ON THE SPECTRAL INSTABILITY FOR WEAK INTERMEDIATE TRIHARMONIC PROBLEMS

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Abstract. We define the weak intermediate boundary conditions for the triharmonic operator $-\Delta^3$. We analyse the sensitivity of this type of boundary conditions upon domain perturbations. We construct a perturbation $(\Omega_\epsilon)_{\epsilon>0}$ of a smooth domain Ω of $\mathbb{R}^N$ for which the weak intermediate boundary conditions on $\partial\Omega_\epsilon$ are not preserved in the limit on $\partial\Omega$, analogously to the Babuška paradox for the hinged plate. Four different boundary conditions can be produced in the limit, depending on the convergence of $\partial\Omega_\epsilon$ to $\partial\Omega$. In one particular case, we obtain a "strange" boundary condition featuring a microscopic energy term related to the shape of the approaching domains. Many aspects of our analysis could be generalised to an arbitrary order elliptic differential operator of order $2m$ and to more general domain perturbations.

1. Introduction

Let W be a smooth bounded domain of $\mathbb{R}^{N-1}$, $b \in C^4(W)$ be a periodic, positive function with period $Y = (-1/2, 1/2)^{N-1}$. Let $\alpha \in (0, +\infty)$ be fixed, and define

$$\begin{aligned} \Omega_\epsilon &:= \left\{ x = (\bar{x}, x_N) \in \Omega : \bar{x} \in W, -1 < x_N < g_\epsilon(\bar{x}) = \epsilon^\alpha b\left(\frac{\bar{x}}{\epsilon}\right) \right\} \\ \Omega &:= W \times (-1, 0), \end{aligned} \tag{1.1}$$

for $\epsilon \in (0, 1]$. We consider the weak intermediate problem for the triharmonic operator $A_\epsilon = (-\Delta)^3 + I$ in Ω_ϵ, given by

$$\int_{\Omega_\epsilon} \left(D^3 u_\epsilon : D^3 \varphi + u_\epsilon \varphi \right) = \lambda(\Omega_\epsilon) \int_{\Omega_\epsilon} u_\epsilon \, \varphi, \quad \varphi \in H^3(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon), \tag{1.2}$$

where $D^3 f : D^3 g = \sum_{i,j,k=1,\dots N} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} \frac{\partial^3 g}{\partial x_i \partial x_j \partial x_k}$ is the Frobenius product of the two tensors $D^3 f$ and $D^3 g$, $\lambda(\Omega_\epsilon)$ is the eigenvalue and $u_\epsilon \in H^3(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon)$ is the eigenfunction. Here and in the sequel H^k, H_0^k denote the standard Sobolev spaces with regularity index k and integrability index 2.

We are interested in the behaviour of the solutions u_ϵ and of the eigenvalues $\lambda(\Omega_\epsilon)$ of (1.2) as $\epsilon \to 0$. Note that Ω_ϵ approaches Ω as $\epsilon \to 0$ in a rather singular way, since the function g_ϵ oscillates with very large frequency as $\epsilon \to 0$. It is worth noting that if $\alpha < 3$, it is not possible to construct a family of smooth diffeomorphisms $\Phi_\epsilon : \Omega \to \Omega_\epsilon$ such that $\|\Phi_\epsilon - I\|_{C^3(\mathbb{R}^N, \mathbb{R}^N)} \to 0$ as $\epsilon \to 0$. Therefore classical and elegant techniques based on the direct comparison of the Rayleigh quotients associated to $\lambda(\Omega_\epsilon)$ and $\lambda(\Omega)$ do not work in general in the singular setting described in (1.1).

Polyharmonic operators $(-\Delta)^m$ with intermediate or Neumann boundary conditions are known to be rather sensitive to variation of the domains in $\mathbb{R}^N$, $N > 1$. See for example [8] for regular perturbations, and [2, 23, 1] for more singular settings. When $m = 1$, unexpected limiting behaviour of the eigenvalues of the Neumann Laplace operator $-\Delta_{\text{neu}}$ is well-known since the 'dumbbell' example in [17], where $\lambda_2(\Omega_\epsilon) \to 0$ as $\epsilon \to 0$ instead of converging to $\lambda_2(\Omega) > 0$. More in general, let R_ϵ

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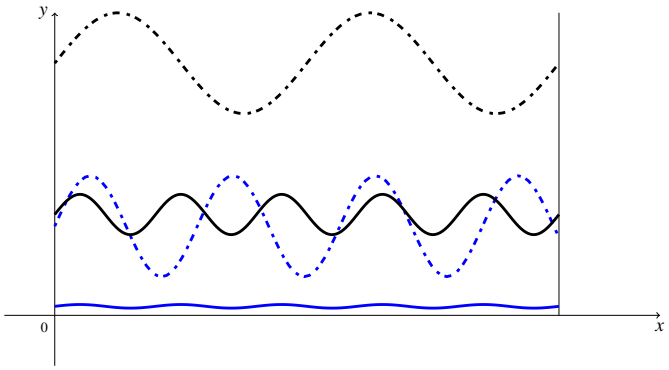

FIGURE 1. Graph of $g_\epsilon(x) = \epsilon^\alpha b(x/\epsilon)$ with $b(y) = 10+2\sin(\pi y/5)$. Black colour corresponds to $\alpha = 1$, blue to $\alpha = 5/2$. The dashed line corresponds to $\epsilon = 0.5$, the thick line to $\epsilon = 0.2$. The blue graph flattens out much faster than the black one as $\epsilon \to 0$.

