A FULLY-NONLINEAR FLOW AND QUERMASSINTEGRAL INEQUALITIES IN THE SPHERE

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Dedicated to Joseph Kohn on the occasion of his 90th birthday

Abstract. This expository paper presents the current knowledge of particular fully nonlinear curvature flows with local forcing term, so-called locally constrained curvature flows. We focus on the spherical ambient space. The flows are designed to preserve a quermassintegral and to de-/increase the other quermassintegrals. The convergence of this flow to a round sphere would settle the full set of quermassintegral inequalities for convex domains of the sphere, but a full proof is still missing. Here we collect what is known and hope to attract wide attention to this interesting problem.

1. Introduction

Let M^n be a smooth, closed and connected manifold and let $X : M^n \hookrightarrow \mathbb{S}^{n+1}$ be the embedding of a strictly convex hypersurface. Let $p \in \mathbb{S}^{n+1}$ be a point in the interior of the convex body enclosed by M, such that M lies in the interior of the hemisphere determined by p and denote by

$$ds^2 = d\rho^2 + \phi(\rho)^2 dz^2 \tag{1.1}$$

the metric in polar coordinates around p, where $\phi(\rho) = \sin(\rho)$, $\rho \in [0, \frac{\pi}{2})$, is the radial distance, and dz^2 is the induced standard metric on $\mathbb{S}^n$.

We consider the following locally constrained curvature flow in the sphere:

$$\begin{aligned} \frac{\partial X}{\partial t} &= (c_{n,k}\phi'(\rho) - u\frac{\sigma_{k+1}(\lambda)}{\sigma_k(\lambda)})\nu, \\ X(\cdot, 0) &= X_0(\cdot). \end{aligned} \tag{1.2}$$

where $X(x,t) \in \mathbb{S}^{n+1}$ is the position vector of the evolving hypersurface $M(t)$, ν the outward unit normal, $u = \langle \phi(\rho)\frac{\partial}{\partial \rho}, \nu\rangle$, $\lambda = (\lambda_1, \cdots, \lambda_n)$ the principal curvatures, $X_0 : M_0 \hookrightarrow \mathbb{S}^{n+1}$ the initial embedded hypersurface, σ_k the k-th elementary symmetric function, and $c_{n,k} = \frac{\sigma_{k+1}(I)}{\sigma_k(I)} = \frac{n-k}{k+1}$, $I = (1, \cdots, 1)$.

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The particular interest in these flows stems from its monotonicity properties with respect to the quermassintegrals for convex bodies Ω in the sphere. Let $M = \partial\Omega$, set

$$
\begin{aligned}
\mathcal{A}_{-1} =& \mathrm{Vol}(\Omega), \quad \mathcal{A}_0 = \int_M d\mu_g, \\
\mathcal{A}_1 =& \int_M \sigma_1(\lambda) d\mu_g + n\mathrm{Vol}(\Omega), \\
\mathcal{A}_m =& \int_M \sigma_m(\lambda) d\mu_g + \frac{n-m+1}{m-1}\mathcal{A}_{m-2},
\end{aligned}
\tag{1.3}
$$

where $2 \le m \le n$. Here g is the induced metric on M and $d\mu_g$ is the associated volume element.

The monotonicity properties of those functionals along the flow (1.2) follow from the following Hsiung-Minkowski identities (see Proposition 2.6):

$$
(m+1)\int_M u\sigma_{m+1}(\lambda) = (n-m)\int_M \phi(\rho)'\sigma_m(\lambda), \quad 0 \le m \le n-1. \tag{1.4}
$$

With the help of the evolution equations (see Proposition 2.9):

$$
\partial_t \mathcal{A}_{-1} = \int_M \left(c_{n,k}\phi'(\rho) - u\frac{\sigma_{k+1}(\lambda)}{\sigma_k(\lambda)}\right) d\mu_g, \tag{1.5}
$$

and

$$
\partial_t \mathcal{A}_l = (l+1)\int_M \sigma_{l+1}(\lambda)\left(c_{n,k}\phi'(\rho) - u\frac{\sigma_{k+1}(\lambda)}{\sigma_k(\lambda)}\right) d\mu_g, \tag{1.6}
$$

we deduce that along the flow (1.2) for $0 \le k \le n-1$, the following monotonicity relations hold:

$$
\partial_t \mathcal{A}_l \begin{cases} \ge 0, & \text{if } l < k-1; \\ = 0, & \text{if } l = k-1; \\ \le 0, & \text{if } l > k-1. \end{cases} \tag{1.7}
$$

Hence, if one can prove that the flow (1.2) moves an arbitrary convex hypersurface to a round sphere, then the following conjecture would turn into a theorem:

Conjecture 1.1.

$$
\mathcal{A}_l \le \xi_{l,k}(\mathcal{A}_k), \quad \forall -1 \le l < k \le n, \tag{1.8}
$$

where $\xi_{l,k}$ is the unique positive function defined on $(0, \infty)$ such that " $=$ " holds when M is a geodesic sphere. " $=$ " holds if and only if M is a geodesic sphere.

Flow (1.2) is another example of hypersurface flows which have been introduced recently with goals to establish optimal geometric inequalities [7, 3, 8, 9, 11] for hypersurfaces in space forms. These locally constrained flows are associated to the optimal solutions to the problems of calculus of variations in geometric setting. The counterpart of (1.2) in $\mathbb{R}^{n+1}$ was considered in [8, 9], where the longtime existence and convergence were proved by transforming the equation to corresponding inverse type PDE on $\mathbb{S}^n$ for the support function. In the case of $\mathbb{S}^{n+1}$, up to several special values of k and l, this conjecture is open until today, see for example [3, 4, 13]. The main issue is that so far we can not control the curvature along the flow (1.2) from above (except the case $k = 0$ [8]). All the other a priori estimates for this flow are in place and this note is supposed to collect those estimates.

The rest of this article is organized as follows. In section 2, we list some basic facts for k-th elementary symmetric functions, hypersurfaces in $\mathbb{S}^{n+1}$ and evolution equations. In section 3, we prove the C^0, C^1 a priori estimates and uniform bounds of $F = \frac{\sigma_{k+1}(\lambda)}{\sigma_k(\lambda)}$. In section 4, we prove the strict convexity of $M(t)$ along the flow (1.2) if M_0 is convex. In the last section, we give a discussion of the C^2 estimate.

2. Preliminary

We first recall some well-known facts about k-th elementary symmetric functions, hypersurfaces in $\mathbb{S}^{n+1}$, and then give some evolution equations along the flow (1.2).

2.1. **Elementary symmetric functions.** For any $k = 1, \cdots, n$, and $\lambda = (\lambda_1, \cdots, \lambda_n)$, the k-th elementary symmetric function is defined as follows

$$\sigma_k(\lambda) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}. \tag{2.1}$$

and $\sigma_0 = 1$. First, we denote by $\sigma_k(\lambda\,|i\,)$ the symmetric function with $\lambda_i = 0$ and $\sigma_k(\lambda\,|ij\,)$ the symmetric function with $\lambda_i = \lambda_j = 0$.

Proposition 2.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $k = 1, \cdots, n$, then*

$$\begin{aligned} &\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|i), \quad \forall\, 1 \le i \le n, \\ &\sum_i \lambda_i \sigma_{k-1}(\lambda|i) = k\sigma_k(\lambda), \\ &\sum_i \sigma_k(\lambda|i) = (n-k)\sigma_k(\lambda). \end{aligned}$$

Viewing σ_k as a function on symmetric matrices, we also denote by $\sigma_k(W\,|i\,)$ the symmetric function with W deleting the i-row and i-column and $\sigma_k(W\,|ij\,)$ the symmetric function with W deleting the i, j-rows and i, j-columns. Then we have the following identities.

