SCALING LIMITS AND STOCHASTIC HOMOGENIZATION FOR SOME NONLINEAR PARABOLIC EQUATIONS

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ABSTRACT. The aim of this paper is twofold. The first is to study the asymptotics of a parabolically scaled, continuous and space-time stationary in time version of the well-known Funaki-Spohn model in Statistical Physics. After a change of unknowns requiring the existence of a space-time stationary eternal solution of a stochastically perturbed heat equation, the problem transforms to the qualitative homogenization of a uniformly elliptic, space-time stationary, divergence form, nonlinear partial differential equation, the study of which is the second aim of the paper. An important step is the construction of correctors with the appropriate behavior at infinity.

1. Introduction

The first aim of the paper is to study the limit, as $\varepsilon \to 0$, of the stochastic partial differential equation (SPDE for short)

$$\begin{cases} d_t U_t^\varepsilon = \operatorname{div}\mathcal{A}(DU_t^\varepsilon, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}, \omega_1)dt + \dfrac{1}{\varepsilon}\sum_{k \in \mathbb{Z}^d} A(\dfrac{x-k}{\varepsilon})dB_t^k \ \ \text{in} \ \ \mathbb{R}^d \times (0, +\infty), \\ U_0^\varepsilon = u_0. \ \ \text{in} \ \ \mathbb{R}^d. \end{cases} \tag{1.1}$$

In the above equation, $(B^k)_{k \in \mathbb{Z}^d}$ is a sequence of independent $d-$dimensional Brownian motions in a probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ with $\Omega_0 = (C^0(\mathbb{R}, \mathbb{R}^d))^{\mathbb{Z}^d}$, and $A : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth map with a compact support. Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ be another probability space endowed with a space-time ergodic group of measure preserving transformations. The vector field $\mathcal{A} : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \Omega_1 \to \mathbb{R}$ is assumed to be smooth, uniformly elliptic and space-time stationary in $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$, and is independent of the Brownian motions. The precise assumptions are listed in section 4.

A reformulation of (1.1) led us to the second aim of the paper. This is the study of the qualitative (stochastic) homogenization of the divergence form quasilinear partial differential equation (PDE for short)

$$u_t^\varepsilon - \operatorname{div} a(Du^\varepsilon, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}, \omega) = f \ \ \text{in} \ \ \mathbb{R}^d \times (0, \infty) \quad u^\varepsilon(\cdot, 0) = u_0, \tag{1.2}$$

where $a : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}^d$ is strongly monotone, Lipschitz continuous and space-time stationary in an ergodic with respect to $\mathbb{Z}^d \times \mathbb{R}$-action random environment, which we denote again by $(\Omega, \mathcal{F}, \mathbb{P})$ although it is different than the one for (1.1), and f and u_0 are square integrable. All the assumptions are made precise in section 3.

The result is that, in either case, there exists a strongly monotone map $\bar{a} : \mathbb{R}^d \to \mathbb{R}^d$ such that the solutions of (1.1) and (1.2) converge either a.s. or in expectation and in an appropriately weighted L^2-space in space-time to the unique solution $\bar{u}$ of the initial value problem

$$\bar{u}_t - \operatorname{div} \bar{a}(D\bar{u}) = f \ \ \text{in} \ \ \mathbb{R}^d \times (0, \infty) \quad \bar{u}(\cdot, 0) = u_0. \tag{1.3}$$

The link between (1.1) and (1.2) is made writing U^ε as

$$U_t^\varepsilon(x) = \varepsilon \widetilde{V}_{\frac{t}{\varepsilon^2}}\left(\frac{x}{\varepsilon}\right) + \widetilde{W}_t^\varepsilon(x),$$

with $\widetilde{V}$ and W^ε been respectively the unique up to constants eternal, space-time stationary solution of the stochastically perturbed heat equation

$$d\widetilde{V}_t = \Delta \widetilde{V}_t dt + \sum_{k \in \mathbb{Z}^d} A(x - k) dB_t^k \ \ \text{in} \ \ \mathbb{R}^d \times \mathbb{R}, \tag{1.4}$$

and the solution of the uniformly elliptic, divergence form PDE

$$\partial_t \widetilde{W}_t^\varepsilon = \operatorname{div}\left(\widetilde{a}(D\widetilde{W}^\varepsilon, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega)\right) \ \ \text{in} \ \ \mathbb{R}^d \times (0, +\infty) \quad \widetilde{W}_0^\varepsilon = u_0 \ \ \text{in} \ \ \mathbb{R}^d, \tag{1.5}$$

with the random nonlinearity

$$\widetilde{a}(p, x, t, \omega) = \mathcal{A}(p + D\widetilde{V}_t(x, \omega_0), t, x, \omega_1) - D\widetilde{V}_t(x, \omega_0) \tag{1.6}$$

space-time stationary, strongly monotone and Lipschitz continuous.

The existence and properties of $\widetilde{V}$ are the topic of section 2. The construction is based on solving the problem in $\mathbb{R}^d \times [-n^{-2}, \infty)$ and then letting $n \to \infty$. To prove, however, the convergence to a unique up to constants stationary solution, it is necessary to obtain suitable gradient bounds. This requires, among others, the quantitative understanding the long-space decorrelation properties of the gradients. For the latter, it is necessary to study in detail the properties of the gradients of localized versions of the stochastically perturbed heat equation, which depend on finitely many Brownian motions in balls of radius R, as $R \to \infty$.

The study of the qualitative homogenization of (1.2), which is developed in section 3, is based on the existence, for each $p \in \mathbb{R}^d$, of space-time stationary solutions $\chi^p = \chi(y, \tau, \omega; p)$ of

$$\partial_\tau \chi^p - \operatorname{div}(a(p + D\chi^p, y, \tau, \omega)) = 0 \ \ \text{in} \ \ \mathbb{R}^d, \tag{1.7}$$

such that, as $\varepsilon \to 0$, $\chi^\varepsilon(x, t; p, \omega) = \varepsilon \chi^p(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega) \to 0$ in $L^2_{loc}(\mathbb{R}^{d+1})$, $\mathbb{P}$−a.s. and in expectation.

The existence of correctors in our setting is, to the best of our knowledge, new. The difficulty arises from the unbounded domain and the lack of regularity in time. Overcoming it, requires the development of new and sharp results.

Once correctors are established, the homogenization follows, at least formally, using, at the level of test functions ϕ, the expansion

$$\phi^\varepsilon(x,t) = \phi(x,t) + \varepsilon\chi(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega; D\phi(x,t)),$$

the justification of which creates additional problems due to the low available regularity of χ^p in p. To overcome it, it is necessary to introduce yet another level of approximation involving "piecewise gradient correctors" corresponding to piecewise constant approximations of $D\phi$.

Funaki and Spohn showed in [8] the convergence of a system of interacting diffusion processes, modeling the height of a surface in $\mathbb{R}^d$, to a deterministic limit equation.

More precisely, for any cube $\Lambda \subseteq \mathbb{Z}^d$, they considered processes of the form

$$d\Phi_t(x) = -\sum_{|x-y|_1=1} V'(\Phi_t(x) - \Phi_t(y)) + \sqrt{2}dB_t(x) \text{ for } x \in \Lambda \subset \mathbb{Z}^d. \tag{1.8}$$

The fields Φ live on a discrete lattice and take values in $\mathbb{R}^d$, $B_t(x)$ are i.i.d. Brownian motions, V' is the derivative of a strictly convex symmetric function, and $|\cdot|_1$ is the l^1-norm. Note that the drift term in (1.8) is simply the discrete divergence of the vector field $(V'(D_i^+\Phi))_{i=1,\ldots,d}$, where $D_i^+\Phi(x) = \Phi(x+e_j) - \Phi(x)$ is the discrete forward partial derivative in direction i.

The result in [8] is that the rescaled fields

$$\Phi^\varepsilon(r,t) = \varepsilon\Phi_{\varepsilon^{-2}t}(x) \text{ for } r \in [x - \varepsilon/2, x + \varepsilon/2)^d \text{ with } N = [\varepsilon^{-1}] \tag{1.9}$$

converge to the solution h of the nonlinear, divergence form deterministic PDE

$$\partial_t h(r,t) = \text{div}(D\sigma(\nabla h)) \text{ in } \mathbb{R}^d \times (0,\infty).$$

A crucial step in the proof in [8] is the existence of unique gradient Gibbs measures, that is, invariant measures for the discrete gradient of the fields which on finite subsets $\Lambda \subseteq \mathbb{Z}^d$ defined by

$$\frac{1}{Z}e^{\beta\sum_{x\in\Lambda} V(\nabla_i\Phi(x))}\prod_{x\in\Lambda} d\Phi(x).$$

The SPDE (1.1) we are considering here can be seen as a continuous version of the equation satisfied by Φ^ε in (1.9). Our proof of the existence of the limit is purely dynamic, that is, it does not use the existence of invariant measures of a certain structure. Instead, we use the eternal solutions of a linear SPDE, which allows to transform the problem to one like (1.2) with an appropriately defined field a.

Although it may appear so, results about the convergence of the solution of U_t^ε and Φ^ε are not, in any sense, equivalent. For example, the effective nonlinearities $\bar{a}$ and h are, in general, not the same. To be able to compare the limit problems, it is necessary to understand in precise way how (1.1) with $\varepsilon = 1$ is the continuous (mescopic) limit of (1.8).

The qualitative stochastic homogenization result is new. We are, of course, aware of earlier works of Efentiev and Panov [5, 6] and Efendiev, Jiang and Pankov [4], which, however, do not apply to the general space-time stationary setting we are considering here. The crucial part of the proof is the existence of a space-time stationary corrector, which requires overcoming the low regularity in time. Beside the references [4, 5, 6] already quoted,

the literature on the space-time homogenization of parabolic equations in a random setting is scarce and mostly devoted to linear equations, starting with the pioneering work of Zhikov, Kozlov and Oleinik [16]: Landim, Olla and Yau [13] provide an invariance invariance principle for diffusion in space-time random environment with a bounded stream matrix; Fannjiang and Komorowski [9] generalize the result to the case of unbounded stationary vector potentials while Komorowski and Olla [11] investigate the problem for divergence free vector fields; Rhodes [15] and Delarue and Rhodes [3] study the homogenization of degenerate diffusions; more recently, Armstrong, Bordas and Mourratï¿½[1] provide a convergence rate for the homogenization of parabolic equation in space-time random environment under a finite range condition by using a variational structure for the equation.

Organization of the paper. Section 2 is devoted to the study of the linear problem (1.4). In section 3 we concentrate on (1.2). The result about (1.1) is presented in section 4. Each of section consists of several subsections which are outlined there. Finally, in the Appendix we include some results about functions with stationary gradients that we use throughout the paper.

Notation. Given $x_0 \in \mathbb{R}^d$, $Q_R(x_0) = x_0 + (-R/2, R/2)^d$ and $B_r(x_0)$ is the open ball in $\mathbb{R}^d$ centered at x_0 and radius r. Moreover, $Q_R = Q_R(0)$, $I_R = (-R/2, R/2)$, $B_r = B_r(0)$, and $\widetilde{Q}_R = Q_R \times I_R = (-R/2, R/2)^{d+1} \subset \mathbb{R}^{d+1}$, while $\widetilde{Q}$ and $\widehat{Q}$ are used for any cube in R^{d+1}. If $a, b \in \mathbb{R}$, $a \wedge b = \min\{a, b\}$ and $a \lesssim_\alpha b$ means that there exists a constant $C = C(\alpha) > 0$ such that $a \leq Cb$. We write $a \underset{\alpha}{\sim} b$ if $a \lesssim_\alpha b$ and $b \lesssim_\alpha a$. The integer part of $s \in \mathbb{R}$ is $\lfloor s \rfloor$. Given $x \in \mathbb{R}^d$, $|x|_\infty = \max\{|x_i| : i = 1, \ldots, d\}$. We write $\mathbf{1}_A$ for the characteristic function of a set A and, finally, Int B is the topological interior of $B \subset \mathbb{R}^k$.

Terminology. We say that a vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ is strongly monotone and Lipschitz continuous if the there exists $C_0 > 0$ such that, respectively and for all $p, q \in \mathbb{R}^d$,

$$(b(p) - b(q)) \cdot (p - q) \geq C_0^{-1}|p - q|^2, \tag{1.10}$$

and

$$|b(p) - b(q)| \leq C_0|p - q|. \tag{1.11}$$

2. The linear problem (1.4)

The goal here is to construct space-time stationary solutions of the linear SPDE

$$dV_t = \Delta V_t dt + \sum_{k \in \mathbb{Z}^d} A(x - k)dB_t^k. \tag{2.1}$$

A building block is the properties of the solutions of the initial value problem

$$\begin{cases} dV_t = \Delta V_t dt + \sum_{k \in \mathbb{Z}^d} A(x - k)dB_t^k & \text{in } \mathbb{R}^d \times (0, +\infty), \\ V_0 = 0 & \text{in } \mathbb{R}^d, \end{cases} \tag{2.2}$$

since the solution of (2.1) is going to be obtained as the limit of solutions of

$$\begin{cases} dV_t^n(x) = \Delta V_t^n(x)dt + \sum_{k \in \mathbb{Z}^d} A(x - k)dB_t^k & \text{in } \mathbb{R}^d \times (-n^2, \infty), \\ V_{-n^2}^n = 0 & \text{in } \mathbb{R}^d. \end{cases} \tag{2.3}$$

It is immediate that V^n satisfy bounds similar to the ones of the solution of (2.2).

We divide the presentation into a a number of subsections. In subsection 2.1 we introduce the assumptions we need to study the problem and state the result. In subsection 2.2 we prove a number of basic estimates for the solution of (2.2). These estimates are not sufficiently strong in order to let $n \to \infty$ in (2.3). In subsection 2.3 we obtain some new stronger estimates taking advantage of the independence at large distances of the Brownian motions. The proof of Theorem 2.1 is presented in subsection 2.4.

2.1. **The assumptions and result.** We assume that

$$\begin{cases} \text{The family } (B_k)_{k \in \mathbb{Z}} \text{ consists of continuous and independent } d-\text{dimensional} \\ \text{processes defined on the probability space } (\Omega_0, \mathcal{F}_0, \mathbb{P}_0) \text{ with} \\ \Omega_0 = (C(\mathbb{R}; \mathbb{R}^d))^{\mathbb{Z}^d} \text{ such that, for any } t_0 \in \mathbb{R}, (B_t^k - B_{t_0}^k)_{t \geq t_0} \\ \text{is a Brownian motion,} \end{cases} \tag{2.4}$$

and

$$\begin{cases} \text{the map } A : \mathbb{R}^d \to \mathbb{R}^d \text{ is smooth and has compact support} \\ \text{in the ball } B_{R_0} \text{ for some } R_0 > 0. \end{cases} \tag{2.5}$$

The assumptions on A are made for simplicity and can be relaxed. Moreover, since the coefficients of the noise in (2.1) are deterministic, the question of whether we need to use Itô's or Stratonovich stochastic differential does not arise here.

In the context of (2.1), a process is stationary, if it is adapted to the filtration generated by the $(B^k)_{k \in \mathbb{Z}^d}$ with a law which is invariant by translation in time and integer translation in space.

The existence of a unique up to constants stationary solution of (2.1) is the subject of the next theorem. In what follows by solution we mean a map $Z : \mathbb{R}^d \times \mathbb{R} \times \Omega_0 \to \mathbb{R}$ such that, for any $x \in \mathbb{R}^d$, $s, t \in \mathbb{R}$ with $s < t$ and $\mathbb{P}-$ a.s. $\omega_0 \in \Omega_0$,

$$Z_t(x, \omega_0) = Z_s(x, \omega_0) + \sum_{k \in \mathbb{Z}^d} \int_0^t \int_{\mathbb{R}^d} p(x - y, t - r) A(y - k) dy dB_r^k(\omega_0), \tag{2.6}$$

where $p = p(x, t)$ the heat kernel, that is, the fundamental solution to the heat equation in $\mathbb{R}^d \times (0, \infty)$.

Theorem 2.1. *Assume (2.4) and (2.5). There exists a unique process $Z : \Omega_0 \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ with*

$$\mathbb{E}\left[\int_{\widetilde{Q}_1} |Z_t(x)|^2 dx dt\right] < \infty,$$

satisfying, for any $i = 1, \ldots, d$,

$$dZ_{i,t}(x) = \Delta Z_{i,t}(x) dt + \sum_{k \in \mathbb{Z}^d} D_{x_i} A(x - k) dB_t^k.$$

In addition, Z is an attractor for (2.1) in the sense that, if V is a solution of (2.1) in $\mathbb{R}^d \times (0, \infty)$ such that $V(\cdot, 0) = 0$, then

$$\lim_{t \to +\infty} \mathbb{E}\left[\int_{Q_1} |DV_t(x) - Z_t(x)|^2 dx\right] = 0. \tag{2.7}$$

Moreover, for $d \geq 3$, there exists a unique up to constants space-time stationary adapted process $V : \Omega \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ solving (2.1) in $\mathbb{R}^d \times \mathbb{R}$ such that

$$\mathbb{E}\left[\iint_{\widetilde{Q}_1} |V_t(x)|^2 dxdt\right] < \infty.$$

We remark that, when $d \leq 2$, the correctors have stationary gradients but are not themselves stationary.

2.2. **Auxiliary results.** We concentrate here on the properties of the solutions of the auxiliary initial value problem.

The first result is about a representation formula for the solution of (2.2) as well as preliminary integral bounds on its derivatives.

Note that the forcing term in (2.2) is periodic only in law and not pointwise. Hence, all the estimates need involve expectation.

Lemma 2.2. *Assume (2.4) and (2.5). Then*

$$V_t(x) = \sum_{k\in\mathbb{Z}^d} \int_0^t \int_{\mathbb{R}^d} p(x-y, t-s)A(y-k)dydB_s^k \tag{2.8}$$

is a stationary in space with respect to integer translations solutions, solution V of (2.2). Moreover, for all $t \geq 0$,

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}[|DV_t(x)|^2] + \mathbb{E}[|D^2V_t(x)|^2] \lesssim_{A,d} (t \wedge 1), \tag{2.9}$$

and

$$\mathbb{E}[|V_t(x)|^2] \lesssim_{A,d} t \wedge \begin{cases} 1 & \text{if } d \geq 3, \\ \log(t+1) & \text{if } d = 2, \\ t^{1/2} & \text{if } d = 1. \end{cases} \tag{2.10}$$

Proof. It is immediate that the V given in (2.8) satisfies (2.6) for any $0 < s < t$ and, hence, is a solution of (2.2). It also follows from (2.8) and the fact that the B^k's are identically distributed that V is stationary in space under integer translations. Hence, we only need to prove the estimates for $x \in Q_1$.

Itô's isometry and (2.8) yield

$$\mathbb{E}\left[|DV_t(x)|^2\right] = \sum_{k\in\mathbb{Z}^d} \int_0^t \left|\int_{\mathbb{R}^d} Dp(x-y, t-s)A(y-k)dy\right|^2 ds.$$

For $k \in \mathbb{Z}$ and $s \geq 0$, let

$$F_k(s) = \left|\int_{\mathbb{R}^d} Dp(x-y, t-s)A(y-k)dy\right| = \left|\int_{\mathbb{R}^d} p(x-y, t-s)DA(y-k)dy\right|. \tag{2.11}$$

To proceed we need the following lemma. Its proof is presented after the end of the ongoing one.

Lemma 2.3. *Assume* (2.4) *and* (2.5) *and, for* $k \in \mathbb{Z}$ *and* $s \geq 0$, *let* $F_k(s)$ *be given by* (2.11). *Then*

$$\sum_{k \in \mathbb{Z}^d} F_k(s)^2 \lesssim_{A,d} (t-s)^{-(1+d/2)}, \tag{2.12}$$

$$\sum_{k \in \mathbb{Z}^d, |k| \geq 2(R_0+2)} F_k(s)^2 \lesssim_{A,d} (t-s)^{-1-d/2} \exp(-4R_0^2/(17(t-s))), \tag{2.13}$$

$$F_k(s)^2 \leq \|DA\|_\infty^2. \tag{2.14}$$

We continue with the proof of Lemma 2.2.

The arguments depend on whether $t \geq 1$ or $t < 1$.