be a smooth domain of $\mathbb{R}^N$ converging (in Hausdorff sense) to a lower dimensional set $D \subset \mathbb{R}^d$, $d < N$, and let Ω_ϵ be the smooth domain obtained by attaching R_ϵ to a bounded smooth domain $\Omega \subset \mathbb{R}^N$. Then, the eigenvalues of $-\Delta_{\text{neu}}$ in Ω_ϵ will not converge only to the respective eigenvalues in Ω, but also to the eigenvalues of a differential problem in D. Indeed, the eigenvalues of $-\Delta_{\text{neu}}$ on R_ϵ are known to converge to the eigenvalues of the Laplace-Beltrami operator on D, see e.g., [31]. See also [24] for more results for reaction-diffusion operators on thin domains, [25, 26] for dumbbell-type domains,[3, 4] for domains with fast oscillating boundaries and [16, 18] for domains with small holes. When $m = 2$ the Babuška Paradox for the biharmonic operator Δ^2_{SBC} shows that intermediate boundary conditions are not stable under polygonal approximation of a smooth domain in $\mathbb{R}^2$, see [30] and the introduction to [21] for further details. Elliptic operators of order $2m$ with $m \geq 2$ and diverse boundary conditions have been recently considered in [15, 7] and in the preprint [22] where it is shown that the eigenvalues of the biharmonic operator with Neumann boundary conditions on a thin domain converge, as the size of the domain tends to zero, to the eigenvalues of a system of equations on the boundary. The common thread in these examples is the lack of *spectral stability*, see Def. 6; roughly speaking, a sequence of operators $(A_n)_n$ satisfying *the same* boundary conditions is spectrally stable if it is spectrally exact (in the sense of [5]) and the limiting operator A satisfies the same boundary conditions as the operators A_n, $n \in \mathbb{N}$.

Problem (1.2) is an interesting example of spectral instability, which, according to [2, 21], can be regarded as a smooth version of the Babuška paradox. In order to describe our main result, it is convenient to define a class of triharmonic problems with different boundary conditions. For every $\epsilon > 0$, let $V(\Omega_\epsilon)$ be a linear subspace of $H^3(\Omega_\epsilon)$ containing $H^3_0(\Omega_\epsilon)$. Assume that $V(\Omega_\epsilon)$ is compactly embedded in $L^2(\Omega_\epsilon)$, and it is complete with respect to the $H^3(\Omega_\epsilon)$ norm, which is induced by the quadratic form

$$Q_{\Omega_\epsilon}(u) = \int_{\Omega_\epsilon} |D^3 u|^2 + |u|^2, \qquad u \in V(\Omega_\epsilon).$$

We then define

$$\int_{\Omega_\epsilon} \left(D^3 u_\epsilon : D^3 \varphi + u_\epsilon \varphi\right) = \lambda(\Omega_\epsilon) \int_{\Omega_\epsilon} u_\epsilon \varphi, \quad \varphi \in V(\Omega_\epsilon). \tag{1.3}$$

By the second representation theorem [27, Theorem 2.23, VI.2], the sesquilinear form in (1.3) is associated to a positive self-adjoint operator $A_{V(\Omega_\epsilon)} := (-\Delta)^3_{V(\Omega_\epsilon)} + I$. The inverse $A^{-1}_{V(\Omega_\epsilon)}$ is a compact operator in $L^2(\Omega_\epsilon)$, due to the compact embedding of $V(\Omega_\epsilon)$ in $L^2(\Omega_\epsilon)$. Thus, the spectrum associated to (1.3) is discrete and consists of an unbounded sequence of positive eigenvalues of finite multiplicity $(\lambda_j(\Omega_\epsilon))_{j\in\mathbb{N}}$.

It is shown in [21] that if $V(\Omega_\epsilon) = H^3(\Omega_\epsilon) \cap H^2_0(\Omega_\epsilon)$ for all $\epsilon \in (0,1]$ then the spectrum $(\lambda_j(\Omega_\epsilon))_{j\in\mathbb{N}}$ of problem (1.3) approaches the spectrum $(\lambda_k(\Omega))_{k\in\mathbb{N}}$ of the same problem (1.3) with Ω in place of Ω_ϵ, provided that $\alpha > \frac{3}{2}$. If instead $0 < \alpha < \frac{3}{2}$ the limiting problem satisfies Dirichlet boundary conditions (corresponding to $V(\Omega) = H^3_0(\Omega)$ in (1.3)) on $W \times \{0\}$. It is also shown that this Babuška-type paradox is shared by all the polyharmonic operators $(-\Delta)^m_{\mathrm{SBC}}$ with strong intermediate boundary conditions, shortened SBC, for which $V(\Omega_\epsilon) = H^m(\Omega_\epsilon) \cap H^{m-1}_0(\Omega_\epsilon)$ in the polyharmonic analogous of (1.3). In other words, polyharmonic operators with SBC are spectrally stable on $(\Omega_\epsilon)_{\epsilon\in[0,1]}$ provided that $\alpha > 3/2$.

As already pointed out in [21], there is another possible choice of intermediate boundary conditions for the triharmonic operator, the weak intermediate boundary conditions (shortened WBC) defined implicitly by (1.2). From the spectral stability result [21, Theorem 4] we know that if Ω_ϵ and Ω are as in (1.1) the sequence of operators $A_\epsilon = (-\Delta^3_{\mathrm{WBC}} + I))_{\epsilon\in[0,1]}$ associated to (1.2) is spectrally stable provided that $\alpha > 5/2$.

