \le 14$, $0 \le t \le 13$.

t\r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	1	1	1	1	1	1	2	9	37	121	331	793	1717	3434
1	0	1	2	3	4	5	6	7	9	18	55	176	507	1300	3017
2	0	0	1	3	6	10	15	21	28	37	55	110	286	793	2093
3	0	0	0	1	4	10	20	35	56	84	121	176	286	572	1365
4	0	0	0	0	1	5	15	35	70	126	210	331	507	793	1365
5	0	0	0	0	0	1	6	21	56	126	252	462	793	1300	2093
6	0	0	0	0	0	0	1	7	28	84	210	462	924	1717	3017
7	1	1	1	1	1	1	1	2	9	37	121	331	793	1717	3434
8	0	1	2	3	4	5	6	7	9	18	55	176	507	1300	3017
9	0	0	1	3	6	10	15	21	28	37	55	110	286	793	2093
10	0	0	0	1	4	10	20	35	56	84	121	176	286	572	1365
11	0	0	0	0	1	5	15	35	70	126	210	331	507	793	1365
12	0	0	0	0	0	1	6	21	56	126	252	462	793	1300	2093
13	0	0	0	0	0	0	1	7	28	84	210	462	924	1717	3017

Table E.3: $f_{10}(r,t,6)$, $0 \le r \le 13$, $0 \le t \le 11$.

t\r	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	1	1	1	1	1	0	−6	−27	−83	−209	−461	−922	−1702
1	0	1	2	3	4	5	6	6	0	−27	−110	−319	−780	−1702
2	0	0	1	3	6	10	15	21	27	27	0	−110	−429	−1209
3	0	0	0	1	4	10	20	35	56	83	110	110	0	−429
4	0	0	0	0	1	5	15	35	70	126	209	319	429	429
5	0	0	0	0	0	1	6	21	56	126	252	461	780	1209
6	−1	−1	−1	−1	−1	−1	0	6	27	83	209	461	922	1702
7	0	−1	−2	−3	−4	−5	−6	−6	0	27	110	319	780	1702
8	0	0	−1	−3	−6	−10	−15	−21	−27	−27	0	110	429	1209
9	0	0	0	−1	−4	−10	−20	−35	−56	−83	−110	−110	0	429
10	0	0	0	0	−1	−5	−15	−35	−70	−126	−209	−319	−429	−429
11	0	0	0	0	0	−1	−6	−21	−56	−126	−252	−461	−780	−1209

Table E.4: $f_{11}(r, t, 7)$, $0 \le r \le 13$, $0 \le t \le 13$.

t\r	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	1	1	1	1	1	1	0	−7	−35	−119	−329	−791	−1715
1	0	1	2	3	4	5	6	7	7	0	−35	−154	−483	−1274
2	0	0	1	3	6	10	15	21	28	35	35	0	−154	−637
3	0	0	0	1	4	10	20	35	56	84	119	154	154	0
4	0	0	0	0	1	5	15	35	70	126	210	329	483	637
5	0	0	0	0	0	1	6	21	56	126	252	462	791	1274
6	0	0	0	0	0	0	1	7	28	84	210	462	924	1715
7	−1	−1	−1	−1	−1	−1	−1	0	7	35	119	329	791	1715
8	0	−1	−2	−3	−4	−5	−6	−7	−7	0	35	154	483	1274
9	0	0	−1	−3	−6	−10	−15	−21	−28	−35	−35	0	154	637
10	0	0	0	−1	−4	−10	−20	−35	−56	−84	−119	−154	−154	0
11	0	0	0	0	−1	−5	−15	−35	−70	−126	−210	−329	−483	−637
12	0	0	0	0	0	−1	−6	−21	−56	−126	−252	−462	−791	−1274
13	0	0	0	0	0	0	−1	−7	−28	−84	−210	−462	−924	−1715

Table E.5: $F_{00}(r, t_6, 6)$, $0 \le r \le 13$, $0 \le t \le 5$.

t\r	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	1	1	1	1	1	2	8	29	85	211	463	926	1730
1	0	−1	−2	−3	−4	−5	−6	−8	−16	−45	−130	−341	−804	−1730
2	0	0	1	3	6	10	15	21	29	45	90	220	561	1365
3	0	0	0	−1	−4	−10	−20	−35	−56	−85	−130	−220	−440	−1001
4	0	0	0	0	1	5	15	35	70	126	211	341	561	1001
5	0	0	0	0	0	−1	−6	−21	−56	−126	−252	−463	−804	−1365

Table E.6: $F_{01}(r, t_7, 7)$, $0 \le r \le 13$, $0 \le t \le 6$.

t\r	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	1	1	1	1	1	1	−2	−9	−37	−121	−331	−793	−1717
1	0	−1	−2	−3	−4	−5	−6	−7	−9	18	55	176	507	1300
2	0	0	1	3	6	10	15	21	28	37	55	−110	−286	−793
3	0	0	0	−1	−4	−10	−20	−35	−56	−84	−121	−176	−286	572
4	0	0	0	0	1	5	15	35	70	126	210	331	507	793
5	0	0	0	0	0	−1	−6	−21	−56	−126	−252	−462	−793	−1300
6	0	0	0	0	0	0	1	7	28	84	210	462	924	1717

Table E.7: $F_{10}(r, t_6, 6)$, $0 \le r \le 13$, $0 \le t \le 5$.

