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The classical Minkowski inequality in the Euclidean space provides a lower bound on the total mean curvature of a hypersurface in terms of the surface area, which is optimal on round spheres. We employ a locally constrained inverse mean curvature flow to prove a properly defined analogue in the Lorentzian de Sitter space.

1. Introduction

The main results. In this paper we prove an optimal Minkowski inequality for compact, spacelike and mean convex hypersurfaces in the upper branch of the $(n+1)$-dimensional Lorentzian de Sitter space. To state the main result, we provide the involved terminology. Let $n \geq 2$ and denote by $\mathbb{M}_1^{n+2}$ the $(n+2)$-dimensional Minkowski space, i.e., $\mathbb{R}^{n+2}$ with metric

$$\langle v, w\rangle = -v^0 w^0 + \sum_{\alpha=1}^{n+1} v^\alpha w^\alpha.$$

We define the upper branch of the de Sitter space by

$$\bar{\mathbb{S}}_1^{n+1} = \{y \in \mathbb{M}_1^{n+2} : \langle y, y\rangle = 1,\ y^0 > 0\}$$

and understand this submanifold to be equipped with the induced metric. Then $\bar{\mathbb{S}}_1^{n+1}$ is a Lorentzian manifold with constant sectional curvature 1; see Section 2 for a detailed discussion. The following theorem is the main result of this paper.

Theorem 1.1. *Let $\Sigma \subset \bar{\mathbb{S}}_1^{n+1}$ be a spacelike, compact, connected and mean-convex hypersurface. Then there holds*

$$\int_\Sigma H_1 \leq \operatorname{vol}(\widehat{\Sigma}) + \varphi(|\Sigma|)$$

with equality precisely if Σ is totally umbilic. Here $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is the strictly increasing function which gives equality on the y^0-slices.

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We review the standard terminology used here very briefly and refer to Section 2 for a more detailed discussion. A hypersurface $\Sigma \subset \overline{\mathbb{S}}^{n+1}_1$ is called spacelike, if its induced metric is Riemannian. Σ is called mean-convex, if for a suitable unit normal vector field ν the normalized mean curvature H_1 with respect to $-\nu$ is positive. $\widehat{\Sigma}$ denotes the region enclosed by the slice $\{y^0 = 0\}$ and Σ, while $\operatorname{vol}(\widehat{\Sigma})$ denotes its enclosed volume; see (2-6). Finally, $|\Sigma|$ denotes the surface area of Σ.

To prove Theorem 1.1, we employ a locally constrained inverse mean curvature flow. This flow is designed to preserve the surface area and to increase the quantity

$$W_2(\Sigma) = \int_\Sigma H_1 - \operatorname{vol}(\widehat{\Sigma}).$$

We will prove that it converges smoothly to a coordinate slice, which will imply Theorem 1.1. To state the result about the curvature flow, we introduce some more notation. The space $\overline{\mathbb{S}}^{n+1}_1$ is isometric to the space $\mathbb{R}_+ \times \mathbb{S}^n$ with the warped product metric

$$\bar{g} = -dr^2 + \vartheta^2(r)\sigma,$$

where σ is the round metric on $\mathbb{S}^n$ and $\vartheta = \cosh$; see Lemma 2.1. For a hypersurface Σ we define

$$u = -\bar{g}(\vartheta\,\partial_r, \nu)$$

to be the support function, where ν is the future directed normal vector field on Σ, where the time orientation is inherited from $\mathbb{M}^{n+2}_1$.

Theorem 1.2. *Let $\Sigma \subset \overline{\mathbb{S}}^{n+1}_1$ be a spacelike, compact, connected and mean-convex hypersurface. Then there exists a unique immortal solution*

$$x : [0, \infty) \times \mathbb{S}^n \to \overline{\mathbb{S}}^{n+1}_1,$$

which satisfies

$$\begin{aligned} \dot{x} &= \Big(u - \frac{\vartheta'}{H_1}\Big)\nu, \\ x(0, \mathbb{S}^n) &= x_0, \end{aligned} \tag{1-1}$$

where x_0 is an embedding of Σ. As $t \to \infty$, the embeddings $x(t, \cdot)$ converge smoothly to a coordinate slice, which is uniquely determined by $|\Sigma|$.

Background.

Minkowski inequality. For a closed and convex hypersurface in the Euclidean space $\mathbb{R}^{n+1}$, the Minkowski inequality states that

$$\int_\Sigma H_1 \geq |\mathbb{S}^n|^{\frac{1}{n}} |\Sigma|^{\frac{n-1}{n}} \tag{1-2}$$

with equality precisely when Σ is a round sphere. For surfaces $n = 2$ this was originally proved by Minkowski [1903]. Using inverse curvature flows, (1-2) was, among more general estimates called *Alexandrov–Fenchel inequalities*, generalized to starshaped and mean-convex hypersurfaces in [Guan and Li 2009]. Using Huisken's and Ilmanen's weak inverse mean curvature flow [2001] one can replace the starshapedness by outward minimality. It is open until today whether (1-2) holds for general mean-convex hypersurfaces. The Michael–Simon–Sobolev inequality [Michael and Simon 1973] gives an estimate on the total mean curvature, however its optimal constant is not the desired optimal constant in (1-2), also compare [Brendle 2021]. In case $n = 2$, there is an L^2-stability result for immersed surfaces [Kuwert and Scheuer 2020], namely there holds

$$\left| \frac{1}{\sqrt{|\Sigma|}} \int_\Sigma H_1 - 2\sqrt{\pi} \right| \le c \|\mathring{A}\|^2_{L^2(\Sigma)},$$

where $\mathring{A}$ is the trace-free part of the second fundamental form. Also see [Dalphin et al. 2016] for a comprehensive overview over related results for the case of closed hypersurfaces of the Euclidean space. For hypersurfaces with free boundary on a cone there are similar results [Cruz 2019]. Both sides of (1-2) can be considered as special cases of the quermassintegrals $W_k(\Sigma)$. We refer to [Gallego and Solanes 2005; Gao et al. 2001–02; Solanes 2006] for a comprehensive introduction and useful further references. In the Euclidean case, (1-2) then reads

$$W_2^{\mathbb{R}^{n+1}}(\Sigma) \ge c_n W_1^{\mathbb{R}^{n+1}}(\Sigma)^{\frac{n-1}{n}}.$$

The superscript is added to indicate the special structure of $W_k(\Sigma)$ in the Euclidean case.

There are analogues of these quantities in the hyperbolic and spherical spaces $\mathbb{H}^{n+1}$ and $\mathbb{S}^{n+1}$. However, the $W_k(\Sigma)$ are then not given by curvature integrals, but by linear combinations of them. While the quantities $W_1(\Sigma)$ are up to dimensional constants given by surface area in all of the three spaceforms, the values of $W_2(\Sigma)$ in the nonflat spaces are (up to dimensional constants)

$$W_2^{\mathbb{H}^{n+1}}(\Sigma) = \int_\Sigma H_1 - \mathrm{vol}(\widehat{\Sigma}), \qquad W_2^{\mathbb{S}^{n+1}}(\Sigma) = \int_\Sigma H_1 + \mathrm{vol}(\widehat{\Sigma}),$$

where $\widehat{\Sigma}$ is the region enclosed by Σ.

A possible generalization of (1-2) to other ambient manifolds would then be to derive an inequality between $W_2(\Sigma)$ and $W_1(\Sigma)$ (we drop the superscript again for the following informal discussion).

In the hyperbolic space, [Gallego and Solanes 2005] treats the case of convex hypersurfaces through a rough estimate which does not characterize the case of equality, while in [Wang and Xia 2014] an optimal inequality was proved in the class of horospherically convex hypersurfaces. Various inequalities between W_i, $1 \le i \le 3$, were provided in [Brendle et al. n.d.; Guan and Li 2021; Li et al. 2014].