Proposition 2.2. *Suppose $W = (W_{ij})$ is diagonal, and m is a positive integer, then*

$$\frac{\partial \sigma_m(W)}{\partial W_{ij}} = \begin{cases} \sigma_{m-1}(W\,|i\,), & \text{if } i = j, \\ 0, & \text{if } i \ne j. \end{cases}$$

and

$$\frac{\partial^2 \sigma_m(W)}{\partial W_{ij} \partial W_{kl}} = \begin{cases} \sigma_{m-2}(W\,|ik\,), & \text{if } i = j, k = l, i \ne k, \\ -\sigma_{m-2}(W\,|ik\,), & \text{if } i = l, j = k, i \ne j, \\ 0, & \text{otherwise }. \end{cases}$$

Recall that the Gårding's cone is defined as

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0, \forall 1 \leq i \leq k\}. \tag{2.2}$$

The following properties are well known.

Proposition 2.3. *Let $\lambda \in \Gamma_k$ and $k \in \{1, 2, \cdots, n\}$. Suppose that*

$$\lambda_1 \geq \cdots \geq \lambda_k \geq \cdots \geq \lambda_n,$$

then we have

$$\sigma_{k-1}(\lambda|n) \geq \sigma_{k-1}(\lambda|n-1) \geq \cdots \geq \sigma_{k-1}(\lambda|k) \geq \cdots \geq \sigma_{k-1}(\lambda|1) > 0; \tag{2.3}$$

$$\lambda_1 \geq \cdots \geq \lambda_k > 0, \quad \sigma_k(\lambda) \leq C_n^k \lambda_1 \cdots \lambda_k; \tag{2.4}$$

$$\sigma_k(\lambda) \geq \lambda_1 \cdots \lambda_k, \quad \text{if } \lambda \in \Gamma_{k+1}; \tag{2.5}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$.

The generalized Newton-MacLaurin inequality is as follows, which will be used all the time. See [17].

Proposition 2.4. *For $\lambda \in \Gamma_k$ and $k > l \geq 0$, $r > s \geq 0$, $k \geq r$, $l \geq s$, we have*

$$\left[\frac{\sigma_k(\lambda)/C_n^k}{\sigma_l(\lambda)/C_n^l}\right]^{\frac{1}{k-l}} \leq \left[\frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s}\right]^{\frac{1}{r-s}}. \tag{2.6}$$

2.2. **Hypersurfaces in $\mathbb{S}^{n+1}$.** The following lemma is well known, e.g. [8].

Lemma 2.5. *Let $M^n \subset \mathbb{S}^{n+1}$ be a closed hypersurface with induced metric g. Let $\Phi = \Phi(\rho) = \int_0^\rho \phi(r)dr$, then*

$$\nabla_i \nabla_j \Phi = \phi'(\rho) g_{ij} - h_{ij} u, \tag{2.7}$$

recall $u = \langle \phi(\rho) \frac{\partial}{\partial \rho}, \nu \rangle$.

We have the following Hsiung-Minkowski identities, see [8].

Proposition 2.6. *Let M be a closed hypersurface in $\mathbb{S}^{n+1}$. Then, for $m = 0, 1, \cdots, n-1$,*

$$(m+1) \int_M u \sigma_{m+1}(\lambda) = (n-m) \int_M \phi'(\rho) \sigma_m(\lambda), \tag{2.8}$$

where we use the convention that $\sigma_0 = 1$.

Next, we state the gradient and hessian of the support function $u = \langle \phi(\rho) \frac{\partial}{\partial \rho}, \nu \rangle$ under the induced metric g on M, see [8].

Lemma 2.7. *The support function u satisfies*

$$\nabla_i u = g^{ml} h_{im} \nabla_l \Phi, \tag{2.9}$$

$$\nabla_i \nabla_j u = g^{ml} \nabla_m h_{ij} \nabla_l \Phi + \phi' h_{ij} - (h^2)_{ij} u, \tag{2.10}$$

where $(h^2)_{ij} = g^{ml} h_{im} h_{jl}$.

2.3. **Evolution equations.** Let $M(t)$ be a smooth family of closed hypersurfaces in $\mathbb{S}^{n+1}$, and $X(\cdot,t)$ denote a point on $M(t)$. The following basic evolution equations for normal variations are well known, e.g. [6].

Proposition 2.8. *Under the flow $\partial_t X = f(X(\cdot,t))\nu$ in the sphere we have the following evolution equations*

$$\partial_t g_{ij} = 2fh_{ij}, \tag{2.11}$$

$$\partial_t d\mu_g = f\sigma_1(\lambda)d\mu_g, \tag{2.12}$$

$$\partial_t h_{ij} = -\nabla_i\nabla_j f + f(h^2)_{ij} - fg_{ij}, \tag{2.13}$$

$$\partial_t h^i_j = -\nabla^i\nabla_j f - fg^{im}(h^2)_{mj} - f\delta^i_j. \tag{2.14}$$

From Proposition 2.8, we can obtain the evolution of the quermassintegrals in the sphere.

Proposition 2.9. *Along the flow $\partial_t X = f(X(\cdot,t))\nu$ in the sphere, we have for $0 \le l \le n-1$*

$$\partial_t \mathcal{A}_l = (l+1)\int_M \sigma_{l+1}(\lambda) f d\mu_g, \tag{2.15}$$

and

$$\partial_t \mathcal{A}_{-1} = \int_M f d\mu_g, \tag{2.16}$$

where Ω is the domain enclosed by the closed hypersurface. Moreover, if the flow is (1.2) *and $h(t) \in \Gamma_{k+1}$, then we have*

$$\partial_t \mathcal{A}_l \begin{cases} \ge 0, & \text{if } l < k-1; \\ = 0, & \text{if } l = k-1; \\ \le 0, & \text{if } l > k-1. \end{cases} \tag{2.17}$$

Proof. (2.15) and (2.16) follow directly form Proposition 2.8. (2.17) follows from (2.8), (2.15) and the Newton-MacLaurin inequality. □

Let (M,g) be a hypersurface in $\mathbb{S}^{n+1}$ with induced metric g. We now give the local expressions of the induced metric, second fundamental form, Weingarten curvatures etc. when M is a graph of a smooth and positive function $\rho(z)$ on $\mathbb{S}^n$. Let $\partial_1,\cdots,\partial_n$ be a local frame along M and ∂_ρ be the vector field along the radial direction. Then the support function, induced metric, inverse metric matrix, second fundamental form can be expressed as follows. For simplicity, all the covariant derivatives with respect to the standard spherical metric e_{ij} will also be denoted as ∇ when there is no confusion in the context.

$$u = \frac{\phi^2}{\sqrt{\phi^2 + |\nabla\rho|^2}}, \tag{2.18}$$

$$g_{ij} = \phi^2 e_{ij} + \rho_i \rho_j, \tag{2.19}$$

$$g^{ij} = \frac{1}{\phi^2}\left(e^{ij} - \frac{\rho^i \rho^j}{\phi^2 + |\nabla\rho|^2}\right), \tag{2.20}$$

$$h_{ij} = \frac{1}{\sqrt{\phi^2 + |\nabla\rho|^2}}(-\phi\nabla_i\nabla_j\rho + 2\phi'\rho_i\rho_j + \phi^2\phi' e_{ij}), \tag{2.21}$$

$$h^i_j = \frac{1}{\phi^2\sqrt{\phi^2 + |\nabla\rho|^2}}\left(e^{im} - \frac{\rho^i\rho^m}{\phi^2 + |\nabla\rho|^2}\right)(-\phi\nabla_m\nabla_j\rho + 2\phi'\rho_m\rho_j + \phi^2\phi' e_{mj}), \tag{2.22}$$

where all the covariant derivatives ∇ and ρ_i are w.r.t. the spherical metric e_{ij} .