The main result of this article, see Thm.1, is the analysis of the case $\alpha \leq 5/2$. We prove that there are three different cases depending on α, that can be summarised as follows

(i) if $\alpha \in \left(\frac{3}{2}, \frac{5}{2}\right)$ the eigenvalues $\lambda_j(\Omega_\epsilon)$ of (1.2) converge in the limit to the eigenvalues of $(-\Delta)^3 + I$ with mixed WBC-SBC.
(ii) if $\alpha \in (0,1)$, the limiting operator $(-\Delta)^3 + I$ satisfies mixed boundary conditions of type WBC-Dirichlet.
(iii) if $\alpha = \frac{5}{2}$ the limiting boundary value problem features a 'strange' boundary conditions which keeps track of the shape of the periodic function b in (1.1).

The case $\alpha \in (1, 3/2]$ is not considered in this article and it is left as an open problem, see Remark 2 for further explanations of why this range of values does not seem treatable with our method of proof.

While Theorem 1 look similar to [21, Theorem 7], we point out that we had to face several new technical difficulties due to the extreme singularity of the perturbation $\Omega \mapsto \Omega_\epsilon$ when $\alpha \leq 5/2$. Indeed, the diffeomorphism $\Phi_\epsilon : \Omega \mapsto \Omega_\epsilon$ that we use in the proof of the main theorem has derivatives with strongly divergent L^2-norms as $\epsilon \to 0$. Furthermore, it is not possible to balance this unboundedness of the derivatives as in [21], where it was pivotal to exploit the vanishing of *both* u_ϵ and $\frac{\partial u}{\partial n}$ at the boundary. Finally, the proof of [21, Theorem 7] in the degenerate case $\alpha \leq 3/2$ relies on [10, Lemma 4.3], for which it is fundamental that the critical threshold for the spectral stability is $\alpha = 3/2$. This condition is clearly not satisfied by weak intermediate problems.

To overcome these additional hurdles, we prove a new, yet rather technical degeneration result, see Lemma 4. Its proof involves a careful analysis of the behaviour of the derivatives of functions $u_\epsilon \in H^3(\Omega_\epsilon) \cap H^1_0(\Omega_\epsilon)$ close to the oscillating boundaries. Broadly speaking, we need a combination of three arguments: (i) the use of the anisotropic unfolding operator to control the L^2-norm of the derivatives of u_ϵ close to the oscillating boundary; (ii) the weighted convergence of the traces of the unfolded functions $\hat{u}_\epsilon$ to the trace of the weak limit u of the original functions u_ϵ; (iii) the use of the standard unfolding operators (which is equivalent to the

so-called two-scale convergence) to deduce additional information on the trace of u when $\alpha \leq 1$ and $1 < \alpha < 2$.
We refer the reader to [14] for more details about homogenisation techniques and to [13] for the unfolding operator. The use of the anisotropic unfolding operator and some of the techniques used in the proof of Lemma 4 were inspired by a careful reading of [10, 11] and by some classical asymptotic analysis techniques in the spirit of [28, 29].
This article is organised in the following way. In Section 2 we introduce the weak intermediate boundary conditions for the triharmonic operator $-\Delta^3$, and we state the main result of the paper, Theorem 1. In Section 3 we collect some standard results about the unfolding operator and the tangential calculus. In Section 4 we recall some definitions and results about the convergence of bounded operators on varying Hilbert spaces, and we give the definitions of spectral exactness and spectral stability. In Section 5 we prove statements (iii) and (iv) of Theorem 1. Section 6 is devoted to the proof of Theorem $1(ii)$, which requires several results from homogenisation theory. In the Appendices we collect some auxiliary results among which the proof of the Triharmonic Green Formula, which is of general interest.

2. Main result

2.1. **Boundary conditions.** Given a bounded domain $\Omega \subset \mathbb{R}^N$, we consider the quadratic form defined by

$$Q_\Omega(u,v) = \int_\Omega D^3u : D^3v \, dx + \int_\Omega uv \, dx, \tag{2.1}$$

for all $u, v \in V(\Omega)$, where $V(\Omega)$ is a linear subspace of $H^3(\Omega)$, $H^3_0(\Omega) \subset V(\Omega)$ and V is complete with respect to the H^3-norm. By the second representation theorem [27, Theorem 2.23, VI.2], there exists a densely defined, non-negative and self-adjoint operator $A_{V(\Omega)}$ with domain $\mathrm{dom}(A_{V(\Omega)}) \subset H^3(\Omega)$ such that

$$Q_\Omega(u,v) = (A^{1/2}_{V(\Omega)}u, A^{1/2}_{V(\Omega)}v),$$

for all $u, v \in V(\Omega)$. Assume that the embedding of $V(\Omega)$ in $L^2(\Omega)$ is compact. Then, $A_{V(\Omega)}$ has compact resolvent, hence it has purely discrete spectrum, made of an increasing sequence of eigenvalues diverging to $+\infty$. Let us consider the eigenvalue problem

$$\int_\Omega D^3u : D^3v \, dx + \int_\Omega uv \, dx = \lambda \int_\Omega uv \, dx, \tag{2.2}$$

in the unknowns λ, $u \in V(\Omega)$ for all $v \in V(\Omega)$. We briefly recall the boundary conditions we are interested in. Their identification is achieved via the Triharmonic Green Formula, stated and proved in Theorem 7. Let $k \in \mathbb{N}$, $0 \leq k \leq 3$ and let us set $V(\Omega) = H^3(\Omega) \cap H^k_0(\Omega)$. If $k = 3$ then $V(\Omega) = H^3_0(\Omega)$ in (2.1). Formula (7.4) implies that $A_{V(\Omega)}$ is the Dirichlet triharmonic operator associated with

$$\begin{cases} -\Delta^3 u + u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = \frac{\partial^2 u}{\partial n^2} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

When $k = 2$, $V(\Omega) = H^3(\Omega) \cap H^2_0(\Omega)$. By (7.4) we deduce that the classical eigenvalue problem associated with (2.2) on $V(\Omega)$ is defined by

$$\begin{cases} -\Delta^3 u + u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \\ \frac{\partial^3 u}{\partial n^3} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

In this case we say that the classical operator $-\Delta^3 u + u$ associated with problem (2.4) satisfies *strong intermediate boundary conditions* on $\partial\Omega$.