If $t \geq 1$, we observe that there are only finitely many k with $|k| < 2(R_0 + 2)$ and we find, using Lemma 2.3, that

$$\mathbb{E}\left[|DV_t(x)|^2\right]$$

$$\leq \sum_{k \in \mathbb{Z}^d} \int_0^{t-1} F_k(s)^2 ds + \sum_{k \in \mathbb{Z}^d, |k| \geq 2(R_0+2)} \int_{t-1}^t F_k(s)^2 ds + \sum_{k \in \mathbb{Z}^d, |k| < 2(R_0+2)} \int_{t-1}^t F_k(s)^2 ds$$

$$\lesssim_{A,d} \left(\int_0^{t-1} (t-s)^{-(1+d/2)} ds + \int_{t-1}^t (t-s)^{-1-d/2} \exp\{-4R_0^2/(17(t-s))\} ds \right)$$

$$+ \sum_{k \in \mathbb{Z}^d, |k| < 2(R_0+2)} \int_{t-1}^t \|DA\|_\infty^2 ds$$

$$\lesssim_{A,d} \left(1 + \int_0^1 s^{-1-d/2} \exp\{-4R_0^2/(17s)\} ds \right) \lesssim_{A,d} 1.$$

If $t \in (0, 1]$, using (2.13) and (2.14), we obtain

$$\mathbb{E}\left[|DV_t(x)|^2\right] \leq \sum_{k \in \mathbb{Z}^d, |k| \geq 2(R_0+2)} \int_0^t F_k(s)^2 ds + \sum_{k \in \mathbb{Z}^d, |k| < 2(R_0+2)} \int_0^t F_k(s)^2 ds$$

$$\lesssim_{A,d} \left(\int_0^t (t-s)^{-1-d/2} \exp\{-4R_0^2/(17(t-s))\} ds + t \right)$$

$$\lesssim_{A,d} \left(\int_0^t s^{-1-d/2} \exp\{-4R_0^2/(17s)\} ds + t \right) \lesssim_{A,d} t.$$

Since the structure of the formula for D^2V is exactly the same as the one for DV, (2.9) is proved similary. The only difference is that now the constants depend on $\|A\|_{C^2}$ too.

To estimate $V_t(x)$, recalling that, for any x, s, t, $\int_{\mathbb{R}^d} p(x - y, t - s) dy = 1$, we find

$$\mathbb{E}\left[|V_t(x)|^2\right] \leq \|A\|_\infty^2 \sum_{k \in \mathbb{Z}^d} \int_0^t \left| \int_{B_{R_0+2}(k)} p(x - y, t - s) dy \right|^2 ds$$

$$\lesssim_{A,d} \left(\sum_{k \in \mathbb{Z}^d} \int_0^{t-1} \int_{B_{R_0+2}(k)} p^2(x - y, t - s) dy ds \right.$$

$$+ \int_{(t-1)\vee 0}^t \sum_{|k| \geq R_0+3} \int_{B_{R_0+2}(k)} p^2(x - y, t - s) dy \, ds + (t - (t - 1) \vee 0) \Big).$$

The first term in the right-hand side can be estimated by

$$\int_0^{(t-1)\vee 0} \sum_{k \in \mathbb{Z}^d} \int_{B_{R_0+2}(k)} p^2(x - y, t - s) dy ds \lesssim_{R_0} \int_0^{(t-1)\vee 0} \int_{\mathbb{R}^d} p^2(x - y, t - s) dy ds$$

$$\lesssim_{R_0} \int_0^{(t-1)\vee 0} (t - s)^{-d/2} ds \lesssim_{R_0,d} \mathbf{1}_{t \geq 1} \begin{cases} 1 & \text{if } d \geq 3, \\ \log(t+1) & \text{if } d = 2, \\ t^{1/2} & \text{if } d = 1. \end{cases}$$

As for second term in the right-hand side, we have

$$\int_{(t-1)\vee 0}^t \sum_{|k| \geq R_0+3} \int_{B_{R_0+2}(k)} p^2(x - y, t - s) dy \, ds$$

$$\lesssim_{R_0} \int_{(t-1)\vee 0}^t \int_{B_1^c} p^2(x - y, t - s) dy \, ds$$

$$\lesssim_{R_0} \int_{(t-1)\vee 0}^t (t - s)^{-d} \int_1^{+\infty} r^{d-1} \exp\{-r^2/(t - s)\} dr ds$$

$$\lesssim_{R_0,d} \int_{(t-1)\vee 0}^t (t - s)^{1-d} \exp\{-1/(2(t - s))\} ds \lesssim_{R_0,d} (t \wedge 1).$$

The proof of (2.10) is now complete.

$\square$

We present now the proof of Lemma 2.3.

Proof of Lemma 2.3. It follows from (2.5) and the fact that $x \in Q_1 \subset B_2$ that

$$F_k(s) \leq \int_{\mathbb{R}^d} |Dp(x - y, t - s) A(y - k)| \, dy \leq \|A\|_\infty \int_{B_{R_0+2}} |Dp(k - y, t - s)| \, dy$$

$$\lesssim_A (t - s)^{-1-d/2} \int_{B_{R_0+2}} |k - y| \exp\{-|k - y|^2/(2(t - s))\} dy.$$

If $|k| \geq 2(R_0 + 2)$, then, for any $y \in B_{R_0+2}$, we have

$$|k - y| \exp\{-|k - y|^2/(2(t - s))\}$$

$$\leq (|k| + (R_0 + 2)) \exp\{-|k|^2/(4(t - s)) + (R_0 + 2)^2/(2(t - s))\}$$

$$\leq (|k| + (R_0 + 2)) \exp\{-|k|^2/(8(t - s))\}$$

$$\lesssim_A |k| \exp\{-|k|^2/(16(t - s))\}.$$

Thus,

$$F_k(s) \lesssim_{A,d} \begin{cases} (t-s)^{-1-d/2}|k|\exp\{-|k|^2/(16(t-s))\} & \text{if } |k| \geq 2(R_0+2), \\ (t-s)^{-1-d/2} \text{ if } |k| \leq 2(R_0+2). \end{cases}$$

Then (2.12) follows, since

$$\sum_{k\in\mathbb{Z}^d} F_k(s)^2 \lesssim_{A,d} (t-s)^{-2-d}(1+\sum_{k\in\mathbb{Z}^d}|k|^2\exp\{-|k|^2/(8(t-s))\})$$

$$\lesssim_{A,d} (t-s)^{-2-d}(1+\int_{\mathbb{R}^d}|z|^2\exp\{-|z|^2/(16(t-s))\}dz) \lesssim_{A,d} (t-s)^{-(1+d/2)}.$$

For (2.13), using that, for all $r \geq 0$, $r^{d+1}\exp\{-r^2/16\} \lesssim r\exp\{-r^2/17\}$, we get

$$\sum_{k\in\mathbb{Z}^d, |k|\geq 2(R_0+2)} F_k(s)^2 \lesssim_{A,d} (t-s)^{-2-d} \sum_{k\in\mathbb{Z}^d, |k|\geq 2(R_0+2)} |k|^2\exp\{-|k|^2/(8(t-s))\}$$

$$\lesssim_{A,d} (t-s)^{-2-d}\int_{B_{2R_0}^c}|z|^2\exp\{-|z|^2/(16(t-s))\}dz,$$

$$\lesssim_{A,d} (t-s)^{-1-d/2}\int_{2R_0(t-s)^{-1/2}}^{+\infty} r^{d+1}\exp\{-r^2/16\}dr,$$

$$\lesssim_{A,d} (t-s)^{-1-d/2}\exp(-4R_0^2/(17(t-s))).$$

Finally, (2.14) is straightforward, since

$$F_k(s) \leq \|DA\|_\infty \int_{\mathbb{R}^d} p(x-y, t-s)dy = \|DA\|_\infty.$$

$\square$

2.3. **The decorrelation estimates.** We show that the solution V of (2.2) decorrelates in space.

To quantify this property, we consider solutions of a localized versions of (2.2), that is, problems that depend only on the Brownian motions in a certain ball.

For $l \in \mathbb{Z}$ and $R \geq 1$, let $V^{l,R}$ be the solution to

$$\begin{cases} dV_t^{l,R} = \Delta V_t^{l,R}dt + \sum_{k\in\mathbb{Z}^d, \, |k-l|\leq R} A(x-k)dB_t^k & \text{in } \mathbb{R}^d \times (0, +\infty), \\ V_0^{l,R} = 0 & \text{in } \mathbb{R}^d. \end{cases} \tag{2.15}$$

Lemma 2.4. *Assume* (2.4) *and* (2.5) *and let V be the solution to* (2.2). *Then there exists $R_1 > 0$ such that, for any $R \geq R_1$, $l \in \mathbb{Z}^d$ and $x \in Q_1(l)$,*

$$\mathbb{E}\left[|DV_t(x) - DV_t^{l,R}(x)|^2\right] \lesssim_{A,d} \begin{cases} R^{-d} & \text{if } R^2/t \leq 1, \\ \exp\{-R^2/(5t)\} & \text{otherwise,} \end{cases} \tag{2.16}$$

and

$$\sup_{t\geq 0}\mathbb{E}\left[|DV_t^{l,R}(x)|^2\right] \lesssim_{A,d} 1. \tag{2.17}$$

If $d \geq 3$, then, for all $R \geq R_1$,

$$\mathbb{E}\left[\left(V_t(x) - V_t^{l,R}(x)\right)^2\right] \lesssim_{A,d} \begin{cases} R^{2-d} & \text{if } R^2/t \leq 1, \\ \exp\{-R^2/(9t)\} & \text{otherwise}, \end{cases} \tag{2.18}$$

and

$$\sup_{t \geq 0} \mathbb{E}\left[(V_t^{l,R}(x))^2\right] \lesssim 1. \tag{2.19}$$

For later use, we note that $V_t^{R,l}(x)$ and $V_{t'}^{R,l'}(x')$ are independent, for any t, t' and x, x', as soon as $|l - l'| > 2R$. For this reason, we consider Lemma 2.4 as a decorrelation property of V.

Proof. Using the representation formulae of DV_t and $DV_t^{l,R}$, we find

$$D(V_t - V^{l,R})(x) = \sum_{|k-l|>R} \int_0^t \int_{\mathbb{R}^d} Dp(x-y, t-s)A(y+k)dydB_s^k.$$

Then (2.5), Itô's isometry and Cauchy-Schwartz inequality yield

$$\mathbb{E}\left[|D(V_t - V^{l,R})(x)|^2\right] \leq \sum_{|k-l|>R} \int_0^t \left|\int_{\mathbb{R}^d} Dp(x-y, t-s)A(y+k)dy\right|^2 ds$$

$$\lesssim_A \sum_{|k-l|>R} \int_0^t \left|\int_{B_{R_0}(k)} (t-s)^{-1-d/2}|x-y|\exp\{-|x-y|^2/(2(t-s))\}dy\right|^2 ds$$

$$\lesssim_A \sum_{|k-l|>R} \int_0^t \int_{B_{R_0}(k)} (t-s)^{-2-d}|x-y|^2 \exp\{-|x-y|^2/(t-s)\}dyds.$$

Therefore, for $x \in Q_1(l)$, we get

$$\mathbb{E}\left[|D(V_t - V^{l,R})(x)|^2\right] \lesssim_A \int_0^t \int_{B^c_{((R-R_0)-1)}(0)} (t-s)^{-2-d}|y|^2 \exp\{-|y|^2/(2(t-s))\}dyds$$

$$\lesssim_A \int_0^t \int_{((R-R_0)-1)(t-s)^{-1/2}}^{+\infty} (t-s)^{-1-d/2}\rho^{d+1}\exp\{-\rho^2/2\}d\rho ds.$$

Choosing R large enough, we can assume that $(R - R_0 - 1) \geq R/2$, and using that, for $\rho \geq 0$, $\rho^{d+1}\exp\{-\rho^2/2\} \lesssim \rho\exp\{-\rho^2/4\}$, integrating in space and an elementary change of variables we find

$$\mathbb{E}\left[|D(V_t - V^{l,R})(x)|^2\right] \lesssim_A \int_0^t (t-s)^{-1-d/2}\exp\{-R^2/(4(t-s))\}ds$$

$$\lesssim_A R^{-d} \int_{R^2/t}^{+\infty} \tau^{-1+d/2}\exp\{-\tau/4\}d\tau,$$

and, hence, (2.16).

The proof of (2.17) is then follows using (2.16) combined and Lemma 2.2.

Next we assume that $d \geq 3$. Then

$$(V_t - V^{l,R})(x) = \sum_{|k-l|>R} \int_0^t \int_{\mathbb{R}^d} p(x-y,t-s)A(y+k)dydB_s^k,$$

and, again, (2.5), Itô's isometry and an application of the Cauchy-Schwartz inequality imply that

$$\mathbb{E}\left[((V_t - V^{l,R})(x))^2\right] \leq \sum_{|k-l|>R} \int_0^t \left|\int_{B_{R_0+2}(k)} p(x-y,t-s)A(y+k)dy\right|^2 ds$$

$$\lesssim_A \sum_{|k-l|>R} \int_0^t \int_{B_{R_0+2}(k)} (t-s)^{-d}\exp\{-|x-y|^2/(t-s)\}dyds.$$

Therefore, if $x \in Q_1(l)$,

$$\mathbb{E}\left[((V_t - V^{l,R})(x))^2\right] \lesssim_A \int_0^t \int_{B^c_{((R-R_0)-1)}(0)} (t-s)^{-d}\exp\{-|y|^2/(t-s)\}dyds$$

$$\lesssim_A \int_0^t \int_{(R-R_0-3)(t-s)^{-1/2}}^{+\infty} (t-s)^{-d/2}\rho^{d-1}\exp\{-\rho^2\}d\rho ds.$$

Assuming that R is large so that $(R-R_0)-1 \geq R/2$, using that, $\rho \geq 0$, $\rho^{d-1}\exp\{-\rho^2\} \lesssim \rho\exp\{-\rho^2/2\}$ and integrating in space, we get

$$\mathbb{E}\left[((V_t - V^{l,R})(x))^2\right] \leq C\int_0^t (t-s)^{-d/2}\exp\{-R^2/(8(t-s))\}ds$$

$$\leq CR^{2-d}\int_{R^2/t}^{+\infty} \tau^{-2+d/2}\exp\{-\tau/8\}d\tau.$$

Then (2.18) follows easily and the proof of (2.19) is then an application of (2.18) combined with Lemma 2.2.

□

2.4. **The proof of Theorem 2.1.** To prove the existence of a stationary solution of (1.4), we consider the sequence of solutions V^n of (2.3).

The main step is to show that $(DV^n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Lemma 2.5. *Assume (2.4) and (2.5). Then, for any $r > 0$ and any $T > 0$, the sequence $(DV^n)_{n\in\mathbb{N}}$ is Cauchy in $L^2(B_r \times [-T,T] \times \Omega)$, that is, for any $n, m \in \mathbb{N}$ and $t \in \mathbb{R}$ with $m > n$ and $t \in [-n^2+1, m-1]$,*

$$\mathbb{E}\left[|D(V_t^m - V_t^n)(x)|^2\right] \lesssim_{A,d} C(t+n^2)^{-(1\wedge(d/4))}.$$

Proof. Fix $n < m$ and $t \in [-n^2, m-1]$. Since $V^m - V^n$ solves the heat equation on $[-n^2, t]$ with initial condition $V_{-n^2}^m$, we have

$$V_t^m(x) - V_t^n(x) = \int_{\mathbb{R}^d} p(x-y,t+n^2)V_{-n^2}^m(y)dy.$$

Hence,

$$\mathbb{E}\left[|D(V_t^m - V_t^n)(x)|^2\right] = \int_{\mathbb{R}^{2d}} p(x-y, t+n^2) p(x-y', t+n^2) \mathbb{E}\left[DV_{-n^2}^m(y) \cdot DV_{-n^2}^m(y')\right] dy dy'$$

$$= \sum_{k,k' \in \mathbb{Z}^d} \int_{Q_1(k) \times Q_1(k')} p(x-y, t+n^2) p(x-y', t+n^2) \mathbb{E}\left[DV_{-n^2}^m(y) \cdot DV_{-n^2}^m(y')\right] dy dy'.$$

Fix $R = \lfloor (t+1+n^2)^{1/4} \rfloor$, and consider, for $l \in \mathbb{Z}^d$, the solution $V^{m,l,R}$ of

$$\begin{cases} dV_t^{m,l,R} = \Delta V_t^{m,l,R} dt + \displaystyle\sum_{k \in \mathbb{Z}^d, \; |k-l| \leq R} A(x-k) dB_t^k & \text{in } \mathbb{R}^d \times (-m^2, +\infty) \\[2mm] V_{-m^2}^{m,l,R} \equiv 0. \end{cases}$$

For any $y \in Q_1(l)$, Lemma 2.4 gives

$$\mathbb{E}\left[|DV_s^m(y) - DV_s^{m,l,R}(y)|^2\right] \lesssim_{A,d} \begin{cases} R^{-d} & \text{if } R^2/(s+m^2) \leq 1, \\ \exp\{-R^2/(5(s+m^2))\} & \text{otherwise.} \end{cases} \tag{2.20}$$

We replace $DV_{-n^2}^m(y)$ by $DV_{-n^2}^{m,k,R}(y)$ in $Q_1(k)$. In order to apply (2.20), we note that the assumption on n, m and t, the choice of R, and the facts that $m-n \geq 1$ and that $t+1 \leq m$ imply that $m^2 - n^2 \geq m+n \geq (t+1+n^2)^{1/2} \geq R^2$.

Hence, $R^2/(m^2-n^2) \leq 1$ and, in view of (2.20), we have

$$\mathbb{E}\left[|D(V_t^m - V_t^n)(x)|^2\right] \lesssim_{A,d} \left(\sum_{k,k' \in \mathbb{Z}^d} A_{k,k'} + B_{k,k'} \right), \tag{2.21}$$

where

$$A_{k,k'} = \int_{Q_1(k) \times Q_1(k')} p(x-y, t+n^2) p(x-y', t+n^2) \mathbb{E}\left[DV_{-n^2}^{m,k,R}(y) \cdot DV_{-n^2}^{m,k',R}(y')\right] dy dy'$$

and

$$B_{k,k'} = \int_{Q_1(k) \times Q_1(k')} p(x-y, t+n^2) p(x-y', t+n^2)$$
$$R^{-d/2} \left(\mathbb{E}^{1/2}\left[|DV_{-n^2}^{m,k,R}(y')|^2\right] + \mathbb{E}^{1/2}\left[|DV_{-n^2}^m(y)|^2\right] \right) dy dy'.$$

Using (2.9) and (2.17) we find

$$\sum_{k,k' \in \mathbb{Z}^d} B_{k,k'} \lesssim_{A,d} R^{-d/2} \sum_{k,k' \in \mathbb{Z}^d} \int_{Q_1(k) \times Q_1(k')} p(x-y, t+n^2) p(x-y', t+n^2) dy dy' \underset{A}{\sim} R^{-d/2}.$$

To estimate $A_{k,k'}$ note that, if $|k-k'| \geq 2R+2$, then $DV^{m,k,R}$ and $DV^{m,k',R}$ are independent, and, hence,

$$A_{k,k'} = \int_{Q_1(k) \times Q_1(k')} p(x-y, t+n^2) p(x-y', t+n^2) \mathbb{E}\left[DV_{-n^2}^{m,k,R}(y)\right] \mathbb{E}\left[DV_{-n^2}^{m,k',R}(y')\right] dy dy'.$$

Recall that, since V^m is stationary in space, we have

$$\mathbb{E}\int_{Q_1(k)} DV_{-n^2}^m(y) dy = \mathbb{E}\int_{Q_1(k')} DV_{-n^2}^m(y) dy = 0. \tag{2.22}$$

To make use of this property, we replace $A_{k,k'}$ by $A'_{k,k'}$ given by

$$A'_{k,k'} = \int_{Q_1(k) \times Q_1(k')} p(x - y, t + n^2) p(x - y', t + n^2) \mathbb{E}\left[DV^m_{-n^2}(y)\right] \mathbb{E}\left[DV^m_{-n^2}(y')\right] dy dy',$$

and we note that, with an argument similar to the one above, we have

$$\sum_{|k-k'| \geq 2R+2} A_{k,k'} - \sum_{|k-k'| \geq 2R+2} A'_{k,k'} \lesssim_{A,d} R^{-d/2}.$$

Next we replace $p(x - y, t + n^2)$ by $p(x - k, t + n^2)$ and $p(x - y, t + n^2)$ by $p(x - k', t + n^2)$ in $A'_{k,k'}$, noting that

$$\max\{|p(x - y, t + n^2) - p(x - k, t + n^2)|, \ |p(x - y', t + n^2) - p(x - k', t + n^2)|\}$$

$$\lesssim (t + n^2)^{-1-d/2} \exp\{-|x - k|^2/(4(t + n^2))\}.$$

Since (2.22) and Lemma 2.2 to control the remaining terms, we obtain

$$A'_{k,k'} \lesssim_{A,d} C(t + n^2)^{-1-d} \exp\{-(|x - k|^2 + |x - k'|^2)/(4(t + n^2))\},$$

Summing the terms $A'_{k,k'}$ with $|k - k'| \geq 2R + 2$, we find

$$\sum_{|k-k'| \geq 2R+2} A'_{k,k'} \lesssim_{A,d} (t + n^2)^{-1-d} \left(\sum_{k \in \mathbb{Z}^d} \exp\{-|x - k|^2/(4(t + n^2))\} \right)^2 \lesssim_{A,d} (t + n^2)^{-1}.$$

On the other hand, if $|k - k'| \leq 2R + 2$, then, (2.17) yields

$$A_{k,k'} \lesssim_{A,d} \int_{Q_1(k) \times Q_1(k')} p(x - y, t + n^2) p(x - y', t + n^2),$$

and, hence,

$$\sum_{|k-k'| \leq 2R+2} A_{k,k'} \lesssim_{A,d} \int_{|y-y'| \leq 2R+4} p(x - y, t + n^2) p(x - y', t + n^2)$$

$$\lesssim_{A,d} (t + n^2)^{-d} \int_{|y-y'| \leq 2R+4} \exp\{-(|x - y|^2 + |x - y'|^2)/(2(t + n^2))\}$$

$$\lesssim_{A,d} \frac{R^d}{(t + n^2)^{d/2}}$$

It follows that

$$\mathbb{E}\left[|D(V^m_t - V^n_t)(x)|^2\right] \lesssim_{A,d} \left(R^{-d/2} + (t + n^2)^{-1} + \frac{R^d}{(t + n^2)^{d/2}} \right),$$

which completes the proof since $R = [(t + 1 + n^2)^{1/4}]$.