We now consider the flow equation (1.2) of radial graphs over $\mathbb{S}^n$ in $\mathbb{S}^{n+1}$. Let

$$\omega = \frac{\phi}{\sqrt{\phi^2 + |\nabla\rho|^2}}.$$

It is known that if a family of radial graphs satisfy $\partial_t X = f\nu$, then the evolution of the scalar function $\rho = \rho(X(z,t),t)$ satisfies

$$\partial_t \rho = f\omega. \tag{2.23}$$

The following is a well known commutator identity.

Lemma 2.10. *Let h_{ij} be the second fundamental form and g_{ij} be the induced metric of a hypersurface in $\mathbb{S}^{n+1}$. Then*

$$\begin{aligned} \nabla_i\nabla_j h_{ml} = {} & \nabla_m\nabla_l h_{ij} + h_{ij}(h^2)_{ml} - (h^2)_{ij}h_{ml} + h_{il}(h^2)_{mj} - (h^2)_{il}h_{mj} \\ & + [h_{ml}g_{ij} - h_{ij}g_{ml} + h_{mj}g_{il} - h_{il}g_{mj}]. \end{aligned} \tag{2.24}$$

Lemma 2.11. *Along the flow (1.2) in $\mathbb{S}^{n+1}$, the graph function ρ and the support function $u = \langle \phi(\rho)\frac{\partial}{\partial\rho}, \nu\rangle$ evolve as follows*

$$\partial_t\rho - uF^{ij}\nabla_i\nabla_j\rho = \frac{\phi'}{\phi}u(c_{n,k} - F^{ij}g_{ij}) + \frac{\phi'}{\phi}uF^{ij}\rho_i\rho_j \tag{2.25}$$

$$\partial_t u - uF^{ij}\nabla_i\nabla_j u = -c_{n,k}\nabla\Phi\nabla\phi' + F\nabla\Phi\nabla u + (c_{n,k}\phi' - 2uF)\phi' + u^2F^{ij}(h^2)_{ij}, \tag{2.26}$$

where $F = \frac{\sigma_{k+1}(\lambda)}{\sigma_k(\lambda)}$ and $F^{ij} = \frac{\partial F}{\partial h_{ij}}$.

Proof. The function ρ satisfies

$$\partial_t\rho = (c_{n,k}\phi' - uF)\omega,$$

and

$$uF^{ij}\rho_{ij} = -u\omega F + \frac{\phi'}{\phi}uF^{ij}g_{ij} - \frac{\phi'}{\phi}uF^{ij}\rho_i\rho_j.$$

Hence (2.25) holds.

Applying Lemma 2.7, we can obtain

$$\begin{aligned}\partial_t u - uF^{ij}u_{ij} &= f\phi' - \nabla\Phi\nabla f - uF^{ij}[\nabla h_{ij}\nabla\Phi + \phi' h_{ij} - (h^2)_{ij}u]\\ &= (c_{n,k}\phi' - uF)\phi' - \nabla\Phi\nabla(c_{n,k}\phi' - uF)\\ &\quad -u[\nabla F\nabla\Phi + \phi' F - F^{ij}(h^2)_{ij}u]\\ &= -c_{n,k}\nabla\Phi\nabla\phi' + F\nabla\Phi\nabla u + (c_{n,k}\phi' - 2uF)\phi' + u^2F^{ij}(h^2)_{ij}.\end{aligned}$$

□

Lemma 2.12. *Let h_{ij} be the second fundamental form and g_{ij} be the induced metric of a hypersurface in $\mathbb{S}^{n+1}$ and $F = \frac{\sigma_{k+1}(\lambda)}{\sigma_k(\lambda)}$. Then*

$$\begin{aligned}\nabla_i\nabla_j F =& F^{\alpha\beta}\nabla_\alpha\nabla_\beta h_{ij} + F^{\alpha\beta,\gamma\eta}\nabla_i h_{\alpha\beta}\nabla_j h_{\gamma\eta}\\ &+ [F^{\alpha\beta}(h^2)_{\alpha\beta} - F^{\alpha\alpha}]h_{ij} - F[(h^2)_{ij} - g_{ij}],\end{aligned} \tag{2.27}$$

where $F^{\alpha\beta} = \frac{\partial F}{\partial h_{\alpha\beta}}$ and $F^{\alpha\beta,\gamma\eta} = \frac{\partial^2 F}{\partial h_{\alpha\beta}\partial h_{\gamma\eta}}$.

Proof.

$$\begin{aligned}\nabla_i\nabla_j F =& F^{\alpha\beta}\nabla_i\nabla_j h_{\alpha\beta} + F^{\alpha\beta,\gamma\eta}\nabla_i h_{\alpha\beta}\nabla_j h_{\gamma\eta}\\ =& F^{\alpha\beta}\nabla_\alpha\nabla_\beta h_{ij} + F^{\alpha\beta,\gamma\eta}\nabla_i h_{\alpha\beta}\nabla_j h_{\gamma\eta}\\ &+ F^{\alpha\beta}[h_{ij}(h^2)_{\alpha\beta} - (h^2)_{ij}h_{\alpha\beta} + h_{i\beta}(h^2)_{\alpha j} - (h^2)_{i\beta}h_{\alpha j}\\ &\qquad + (h_{\alpha\beta}g_{ij} - h_{ij}g_{\alpha\beta} + h_{i\beta}g_{\alpha j} - h_{\alpha j}g_{i\beta})]\\ =& F^{\alpha\beta}\nabla_\alpha\nabla_\beta h_{ij} + F^{\alpha\beta,\gamma\eta}\nabla_i h_{\alpha\beta}\nabla_j h_{\gamma\eta}\\ &+ [F^{\alpha\beta}(h^2)_{\alpha\beta} - F^{\alpha\alpha}]h_{ij} - F[(h^2)_{ij} - g_{ij}].\end{aligned} \tag{2.28}$$

□

Lemma 2.13. *Let h_{ij} be the second fundamental form and g_{ij} be the induced metric of a hypersurface in $\mathbb{S}^{n+1}$ and $F = \frac{\sigma_{k+1}(\lambda)}{\sigma_k(\lambda)}$. Then along the flow* (1.2)

$$\begin{aligned}\partial_t h^i_j - uF^{ml}\nabla_m\nabla_l h^i_j =& uF^{ml,pq}\nabla^i h_{ml}\nabla_j h_{pq} + \nabla^i u\nabla_j F + \nabla_j u\nabla^i F + F\nabla h^i_j\nabla\Phi\\ &- (c_{n,k}\phi' + uF)(h^2)^i_j\\ &+ h^i_j[u(F^{ml}(h^2)_{ml} - F^{mm}) + \phi' F - c_{n,k}u]\\ &+ 2uF\delta^i_j,\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}\partial_t F - uF^{ml}\nabla_m\nabla_l F =& 2F^{ml}\nabla_m u\nabla_l F + F\nabla\Phi\nabla F\\ &- [c_{n,k}F^{ml}(h^2)_{ml} - F^2]\phi' + uF[\sum F^{mm} - c_{n,k}].\end{aligned} \tag{2.30}$$