Other variants of such estimates involving total mean curvature, surface area and possibly other quantities, are contained for example in [Borisenko and Miquel 1999; de Lima and Girão 2016; Natário 2015; Wei and Xiong 2015].

In the sphere there are also many variants of (1-2), for example [Girão and Pinheiro 2017; Makowski and Scheuer 2016; Wei and Xiong 2015] and in other ambient spaces there are weighted Minkowski-type inequalities [Brendle et al. 2016; Ge et al. 2015; McCormick 2018; Scheuer and Xia 2019; Wang 2015; Wei 2018; Xia 2016].

In the de Sitter space we are only aware of one similar result, which recently appeared [Andrews et al. 2020]. The Alexandrov–Fenchel inequality proved in this paper is deduced as a corollary from its dual version in the hyperbolic space with the help of a well-known duality method available for convex hypersurfaces of the sphere and hyperbolic/de Sitter space; see [Gao et al. 2001–02; Gerhardt 2015] and Section 2.

Note that the inequality in Theorem 1.1 cannot be deduced by duality from hyperbolic space, as such is not defined for nonconvex hypersurfaces.

Locally constrained curvature flows. The strategy to prove Theorem 1.1 is to employ a suitably defined curvature flow, i.e., a variation of the hypersurface Σ which is defined through its curvature and possibly lower order quantities. This method has become very popular for the deduction of geometric inequalities during the past decades. For example, Huisken [1987] studied the *volume preserving mean curvature flow*, where the mean curvature flow is modified by addition of a global term, namely the averaged mean curvature. Long-time existence and convergence to a round sphere is proven, if the initial datum is convex. This reproved the isoperimetric inequality in all dimensions for convex domains. Similar nonlocal flows have been widely used to prove other geometric inequalities in the Euclidean and hyperbolic space [Andrews 2001; Athanassenas 1997; Cabezas-Rivas and Miquel 2007; Ivaki and Stancu 2013; McCoy 2003; 2004; 2005; Sinestrari 2015]. These nonlocal flows are hard to study due to the nonlocal term involved and the results mentioned above usually required preservation of convexity at the least.

A new kind of volume preserving curvature flow was invented by Pengfei Guan and Junfang Li [2015]. In the Euclidean space it is

$$\dot{x} = \frac{1}{2n}\Delta_\Sigma |x|^2 = (1 - uH_1)\nu, \tag{1-3}$$

where H_1 is the normalized mean curvature and u the support function. This flow obviously preserves the enclosed volume and it can be calculated that it decreases the surface area unless Σ is a round sphere. This gives a further proof of the isoperimetric inequality. The major advantage over the classical volume preserving mean curvature flow is that it preserves the starshapedness and hence the technical obstructions are lower. Hence there has been some recent interest in these so-called *locally constrained curvature flows*. Later, the flow (1-3) was also studied in other ambient manifolds [Guan et al. 2019]. From that idea, it was tempting to modify (1-3) in such a way that the modified flow preserves other geometric quantities. For example, the flow

$$\dot{x} = \left(\frac{1}{H_1} - u\right)\nu \tag{1-4}$$

in $\mathbb{R}^{n+1}$ preserves the surface area and decreases the total mean curvature, giving another proof of the Minkowski inequality for starshaped and mean-convex hypersurfaces. To quickly complete the list of previous literature on these locally constrained flows, we refer to the related results in [Guan and Li 2018; Scheuer et al. 2018; Scheuer and Xia 2019].

Note that while (1-4) is very easy to treat in $\mathbb{R}^{n+1}$ as it is basically a rescaling of the inverse mean curvature flow originally treated in [Gerhardt 1990; Urbas 1990], it seems hard to study the proper modification of this flow in the other spaceforms. In the hyperbolic space there is a partial unpublished result [Brendle et al. n.d.] which requires an additional initial gradient smallness assumption, while in the sphere there is no result.

The purpose of this paper is to prove the full convergence result of the correct version of (1-4) in the Lorentzian de Sitter space under the most natural assumptions, which are spacelikeness and mean convexity, and in turn obtain the Minkowski inequality for such hypersurfaces.

Outline. In Section 2 we spend some time to review the geometry of de Sitter space and to introduce our notation. We take some care here, as we will also deduce some maybe not so well known relations between hypersurfaces of de Sitter space and their duals in the hyperbolic space. In particular we will deduce the dual flow of (1-1) in the hyperbolic space. In Sections 3 and 4 we collect the relevant evolution equations and deduce the required a priori estimates to obtain long-time existence. In Section 5 we prove that the flow converges to a coordinate slice and complete the proof of the Minkowski inequality.

2. Geometry of de Sitter space and duality

We recall some facts about hypersurfaces of semi-Riemannian manifolds as well as basic properties of the de Sitter space, most of which can be found in [O'Neill 1983].

For a semi-Riemannian manifold $(\overline{M}, \langle\cdot,\cdot\rangle)$ with Levi-Civita connection $\overline{D}$ and a hypersurface M we have the Gaussian formula for vector fields V, W on M (which are smoothly extended to $\overline{M}$),

$$\overline{D}_V W = D_V W + \mathrm{II}(V, W),$$

where D is the Levi-Civita connection of the metric induced by the inclusion $\iota : M \to \overline{M}$ and the decomposition is orthogonal. The normal part $\mathrm{II}(\cdot,\cdot)$ is called the vector valued second fundamental form. There holds the Gauss equation [O'Neill 1983, p. 100]

$$\begin{aligned}(2\text{-}1)\quad &\langle R(V,W)X,Y\rangle \\ &\qquad = \langle \overline{R}(V,W)X,Y\rangle + \langle \mathrm{II}(V,X),\mathrm{II}(W,Y)\rangle - \langle \mathrm{II}(V,Y),\mathrm{II}(W,X)\rangle,\end{aligned}$$

where we used the curvature tensor convention from [O'Neill 1983],

$$R(X,Y)Z = D_Y D_X Z - D_X D_Y Z - D_{[Y,X]} Z$$

for all vector fields X, Y, Z on M. The shape operator S of M derived from a normal N, which is defined by

$$\langle S(V), W\rangle = \langle \mathrm{II}(V,W), N\rangle,$$

satisfies the Weingarten equation [O'Neill 1983, p. 107]

$$S(V) = -\overline{D}_V N. \tag{2-2}$$

Let $n \geq 2$ and $\mathbb{M}_1^{n+2}$ be the $(n+2)$-dimensional Minkowski space with metric

$$\langle v, w\rangle = -v^0 w^0 + \sum_{\alpha=1}^{n+1} v^\alpha w^\alpha.$$

The Lorentzian de Sitter space is defined as the hyperquadric

$$\mathbb{S}_1^{n+1} = \{y \in \mathbb{M}_1^{n+2} : \langle y, y\rangle = 1\}.$$

Differentiating $1 = \langle \gamma, \gamma\rangle$ along an arbitrary curve γ in $\mathbb{S}_1^{n+1}$, we obtain that the normal space of $\mathbb{S}_1^{n+1} \subset \mathbb{M}_1^{n+2}$ is spanned by the spacelike position vector field y and thus $\mathbb{S}_1^{n+1}$ has sign 1 in $\mathbb{M}_1^{n+2}$. From (2-2) we also obtain that the shape operator of $\mathbb{S}_1^{n+1}$ is

$$S(V) = -\overline{D}_V y = -V$$

and hence $\mathrm{II}(\cdot,\cdot) = -\langle \cdot,\cdot\rangle y$. We obtain from (2-1) that the Riemann tensor R of $\mathbb{S}_1^{n+1}$ satisfies

$$\langle R(V,W)X,Y\rangle = \langle V,X\rangle\langle W,Y\rangle - \langle V,Y\rangle\langle W,X\rangle.$$

It follows that $\mathbb{S}_1^{n+1}$ has constant sectional curvature

$$K(e_i,e_j) = \frac{\langle R(e_i,e_j)e_i,e_j\rangle}{\langle e_i,e_i\rangle\langle e_j,e_j\rangle} = \frac{\langle e_i,e_i\rangle\langle e_j,e_j\rangle}{\langle e_i,e_i\rangle\langle e_j,e_j\rangle} = 1$$

for every orthogonal unit vectors e_i, e_j [O'Neill 1983, p. 77].