$\square$

We have now all the ingredients needed to prove the main result.

Proof of Theorem 2.1. In view of Lemma 2.5, the sequence $(DV^n)_{n \in \mathbb{N}}$ converges along subsequences in $L^2(\Omega, L^2_{loc}(\mathbb{R}^d \times \mathbb{R}))$ to some Z, which is stationary in space, and solves

$$dZ_t(x) = \Delta Z_t(x) dt + \sum_{k \in \mathbb{Z}^d} DA(x - k) dB^k_t \quad \text{in} \ \ \mathbb{R}^d \times \mathbb{R}, \tag{2.23}$$

and, thus, is continuous in time and smooth in space.

Moreover, in view of the bound on DV^n in Lemma 2.2, for any $x \in \mathbb{R}^d$, we have

$$\sup_{t\in\mathbb{R}} \mathbb{E}\left[|Z_t(x)|^2\right] + \sup_{t\in\mathbb{R}} \mathbb{E}\left[|DZ_t(x)|^2\right] \lesssim_{A,d} 1.$$

Fix $t_0 \in \mathbb{R}$, let V_{t_0} be the smooth antiderivative of Z_{t_0} with, for definiteness, $V_{t_0}(0) = 0$, which exists since Z is the limit of gradients, and V the solution of (2.1) in $\mathbb{R}^d \times [t_0, +\infty)$ with initial condition V_{t_0}. It is immediate that $DV = Z$.

Next we prove that Z is the unique process satisfying (2.23) which is stationary in space and satisfies the bounds $\sup_t \mathbb{E}[|Z_t|^2] < +\infty$.

Let Z' be another such process. Then, for any $i \in \{1, \ldots, d\}$, $u(x,t) = (Z - Z')_i$ is an entire solution of the heat equation. It follows from a classical estimate on the heat equation (see, for example, [7]) that there exists $C > 0$ such that, for any $r \in \mathbb{R}$,

$$\max_{(y,s)\in Q_{r/2}\times[t-r^2/4,t]} |Du(y,s)| \leq \frac{C}{r^{d+3}} \int_{t-r^2}^t \int_{Q_r(x)} |u(y,s)| dy ds,$$

and, hence,

$$\max_{y\in Q_1(0)} |Du(y,t)|^2 \leq \frac{C}{r^{d+4}} \int_{t-r^2}^t \int_{Q_r(x)} |u(y,s)|^2 dy ds.$$

Taking expectation and using the stationarity of u and the L^2 bound, we find

$$\mathbb{E}\left[\max_{y\in Q_1(0)} |Du(y,t)|^2\right] \leq \frac{C}{r^2}.$$

This proves that $Z_i(\cdot,t) \equiv Z'_i(\cdot,t)$ for a fixed t, and, since $\mathbb{E}[\int_{Q_1} Z(x,t)dx] = \mathbb{E}[\int_{Q_1} Z'(x,t)dx] = 0$, it follows that $Z \equiv Z'$.

Finally, we prove that Z is stationary in time. For this, we note that the map $t \to Z_t$ is measurable with respect to the $\sigma-$algebra generated by $(B^i_{s\wedge t})_{s\leq t}$ because this is the case for the maps $t \to DV_t^n$. Therefore, there exists a measurable $\mathcal{Z}$ such that $Z_t(x) = \mathcal{Z}(t, x, (B^i_{\cdot\wedge t})_{i\in\mathbb{Z}^d})$.

Next we note that, for any $s \in \mathbb{R}$, $Z_{\cdot+s}(\cdot)$ solves the same equation as Z with Brownian motions shifted in time by s. Hence, by the uniqueness of the solution, $Z_{t+s}(\cdot) = \mathcal{Z}(t, x, (B^i_{(\cdot+s)\wedge t})_{i\in\mathbb{Z}^d})$, which has the same law as $\mathcal{Z}(t, x, (B^i_{\cdot\wedge t})_{i\in\mathbb{Z}^d})$. It follows that the law of Z. is the same as the law of $Z_{\cdot+s}$, thus, Z is the stationary in time.

The attractor property of Z, that is, (2.7), is a straightforward consequence of Lemma 2.5. Indeed, choose $n = 0$, $t > 0$ and m larger than $t + 1$ in the lemma. Then

$$\mathbb{E}\left[|D(V_t^m - V_t)(x)|^2\right] \lesssim_{A,d} t^{-(1\wedge(d/4))}.$$

Letting $m \to +\infty$, the construction of Z, gives

$$\mathbb{E}\left[|Z_t(x) - DV_t(x)|^2\right] \lesssim_{A,d} t^{-(1\wedge(d/4))}.$$

Integrating over Q_1 yields (2.7).

$$\square$$

3. Random homogenization of uniformly elliptic nonlinear parabolic equation in divergence form

In this section we investigate the random homogenization of

$$\partial_t u^\varepsilon - \mathrm{div}(a(Du^\varepsilon, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega)) = f \ \text{ in } \ \mathbb{R}^d \times (0, T) \quad u^\varepsilon(\cdot, 0) = u_0. \tag{3.1}$$

We start with the description of the environment in subsection 3.1 and state the assumptions on the vector field in subsection 3.2. Subsection 3.3 is about the existence of the corrector and introduces the effective vector field. The homogenization result is developed in subsection 3.4.

3.1. **Description of the environment.** We fix an ergodic environment probability, that is, assume that

$$\begin{cases} (\Omega, \mathcal{F}, \mathbb{P}) \text{ is a probability space endowed with an ergodic semigroup} \\ \tau : \mathbb{Z}^d \times \mathbb{R} \times \Omega \to \Omega \ \text{ of measure preserving maps,} \end{cases} \tag{3.2}$$

and we denote by $\mathbf{L^2}$ the set of stationary maps $u = u(x, t, \omega)$ meaning

$$u(x + k, t + s, \omega) = u(x, t, \tau_{(k,s)}\omega) \ \text{ for all } \ (k, s, \omega) \in \mathbb{Z}^d \times \mathbb{R} \times \Omega, \tag{3.3}$$

and such that

$$\|u\|_{\mathbf{L^2}} = \mathbb{E}\left[\int_{\widetilde{Q}_1} u^2\right] < +\infty. \tag{3.4}$$

Note that, if $u \in \mathbf{L^2}$, the stationarity in time implies that the quantity

$$\mathbb{E}\left[\int_{t_1}^{t_2} \int_O u(x, s)dxds\right],$$

where O is a bounded measurable subset of $\mathbb{R}^d$, is affine in $t_2 - t_1$, and, therefore, the limit

$$\mathbb{E}\left[\int_O u(x, t)dx\right] = \lim_{h \to 0^+} \mathbb{E}\left[\frac{1}{2h} \int_O \int_{t-h}^{t+h} u(x, s)dxds\right]$$

exists for any $t \in \mathbb{R}$ and is independent of t.

Let $\mathcal{C}$ be the subset of $\mathbf{L^2}$ of maps with smooth and square integrable space and time derivatives of all order belonging to $\mathbf{L^2}$. A simple argument using mollification in $\mathbb{R}^{d+1}$ yields that $\mathcal{C}$ is dense in $\mathbf{L^2}$ with respect to the norm in (3.4).

We denote by $\mathbf{H^1}$ the closure of $\mathcal{C}$ with respect to the norm

$$\|u\|_{\mathbf{H^1}} = (\|u\|_{\mathbf{L^2}}^2 + \|\partial_t u\|_{\mathbf{L^2}}^2 + \|Du\|_{\mathbf{L^2}}^2)^{1/2},$$

while $\mathbf{H_x^1}$ the closure of $\mathcal{C}$ with respect to the norm

$$\|u\|_{\mathbf{H_x^1}} = (\|u\|_{\mathbf{L^2}}^2 + \|Du\|_{\mathbf{L^2}}^2)^{1/2}$$

and $\mathbf{H_x^{-1}}$ is its dual space.

Moreover, $\mathbf{L_{pot}^2}$ is the closure with respect to the $\mathbf{L^2}$-norm of $\{Du : u \in \mathcal{C}\}$ in $(\mathbf{L^2}(\Omega))^d$.

For later use, we also note that, in view of the stationarity, for all $u, v \in \mathbf{H^1}$ and $i = 1, \ldots, d$,

$$\mathbb{E}\left[\int_{\widetilde{Q}_1} u\partial_{x_i}v\right] = -\mathbb{E}\left[\int_{\widetilde{Q}_1} v\partial_{x_i}u\right],$$

and

$$\mathbb{E}\left[\int_{\widetilde{Q}_1} u\partial_t v\right] = -\mathbb{E}\left[\int_{\widetilde{Q}_1} v\partial_t u\right].$$

Finally, given a nonnegative weight ρ, we write $\mathbf{L}^2_\rho$, $\mathbf{H}^1_\rho$ and $\mathbf{H}^{-1}_\rho$ for the spaces in which the norm is evaluated against ρ.

Finally, we note that, whenever an equation is said to be solved in the sense of distributions, then the pairing is the standard and not the weighted one.

3.2. **The assumptions on the vector field.** We assume that the vector field $a : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}^d$ is

$$\begin{cases} \text{space-time stationary, and,} \\ \text{strongly monotone and Lipschitz continuous uniformly in } x, t \text{ and } \omega. \end{cases} \tag{3.5}$$

Moreover, it is assumed that

$$|a(0,\cdot,\cdot,\cdot)| : \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R} \in \mathbf{L^2}, \tag{3.6}$$

and, hence,

$$\mathbb{E}\left[\int_{\widetilde{Q}_1} |a(0,x,t)|^2 dx dt\right] < +\infty. \tag{3.7}$$

3.3. **The existence of a corrector and the effective nonlinearity.** We prove here the existence, for each $p \in \mathbb{R}^d$, of a corrector, that is a map $\chi^p : \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}$ with $\partial_t \chi^p$ and $D\chi^p$ stationary and of mean 0 and such that

$$\partial_t \chi^p - \text{div}(a(p + D\chi^p, x, t, \omega)) = 0 \ \text{ in } \ \mathbb{R}^d \times \mathbb{R},$$

and use it to define the effective vector field $\bar{a} : \mathbb{R}^d \to \mathbb{R}^d$.

The result is stated next.

Theorem 3.1. *Assume* (3.2), (3.5) *and* (3.6). *For any* $p \in \mathbb{R}^d$, *there exists a unique map* $\chi^p : \mathbb{R}^{d+1} \times \Omega \to \mathbb{R}$ *such that*

$$\int_{\widetilde{Q}_1} \chi^p(x,t,\omega) dx dt = 0 \ \ \mathbb{P}-a.s., \ \ D\chi^p \in \mathbf{L^2_{pot}}, \ \ \partial_t \chi^p \in \mathbf{H^{-1}_x},$$

and

$$\partial_t \chi^p - \text{div}(a(p + D\chi^p, x, t, \omega)) = 0 \ \text{ in } \ \mathbf{H^{-1}_x}. \tag{3.8}$$

Moreover, as $\varepsilon \to 0$ *and* $\mathbb{P}-a.s.$ *and in expectation,*

$$\chi^\varepsilon(x,t;p,\omega) = \varepsilon\chi^p(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega) \to 0 \ \ \text{ in } \ L^2_{loc}(\mathbb{R}^{d+1}).$$

In addition, the vector field $\bar{a} : \mathbb{R}^d \to \mathbb{R}^d$ *defined by*

$$\bar{a}(p) = \mathbb{E}\left[\int_{\widetilde{Q}_1} a(p + D\chi^p, y, \tau, \omega) dy d\tau\right] \tag{3.9}$$

is monotone and Lipschitz continuous.

The proof of Theorem 3.1 is long and technical. At first look, its structure appears to be similar to the ones of the analogous results for periodic and almost periodic media. The standard approach is to consider the solution (approximate corrector) of a regularized version of the corrector equation with small second derivative in time to make the problem uniformly elliptic set in a bounded domain and small discount factor to guarantee the solvability. The next step is to obtain uniformly apriori bounds for the space and time derivatives of the approximate corrector and to pass to the weak limit, which yields an equation involving the weak limit of the time derivative and the divergence of the weak limit of the vector field. Note that, due to the unboundedness of the domain it is necessary to use weighted space, a fact that introduces another layer of approximations and technicalities.

The proof of Theorem 3.1 is organized in a number of lemmata, which provide incremental information leading to the final argument.

Throughout the proof, to justify repeated integration by parts and to deal with the unbounded domain, we use the exponential exponential weight $\widehat{\rho}_\theta$, which, for $\theta > 0$, is given by

$$\widehat{\rho}_\theta(x,t) = \exp\{-\theta(1 + |x|^2 + t^2)^{1/2}\}.$$

The first lemma is about the existence of as well as some apriori bounds for the approximate corrector in a bounded domain.

Lemma 3.2. *Assume (3.2), (3.5) and (3.6). For any $\omega \in \Omega$, $\lambda > 0$ and $L > 0$, let $u_L \in H^1_0(\widetilde{Q}_L)$ be the solution of*

$$\lambda u_L - \lambda \partial_{tt} u_L + \partial_t u_L - \operatorname{div}(a(Du_L + p, \omega)) = 0 \quad in \ \widetilde{Q}_L \quad u_L = 0 \quad in \ \partial \widetilde{Q}_L. \quad (3.10)$$

There exists $\theta_0 > 0$, which depends on λ but not on L or ω, such that, for any $\theta \in (0, \theta_0]$ and $\mathbb{P}-a.s.$,

$$\int_{\widetilde{Q}_L} \left(\lambda u_L^2 + \lambda(\partial_t u_L)^2 + |Du_L|^2\right) \widehat{\rho}_\theta \lesssim_{p,\lambda,\theta} \left(1 + \int_{\widetilde{Q}_L} |a(0)|^2 \widehat{\rho}_\theta\right). \quad (3.11)$$

Note that the integral $\int_{\widetilde{Q}_L} |a(0)|^2 \widehat{\rho}_\theta$ in the right-hand side of (3.11) is random and that the implicit constant does not depend on either ω or L

Proof. Using $\widehat{\rho}_\theta u_L$ as a test function in (3.10), the monotonicity and Lipschitz continuity of a and the fact that $|D\widehat{\rho}_\theta| + |\partial_t \widehat{\rho}_\theta| \lesssim \theta \widehat{\rho}_\theta$, we find

$$\int_{\widetilde{Q}_L} \left(\lambda u_L^2 + \lambda(\partial_t u_L)^2 + C_0^{-1}|Du_L + p|^2\right) \widehat{\rho}_\theta$$

$$\leq -\int_{\widetilde{Q}_L} \left(\lambda \partial_t u_L u_L \frac{\partial_t \widehat{\rho}_\theta}{\widehat{\rho}_\theta} - \frac{(u_L)^2 \partial_t \widehat{\rho}_\theta}{2\widehat{\rho}_\theta} - a(Du_L + p) \cdot p + u_L a(Du_L + p) \cdot \frac{D\widehat{\rho}_\theta}{\widehat{\rho}_\theta}\right) \widehat{\rho}_\theta$$

$$\lesssim \int_{\widetilde{Q}_L} (\lambda \theta |\partial_t u_L||u_L| + \theta u_L^2 + (|a(0)| + |Du_L + p|)(|p| + \theta|u_L|)) \widehat{\rho}_\theta,$$

and, hence, the claim.

$\square$

Next, we use Lemma 3.11 to obtain the existence and bounds for approximate solutions of the approximate regularized problem in all of $\mathbb{R}^{d+1}$.

Lemma 3.3. *Assume* (3.2), (3.5), *and* (3.6). *For any* $p \in \mathbb{R}^d$, $\lambda > 0$ *and* $\theta \in (0, \theta_0)$, *there exists*, $\mathbb{P}-a.s.$ *and in the sense of distributions, a unique stationary solution* $\chi^{\lambda, p} \in \mathbf{H}^1_{\widehat{\rho}_\theta}$ *of*

$$\lambda \chi^{\lambda, p} - \lambda \partial_{tt} \chi^{\lambda, p} + \partial_t \chi^{\lambda, p} - \operatorname{div}(a(D\chi^{\lambda, p} + p, \omega)) = 0 \quad \text{in} \quad \mathbb{R}^{d+1}, \tag{3.12}$$

which is independent of $\theta \in (0, \theta_0)$, *belongs to* $\mathbf{H}^1$ *and, in addition,*

$$\mathbb{E}\left[\int_{\widetilde{Q}_1} \lambda(\chi^{\lambda, p})^2 + \lambda(\partial_t \chi^{\lambda, p})^2 + |D\chi^{\lambda, p}|^2\right] \lesssim_p 1 \tag{3.13}$$

and, for all $\phi \in \mathbf{H}^1$,

$$\mathbb{E}\left[\int_{\widetilde{Q}_1} \partial_t \chi^{\lambda, p} \phi\right] \lesssim_p \lambda^{1/2} \|\phi\|_{\mathbf{H}^1} + \|D\phi\|_{\mathbf{L}^2}, \tag{3.14}$$

both estimates being independent of λ.

Proof. Let u_L be as in Lemma 3.2. The stationarity of a and (3.7) yield

$$\mathbb{E}\left[\int_{\mathbb{R}^{d+1}} |a(0)|^2 \widehat{\rho}_\theta\right] \lesssim_{\theta, d} \mathbb{E}\left[\int_{\widetilde{Q}_1} |a(0)|^2\right] < +\infty,$$

Let $\omega \in \Omega$ be such that

$$\int_{\mathbb{R}^{d+1}} |a(0, x, t, \omega)|^2 \widehat{\rho}_\theta(x, t) dx dt < \infty$$

for a countable sequence of $\theta \to 0$ and, thus, for any $\theta \in (0, 1]$. Cleary, the set of such ω has probability 1.

Fix such ω. It follows from Lemma 3.2 that the family $(u_L)_{L \in (0, \infty)}$ is bounded in $\mathbf{H}^1_{\widehat{\rho}_\theta}$ for any $\theta \in (0, \theta_0]$. A diagonal argument then yields a subsequence, which, to keep the notation simple, is denoted as the family, and some $u \in \bigcap_{\theta' \in (0, \theta_0]} H^1_{\widehat{\rho}_{\theta'}}$, such that, as $L \to \infty$, $u_L \rightharpoonup u$ in $H^1_{\widehat{\rho}_\theta}$ for any $\theta \in (0, \theta_0]$.

In particular, $u_L \to u$ in $L^2(\widetilde{Q}_R)$ for any $R > 0$ and, therefore, in $L^2_{\widehat{\rho}_\theta}$ for all $\theta \in (0, \theta_0)$, since, for any $R > 0$,

$$\|u_L - u\|_{L^2_{\rho_\theta}(\mathbb{R}^{d+1})} \leq \|u_L - u\|_{L^2_{\rho_\theta}(\widetilde{Q}_R)} + \Big(\sup_{\mathbb{R}^{d+1} \setminus \widetilde{Q}_R} \frac{\rho_\theta}{\rho_{\theta_0}}\Big) \|u_L - u\|_{L^2_{\rho_{\theta_0}}(\mathbb{R}^{d+1} \setminus \widetilde{Q}_R)}$$

$$\leq \|u_L - u\|_{L^2(\widetilde{Q}_R)} + \Big(\sup_{\mathbb{R}^{d+1} \setminus \widetilde{Q}_R} \frac{\rho_\theta}{\rho_{\theta_0}}\Big) \big(\|u_L\|_{L^2_{\rho_{\theta_0}}(\mathbb{R}^{d+1})} + \|u\|_{L^2_{\rho_{\theta_0}}(\mathbb{R}^{d+1})}\big).$$

Note that above the first term in the right-hand side tends to 0 as $L \to \infty$ and the second one tends to 0, uniformly in L, as $R \to +\infty$.

We can also assume that, as $L \to \infty$, $a(Du_L + p, \omega) \rightharpoonup \xi \in \bigcap_{\theta' \in (0, \theta_0]} L^2_{\widehat{\rho}_{\theta'}}$.