Finally, when $k = 1$, $V(\Omega) = H^3(\Omega) \cap H_0^1(\Omega)$. By (7.4) we deduce that

$$\begin{cases} -\Delta^3 u + u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \left((n^T D^3 u)_{\partial\Omega} : D_{\partial\Omega} n\right) - \frac{\partial^2(\Delta u)}{\partial n^2} - 2\,\mathrm{div}_{\partial\Omega}(D^3 u[n \otimes n])_{\partial\Omega} = 0, & \text{on } \partial\Omega, \\ \frac{\partial^3 u}{\partial n^3} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

where we have denoted by $(\cdot)_{\partial\Omega}$ the tangential part of a tensor (which can be defined formally exactly as the tangential Hessian, see Def. 3 below) , $D_{\partial\Omega}$ is the tangential Jacobian, n is the outer unit normal to $\partial\Omega$, $\mathrm{div}_{\partial\Omega}$ is the tangential divergence, $[n \otimes n] = (n_i n_j)_{i,j=1,\dots,N}$. In this case, we say that the classical operator $-\Delta^3 u + u$ associated with problem (2.4) satisfies *weak intermediate boundary conditions* on $\partial\Omega$. Note that the curvature tensor $D_{\partial\Omega} n$ appears non-trivially in the second boundary condition. To the best of our knowledge these boundary conditions were never defined before in this form.

2.2. **Main theorem.** Let Ω_ϵ, $\epsilon > 0$ and Ω be as in (1.1). Set $\Gamma := \overline{W} \times \{0\}$. Let A_{Ω_ϵ} be the operator associated to (1.2), $\epsilon > 0$, and define A_Ω in a analogous way by replacing Ω_ϵ with Ω. Let $A_{\Omega,S}$ be the operator associated to

$$\begin{cases} A_{\Omega,S} u := -\Delta^3 u + u = \lambda u, & \text{in } \Omega, \\ (WBC), & \text{on } \partial\Omega \setminus \Gamma \\ (SBC), & \text{on } \Gamma, \end{cases} \tag{2.6}$$

where (WBC) are the boundary conditions in (2.5), (SBC) those in (2.4). Let $A_{\Omega,D}$ be the operator associated to

$$\begin{cases} A_{\Omega,D} u := -\Delta^3 u + u = \lambda u, & \text{in } \Omega, \\ (WBC), & \text{on } \partial\Omega \setminus \Gamma \\ (DBC), & \text{on } \Gamma, \end{cases} \tag{2.7}$$

where (DBC) are the Dirichlet boundary conditions defined in (2.3). Finally, let $\hat{A}_\Omega$ be the operator associated to

$$\begin{cases} \hat{A}_\Omega u := -\Delta^3 u + u = \lambda u, & \text{in } \Omega, \\ (WBC), & \text{on } \partial\Omega \setminus \Gamma \\ u = \partial_{x_N^3} u = 0, & \text{on } \Gamma, \\ \Delta(\partial_{x_N^2} u) + 2\Delta_{N-1}(\partial_{x_N^2} u) + K_1 \partial_{x_N} u = 0, & \text{on } \Gamma, \end{cases} \tag{2.8}$$

where $K_1 > 0$ is given by

$$\begin{aligned} K_1 &= \int_Y \left(\Delta^2 \left(\frac{\partial V}{\partial y_N} \right) + \Delta_{N-1} \left(\frac{\partial(\Delta V)}{\partial y_N} \right) + \Delta_{N-1}^2 \left(\frac{\partial V}{\partial y_N} \right) \right) b(\bar{y}) \mathrm{d}\bar{y} \\ &= \int_{Y \times (-\infty, 0)} |D^3 V|^2 \, \mathrm{d}y, \end{aligned} \tag{2.9}$$

where the function V is Y-periodic in the variables $\bar{y}$ and satisfies the following microscopic problem

$$\begin{cases} \Delta^3 V = 0, & \text{in } Y \times (-\infty, 0), \\ V(\bar{y}, 0) = b(\bar{y}), & \text{on } Y, \\ -\partial_{y_N^2}(\Delta V) + 2\partial_{y_N^2}(\Delta_{N-1} V) = 0, & \text{on } Y, \\ \partial_{y_N^3} V = 0, & \text{on } Y. \end{cases} \tag{2.10}$$

Then we have the following

Theorem 1. *For $\epsilon \geq 0$ let $\Omega_\epsilon \subset \mathbb{R}^N$ be defined by (1.1). Let A_{Ω_ϵ}, $\epsilon > 0$, A_Ω, $A_{\Omega,S}$, $A_{\Omega,D}$, $\hat{A}_\Omega$ be the operators defined above in (2.6), (2.7), (2.8). Then:*