For the calculations in this paper it will be convenient to have a warped product structure for $\mathbb{S}_1^{n+1}$ and hence we prove the following lemma.

Lemma 2.1. *The Lorentzian manifold* $\mathbb{S}^{n+1}_1$ *is isometric to the warped product* $\mathbb{R}\times\mathbb{S}^n$ *with metric*

$$\bar{g} = -dr^2 + \vartheta^2(r)\sigma,$$

where σ *is the round metric on* $\mathbb{S}^n$ *and* $\vartheta = \cosh$. *The hyperbolic space*

$$\mathbb{H}^{n+1} := \{\tilde{y}\in\mathbb{M}^{n+2}_1 : \langle\tilde{y},\tilde{y}\rangle = -1,\ \tilde{y}^0 > 0\}$$

is diffeomorphic to $\mathbb{R}_+\times\mathbb{S}^n$ *with metric*

$$\tilde{\bar{g}} = d\tilde{r}^2 + \tilde{\vartheta}^2(\tilde{r})\sigma,$$

where $\tilde{\vartheta} = \sinh$.

Proof. It follows from [O'Neill 1983, p. 111] that the map

$$\phi : \mathbb{R}\times\mathbb{S}^n \to \mathbb{S}^{n+1}_1 \subset \mathbb{M}^{n+2}_1, \quad (r, p)\mapsto(\vartheta'(r), \vartheta(r)p)$$

is a diffeomorphism. Let x^i be local coordinates on $\mathbb{S}^n$. The pullback metric of ϕ is calculated as follows:

$$\begin{aligned}\bar{g}(\partial_r,\partial_r) &= |(\vartheta,\vartheta' p)|^2 = -1,\\ \bar{g}(\partial_r,\partial_{x^i}) &= 0,\end{aligned}$$

since p is normal to $\partial_{x^i}p$ in $\mathbb{R}^{n+1}$ and

$$\bar{g}(\partial_{x^i},\partial_{x^j}) = \vartheta^2(r)\sigma_{ij}$$

by definition. The hyperbolic case is proven by replacing ϑ by $\tilde{\vartheta}$ in the above proof. □

In order to employ the duality between $\mathbb{S}^{n+1}_1$ and $\mathbb{H}^{n+1}$ we need the following formulae. The first follows from the previous proof and the second is a straightforward computation of the push-forward of $\vartheta\,\partial_r$. Recall that $e = e_0$ is the future-directed timelike standard unit vector in $\mathbb{M}^{n+2}_1$.

Lemma 2.2. *As quantities of the ambient manifold, the function* $\vartheta'(r)$ *is given by*

$$\vartheta'(r) = -\langle y, e\rangle$$

and the vector field $\vartheta\,\partial_r$ *is*

$$\vartheta\,\partial_r = e - \langle e, y\rangle y,$$

where we identified (r, p) *with* $y\in\mathbb{S}^{n+1}_1\subset\mathbb{R}^{n+2}_1$.

Similarly, the function $\tilde{\vartheta}(\tilde{r})$ *is given by*

$$\tilde{\vartheta}'(\tilde{r}) = -\langle\tilde{y}, e\rangle$$

while the vector field $\tilde{\vartheta}\,\partial_{\tilde{r}}$ *is given by*

$$\tilde{\vartheta}\,\partial_{\tilde{r}} = -e - \langle e,\tilde{y}\rangle\tilde{y}.$$

Remark 2.3. We will see later that the curvature flow is only parabolic if $\vartheta' > 0$. Hence we will restrict ourselves to the upper branch $\{r > 0\} \subset \mathbb{S}^{n+1}_1$.

Spacelike hypersurfaces of de Sitter space. Let $\Sigma \subset \mathbb{S}^{n+1}_1$ be a spacelike, compact, connected hypersurface. The manifold $\mathbb{S}^{n+1}_1$ is globally hyperbolic [Gerhardt 2006, Theorem 1.4.2] and

$$\mathcal{S}_0 = \{0\} \times \mathbb{S}^n$$

a Cauchy hypersurface, terminologies which are defined, e.g., in [Gerhardt 2006, Definitions 1.3.7, 1.3.8]. From [Gerhardt 2006, Proposition 1.6.3, Remark 1.6.4] we obtain that Σ is a smooth graph over $\mathcal{S}_0$, i.e., in the coordinates from Lemma 2.1 we have

$$\Sigma = \{(\rho(x^i), x^i) : (x^i) \in \mathcal{S}_0\},$$

where in the sequel latin indices range between 1 and n, while greek indices range from 0 to n. As mentioned above, we assume $\rho > 0$. In order to shorten notation we often write

$$x^0 = r$$

and hence a point $x \in \mathbb{S}^{n+1}_1$ has the coordinate representation $x = (x^\alpha)$ in a given coordinate system.

Now we describe the geometry of Σ in terms of the ambient geometry and ρ, starting with the introduction of some notation. Fix a local coordinate system (ξ^i) for Σ. For tensors on Σ we use the coordinate notation, e.g., the induced metric g is written as $g = (g_{ij})$, where

$$g_{ij} = g(\partial_{\xi^i}, \partial_{\xi^j}).$$

If ∇ denotes the Levi-Civita connection of g, in order to shorten the appearance of evolution equations, we will denote the coordinate functions of covariant derivatives of tensors by the use of semicolons: If $A = (a^{i_1\dots i_k}_{j_1\dots j_l})$ is a k-contravariant and l-covariant tensor (or merely a function), its covariant derivative is denoted by

$$\nabla_{\partial_{x^m}} A = (a^{i_1\dots i_k}_{j_1\dots j_l;m}).$$

For spacelike hypersurfaces Σ we define ν to be the future directed (timelike) normal, i.e.,

$$\bar{g}(\partial_r, \nu) < 0.$$

We derive the shape operator $S = (h^i_j)$ of Σ from $-\nu$,

$$h_{ij} := g_{ik} h^k_j = -\bar{g}(\mathrm{II}(\partial_{\xi^i}, \partial_{\xi^j}), \nu)$$

and call the tensor (h_{ij}) the second fundamental form of Σ. The eigenvalues $(\kappa_i)_{1\le i\le n}$ of the shape operator are called the principal curvatures of Σ. This definition implies that

$$\mathrm{II}(\partial_{\xi^i}, \partial_{\xi^j}) = h_{ij}\nu.$$

This is in accordance with the convention in [Gerhardt 2006, Theorem 1.1.2] and we use this reference for further formulae.

From [Gerhardt 2006, (1.6.13)] we obtain that the second fundamental form of the slices $\{x^0 = r\}$ is given by

$$\bar{h}_{ij} := \frac{\vartheta'(r)}{\vartheta(r)} \bar{g}_{ij}.$$

The induced metric in terms of ρ is

$$g_{ij} = -\rho_{;i}\rho_{;j} + \vartheta^2(\rho)\sigma_{ij} = -\rho_{;i}\rho_{;j} + \bar{g}_{ij}.$$

Defining

$$v^2 = 1 - \vartheta^{-2}\sigma^{ij}\rho_{;i}\rho_{;j},$$

the second fundamental form satisfies

$$v^{-1}h_{ij} = \rho_{;ij} + \bar{h}_{ij}, \tag{2-3}$$

[Gerhardt 2006, (1.6.11)], where we note that in this reference the past directed normal is used. Also note that in order to control the property of a hypersurface of being spacelike, one has to ensure $v^2 > 0$.

