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Integer Points close to Convex Surfaces

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1 INTRODUCTION

Let C be the boundary surface of a strictly convex bounded three-dimensional body. Strictly convex means that if P and Q are points on C , then points on the line segment PQ between P and Q lie in the convex body, but not on its boundary C . Let MC denote the dilation of C by a factor M . Andrews [1] proved a general result which in three dimensions gives the number of points of the integer lattice on MC to be

$$O\left(M^{\frac{3}{2}}\right), \quad (1)$$

as M tends to infinity. Strict convexity is necessary because a part of a 2-dimensional plane in the boundary C can give as many as a constant times M^2 integer points for infinitely many values of M .

We consider the integer points within a distance δ of the surface MC . The two-dimensional case has been well-studied [12], [5], [9], [6], [10] and [11]. Introducing δ requires some uniform approximability condition on the surface C , usually expressed in terms of upper and lower bounds for derivatives and determinants of derivatives. Let A be the 2-dimensional area of C . The search region has 3-dimensional volume

$$(2A\delta + O(\delta^2)) M^2, \quad (2)$$

and this is known to be the number of integer points on average over translations of the surface MC . To obtain an asymptotic formula one considers the Fourier transform of the convex body, with conditions at least as far as the 18th

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derivatives in order to estimate the multiple exponential integrals [7]. Hlawka [8] obtained a general dimensional asymptotic formula, which in three-dimensions yields an error of size (1); see also Krätzel [13]. Under the C^∞ hypothesis of a convergent Taylor series, the error term in the asymptotic formula has been improved, most recently by Müller [18].

We derive an upper bound for the number of integer points within a distance δ of the surface. We require only that C has a tangent plane at every point, and that any two-dimensional cross-section through the normal at some point P consists (in a neighbourhood of P) of a plane curve C' with continuous radius of curvature ρ' satisfying a condition

$$\frac{1}{M} < c_0 < \rho' < c_1.$$

This condition involves derivatives up to the second order. We actually suppose that M is large and we assume the condition

$$c_0 M + \frac{1}{2} \leq \rho = \rho' M \leq c_1 M - \frac{1}{2}, \quad (3)$$

with C contained in a sphere of radius $c_1 M$.

Our upper bound has two terms whose orders of magnitude correspond to (1) and (2). The constant factor in the second term is larger than $2A$.

In Müller [18] the differential inequality assumed is that the Gaussian curvature does not vanish. We can regard (3) as a corresponding quantitative bound.

Under the curvature conditions we prove that an upper bound for the number of integer points N lying on or within a distance δ from the surface C is given by

$$N \leq \left(\frac{c_1}{c_0}\right)^2 2^{16} \left((c_1 M)^{\frac{3}{2}} + 2^9 \delta (c_1 M)^2\right).$$

In section 3 we consider the convex hull of the integer points in a d -dimensional convex body satisfying the Curvature Condition, and find explicit bounds for the number of faces in different dimensions.

There is a vast literature on the lattice points in a convex polytope. We have been able to use some parts of this theory [1], [2], [3], [4], [16], [17] and [19]. Where possible, this theory has been stated in the general d -dimensional case. To extend all the results of this paper to d dimensions requires more investigation of configurations in intermediate dimensions, and of the distribution of large faces of the convex hull with short normal vectors. This will be the subject of a following paper [14].

2 SHELLS AND CURVATURE

Let C_0 be the locus of points at distance δ from C measured along the interior normals to C , and let C_1 be the locus of points at distance δ measured along the exterior normals. Let E be the d -dimensional shell bounded by C_0 and C_1 so that E has thickness 2δ . Let S be the set of integer points in E , and let H be the convex hull of S , so that H is a d -dimensional convex polytope. All points of S lie in H , but not all integer points on the boundary of H lie in S .

CURVATURE CONDITION (with size parameter M). For any point P on C and any two-plane Π through the normal to C at P , let $C(\Pi, P)$ be the closed plane curve $C \cap \Pi$. Then $C(\Pi, P)$ is a twice differentiable plane curve with radius of curvature ρ lying in the range

$$c_0 M + \frac{1}{2} \leq \rho \leq c_1 M - \frac{1}{2}, \quad (4)$$

where the constants c_0, c_1 and δ satisfy

$$\frac{1}{M} < c_0 \leq 1 \leq c_1, \text{ and } \delta < \frac{1}{4}. \quad (5)$$

As an immediate consequence of the Curvature Condition we have the following Lemma.

LEMMA 2.1. *Let C satisfy the Curvature Condition. For both of the boundary hypersurfaces C_0 and C_1 of the shell E , at each point Q of the hypersurface there is a tangent hyperplane. The 2-dimensional section $C(\Pi, Q)$ by a 2-plane normal to the tangent hyperplane is twice differentiable. The radius of curvature of $C(\Pi, Q)$ lies in the range*

$$c_0 M \leq \rho \leq c_1 M. \quad (6)$$

The proof follows by direct consideration of points on $C_0(\Pi, Q)$ and $C_1(\Pi, Q)$ with respect to $C(\Pi, Q)$.

By the condition (4), S , the set of integer points, lies in a d -hypersphere of radius $R = c_1 M$, and we recall the formulae for V_d , the volume, and for S_d , the surface content, of a d -dimensional sphere [19].

$$V_d = \alpha_d R^d, \quad S_d = d \alpha_d R^{d-1}, \quad (7)$$

where

$$\alpha_{2k} = \frac{\pi^k}{k!}, \quad \alpha_{2k+1} = \frac{2^{2k+1} \pi^k k!}{(2k+1)!}, \quad \alpha_d \leq 6. \quad (8)$$

3 CONVEX POLYTOPES

In this section we again consider the general d -dimensional case, so that the convex hull H of the set of integer points S is a d -dimensional convex polytope, where $d \geq 2$.

LEMMA 3.1. *To each hypersurface face of the convex polytope H we assign a standard normal vector; this is the unique outward normal integer vector $(A_1, A_2, A_3, \dots, A_d)$, which is primitive in the sense that $\text{hcf}(A_1, A_2, A_3, \dots, A_d) = 1$. Then for each $N \geq 1$ there are*

$$\leq 3^d N^d \quad (9)$$

hyperfaces of H whose standard normal vector has length at most N .