(i) [Spectral stability] *If $\alpha > 5/2$, then $A_{\Omega_\epsilon}^{-1} \overset{\mathcal{C}}{\to} A_\Omega^{-1}$.*

(ii) [Strange term] *If $\alpha = 5/2$, then $A_{\Omega_\epsilon}^{-1} \overset{\mathcal{C}}{\to} \hat{A}_\Omega^{-1}$.*

(iii) [Mild instability] *If $3/2 < \alpha < 5/2$, then $A_{\Omega_\epsilon}^{-1} \overset{\mathcal{C}}{\to} A_{\Omega,S}^{-1}$.*

(iv) [Strong instability] *If $\alpha \leq 1$, then $A_{\Omega_\epsilon}^{-1} \overset{\mathcal{C}}{\to} A_{\Omega,D}^{-1}$.*

In particular, the eigenvalues $\lambda_j(\Omega_\epsilon)$, $j \geq 1$, of (1.2) converge as $\epsilon \to 0$ to the eigenvalues of A_Ω in case (i), $\hat{A}_\Omega$ in case (ii), $A_{\Omega,S}$ in case (iii) and $A_{\Omega,D}$ in case (iv).

The compact convergence $\overset{\mathcal{C}}{\to}$ in the previous theorem is defined in Definition 4. The novelty of Theorem 1 lies in the identification of the double instability effect, namely a first degeneration to SBC when $\alpha \in (\frac{3}{2}, \frac{5}{2})$ and a further degeneration to Dirichlet when $\alpha \leq 1$. We immediately give a proof of item (i):

Proof of Thm 1(i). The follows from [21, Theorem 4], with $m = 3$, $k = 1$. □

Remark 1. *The results in Theorem 1 can be easily generalised to the case where Ω has a piecewise flat boundary, Ω_ϵ, Ω belong to the same atlas class in the sense of* [9, Definition 2.4] *for all $\epsilon > 0$, and have an oscillating boundary locally described by* (1.1). *Indeed, if V is one of the chart in the common atlas class, using a partition of unity we can directly assume that*

$$\Omega_\epsilon \cap V = \{(\bar{x}, x_N) \in \mathbb{R}^N : \bar{x} \in W, -1 < x_N < g_{V,\epsilon}(\bar{x})\}.$$

It is clear that if we allow $\alpha > 0$ to be chart dependent, we may find a limiting boundary value problem with mixed boundary conditions. Nevertheless, the passage to the limit can be treated locally exactly as in Theorem 1. As an example, assume that the sequence of open sets $(\Omega_\epsilon)_{\epsilon>0}$ has a common atlas given by three charts V_1, V_2 and V_3. Then up to a possible rotation and translation

$$\Omega_\epsilon \cap V_1 = \{(\bar{x}, x_N) \in \mathbb{R}^N : \bar{x} \in W, -1 < x_N < \epsilon^{\alpha_1} b_1(\bar{x}/\epsilon)\},$$

$$\Omega_\epsilon \cap V_2 = \{(\bar{x}, x_N) \in \mathbb{R}^N : \bar{x} \in W, -1 < x_N < \epsilon^{\alpha_2} b_2(\bar{x}/\epsilon)\},$$

$$\Omega_\epsilon \cap V_3 = \Omega \cap V_3, \quad \epsilon > 0,$$

with $\alpha_1 > 5/2$, $\alpha_2 \leq 1$. Then the limiting boundary value problem in Ω will be in the form

$$\begin{cases} -\Delta^3 u + u = \lambda u, & \text{in } \Omega, \\ (WBC), & \text{in } \Gamma_1 \cup \Gamma_3, \\ (DBC), & \text{in } \Gamma_2, \end{cases}$$

where $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, Γ_j being the boundary of Ω inside V_j, $j = 1, 2, 3$.

Remark 2. *The case $\alpha \in (1, 3/2)$ in Theorem 1 remains at the moment open. The proof of Theorem 1 seems to suggest that $\alpha = 3/2$ is not a critical threshold; in other words, we do not expect degeneration to the Dirichlet problem at $\alpha = 3/2$. The main difficulty is that the derivatives of $T_\epsilon\varphi$, $\varphi \in L^2(\Omega)$, where T_ϵ is the pullback operator defined in* (3.2) *have singularities that are balanced by neither the shrinking of the set Ω_ϵ when $\alpha \in (1, 3/2)$, nor by the vanishing of the traces of the eigenfunctions at the boundary. The construction of a more efficient extension operator $T_\epsilon : H^3(\Omega) \cap H_0^1(\Omega) \to H^3(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon)$ is however even more challenging: note that the classically used Sobolev extension operators do not work here since they do not preserve the boundary conditions.*

3. Auxiliary results

• A diffeomorphism between Ω and Ω_ϵ.
Let us define a diffeomorphism Φ_ϵ from Ω_ϵ to Ω by

$$\Phi_\epsilon(\bar{x}, x_N) = (\bar{x}, x_N - h_\epsilon(\bar{x}, x_N)), \quad \text{for all } x = (\bar{x}, x_N) \in \Omega_\epsilon, \tag{3.1}$$

where h_ϵ is defined by

$$h_\epsilon(\bar{x}, x_N) = \begin{cases} 0, & \text{if } -1 \le x_N \le -\epsilon, \\ g_\epsilon(\bar{x})\Big(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\Big)^4, & \text{if } -\epsilon \le x_N \le g_\epsilon(\bar{x}). \end{cases}$$

By standard calculus one can prove the following

Lemma 1. *The map Φ_ϵ is a diffeomorphism of class C^3 and there exists a constant $c > 0$ independent of ϵ such that $|h_\epsilon| \le c\epsilon^\alpha$ and $\left|D^l h_\epsilon\right| \le c\epsilon^{\alpha-l}$, for all $l = 1, \dots, 3$, $\epsilon > 0$ sufficiently small.*

We then introduce the pullback operator

$$T_\epsilon : L^2(\Omega) \to L^2(\Omega_\epsilon), \quad T_\epsilon u = u \circ \Phi_\epsilon \tag{3.2}$$

for all $u \in L^2(\Omega)$.