Proof. By the tensorial property, we do not distinguish upper and lower indexes in this proof whenever applicable. We need the fact that $\nabla\phi' = -\nabla\Phi$, and then we can obtain

$$\begin{aligned}
\partial_t h^i_j =& -\nabla_i\nabla_j(c_{n,k}\phi' - uF) - (c_{n,k}\phi' - uF)(h^2)_{ij} - (c_{n,k}\phi' - uF)\delta^i_j\\
=& -c_{n,k}\nabla_i\nabla_j\phi' + u\nabla_i\nabla_j F + \nabla_i u\nabla_j F + \nabla_j u\nabla_i F + F\nabla_i\nabla_j u\\
&-(c_{n,k}\phi' - uF)(h^2)_{ij} - (c_{n,k}\phi' - uF)\delta^i_j\\
=& c_{n,k}(\phi' g_{ij} - uh_{ij}) + \nabla_i u\nabla_j F + \nabla_j u\nabla_i F\\
&+F[\nabla h_{ij}\nabla\Phi + \phi' h_{ij} - u(h^2)_{ij}]\\
&+u[F^{ml}\nabla_m\nabla_l h_{ij} + F^{ml,pq}\nabla_i h_{ml}\nabla_j h_{pq} + (F^{ml}(h^2)_{ml} - F^{mm})h_{ij} - F((h^2)_{ij} - g_{ij})]\\
&-(c_{n,k}\phi' - uF)(h^2)_{ij} - (c_{n,k}\phi' - uF)\delta^i_j\\
=& u[F^{ml}\nabla_m\nabla_l h_{ij} + F^{ml,pq}\nabla_i h_{ml}\nabla_j h_{pq}] + \nabla_i u\nabla_j F + \nabla_j u\nabla_i F + F\nabla h_{ij}\nabla\Phi\\
&-(c_{n,k}\phi' + uF)(h^2)_{ij}\\
&+h_{ij}[u(F^{ml}(h^2)_{ml} - F^{mm}) + \phi' F - c_{n,k}u]\\
&+2uF\delta^i_j.
\end{aligned}$$

Finally, (2.30) follows from

$$\partial_t F = F^i_j\partial_t h^j_i.$$

□

Lemma 2.14. *Let h_{ij} be the second fundamental form and g_{ij} be the induced metric of a hypersurface in $\mathbb{S}^{n+1}$ and $F = \frac{\sigma_{k+1}(\lambda)}{\sigma_k(\lambda)}$. Then along the flow* (1.2)

$$\begin{aligned}
\partial_t(uF) - uF^{ml}\nabla_m\nabla_l(uF) =& F[c_{n,k}|\nabla\Phi|^2 + (c_{n,k}\phi' - 2uF)\phi' + u^2F^{ml}(h^2)_{ml}]\\
&+u[uF(\sum F^{mm} - c_{n,k}) - (c_{n,k}F^{ml}(h^2)_{ml} - F^2)\phi']\\
&+F\nabla\Phi\nabla(uF),
\end{aligned}\tag{2.31}$$

and

$$\begin{aligned}
\partial_t(\frac{h^i_i}{u}) - uF^{ml}\nabla_m\nabla_l(\frac{h^i_i}{u}) =& F^{ml,pq}\nabla^i h_{ml}\nabla_i h_{pq} + \frac{2}{u}\nabla^i u\nabla_i F + F\nabla\Phi\nabla(\frac{h^i_i}{u})\\
&+2F^{ml}\nabla_m u\nabla_l(\frac{h^i_i}{u}) - (c_{n,k}\frac{\phi'}{u} + F)(h^2)^i_i + 2F\\
&+h^i_i[F^{mm} - (c_{n,k}\frac{\phi'}{u^2} - 3\frac{F}{u})\phi' - c_{n,k} + \frac{c_{n,k}}{u^2}\nabla\Phi\nabla\phi'].
\end{aligned}\tag{2.32}$$

3. A priori estimates

Since M_0 is strictly convex, there is $T > 0$ such that flow (1.2) exists and the solution $M(t)$ is strictly convex for all $0 \le t < T$. *This will be assumed in the rest of this section.*

3.1. C^0 estimate.

Theorem 3.1. *Let M_0 be a strictly convex, radial graph of positive function ρ_0 over $\mathbb{S}^n$ embedded in $\mathbb{S}^{n+1}$. If $M(t)$ solves the flow (1.2) with the initial value M_0, then for any $(z,t) \in \mathbb{S}^n \times [0,T)$*

$$\min_{z\in\mathbb{S}^n} \rho_0(z) \leq \rho(z,t) \leq \max_{z\in\mathbb{S}^n} \rho_0(z). \tag{3.1}$$

Proof. At critical points of ρ, we have the following critical point conditions,

$$\nabla\rho = 0, \quad \omega = 1, \quad u = \phi, \tag{3.2}$$

and then the Weingarten curvature is

$$h^i_j = \frac{1}{\phi^2}(-\rho_{ij} + \phi\phi' e_{ij}), \tag{3.3}$$

and then

$$F = \frac{\sigma_{k+1}(\lambda)}{\sigma_k(\lambda)} = \frac{\phi'}{\phi}\frac{\sigma_{k+1}(-\frac{\rho_{ij}}{\phi\phi'} + e_{ij})}{\sigma_k(-\frac{\rho_{ij}}{\phi\phi'} + e_{ij})}. \tag{3.4}$$

So at the critical point, we have

$$\partial_t\rho = c_{n,k}\phi' - uF = c_{n,k}\phi' - \phi'\frac{\sigma_{k+1}(-\frac{\rho_{ij}}{\phi\phi'} + e_{ij})}{\sigma_k(-\frac{\rho_{ij}}{\phi\phi'} + e_{ij})}. \tag{3.5}$$

By standard maximum principle, this proves the upper and lower bounds for ρ. □

3.2. C^1 estimates.

Lemma 3.2. *Let $F(\lambda) := \frac{\sigma_{k+1}(\lambda)}{\sigma_k(\lambda)}$, and $c_{n,k} = F(I) = \frac{n-k}{k+1}$ where $I = (1,\cdots,1)$. Then*

$$\sum_i F^{ii}\lambda_i^2 \geq \frac{F^2}{c_{n,k}}, \quad \sum_i F^{ii} \geq c_{n,k}, \quad \forall\lambda\in\Gamma_k. \tag{3.6}$$

Moreover, if $\lambda \in \bar{\Gamma}_{k+1}$, then $\sum F^{ii} \leq n-k$.

Proof. The proof below is from [3] We first derive

$$\begin{aligned}
\sum F^{ii}\lambda_i^2 = & \sum \tfrac{\sigma_k(\lambda|i)\lambda_i^2\sigma_k - \sigma_{k+1}\sigma_{k-1}(\lambda|i)\lambda_i^2}{\sigma_k^2} \\
= & \tfrac{[\sigma_1\sigma_{k+1} - (k+2)\sigma_{k+2}]\sigma_k - \sigma_{k+1}[\sigma_1\sigma_k - (k+1)\sigma_{k+1}]}{\sigma_k^2} \\
= & \tfrac{(k+1)\sigma_{k+1}^2 - (k+2)\sigma_{k+2}\sigma_k}{\sigma_k^2} \\
\geq & \tfrac{k+1}{n-k}F^2 = \tfrac{1}{c_{n,k}}F^2,
\end{aligned} \tag{3.7}$$

where the last inequality follows from Newton-McLaurin inequality.