Proof. There are $(2N + 1)$ possibilities for each vector entry so that the total possible number of vectors is

$$(2N + 1)^d \leq 3^d N^d. \quad \square$$

LEMMA 3.2. *Let U be a set of K integer points in d -dimensional space that do not all lie on some hyperplane. Then there is a simplicial complex of at least $(K - d)$ non-overlapping simplices whose vertices are the K points of U .*

Proof. This Lemma may be proved from the outside in, by successively removing vertices of the convex hull of U , or from the inside out by choosing simplices of minimal non-zero volume [15]. \square

LEMMA 3.3. *Let H be a convex polytope contained in a hypersphere radius R , whose vertices are integer points. Then the number of $(d - 1)$ -hyperplane faces of H whose standard normal vector has length greater than N is*

$$\leq \frac{\alpha_d R^{d-1} d!}{N}. \quad (10)$$

Proof. Consider d integer points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_d$ lying on a hyperplane face with primitive normal vector $(A_1, A_2, A_3, \dots, A_d)$, where the d -integer points form a simplex with $(d - 1)$ -dimensional volume $V^{(d-1)}$, and \mathbf{x}_{d+1} , an integer point lying off the hyperplane face. The perpendicular distance from \mathbf{x}_{d+1} to the hyperplane face is

$$D = \frac{k}{\sqrt{(A_1^2 + A_2^2 + A_3^2 + \dots + A_d^2)}}, \quad (11)$$

for some positive integer k . We chose \mathbf{x}_{d+1} so that the distance is minimal and so $k = 1$. Then, the d -dimensional volume $V^{(d)}$ of the convex hull of these $d + 1$ points satisfies

$$V^{(d)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_{d+1}) = \frac{1}{d} D V^{(d-1)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_d).$$

Since the volume of a d -simplex whose vertices are integer points is at least $1/d!$, we have

$$\begin{aligned} V^{(d-1)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_d) &\geq \frac{d}{(d)!} \frac{1}{D} = \frac{1}{(d-1)!} \sqrt{(A_1^2 + A_2^2 + A_3^2 + \dots + A_d^2)} \\ &\geq \frac{N}{(d-1)!} \end{aligned} \quad (12)$$

by the conditions of the Lemma.

The $(d-1)$ -dimensional hypervolume of the hyperplane faces of the convex polytope must be less than or equal to the $(d-1)$ -hypervolume of the surface of the d -dimensional hypersphere enclosing it. Let A_i be the hypervolume of each hyperplane face of the polytope then by equation (7) we have

$$\sum A_i \leq S_d = d\alpha_d R^{d-1}. \quad (13)$$

We obtain an upper bound for the number of large hyperplane faces of the convex polytope by dividing the lower bound (12) into the upper bound (13) to obtain

$$\leq \frac{d\alpha_d R^{d-1} (d-1)!}{N}. \quad (14)$$

□

THEOREM 3.4. *Let H be a convex polytope contained in a d -sphere of radius R . Then H has at most*

$$2(3\alpha_d d!)^{\frac{d}{d+1}} R^{\frac{d(d-1)}{d+1}} \quad (15)$$

hyperplane faces.

Proof. We take

$$N = \left(\frac{\alpha_d d!}{3^d} \right)^{\frac{1}{d+1}} R^{\frac{d-1}{d+1}}.$$

in (9) of Lemma 3.1 and (10) of Lemma 3.3. The total number of hyperplane faces is the sum of bounds for those with long normal vectors in (9) and those with short normal vectors in (10)

$$\leq \frac{\alpha_d R^{d-1} d!}{N} + (3N)^d = 2(3\alpha_d d!)^{\frac{d}{d+1}} R^{\frac{d(d-1)}{d+1}}.$$

□

LEMMA 3.5. *Let H be a convex d -polytope with vertices at integer points. From each j -face F_i of H , we pick out $(j+1)$ vertices $\mathbf{v}_{i,1}, \mathbf{v}_{i,2}, \dots, \mathbf{v}_{i,j+1}$ that do not all lie on a $(j-1)$ -dimensional plane. Let \mathbf{w}_i be the centroid of these vertices*

$$\mathbf{w}_i = \frac{1}{j+1} (\mathbf{v}_{i,1} + \mathbf{v}_{i,2} + \dots + \mathbf{v}_{i,(j+1)}). \quad (16)$$

Let $T = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_h\}$ be the set of centroids associated with all the j -faces of H . For a set U , let $\text{conv}\{U\}$ denote the smallest convex set containing all the elements of U . Then the centroids \mathbf{w}_i are true vertices of $\text{conv}\{T\}$, in the sense that for any $t = 1, \dots, h$

$$\text{conv}\{T \setminus \{\mathbf{w}_h\}\} \neq \text{conv}\{T\}.$$

Proof. We must rule out the possibility that

$$\mathbf{w}_i = \sum_{g=1}^h \lambda_g \mathbf{w}_g, \quad (17)$$

with

$$0 \leq \lambda_g \leq 1, \quad \sum_{g=1}^h \lambda_g = 1. \quad (18)$$

Substituting for \mathbf{w}_g using (16) and multiplying by $(j+1)$ to clear fractions yields

$$\mathbf{v}_{i,1} + \mathbf{v}_{i,2} + \dots + \mathbf{v}_{i,(j+1)} = \sum_{g=1}^h \sum_{f=1}^{j+1} \lambda_g \mathbf{v}_{g,f}. \quad (19)$$

Each j -face F_i is the intersection of at least $(d-j)$ facets or hyperplanes of H and our $(j+1)$ vertices of F_i are also vertices of each of these hyperplanes. We label these hyperplanes $\Pi_1, \Pi_2, \dots, \Pi_k, \dots, \Pi_t$ with primitive integer normal vectors \mathbf{n}_k , so that any point \mathbf{r} lying on Π_k satisfies the equation

$$\mathbf{r} \cdot \mathbf{n}_k = D_k.$$

As H is convex, all the Π_k are supporting hyperplanes of P . Hence, for any point \mathbf{x} in H we have

$$\mathbf{x} \cdot \mathbf{n}_k \leq D_k, \quad (20)$$

where we have assumed (using a suitable integer vector translation) that H contains the origin. Applying (20) to (19) yields

$$\begin{aligned} (\mathbf{v}_{i,1} + \mathbf{v}_{i,2} + \dots + \mathbf{v}_{i,(j+1)}) \cdot \mathbf{n}_k &= D_k(j+1) = \sum_{g=1}^h \sum_{f=1}^{j+1} \lambda_g \mathbf{v}_{g,f} \cdot \mathbf{n}_k \\ &\leq (j+1) \sum_{g=1}^h \lambda_g D_k = D_k(j+1), \end{aligned}$$

implying that

$$D_k(j+1) = \sum_{g=1}^h \lambda_g \sum_{f=1}^{j+1} \mathbf{v}_{g,f} \cdot \mathbf{n}_k \leq D_k(j+1). \quad (21)$$

This equality is only satisfied if all of the vertices $\mathbf{v}_{g,f}$ for which $\lambda_g \neq 0$ are on the hyperplanes $\Pi_k, 1 \leq k \leq t$.

Now any j -face F_i of a convex d -polytope H can be defined as the intersection of the q -faces that contain $F_i, j \leq q \leq (d-1)$. Therefore, as the vertices $\mathbf{v}_{g,f}$ lie on such an intersection with $q = (d-1)$, we deduce that the vertices $\mathbf{v}_{g,f}$ for which $\lambda_g \neq 0$ are all vertices of our j -face F_i . That is, $\mathbf{v}_{g,1}, \mathbf{v}_{g,2}, \dots, \mathbf{v}_{g,j+1}$ are vertices of F_i .