It follows that, in the sense of distributions,

$$\lambda u - \lambda \partial_{tt} u + \partial_t u - \operatorname{div}(\xi) = 0 \quad \text{in} \quad \mathbb{R}^{d+1}, \tag{3.15}$$

and, for all $\theta \in (0, \theta_0]$,

$$\int_{\mathbb{R}^{d+1}} \big(\lambda u^2 + \lambda(\partial_t u)^2 + |Du|^2\big) \widehat{\rho}_\theta \lesssim_{p, \lambda, \theta)} (1 + \int_{\mathbb{R}^{d+1}} |a(0)|^2 \widehat{\rho}_\theta). \tag{3.16}$$

Next we check that u is a solution of (3.12). In what follows, we use that $u \in \bigcap_{\theta' \in (0, \theta_0]} H^1_{\widehat{\rho}_{\theta'}}$.

Let $\phi \in C_c^\infty(\mathbb{R}^{d+1})$. The strong monotonicity of a gives, for L large enough,

$$\int_{\widetilde{Q}_L} \Big(\lambda(u_L - \phi)^2 + \lambda(\partial_t u_L - \partial_t \phi)^2 + (a(Du_L + p) - a(D\phi + p)) \cdot D(u_L - \phi) \Big) \widehat{\rho}_\theta \geq 0.$$

Moreover, using $u_L \widehat{\rho}_\theta$ as a test function for the equation of u_L, we find

$$\int_{\widetilde{Q}_L} \Big(\lambda u_L^2 + \lambda(\partial_t u_L)^2 + \lambda \partial_t u_L u_L \frac{\partial_t \widehat{\rho}_\theta}{\widehat{\rho}_\theta} + \partial_t u_L u_L + a(Du_L + p) \cdot Du_L$$
$$+ u_L a(Du_L + p) \cdot \frac{D\widehat{\rho}_\theta}{\widehat{\rho}_\theta} \Big) \widehat{\rho}_\theta = 0.$$

Hence,

$$\int_{\widetilde{Q}_L} \Big(-2\lambda u_L \phi + \lambda \phi^2 - 2\lambda \partial_t u_L \partial_t \phi + \lambda(\partial_t \phi)^2 - a(Du_L + p) \cdot D\phi$$
$$- a(D\phi + p) \cdot D(u_L - \phi) - \partial_t u_L u_L - \lambda \partial_t u_L u_L \frac{\partial_t \widehat{\rho}_\theta}{\widehat{\rho}_\theta} - u_L a(Du_L + p) \cdot \frac{D\widehat{\rho}_\theta}{\widehat{\rho}_\theta} \Big) \widehat{\rho}_\theta \geq 0.$$

Letting $L \to +\infty$ and recalling that, as $L \to \infty$, $u_L \to u$, $\partial_t u_L \rightharpoonup \partial_t u$, $Du_L \rightharpoonup Du$ and $a(Du_L + p) \rightharpoonup \xi$ in $L_{\widehat{\rho}_\theta}^2$ and, hence, in L_{loc}^2, we obtain

$$\int_{\mathbb{R}^{d+1}} \Big(-2\lambda u \phi + \lambda \phi^2 - 2\lambda \partial_t u \partial_t \phi + \lambda(\partial_t \phi)^2 - \xi \cdot D\phi - a(D\phi + p) \cdot D(u - \phi)$$
$$- \partial_t u u - \lambda \partial_t u u \frac{\partial_t \widehat{\rho}_\theta}{\widehat{\rho}_\theta} - u \xi \cdot \frac{D\widehat{\rho}_\theta}{\widehat{\rho}_\theta} \Big) \widehat{\rho}_\theta \geq 0.$$

On the other hand, integrating (3.15) against $\phi \widehat{\rho}_\theta$, we get

$$\int_{\mathbb{R}^{d+1}} \Big(\lambda u \phi + \lambda \partial_t u \partial_t \phi + \lambda \partial_t u \phi \frac{\partial_t \widehat{\rho}_\theta}{\widehat{\rho}_\theta} + \partial_t u \phi + \xi \cdot D\phi + \phi \xi \cdot \frac{D\widehat{\rho}_\theta}{\widehat{\rho}_\theta} \Big) \widehat{\rho}_\theta = 0.$$

Inserting the last equality into the inequality above gives

$$\int_{\mathbb{R}^{d+1}} \Big(-\lambda \phi(u - \phi) - \lambda \partial_t \phi(\partial_t u - \partial_t \phi) - a(D\phi + p) \cdot D(u - \phi)$$
$$- \partial_t u(u - \phi) - \lambda \partial_t u(u - \phi) \frac{\partial_t \widehat{\rho}_\theta}{\widehat{\rho}_\theta} - (u - \phi)\xi \cdot \frac{D\widehat{\rho}_\theta}{\widehat{\rho}_\theta} \Big) \widehat{\rho}_\theta \geq 0.$$

Using $\phi = u + h\psi$ for $h > 0$ small and $\psi \in C_c^\infty(\mathbb{R}^{d+1})$, something that can be done using standard approximation arguments, yields, after dividing by h and letting $h \to 0$,

$$\int_{\mathbb{R}^{d+1}} \Big(\lambda u \psi + \lambda \partial_t u \partial_t \psi + a(Du + p) \cdot D\psi + \partial_t u \psi + \lambda \partial_t u \psi \frac{\partial_t \widehat{\rho}_\theta}{\widehat{\rho}_\theta} + \psi \xi \cdot \frac{D\widehat{\rho}_\theta}{\widehat{\rho}_\theta} \Big) \widehat{\rho}_\theta \geq 0.$$

The facts that ψ has a compact support, u and its derivatives are locally integrable and, as $\theta \to 0$, the derivatives of $\widehat{\rho}_\theta$ tend to 0 locally uniformly, gives, after letting $\theta \to 0$,

$$\int_{\mathbb{R}^{d+1}} \lambda u \psi + \lambda \partial_t u \partial_t \psi + a(Du + p) \cdot D\psi + \partial_t u \psi \geq 0.$$

Since $\psi \in C_c^\infty(\mathbb{R}^{d+1})$ is arbitrary, the last inequality implies that u is a solution of (3.12) in the sense of distributions.

Next we check that u is unique among weak solutions of (3.12) in $H_{\widehat{\rho}_\theta}^1$ for some $\theta > 0$.

Let u_1, u_2 be two solutions and set $\widetilde{u} = u_1 - u_2$. Using $\widetilde{u}\widehat{\rho}_\theta$ as a test function in the equation for $\widetilde{u}$, we find

$$\int_{\mathbb{R}^{d+1}} (\lambda \widetilde{u}^2 + \lambda(\partial_t \widetilde{u})^2 + (a(p + Du_1) - a(p + Du_2)) \cdot D\widetilde{u})\widehat{\rho}_\theta$$

$$= -\int_{\mathbb{R}^{d+1}} \lambda \widetilde{u}\partial_t \widetilde{u}\partial_t \widehat{\rho}_\theta + \widetilde{u}(a(p + Du_1) - a(p + Du_2)) \cdot D\widehat{\rho}_\theta$$

$$\lesssim_a \theta \int_{\mathbb{R}^{d+1}} (\lambda |\widetilde{u}||\partial_t \widetilde{u}| + C_0|D\widetilde{u}||\widetilde{u}|)\widehat{\rho}_\theta.$$

Then a standard argument based on Cauchy-Schwartz inequality implies that, for θ small enough, $\widetilde{u} \equiv 0$.

Since (3.12) has a unique solution in $H^1_{\widehat{\rho}_\theta}$ for some $\theta > 0$, the whole family u_L converges to u as $L \to +\infty$. It follows that u is measurable in Ω. Moreover, the stationarity of the equation and the uniqueness of the solution imply that u is also stationary.

To establish the bounds claimed, we test the equation for u against $\widehat{\rho}_\theta u$. Using the monotonicity of a and arguing as above we get

$$\int_{\mathbb{R}^{d+1}} \left(\lambda u^2 + \lambda(\partial_t u)^2 + C_0^{-1}|Du + p|^2\right) \widehat{\rho}_\theta$$

$$\lesssim_a \int_{\mathbb{R}^{d+1}} (\lambda \theta |\partial_t u||u| + \theta u^2 + (|a(0)| + |Du + p|)(|p| + \theta |u|))\widehat{\rho}_\theta,$$

It follows that, for θ small enough depending on p but independent of ω,

$$\int_{\mathbb{R}^{d+1}} (\lambda u^2 + \lambda(\partial_t u)^2 + C_0^{-1}|Du + p|^2)\widehat{\rho}_\theta \lesssim_p \int_{\mathbb{R}^{d+1}} |a(0)|^2\widehat{\rho}_\theta.$$

Taking expectations and using (3.6) and the fact u is stationary and (3.6) gives gives (3.13).

Finally, to obtain (3.14) we use the equation and (3.13).

$\square$

In order to proceed, we need the following remark about the reconstruction of a map from its derivatives.

Lemma 3.4. *Assume* (3.2) *and let* $\theta \in \mathbf{H}_\mathbf{x}^{-1}$ *and* $w \in \mathbf{L}_{\mathbf{pot}}^2$ *satisfy, for all* $\phi \in \mathcal{C}$ *and* $i = 1, \ldots, d$, *the compatibility condition*

$$\langle \theta, \partial_{x_i}\phi \rangle_{\mathbf{H}_\mathbf{x}^{-1}, \mathbf{H}_\mathbf{x}^1} = \mathbb{E}[w_i \partial_t \phi] \tag{3.17}$$

Then there exists a measurable map $u : \mathbb{R}^{d+1} \times \Omega \to \mathbb{R}$ *such that, a.s.,* $\int_{\widetilde{Q}_1} u(x, t, \omega)dxdt = 0$, $Du = w$ *and* $\partial_t u = \theta$ *in the sense of distributions.*

For the proof, we need to use regularizations (convolutions) with a kernel $K^\varepsilon(x, t) = \varepsilon^{-(d+1)}K(x/\varepsilon, t/\varepsilon)$ for $K : \mathbb{R}^{d+1} \to [0, +\infty)$ smooth, nonnegative, symmetric, compactly supported and such that $\int_{\mathbb{R}^{d=1}} K\,dxdt = 1$.

For $u \in \mathbf{L}^2$, define

$$K^\varepsilon * u(x, t, \omega) = \int_{\mathbb{R}^{d+1}} K^\varepsilon(x - y, t - s)u(y, s, \omega)dyds.$$

It is a classical fact that $K^\varepsilon * u$ belongs to $\mathcal{C}$ and that

$$\lim_{\varepsilon \to 0} \|u - K^\varepsilon * u\|_{\mathbf{L}^2} = 0.$$

The proof of Lemma 3.17. Fix $\varepsilon > 0$ and define $\theta^\varepsilon \in \mathbf{H}_\mathbf{x}^{-1}$ and w^ε so that, for all $\phi \in \mathbf{H}_\mathbf{x}^1$ and $w^\varepsilon = K^\varepsilon * w$,

$$\langle \theta^\varepsilon, \phi \rangle_{\mathbf{H}_\mathbf{x}^{-1}, \mathbf{H}_\mathbf{x}^1} = \langle \theta, K^\varepsilon * \phi \rangle_{\mathbf{H}_\mathbf{x}^{-1}, \mathbf{H}_\mathbf{x}^1} \tag{3.18}$$

It is immediate that $\theta^\varepsilon, w^\varepsilon$ belong to $\mathcal{C}$ and, in view of (3.17), for all $i = 1, \ldots, d$,

$$\partial_{x_i} \theta^\varepsilon = \partial_t w_i^\varepsilon.$$

It follows that there exists a measurable and smooth in x, t map $u^\varepsilon : \mathbb{R}^{d+1} \times \Omega \to \mathbb{R}$ such that $\partial_t u^\varepsilon = \theta^\varepsilon$, $Du^\varepsilon = w^\varepsilon$, and, without loss of generality, $\int_{\widetilde{Q}_1} u^\varepsilon = 0$.

For any $R \geq 1$, Poincaré's inequality gives (see, for instance, the proof of Lemma 4.2.1 in [12])

$$\|u^\varepsilon(\cdot, \cdot, \omega)\|_{L^2(\widetilde{Q}_R)} \lesssim_{d,R} \|Du^\varepsilon(\cdot, \cdot, \omega)\|_{L^2(\widetilde{Q}_R)} + \|\partial_t u^\varepsilon(\cdot, \cdot, \omega)\|_{L^2(I_R, H^{-1}(Q_R))},$$

and, thus,

$$\mathbb{E}\left[\|u^\varepsilon(\cdot, \cdot, \omega)\|_{L^2(\widetilde{Q}_R)}^2\right] \lesssim_{d,R} \mathbb{E}\left[\|w^\varepsilon\|_{L^2}^2 + \|\theta^\varepsilon\|_{L^2(I_R, H^{-1}(Q_R))}^2\right].$$

Using a diagonal argument, we can find $\varepsilon_n \to 0$ and $u \in L^2_{loc}(\mathbb{R}^{d+1} \times \Omega)$ such that, for any R, $u^{\varepsilon_n} \rightharpoonup u$ in $L^2(\widetilde{Q}_R \times \Omega)$.

It is, then, easy to check that $Du = w$, $\partial_t u = \theta$ and $\int_{\widetilde{Q}_1} u = 0$.

$\square$

We use Lemma 3.4 to obtain the following result which is one of the most crucial steps for the construction of the corrector.

Lemma 3.5. *Assume* (3.2). *If* $\theta \in \mathbf{H}_\mathbf{x}^{-1}$, $w \in \mathbf{L}_{\mathbf{pot}}^2$ *and* $\xi \in \mathbf{L}^2$ *satisfy the compatibility condition* (3.17) *and*

$$\theta - \operatorname{div}(\xi) = 0 \quad \text{in} \quad \mathbf{H}_\mathbf{x}^{-1},$$

then

$$\mathbb{E}\left[\int_{\widetilde{Q}_1} w \cdot \xi\right] = 0.$$

Proof. Let θ^ε be defined by (3.18), $w^\varepsilon = K^\varepsilon * w$ and $\xi^\varepsilon = K^\varepsilon * \xi$. Then,

$$\theta^\varepsilon - \operatorname{div}(\xi^\varepsilon) = 0.$$

Lemma 3.4 and its proof yield a measurable in ω and smooth in (x, t) map $u^\varepsilon : \mathbb{R}^{d+1} \times \Omega \to \mathbb{R}$ such that $Du^\varepsilon = w^\varepsilon$ and $\partial_t u^\varepsilon = \theta^\varepsilon$, and, in the classical sense,

$$\partial_t u^\varepsilon - \operatorname{div}(\xi^\varepsilon) = 0 \quad \text{in} \quad \mathbb{R}^{d+1} \times \Omega. \tag{3.19}$$

Arguing by contradiction, we assume that

$$\widehat{\kappa} = \mathbb{E}\left[\int_{\widetilde{Q}_1} \xi \cdot w\right] > 0.$$

Since the map $t \to \mathbb{E}\left[\int_{Q_1} w(x, t) \cdot \xi(x, t) dx\right]$ is well-defined and constant, we actually have, for all $t \in \mathbb{R}$,

$$\mathbb{E}\left[\int_{Q_1} w(x, t) \cdot \xi(x, t) dx\right] = \widehat{\kappa} > 0. \tag{3.20}$$

In view of the stationarity of w and ξ, (3.20) implies that there exist $\varepsilon_0 > 0$ and $0 < \kappa < \widehat{\kappa}$ such that, for all $t \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0)$ and $R > 0$,

$$\mathbb{E}\left[\int_{Q_R} w^\varepsilon(x,t) \cdot \xi^\varepsilon(x,t)dx\right] \geq \kappa R^d. \tag{3.21}$$

Fix $R > 0$ and let $\psi = \psi_R \in C^1(\mathbb{R}^d \times [0, +\infty))$ be such that

$$\begin{cases} \psi(x, R) = 0 & \text{in } \mathbb{R}^d \backslash Q_{R+1}, \quad \psi(x, R) = 1 \quad \text{in } Q_R, \quad \|D\psi\|_\infty + \|\partial_R \psi\|_\infty \lesssim_d 1, \\ \text{and } |D\psi(x, R)| \lesssim_d \partial_R \psi(x, R). \end{cases}$$

Note that such ψ can be constructed by convolving in space the map $x \to \mathbf{1}_{Q_{R+1/2}}(x)$ with a nonnegative kernel with sufficiently small support.

Finally, for some $c_0 > 0$ and T sufficiently large to be chosen later, set $R(t) = (T - c_0 t)^{1/2}$. Then

$$\frac{d}{dt}\int_{\mathbb{R}^d} \frac{(u^\varepsilon)^2(x,t)}{2}\psi(x, R(t))dx$$

$$= R'(t)\int_{\mathbb{R}^d} \frac{(u^\varepsilon)^2(x,t)}{2}\partial_R\psi(x, R(t))dx + \int_{\mathbb{R}^d} u^\varepsilon(x,t)\partial_t u^\varepsilon(x,t)\psi(x, R(t))dx$$

$$= R'(t)\int_{\mathbb{R}^d} \frac{(u^\varepsilon)^2(x,t)}{2}\partial_R\psi(x, R(t))dx - \int_{\mathbb{R}^d} \xi^\varepsilon(x,t) \cdot w^\varepsilon(x,t)\psi(x, R(t))dx$$

$$- \int_{\mathbb{R}^d} u^\varepsilon(x,t)w^\varepsilon(x,t) \cdot D\psi(x, R(t))dx.$$

Young's inequality yields, for any $\alpha > 0$,

$$\frac{d}{dt}\int_{\mathbb{R}^d} \frac{(u^\varepsilon)^2(x,t)}{2}\psi(x, R(t))dx$$

$$\leq R'(t)\int_{\mathbb{R}^d} \frac{(u^\varepsilon)^2(x,t)}{2}\partial_R\psi(x, R(t))dx - \int_{Q_{R(t)}} \xi^\varepsilon(x,t) \cdot w^\varepsilon(x,t)dx$$

$$+ \int_{Q_{R(t)+1}\backslash Q_{R(t)}} |\xi^\varepsilon(x,t)|\,|w^\varepsilon(x,t)|dx + \alpha|R'(t)|\int_{\mathbb{R}^d} \frac{(u^\varepsilon)^2(x,t)}{2}|D\psi(x, R(t))|dx$$

$$+ \frac{C}{\alpha|R'(t)|}\int_{Q_{R(t)+1}\backslash Q_{R(t)}} |w^\varepsilon(x,t)|^2dx.$$

Recall that, by construction, $R' < 0$, $\|D\psi\|_\infty \lesssim_d C$ and $|D\psi| \lesssim_d \partial_R\psi$

Hence, choosing from now on α small enough depending only on d, taking expectations and using (3.21), we find

$$\frac{d}{dt}\mathbb{E}\left[\int_{\mathbb{R}^d} \frac{(u^\varepsilon)^2(x,t)}{2}\psi(x, R(t))dx\right] + \kappa(R(t))^d - \mathbb{E}\left[\int_{Q_{R(t)+1}\backslash Q_{R(t)}} |\xi^\varepsilon(x,t)|\,|w^\varepsilon(x,t)|dx\right]$$

$$\lesssim_\alpha \frac{1}{|R'(t)|}\mathbb{E}\left[\int_{Q_{R(t)+1}\backslash Q_{R(t)}} |w^\varepsilon(x,t)|^2dx\right]. \tag{3.22}$$

We use next the stationarity of w^ε and ξ^ε, and the facts that $|Q_{R(t)+1}\backslash Q_{R(t)}| \lesssim C(R(t))^{d-1}$, $\xi^\varepsilon, w^\varepsilon \in \mathbf{L^2}$, and $R'(t) = c_0(R(t))^{-1}$ to get, for some $C > 0$,

$$\frac{d}{dt}\mathbb{E}\left[\int_{\mathbb{R}^d}\frac{(u^\varepsilon)^2(x,t)}{2}\psi(x,R(t))dx\right] \le -(R(t))^d\kappa + C(|R'(t)|^{-1}+1)(R(t))^{d-1}$$
$$= -(R(t))^d(\kappa - Cc_0^{-1} - C(R(t))^{-1}),$$

Choosing $c_0 > 1$ large so that $\kappa - Cc_0^{-1} \ge \kappa/2$ and $t \le t_T = T - 16C^2\kappa^{-2}c_0^{-1}$, in order to have $C(R(t))^{-1} \le \kappa/4$ on $[0, t_T]$, we find, for all $t \in [0, t_T]$,

$$\frac{d}{dt}\mathbb{E}\left[\int_{\mathbb{R}^d}\frac{(u^\varepsilon)^2(x,t)}{2}\psi(x,R(t))dx\right] \le -(R(t))^d\frac{\kappa}{4}.$$

Integration in time over $t \in [h, c_0^{-1}T]$ for $h \in [0, T^{1/2}]$ (note that, if c_0 and T are large enough, $c_0^{-1}T < t_T$) and the fact that $\psi \ge 0$ give

$$\mathbb{E}\left[\int_{\mathbb{R}^d}\frac{(u^\varepsilon)^2(x,h)}{2}\psi(x,R(h))dx\right] \ge \frac{\kappa}{4}\int_h^{c_0^{-1}T}(R(t))^d dt.$$

Integrating once more in time over $h \in [0, T^{1/2}]$ and noting that, since $R(h) \le T^{1/2}$, $\psi(x, R(h)) \le 1_{Q_{T^{1/2}+1}}$, we get

$$\mathbb{E}\left[\int_0^{T^{1/2}}\int_{Q_{T^{1/2}+1}}\frac{(u^\varepsilon)^2(x,h)}{2}dxdh\right] \ge C^{-1}\kappa T^{(d+3)/2}.$$

Our goal is to apply A.2 in the Appendix. For this, we note that, since $Du^\varepsilon = w^\varepsilon \in \mathbf{L}^2_{\mathbf{pot}}$, $\mathbb{E}[\int_{Q_1} Du^\varepsilon(\cdot, t)] = 0$. Moreover, in view of (3.19) and the fact that $\xi^\varepsilon \in \mathbf{L}^2$ is stationary,

$$\mathbb{E}\left[\int_{\widetilde{Q}_1}\partial_t u^\varepsilon\right] = \langle\partial_t u^\varepsilon, 1\rangle_{\mathbf{H}_x^{-1},\mathbf{H}_x^1} = \langle\mathrm{div}(\xi^\varepsilon), 1\rangle_{\mathbf{H}_x^{-1},\mathbf{H}_x^1} = 0$$

Hence, we can apply Lemma A.2 which implies that, for any $\delta > 0$, there exists R_δ such that, for all $R \ge R_\delta$,

$$\mathbb{E}\left[\int_0^R\int_{Q_R}(u^\varepsilon(x,h))^2 dxdh\right] \le \delta R^{d+3}.$$

Choosing $R = T^{1/2} + 1$ and T large, we obtain

$$\frac{\delta}{2}(T^{1/2}+1)^{d+3} \ge \mathbb{E}\left[\int_0^{T^{1/2}}\int_{Q_{T^{1/2}+1}}\frac{(u^\varepsilon)^2(x,h)}{2}dxdh\right] \ge C^{-1}\kappa T^{\frac{d+3}{2}},$$

which yields a contradiction if δ is small enough and T is large enough.