Similarly, we have

$$\begin{aligned} \sum F^{ii} &= \sum \tfrac{\sigma_k(\lambda|i)\sigma_k(\lambda)-\sigma_{k+1}(\lambda)\sigma_{k-1}(\lambda|i)}{\sigma_k^2} \\ &= \tfrac{(n-k)\sigma_k^2-(n-k+1)\sigma_{k+1}\sigma_{k-1}}{\sigma_k^2} \\ &\geq \tfrac{n-k}{k+1} = c_{n,k}, \end{aligned} \tag{3.8}$$

where the last inequality follows from Newton-McLaurin inequality. If $\lambda \in \bar{\Gamma}_{k+1}$, then $\sigma_{k+1}\sigma_{k-1} \geq 0$ and we conclude that $\sum F^{ii} \leq n-k$ from the second identity in (3.8). □

Theorem 3.3. *Let M_0 be a strictly convex, radial graph of positive function ρ_0 over $\mathbb{S}^n$ embedded in $\mathbb{S}^{n+1}$. If $M(t)$ solves the flow (1.2) with the initial value M_0, then for any $(z,t) \in \mathbb{S}^n \times [0,T)$*

$$u(z,t) \geq \min_{z\in\mathbb{S}^n} u(z,0). \tag{3.9}$$

As a consequence, we have C^1 bound for ρ, that is,

$$|\rho|_{C^1(\mathbb{S}^n)} \leq C, \tag{3.10}$$

where C depends only on the initial data.

Proof. For any fixed $t \in (0,T)$, we have at the minimum point of $u(z,t)$,

$$\nabla u = 0.$$

So from Lemma 2.11, we have

$$\begin{aligned} \partial_t u - uF^{ij}u_{ij} =& c_{n,k}|\nabla\Phi|^2 + (c_{n,k}\phi' - 2uF)\phi' + u^2F^{ij}(h^2)_{ij} \\ \geq& c_{n,k}|\nabla\Phi|^2 + c_{n,k}(\phi' - \frac{1}{c_{n,k}}uF)^2 \geq 0, \end{aligned} \tag{3.11}$$

which finishes the proof of (3.9). □

3.3. **Uniform bounds of F.**

Lemma 3.4. *Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_n$, and $\lambda_1 \geq \cdots \geq \lambda_n$. Then for $1 \leq m \leq n-1$, we have*

$$\frac{m(n-m)\sigma_m(\lambda)^2 - (m+1)(n-m+1)\sigma_{m+1}(\lambda)\sigma_{m-1}(\lambda)}{\sigma_m(\lambda)^2} \sim \frac{(\lambda_1-\lambda_n)^2}{{\lambda_1}^2}, \tag{3.12}$$

where $f \sim g$ means $\frac{1}{C(n,m)}g \leq f \leq C(n,m)g$ for some constant $C(n,m) > 0$.

Proof. The case $m = 1$ is trivial, as

$$(n-1)\sigma_1(\lambda)^2 - 2n\sigma_2(\lambda) = \sum_{i<j}(\lambda_i - \lambda_j)^2.$$

We may assume $1 < m < n$. By direct computation we can derive

$$\begin{aligned}
&m(n-m)\sigma_m(\lambda)^2 - (m+1)(n-m+1)\sigma_{m+1}(\lambda)\sigma_{m-1}(\lambda)\\
=&[m\sigma_m(\lambda)][(n-m)\sigma_m(\lambda)] - [(m+1)\sigma_{m+1}(\lambda)][(n-m+1)\sigma_{m-1}(\lambda)]\\
=&[\sum_i \lambda_i\sigma_{m-1}(\lambda|i)][\sum_j \sigma_m(\lambda|j)] - [\sum_j \lambda_j\sigma_m(\lambda|j)][\sum_i \sigma_{m-1}(\lambda|i)]\\
=&\sum_{i,j}(\lambda_i-\lambda_j)\sigma_{m-1}(\lambda|i)\sigma_m(\lambda|j)\\
=&\sum_{i<j}(\lambda_i-\lambda_j)[\sigma_{m-1}(\lambda|i)\sigma_m(\lambda|j) - \sigma_{m-1}(\lambda|j)\sigma_m(\lambda|i)]\\
=&\sum_{i<j}(\lambda_i-\lambda_j)[(\sigma_{m-1}(\lambda|ij)+\lambda_j\sigma_{m-2}(\lambda|ij))\,(\sigma_m(\lambda|ij)+\lambda_i\sigma_{m-1}(\lambda|ij))\\
&-(\sigma_{m-1}(\lambda|ij)+\lambda_i\sigma_{m-2}(\lambda|ij))\,(\sigma_m(\lambda|ij)+\lambda_j\sigma_{m-1}(\lambda|ij))]\\
=&\sum_{i<j}(\lambda_i-\lambda_j)\cdot(\lambda_i-\lambda_j)[\sigma_{m-1}(\lambda|ij)^2 - \sigma_{m-2}(\lambda|ij)\sigma_m(\lambda|ij)]\\
\sim&\sum_{i<j}(\lambda_i-\lambda_j)^2\sigma_{m-1}(\lambda|ij)^2,
\end{aligned} \tag{3.13}$$

where the last $\sim$ follows from Newton-MacLaurin inequality. Then we can obtain

$$\begin{aligned}
&\frac{m(n-m)\sigma_m(\lambda)^2 - (m+1)(n-m+1)\sigma_{m+1}(\lambda)\sigma_{m-1}(\lambda)}{\sigma_m(\lambda)^2}\\
\sim&\sum_{i<j}(\lambda_i-\lambda_j)^2\frac{\sigma_{m-1}(\lambda|ij)^2}{\sigma_m(\lambda)^2}\\
\sim&\sum_{m\le i<j}(\lambda_i-\lambda_j)^2\frac{(\lambda_1\cdots\lambda_{m-1})^2}{(\lambda_1\cdots\lambda_m)^2} + \sum_{i<m<j}(\lambda_i-\lambda_j)^2\frac{(\lambda_1\cdots\widehat{\lambda_i}\cdots\lambda_m)^2}{(\lambda_1\cdots\lambda_m)^2}\\
&+\sum_{i<j\le m}(\lambda_i-\lambda_j)^2\frac{(\lambda_1\cdots\widehat{\lambda_i}\cdots\widehat{\lambda_j}\cdots\lambda_{m+1})^2}{(\lambda_1\cdots\lambda_m)^2}\\
\sim&\sum_{m\le i<j}\frac{(\lambda_i-\lambda_j)^2}{\lambda_m^2} + \sum_{i<m<j}\frac{(\lambda_i-\lambda_j)^2}{\lambda_i^2} + \sum_{i<j\le m}\frac{(\lambda_i-\lambda_j)^2\lambda_{m+1}^2}{(\lambda_i\lambda_j)^2}\\
\sim&\frac{(\lambda_m-\lambda_n)^2}{\lambda_m^2} + \frac{(\lambda_1-\lambda_n)^2}{\lambda_1^2} + \frac{(\lambda_1-\lambda_m)^2}{\lambda_1^2}\frac{\lambda_{m+1}^2}{\lambda_m^2}\\
\sim&\frac{(\lambda_1-\lambda_n)^2}{\lambda_1^2},
\end{aligned} \tag{3.14}$$

where $\widehat{\lambda_i}$ means that λ_i is omitted. The proof is finished. □

Due to Lemma 3.4, we obtain the uniform bound of F as follows.