This implies that for $g \neq i$ in equation (19) we must have $\lambda_g = 0$, as two distinct j -faces of H cannot share $(j+1)$ vertices. Hence there is only one term, λ_g , with $g = i$ and $\lambda_i = 1$ yielding the trivial expression, right hand side is identical to left hand side in equation (19).

Therefore, \mathbf{w}_i has only one expression as a convex sum of

$$T = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_h\},$$

and thus \mathbf{w}_i is not in the convex hull of $T - \mathbf{w}_i$. \square

Theorem 3.6 is a version of Andrew's Theorem [1] with explicit constants. The second statement regarding the number of faces was not stated in [1]. McMullen [17] has upper bounds for the number of faces in terms of the vertices. These bounds can be attained by polytopes with integer vertices lying on a twisted quantum curve, but the parameter R is very large. Hence, for a spherically contained convex d -polytope, there exist triples, (d, f_0, j) , for which the second statement of Theorem 3.6 is an improvement on the general upper bound stated by McMullen in [17].

THEOREM 3.6. *In d -dimensional space, a convex polytope H with f_0 vertices, all at integer points, contained in a hypersphere of radius R satisfies*

$$f_0 \leq 2(3\alpha_d d!)^{\frac{d}{d+1}} (2R)^{\frac{d(d-1)}{d+1}} \leq 36d!(2R)^{\frac{d(d-1)}{d+1}}. \quad (22)$$

Let $1 \leq j \leq d-2$. Under the conditions of the theorem, the number f_j of j -faces of P satisfies

$$f_j \leq 2(3\alpha_d d!)^{\frac{d}{d+1}} (2(j+1)R)^{\frac{d(d-1)}{d+1}}. \quad (23)$$

Proof. Let T be the set of midpoints of edges of H , and let H' be the convex hull of T . By Lemma 3.5 each point of T is a vertex of H' . Let V be the vertex of H where edges e_1, e_2, \dots, e_r meet and let W_1, W_2, \dots, W_r be the respective midpoints of these edges. The W_1, W_2, \dots, W_r are all vertices of H' but not necessarily of the same facet.

By construction, each vertex V of H is truncated by a facet F of H' and we say that V belongs to the facet F . Geometrically we can think of V as lying

above the facet F . The supporting hyperplane Π of H' containing F cuts H in a $(d - 1)$ -dimensional convex polytope Q . The join of V to any other vertex V' of H cuts Π within this convex polytope. We now show that V' cannot lie above the facet F . The vertices of Q are points X_1, X_2, \dots, X_r on e_1, e_2, \dots, e_r and X_i is either W_i , the midpoint of e_i , or between V and W_i . Therefore, if V' lies above F , then V' lies in $\text{conv}(Q, V)$ and so V' lies in $\text{conv}(V, X_1, X_2, \dots, X_r)$. The only vertex of H in this list is V , so $V' = V$.

This implies that the number of facets of H' is greater than or equal to the number of vertices of H .

Now $2H'$ is a polytope with integer vertices lying in a d -sphere radius $2R$, so the number of faces of H' is given by (15) of Theorem 3.4, but with a larger implied constant. We deduce the result (22).

For each j -face G of H we choose $j + 1$ vertices that do not all lie on the same $(j - 1)$ -plane and construct $C(G)$, the centroid of the $j + 1$ vertices. Since $C(G)$ does not lie on the $(j - 1)$ -dimensional boundary of G , $C(G)$ cannot lie on any other j -face. Let U be the set of centroids $C(G)$ constructed from the j -faces of H .

By Lemma 3.5, U is a strictly convex set and we define H'' to be the convex hull of the points $C(G)$ in U . Then $(j + 1)H''$ is a polytope with integer point vertices lying in a sphere radius $(j + 1)R$, so that the number of vertices of H'' is given by equation (22), but with a larger implied constant. Each j -face G gives a distinct point $C(G)$ in U which is a vertex of the convex polytope H'' . We deduce the result (23). \square

4 MAJOR ARCS AND LATTICES

Throughout this section H is the convex hull of the set S of integer points inside the shell E . It is helpful in many problems to separate “major arcs”, regions where there is good Diophantine approximation, from “minor arcs”, regions where there is not. In this paper a major arc can be described informally as a region U of the shell E such that the convex hull of all the integer points in U is contained in the intersection of E with some hyperplane. In three dimensions, major arcs on the plane faces and edges of the convex hull H can have dimension 1 or 2.

LEMMA 4.1. *The maximum length of a straight line segment in E is*

$$\leq 4\sqrt{\delta c_1 M}. \quad (24)$$

A chord AB of C_1 tangent to C_0 has length

$$4\sqrt{\delta c_0 M} \leq AB \leq 4\sqrt{\delta c_1 M} \quad (25)$$

Proof. Let $R = c_1 M$ and let Π be a two-dimensional plane containing the normal vector at a point X on C_1 and any other point A also on the outer boundary C_1 . The two-dimensional section E^* of the shell E in Π is depicted in Figure 1. We consider the line segment AB when it is wholly contained within E^* and so the perpendicular distance XY from AB to X must be $\leq 2\delta$. Applying circular geometry to the circle radius R with respect to the mid-point Y of chord AB we find that

$$\left(\frac{AB}{2}\right)^2 = AY^2 = XY(2R - XY), \quad (26)$$

and for fixed R , this is maximal when $XY = R$. Hence we take XY as large as possible in (26), yielding the required result.

The lower bound in (25) corresponds to the case when the cross-sectional curve is a circle of minimal radius of curvature $R = c_0 M + 1/2$. In this case, by (26)

$$AB = 2AY = 4\sqrt{\delta(R - \delta)} \geq 4\sqrt{\delta c_0 M},$$

by (4) and (5). \square

LEMMA 4.2. *Let $R = c_1 M$ and let F be a facet or hyperplane face of H that lies in a hyperplane Ψ with outward normal \mathbf{n} . Let X be the point of C_1 at which \mathbf{n} is the outward normal. Let h be the distance from X along the inward normal to the nearest point Y on the hyperplane Ψ . Let E' be the $(d - 1)$ -dimensional section of E contained in Ψ , so that E' contains all parts of the face F that lie in the shell E . Then the $(d - 1)$ -dimensional volume V of E' is bounded above by*

$$V \leq 2^{\frac{d+9}{2}} d \delta R^{\frac{d-1}{2}} h^{\frac{d-3}{2}}. \quad (27)$$

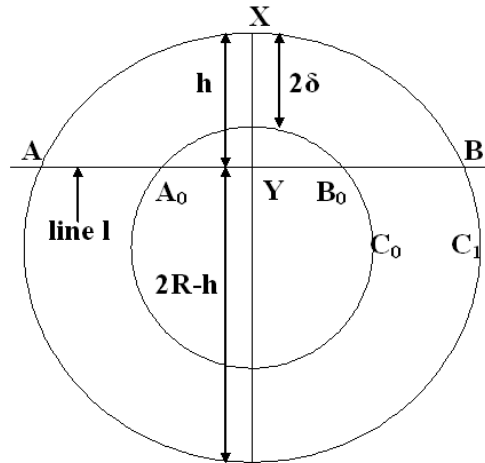


Figure 1: (section by 2-plane Π through l and X).