It follows that we must have

$$\mathbb{E}\left[\int_{\widetilde{Q}_1}\xi\cdot w\right] \le 0.$$

Arguing similarly for negative t gives the opposite inequality.

$\square$

The next lemma is the step that provides the sought after corrector as well as the properties (monotonicity and Lipschitz continuity) of $\bar{a}$.

Lemma 3.6. *Assume* (3.2), (3.5), *and* (3.6). *For any* $p \in \mathbb{R}^d$ *there exists a unique pair* $(\theta^p, w^p) \in \mathbf{H_x^{-1}} \times \mathbf{L_{pot}^2}$ *satisfying* (3.17) *and*

$$\theta^p - \operatorname{div}(a(w^p + p, x, t, \omega)) = 0 \ \ \text{in} \ \ \mathbf{H_x^{-1}}. \tag{3.23}$$

Moreover, for all $p, p' \in \mathbb{R}^d$,

$$\|w^p - w^{p'}\|_{\mathbf{L^2}} \lesssim_a |p - p'|. \tag{3.24}$$

Finally, the vector field $\bar{a}$ *defined by* (3.9) *is monotone and Lipschitz continuous.*

Proof. Let $\chi^{\lambda,p}$ be given by Lemma 3.3. In view of (3.13) and (3.14), there exist a subsequence $\lambda_n \to 0$, $w \in \mathbf{L_{pot}^2}$, $\theta \in \mathbf{H^{-1}}$ and $\xi \in \mathbf{L^2}$ such that $D\chi^{\lambda_n,p} \rightharpoonup w$, $\partial_t \chi^{\lambda_n,p} \rightharpoonup \theta$, and $a(D\chi^{\lambda_n,p} + p) \rightharpoonup \xi$ in their respective spaces.

Moreover, in view of (3.14), for all $\phi \in \mathbf{H^1}$,

$$\langle \theta, \phi \rangle_{\mathbf{L^2}} \lesssim_p \|D\phi\|_{\mathbf{L^2}}$$

which means that, in fact, $\theta \in \mathbf{H_x^{-1}}$.

Note also that, since the pair $(\partial_t \chi^{\lambda,p}, D\chi^{\lambda,p})$ satisfies (3.17), so does (θ, w).

Finally, (3.12) implies

$$\theta - \operatorname{div}(\xi) = 0 \ \ \text{in} \ \ \mathbf{H_x^{-1}}. \tag{3.25}$$

It remains to check that (3.23) holds. As we show below, this is a consequence of the monotonicity of a, which gives that, for any test function $\phi \in \mathcal{C}$,

$$\mathbb{E}\left[\int_{\widetilde{Q}_1} \lambda(\chi^{\lambda,p} - \phi)^2 + \lambda(\partial_t \chi^{\lambda,p} - \partial_t \phi)^2 + (a(D\chi^{\lambda,p} + p) - a(D\phi + p)) \cdot (D\chi^{\lambda,p} - D\phi) \right] \geq 0.$$

Multiplying (3.12) by $\chi^{\lambda,p}$ and taking expectation, we find

$$\mathbb{E}\left[\int_{\widetilde{Q}_1} \lambda(\chi^{\lambda,p})^2 + \lambda(\partial_t \chi^{\lambda,p})^2 + a(D\chi^{\lambda,p} + p) \cdot D\chi^{\lambda,p} \right] = 0,$$

and, thus,

$$\mathbb{E}\Big[\int_{\widetilde{Q}_1} \lambda(-2\chi^{\lambda,p}\phi + \phi^2) + \lambda(-2\partial_t \chi^{\lambda,p}\partial_t \phi + (\partial_t \phi)^2) - a(D\chi^{\lambda,p} + p) \cdot D\phi$$
$$-a(D\phi + p) \cdot (D\chi^{\lambda,p} - D\phi))) \Big] \geq 0.$$

Passing to the limit $\lambda_n \to 0$, in view of the estimates on $\chi^{\lambda,p}$ we get

$$\mathbb{E}\left[\int_{\widetilde{Q}_1} -\xi \cdot D\phi - a(D\phi + p) \cdot (w - D\phi) \right] \geq 0.$$

Since this last inequality holds for any $\phi \in \mathcal{C}$, we also have, for any $z \in \mathbf{L_{pot}^2}$,

$$\mathbb{E}\left[\int_{\widetilde{Q}_1} -\xi \cdot z - a(z + p) \cdot (w - z) \right] \geq 0.$$

Choose $z = w + \theta z'$ with $z' \in \mathbf{L_{pot}^2}$. Then, after dividing by θ and letting $\theta \to 0$, in view of Lemma (3.5), we get

$$\mathbb{E}\left[\int_{\widetilde{Q}_1} -\xi \cdot z' + a(w + p) \cdot z' \right] \geq \limsup_{\theta \to 0} \frac{1}{\theta} \mathbb{E}\left[\int_{\widetilde{Q}_1} \xi \cdot w \right] = 0,$$

Since the last inequality holds for any $z' \in \mathbf{L}^2_{\mathbf{pot}}$, we infer that

$$\mathbb{E}\left[\int_{\widetilde{Q}_1} -\xi \cdot z' + a(w+p) \cdot z'\right] = 0. \tag{3.26}$$

Going back to (3.25), (3.26) implies that, for any $\phi \in \mathbf{H}^1_{\mathbf{x}}$,

$$\langle \theta - \operatorname{div}(a(w+p)), \phi\rangle_{\mathbf{H}^{-1}_{\mathbf{x}}, \mathbf{H}^1_{\mathbf{x}}} = \mathbb{E}\left[\int_{\widetilde{Q}_1} \xi \cdot D\phi - a(w+p) \cdot D\phi\right] = 0,$$

and, hence, (θ, w) satisfies (3.23).

Next we prove at the same time the uniqueness of (θ, w) and the monotonicity of $\bar{a}$.

Let $p^1 \in \mathbb{R}^d$ and (θ^1, w^1) be a solution associated with p^1, and set $\xi^1 = a(w^1 + p^1)$. Then

$$\theta - \theta^1 - \operatorname{div}(\xi - \xi^1) = 0.$$

Applying Lemma 3.5 to the pair $(\theta - \theta^1, w - w^1)$, we find

$$\mathbb{E}\left[\int_{\widetilde{Q}_1} (\xi - \xi^1) \cdot (w - w^1)\right] = 0.$$

The monotonicity of $\bar{a}$ follows from the following calculation that uses the fact that, since $w - w^1 \in \mathbf{L}^2_{\mathbf{pot}}$, we have $\mathbb{E}[\int_{\widetilde{Q}_1} w - w^1] = 0$:

$$(\bar{a}(p) - \bar{a}(p^1)) \cdot (p - p^1) = \mathbb{E}\left[\int_{\widetilde{Q}_1} (a(w+p) - a(w^1 + p^1)) \cdot (w + p - w^1 - p^1)\right]$$

$$\geq C_0^{-1}\mathbb{E}\left[\int_{\widetilde{Q}_1} |w + p - w^1 - p^1|^2\right] = C_0^{-1}(\mathbb{E}\left[\int_{\widetilde{Q}_1} |w - w^1|^2\right] + |p - p^1|^2).$$

The uniqueness of (θ, w) also follows from the inequality above. Indeed set $p^1 = p$. It follows that $w = w^1$, which in turn implies that $\theta = \theta^1$.

The Lipschitz continuity follows from the observation that

$$|\bar{a}(p) - \bar{a}(p^1)| \leq \mathbb{E}\left[\int_{\widetilde{Q}_1} |a(w+p) - a(w^1 + p^1)|\right] \leq C_0(\mathbb{E}^{1/2}\left[\int_{\widetilde{Q}_1} |w - w^1|^2\right] + |p - p^1|)$$

$$\leq C_0(((\bar{a}(p) - \bar{a}(p^1)) \cdot (p - p^1))^{1/2} + |p - p^1|) \leq \frac{1}{2}|\bar{a}(p) - \bar{a}(p^1)| + C|p - p^1|.$$

Note that the above also yields (3.24).

$\square$

We have now all the necessary ingredients to prove Theorem 3.1.

Proof of Theorem 3.1. Fix $p \in \mathbb{R}^d$ and let (θ^p, w^p) and χ^p be given respectively by Lemma 3.6 and Lemma 3.4.

Then, for $\chi^\varepsilon(x, t; p, \omega) = \varepsilon \chi^p(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega)$ and $a^\varepsilon(p, x, t, \omega) = a(p, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega)$, we have

$$\partial_t \chi^\varepsilon - \operatorname{div}(a^\varepsilon(p + D\chi^\varepsilon, x, t)) = 0 \quad \text{in} \quad R^d \times \mathbb{R}. \tag{3.27}$$

First we show that there exists a universal constant C_0 such that, $\mathbb{P}$−a.s. and for any $R, T > 0$,

$$\limsup_{\varepsilon \to 0} \int_0^T \int_{Q_R} (\chi^\varepsilon(x, t))^2 dx dt \leq C_0 T^3 R^{d-2} \mathbb{E}\left[\int_{\widetilde{Q}_1} |a(D\chi + p)|^2\right]. \tag{3.28}$$

Fix $\xi \in C^\infty(\mathbb{R}; [0,1])$ such that $\xi \equiv 0$ in $(-\infty, -1)$, $\xi \equiv 1$ in $[0, +\infty)$ and $\xi' \leq 2$ and set
$$\phi(x, s, t) = \xi\big((3 - 2t^{-1}s) - R^{-1}|x|_\infty\big).$$
We note for later use that, since $1 \leq (3 - 2t^{-1}s) \leq 3$ for $s \in [0, t]$, $\phi(x, s, t) = 1$ in Q_R, while $\phi(x, s, t) = 0$ in $\mathbb{R}^d \backslash Q_{4R}$.

Using the equation satisfied by χ^ε and Young's inequality, we find, for any $t > 0$ fixed and any $s \in (0, t)$,
$$\frac{d}{ds} \int_{\mathbb{R}^d} \frac{1}{2} (\chi^\varepsilon(s))^2 \phi(x, s, t) = \int_{\mathbb{R}^d} \frac{1}{2} (\chi^\varepsilon)^2 \partial_s \phi - (a^\varepsilon D\chi^\varepsilon \phi + \chi^\varepsilon a^\varepsilon D\phi).$$

$$\leq \int_{\mathbb{R}^d} \frac{1}{2} (\chi^\varepsilon)^2 \partial_s \phi - a^\varepsilon D\chi^\varepsilon \phi + \frac{1}{2} R^{-1} t |a^\varepsilon|^2 |D\phi| + \frac{1}{2} Rt^{-1} (\chi^\varepsilon)^2 |D\phi|.$$

The computation above, which here is made at a formal level, can be easily be rigorous by regularizing χ^ε by convolution.

Since $\partial_s \phi = -2t^{-1} \xi'$ while $|D\phi| \leq R^{-1} \xi'$, we can absorb the last term in the righthand side into the first one to obtain
$$\frac{d}{ds} \int_{\mathbb{R}^d} \frac{1}{2} (\chi^\varepsilon(s))^2 \phi(x, s, t) \leq \int_{\mathbb{R}^d} -a^\varepsilon D\chi^\varepsilon \phi + R^{-2} t \int_{Q_{4R}} |a^\varepsilon|^2.$$

Integrating the above inequality in time, between 0 and t and using the definition of ϕ we get
$$\int_{Q_R} \frac{1}{2} (\chi^\varepsilon(t))^2 dx \leq \int_{Q_{4R}} \frac{1}{2} (\chi^\varepsilon(0))^2 dx$$
$$- \int_0^t \int_{\mathbb{R}^d} a^\varepsilon(s) D\chi^\varepsilon(s) \phi(x, s, t) dx ds + R^{-2} t \int_0^t \int_{Q_{4R}} |a^\varepsilon(s)|^2 dx ds.$$

A second integration in $t \in (0, T)$ gives
$$\int_0^T \int_{Q_R} \frac{1}{2} (\chi^\varepsilon(t))^2 dx dt \leq T \int_{Q_{4R}} \frac{1}{2} (\chi^\varepsilon(0))^2 dx$$
$$- \int_0^T \int_0^t \int_{\mathbb{R}^d} a^\varepsilon(s) D\chi^\varepsilon(s) \phi(x, s, t) dx ds dt$$
$$+ R^{-2} \int_0^T t \int_0^t \int_{Q_{4R}} |a^\varepsilon(s)|^2 dx ds dt.$$

We now let $\varepsilon \to 0$. It follows from Lemma A.2 and the ergodic theorem that, $\mathbb{P}-$a.s.,
$$\limsup_{\varepsilon \to 0} \int_0^T \int_{Q_R} \frac{1}{2} (\chi^\varepsilon(t))^2 dx dt \leq \tag{3.29}$$
$$- \int_0^T \int_0^t \int_{\mathbb{R}^d} \mathbb{E}\left[\int_{\tilde{Q}_1} a(D\chi + p) \cdot D\chi\right] \phi(x, s, t) dx ds dt$$
$$+ R^{-2} T \int_0^T \int_0^t \int_{Q_{4R}} \mathbb{E}\left[\int_{\tilde{Q}_1} |a(D\chi + p)|^2\right] dx ds dt. \tag{3.30}$$

Lemma 3.5 gives that the first term in the right-hand side vanishes. Thus,
$$\limsup_{\varepsilon \to 0} \int_0^T \int_{Q_R} \frac{1}{2} (\chi^\varepsilon(t))^2 dx dt \lesssim R^{d-2} T^3 \mathbb{E}\left[\int_{\tilde{Q}_1} |a(D\chi + p)|^2\right],$$

and, hence, (3.28).

A symmetric argument yields that, $\mathbb{P}-$a.s.,

$$\limsup_{\varepsilon \to 0} \int_{-T}^{T} \int_{Q_R} (\chi^\varepsilon(x,t))^2 dx dt \le C_0 T^3 R^{d-2} \mathbb{E}\left[\int_{\widetilde{Q}_1} |a(D\chi + p)|^2\right]. \tag{3.31}$$

Next we show the convergence of (χ^ε) to 0.

Let $\omega \in \Omega$ be such that (3.31) holds for any $T, R > 0$. Then, in view of (3.31), the families $(\chi^\varepsilon)_{\varepsilon>0}$, $(D\chi^\varepsilon)_{\varepsilon>0}$ and $(\partial_t \chi^\varepsilon)_{\varepsilon>0}$ are respectively is bounded in $L^2_{loc}(\mathbb{R}^d \times \mathbb{R})$, $L^2_{loc}(\mathbb{R}^d \times \mathbb{R})$ and $L^2_{loc}(H^{-1})$. Hence, in view of the classical Lions-Aubin Lemma [2, 14], the family $(\chi^\varepsilon)_{\varepsilon>0}$ is relatively compact in $L^2_{loc}(\mathbb{R}^{d+1})$.

Let (χ^{ε_n}) be any converging subsequence with limit χ in $L^2_{loc}(\mathbb{R}^d \times \mathbb{R})$. Since a and $D\chi^\varepsilon$ are stationary in an ergodic environment, $a^\varepsilon(D\chi^\varepsilon + p)$ converges weakly to a constant. Thus, in view of (3.27), χ solves $\partial_t \chi = 0$ in $\mathbb{R}^d \times \mathbb{R}$. Dividing (3.31) and letting $T \to 0$ yields that $\chi(\cdot, 0) = 0$.

Therefore $\chi \equiv 0$, and, hence, $\chi^{\varepsilon_n} \to 0$ in $L^2_{loc}(\mathbb{R}^d \times \mathbb{R})$.

$\square$

3.4. **Homogenization.** We now turn to the homogenization of (3.1). The aim is to show that the family $(u^\varepsilon)_{\varepsilon>0}$ converges to the solution u of the homogenous equation

$$\partial_t \overline{u} - \text{div}(\overline{a}(D\overline{u})) = f(x,t) \text{ in } \mathbb{R}^d \times (0,T) \quad \overline{u}(\cdot, 0) = u_0 \text{ in } \mathbb{R}^d, \tag{3.32}$$

where $\overline{a} : \mathbb{R}^d \to \mathbb{R}$ is defined by (3.9), see below for a precise statement.

For the statement and the proof of the result we will use again the weight

$$\rho_\theta(x) := \exp\{-\theta(1 + |x|^2)^{1/2}\} \tag{3.33}$$

and we will work in the weighted spaces $L^2_{\rho_\theta} = L^2_{\rho_\theta}(\mathbb{R}^d)$, $H^1_{\rho_\theta} = H^1_{\rho_\theta}(\mathbb{R}^d)$, etc...

The homogenization result is stated next.

Theorem 3.7. *Assume* (3.2), (3.5), *and* (3.6) *and let* $\overline{a} : \mathbb{R}^d \to \mathbb{R}^d$ *be the monotone and Lipschitz continuous vector field defined by* (3.9). *Then, for every* $T > 0$, $u_0 \in L^2(\mathbb{R}^d)$ *and* $f \in L^2(R^d \times (0,T))$, *if* u^ε *and* $\overline{u}$ *solve respectively* (3.1) *and* (3.32), *then,* $\mathbb{P}-$*a.s. and in expectation,* $u^\varepsilon(\cdot, t) \to \overline{u}(\cdot, t)$ *in* $L^2_{\rho_\theta}(\mathbb{R}^d \times (0,T))$ *for any* $\theta > 0$.

The argument is long. To help the reader we split it in several parts (subsubsections). In the first subsubsection we prove a refined energy estimate for solutions of (3.1). Then, in subsubsection 3.4.2 we identify $\Omega_0 \subset \Omega$ of full measure where the homogenization takes place. In subsubsection 3.4.3 we extract a subsequence $\varepsilon_n \to 0$ along which u^{ε_n} has a limit. To show that this limit satisfies the effective PDE, we construct a special test function in subsubsection 3.4.4. Theorem 3.7 is proved in subsubsection 3.4.5. The last three subsubsections are devoted to the proof of some technical parts used in subsubsection 3.4.5.

3.4.1. *Preliminary estimates.* A solution to (3.1) is a measurable map $u^\varepsilon : \mathbb{R}^d \times [0,T] \times \Omega \to \mathbb{R}$ such that, $\mathbb{P}-$a.s., $u^\varepsilon(\cdot, \cdot, \omega) \in L^2([0,T], H^1_{\rho_\theta}) \cap C^0([0,T], L^2_{\rho_\theta})$ which satisfies the equation in the sense of distributions. Since, $\mathbb{P}-$a.s., $a(0, \cdot, \cdot, \omega) \in L^2_{loc}(\mathbb{R}^d \times (0,T])$, $u^\varepsilon(\cdot, \cdot, \omega)$ exists and is unique.

In the next lemma we sharpen the standard energy estimate for solutions of (3.1).