The function

$$u := \frac{\vartheta}{v} = -\bar{g}(\vartheta\partial_r, \nu)$$

is of special interest and can be regarded as a generalized support function.

Graphical hypersurfaces of hyperbolic space. Let $\widetilde{\Sigma} \subset \mathbb{H}^{n+1}$ be starshaped with respect to the origin in the coordinates given by Lemma 2.1,

$$\widetilde{\Sigma} = \{(\tilde{\rho}(x^i), x^i) : (x^i) \in \mathcal{S}_0\}.$$

With the corresponding notation as for the de Sitter space, we always chose the normal $\tilde{\nu}$ to Σ to point in the same direction as $\partial_{\tilde{r}}$, i.e.,

$$\tilde{\bar{g}}(\partial_{\tilde{r}}, \tilde{\nu}) > 0.$$

We derive the shape operator of Σ from $-\tilde{\nu}$ and obtain from [Gerhardt 2006, (1.5.10)] that

$$\tilde{h}_{ij}\tilde{v}^{-1} = -\tilde{\rho}_{;ij} + \tilde{\bar{h}}_{ij}, \tag{2-4}$$

where

$$\tilde{v}^2 = 1 + \tilde{\vartheta}^{-2}\sigma^{ij}\tilde{\rho}_{;i}\tilde{\rho}_{;j}.$$

We also may define the support function of $\widetilde{\Sigma}$ to be

$$\tilde{u} = \tilde{\bar{g}}(\tilde{\vartheta}\partial_{\tilde{r}}, \tilde{\nu}).$$

Duality. There is an important relation between strictly convex hypersurfaces of the hyperbolic space containing the origin and strictly convex hypersurfaces of the upper branch of de Sitter space. It states that these two sets are in one-to-one relation through the idempotent Gauss maps

$$\tilde{x} := \nu, \quad x := \tilde{\nu}.$$

Furthermore the respective principal curvatures satisfy the relation

$$\tilde{\kappa}_i = \kappa_i^{-1}$$

and the induced metric of the dual hypersurfaces are given by

$$\tilde{g}_{ij} = h_{ik}h^k_j, \quad g_{ij} = \tilde{h}_{ik}\tilde{h}^k_j. \tag{2-5}$$

We refer to [Gerhardt 2006, Theorems 10.4.4, 10.4.5, 10.4.9] for a more thorough discussion. In the theory of curvature flows, the method of duality has successfully been employed several times, e.g., [Bryan et al. 2020; Gerhardt 2015; Yu 2017].

Later we will employ the following relations between the support functions and the respective height functions of the dual hypersurfaces.

Lemma 2.4. *Let $\Sigma \subset \bar{\mathbb{S}}^{n+1}_1$ be a strictly convex, compact and spacelike hypersurface. Then its dual $\widetilde{\Sigma} \subset \mathbb{H}^{n+1}$ is starshaped with respect to the origin and the height/support functions satisfy*

$$\vartheta'(r) = \tilde{u}, \quad \tilde{\vartheta}'(\tilde{r}) = u.$$

Proof. We use Lemma 2.2 and $\langle x, \tilde{x}\rangle = 0$. There holds

$$\begin{aligned} u &= -\langle e - \langle e, x\rangle x, \tilde{x}\rangle = -\langle e, \tilde{x}\rangle = \tilde{\vartheta}'(\tilde{r}), \\ \tilde{u} &= -\langle e + \langle e, \tilde{x}\rangle \tilde{x}, x\rangle = -\langle e, x\rangle = \vartheta'(r). \end{aligned}$$

□

Volume, surface area and Minkowski identities. A function f on a spacelike hypersurface $\Sigma \subset \mathbb{S}^{n+1}_1$ is in $L^1(\Sigma)$, if the differential n-form $|f| d\omega_g$ has a finite integral over Σ, in which case we write

$$\int_\Sigma f := \int_{\mathbb{S}^n} f \, d\omega_g.$$

Here $d\omega_g$ is the Riemannian volume form on Σ. For a spacelike hypersurface $\Sigma = \operatorname{graph} \rho$ as above we define the *enclosed volume* as in [Makowski 2013, Section 4] by

$$\operatorname{vol}(\widehat{\Sigma}) = \int_{\mathbb{S}^n} \int_0^{\rho(\cdot)} \frac{\sqrt{\det(\bar{g}_{ij}(s, \cdot))}}{\sqrt{\det(\sigma_{ij})}} \, ds \tag{2-6}$$

and the surface area by

$$|\Sigma| = \int_\Sigma 1.$$

Now we recall some crucial integral identities, which are known as Minkowski identities in the Riemannian context. We restrict to those we need, while more general versions were proved in [Kwong 2016]. First of all let

$$H_k(\kappa) = \frac{1}{\binom{n}{k}} \sum_{1\le i_1\le\cdots\le i_k\le n} \kappa_{i_1}\dots\kappa_{i_k}, \quad 1\le k\le n,$$

be the normalized elementary symmetric polynomials, where we also define $H_0 = 1$. We calculate

$$\vartheta'(\rho)_{;ij} = \vartheta'\rho_{;i}\rho_{;j} + \vartheta\rho_{;ij} = \vartheta'\rho_{;i}\rho_{;j} + \frac{\vartheta}{v}h_{ij} - \vartheta'\bar{g}_{ij} = -\vartheta' g_{ij} + \frac{\vartheta}{v}h_{ij}. \tag{2-7}$$

Regarding the functions $H_k = H_k(g_{ij}, h_{ij})$ in dependence of the metric and the second fundamental form, in spaceforms the tensors

$$H_k^{ij} = \frac{\partial H_k}{\partial h_{ij}}$$

are divergence free. Taking into account that

$$g_{ij}H_k^{ij} = kH_{k-1}, \quad 1\le k\le n,$$

we obtain from tracing (2-7) with respect to H_k^{ij} and integration that

$$\int_\Sigma \vartheta' H_{k-1} = \int_\Sigma uH_k, \quad 1\le k\le n. \tag{2-8}$$

In the hyperbolic space the same relations can be deduced from (2-4).

3. Evolution equations

To prove the Minkowski inequality for Σ, we make use of normal variations and therefore we recall some known variational formulae and deduce the ones we specifically need here.

A quite general treatment also appears in [Gerhardt 2006, Chapter 2], while our framework does not fit into the setting discussed there. Hence we take some more care in this section.

General evolution equations. Let $T > 0$, M compact and connected and

$$x : (0, T)\times M \to \bar{\mathbb{S}}_1^{n+1}$$

a time-dependent family of embeddings of the spacelike hypersurfaces

$$\Sigma_t = x(t, M).$$

Let us denote their normal velocity by f, i.e.,

$$\dot{x} = f\nu.$$

From [Gerhardt 2006, p. 94] we get

$$\partial_t g_{ij} = 2 f h_{ij} \quad \text{and} \quad \dot{\nu} = f_{;k} g^{kj} x_{;j}$$

where (g^{kj}) is the inverse of g and a dot denotes the covariant time derivative of a tensor field along the curve $x(\cdot, \xi)$.

As [Gerhardt 2006] uses a different convention for the Riemann tensor, for the reader's convenience we deduce the evolution of the shape operator here.