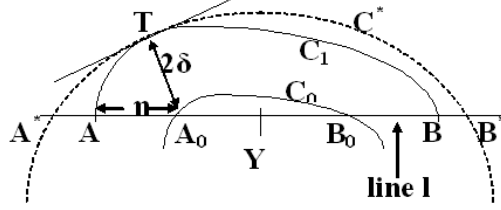


Figure 2: (section by 2-plane Π_1 through l and T).

Proof. Let Π be a two-dimensional plane through XY , and let E^* be the two-dimensional section of E by Π (Figure 1). Then Π cuts Ψ in a straight line l which meets C_1 in two distinct points A and B . The points A and B lie inside the circle radius R through X with \mathbf{n} as outward normal at X . For clarity, the curves C_0 and C_1 in Figure 1 are drawn as circles. From (26) in the proof of Lemma 4.1 we have

$$AY \leq \sqrt{h(2R - h)} = k. \quad (28)$$

Hence the set $E' = E \cap \Psi$ lies within a $(d - 1)$ -sphere centre Y radius $\leq \sqrt{2Rh}$.

Case 1. When $h \leq 2\delta$ the plane Ψ does not cut C_0 and by (6), the diameter of E' satisfies (24). This implies that the whole of the facet F is contained within the shell E . Therefore, the $(d - 1)$ -dimensional volume V of E' is less than or equal to that of a $(d - 1)$ -sphere radius $\sqrt{2hR}$. Applying (7) yields

$$V \leq \alpha_{d-1}(2hR)^{\frac{d-1}{2}} \leq 2^{\frac{d+5}{2}}(hR)^{\frac{d-1}{2}}. \quad (29)$$

Case 2. When $h > 2\delta$ the hyperplane Ψ meets C_0 , and the line l in the two-dimensional plane Π cuts C_0 in two distinct points A_0 and B_0 . Let A_0T be the normal from A_0 to C_1 , so the distance A_0T is 2δ , and let C^* be the hypersphere radius R touching C_1 at T . Let Π_1 be the two-dimensional plane through the line l and the point T (Figure 2). Then C_1 and the shell E are contained within C^* . The line l cuts C^* at A^* and B^* , so that by the geometry of circles

$$AA_0 \cdot A_0B \leq A^*A_0 \cdot A_0B^* = 2\delta(2R - 2\delta) \leq 4\delta R, \quad (30)$$

On the line l , the point A lies between A^* and A_0 , with $AA_0 = \eta$ (say) and $\eta > 0$. Hence

$$\eta \leq A^*A_0. \quad (31)$$

We also have

$$A_0 B^* \geq Y B^* = k = \sqrt{h(2R - h)}. \quad (32)$$

Each point of E' lies within a distance η of the $(d - 2)$ -dimensional surface of $C_1 \cap \Psi$. The $(d - 2)$ -dimensional volume of $C_1 \cap \Psi$ is at most the surface content of a $(d - 1)$ -dimensional sphere radius k , which by (7) is equal to

$$(d - 1)\alpha_{d-1}k^{d-2}.$$

Therefore, the $(d - 1)$ -dimensional volume V of E' satisfies

$$V \leq (d - 1)\alpha_{d-1}\eta k^{d-2} \quad (33)$$

From (30), (31) and (32) we have

$$\eta k \leq A^* A_0 A_0 B^* \leq 4\delta R. \quad (34)$$

Hence we can write

$$V \leq (d - 1)\alpha_{d-1}(4\delta R)k^{d-3}$$

which simplifies to

$$V \leq 2^{\frac{d+7}{2}} (d - 1)\delta R^{\frac{d-1}{2}} h^{\frac{d-3}{2}}. \quad (35)$$

Combining (29) and (35) yields

$$V \leq 2^{\frac{d+9}{2}} d\delta R^{\frac{d-1}{2}} h^{\frac{d-3}{2}}.$$

and hence the result. \square

LEMMA 4.3. *In d -dimensional space, the number of integer points of S in E that lie strictly inside the convex hull H of S is*

$$\leq 2\delta d!\alpha_d(c_1 M)^{d-1}. \quad (36)$$

In particular, if $d = 3$, then the number of integer points lying within a short distance δ of the convex hull H is

$$\leq 48\pi\delta(c_1 M)^2. \quad (37)$$

Proof. Given that the integer point vertices of our convex hull H lie within a distance δ from the closed convex hypersurface C , we can associate a hyperslab of width 2δ with each facet of the polytopal convex hull where the hyperslabs will overlap.

Any integer points $H \cap E$ must lie within a distance 2δ of the nearest polytope facet F_i with hypersurface area A_i . The internal or “dihedral” angles between facets are $\leq 180^\circ$ due to convexity. Let P be such a point with nearest hyperface

F_i , so that the perpendicular from P to the hyperplane F_i actually hits F_i . If not, then some other hyperplane is nearer (F_j say) under the distance equation (11) defined in Lemma 3.3.

Therefore each integer point P lying inside the convex hull can be associated uniquely with a nearest hyperface F_i .

Corresponding to each hyperface F_i we have a hyperslab S_i consisting of two completely parallel hyperfaces F_i and F_i shifted by 2δ in the normal direction to the hyperplane. The hypervolume of $S_i = 2\delta A_i$ where A_i is the hypersurface area of F_i .

We know from Lemma 3.2 that in d -dimensions, K points that do not all lie on the same hyperplane form at least $(K - d)$ non-overlapping simplices. Each simplex has hypervolume $1/d!$ multiplied by an integer so that each of these simplices has hypervolume $\geq 1/d!$.

Therefore, if K_i is the number of internal integer points associated uniquely with the hyperface F_i , which itself has at least d integer point vertices, then the total number of internal and boundary integer points of the hyperface is

$$\geq d + K_i,$$

so that we have at least K_i non-overlapping simplices, yielding

$$\frac{K_i}{d!} \leq 2\delta A_i,$$

so that

$$K_i \leq 2d!\delta A_i.$$

Hence the total number of integer points lying within a short distance δ of the convex hull H is

$$\leq \sum_i K_i \leq \sum_i 2d!\delta A_i.$$

The boundary content of our convex d -polytope H is less than or equal to that of the hypersphere with radius of curvature $c_1 M$ enclosing it. Therefore

$$\sum_i K_i \leq 2d!\delta \alpha_d d (c_1 M)^{d-1}. \quad \square$$

The next two lemmas are commonplaces of integer geometry, so we merely sketch the proofs.

LEMMA 4.4. *Let Π be a hyperplane with equation*

$$\mathbf{n} \cdot \mathbf{x} = D,$$

where \mathbf{n} is a primitive integer vector, and D is an integer. Then the integer points of Π form a lattice with determinant $|\mathbf{n}|$.