t\r	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	1	1	1	1	1	0	−6	−27	−83	−209	−461	−922	−1702
1	0	−1	−2	−3	−4	−5	−6	−6	0	27	110	319	780	1702
2	0	0	1	3	6	10	15	21	27	27	0	−110	−429	−1209
3	0	0	0	−1	−4	−10	−20	−35	−56	−83	−110	−110	0	429
4	0	0	0	0	1	5	15	35	70	126	209	319	429	429
5	0	0	0	0	0	−1	−6	−21	−56	−126	−252	−461	−780	−1209

Table E.8: $F_{11}(r, t_7, 7)$, $0 \le r \le 13$, $0 \le t \le 6$.

t\r	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	1	1	1	1	1	1	0	−7	−35	−119	−329	−791	−1715
1	0	−1	−2	−3	−4	−5	−6	−7	−7	0	35	154	483	1274
2	0	0	1	3	6	10	15	21	28	35	35	0	−154	−637
3	0	0	0	−1	−4	−10	−20	−35	−56	−84	−119	−154	−154	0
4	0	0	0	0	1	5	15	35	70	126	210	329	483	637
5	0	0	0	0	0	−1	−6	−21	−56	−126	−252	−462	−791	−1274
6	0	0	0	0	0	0	1	7	28	84	210	462	924	1715

Appendix F

F.1 Tables of values of $\mathcal{L}_{1;abc}(r,t,q)$ for $q=6$ and $q=7$

Table F.1: $L_{000}(r,t,6)$, $0 \le r \le 10$, $0 \le t \le 3$,   (and $L_{000} = (-1)^{r+t+1}l_{000}$).

t\r	0	1	2	3	4	5	6	7	8	9	10
0	−2	6	−20	70	−252	926	−3460	13110	−50252	194446	−758100
1	1	−4	15	−56	211	−804	3095	−12016	46971	−184604	728575
2	0	1	−6	29	−130	561	−2366	9829	−40410	164921	−669526
3	0	0	2	−16	90	−440	2002	−8736	37130	−155080	640002

Table F.2: $L_{001}(r,t,6)$, $0 \le r \le 10$, $1 \le t \le 3$,   (and $L_{001} = (-1)^{r+t}l_{001}$).

t\r	0	1	2	3	4	5	6	7	8	9	10
1	−1	3	−10	35	−126	463	−1730	6555	−25126	97223	−379050
2	0	−1	5	−21	85	−341	1365	−5461	21845	−87381	349525
3	0	0	−1	8	−45	220	−1001	4368	−18565	77540	−320001

Table F.3: $L_{010}(r,t,7)$, $0 \le r \le 10$, $0 \le t \le 3$,   (and $L_{010} = (-1)^{r+t+1}l_{010}$).

t\r	0	1	2	3	4	5	6	7	8	9	10
0	−2	6	−20	70	−252	924	−3434	12902	−48926	187036	−720062
1	1	−4	15	−56	210	−793	3017	−11561	44592	−172995	674520
2	0	1	−6	28	−121	507	−2093	8568	−34885	141494	−572264
3	0	0	1	−9	55	−286	1365	−6188	27132	−116281	490337

Table F.4: $L_{011}(r,t,7)$, $0 \le r \le 10$, $1 \le t \le 4$, (and $L_{011} = (-1)^{r+t}l_{011}$).

t\r	0	1	2	3	4	5	6	7	8	9	10
1	-1	3	-10	35	-126	462	-1717	6451	-24463	93518	-360031
2	0	-1	5	-21	84	-331	1300	-5110	20129	-79477	314489
3	0	0	-1	7	-37	176	-793	3458	-14756	62017	-257775
4	0	0	0	-2	18	-110	572	-2730	12376	-54264	232562

Table F.5: $L_{100}(r,t,6)$, $0 \le r \le 10$, $0 \le t \le 3$, (and $L_{100} = (-1)^{r+t+1}l_{100}$).

t\r	0	1	2	3	4	5	6	7	8	9	10
0	-2	6	-20	70	-252	922	-3404	12630	-46988	175066	-652764
1	1	-4	15	-56	209	-780	2911	-10864	40545	-151316	564719
2	0	1	-6	27	-110	429	-1638	6187	-23238	87021	-325358
3	0	0	0	0	0	0	0	0	0	0	0

Table F.6: $L_{101}(r,t,6)$, $0 \le r \le 10$, $1 \le t \le 3$, (and $L_{101} = (-1)^{r+t}l_{101}$).

t\r	0	1	2	3	4	5	6	7	8	9	10
1	-1	3	-10	35	-126	461	-1702	6315	-23494	87533	-326382
2	0	-1	5	-21	83	-319	1209	-4549	17051	-63783	238337
3	0	0	-1	6	-27	110	-429	1638	-6187	23238	-87021

Table F.7: $L_{110}(r,t,7)$, $0 \le r \le 10$, $0 \le t \le 3$, (and $L_{110} = (-1)^{r+t+1}l_{110}$).

t\r	0	1	2	3	4	5	6	7	8	9	10
0	-2	6	-20	70	-252	924	-3430	12838	-48314	182476	-690802
1	1	-4	15	-56	210	-791	2989	-11319	42924	-162925	618772
2	0	1	-6	28	-119	483	-1911	7448	-28763	110446	-422576
3	0	0	1	-7	35	-154	637	-2548	9996	-38759	149205

Table F.8: $L_{111}(r,t,7)$, $0 \le r \le 10$, $1 \le t \le 3$, (and $L_{111} = (-1)^{r+t}l_{111}$).

t\r	0	1	2	3	4	5	6	7	8	9	10
1	-1	3	-10	35	-126	462	-1715	6419	-24157	91238	-345401
2	0	-1	5	-21	84	-329	1274	-4900	18767	-71687	273371
3	0	0	-1	7	-35	154	-637	2548	-9996	38759	-149205