Theorem 3.5. *Let M_0 be a strictly convex, radial graph of positive function ρ_0 over $\mathbb{S}^n$ embedded in $\mathbb{S}^{n+1}$. If $M(t)$ solves the flow (1.2) with the initial value M_0, then for any $(z,t) \in \mathbb{S}^n \times [0,T)$*

$$\frac{1}{C} \leq F \leq C, \tag{3.15}$$

where C depends only on n, k and the initial data.

Proof. For any fixed $t \in (0,T)$, we have at the critical points of F,

$$\nabla F = 0.$$

So from Lemma 2.13, we can get

$$\partial_t F - uF^{ij}\nabla_i\nabla_j F = -c_{n,k}\phi' F^2[\frac{F^{ij}(h^2)_{ij}}{F^2} - \frac{1}{c_{n,k}}] + uF[\sum F^{ii} - c_{n,k}]. \tag{3.16}$$

In the following, we divide the proof of (3.15) into three cases.

Firstly, for the Case: $1 \leq k \leq n-2$, we know from Lemma 3.4,

$$\frac{F^{ij}(h^2)_{ij}}{F^2} - \frac{1}{c_{n,k}} = \frac{(k+1)\frac{n-k-1}{n-k}\sigma_{k+1}^2 - (k+2)\sigma_{k+2}\sigma_k}{\sigma_{k+1}^2} \sim \frac{(\lambda_1 - \lambda_n)^2}{\lambda_1^2}, \tag{3.17}$$

and

$$\sum F^{ii} - c_{n,k} = \frac{\frac{k}{k+1}(n-k)\sigma_k^2 - (n-k+1)\sigma_{k+1}\sigma_{k-1}}{\sigma_k^2} \sim \frac{(\lambda_1 - \lambda_n)^2}{\lambda_1^2}. \tag{3.18}$$

Thus

$$\frac{F^{ij}(h^2)_{ij}}{F^2} - \frac{1}{c_{n,k}} \sim \sum F^{ii} - c_{n,k}. \tag{3.19}$$

Hence (3.15) holds.

Secondly, for the Case: $k = 0$, we can directly get,

$$\frac{F^{ij}(h^2)_{ij}}{F^2} - \frac{1}{c_{n,k}} \geq 0, \tag{3.20}$$

and

$$\sum F^{ii} - c_{n,k} = 0, \tag{3.21}$$

so we can get $F \leq C$ from (3.16). To prove the lower bound of F, we consider the minimum point of uF, and then we can get $F \geq \frac{1}{C}$ from (2.31). Hence (3.15) holds.

Lastly, we consider the Case: $k = n-1$ in the following. It is easy to know

$$\frac{F^{ij}(h^2)_{ij}}{F^2} - \frac{1}{c_{n,k}} = 0, \tag{3.22}$$

and

$$(n-k) - c_{n,k} \geq \sum F^{ii} - c_{n,k} \geq 0, \tag{3.23}$$

so we can get $F \geq \frac{1}{C}$ from (3.16).

To prove the upper bound of F, we consider the maximum point of $P =: F+\frac{1}{u}$. At the maximum point of P, we have

$$0 = \nabla_i P = \nabla_i F - \frac{1}{u^2}\nabla_i u, \tag{3.24}$$

and then

$$\begin{aligned}
\partial_t P - uF^{ij}\nabla_i\nabla_j P =& \partial_t F - uF^{ij}\nabla_i\nabla_j F\\
&- \frac{1}{u^2}\left[\partial_t u - uF^{ij}\nabla_i\nabla_j u\right] - uF^{ij}\cdot\frac{2}{u^3}\nabla_i u\nabla_j u\\
=& 2F^{ij}\nabla_i u\nabla_j F + F\nabla\Phi\nabla F + uF[\sum F^{ii} - c_{n,k}]\\
&- \frac{1}{u^2}\left[-c_{n,k}|\nabla\Phi|^2 + F\nabla\Phi\nabla u + (c_{n,k}\phi' - 2uF)\phi' + u^2F^{ij}(h^2)_{ij}\right]\\
&- uF^{ij}\cdot\frac{2}{u^3}\nabla_i u\nabla_j u\\
=& uF[\sum F^{ii} - c_{n,k}] - \frac{1}{u^2}\left[-c_{n,k}|\nabla\Phi|^2 + (c_{n,k}\phi' - 2uF)\phi' + u^2\frac{F^2}{c_{n,k}}\right]. \qquad (3.25)
\end{aligned}$$

So we can get $F \le C$ from (3.25). Hence (3.15) holds.

□

4. Preserving convexity

In this section, we prove the flow (1.2) preserves convexity in $\mathbb{S}^{n+1}$. Denote $T > 0$ to be largest time, up to which all flow hypersurfaces are strictly convex. Furthermore denote by T^* is the largest time of existence of a smooth solution to (1.2). Recall that Γ_k is the k-th Garding cone, see (2.2).

Theorem 4.1. *Let $M(t)$ be an oriented immersed connected hypersurface in $\mathbb{S}^{n+1}$ with a positive semi-definite second fundamental form $h(t) \in \Gamma_{k+1}$ satisfying equation* (1.2) *for $t \in [0, T^*)$, then $M(t)$ is strictly convex for all $t \in (0, T^*)$.*

Here we provide two proofs. The first proof follows from the following lemma.

Lemma 4.2. *Along the solution of* (1.2) *with a strictly convex initial hypersurface $M_0 \subset \mathbb{S}^{n+1}$ all flow hypersurfaces $M(t) = X(M, t)$ are strictly convex up to T^*, i.e. $T = T^*$, with a uniform estimate*

$$h^i_j \ge c\delta^i_j,$$

where $c = c(\sup_{M_0}\rho, \inf_{M_0}\rho, n, k, T^*)$.

Proof. We calculate the evolution equation of the inverse $\{b^i_j\} = \{h^j_i\}^{-1}$, which is well defined up to T. We suppose that $T < T^*$.

$$\begin{aligned}
\partial_t b^m_m =& - b^m_r\partial_t h^r_s b^s_m,\\
F^{ij}\nabla_i\nabla_j b^m_m =& 2F^{ij}b^m_r\nabla_i h^r_s b^s_p\nabla_j h^p_q b^q_m - F^{ij}b^m_r\nabla_i\nabla_j h^r_s b^s_m,\\
\nabla_i u =& h^j_i\nabla_j\Phi,
\end{aligned}$$

and by the evolution equation for h_i^j, we deduce

$$
\begin{aligned}
&\partial_t b_m^m - uF^{ij}\nabla_i\nabla_j b_m^m - F\nabla^i\Phi\nabla_i b_m^m \\
=& - u(F^{pq,rs} + 2F^{qs}b^{pr})\nabla_i h_{pq}\nabla^j h_{rs} b_j^m b_m^i - b_j^m\nabla^j F\nabla_m\Phi - b_m^i\nabla_i F\nabla^m\Phi \\
& - uF^{ij}h_{ir}h_j^r b_m^m + (c_{n,k}\phi' + uF) - \phi' F b_m^m \qquad (4.1)\\
& + uF^{ij}g_{ij}b_m^m + c_{n,k}ub_m^m - 2uFb_r^m b_m^r \\
\leq& - \frac{2u}{F}\nabla_i F\nabla^j F b_j^m b_m^i - b_j^m\nabla^j F\nabla_m\Phi - b_m^i\nabla_i F\nabla^m\Phi \\
& + \psi_1(t)b_m^m + \psi_2(t) - 2uFb_r^m b_m^r,
\end{aligned}
$$

where we used the inverse concavity of F, cf. Theorem 2.3 in [1] and where ψ_i are smooth functions which are uniformly bounded up to T, due to the uniform upper and lower bounds of F.