Proof. The lattice of integers on Π is congruent to the lattice of integers on the plane $\mathbf{n} \cdot \mathbf{x} = 0$. Let $m = \mathbf{n} \cdot \mathbf{n}$. The lattice of integer vectors with $\mathbf{n} \cdot \mathbf{x} \equiv 0 \pmod{m}$ consists of $(d-1)$ -dimensional lattices on the plane $\mathbf{n} \cdot \mathbf{x} = 0$ and on parallel planes at distance $|\mathbf{n}|$. \square

LEMMA 4.5. *Let Λ be a j -dimensional lattice of determinant n , $1 \leq j \leq d$. Let U be a convex set in the j -plane of Λ , with j -dimensional volume V , containing K points of the lattice Λ . Then one of the following two cases holds.*

(1) *Major case. All the points of Λ in the set U lie on a $(j-1)$ -dimensional plane.*

(2) *Minor case.*

$$K \leq j! \frac{V}{n} + j \leq (j+1)! \frac{V}{n}.$$

Proof. In the minor case, by Lemma 3.2, there is a simplicial complex of at least $K-j$ non-overlapping simplices, each of volume at least $n/j!$, whose vertices are the K points of U . The union of these simplices lies inside U and this gives the first inequality. There is at least one such simplex, so $V \geq n/j!$, and we deduce the second inequality. \square

5 VERTEX COMPONENTS

From this point on, we are restricted to three dimensions.

For each point P in our shell E , there exists a normal to the outer boundary surface C_1 , meeting C_1 at a point R_1 . We call R_1 the normal projection of P onto C_1 and R_0 the normal projection of P onto C_0 . The vertices of our polyhedron, must, by definition lie in E and for every other non vertex integer point in E there must exist a nearest vertex. We now formalise this concept with the following definition.

DEFINITION. Let P be a point of S in the shell E and R_1 be the normal projection of P onto C_1 . Let V be a vertex of the convex polyhedral hull H and E' be the plane sectional strip of E containing V , P and R_1 . If the line segment R_1V lies entirely within the closed strip E' , then we say that P lies in the component $S(V)$ of S .

LEMMA 5.1. *Every point P of S belongs to some vertex component $S(V)$.*

Proof. The line segment PR_1 cuts the boundary of the polyhedral hull H at some point Q between P and R_1 inside E , so that Q lies in some plane face F of H . If Q is a vertex of H then P belongs to $S(Q)$ as QR_1 will lie on the line segment R_0R_1 inside E .

We now assume that Q is not a vertex of H and triangulate the face F of H containing Q so that Q lies in some triangle $W = V_1V_2V_3$. If QV_i does not enter the interior of the convex set bounded by C_0 then neither does R_1V_i , implying that P lies in $S(V_i)$.

If P lies in no $S(V_i)$ then each line segment QV_i on F cuts the interior of C_0 in some point Q_i also on F but not in E . The whole convex triangle $Q_1Q_2Q_3$ therefore lies strictly inside C_0 and contains Q . Hence, Q is not in E which is impossible, since Q lies on the line segment R_0R_1 , which is strictly inside E . This contradiction shows that for some i , the line segment V_iQ lies in E and so V_iR_1 lies in E and P is in the component corresponding to V_i . \square

LEMMA 5.2 (spacing lemma). *Let V be a vertex of the convex hull H . Let P be a point of S not in the component $S(V)$ of V . Let R_1 and R_2 be the respective normal projections of P and V onto C_1 . Then*

$$R_1R_2 > \sqrt{c_0\delta M} \quad (38)$$

and the angle between the normals to C_1 at R_1 and R_2 is

$$> \frac{1}{c_1} \sqrt{\frac{c_0\delta}{M}} \quad (39)$$

Proof. Since P is not in the component of V , the line R_1V cuts C_0 in two points W_1 and W_2 . Let E' be the plane sectional closed strip of E defined by the line R_1V and the point R_2 , so that E' also contains the points W_1 and W_2 . Between W_1 and W_2 on C_0 is a point W where the tangent to C_0 in E' passes through R_1 . Then

$$R_1V > R_1W_2 > R_1W \geq 2\sqrt{\delta c_0 M}$$

by (25). Hence

$$\begin{aligned} R_1R_2 &\geq R_1V - 2\delta > 2\sqrt{\delta c_0 M} - 2\delta \\ &\geq 2\sqrt{\delta c_0 M} - \sqrt{\delta c_0 M} = \sqrt{\delta c_0 M}, \end{aligned}$$

by (4) and (5), which is (38).

To obtain (39) we consider the sphere B with centre on R_2V produced, radius c_1M , touching C_1 at R_2 . There is a point R'_1 on B where the outward normal is parallel to the outward normal to C_1 at R_1 , making some angle θ with the outward normal at R_2 . Since C_1 has sectional radius of curvature less than or equal to c_1M , the radius of B , we have

$$R_1R_2 \leq R'_1R_2.$$

The shortest distance from R'_1 to R_2 along the surface of B is θc_1M , so

$$\theta c_1M \geq R'_1R_2 \geq R_1R_2 > \sqrt{c_0\delta M},$$

$$\theta > \frac{1}{c_1} \sqrt{\frac{c_0 \delta}{M}}$$

as required. \square

As each integer point P in S belongs to at least one component $S(V)$ labelled by some vertex V of the convex hull H , components labelled by different vertices may well overlap and different vertices of the convex hull may be close together. We pick a well-spaced set of vertices of H as follows. Pick a vertex V_1 , and let the enlarged component $S'(V_1)$ be the union of all components $S(V)$ with V in $S(V_1)$.

Now pick a vertex V_2 not in $S'(V_1)$, and form the enlarged component $S'(V_2)$. We pick V_{i+1} not in $S'(V_1), S'(V_2), \dots, S'(V_i)$, and so on until all of the vertices V of the convex hull H lie in some enlarged component.

LEMMA 5.3 (thickness lemma). *Let $S'(V)$ be an enlarged component and let R_2 be the normal projection of V onto C_1 . Let P be a point in $S'(V)$. Then the distance h of P from the tangent plane at R_2 satisfies*

$$h \leq \frac{52\delta c_1}{c_0}. \quad (40)$$

Proof. The integer point P lies in some component $S(V')$ with V' in $S'(V)$. Let R_1 and R'_2 be the respective normal projections of P and V' onto C_1 . The line segments R_1V' and R'_2V lie inside the shell E , so by Lemma 4.1

$$R_1V' \leq 4\sqrt{\delta c_1 M}, \quad R'_2V \leq 4\sqrt{\delta c_1 M}.$$

The distances $V'R'_2$ and VR_2 are at most 2δ , so

$$R_1R_2 \leq R_1V' + V'R'_2 + R'_2V + VR_2 \leq 8\sqrt{\delta c_1 M} + 4\delta \leq 10\sqrt{\delta c_1 M}, \quad (41)$$

where we have used (4) and (5).