Lemma 3.1. *There holds*

$$\partial_t h_i^j = f_{;i}{}^j - f h_k^j h_i^k + f \delta_i^j.$$

Proof. Let $\overline{\nabla}$ denote the Levi-Civita connection of $\bar{g}$. We differentiate the Weingarten equation

$$\overline{\nabla}_{x_{;i}} \nu = h_i^k x_{;k}$$

covariantly with respect to time and get

$$\partial_t h_i^k x_{;k} + h_i^k \dot{x}_{;k} = \overline{\nabla}_{\dot{x}} \overline{\nabla}_{x_{;i}} \nu = \overline{\nabla}_{x_{;i}} \overline{\nabla}_{\dot{x}} \nu + \bar{R}(x_{;i}, \dot{x}) \nu = \overline{\nabla}_{x_{;i}} \overline{\nabla}_{\dot{x}} \nu + f x_{;i}$$

and hence, after multiplying with $x_{;j}$,

$$\begin{aligned} g_{kj} \partial_t h_i^k &= f g_{ij} - f h_i^k h_{kj} + \bar{g}(\overline{\nabla}_{x_{;i}}(f_{;k} g^{kl} x_{;l}), x_{;j}) \\ &= f g_{ij} - f h_i^k h_{kj} + f_{;ij}. \end{aligned}$$

□

Lemma 3.2. *Volume, surface area and total mean curvature evolve by*

$$\partial_t \operatorname{vol}(\widehat{\Sigma}_t) = \int_{\Sigma_t} f, \quad \partial_t |\Sigma_t| = n \int_{\Sigma_t} f H_1$$

and

$$\partial_t \int_{\Sigma_t} H_1 = (n-1) \int_{\Sigma_t} f H_2 + \partial_t \operatorname{vol}(\widehat{\Sigma}_t).$$

Proof. According to [Gerhardt 2006, (2.4.21)], the radial function ρ satisfies

$$\frac{\partial \rho}{\partial t} = f v,$$

where we note again the flip of our normal compared to that reference. Hence

$$\partial_t \operatorname{vol}(\widehat{\Sigma}_t) = \int_{\mathbb{S}^n} f v \frac{\sqrt{\det(\bar{g}_{ij}(\rho(\cdot), \cdot))}}{\sqrt{\det(\sigma_{ij})}} = \int_{\mathbb{S}^n} f \frac{\sqrt{\det(g_{ij})}}{\sqrt{\det(\sigma_{ij})}} = \int_{\Sigma} f.$$

There holds

$$\partial_t \sqrt{\det(g_{ij})} = \frac{\det(g_{ij}) g^{kl} \partial_t g_{kl}}{2\sqrt{\det(g_{ij})}} = n f H_1 \sqrt{\det(g_{ij})}.$$

The second claim follows. We calculate

$$\begin{aligned}\partial_t \int_{\Sigma_t} H_1 &= \frac{1}{n}\int_{\Sigma_t} f(n^2H_1^2 - |S|^2 + n)\\ &= \frac{2}{n}\binom{n}{2}\int_{\Sigma_t} fH_2 + \int_{\Sigma_t} f. \qquad \square\end{aligned}$$

Specific evolution equations. In order to prove the Minkowski inequality, we want to build a flow that preserves the surface area and increases the quantity

$$W_2(\Sigma) = \int_\Sigma H_1 - \mathrm{vol}(\widehat{\Sigma}).$$

A natural flow to consider is the locally constrained inverse mean curvature flow

$$\dot{x} = \Big(u - \frac{\vartheta'(\rho)}{H_1}\Big)\nu, \tag{3-1}$$

which indeed has the desired properties:

Lemma 3.3. *Along (3-1) the surface area is preserved and the quantity*

$$W_2(\Sigma_t) = \int_{\Sigma_t} H_1 - \mathrm{vol}(\widehat{\Sigma}_t)$$

is nondecreasing and strictly increasing unless the flow hypersurfaces are totally umbilic.

Proof. From Lemma 3.2 and (2-8) we obtain

$$\frac{1}{n}\partial_t|\Sigma_t| = \int_{\Sigma_t}(uH_1 - \vartheta') = 0$$

and

$$\begin{aligned}\partial_t W_2(\Sigma_t) &= (n-1)\int_{\Sigma_t}\Big(uH_2 - \frac{\vartheta' H_2}{H_1}\Big)\\ &= (n-1)\int_{\Sigma_t}\Big(\vartheta' H_1 - \frac{\vartheta' H_2}{H_1}\Big) \geq 0,\end{aligned}$$

where we used

$$H_2 = H_1^2 - \|\mathring{A}\|^2,$$

where $\mathring{A}$ is the trace-free part of the second fundamental form. $\square$

We need the evolution equation for the radial and support function and so define

$$\mathcal{L} = \partial_t - \frac{\vartheta'}{nH_1^2}\Delta - \vartheta\rho_;{}^k\partial_k.$$

Lemma 3.4. (i) *The radial function ρ satisfies*

$$\mathcal{L}\rho = \vartheta - \frac{2\vartheta'}{H_1}v^{-1} + \frac{\vartheta'^2}{\vartheta H_1^2} + \frac{\vartheta'^2}{n\vartheta H_1^2}\|\nabla\rho\|^2.$$

(ii) *The support function u satisfies*

$$\mathcal{L}u = -\frac{\vartheta'}{nH_1^2}\|\mathring{A}\|^2 u - \frac{\vartheta^2}{H_1}\|\nabla\rho\|^2. \tag{3-2}$$

Proof. (i) The choice of the normal implies

$$\partial_t\rho = \left(u - \frac{\vartheta'}{H_1}\right)v^{-1}.$$

From (2-3) we obtain

$$\begin{aligned}\partial_t\rho - \frac{\vartheta'}{nH_1^2}\Delta\rho &= \left(u - \frac{\vartheta'}{H_1}\right)v^{-1} - \frac{\vartheta'}{nH_1^2}g^{ij}(v^{-1}h_{ij} - \bar{h}_{ij})\\ &= \left(u - \frac{\vartheta'}{H_1}\right)v^{-1} - \frac{\vartheta'}{H_1}v^{-1} + \frac{\vartheta'^2}{n\vartheta H_1^2}g^{ij}\bar{g}_{ij}\\ &= \vartheta v^{-2} - \frac{2\vartheta'}{H_1}v^{-1} + \frac{\vartheta'^2}{\vartheta H_1^2} + \frac{\vartheta'^2}{n\vartheta H_1^2}\|\nabla\rho\|^2.\end{aligned}$$

The result follows from

$$\|\nabla\rho\|^2 = \left(\bar{g}^{ij} + \frac{\bar{g}^{ik}\rho_{;k}\bar{g}^{jl}\rho_{;l}}{v^2}\right)\rho_{;i}\rho_{;j} = 1 - v^2 + \frac{(1-v^2)^2}{v^2} = \frac{1-v^2}{v^2}.$$

(ii) We calculate that the vector field $\vartheta\partial_{x^0}$ is conformal:

$$(\vartheta\partial_{x^0})_{;\alpha} = \vartheta' r_{;\alpha}\partial_{x^0} + \vartheta\bar{\Gamma}^{\beta}_{0\alpha}\partial_{x^\beta}.$$

The Christoffel-symbols of $\bar{g}$ are

$$\begin{aligned}\bar{\Gamma}^{\beta}_{0\alpha} &= \frac{1}{2}\bar{g}^{\beta\delta}\left(\frac{\partial}{\partial x^0}\bar{g}_{\alpha\delta} + \frac{\partial}{\partial x^\alpha}\bar{g}_{0\delta} - \frac{\partial}{\partial x^\delta}\bar{g}_{0\alpha}\right)\\ &= \begin{cases} 0 & \text{if } \beta = 0\\ \frac{\vartheta'}{\vartheta}\delta^i_\alpha & \text{if } \beta = i,\end{cases}\end{aligned}$$

and hence

$$(\vartheta\partial_{x^0})_{;\alpha} = \vartheta' r_{;\alpha}\partial_{x^0} + \vartheta'\delta^i_\alpha\partial_{x^i} = \vartheta'\delta^0_\alpha\partial_{x^0} + \vartheta'\delta^i_\alpha\partial_{x^i} = \vartheta'\delta^\beta_\alpha\partial_{x^\beta} = \vartheta'\partial_{x^\alpha}.$$