Lemma 3.8. *Assume* (3.2), (3.5), *and* (3.6), $u_0 \in L^2(\mathbb{R}^d)$ *and* $f \in L^2(R^d \times (0,T))$. *There exists* $C_\theta^\varepsilon(\omega) > 0$, *which is* $\mathbb{P}-a.s.$ *finite, converges, as* $\varepsilon \to 0$, *in* $L^1(\Omega)$, *and depends on* θ, T, $\|f\|_2$ *and the monotonicity and Lipschitz constants of* a *such that*

$$\sup_{t \in [0,T]} \|u^\varepsilon(\cdot,t)\|_{L^2_{\rho_\theta}}^2 + \int_0^T \|Du^\varepsilon(\cdot,t)\|_{L^2_{\rho_\theta}}^2 \, dt + \int_0^T \|\partial_t u^\varepsilon\|_{H^{-1}_{\rho_\theta}}^2 \le C_\theta^\varepsilon(\omega). \tag{3.34}$$

Proof. Throughout the proof, to simplify the notation, in place of $a(Du^\varepsilon, \dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}, \omega)$, we write $a^\varepsilon(Du^\varepsilon)$.

It is immediate that, for a.e. $t \in (0,T]$, u^ε satisfies the standard energy inequality

$$\int_{\mathbb{R}^d} \frac{1}{2} u^\varepsilon(t)^2 \rho_\theta - \int_{\mathbb{R}^d} \frac{1}{2} u_0^2 \rho_\theta = \int_0^t \int_{\mathbb{R}^d} -a^\varepsilon(Du^\varepsilon) \cdot (Du^\varepsilon \rho_\theta + u^\varepsilon D\rho_\theta) + fu^\varepsilon \rho_\theta$$

$$\le \int_0^t \int_{\mathbb{R}^d} (-C_0^{-1}|Du^\varepsilon|^2 + |a^\varepsilon(0)||Du^\varepsilon| + \theta|u^\varepsilon|(|a^\varepsilon(0)| + C_0|Du^\varepsilon|) + |f||u^\varepsilon|)\rho_\theta$$

$$\le \int_0^t \int_{\mathbb{R}^d} (-\frac{1}{2C_0}|Du^\varepsilon|^2 + C(|a^\varepsilon(0)|^2 + |u^\varepsilon|^2 + |f|^2)\rho_\theta$$

$$\le \int_0^t \int_{\mathbb{R}^d} (-\frac{1}{2C_0}|Du^\varepsilon|^2 + C|u^\varepsilon|^2)\rho_\theta + C_\theta(\widetilde{C}_\theta^\varepsilon(\omega) + 1),$$

where C_θ is a constant which depends only on θ, T, $\|f\|_2$ and C_0 in (3.5) (and might change from line to line) and

$$\widetilde{C}_\theta^\varepsilon(\omega) = \int_0^T \int_{\mathbb{R}^d} |a^\varepsilon(0,x,t)|^2 \rho_\theta(x) dx.$$

It then follows from Gronwall's Lemma that

$$\sup_{t \in [0,T]} \|u^\varepsilon(\cdot,t)\|_{L^2_{\rho_\theta}}^2 + \int_0^T \|Du^\varepsilon(\cdot,t)\|_{L^2_{\rho_\theta}}^2 \, dt \le C_\theta(1 + \widetilde{C}_\theta^\varepsilon(\omega)).$$

To estimate $\partial_t u^\varepsilon$, we use $\phi \rho_\theta$ with $\phi \in C_c^\infty(\mathbb{R}^d \times [0,T])$ as a test function in (3.1) and get

$$\int_0^T \langle \partial_t u^\varepsilon, \phi \rangle_{H^{-1}_{\rho_\theta}(\mathbb{R}^d), H^1_{\rho_\theta}(\mathbb{R}^d)} = \int_0^T \int_{\mathbb{R}^d} -a^\varepsilon(Du^\varepsilon) \cdot D\phi \rho_\theta - a^\varepsilon(Du^\varepsilon) \cdot D\rho_\theta \phi + f\phi \rho_\theta$$

$$\le C_\theta(\|a^\varepsilon(0)\|_{L^2_{\rho_\theta}} + C_0\|u^\varepsilon\|_{L^2(H^1_{\rho_\theta})} + \|f\|_{L^2_{\rho_\theta}})\|\phi\|_{L^2(H^1_{\rho_\theta})},$$

and, in view of the previous estimate on u^ε,

$$\int_0^T \|\partial_t u^\varepsilon\|_{H^{-1}_{\rho_\theta}}^2 dt \le C_\theta(\widetilde{C}_\theta^\varepsilon + 1).$$

To complete the proof, we note that the ergodic Theorem implies that $\widetilde{C}_\theta^\varepsilon$ converges, $\mathbb{P}-a.s.$ and in $L^1(\Omega)$, to

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} |a(0,x,t)|^2 \rho_\theta(x) dx dt\right] < +\infty.$$

$\square$

3.4.2. *The identification of Ω_0.* Let $\chi^\varepsilon(x, t; p, \omega) = \varepsilon\chi(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}; p, \omega)$, where $\chi(y, \tau; p, \omega)$ is the corrector found in Theorem 3.1. We know from Theorem 3.1 that χ^ε solves in the sense of distributions the corrector equation

$$\partial_t \chi^\varepsilon - \mathrm{div}(a(p + D\chi^\varepsilon, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega)) = 0 \quad \text{in} \quad \mathbb{R}^d \times \mathbb{R},$$

and satisfies

$$\lim_{\varepsilon \to 0} \int_{\widetilde{Q}_R} |\chi^\varepsilon|^2 = 0 \quad \mathbb{P} - \text{a.s..} \tag{3.35}$$

In addition, since, for each $p \in \mathbb{R}^d$, $a(p + D\chi, \cdot, \cdot, \cdot) \in \mathbf{L}^2$ and stationary, the ergodic theorem yields, for any cube $\widetilde{Q}$ and any $g \in L^2(\widetilde{Q}, \mathbb{R}^d)$ and $\mathbb{P}$−a.s.,

$$\int_{\widetilde{Q}} g(x, t) \cdot a(p + D\chi^\varepsilon(x, t; p), \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega) dx dt \underset{\varepsilon \to 0}{\to} \int_{\widetilde{Q}} g(x, t) \cdot \overline{a}(p) dx dt. \tag{3.36}$$

Similarly, in view of the stationarity of $D\chi$, for any $g \in L^2(\widetilde{Q})$ and $\mathbb{P}$−a.s.,

$$\int_{\widetilde{Q}} g(x, t)|D\chi^\varepsilon(x, t; p)|^2 dx dt \underset{\varepsilon \to 0}{\to} \mathbb{E}\left[\int_{\widetilde{Q}} g(x, t)|D\chi(x, t; p)|^2\right]. \tag{3.37}$$

Finally, Lemma 3.5 yields

$$\int_{\widetilde{Q}} g(x, t)a^\varepsilon(p + D\chi^\varepsilon(x, t), x, t) \cdot D\chi^\varepsilon(x, t; p) dx dt \underset{\varepsilon \to 0}{\to} 0. \tag{3.38}$$

Hence, given a countable family $\mathcal{E}$ dense in $\mathbb{R}^d$ and the (countable) family $\mathcal{Q}$ of cubes with rational coordinates, we can find using a diagonal argument a set Ω_1 of full probability such that, for any $\omega \in \Omega_1$, any $p \in \mathcal{E}$ and $\widetilde{D} \in \mathcal{Q}$, (3.35), (3.36), (3.37) and (3.38) hold.

Let Ω_2 be the full measure subset of Ω such that, for any $\omega \in \Omega_2$, the limit of the constant $C_\theta^\varepsilon(\omega)$ in (3.34) exists and is finite for any (rational) $\theta > 0$ and such that, for $\varepsilon_0 = \varepsilon_0(\omega) > 0$ small enough and every $R > 0$,

$$\sup_{\varepsilon \in (0, \varepsilon_0)} \int_{\widetilde{Q}_R} (|a^\varepsilon(0, x, t)|^2 + |D\chi^\varepsilon(x, t; p)|^2) dx dt < +\infty. \tag{3.39}$$

The full measure subset of Ω in which homogenization takes place is $\Omega_0 = \Omega_1 \cap \Omega_2$. Heretofore, we always work with $\omega \in \Omega_0$.

3.4.3. *Extracting a subsequence.* Fix $\omega \in \Omega_0$. In view of (3.34), we know that the family $(u^\varepsilon)_{\varepsilon > 0}$ is compact in $L^2_{loc}(\mathbb{R}^{d+1})$.

Let $(u^{\varepsilon_n})_{n \in \mathbb{N}}$ be a converging sequence with limit u. Then, for any $\theta > 0$,

$$\begin{cases} u^{\varepsilon_n} \to u \text{ in } L^2_{\rho_\theta}(\mathbb{R}^d \times (0, T)), \quad Du^{\varepsilon_n} \rightharpoonup Du \text{ in } L^2_{\rho_\theta}(\mathbb{R}^d \times (0, T)), \quad \text{and} \\ \qquad a^{\varepsilon_n}(Du^{\varepsilon_n}, \frac{x}{\varepsilon_n}, \frac{t}{\varepsilon_n^2}, \omega) \rightharpoonup \xi \text{ in } L^2_{\rho_\theta}(\mathbb{R}^d \times (0, T)). \end{cases} \tag{3.40}$$

The aim is to prove that u is the unique solution to (3.32), which will then yield the a.s. convergence of u^ε to u.

Heretofore, we work along this particular subsequence ε_n, which we denote by ε to simplify the notation. Note that, in view of (3.1), we have

$$\partial_t u - \mathrm{div}(\xi) = f(x, t) \quad \text{in} \quad \mathbb{R}^d \times (0, T) \quad u(\cdot, 0) = u_0 \quad \text{in} \quad \mathbb{R}^d. \tag{3.41}$$

In addition, in view of (3.34), for any $\theta > 0$, we have

$$\sup_{t \in [0,T]} \|u(\cdot, t)\|_{L^2_{\rho_\theta}}^2 + \int_0^T \|Du(\cdot, t)\|_{L^2_{\rho_\theta}}^2 \, dt + \int_0^T \|\partial_t u\|_{H^{-1}_{\rho_\theta}}^2 \leq C_\theta(\omega), \qquad (3.42)$$

where $C_\theta(\omega) = \sup_{\varepsilon \in (0, \varepsilon_0]} C_\theta^\varepsilon(\omega)$ is finite, for ε_0 small, since by the construction of Ω_0, $C_\theta^\varepsilon(\omega)$ has a limit as $\varepsilon \to 0$. Similarly to the construction of the corrector, we need to prove that we can replace ξ by $\bar{a}(Du)$ in (3.41).

3.4.4. *The test functions.* Following the usual approach to prove homogenization for divergence form elliptic equations, given a test function $\phi \in C_c^\infty(\mathbb{R}^d \times [0, T))$, we need to consider, for each $\varepsilon > 0$, the corrector $\chi^\varepsilon(x, t, \omega) = \chi(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, D\phi(x, t), \omega)$ and work with $D\chi^\varepsilon$. The dependence on $D\phi$ creates technical problems since we do not have enough information about the regularity of the map $p \to \chi(\cdot, \cdot, p, \omega)$.

To circumvent this difficulty, we introduce a localization argument for the gradient of the corrector, which is based on a piecewise constant approximation of $D\phi$.

Fix $\delta \in (0, 1)$ and consider a locally finite family $(\widehat{Q}_k)_{k \in \mathbb{N}}$ of disjoint cubes $\widehat{Q}_k = Q_{R_k}(x_k) \times (t_k - T_k, t_k + T_k)$ in $\mathcal{Q}$ with $T_k + R_k \leq \delta$ covering $\mathbb{R}^d \times [0, T]$ up to a set of 0 Lebesgue measure.

Let

$$p_k = \fint_{\widetilde{Q}_k} D\phi(x, t) dx dt,$$

and, for each k, choose p_k^δ in the countable family $\mathcal{E}$ defined in subsection 3.4.2 and is such that $|p_k - p_k^\delta| \leq \delta$.

The localizations of $D\phi$ and $D\chi^\varepsilon$ are

$$D\widetilde{\phi}(x, t) = \sum_k p_k^\delta \mathbf{1}_{\widehat{Q}_k} \quad and \quad D\widetilde{\chi}^\varepsilon(x, t, \omega) = \sum_k D\chi^\varepsilon(x, t; p_k^\delta). \qquad (3.43)$$

Note that above we abused notation, since neither $D\widetilde{\phi}$ nor $D\widetilde{\chi}^\varepsilon$ are gradients. We use, however, the gradient symbol in order to stress the fact that they are respectively close to $D\phi$ and $D\chi^\varepsilon$. Indeed, we note, for later use, that $D\widetilde{\phi}$ and $D\widetilde{\chi}^\varepsilon$ depend on δ and that $D\widetilde{\phi}$ converges, as $\delta \to 0^+$, uniformly to $D\phi$.

Finally, we fix a smooth nonincreasing function $\zeta : [0, T] \to \mathbb{R}$ such that $\zeta(0) = 1$ and $\zeta(1) = 0$.

3.4.5. *The proof of Theorem 3.7.* We prove that $\xi = \bar{a}(D\bar{u})$ in (3.41).

We write for simplicity below $a^\varepsilon(p, x, t)$ for $a(p, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega)$.

The monotonicity of a gives

$$\int_0^T \int_{\mathbb{R}^d} \left(a^\varepsilon\big(Du^\varepsilon(x, t), x, t\big) - a^\varepsilon\big(D\widetilde{\phi}(x, t) + D\widetilde{\chi}^\varepsilon(x, t), x, t\big) \right)$$

$$\cdot (Du^\varepsilon(x, t) - D\widetilde{\phi}(x, t) - D\widetilde{\chi}^\varepsilon(x, t)) \rho_\theta(x) \zeta(t) dx dt \geq 0.$$

Multiplying (3.1) by $u^\varepsilon \rho_\theta \zeta$ and integrating in space and time we find

$$
- \int_{\mathbb{R}^d} \frac{u_0^2(x)}{2} \rho_\theta(x) dx - \int_0^T \int_{\mathbb{R}^d} \frac{(u^\varepsilon(x,t))^2}{2} \rho_\theta(x) \zeta'(t) dx dt
$$

$$
+ \int_0^T \int_{\mathbb{R}^d} a^\varepsilon(Du^\varepsilon(x,t), x, t) \cdot Du^\varepsilon(x,t) \rho_\theta(x) \zeta(t) dx dt
$$

$$
+ \int_0^T \int_{\mathbb{R}^d} u^\varepsilon a^\varepsilon(Du^\varepsilon(x,t), x, t) \cdot D\rho_\theta \zeta dx dt = \int_0^T \int_{\mathbb{R}^d} f(x,t) u^\varepsilon(x,t) \rho_\theta(x) \zeta(t).
$$

Subtracting the last two expressions we obtain

$$
\int_{\mathbb{R}^d} \frac{u_0^2}{2} \rho_\theta + \int_0^T \int_{\mathbb{R}^d} \frac{(u^\varepsilon)^2}{2} \rho_\theta \zeta' + \int_0^T \int_{\mathbb{R}^d} \left(-a^\varepsilon(Du^\varepsilon) \cdot (D\widetilde{\phi} + D\widetilde{\chi}^\varepsilon) \right.
$$

$$
(3.44)
$$

$$
\left. - a^\varepsilon(D\widetilde{\phi} + D\widetilde{\chi}^\varepsilon) \cdot (Du^\varepsilon - D\widetilde{\phi} - D\widetilde{\chi}^\varepsilon) + f u^\varepsilon \right) \rho_\theta \zeta - \int_0^T \int_{\mathbb{R}^d} u^\varepsilon a^\varepsilon(Du^\varepsilon) \cdot D\rho_\theta \zeta \geq 0.
$$

To let $\varepsilon \to 0$ in the above inequality, we first note that, in view of (3.40),

$$
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \frac{u_0^2}{2} \rho_\theta + \int_0^T \int_{\mathbb{R}^d} \frac{(u^\varepsilon)^2}{2} \rho_\theta \zeta' + \int_0^T \int_{\mathbb{R}^d} \left(-a^\varepsilon(Du^\varepsilon) \cdot D\widetilde{\phi} + f u^\varepsilon \right) \rho_\theta \zeta
$$

$$
- \int_0^T \int_{\mathbb{R}^d} u^\varepsilon a^\varepsilon(Du^\varepsilon) \cdot D\rho_\theta \zeta
$$

$$
= \int_{\mathbb{R}^d} \frac{u_0^2}{2} \rho_\theta + \int_0^T \int_{\mathbb{R}^d} \frac{u^2}{2} \rho_\theta \zeta' + \int_0^T \int_{\mathbb{R}^d} \left(-\xi \cdot D\widetilde{\phi} + f u \right) \rho_\theta \zeta
$$

$$
- \int_0^T \int_{\mathbb{R}^d} u \xi \cdot D\rho_\theta \zeta.
$$

We claim that

$$
\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}^d} a^\varepsilon(D\widetilde{\phi}(x,t) + D\widetilde{\chi}^\varepsilon(x,t), x, t) \cdot (D\widetilde{\phi}(x,t) + D\widetilde{\chi}^\varepsilon(x,t)) \rho_\theta(x) \zeta(t) dx dt
$$

$$
= \int_0^T \int_{\mathbb{R}^d} \overline{a}(D\widetilde{\phi}(x,t)) \cdot D\widetilde{\phi}(x,t) \rho_\theta(x) \zeta(t) dx dt, \qquad (3.45)
$$

and

$$
\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}^d} a^\varepsilon(Du^\varepsilon(x,t), x, t) \cdot D\widetilde{\chi}^\varepsilon(x,t) \rho_\theta(x) \zeta(t) dx dt \qquad (3.46)
$$

$$
+ \int_0^T \int_{\mathbb{R}^d} a^\varepsilon(D\widetilde{\phi}(x,t) + D\widetilde{\chi}^\varepsilon(x,t), x, t) \cdot Du^\varepsilon(x,t) \rho_\theta(x) \zeta(t) dx dt
$$

$$
= \int_0^T \int_{\mathbb{R}^d} \overline{a}(D\widetilde{\phi}(x,t)) \cdot Du(x,t) \rho_\theta(x) \zeta(t) dx dt.
$$

Assuming (3.45) and (3.46), we proceed with the ongoing proof. Passing to the $\varepsilon \to 0$ limit in (3.44), we find

$$\int_{\mathbb{R}^d} \frac{u_0^2}{2} \rho_\theta + \int_0^T \int_{\mathbb{R}^d} \frac{u^2}{2} \rho_\theta \zeta' + \int_0^T \int_{\mathbb{R}^d} \left(-\xi \cdot D\widetilde{\phi} - \overline{a}(D\widetilde{\phi}) \cdot (Du - D\widetilde{\phi}) + fu\right) \rho_\theta \zeta$$
$$- \int_0^T \int_{\mathbb{R}^d} u\xi \cdot D\rho_\theta \zeta \geq 0.$$

Next we let $\delta \to 0$. Since, $D\widetilde{\phi} \to D\phi$ uniformly, we obtain,

$$\int_{\mathbb{R}^d} \frac{u_0^2}{2} \rho_\theta + \int_0^T \int_{\mathbb{R}^d} \frac{u^2}{2} \rho_\theta \zeta' + \int_0^T \int_{\mathbb{R}^d} \left(-\xi \cdot D\phi - \overline{a}(D\phi) \cdot (Du - D\phi) + fu\right) \rho_\theta \zeta$$
$$- \int_0^T \int_{\mathbb{R}^d} u\xi \cdot D\rho_\theta \zeta \geq 0. \tag{3.47}$$

while using $\phi \rho_\theta \zeta$ as a test function in (3.41) yields

$$0 = -\int_{\mathbb{R}^d} u_0 \phi(0) \rho_\theta + \int_0^T \int_{\mathbb{R}^d} u(-\partial_t \phi \rho_\theta \zeta - \phi \rho_\theta \zeta') + \xi \cdot (D\phi \rho_\theta \zeta + \phi D\rho_\theta \zeta) - f\phi \rho_\theta \zeta.$$

Combining the equation above and (3.47) we get

$$\int_{\mathbb{R}^d} (\frac{u_0^2}{2} - u_0 \phi(0)) \rho_\theta + \int_0^T \int_{\mathbb{R}^d} u(\frac{u}{2} - \phi) \rho_\theta \zeta'$$
$$+ \int_0^T \int_{\mathbb{R}^d} \left(-\overline{a}(D\phi) \cdot (Du - D\phi) + f(u - \phi) - u\partial_t \phi\right) \rho_\theta \zeta \tag{3.48}$$
$$- \int_0^T \int_{\mathbb{R}^d} (u - \phi)\xi \cdot D\rho_\theta \zeta \geq 0.$$

We choose $\phi = u^\sigma + s\psi$ where $s > 0$, $\psi \in C_c^\infty(\mathbb{R}^d \times [0, T))$ and u^σ is a smooth approximation of u with compact support in $\mathbb{R}^d \times [0, T]$ such that, as $\sigma \to 0$,

$$u^\sigma(\cdot, 0) \to u_0, \quad u^\sigma \to u \quad \text{and} \quad Du^\sigma \to Du \quad \text{in} \quad L_{\rho_\theta}^2, \quad \text{and} \quad \partial_t u^\sigma \to \partial_t u \quad \text{in} \quad L^2(H_{\rho_\theta}^{-1});$$

note that such an approximation is possible in view of (3.42).