Let E' be the plane sectional strip of E defined by R_1 , V and the normal projection R_2 of V onto C_1 . Let C' be the convex curve defined by the intersection of C_1 and E' .

For fixed distance $R_1R_2 = D$, the distance of R_1 from the tangent to C' at R_2 in E' is greatest when the radius of curvature is least, which is when C' is an arc of a circle radius c_0M . Let α be the angle between R_1R_2 and the tangent at R_2 . In the extreme case when C' is a circle radius c_0M , the chord R_1R_2 subtends an angle 2α at the centre of the circle, so

$$D = 2c_0M \sin \alpha,$$

and the distance of R_1 from the tangent at R_2 is

$$D \sin \alpha = \frac{D^2}{2c_0M} \leq \frac{100\delta c_1 M}{2c_0M} = \frac{50\delta c_1}{c_0}.$$

The distance of P from the tangent plane to C_1 at R_2 is therefore

$$\leq \frac{50\delta c_1}{c_0} + 2\delta \leq \frac{52\delta c_1}{c_0}. \quad \square$$

REMARK. If we can obtain a bound valid for δ sufficiently small, then we can deduce a possible weaker bound for large δ by dividing the shell E into concentric shells E_r , $1 \leq r \leq R$ of thickness δ_0 , bounded by shrunken copies of the exterior surface C_1 of E . By inequality (6), we have a uniform upper bound of $c_1 M$ for the sectional radius of curvature at any point on each shell E_r . Hence, when regarding maximum sectional radius of curvatures, we can work within the general shell boundary C_1 , whose sectional radius of curvature is also $\leq c_1 M$.

LEMMA 5.4 (flatness lemma). *Let $S'(V)$ be an enlarged vertex component of our convex polyhedral hull H . If*

$$\delta < \delta_0 = \sqrt{\frac{c_0}{273.5^2 13c_1}} \cdot \frac{1}{\sqrt{c_1 M}}, \quad (42)$$

then all the points of $S'(V)$ lie on a plane through the vertex V .

Proof. Let R_2 be the normal projection of V onto C_1 . All points P of $S'(V)$ lie within a distance $52\delta c_1/c_0$ from the tangent plane at R_2 and by (41)

$$PV \leq PR_1 + R_1 V' + V' R_2 + R_2 V \leq 8\sqrt{\delta c_1 M} + 4\delta \leq 10\sqrt{\delta c_1 M}.$$

Hence, the set of integer points $S'(V)$ all lie within a rectangular box L , of volume $\text{Vol}(L)$, with

$$\text{Vol}(L) \leq \frac{52\delta c_1}{c_0} \left(20\sqrt{\delta c_1 M}\right)^2 < \frac{1}{6}, \quad (43)$$

where we have used the assumption (42). Therefore, by Lemma 4.5 the major arc case holds, and all points of the enlarged vertex component $S'(V)$, including V itself, lie on a plane. \square

LEMMA 5.5 (approximate tangency). *Let $S'(V)$ be an enlarged component. Let T be the point of C_1 closest to V . Let P be another point of $S'(V)$, and let \mathbf{g} be the integer vector VP . Then the angle α between VP and the normal to C_1 at T satisfies*

$$|\cos \alpha| \leq \frac{52\delta c_1}{c_0 |\mathbf{g}|}. \quad (44)$$

Proof. Let Π be the plane through P and the normal to C_1 at T through V . Then C_1 will appear in Π as a convex curve C' . Let l be the tangent to C' at T , and let U be the foot of the perpendicular from P to l in Π . If W is the foot of the perpendicular from V to PU then $VTUW$ is a rectangle in Π .

By Lemma 5.3 we have

$$PU \leq \frac{52\delta c_1}{c_0}.$$

Now if P is between W and U , then

$$VP|\cos \alpha| = PW \leq WU = VT \leq 2\delta$$

and if W is between P and U then

$$VP|\cos \alpha| = PW \leq PU \leq \frac{52\delta c_1}{c_0}.$$

The inequality (44) holds in both cases. \square

LEMMA 5.6 (sums of reciprocals). *We have*

$$\sum_{1 \leq |\mathbf{e}| \leq E} \frac{1}{|\mathbf{e}|} \leq 2^6 E^2. \quad (45)$$

Proof. Applying the Cauchy condensation method, we divide the normal vectors into ranges

$$\frac{F}{2} < |\mathbf{e}| \leq F, \quad F = 1, 2, 4, \dots, 2^K,$$

where 2^K is the largest power of 2 less than or equal to E . The number of integer vectors in this range is

$$\leq (2F + 1)^3 - (F + 1)^3 \leq 19F^3$$

so that

$$\sum_{F/2 < |\mathbf{e}| \leq F} \frac{1}{|\mathbf{e}|} \leq 19F^3 \cdot \frac{2}{F} = 38F^2.$$

Summing over the ranges for F , we have

$$\begin{aligned} \sum \frac{1}{|\mathbf{e}|} &\leq 38(1 + 4 + 16 + \dots + 2^{2K}) \\ &\leq \frac{39(2^{2K+2} - 1)}{4 - 1} \leq 13.4(2^{2K}) \\ &\leq 2^6 E^2. \end{aligned}$$

\square

6 PLANE FACES AND EDGES

In Lemma 4.3, we counted all of the integer points in the extended vertex components that lie strictly inside the convex hull H . Therefore we need only consider the set $S(H)$ of integer points in our extended vertex components that lie strictly on the plane faces and edges of H in S .

Let $S^*(V_i)$ be the subset of $S'(V_i)$ consisting of integer points on the boundary of H . We will call this a boundary component. We have shown that for each extended vertex component $S'(V_i)$, if δ is sufficiently small then $S'(V_i)$ lies in a plane and so $S^*(V_i)$ lies in the same plane.

LEMMA 6.1. *The number of integer points on 1-dimensional boundary components is estimated by*

$$\sum_{\dim S^*(V_i)=1} |S^*(V_i)| \leq \left(\frac{2^{16} 3^3 \pi c_1}{c_0^2} \right) \delta (c_1 M)^2. \quad (46)$$

Proof. First we note that at most two one-dimensional components can lie on the same straight line. If this were not the case and there were more than two, then there would exist at least two separate sections of a straight line segment that do not lie in the shell E . This in turn means that there exists a straight line segment that cuts C_0 or C_1 more than twice which contradicts the convex property assumed.