• Unfolding method.
We recall the following notation and results from [21] regarding the unfolding method. For any $k \in \mathbb{Z}^{N-1}$ and $\epsilon > 0$ we define

$$\begin{cases} C_\epsilon^k = \epsilon k + \epsilon Y, \\ I_{W,\epsilon} = \{k \in \mathbb{Z}^{N-1} : C_\epsilon^k \subset W\}, \\ \widehat{W}_\epsilon = \displaystyle\bigcup_{k \in I_{W,\epsilon}} C_\epsilon^k. \end{cases} \tag{3.3}$$

Definition 1. *Let u be a real-valued function defined in Ω. For any $\epsilon > 0$ sufficiently small the unfolding $\hat{u}$ of u is the real-valued function defined on $\widehat{W}_\epsilon \times Y \times (-1/\epsilon, 0)$ by*

$$\hat{u}(\bar{x}, \bar{y}, y_N) = u\Big(\epsilon\Big[\frac{\bar{x}}{\epsilon}\Big] + \epsilon\bar{y}, \epsilon y_N\Big),$$

for almost all $(\bar{x}, \bar{y}, y_N)) \in \widehat{W}_\epsilon \times Y \times (-1/\epsilon, 0)$, where $[\bar{x}\epsilon^{-1}]$ denotes the integer part of the vector $\bar{x}\epsilon^{-1}$ with respect to Y, i.e., $[\bar{x}\epsilon^{-1}] = k$ if and only if $\bar{x} \in C_\epsilon^k$.

The following lemma will be often used in the sequel. For a proof we refer to [12, Proposition 2.5(i)].

Lemma 2. *Let $a \in [-1, 0[$ be fixed. Then*

$$\int_{\widehat{W}_\epsilon \times (a,0)} u(x)dx = \epsilon \int_{\widehat{W}_\epsilon \times Y \times (a/\epsilon, 0)} \hat{u}(\bar{x}, y)d\bar{x}dy \tag{3.4}$$

for all $u \in L^1(\Omega)$ and $\epsilon > 0$ sufficiently small. Moreover

$$\int_{\widehat{W}_\epsilon \times (a,0)} \left|\frac{\partial^l u(x)}{\partial x_{i_1} \cdots \partial x_{i_l}}\right|^2 dx = \epsilon^{1-2l} \int_{\widehat{W}_\epsilon \times Y \times (a/\epsilon, 0)} \left|\frac{\partial^l \hat{u}}{\partial y_{i_1} \cdots \partial y_{i_l}}(\bar{x}, y)d\bar{x}\right|^2 dy,$$

for all $l \le 3$, $u \in H^3(\Omega)$ and $\epsilon > 0$ sufficiently small.

Let $H^3_{\mathrm{Per}_Y,\mathrm{loc}}(Y \times (-\infty, 0))$ be the subspace of $H^3_{\mathrm{loc}}(\mathbb{R}^{N-1} \times (-\infty, 0))$ containing Y-periodic functions in the first $(N-1)$ variables $\bar{y}$. We then define $H^3_{loc}(Y \times$

$(-\infty,0))$ to be the space of functions in $H^3_{\mathrm{Per}_Y,\mathrm{loc}}(Y\times(-\infty,0))$ restricted to $Y\times(-\infty,0)$. Finally we set

$$w^{3,2}_{\mathrm{Per}_Y}(Y\times(-\infty,0)) := \big\{u\in H^3_{\mathrm{Per}_Y,\mathrm{loc}}(Y\times(-\infty,0)) \\ : \|D^\gamma u\|_{L^2(Y\times(-\infty,0))}<\infty, \forall|\gamma|=3\big\}. \quad (3.5)$$

For any $d<0$, let $\mathcal{P}^l_{hom,y}(Y\times(d,0))$ be the space of homogeneous polynomials of degree at most l restricted to the domain $(Y\times(d,0))$. Let $\epsilon>0$ be fixed. We define the projectors P_i from $L^2(\widehat{W}_\epsilon,H^3(Y\times(-1/\epsilon,0)))$ to $L^2(\widehat{W}_\epsilon,\mathcal{P}^i_{hom,y}(-1/\epsilon,0))$ by setting

$$P_i(\psi)=\sum_{|\eta|=i}\int_Y D^\eta\psi(\bar{x},\bar{\zeta},0)d\bar{\zeta}\frac{y^\eta}{\eta!}$$

for all $i=0,1,2$. We now set $Q_2=P_2$, $Q_1=P_1(\mathbb{I}-Q_1)$,$Q_0=P_0\big(\mathbb{I}-\sum_{j=1}^2 Q_j\big)$. Note that Q_{3-j}, $j=1,\dots,3$ is a projection on the space of homogeneous polynomials of degree $3-j$, with the property that $Q_{3-k}(p)=0$ for all polynomials p of degree $3-k$ with $k\neq j$. We finally set

$$\mathcal{P}=Q_0+Q_1+Q_2, \quad (3.6)$$

which is a projector on the space of polynomials in y of degree at most 2. Note that $D^\beta_y\mathcal{P}(\psi)(\bar{x},\bar{y},0)=\int_Y D^\beta_y\psi(\bar{x},\bar{y},0)d\bar{y}$ for all $|\beta|=0,\dots,2$.