We use a well known trick to estimate the maximal eigenvalue of b, e.g. compare Lemma 6.1 [5]. Let

$$Q = \sup\{b_{ij}\eta^i\eta^j \mid \quad g_{ij}\eta^i\eta^j = 1\},$$

and suppose this function attains a maximum at (t_0, ξ_0) with $t_0 < T$, i.e.

$$Q(t_0, \xi_0) = \sup_{[0,t_0]\times M} Q.$$

Choose coordinates in (t_0, ξ_0) with

$$g_{ij} = \delta_{ij}, \quad b_{ij} = \lambda_i^{-1}\delta_{ij}, \quad \lambda_1^{-1} \leq \cdots \leq \lambda_n^{-1}.$$

Let η be the vector field $\eta = (0, \ldots, 0, 1)$ and define

$$\tilde{Q} = \frac{b_{ij}\eta^i\eta^j}{g_{ij}\eta^i\eta^j},$$

then locally around (t_0, ξ_0) we have $\tilde{Q} \leq Q$ and the derivatives coincide. Thus at (t_0, ξ_0) the function $\tilde{Q}$ and b_n^n satisfy the same evolution equation and we may show that the right hand side of (4.1) is negative at the point (t_0, ξ_0) in these coordinates, yielding a contradiction.

In these coordinates we obtain

$$
\begin{aligned}
&\partial_t b_n^n - uF^{ij}\nabla_i\nabla_j b_n^n - F\nabla^i\Phi\nabla_i b_n^n \\
\leq& - \frac{2u}{F}(\nabla_n F)^2\lambda_n^{-2} - 2\lambda_n^{-1}\nabla_n F\nabla_n\Phi + \psi_1(t)\lambda_n^{-1} + \psi_2(t) - 2uF\lambda_n^{-2} \\
\leq& - \frac{2u}{F}(\nabla_n F)^2\lambda_n^{-2} + \epsilon\lambda_n^{-2}(\nabla_n F)^2 + c_\epsilon(\nabla_n\Phi)^2 + \psi_1(t)\lambda_n^{-1} + \psi_2(t) - 2uF\lambda_n^{-2} \\
<& 0, \qquad (4.2)
\end{aligned}
$$

for small ϵ and large λ_n^{-2}. Hence we obtain that λ_n^{-1} does *not* blow up at T, in contradiction to the definition of $T < T^*$. Hence we must have $T = T^*$, with a uniform lower bound on h_{ij} on finite intervals. The proof is complete. □

The second proof is from the Constant Rank Theorem in Bian-Guan [2] , along the lines of proof of Theorem 6.1 in [8] (where the constant rank theorem was proved for a general flow in $\mathbb{R}^{n+1}$). For (1.2) in $\mathbb{S}^{n+1}$, we have an extra good term which is associated to the curvature $K = 1$. We outline the arguments here with necessary modification.

Proof. Let $W = (g^{im}h_{mj})$, and $l(t)$ be the minimal rank of W. Suppose at W is degenerate (x_0, T), such that $W(T)$ attains minimal rank $l < n$ at x_0. Set

$$\varphi(x,t) = \sigma_{l+1}(W(x,t)) + \frac{\sigma_{l+2}(W(x,t))}{\sigma_{l+1}(W(x,t))}.$$

It is proved in section 2 in [2] that φ is in $C^{1,1}$.

As in Bian-Guan [2], near (x_0, T), the index set $\{1, 2, \cdots, n\}$ can be divided in to two subsets B, G, where for $i \in B$, the eigenvalues of $\{W_{ij}\}$, λ_i is small and for $j \in G$, λ_j is strictly positive away from 0. As in [2], we may assume at each point of computation, $\{W_{ij}\}$ is diagonal. Notice that $W_{ii} \le C\varphi$ for all $i \in B$.

Denote $G = uF - c_{n,k}\phi'$ and $F = \frac{\sigma_{k+1}(\lambda)}{\sigma_k(\lambda)}$. From (2.14) we recall

$$\partial_t h_i^i = \nabla^i\nabla_i G + Gg^{ik}(h^2)_{ki} + G. \tag{4.3}$$

So we have the following equality

$$\begin{aligned}
&\sum G^{\alpha\beta}\varphi_{\alpha\beta} - \varphi_t \\
=\ & O(\varphi + \sum_{i,j\in B}|\nabla W_{ij}|) - \frac{1}{\sigma_1(B)}\sum_{\alpha\beta}\sum_{i\neq j\in B} G^{\alpha\beta}W_{ij,\alpha}W_{ij,\beta} \\
& - \frac{1}{\sigma_1(B)^3}\sum_{\alpha\beta}\sum_{i\in B} G^{\alpha\beta}(W_{ii,\alpha}\sigma_1(B) - W_{ii}\sum_{j\in B}W_{jj,\alpha})(W_{ii,\beta}\sigma_1(B) - W_{ii}\sum_{j\in B}W_{jj,\beta}) \\
& - 2\sum_{i\in B}[\sigma_l(G) + \frac{\sigma_1(B|i)^2 - \sigma_2(B|i)}{\sigma_1(B)^2}]\sum_{\alpha\beta}\sum_{j\in G}G^{\alpha\beta}\frac{W_{ij,\alpha}W_{ij,\beta}}{W_{jj}} \\
& + \sum_{i\in B}[\sigma_l(G) + \frac{\sigma_1(B|i)^2 - \sigma_2(B|i)}{\sigma_1(B)^2}]\left(\sum_{\alpha\beta}G^{\alpha\beta}W_{ii,\alpha\beta} - \partial_t W_{ii}\right)
\end{aligned} \tag{4.4}$$

By (4.3) and (2.29),

$$\partial_t h_i^i = u[F^{\alpha\beta}\nabla_\alpha\nabla_\beta h_{ii} + F^{\alpha\beta,\gamma\eta}\nabla_i h_{\alpha\beta}\nabla_i h_{\gamma\eta}] + O(\varphi + \sum_{i,j\in B}|\nabla W_{ij}|) + 2uF, \tag{4.5}$$

where we used the facts $\nabla_i u = O(\varphi)$, $\nabla_i\nabla_i u = O(\varphi + |\nabla W_{ii}|)$ and $\nabla_i\nabla_i\Phi = \phi' + O(\varphi)$, $\forall i \in B$.

As $G^{\alpha\beta} = uF^{\alpha\beta}$,

$$\begin{aligned}
\sum_{\alpha\beta}G^{\alpha\beta}W_{ii,\alpha\beta} - \partial_t W_{ii} =& u\sum_{\alpha\beta}F^{\alpha\beta}\nabla_\alpha\nabla_\beta h_{ii} - \partial_t h_i^i \\
=& O(\varphi + \sum_{i,j\in B}|\nabla W_{ij}|) - uF^{\alpha\beta,\gamma\eta}W_{\alpha\beta,i}W_{\gamma\eta,i} - 2uF,
\end{aligned} \tag{4.6}$$

Since F satisfies the structure condition in [2]

$$F^{\alpha\beta,\gamma\eta}W_{\alpha\beta,i}W_{\gamma\eta,i}+2\sum_{\alpha\beta}\sum_{j\in G}F^{\alpha\beta}\frac{W_{ij,\alpha}W_{ij,\beta}}{W_{jj}}\geq 0.$$

We obtain

$$\sum G^{\alpha\beta}\varphi_{\alpha\beta}-\varphi_t\leq C(\varphi+|\nabla\varphi|)-C\sum_{i,j\in B}|\nabla W_{ij}|-2uF\sum_{i\in B}[\sigma_l(G)+\frac{\sigma_1(B|i)^2-\sigma_2(B|i)}{\sigma_1(B)^2}].$$

Following the analysis in the proof of Theorem 3.2 in [2], it yields

$$\sum G^{\alpha\beta}\varphi_{\alpha\beta}-\varphi_t\leq C(\varphi+|\nabla\varphi|)-2uF\sigma_l(G).$$

This is a contradiction from the standard strong maximum principle for parabolic equations. $\square$

Remark 4.3. The preservation convexity of flow (1.2) when $k=0$ was first observed by JS and Chao Xia [15].