Therefore

$$\partial_t u = -\bar{g}(\overline{\nabla}_{\dot{x}}(\vartheta\partial_r), \nu) - \bar{g}(\vartheta\partial_r, \dot{\nu}) = f\vartheta' - \bar{g}(\vartheta\partial_r, x_{;j}) f_{;k} g^{kj},$$
$$u_{;i} = -h_i^k \bar{g}(\vartheta\partial_r, x_{;k})$$

and

$$u_{;ij} = -h^k_{i;j}\bar{g}(\vartheta\partial_r, x_{;k}) - \vartheta' h_{ij} + h_i^k h_{kj} u.$$

Hence

$$\begin{aligned}\partial_t u - \frac{\vartheta'}{nH_1^2}\Delta u &= -\frac{\vartheta'}{nH_1^2}\|A\|^2 u + \vartheta'\left(u - \frac{\vartheta'}{H_1}\right) + \frac{\vartheta'^2}{H_1} - \bar{g}(\vartheta\partial_r, x_{;j})\left(u - \frac{\vartheta'}{H_1}\right)_{;k} g^{kj} \\ &\qquad + \frac{\vartheta'}{H_1^2} H_{1;}{}^{k}\bar{g}(\vartheta\partial_r, x_{;k}) \\ &= -\frac{\vartheta'}{nH_1^2}(\|A\|^2 - nH_1^2)u - \bar{g}(\vartheta\partial_r, x_{;j})u_{;k}g^{kj} + \bar{g}(\vartheta\partial_r, x_{;j})\frac{\vartheta}{H_1}\rho_{;}{}^{j} \\ &= -\frac{\vartheta'}{nH_1^2}\|\mathring{A}\|^2 u + \vartheta g(\nabla\rho, \nabla u) - \frac{\vartheta^2}{H_1}\|\nabla\rho\|^2. \qquad \square\end{aligned}$$

We conclude this section with the full evolution of the shape operator.

Lemma 3.5. *There holds*

$$\begin{aligned}\mathcal{L}h^i_j &= -\vartheta'\left(\frac{\|A\|^2}{nH_1^2}h_i^j - \frac{2}{H_1}h_i^m h_m^j + h_i^j\right) + u\left(\delta_i^j - \frac{h_i^j}{H_1}\right) + \frac{\vartheta'}{H_1^2}(H_1\delta_i^j - h_i^j) \\ &\qquad + \frac{\vartheta}{H_1^2}H_{1;}{}^{j}\rho_{;i} + \frac{\vartheta}{H_1^2}H_{1;i}\rho_{;}{}^{j} - 2\frac{\vartheta'}{H_1^3}H_{1;i}H_{1;}{}^{j}.\end{aligned} \tag{3-3}$$

Proof. From Lemma 3.1 we have

$$\partial_t h_i^j = \left(u - \frac{\vartheta'}{H_1}\right)_{;i}^{\;\;j} - \left(u - \frac{\vartheta'}{H_1}\right)h_k^j h_i^k + \left(u - \frac{\vartheta'}{H_1}\right)\delta_i^j.$$

We have to expand the second order term. There holds

$$\left(u - \frac{\vartheta'}{H_1}\right)_{;i} = u_{;i} - \frac{\vartheta}{H_1}\rho_{;i} + \frac{\vartheta'}{H_1^2}H_{1;i}$$

and

$$\begin{aligned}\left(u - \frac{\vartheta'}{H_1}\right)_{;ij} &= u_{;ij} - \frac{\vartheta'}{H_1}\rho_{;i}\rho_{;j} + \frac{\vartheta}{H_1^2}H_{1;j}\rho_{;i} - \frac{\vartheta}{H_1}\rho_{;ij} + \frac{\vartheta}{H_1^2}H_{1;i}\rho_{;j} \\ &\qquad - 2\frac{\vartheta'}{H_1^3}H_{1;i}H_{1;j} + \frac{\vartheta'}{H_1^2}H_{1;ij}.\end{aligned}$$

In order to replace the term $H_{1;ij}$, we have to use the Codazzi and Gauss equation (2-1). As in [O'Neill 1983, p. 76] we define

$$R(\partial_k, \partial_l)\partial_j = R^m_{jkl}\partial_m.$$

There holds

$$\begin{aligned} nH_{1;ij} &= h^k_{k;ij} = h^k_{i;kj} \\ &= h^k_{i;jk} + h^m_i R^k_{mkj} - h^k_m R^m_{ikj} \\ &= h_{ij;k}{}^k + h^m_i R^k_{mkj} - h^k_m R^m_{ikj}. \end{aligned}$$

We use (2-1) and $\bar{g}(\nu, \nu) = -1$ to deduce

$$\begin{aligned} R^i_{jkl} &= g(R(\partial_k, \partial_l)\partial_j, \partial_m)g^{im} \\ &= \delta^i_l g_{jk} - \delta^i_k g_{jl} - h^i_l h_{jk} + h^i_k h_{jl} \end{aligned}$$

and hence

$$\begin{aligned} nH_{1;ij} &= h_{ij;k}{}^k + h^m_i R^k_{mkj} - h^k_m R^m_{ikj} \\ &= h_{ij;k}{}^k + h^m_i(\delta^k_j g_{mk} - ng_{mj} - h^k_j h_{mk} + h^k_k h_{mj}) \\ &\qquad - h^k_m(\delta^m_j g_{ik} - \delta^m_k g_{ij} - h^m_j h_{ik} + h^m_k h_{ij}) \\ &= h_{ij;k}{}^k - (n-1)h_{ij} - h^m_i h_{mk} h^k_j + nH_1 h^m_i h_{mj} \\ &\qquad - h_{ij} + nH_1 g_{ij} + h^k_m h^m_j h_{ik} - \|A\|^2 h_{ij} \\ &= h_{ij;k}{}^k - nh_{ij} + nH_1 h^m_i h_{mj} + nH_1 g_{ij} - \|A\|^2 h_{ij}. \end{aligned}$$

From (2-3) we obtain

$$\begin{aligned} \left(u - \frac{\vartheta'}{H_1}\right)_{;ij} &= -h^k_{i;j}\bar{g}(\vartheta\partial_r, x_{;k}) - \vartheta' h_{ij} + h^k_i h_{kj} u - \frac{\vartheta'}{H_1}\rho_{;i}\rho_{;j} + \frac{\vartheta}{H_1^2}H_{1;j}\rho_{;i} \\ &\quad - \frac{\vartheta}{H_1}(v^{-1}h_{ij} - \bar{h}_{ij}) + \frac{\vartheta}{H_1^2}H_{1;i}\rho_{;j} - 2\frac{\vartheta'}{H_1^3}H_{1;i}H_{1;j} \\ &\quad + \frac{\vartheta'}{nH_1^2}\bigl(h_{ij;k}{}^k - nh_{ij} + nH_1 h^m_i h_{mj} + nH_1 g_{ij} - \|A\|^2 h_{ij}\bigr) \\ &= -h^k_{i;j}\bar{g}(\vartheta\partial_r, x_{;k}) - \vartheta' h_{ij} + h^k_i h_{kj} u + \frac{\vartheta}{H_1^2}H_{1;j}\rho_{;i} - \frac{u}{H_1}h_{ij} + \frac{2\vartheta'}{H_1}g_{ij} \\ &\quad + \frac{\vartheta}{H_1^2}H_{1;i}\rho_{;j} - 2\frac{\vartheta'}{H_1^3}H_{1;i}H_{1;j} + \frac{\vartheta'}{nH_1^2}h_{ij;k}{}^k - \frac{\vartheta'}{H_1^2}h_{ij} \\ &\quad + \frac{\vartheta'}{H_1}h^m_i h_{mj} - \frac{\vartheta'}{nH_1^2}\|A\|^2 h_{ij}. \end{aligned}$$