We prove below that

$$\lim_{\sigma \to 0} \int_{\mathbb{R}^d} (\frac{u_0^2}{2} - u_0 u^\sigma(0)) \rho_\theta + \int_0^T \int_{\mathbb{R}^d} u(\frac{u}{2} - u^\sigma) \rho_\theta \zeta' - \int_0^T \int_{\mathbb{R}^d} u\partial_t u^\sigma \rho_\theta \zeta = 0. \tag{3.49}$$

Thus, in the limit $\sigma \to 0$, (3.48) becomes

$$- s\int_{\mathbb{R}^d} u_0 \psi(0) \rho_\theta - s\int_0^T \int_{\mathbb{R}^d} u\psi \rho_\theta \zeta' + s\int_0^T \int_{\mathbb{R}^d} \left(\overline{a}(Du + sD\psi) \cdot D\psi - f\psi - u\partial_t \psi\right) \rho_\theta \zeta$$
$$+ s\int_0^T \int_{\mathbb{R}^d} \psi \xi \cdot D\rho_\theta \zeta \geq 0.$$

Then, we divide by s and let $s \to 0$ to get

$$-\int_{\mathbb{R}^d} u_0 \psi(0) \rho_\theta - \int_0^T \int_{\mathbb{R}^d} u\psi \rho_\theta \zeta' + \int_0^T \int_{\mathbb{R}^d} \left(\overline{a}(Du) \cdot D\psi - f\psi - u\partial_t \psi\right) \rho_\theta \zeta$$
$$+ \int_0^T \int_{\mathbb{R}^d} \psi \xi \cdot D\rho_\theta \zeta \geq 0.$$

Finally, letting $\zeta \to 1$ and $\theta \to 0$, so that $\zeta' \to 0$ and $\rho_\theta \to 1$ while $D\rho_\theta \to 0$ locally uniformly, we get

$$-\int_{\mathbb{R}^d} u_0 \psi(0) + \int_0^T \int_{\mathbb{R}^d} \left(\bar{a}(Du) \cdot D\psi - f\psi - u\partial_t \psi\right) \geq 0,$$

which, since ψ is arbitrary, yields that u is a weak solution to (3.32) since ψ is arbitrary.

The proof of the $\mathbb{P}$−a.s. convergence of the family $(u^\varepsilon)_{\varepsilon>0}$ to u in $L^2_{\rho_\theta}(\mathbb{R}^d \times [0,T])$ for any $\theta > 0$ is now complete. Moreover, in view of the estimates in (3.34), where C_θ^ε converges in expectation, the L^2 convergence of u^ε to u also holds in expectation.

In the next subsections, we prove (3.45), (3.46) and (3.49) hold.

3.4.6. *The proof of* (3.45). The definition of $D\widetilde{\phi}$ and $D\widetilde{\chi}^\varepsilon$ gives

$$\int_0^T \int_{\mathbb{R}^d} a^\varepsilon(D\widetilde{\phi}(x,t) + D\widetilde{\chi}^\varepsilon(x,t), x, t) \cdot (D\widetilde{\phi}(x,t) + D\widetilde{\chi}^\varepsilon)\rho_\theta(x)\zeta(t)dxdt$$

$$= \sum_k \int_{\widehat{Q}_k} a^\varepsilon(p_k^\delta + D\chi^\varepsilon(x,t;p_k^\delta), x, t) \cdot (p_k^\delta + D\chi^\varepsilon(x,t;p_k^\delta))\rho_\theta(x)\zeta(t)dxdt.$$

Since, in view of the choice of p_k^δ and of $\widehat{Q}_k$, (3.36) and (3.38) hold, we get

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}^d} a^\varepsilon(D\widetilde{\phi}(x,t) + D\widetilde{\chi}^\varepsilon(x,t), x, t) \cdot D\widetilde{\phi}(x,t))\rho_\theta(x)\zeta(t)dxdt$$

$$= \sum_k \int_{\widehat{Q}_k} \bar{a}(p_k^\delta) \cdot p_k^\delta \rho_\theta(x)\zeta(t)dxdt = \int_0^T \int_{\mathbb{R}^d} \bar{a}(D\widetilde{\phi}(x,t)) \cdot D\widetilde{\phi}(x,t)\rho_\theta(x)\zeta(t)dxdt,$$

which is (3.45).

3.5. **The proof of** (3.46). The argument is longer and more complicated.

Using again the piecewise structure of $D\widetilde{\chi}^\varepsilon$ and $D\widetilde{\phi}$, we find

$$\int_0^T \int_{\mathbb{R}^d} a^\varepsilon(Du^\varepsilon(x,t), x, t) \cdot D\widetilde{\chi}^\varepsilon(x,t)\rho_\theta(x)\zeta(t)dxdt$$

$$+ \int_0^T \int_{\mathbb{R}^d} a^\varepsilon(D\widetilde{\phi}(x,t) + D\widetilde{\chi}^\varepsilon(x,t), x, t)) \cdot Du^\varepsilon(x,t)\rho_\theta(x)\zeta(t)dxdt$$

$$= \sum_k \int_{\widehat{Q}_k} \Big(a^\varepsilon(Du^\varepsilon(x,t), x, t) \cdot D\chi^\varepsilon(x,t;p_k^\delta)$$

$$+ a^\varepsilon(p_k^\delta + D\chi^\varepsilon(x,t;p_k^\delta), x, t)) \cdot Du^\varepsilon(x,t)\Big)\rho_\theta(x)\zeta(t)dxdt.$$

Now we work separately in each cube $\widehat{Q}_k$. To simplify the notation, we denote by $\widehat{Q} = Q_R(x_0) \times (t_0 - T, t_0 + T)$ a generic cube $\widehat{Q}_k$ and let $p = p_k^\delta$, $\chi = \chi(\cdot, \cdot; p)$, and recall that $R + T \leq \delta \leq 1$.

Note that (3.46) follows, if we show that

$$\limsup_{\varepsilon \to 0} \int_{\widehat{Q}} \Big(a^\varepsilon (Du^\varepsilon(x,t), x, t) \cdot D\chi^\varepsilon(x,t)$$
$$+ a^\varepsilon(p + D\chi^\varepsilon(x,t), x, t)) \cdot Du^\varepsilon(x,t) \Big) \rho_\theta(x)\zeta(t) dx dt \qquad (3.50)$$
$$= \int_{\widehat{Q}} \overline{a}(p) \cdot Du(x,t) \rho_\theta(x)\zeta(t) dx dt.$$

To proceed, we need to work with functions which are compactly supported in $\widehat{Q}$. For this, we prove below that, for any $\delta' > 0$, we can choose $\psi \in C_c^\infty(\text{Int}(\widehat{Q}))$ and $\varepsilon_0 > 0$ such that

$$\sup_{\varepsilon \in (0, \varepsilon_0)} \int_{\widehat{Q}} \Big| \Big(a^\varepsilon(Du^\varepsilon(x,t), x, t) \cdot D\chi^\varepsilon(x,t)$$
$$+ a^\varepsilon(p + D\chi^\varepsilon(x,t), x, t) \cdot Du^\varepsilon(x,t) \Big) \rho_\theta(x)\zeta(t) \Big| |1 - \psi(x,t)| dx dt \qquad (3.51)$$
$$+ \int_{\widehat{Q}} |\overline{a}(p) \cdot Du(x,t) \rho_\theta(x)\zeta(t)| |1 - \psi(x,t)| dx dt \le \delta'.$$

Then, we show that, if $\kappa := \rho_\theta \zeta \psi$, then

$$\lim_{\varepsilon \to 0} \int_{\widehat{Q}} \Big(a^\varepsilon(Du^\varepsilon, x, t) \cdot D\chi^\varepsilon + a^\varepsilon(p + D\chi^\varepsilon, x, t) \cdot Du^\varepsilon \Big) \kappa dx dt = \int_{\widehat{Q}} \overline{a}(p) \cdot Du \kappa dx dt. \quad (3.52)$$

Once we know (3.52), we can combine (3.50) and (3.51) to get

$$\limsup_{\varepsilon \to 0} \Big| \int_{\widehat{Q}} \Big(a^\varepsilon(Du^\varepsilon(x,t), x, t) \cdot D\chi^\varepsilon(x,t) + a^\varepsilon(p + D\chi^\varepsilon(x,t), x, t)) \cdot Du^\varepsilon(x,t) \Big) \rho_\theta(x)\zeta(t) dx dt$$
$$- \int_{\widehat{Q}} \overline{a}(p) \cdot Du(x,t) \rho_\theta(x)\zeta(t) dx dt \Big| \le 2\delta',$$

which gives the result since δ' is arbitrary.

We now prove (3.52). Using $\chi^\varepsilon \kappa$ as a test function in (3.1) $u^\varepsilon \kappa$ as a test function in the equation satisfied by χ^ε we get

$$\int_{t_0 - T}^{t_0 + T} \langle \partial_t u^\varepsilon, \chi^\varepsilon \kappa \rangle_{H^{-1}, H^1} + \int_{\widehat{Q}} a(Du^\varepsilon) \cdot (D\chi^\varepsilon \kappa + D\kappa \chi^\varepsilon) = \int_{\widehat{Q}} f \chi^\varepsilon \kappa.$$

and

$$\int_{t_0 - T}^{t_0 + T} \langle \partial_t \chi^\varepsilon, u^\varepsilon \kappa \rangle_{H^{-1}, H^1} + \int_{\widehat{Q}} a(p + D\chi^\varepsilon) \cdot (Du^\varepsilon \kappa + D\kappa u^\varepsilon) = 0,$$

and, hence, after using an easy regularization argument, we find

$$\int_{\widehat{Q}} f\chi^\varepsilon \kappa = \int_{t_0 - T}^{t_0 + T} \langle \partial_t u^\varepsilon, \chi^\varepsilon \kappa \rangle_{H^{-1}, H^1} + \int_{t_0 - T}^{t_0 + T} \langle \partial_t \chi^\varepsilon, u^\varepsilon \kappa \rangle_{H^{-1}, H^1}$$
$$+ \int_{\widehat{Q}} (a(Du^\varepsilon) \cdot (D\chi^\varepsilon \kappa + D\kappa \chi^\varepsilon) + a(p + D\chi^\varepsilon) \cdot (Du^\varepsilon \kappa + D\kappa u^\varepsilon))$$
$$= - \int_{\widehat{Q}} (u^\varepsilon \chi^\varepsilon) \partial_t \kappa$$
$$+ \int_{\widehat{Q}} (a(Du^\varepsilon) \cdot (D\chi^\varepsilon \kappa + D\kappa \chi^\varepsilon) + a(p + D\chi^\varepsilon) \cdot (Du^\varepsilon \kappa + D\kappa u^\varepsilon)),$$

Recalling (3.36), (3.40) and that, in view of (3.35), $\chi^\varepsilon \to 0$ in L^2_{loc}, we pass to the limit $\varepsilon \to$ in the last equalities and get

$$\lim_\varepsilon \int_{\widehat{Q}} (a(Du^\varepsilon) \cdot D\chi^\varepsilon + a(p + D\chi^\varepsilon) \cdot Du^\varepsilon)\kappa + \bar{a}(p)D\kappa u = 0$$

An integration by parts then yields (3.52).

To complete the proof, we show that it is possible to build ψ with values in $[0, 1]$ in such a way that (3.51) holds. Indeed, choose an increasing family of cubes $(\widehat{Q}_n)_{n \in \mathbb{N}}$ in $\mathcal{Q}$ (recall was $\mathcal{Q}$ is defined in subsubsection 3.4.2) such that $|\widehat{Q} \backslash \widehat{Q}_n| \to 0$. Then, given $\gamma > 0$ to be chosen below, in view of (3.37) and for n large enough, we have

$$\lim_\varepsilon \int_{\widehat{Q} \backslash \widehat{Q}_n} |D\chi^\varepsilon|^2 dx dt = \mathbb{E}\left[\int_{\widetilde{Q}_1} |D\chi|^2\right] |\widetilde{Q} \backslash \widetilde{Q}_n| \le \gamma^2/2.$$

Hence, there exists ε_0 such that

$$\sup_{\varepsilon \in (0,\varepsilon_0]} \int_{\widehat{Q} \backslash \widehat{Q}_n} |D\chi^\varepsilon|^2 dx dt \le \gamma^2.$$

Choose $\psi \in C^\infty_c(\text{Int}(\widehat{Q}); [0, 1])$ such that $\psi = 1$ in $\widehat{Q}_n$. Then, for any $\varepsilon \in (0, \varepsilon_0]$,

$$\int_{\widehat{Q}} \left|a^\varepsilon(Du^\varepsilon) \cdot D\chi^\varepsilon \rho_\theta(x)\zeta(t)\right| |1 - \psi(x,t)| dx dt$$

$$\le \|\rho_\theta\zeta\|_\infty \|a^\varepsilon(Du^\varepsilon)\|_{L^2(\widehat{Q})} \|D\chi^\varepsilon|1 - \psi|\|_{L^2(\widehat{Q})}$$

$$\le \|\rho_\theta\zeta\|_\infty (\|a^\varepsilon(0)\|_{L^2(\widehat{Q})} + C_0 \|Du^\varepsilon\|_{L^2(\widehat{Q})}) \|D\chi^\varepsilon|1 - \psi|\|_{L^2(\widehat{Q})}$$

$$\lesssim_\omega \|D\chi^\varepsilon\|_{L^2(\widehat{Q} \backslash \widehat{Q}_n)} \lesssim_\omega \gamma;$$

the dependence on ω is through the constants in (3.34) and in (3.39).

Treating the other terms in (3.51) similarly we obtain, for γ small enough,

$$\sup_{\varepsilon \in (0,\varepsilon_0)} \int_{\widehat{Q}} \left|\left(a^\varepsilon(Du^\varepsilon(x,t), x, t) \cdot D\chi^\varepsilon(x,t)\right.\right.$$

$$\left.\left. + a^\varepsilon(p + D\chi^\varepsilon(x,t), x, t) \cdot Du^\varepsilon(x,t)\right)\rho_\theta(x)\zeta(t)\right| |1 - \psi(x,t)| dx dt$$

$$+ \int_{\widehat{Q}} |\bar{a}(p) \cdot Du(x,t)\rho_\theta(x)\zeta(t)||1 - \psi(x,t)| dx dt \lesssim_\omega C\gamma \le \delta'.$$

3.5.1. *The proof of* (3.49). Note first that

$$\lim_{\sigma \to 0} \int_{\mathbb{R}^d} (\frac{u_0^2}{2} - u_0 u^\sigma(0))\rho_\theta + \int_0^T \int_{\mathbb{R}^d} u(\frac{u}{2} - u^\sigma)\rho_\theta\zeta' = \int_{\mathbb{R}^d} -\frac{u_0^2}{2}\rho_\theta - \int_0^T \int_{\mathbb{R}^d} \frac{u^2}{2}\rho_\theta\zeta'. \quad (3.53)$$

On the other hand, the weak convergence, as $\sigma \to 0$, of $\partial_t u^\sigma$ to $\partial_t u$ yields

$$\int_0^T \|\partial_t u^\sigma\|^2_{H^{-1}_{\rho_\theta}} dt \lesssim 1.$$

Thus, as $\sigma \to 0$,

$$\left| \int_0^T \int_{\mathbb{R}^d} (u - u^\sigma) \partial_t u^\sigma \rho_\theta \zeta \right| \leq \left(\int_0^T \|\partial_t u^\sigma\|_{H_{\rho_\theta}^{-1}}^2 dt \right)^{1/2} \left(\int_0^T \|(u - u^\sigma)\zeta\|_{H_{\rho_\theta}^1}^2 \right)^{1/2}$$

$$\lesssim \left(\int_0^T \|(u - u^\sigma)\|_{H_{\rho_\theta}^1}^2 \right)^{1/2} \to 0.$$

Therefore,

$$\lim_{\sigma \to 0} \int_0^T \int_{\mathbb{R}^d} u \partial_t u^\sigma \rho_\theta \zeta = \lim_{\sigma \to 0} \int_0^T \int_{\mathbb{R}^d} u^\sigma \partial_t u^\sigma \rho_\theta \zeta$$

$$= \lim_{\sigma \to 0} - \int_{\mathbb{R}^d} \frac{(u^\sigma)^2(0)}{2} - \int_0^T \int_{\mathbb{R}^d} \frac{(u^\sigma)^2}{2} \rho_\theta \zeta' = - \int_{\mathbb{R}^d} \frac{u_0^2(0)}{2} - \int_0^T \int_{\mathbb{R}^d} \frac{u^2}{2} \rho_\theta \zeta'. \quad (3.54)$$

Combining (3.53) and (3.54) gives (3.49).

4. The Homogenization of (1.1)

We use the results of the two previous sections to study the behavior, as $\varepsilon \to$, of (1.1).

We begin with the assumptions. As far the $(B^k)_{k \in \mathbb{Z}^d}$ and A are concerned we assume (2.4) and (2.5).

We also assume

$$\begin{cases} (\Omega_1, \mathcal{F}_1, \mathbb{P}_1) \text{is a probability space endowed with an ergodic} \\ \text{measure-preserving group of transformations } \tau : \mathbb{Z}^d \times \mathbb{R} \times \Omega_1 \to \Omega_1, \end{cases} \quad (4.1)$$

and

$$\begin{cases} \mathcal{A} : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \Omega_1 \to \mathbb{R}^d \text{ is a smooth and stationary in } (\Omega_1, \mathcal{F}_1, \mathbb{P}_1) \\ \text{vector field, which is strongly monotone and Lipschitz continuous} \\ \text{in the first variable, uniformly with respect to the other variables;} \end{cases} \quad (4.2)$$

note that the family $(B^k)_{k \in \mathbb{Z}^d}$ and the vector field $\mathcal{A}$ are defined in different probability spaces.

Finally, for the random environment we assume that

$$(\Omega, \mathcal{F}, \mathbb{P}) \text{ is the product probability space of } (\Omega_0, \mathcal{F}_0, \mathbb{P}_0) \text{ and } (\Omega_1, \mathcal{F}_1, \mathbb{P}_1), \quad (4.3)$$

that is, $\Omega = \Omega_0 \times \Omega_1$, $\mathcal{F} = \mathcal{F}_0 \otimes \mathcal{F}_1$ and $\mathbb{P} = \mathbb{P}_0 \otimes \mathbb{P}_1$.

We continue making precise the meaning of a solution of (1.1). A field U^ε solves (1.1) if

$$U_t^\varepsilon(x, \omega) = \varepsilon V_{\frac{t}{\varepsilon^2}}(\frac{x}{\varepsilon}, \omega_0) + W^\varepsilon(x, t, \omega) = V_t^\varepsilon(x, \omega_0) + W^\varepsilon(x, t, \omega), \quad (4.4)$$

with V and W^ε solving respectively (2.2) and

$$\partial_t W_t^\varepsilon = \text{div}\,(\widehat{a}^\varepsilon(DW_t^\varepsilon, x, t, \omega)) \quad \text{in } \mathbb{R}^d \times (0, +\infty) \quad W_0^\varepsilon = u_0 \quad \text{in } \mathbb{R}^d, \quad (4.5)$$

where

$$\widehat{a}^\varepsilon(p, x, t, \omega) = \widehat{a}(p, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega), \quad (4.6)$$

and

$$\widehat{a}(p, x, t, \omega) = \mathcal{A}(p + DV_t(x, \omega_0), x, t, \omega_1) - DV_t(x, \omega_0). \quad (4.7)$$

Note that $\widehat{a}$ is strongly monotone and Lipschitz continuous in the first variable, uniformly with respect to the other variables, and satisfies (3.7) (thanks to Lemma 2.2).

We say that $W^\varepsilon : \mathbb{R}^d \times [0,T] \times \Omega \to \mathbb{R}$ is a solution of (4.5), if it is measurable in ω for each (x,t), $W_\cdot^\varepsilon(\cdot,\omega) \in L^2([0,T], H^1_{\rho_\theta}) \cap C^0([0,T], L^2_{\rho_\theta})$ $\mathbb{P}$−a.s. with ρ_θ defined in (3.33), and it satisfies (4.5) in the sense of distributions. It is easily checked that such a solution exists and is unique.

Theorem 4.1. *Assume (2.4), (2.5), (4.1), and (4.2). Then there exists a strongly monotone and Lipschitz continuous vector filed $\overline{a} : \mathbb{R}^d \to \mathbb{R}^d$ such that, for any $u_0 \in L^2(\mathbb{R}^d)$, the solution U^ε of (1.1) converges to the solution of the homogenized problem*

$$\partial_t \overline{u} = \mathrm{div}(\overline{a}(D\overline{u})) \ \ \text{in } \mathbb{R}^d \times (0,\infty) \quad \overline{u}(\cdot,0) = u_0 \ \ \text{in } \mathbb{R}^d, \tag{4.8}$$

in the sense that, for any $T > 0$,

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} |U_t^\varepsilon(x) - u(x,t)|^2 \rho_\theta(x) dx dt\right] = 0,$$

where $\rho_\theta(x) = \exp\{-\theta(1 + |x|^2)^{1/2}\}$.

The proof is a combination of the results of the previous sections.