We consider all the boundary components $S^*(V_i)$ which are one-dimensional lying parallel to some primitive integer vector \mathbf{e} . Suppose that the component contains l points of $S(H)$, where

$$L + 1 \leq l \leq 2L \quad (47)$$

for some L equal to a power of two. We can take $\mathbf{g} = (l-1)\mathbf{e}$ in Lemma 5.5, with

$$|\mathbf{g}| \geq (l-1)|\mathbf{e}| \geq L|\mathbf{e}|.$$

In Lemma 5.5 the angle α between the vector \mathbf{e} and the normal to C_1 at T , the point of C_1 nearest to V , satisfies

$$|\cos \alpha| \leq \frac{52\delta c_1}{c_0 L |\mathbf{e}|}.$$

Hence

$$\left| \frac{\pi}{2} - \alpha \right| \leq \frac{26c_1 \pi \delta}{c_0 L |\mathbf{e}|}. \quad (48)$$

We want to discuss the spacing of the vertices V_i that label the enlarged components $S'(V_i)$ and so the boundary components $S^*(V_i)$. Each V_i has a normal projection T_i on C_1 . Consider a sphere B of radius $c_1 M$. We map T_i on C_1 to

the point W_i on B where the outward normal \mathbf{n} to B is parallel to the outward normal to C_1 at T_i .

Let V_i and V_j be distinct vertices labelling enlarged vertex components. Since $V_j \notin S(V_i)$, we have

$$T_i T_j > \sqrt{c_0 \delta M},$$

by (38) of Lemma 5.2. Since C_1 has sectional radii of curvature at most $c_1 M$,

$$W_i W_j \geq T_i T_j > \sqrt{c_0 \delta M}.$$

Hence balls B_i , radii $\frac{1}{2}\sqrt{c_0 \delta M}$, centred on the points W_i on B , are disjoint.

The three-ball B_i meets the surface of the sphere B in a set A_i which contains the centre W_i of B_i and is a two-ball in spherical geometry. As the B_i are disjoint, the areas A_i , on the surface of the sphere B , are also disjoint and do not overlap. Hence different sets $S'(V_i)$ correspond to disjoint sets A_i , centre W_i , on the surface of the sphere B . The area of A_i is greater than the area of the intersection of a plane through W_i with B_i , which is

$$\frac{\pi c_0 \delta M}{4}. \quad (49)$$

As $V_i \in S^*(V_i)$ and $S^*(V_i) \subseteq S'(V_i)$, different sets $S^*(V_i)$ also correspond to disjoint set A_i , centre W_i , on the surface of the sphere B .

For each vector \mathbf{e} , there is an equatorial plane of the sphere B at right angles to \mathbf{e} . By (48) the point W_i on the surface of B , where the normal is parallel to the normal \mathbf{n} to C_1 at T , lies

$$\leq \frac{26\pi\delta c_1 M}{c_0 L |\mathbf{e}|}$$

from the equatorial plane measured along the surface of B . As stated, the set A_i is the intersection of the surface of B with a three-ball radius $\frac{1}{2}\sqrt{c_0 \delta M}$, so it forms a two-ball in the spherical geometry of the surface of B , whose radius in spherical geometry is

$$\begin{aligned} &\leq \frac{\pi}{2} \cdot \sqrt{\frac{c_0 \delta M}{4}} \leq \pi \sqrt{\frac{c_0 \delta M}{16}} \cdot \frac{4\sqrt{\delta c_1 M}}{L |\mathbf{e}|} \\ &= \frac{\pi \delta c_1 M}{L |\mathbf{e}|} \sqrt{\frac{c_0}{c_1}} \leq \frac{\pi \delta c_1 M}{c_0 L |\mathbf{e}|}, \end{aligned}$$

by equation (4) and Lemma 4.1.

Hence, each point of A_i lies within a distance

$$\leq \frac{26\pi\delta c_1 M}{c_0 L |\mathbf{e}|} + \frac{\pi \delta c_1 M}{c_0 L |\mathbf{e}|} = \frac{27\pi\delta c_1 M}{c_0 L |\mathbf{e}|}$$

from the equatorial plane, measured along the surface of the sphere B .

We consider the “girdle” of one-dimensional boundary components $S^*(V_i)$ which are parallel to the fixed vector \mathbf{e} . The components in the girdle satisfying (47) correspond to points W_i and disjoint sets A_i on the surface of B , such that every point of A_i lies close to the equatorial plane perpendicular to \mathbf{e} . The disjoint sets A_i lie in an annulus whose volume in spherical geometry is at most

$$(2\pi c_1 M) \left(\frac{54\pi\delta c_1 M}{c_0 L |\mathbf{e}|} \right) = \frac{27(2\pi)^2 \delta (c_1 M)^2}{c_0 L |\mathbf{e}|}.$$

By equation (49) the number of disjoint sets A_i in the girdle is at most

$$\begin{aligned} & \frac{2^2}{\pi(c_0 \delta M)} \cdot \frac{27(2\pi)^2 \delta (c_1 M)^2}{c_0 L |\mathbf{e}|} \\ &= \frac{27(4\pi c_1)^2 M}{\pi c_0^2 L |\mathbf{e}|}, \end{aligned} \tag{50}$$

so the boundary components $S^*(V_i)$ in the girdle for which the number l of points is in the range (47) contribute at most

$$\frac{54\pi(4c_1)^2 M}{c_0^2 |\mathbf{e}|} \tag{51}$$

integer points. The estimate (51) refers only to components in the girdle for which l lies in the range (47). We keep the condition (47), and sum over primitive integer vectors \mathbf{e} . Since each component is a straight line segment lying within the strip E , by Lemma 4.1 we have

$$L|\mathbf{e}| \leq (l-1)|\mathbf{e}| \leq 4\sqrt{\delta c_1 M}.$$

We note that if two boundary components lie on the same line, then the vertices V_i which label the boundary components $S^*(V_i)$ must be different, so they are counted separately in this argument. We use Lemma 5.6 to sum over \mathbf{e} , so that the number of points on 1-dimensional boundary components with l in the range (47) is at most

$$\begin{aligned} & \frac{54\pi(4c_1)^2 M}{c_0^2} \cdot 2^6 \left(\frac{4\sqrt{\delta c_1 M}}{L} \right)^2 \\ &= \frac{2^{15} 3^3 c_1 \pi \delta (c_1 M)^2}{c_0^2 L^2}. \end{aligned} \tag{52}$$

Finally we remove the condition (47) by summing L through powers of 2, noting that

$$\left(1 + \frac{1}{2^k} + \frac{1}{4^k} + \frac{1}{8^k} + \dots \right) \leq \frac{2^k}{2^k - 1} \leq 2.$$

Hence the total number of integer points of $S(H)$ which lie on 1-dimensional boundary components is at most

$$\left(\frac{2^{16} 3^3 c_1 \pi}{c_0^2} \right) \delta (c_1 M)^2 \leq \left(\frac{2^{16} 3^3 \pi c_1}{c_0^2} \right) \delta (c_1 M)^2. \quad \square$$

LEMMA 6.2. *The number of integer points lying on the plane boundary components is estimated by*

$$\leq 2^{19}\delta(c_1M)^2. \quad (53)$$

Proof. For each plane boundary component, by (43) of Lemma 5.4, the integer points will all lie in a square of area

$$400\delta c_1M.$$

The boundary components are convex sets lying on the convex hull H . Hence the convex hull of the plane boundary components, H' say, is a convex polyhedron contained within H , and the boundary components lie on the boundary planes of H' . If the boundary of the convex polyhedron H' contains two parallel planes, then the convex body includes all points of the joins of any point on one plane with any point on the other plane. It follows that the outward normal vectors must be in opposite directions. Boundary components lie on boundary planes of H' , so the direction of the outward normal determines the boundary plane.