Thus

$$\begin{aligned}\mathcal{L}h^i_j &= -\vartheta' h_i^j + \frac{\vartheta}{H_1^2} H_{1;}{}^j \rho_{;i} - \frac{u}{H_1} h_i^j + \frac{\vartheta'}{H_1}\delta_i^j + \frac{\vartheta}{H_1^2} H_{1;i}\rho_;{}^j - 2\frac{\vartheta'}{H_1^3} H_{1;i} H_{1;}{}^j \\ &\qquad - \frac{\vartheta'}{H_1^2} h_i^j + 2\frac{\vartheta'}{H_1} h_i^m h_m^j - \frac{\vartheta'}{nH_1^2}\|A\|^2 h_i^j + u\delta_i^j \\ &= -\vartheta'\left(\frac{\|A\|^2}{nH_1^2} h_i^j - \frac{2}{H_1} h_i^m h_m^j + h_i^j\right) + u\left(\delta_i^j - \frac{h_i^j}{H_1}\right) + \frac{\vartheta'}{H_1^2}(H_1\delta_i^j - h_i^j) \\ &\qquad + \frac{\vartheta}{H_1^2} H_{1;}{}^j \rho_{;i} + \frac{\vartheta}{H_1^2} H_{1;i}\rho_;{}^j - 2\frac{\vartheta'}{H_1^3} H_{1;i} H_{1;}{}^j. \qquad \square\end{aligned}$$

4. A priori estimates

We establish C^2-estimates for (3-1) and use standard regularity theory for parabolic equations to conclude smooth convergence of the flow to a round sphere.

We assume throughout this section that $\Sigma \subset \overline{\mathbb{S}}_1^{n+1}$ is a smooth, closed, connected, spacelike and mean-convex hypersurface.

Then the differential operator $\mathcal{L}$ is strictly parabolic at Σ and hence the flow (3-1) has a unique solution for a short time T^* with initial hypersurface $\Sigma_0 = \Sigma$. The a priori estimates of this section refer to this solution.

Estimates up to first order.

Lemma 4.1. *Along the flow* (3-1) *for all* $(t, \xi) \in (0, T^*) \times M$ *there holds*

(i) $$\min_M \rho(0,\cdot) \le \rho(t,\xi) \le \max_M \rho(0,\cdot),$$

(ii) $$u(t,\xi) \le \max_M u(0,\cdot).$$

Proof. (i) The radial function ρ satisfies

$$\partial_t \rho = \left(u - \frac{\vartheta'}{H_1}\right) v^{-1}.$$

From (2-3) we obtain that spatial maximal points of ρ we have

$$0 \ge \Delta\rho = nH_1 - n\frac{\vartheta'}{\vartheta}$$

and hence

$$u - \frac{\vartheta'}{H_1} \le 0.$$

Therefore $\max \rho$ is nonincreasing and the reverse estimate proves that $\min \rho$ is nondecreasing.

(ii) Directly follows from (3-2) and the maximum principle. $\square$

Curvature estimates. Tracing (3-3) yields the evolution of the normalized mean curvature:

$$\mathcal{L}H_1 = \frac{\vartheta'}{nH_1}(\|A\|^2 - nH_1^2) + \frac{2\vartheta}{nH_1^2} g(\nabla H_1, \nabla\rho) - \frac{2\vartheta'}{nH_1^3}\|\nabla H_1\|^2.$$

Lemma 4.2. (i) *For all* $(t,\xi) \in (0, T^*) \times M$ *there holds*

$$H_1(t,\xi) \geq \min_M H_1(0,\cdot).$$

(ii) *There exists a constant* $c = c(M_0)$ *such that*

$$H_1 \leq c.$$

Proof. (i) Follows directly from the maximum principle.

(ii) Define

$$w = \log H_1 + \lambda u - \rho,$$

where λ is determined to be a large number. Then

$$\begin{aligned}
\mathcal{L}w &= \frac{\mathcal{L}H_1}{H_1} + \frac{\vartheta'}{nH_1^2}\|\nabla \log H_1\|^2 + \lambda\mathcal{L}u - \mathcal{L}\rho \\
&= \frac{\vartheta'}{nH_1^2}\|\mathring{A}\|^2(1-\lambda u) + \frac{2\vartheta}{nH_1^2} g(\nabla \log H_1, \nabla\rho) - \frac{\vartheta'}{nH_1^2}\|\nabla \log H_1\|^2 \\
&\qquad - \frac{\lambda\vartheta^2}{H_1}\|\nabla\rho\|^2 - \vartheta + \frac{2\vartheta'}{H_1}v^{-1} - \frac{\vartheta'^2}{\vartheta H_1^2} - \frac{\vartheta'^2}{n\vartheta H_1^2}\|\nabla\rho\|^2 \\
&\leq -\vartheta + \frac{2\vartheta'}{H_1}v^{-1} + \frac{c}{H_1^2}\|\nabla\rho\|^2
\end{aligned}$$

for large λ. If H_1 is too large, this is negative and hence we obtain a bound on H_1 by the maximum principle. □

Lemma 4.3. *There exists a constant* $c = c(M_0)$ *such that*

$$\|A\|^2 \leq c.$$

Proof. We estimate the largest principal curvature κ_n. By a well known trick, see [Gerhardt 2011, p. 500], it suffices to estimate the evolution of $h_n^n = \kappa_n$ at a point where κ_n attains a space-time maximum. At such a point we introduce coordinates such that

$$g_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_i\delta_{ij}.$$

Using

$$0 < H_1 \leq \kappa_n \leq \|A\| = \sqrt{\kappa_1^2 + \cdots + \kappa_n^2},$$

(3-3) and all previously deduced bounds, we obtain constants ϵ and c such that

$$\begin{aligned}\mathcal{L}h_n^n &= -\vartheta'\left(\frac{\|A\|^2}{nH_1^2}\kappa_n - \frac{2}{H_1}\kappa_n^2 + \kappa_n\right) + u\left(1 - \frac{\kappa_n}{H_1}\right) \\ &\qquad + \frac{\vartheta'}{H_1^2}(H_1 - \kappa_n) + \frac{2\vartheta}{H_1^2}H_{1;n}\rho_{;n} - 2\frac{\vartheta'}{H_1^3}(H_{1;n})^2 \\ &\leq -\epsilon\kappa_n^3 + c\kappa_n^2 + c + c|H_{1;n}|\|\nabla\rho\| - \epsilon|H_{1;n}|^2.\end{aligned}$$

After employing Cauchy–Schwarz to absorb $|H_{1;n}|$, we see that the expression on the right-hand side is negative, provided κ_n is sufficiently large. This threshold depends on ϵ and c. The proof is complete due to the maximum principle. □

Corollary 4.4. *The flow (3-1) starting from Σ exists for all times and satisfies uniform estimates in $C^m(\mathbb{S}^n)$ for all $m \geq 0$.*

Proof. After the previously established C^2-estimates this is standard from parabolic regularity [Krylov 1987] applied to the graph function ρ. □

5. Completion of the proof

To complete the proof, we have to show that the flow converges to a round sphere.