The first step consists in replacing the non-stationary in time process DV^ε by the space-time stationary random field Z constructed in Theorem 2.1. To keep the notation in the statement simpler, we introduce the maps $\widetilde{a}^\varepsilon$ and $\widetilde{a}$ which are defined as

$$\widetilde{a}^\varepsilon(p,x,t,\omega) = \widetilde{a}(p, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega), \tag{4.9}$$

and

$$\widetilde{a}(p,x,t,\omega) = \mathcal{A}(p + Z_t(x,\omega_0), x, t, \omega_1) - Z_t(x,\omega_0). \tag{4.10}$$

Lemma 4.2. *Assume (2.4), (2.5), (4.1), and (4.2), and let W^ε and $\widetilde{W}^\varepsilon$ be respectively solutions of (4.5) with $\widehat{a}^\varepsilon$ as in (4.6) and*

$$\partial_t \widetilde{W}^\varepsilon = \mathrm{div}\left(\widetilde{a}^\varepsilon(\widetilde{W}^\varepsilon, x, t, \omega)\right) \ \ \text{in } \mathbb{R}^d \times (0,\infty) \quad \widetilde{W}_0^\varepsilon = u_0 \ \ \text{in } \mathbb{R}^d, \tag{4.11}$$

with $\widetilde{a}^\varepsilon$ given by (4.9). Then, for any $\theta > 0$,

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \mathbb{E}\left[\int_{\mathbb{R}^d} |W^\varepsilon(x,t) - \widetilde{W}^\varepsilon(x,t)|^2 \rho_\theta(x) dx\right] = 0. \tag{4.12}$$

Proof. Let $V^\varepsilon(x,t)) = DV_{\frac{t}{\varepsilon^2}}(\frac{x}{\varepsilon})$ and $Z_t^\varepsilon(x) = Z_{\frac{t}{\varepsilon^2}}(\frac{x}{\varepsilon})$.

Using the strong monotonicity and Lipschitz continuity of $\mathcal{A}$ as well as (4.6) and (4.7) we find, after some routine calculations, that, for some constants $C > 0$,

$$\frac{d}{dt}\mathbb{E}\left[\int_{\mathbb{R}^d} (W_t^\varepsilon - \widetilde{W}_t^\varepsilon)^2 \rho_\theta dx\right] \leq -\mathbb{E}\left[\int_{\mathbb{R}^d} |D(W_t^\varepsilon - \widetilde{W}_t^\varepsilon)|^2 \rho_\theta dx\right]$$

$$+ C\mathbb{E}\left[\int_{\mathbb{R}^d} (W_t^\varepsilon - \widetilde{W}_t^\varepsilon)^2 \rho_\theta dx\right] + C\mathbb{E}\left[\int_{\mathbb{R}^d} |DV_t^\varepsilon - Z_t^\varepsilon|^2 \rho_\theta dx\right].$$

Since DV and Z are stationary in space, we find

$$\mathbb{E}\left[\int_{\mathbb{R}^d} |DV_t^\varepsilon - Z_t^\varepsilon|^2 \rho_\theta dx\right] \leq C_\theta \mathbb{E}\left[\int_{Q_1} |DV_{\varepsilon^{-2}t}(x) - Z_{\varepsilon^{-2}t}(x)|^2 dx\right],$$

with the right hand side bounded and converging, in view of (2.7), to 0 for $t > 0$. We conclude using Gronwall's inequality.

$\square$

The proof of Theorem 4.1. It now remains to show that (1.1) homogenizes.
On Ω we define the ergodic measure preserving group $\widetilde{\tau} : \mathbb{Z}^d \times \mathbb{R} \times \Omega \to \Omega$ by

$$\widetilde{\tau}_{k,s}\omega = (\omega_0^{l+k}(s + \cdot), \tau_{k,s}\omega_1)$$

for any $\omega = (\omega_0, \omega_1) = ((\omega_0^l)_{l \in \mathbb{Z}^d}, \omega_1) \in \Omega = (C^0(\mathbb{R}, \mathbb{R}^d))^{\mathbb{Z}^d} \times \Omega_1$.
Set

$$a(p, x, t, \omega) = \mathcal{A}(p + Z_t(x, \omega_0), x, t, \omega_1) - Z_t(x, \omega)$$

and note that a satisfies (3.5) and (3.6).
Then, in view of Theorem 3.7, the vector field $\overline{a}$ is strongly monotone and Lipschitz continuous and the solution $\widetilde{W}^\varepsilon$ of (4.11) converges, for all $\theta > 0$, $\mathbb{P}-$a.s. in $L^2_{\rho_\theta}(\mathbb{R}^d \times [0, T])$ and in $L^2_{\rho_\theta}(\mathbb{R}^d \times [0, T] \times \Omega)$ to the solution $\overline{u}$ of (4.8).
Finally we return to U^ε. In view of (4.4), for any $\theta > 0$, we have

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} |U_t^\varepsilon(x) - u(x, t)|^2 \rho_\theta(x) dx dt\right]$$

$$\leq 2\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} |\varepsilon V_{\varepsilon^{-2}t}(\varepsilon^{-1}x)|^2 \rho_\theta(x) dx dt\right] + 2\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} |W_t^\varepsilon(x) - \widetilde{W}_t^\varepsilon(x)|^2 \rho_\theta(x) dx dt\right]$$

$$+ 2\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} |\widetilde{W}_t^\varepsilon(x) - u(x, t)|^2 \rho_\theta(x) dx dt\right].$$

In view of (2.10), Lemma 4.2 and Theorem 3.7, the right hand side of the inequality above tends to 0 as $\varepsilon \to 0$.

$\square$

APPENDIX A

We summarize here with proofs results about stationary gradients, which are needed in the paper. Some of them appear in the literature in different structures and with stronger assumptions.

The following is classical in the literature (see for instance the proof of Theorem 5.3 of [10]). We give a proof here because the environment has not exactly the same structure as in [10] and the maps here have lower regularity in time.

Lemma A.1. *Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with an ergodic group of measure preserving maps $\tau : \mathbb{Z}^d \times \mathbb{R} \times \Omega \to \Omega$, and, for $i = 1, \ldots, d$ and $t \in \mathbb{R}$, let $\mathcal{G}_i$ and $\mathcal{G}_t$ be respectively the $\sigma-$algebra of sets $A \in \mathcal{F}$ such that, for any $k \in \mathbb{Z}$, $\mathbb{P}[A \Delta(\tau_{(ke_i, 0)}A)] = 0$, and the $\sigma-$algebra of sets $A \in \mathcal{F}$ such that, for any $s \in \mathbb{R}$, $\mathbb{P}[A \Delta(\tau_{(0,s)}A)] = 0$. If $u : \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}$ has space-time stationary weak derivatives Du and $\partial_t u$ such that*

$$\mathbb{E}[\int_{\widetilde{Q}_1} |Du|^2] < +\infty, \quad \mathbb{E}[\int_{\widetilde{Q}_1} Du] = 0, \quad \mathbb{E}[\int_0^1 \|\partial_t u(\cdot, t)\|^2_{H^{-1}(Q_1)} dt] < +\infty,$$

$$\mathbb{E}[\int_0^1 \langle \partial_t u(\cdot, t), 1 \rangle_{H^{-1}(Q_1), H^1(Q_1)} dt] = 0 \quad and \quad \int_{Q_1} u dx = 0 \quad \mathbb{P} - a.s.,$$

then, for any $i = 1, \ldots, d$ and any $(z, t) \in \mathbb{R}^d \times \mathbb{R}$,

$$\mathbb{E}\left[\int_{Q_1} \partial_{x_i} u(\cdot + z, t) \,\Big|\, \mathcal{G}_i \right] = 0 \quad and \quad \mathbb{E}\left[\langle \partial_t u(\cdot + z, t), 1 \rangle_{H^{-1}(Q_1), H^1(Q_1)} \,\Big|\, \mathcal{G}_t \right] = 0.$$

Proof. To fix the ideas we prove the result for $i = 1$.

Fix $(z, s) \in \mathbb{R}^d \times \mathbb{R}$ and let $\xi : \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}$ be bounded, stationary, and $\mathcal{G}_1$−measurable. For any $n \in \mathbb{N}$ large, we have

$$\int_{\tilde{Q}_1} (u(x + ne_1 + z, t + s) - u(x, t))\xi(x, t)dxdt$$

$$= \sum_{l=0}^{n-1} \int_{\tilde{Q}_1} \int_0^1 \partial_{x_1} u(x + le_1 + re_1 + z, t + s)\xi(x, t)dxdtdr$$

$$+ \int_{\tilde{Q}_1} (u(x + z, t + s) - u(x, t))\xi(x, t)dxdt.$$

It follows from the stationarity of $\partial_{x_1} u$ and the $\mathcal{G}_1$−measurability of ξ that

$$\mathbb{E}\left[\int_{\tilde{Q}_1} (u(x + ne_1 + z, t + s) - u(x, t))\xi(x, t)dxdt \right]$$

$$= n\mathbb{E}\left[\int_0^1 \int_{\tilde{Q}_1} \partial_{x_1} u(x + re_1 + z, t + s)\xi(x, t)dxdtdr \right]$$

$$+ \mathbb{E}\left[\int_{\tilde{Q}_1} (u(x + z, t + s) - u(x, t))\xi(x, t)dxdt \right]$$

$$= n\mathbb{E}\left[\int_{\tilde{Q}_1} \partial_{x_1} u(x + z, t + s)\xi(x, t)dxdt \right] + \mathbb{E}\left[\int_{\tilde{Q}_1} (u(x + z, t + s) - u(x, t))\xi(x, t)dxdt \right],$$

the last two lines following from the $\mathbb{Z}$−periodicity of $s \to \mathbb{E}\left[\partial_{x_1} u(x + se_1 + z, t + s)\xi(x, t)dxdt \right]$.

Hence

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E}\left[\int_{\tilde{Q}_1} (u(x + ne_1 + z, t + s) - u(x, t))\xi(x, t)dxdt \right]$$

$$= \mathbb{E}\left[\int_{\tilde{Q}_1} \partial_{x_1} u(x + z, t + s)\xi(x, t)dxdt \right].$$

On the other hand,

$$\mathbb{E}\left[\int_{\tilde{Q}_1} (u(x + ne_1 + z, t + s) - u(x, t))\xi(x, t)dxdt \right] =$$

$$\mathbb{E}\left[\int_{\tilde{Q}_1} (u(x + ne_1, t) - u(x, t))\xi(x, t)dxdt \right]$$

$$+ \mathbb{E}\left[\int_{\tilde{Q}_1} (u(x + ne_1 + z, t + s) - u(x + ne_1 + z, t))\xi(x, t)dxdt \right]$$

$$+ \mathbb{E}\left[\int_{\tilde{Q}_1} (u(x + ne_1 + z, t) - u(x + ne_1, t))\xi(x, t)dxdt \right].$$

The goal is to divide by n and let $n \to +\infty$. The left-hand side and the first term in the right-hand side have a limit given by the previous equality.

We show next that the two remaining terms after divided by n tend to 0.

In order to use the time regularity of u, we need to regularize in space the indicatrix function of Q_1. Let $\zeta_\delta \in C_c^\infty(Q_1)$ with $\|1 - \zeta_\delta\|_{L^2(Q_1)} \le \delta$.

Then, using the stationarity of $\partial_t u$ and the fact that ξ is $\mathcal{G}_1-$measurable, we find

$$\mathbb{E}\left[\iint_{\widetilde{Q}_1}(u(x+ne_1+z,t+s)-u(x+ne_1+z,t))\zeta_\delta(x)\xi(x,t)dxdt\right]$$

$$=\mathbb{E}\left[\int_{-1/2}^{1/2}\int_0^s\langle\partial_t u(\cdot+ne_1+z,t+s'),\zeta_\delta\xi(\cdot,t)\rangle_{H^{-1},H^1}ds'dt\right]$$

$$=\mathbb{E}\left[\int_{-1/2}^{1/2}\int_0^s\langle\partial_t u(\cdot+z,t+s'),\zeta_\delta\xi(\cdot,t)\rangle_{H^{-1},H^1}ds'dt\right],$$

Thus,

$$\mathbb{E}\left[\iint_{\widetilde{Q}_1}(u(x+ne_1+z,t+s)-u(x+ne_1+z,t))\zeta_\delta(x)\xi(x,t)dxdt\right]$$

$$=\mathbb{E}\left[\iint_{\widetilde{Q}_1}(u(x+z,t+s)-u(x+z,t))\zeta_\delta(x)\xi(x,t)dxdt\right],$$

and, after letting $\delta \to 0$,

$$\mathbb{E}\left[\iint_{\widetilde{Q}_1}(u(x+ne_1+z,t+s)-u(x+ne_1+z,t))\xi(x,t)dxdt\right]$$

$$=\mathbb{E}\left[\iint_{\widetilde{Q}_1}(u(x+z,t+s)-u(x+z,t))\xi(x,t)dxdt\right].$$

Similarly, using the stationarity of Du, we get

$$\mathbb{E}\left[\iint_{\widetilde{Q}_1}(u(x+ne_1+z,t)-u(x+ne_1,t))\xi(x,t)dxdt\right]=\mathbb{E}\left[\iint_{\widetilde{Q}_1}(u(x+z,t)-u(x,t))\xi(x,t)dxdt\right].$$

It follows that, for any $(z,s) \in \mathbb{R}^d \times \mathbb{R}$ and any $\mathcal{G}_1-$measurable ξ,

$$\mathbb{E}\left[\iint_{\widetilde{Q}_1}(\partial_{x_1}u(x+z,t+s)-\partial_{x_1}u(x,t))\xi(x,t)dxdt\right]=0$$

Hence, the map

$$(z,s) \to \mathbb{E}\left[\iint_{\widetilde{Q}_1}(\partial_{x_1}u(x+z,t+s)dxdt \mid \mathcal{G}_1\right]$$

is $\mathbb{P}-$a.s constant. Since it is also stationary in an ergodic environment, it must also be constant in ω and, as it has a zero expectation, it has to be equal to 0.

The proof of the time derivative follows is similar and, hence, we omit it.

$\square$

We discuss next the sublinearity of maps with stationary derivatives.

Lemma A.2. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *and* $u : \mathbb{R}^{d+1} \times \Omega \to \mathbb{R}$ *be as in Lemma A.1. Then,* $\mathbb{P}-$*a.s. and in expectation,*

$$\lim_{R\to\infty}R^{-(d+2)}\int_{Q_R}|u(x,0)|^2dxdt=0 \quad \text{and} \quad \lim_{R\to\infty}R^{-(d+3)}\int_{\widetilde{Q}_R}|u(x,t)|^2dxdt.$$

The above result can also be formulated as follows. Let $u^\varepsilon(x, t, \omega) = \varepsilon u(x/\varepsilon, t/\varepsilon, \omega)$. Then, for any fixed $R > 0$, $\mathbb{P}-$a.s. and in expectation,

$$\lim_{\varepsilon \to 0} \int_{Q_R} |u^\varepsilon(x, 0)|^2 dxdt = 0 \text{ and } \lim_{\varepsilon \to 0} \int_{\widetilde{Q}_R} |u^\varepsilon(x, t)|^2 dxdt = 0.$$

Note that, here, the scaling is hyperbolic in contrast with what we did throughout the paper.

Proof. In view of Lemma A.1, we can apply Theorem 5.3 of [10] to the map $x \to u(x, 0)$ to infer that, for any $R > 0$ and $\mathbb{P}-$a.s.

$$\lim_{\varepsilon \to 0} \int_{Q_R} |u^\varepsilon(x, 0)| dx = 0. \tag{A.1}$$

In [10], the problem is stationary with respect to any (space) translation, while here the problem is $\mathbb{Z}^d-$ stationary. However, a careful inspection of the proof of Theorem 5.3 in [10] shows that the result still holds in our setting, the key point of the proof in [10] being precisely the statement in Lemma A.1.

Let $\xi_R \in C_c^\infty(\mathbb{R}^d; [0, 1])$ be such that $\xi_R = 1$ in Q_R, $\xi_R = 0$ in $\mathbb{R}^{d+1} \setminus Q_{R+1}^c$ and $\|D\xi_R\|_\infty \leq 2$. Then, after an integration by parts in time, we have

$$\int_0^{R/2} \int_{Q_R} u^\varepsilon(x, t)\xi_R(x)dxdt = \int_{Q_R} u^\varepsilon(x, 0)\xi_R(x)dxdt - \int_0^{R/2}(1-t)\langle \partial_t u^\varepsilon(\cdot, t), \xi_R \rangle_{H^{-1}, H^1} dt.$$

In view of (A.1), the first term in the right-hand side tends to 0 as $\varepsilon \to 0$, while, since $q(k, s, \omega) = \langle \partial_t u(\cdot, t), 1 \rangle_{H^{-1}(Q_1), H^1(Q_1)}$ is stationary, the ergodic theorem also implies that the second term in the right-hand side has a $\mathbb{P}-$a.s limit., which again does not depend on ω and, therefore, has to be zero since $\mathbb{E}[\langle \partial_t u, 1 \rangle_{H^{-1}, H^1}] = 0$.

It follows that, $\mathbb{P}-$a.s.,

$$\limsup_{\varepsilon \to 0} \int_0^{R/2} \int_{Q_R} u^\varepsilon(x, t)\xi_R(x)dxdt = 0.$$

Applying similar arguments on the time interval $[-T, 0]$, we also find that, $\mathbb{P}-$a.s.,

$$\lim_{\varepsilon \to 0} \int_{\widetilde{Q}_R} u^\varepsilon(x, t)\xi(x)dxdt = 0. \tag{A.2}$$

Next, we claim that there exists a constant C such that, $\mathbb{P}-$a.s.,

$$\limsup_{\varepsilon \to 0} \int_{\widetilde{Q}_R} (u^\varepsilon(x, t))^2 dxds \leq C.$$

Indeed, set

$$\langle u^\varepsilon \rangle_{\xi_R} := (\int_{\widetilde{Q}_R} \xi_R(x)dxdt)^{-1} \int_{\widetilde{Q}_R} u^\varepsilon(x, t)\xi_R(x)dxdt,$$

and observe that a minor generalization of the classical Poincaré 's inequality yields, for some C_R which depends on ξ_R,

$$\int_{\widetilde{Q}_R} (u^\varepsilon(x, t))^2 dxdt \leq 2 \int_{\widetilde{Q}_R} (u^\varepsilon(x, t) - \langle u^\varepsilon \rangle_{\xi_R})^2 dx + 2R^{d+1}\langle u^\varepsilon \rangle_{\xi_R}^2$$

$$\leq C_R[\int_{\widetilde{Q}_R} |Du^\varepsilon(x, t)|^2 dxdt + \int_{-R/2}^{R/2} \|\partial_t u^\varepsilon(\cdot, t)\|_{H^{-1}(Q_R)}^2 dt] + 2R^{d+1}\langle u^\varepsilon \rangle_{\xi_R}^2,$$

Then, the ergodic Theorem and (A.2), give that, $\mathbb{P}-$a.s.,

$$\limsup_{\varepsilon \to 0} \int_{\widetilde{Q}_R} (u^\varepsilon(x,t))^2 dx \le C_R \mathbb{E} \left[\int_{\widetilde{Q}_1} |Du(x,t)|^2 dx dt + \int_{-1/2}^{1/2} \|\partial_t u(t)\|^2_{H^{-1}(Q_1)} dt \right].$$

To summarize, we have shown that there exists $\Omega_0 \subset \Omega$ on which, for every $R > 0$ and $\mathbb{P}-$a.s.,

$$\text{the family } (u^\varepsilon)_{\varepsilon>0} \text{ is bounded respectively in } L^2([-R,R], H^1(Q_R)), \qquad (A.3)$$

and

$$\text{the family } (\partial_t u^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^2([-R,R], H^{-1}(Q_R)). \qquad (A.4)$$

It follows that, for any $\omega \in \Omega_0$, the family $(u^\varepsilon)_{\varepsilon>0}$ is relatively compact in $L^2_{loc}(\mathbb{R}^{d+1})$.

Let $(u^{\varepsilon_n})_{n \in \mathbb{N}}$ be a sequence which converges in $L^2_{loc}(\mathbb{R}^{d+1})$ to some $u \in L^2([-R,R], H^1(Q_R))$ with $\partial_t u$ in $L^2([-R,R], H^{-1}(Q_R))$. Since, as $\varepsilon \to 0$, $Du^\varepsilon \rightharpoonup 0$ and $\partial_t u^\varepsilon \rightharpoonup 0$, u is a constant, which, in view of (A.2), must be zero.

It follows that, as $n \to \infty$ and in $L^2_{loc}(\mathbb{R}^{d+1})$, $u^{\varepsilon_n} \to 0$, and, therefore that, as $\varepsilon \to 0$ and $\mathbb{P}-$a.s., $u^\varepsilon \to 0$ in $L^2_{loc}(\mathbb{R}^{d+1})$, and, by the estimate above, in expectation.

The claim for $u^\varepsilon(\cdot, 0)$ follows similarly and with a simpler argument, hence, we omit it.

$\square$

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