Since in the case of $k=0$, the longtime existence and convergence was proved in [8], the following sharp inequality follows.

Proposition 4.4. *If $\Omega\subset\mathbb{S}^{n+1}$ is convex, then*

$$\mathrm{Vol}(\Omega)\leq\xi_{l,-1}(\mathcal{A}_l)\quad\forall l\geq 0,$$

with equality holds iff Ω is a convex geodesic ball.

5. Discussion of C^2 estimate

The curvature estimate for flow (1.2) is still open. In the case of $\mathbb{R}^{n+1}$, $\phi'=1$, the corresponding flow is

$$X_t=(c_{n,k}-uF(\lambda))\nu. \tag{5.1}$$

Flow (5.1) has the same curvature estimate issue. In [9, 10] the flow (5.1) was converted to a corresponding inverse type flow for the Euclidean support function u of the evolving convex body parametrized on the outer normals (i.e. on $\mathbb{S}^n$). The longtime existence and convergence of the admissible solution were proved in [9, 10].

Below we will convert flow (1.2) to an evolution of convex bodies in $\mathbb{R}^{n+1}$ and we write down the evolution equation of the corresponding Euclidean support function $\tilde{u}$ on $\mathbb{S}^n$.

Introduce a new variable γ satisfying

$$\frac{d\gamma}{d\rho}=\frac{1}{\phi}. \tag{5.2}$$

Let $\omega = \sqrt{1 + |\nabla\gamma|^2}$, one can compute the unit outward normal $\nu = \frac{1}{\omega}(1, -\frac{\nabla\rho}{\phi^2})$, and

$$u = \frac{\phi}{\omega}, \tag{5.3}$$

$$g_{ij} = \phi^2 e_{ij} + \rho_i\rho_j, \tag{5.4}$$

$$g^{ij} = \frac{1}{\phi^2}(e^{ij} - \frac{\gamma^i\gamma^j}{\omega^2}), \tag{5.5}$$

$$h_{ij} = \frac{\phi}{\omega}(-\gamma_{ij} + \phi'\gamma_i\gamma_j + \phi' e_{ij}), \tag{5.6}$$

$$h^i_j = \frac{1}{\phi\omega}(e^{im} - \frac{\gamma^i\gamma^m}{\omega^2})(-\gamma_{mj} + \phi'\gamma_m\gamma_j + \phi' e_{mj}). \tag{5.7}$$

It follows from (2.23) that the evolution equation for γ is

$$\begin{aligned} \partial_t\gamma =& \frac{1}{\phi}\partial_t\rho = f\frac{\omega}{\phi} \\ =& c_{n,k}\frac{\phi'}{u} - \frac{\sigma_{k+1}(\lambda)}{\sigma_k(\lambda)}. \end{aligned} \tag{5.8}$$

From (5.7),

$$\begin{aligned} h^i_j =& \frac{1}{\phi\omega}(e^{im} - \frac{\gamma^i\gamma^m}{\omega^2})(-\gamma_{mj} + \phi'\gamma_m\gamma_j + \phi' e_{mj}) \\ =& \frac{1}{\phi\omega}\{(e^{im} - \frac{\gamma^i\gamma^m}{\omega^2})(-\gamma_{mj} + \gamma_m\gamma_j + e_{mj}) + (\phi' - 1)\delta^i_j\} \\ =& \frac{e^\gamma}{\phi}\widetilde{h}^i_j + \frac{\phi' - 1}{\phi\omega}\delta^i_j, \end{aligned} \tag{5.9}$$

where

$$\widetilde{h}^i_j = \frac{1}{e^\gamma\omega}\{(e^{im} - \frac{\gamma^i\gamma^m}{\omega^2})(-\gamma_{mj} + \gamma_m\gamma_j + e_{mj}),$$

which is the Weingarten tensor of the graph $\widetilde{M}$ (over $\mathbb{S}^n$) of radial function $\widetilde{\rho} = e^\gamma$ in $\mathbb{R}^{n+1}$. Since $M(t)$ is strictly convex (i.e. $\{h^i_j\} > 0$), and $\phi' - 1 = \cos\rho - 1 < 0$, thus $\{\widetilde{h}^i_j\} > 0$. That is $\widetilde{M}$ is strictly convex.

We have

$$\begin{aligned} \partial_t\widetilde{\rho} =& \widetilde{\rho}\partial_t\gamma \\ =& c_{n,k}\frac{\phi'}{\phi}\widetilde{\rho}\omega - \frac{\widetilde{\rho}^2}{\phi}\frac{\sigma_{k+1}}{\sigma_k}(\widetilde{h}^i_j + \frac{\phi' - 1}{\widetilde{\rho}\omega}\delta^i_j). \end{aligned} \tag{5.10}$$

Let $\widetilde{u}$ be the support function of the strictly convex body $\widetilde{M}$, and z and ν be the unit radial vector and the unit outer normal vector of $\widetilde{M}$, respectively. Then from $\widetilde{\rho}(z,t)(z \cdot \nu) =$

$\widetilde{u}(\nu,t)$, we can get $\log\widetilde{\rho}(z,t)=\log\widetilde{u}(\nu,t)-\log(z\cdot\nu)$ and

$$\begin{aligned}\frac{1}{\widetilde{\rho}(z,t)}\frac{\partial\widetilde{\rho}(z,t)}{\partial t}=&\frac{1}{\widetilde{u}(\nu,t)}[\nabla\widetilde{u}\cdot\nu_t+\widetilde{u}_t]-\frac{z\cdot\nu_t}{z\cdot\nu}\\=&\frac{1}{\widetilde{u}(\nu,t)}\frac{\partial\widetilde{u}(\nu,t)}{\partial t}+\frac{1}{\widetilde{u}(\nu,t)}[(\nabla\widetilde{u}-\widetilde{\rho}(z,t)z)\cdot\nu_t]\\=&\frac{1}{\widetilde{u}(\nu,t)}\frac{\partial\widetilde{u}(\nu,t)}{\partial t}.\end{aligned}$$

Denote $W_{\widetilde{u}}=:\{\widetilde{u}_{ij}+\widetilde{u}\delta_{ij}\}=\{\widetilde{h}^i_j\}^{-1}$ and we can get the evolution equation of $\widetilde{u}$ as follows

$$\begin{aligned}\partial_t\widetilde{u}=&\frac{\widetilde{u}}{\widetilde{\rho}}\partial_t\widetilde{\rho}\\=&c_{n,k}\frac{\phi'}{\phi}\widetilde{u}\omega-\frac{\widetilde{\rho}\widetilde{u}}{\phi}\frac{\sigma_{k+1}}{\sigma_k}(\widetilde{h}^i_j+\frac{\phi'-1}{\widetilde{\rho}\omega}\delta^i_j)\\=:&\ G(W_{\widetilde{u}},\widetilde{u},\nabla\widetilde{u}).\end{aligned}\tag{5.11}$$

This equation is of inverse type, and *the question is whether* (5.11) *exists for all time?*

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