Lemma 3. *The following statements hold:*

(i) Let $v_\epsilon\in H^3(\Omega)$ with $\|\hat{v_\epsilon}\|_{H^3(\Omega)}\le M$, for all $\epsilon>0$. Let V_ϵ be defined by

$$V_\epsilon(\bar{x},y)=\hat{v_\epsilon}(\bar{x},y)-\mathcal{P}(v_\epsilon)(\bar{x},y),$$

for $(\bar{x},y)\in\widehat{W}_\epsilon\times Y\times(-1/\epsilon,0)$, where $\mathcal{P}$ is defined by (3.6) . Then there exists a function $\hat{v}\in L^2(W,w^{3,2}_{\mathrm{Per}_Y}(Y\times(-\infty,0)))$ such that for every $d<0$

(a) $\frac{D^\gamma_y V_\epsilon}{\epsilon^{5/2}}\rightharpoonup D^\gamma_y\hat{v}$ in $L^2(W\times Y\times(d,0))$ as $\epsilon\to0$, for any $\gamma\in\mathbb{N}^N_0$, $|\gamma|\le2$.

(b) $\frac{D^\gamma_y V_\epsilon}{\epsilon^{5/2}}\rightharpoonup D^\gamma_y\hat{v}$ in $L^2(W\times Y\times(-\infty,0))$ as $\epsilon\to0$, for any $\gamma\in\mathbb{N}^N_0$, $|\gamma|=3$,

where it is understood that the functions $V_\epsilon, D^\gamma_y V_\epsilon$ are extended by zero to the whole of $W\times Y\times(-\infty,0)$ outside their natural domain of definition $\widehat{W}_\epsilon\times Y\times(-1/\epsilon,0)$.

(ii) If $\psi\in W^{1,2}(\Omega)$, then $\lim_{\epsilon\to0}\widehat{(T_\epsilon\psi)_{|\Omega}}=\psi(\bar{x},0)$ in $L^2(W\times Y\times(-1,0))$.

• Tangential Calculus.

Recall now the following standard definitions of the tangential differential operators. We refer to [19, Chapter 9] for details and further information. Given $A\subset\mathbb{R}^N$ let d_A be the Euclidean distance function from A, defined by $d_A(x)=\inf_{y\in A}|x-y|$. We define the oriented distance function b_A from A by

$$b_A(x)=d_A(x)-d_{A^C}(x),$$

for all $x\in\mathbb{R}^N$. Let now Ω be an bounded open set of class C^2. In this case b_Ω coincides with the signed distance from $\partial\Omega$. It is well-known that there exists $h>0$ and a tubular neighbourhood $S_{2h}(\partial\Omega)$ of radius h such that $b_\Omega\in C^2(S_{2h}(\partial\Omega))$, see [20]. We define the projection of a point x to $\partial\Omega$ by

$$p(x)=x-b_\Omega(x)\nabla b_\Omega(x), \quad (3.7)$$

for all $x\in S_{2h}(\partial\Omega)$. If $f\in C^0(\partial\Omega)$ we write $(f)_{\partial\Omega}=(f\circ p)|_{\partial\Omega}$.

Definition 2. *Let Ω be an bounded open set of class C^2 and let $h > 0$ be such that $b_\Omega \in C^2(S_{2h}(\partial\Omega))$. Let $f \in C^1(\partial\Omega)$ and let $F \in C^1(S_{2h}(\partial\Omega))$ be a C^1 extension of f to $S_{2h}(\partial\Omega)$ (that is, $F|_{\partial\Omega} = f$). We define the tangential gradient of f on $\partial\Omega$ by*

$$\nabla_{\partial\Omega} f = \nabla F|_{\partial\Omega} - \frac{\partial F}{\partial n} n.$$

Definition 3. *Let $N \geq 1$, $v \in C^1(\partial\Omega)^N$. We define the tangential Jacobian matrix of v by $D_{\partial\Omega} v = D(v \circ p)|_{\partial\Omega}$ and the tangential divergence of v by $\operatorname{div}_{\partial\Omega}(v \circ p)|_{\partial\Omega} = \operatorname{tr}(D_{\partial\Omega} v)$. Assume now Ω is of class C^3 and $f \in C^2(\partial\Omega)$. We define the Laplace-Beltrami operator of f by*

$$\Delta_{\partial\Omega} f = \Delta(f \circ p)|_{\partial\Omega} = \operatorname{div}_{\partial\Omega}(\nabla_{\partial\Omega} f),$$

and similarly we define the tangential Hessian matrix by $D^2_{\partial\Omega} f = D_{\partial\Omega}(\nabla_{\partial\Omega} f)$.