Therefore, either the plane boundary components will all have different outward normal vectors \mathbf{n}_i , or some will share vectors and so form convex sets that all lie on the same plane. In the latter instance, these plane boundary components will all lie in an annulus as described in Lemma 4.2. As each component is convex in this annulus we can apply Lemma 4.5 and summing over all possible normal vectors gives the total number of integer points to be

$$\leq 3!2^6 3\delta c_1M \sum \frac{1}{|\mathbf{n}_i|}. \quad (54)$$

Applying similar logic to the former case yields

$$\leq 3!400\delta c_1M \sum \frac{1}{|\mathbf{n}_i|} \quad (55)$$

integer points. The constant in (55) is greater than that in (54) and for each \mathbf{n}_i only one of the cases can occur. Hence we need only calculate the sum in (55). We note that the sum over all possible short normal vectors (length $\leq N$) will be greater than the sum over all possible long normal vectors (length $\geq N$) and so we consider twice the sum over the short normal vectors, giving

$$\leq 2 \cdot 3!400\delta c_1M \sum_{1 \leq |\mathbf{n}_i| \leq N} \frac{1}{|\mathbf{n}_i|},$$

where, by Theorem 3.4,

$$N = 2^K = \left(\frac{8\pi}{27}\right)^{\frac{1}{4}} (c_1M)^{\frac{1}{2}}.$$

Applying Lemma 5.6 yields

$$\begin{aligned} 2.3!400\delta c_1 M \sum_{1 \leq |\mathbf{n}_i| \leq N} \frac{1}{|\mathbf{n}_i|} &\leq 2^{12} 3.5^2 \delta c_1 M N^2 \\ &\leq 2^{12} 3.5^2 \delta (c_1 M)^2 \leq 2^{19} \delta (c_1 M)^2, \end{aligned}$$

as required. \square

LEMMA 6.3. *The number of integer points on three-dimensional boundary components, when $\delta = \delta_0$, is estimated by*

$$\begin{aligned} \sum_{\dim S^*(V_i)=3} |S^*(V_i)| &\leq 8 (24)^{\frac{3}{4}} (2c_1 M)^{\frac{3}{2}} \\ &\leq 2^9 (c_1 M)^{\frac{3}{2}}. \end{aligned} \tag{56}$$

Proof. From (43), the three-dimensional boundary component $S^*(V_i)$ will have a volume $\text{Vol}(H_i)$, with

$$\text{Vol}(H_i) \leq \frac{52\delta c_1}{c_0} \left(20\sqrt{\delta c_1 M}\right)^2.$$

Since $\delta = \delta_0$ this gives a volume of at most $1/6$. Applying the minor arc case of Lemma 4.5 then gives

$$K_i \leq 24\text{Vol}(H_i) \leq 4, \tag{57}$$

where K_i is the number of integer points contained in $S^*(V_i)$. However, the existence of a 3-dimensional $S^*(V_i)$ in $S'(V_i)$ requires that $K_i \geq 4$, and so if we consider $\delta = \delta_0$, then K_i , the number of integer points in the boundary component, is exactly 4. The number of vertices of the convex hull is

$$\leq 2 (24)^{\frac{3}{4}} (2c_1 M)^{\frac{3}{2}},$$

by (22), in Theorem 3.6. Hence, when $\delta = \delta_0$, the total number of integer points in the three-dimensional boundary components is estimated by

$$\leq 8 (24)^{\frac{3}{4}} (2c_1 M)^{\frac{3}{2}} \leq 2^9 (c_1 M)^{\frac{3}{2}}. \tag{58}$$

\square

We now collect together the terms (22), (46), (53), (56) and (36) to obtain an upper bound for the total number of integer points contributed from the j -dimensional extended vertex components, $0 \leq j \leq 3$, along with the internal integer points, when $\delta \leq \delta_0$. This gives

$$\leq \left(\frac{c_1}{c_0}\right) \left((2^7 + 2^9)(c_1 M)^{\frac{3}{2}} + (2^{19} + 2^{16} 3^3 \pi + 2^8) \delta_0 (c_1 M)^2 \right)$$

$$\leq \left(\frac{c_1}{c_0}\right) \left(2^{10}(c_1M)^{\frac{3}{2}} + 2^{23}\delta_0(c_1M)^2\right). \quad (59)$$

This result is valid for a shell of thickness $\delta = \delta_0$ and consists of terms independent of δ (degree zero), and those with a factor of δ (degree one).

We cover the shell E of all extended vertex components, bounded internally by C_0 and externally by C_1 , by R thinner concentric shells E_1, \dots, E_R of thickness δ_0 . The distance between C_1 and C_0 along any inward normal vector to these two surfaces is 2δ . Hence we choose R to be the smallest such integer with

$$R\delta_0 \geq 2\delta, \quad (R-1)\delta_0 < 2\delta,$$

so that

$$R < \frac{2\delta}{\delta_0} + 1. \quad (60)$$

The shell E_r consists of the points on some inward normal whose distance l from the surface C_1 lies in the range

$$(r-1)\delta_0 \leq l \leq r\delta_0.$$

When we replace δ with $r\delta_0$ in Lemma 2.1, we see that each shell E_r will satisfy the Curvature Condition, so that any plane sectional curve of E_r will lie in the range

$$c_0M \leq \rho \leq c_1M.$$

Therefore, expression (59) gives a uniform upper bound for the number of integer points contributed by any shell E_r . We note that

$$\delta_0\sqrt{c_1M} \leq \left(\frac{c_1}{c_0}\right) \left(\frac{1}{2^8}\right), \quad (61)$$

and

$$\left(\delta_0\sqrt{c_1M}\right)^{-1} \leq \left(\frac{c_1}{c_0}\right) 2^9. \quad (62)$$

THEOREM 6.4. *The number of integer points lying on or within a short distance δ from a convex closed surface that is contained in a sphere of radius c_1M in three-dimensional Euclidean space is*

$$\leq \left(\frac{c_1}{c_0}\right)^2 2^{16} \left((c_1M)^{\frac{3}{2}} + 2^9\delta(c_1M)^2\right).$$

Proof. We multiply the upper bound (59) by the maximum number of shells given by (60). This yields

$$\left(\frac{2\delta}{\delta_0} + 1\right) \left(\frac{c_1}{c_0}\right) \left(2^{10}(c_1M)^{\frac{3}{2}} + 2^{23}\delta_0(c_1M)^2\right).$$

Simplifying using (61) and (62) and combining terms we have at most

$$\left(\frac{c_1}{c_0}\right)^2 2^{16} \left((c_1M)^{\frac{3}{2}} + 2^9\delta(c_1M)^2\right)$$

integer points. □

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