Lemma 5.1. *The flow (3-1) converges to a uniquely determined coordinate slice and hence Theorem 1.2 holds.*

Proof. Along the flow the quantity

$$W_2(\Sigma_t) = \int_{\Sigma_t} H_1 - \operatorname{vol}(\widehat{\Sigma}_t)$$

is clearly bounded and nondecreasing. Hence

$$\partial_t W_2(\Sigma_t) = (n-1)\int_{\Sigma_t} \frac{\vartheta'\|\mathring{A}\|^2}{H_1} \to 0$$

as $t \to \infty$ and thus every subsequential C^∞-limit Σ_∞ must be totally umbilical. As $H_1 > 0$, Σ_∞ is strictly convex. The dual hypersurface $\widetilde{\Sigma}_\infty$, which is given by the Gauss map

$$\tilde{x} = \nu : M \to \mathbb{H}^{n+1} \subset \mathbb{M}_1^{n+2},$$

is thus also totally umbilical, see [Gerhardt 2006, Theorem 10.4.4], and hence a geodesic sphere. From (2-5) we obtain

$$|\Sigma_\infty| = \int_{\widetilde{\Sigma}_\infty} \widetilde{H}_n,$$

which is, up to an additive constant, the $(n-1)$-quermassintegral $\widetilde{W}_{n-1}(\widetilde{\Sigma}_\infty)$ in $\mathbb{H}^{n+1}$. See [Wang and Xia 2014] for a definition. As $|\Sigma_\infty|$ is independent of the subsequential limit, so is $\widetilde{W}_{n-1}(\widetilde{\Sigma}_\infty)$ and hence the radius and the principal curvatures of $\widetilde{\Sigma}_\infty$ are uniquely determined. This implies that H_1 takes the same value $0 < H_* < 1$ for every subsequential limiting hypersurface Σ_∞. In particular, the function H_1 converges uniformly to H_* along the flow. Let

$$r_0 = \tanh^{-1}(H_*).$$

We claim that the flow converges to the slice $\{r = r_0\}$ smoothly and prove this in several steps.

(i) For every subsequential limit Σ_∞ with corresponding radial function ρ_∞ there holds

$$\max_M \rho_\infty \geq r_0,$$

since at a point where the maximum is attained we have

$$0 \geq \Delta\rho_\infty = nH_* - n\tanh(\rho_\infty)$$

and hence $\rho_\infty \geq r_0$ due to the monotonicity of tanh.

(ii) Define

$$\varphi(t) = \max_M \rho(t,\cdot) - r_0 = \rho(t,\xi_t) - r_0.$$

Then φ is Lipschitz and hence differentiable almost everywhere. Let $\epsilon > 0$ and fix some $T_\epsilon > 0$ to be specified later. Let $t > T_\epsilon$ be a point of differentiability of φ and suppose

$$\varphi(t) \geq \epsilon.$$

Note that this condition is only nonvoid for bounded $\epsilon \leq \epsilon_0$, due to the barrier estimates. Then there holds

$$\varphi'(t) = \partial_t\rho(t,\xi_t) = \left(u - \frac{\vartheta'}{H_1}\right)v^{-1}(t,\xi_t) = \vartheta - \frac{\vartheta'}{H_1},$$

where the right-hand side is evaluated at $\rho(t,\xi_t)$. We estimate

$$\begin{aligned}
\varphi'(t) &= \frac{\vartheta'}{H_1}\big(\coth(\rho(t,\xi_t))H_1 - 1\big)\\
&\leq \frac{\vartheta'}{H_1}\big(\coth(r_0+\epsilon)H_* - 1\big) + c_{r_0}|H_1 - H_*|\\
&= -\frac{\vartheta'}{H_1}\frac{\vartheta'(r_0)}{\vartheta(r_0)\vartheta'^2(\eta)}\epsilon + c_{r_0}|H_1 - H_*|\\
&= \left(-\frac{\vartheta'}{H_1}\frac{\vartheta'(r_0)}{\vartheta(r_0)\vartheta'^2(\eta)} + \frac{c_{r_0}}{\epsilon}|H_1 - H_*|\right)\epsilon,
\end{aligned}$$

where we have used the mean value theorem and $\eta \in [r_0, r_0+\epsilon_0]$. As $H_1 \to H_*$, we may now choose $T = T_\epsilon$, which only depends on ϵ and the initial data, such that for all $t > T_\epsilon$ with $\varphi(t) \geq \epsilon$ there holds

$$\varphi'(t) \leq -\delta_\epsilon.$$

From [Scheuer 2015, Lemma 4.2] it follows that

$$\limsup_{t\to\infty} \max_M \rho(t,\cdot) \leq r_0.$$

(iii) Combining (i) and (ii) we obtain

$$\lim_{t\to\infty} \max_M \rho(t,\cdot) = r_0.$$

(iv) A similar argument applied to $\min \rho$ implies

$$\lim_{t\to\infty} \min_M \rho(t,\cdot) = r_0.$$

Hence the unique limit is the slice $\{r = r_0\}$ and the proof is complete. □

Proof. Now we prove Theorem 1.1. So let $\Sigma \subset \bar{\mathbb{S}}^{n+1}_1$ be a spacelike, compact, connected and mean-convex hypersurface. According to Theorem 1.2, we may deform Σ in an infinite amount of time to a coordinate slice S_{r_∞} of radius r_∞. Define

$$\varphi_1(r) = |\{x^0 = r\}|, \quad \varphi_2(r) = W_2(\{x^0 = r\}).$$

From Lemma 3.2 we see that both of these function are strictly increasing functions of r. By Lemma 3.3 there holds

$$W_2(\Sigma) \leq W_2(S_{r_\infty}) = \varphi_2(r_\infty) = \varphi_2 \circ \varphi_1^{-1}(\varphi_1(r_\infty)) = \varphi_2 \circ \varphi_1^{-1}(|\Sigma|).$$

From the proof of Lemma 3.3 we obtain that if we have equality in this inequality, Σ must be totally umbilic, otherwise the flow would increase W_2 strictly.

It remains to show that total umbilicity implies equality. So let Σ be totally umbilic. Now it suffices to prove that this property is preserved along the flow (3-1), since we know that this flow deforms Σ into a coordinate slice on which equality holds. Furthermore, if all flow hypersurfaces are totally umbilic, W_2 is constant. As $|\Sigma_t|$ is constant anyway, the equality must already hold on Σ.

So let us prove that (3-1) preserves total umbilicity. If Σ is totally umbilic, it must be strictly convex. Hence for a short time the flow hypersurfaces Σ_t are strictly convex. As in [Bryan et al. 2020, Section 4] and employing Lemma 2.4 we can calculate, that up to a tangential diffeomorphism the dual hypersurfaces $\widetilde{\Sigma}_t \subset \mathbb{H}^{n+1}$ satisfy the flow equation

$$\dot{\tilde{x}} = \left(u - \frac{\vartheta'}{H_1(\kappa_i)}\right)\tilde{\nu} = \left(\tilde{\vartheta}' - n\tilde{u}\frac{\sigma_n(\tilde{\kappa})}{\sigma_{n-1}(\tilde{\kappa})}\right)\tilde{\nu},$$

where σ_k is the k-th elementary symmetric polynomial. This is a locally constrained curvature flow of contracting type in hyperbolic space. In the Euclidean space such kind of flows were studied in [Guan and Li 2018]. Although in the nonflat spaces no satisfactory convergence results are available in general, we can still show that this flow preserves the total umbilicity. It has the property that it preserves the $(n-1)$-quermassintegral $\widetilde{W}_{n-1}(\widetilde{\Sigma}_t)$ in $\mathbb{H}^{n+1}$, while it decreases $\widetilde{W}_n(\widetilde{\Sigma}_t)$ and we have

$$\partial_t \widetilde{W}_n(\widetilde{\Sigma}_t) < 0 \tag{5-1}$$

at $t > 0$ unless $\widetilde{\Sigma}_t$ is totally umbilic. Since by assumption $\widetilde{\Sigma}$ is totally umbilic, we have

$$\widetilde{W}_n(\widetilde{\Sigma}) = \psi(\widetilde{W}_{n-1}(\widetilde{\Sigma})),$$

see [Wang and Xia 2014, Theorem 1.1], with a suitable function ψ. If $\widetilde{\Sigma}_t$ was not totally umbilic for some $t > 0$, by (5-1) we would obtain

$$\widetilde{W}_n(\widetilde{\Sigma}_t) < \psi(\widetilde{W}_{n-1}(\widetilde{\Sigma}_t)),$$

which violates [Wang and Xia 2014, Theorem 1.1]. Hence the $\widetilde{\Sigma}_t$ must be totally umbilic and, going back to the flow in de Sitter space, Σ_t must be totally umbilic. Hence (3-1) preserves the total umbilicity and the proof is complete. □

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