We conclude this section recalling the following important

Theorem 2 (Tangential Divergence Theorem)**.** *Let Ω be a bounded open set of class C^2 and let $v \in C^1(\partial\Omega)^N$. Let $\mathcal{H}$ be the trace of the second fundamental form of $\partial\Omega$. Then*

$$\int_{\partial\Omega} \operatorname{div}_{\partial\Omega} v \, dS = \int_{\partial\Omega} \mathcal{H}\,(v \cdot n)\, dS. \tag{3.8}$$

Let $f \in C^1(\partial\Omega)$. Then

$$\int_{\partial\Omega} (f \operatorname{div}_{\partial\Omega} v + \nabla_{\partial\Omega} f \cdot v)\, dS = \int_{\partial\Omega} \mathcal{H} f\,(v \cdot n)\, dS. \tag{3.9}$$

Proof. We refer to [19, §5.5 Chapter 9]. □

4. Spectral exactness and spectral stability

Let $(\mathcal{H}_\epsilon)_{\epsilon\in[0,1]}$ be a family of Hilbert spaces. Let $(\mathcal{E}_\epsilon)_{\epsilon\in(0,1]}$ be a connecting system for $(\mathcal{H}_\epsilon)_{\epsilon\in[0,1]}$, that is, $\mathcal{E}_\epsilon \in L(\mathcal{H}_0, \mathcal{H}_\epsilon)$, $\epsilon \in (0,1]$, and

$$\lim_{\epsilon\to 0} \| \mathcal{E}_\epsilon u\|_{\mathcal{H}_\epsilon} = \|u\|_{\mathcal{H}_0}$$

for every $u \in \mathcal{H}_0$.

We recall the following definitions.

Definition 4. *Let $(\mathcal{H}_\epsilon)_{\epsilon\in[0,1]}$ and $\mathcal{E}_\epsilon$ be as above.*

- *(i) Let $u_\epsilon \in \mathcal{H}_\epsilon$, $\epsilon > 0$. We say that u_ϵ $\mathcal{E}$-converges to u as $\epsilon \to 0$ if $\|u_\epsilon - \mathcal{E}_\epsilon u\|_{\mathcal{H}_\epsilon} \to 0$ as $\epsilon \to 0$. We write $u_\epsilon \xrightarrow{\mathcal{E}} u$.*
- *(ii) Let $B_\epsilon \in L(\mathcal{H}_\epsilon)$, $\epsilon > 0$. We say that B_ϵ $\mathcal{E}$ $\mathcal{E}$-converges to a linear operator $B_0 \in L(\mathcal{H}_0)$ if $B_\epsilon u_\epsilon \xrightarrow{\mathcal{E}} B_0 u$ whenever $u_\epsilon \xrightarrow{\mathcal{E}} u \in \mathcal{H}_0$. We write $B_\epsilon \xrightarrow{\mathcal{E}\,\mathcal{E}} B_0$.*
- *(iii) Let $B_\epsilon \in L(\mathcal{H}_\epsilon)$, $\epsilon > 0$. We say that B_ϵ compactly converges to $B_0 \in L(\mathcal{H}_0)$ (and we write $B_\epsilon \xrightarrow{\mathcal{C}} B_0$) if the following two conditions are satisfied:*
  - *(a) $B_\epsilon \xrightarrow{\mathcal{E}\,\mathcal{E}} B_0$ as $\epsilon \to 0$;*
  - *(b) for any family $u_\epsilon \in \mathcal{H}_\epsilon$, $\epsilon > 0$, such that $\|u_\epsilon\|_{\mathcal{H}_\epsilon} = 1$ for all $\epsilon \in (0,1)$, there exists a subsequence $B_{\epsilon_k} u_{\epsilon_k}$ of $B_\epsilon u_\epsilon$ and $\bar{u} \in \mathcal{H}_0$ such that $B_{\epsilon_k} u_{\epsilon_k} \xrightarrow{\mathcal{E}} \bar{u}$ as $k \to \infty$.*

Definition 5. *Let T, T_n be closed operators in $\mathcal{H}$, $\mathcal{H}_n$ respectively, $n \in \mathbb{N}$.*

- *(1) The sequence $(T_n)_{n\in\mathbb{N}}$ is called spectrally inclusive if for every $\lambda \in \sigma(T)$, there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$, $\lambda_n \in \sigma(T_n)$, $n \in \mathbb{N}$ such that $\lambda_n \to \lambda$.*
- *(2) We say that spectral pollution occurs for $(T_n)_{n\in\mathbb{N}}$ if there exists $\lambda \in \varrho(T)$ and $\lambda_n \in \sigma(T_n)$, $n \in \mathbb{N}$ such that $\lambda_n \to \lambda$.*

(3) The sequence $(T_n)_{n\in\mathbb{N}}$ is called spectrally exact if it is spectrally inclusive and no spectral pollution occurs.

Let $(A_\epsilon)_{\epsilon\in[0,1]}$ be a family of closed densely defined linear operators, $A_\epsilon \in C(\mathcal{H}_\epsilon)$, $\epsilon \in [0,1]$. Assume that:

$$\text{(A1)(a): } \exists \lambda_0 \in \bigcap_{\epsilon\in[0,1]} \varrho(A_\epsilon),\ (A_\epsilon - \lambda_0)^{-1} \text{ compact } \epsilon \in [0,1],$$

$$\text{(A1)(b) } (A_\epsilon - \lambda_0)^{-1} \xrightarrow{\mathcal{C}} (A_0 - \lambda_0)^{-1} \text{ as } \epsilon \to 0.$$

Then [6, Theorem 2.6] implies that $(A_\epsilon)_{\epsilon\in(0,1]}$ is a spectrally exact approximation of A_0. Consider now the following setting. Let $m \in \mathbb{N}$, and let Ω be an open set of $\mathbb{R}^N$. Let M be the number of multi-indices $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$ with length $|\alpha| = |\alpha_1| + \cdots + |\alpha_N| = m$. For all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha| = |\beta| = m$ let $c_{\alpha\beta}$ be bounded measureable real-valued functions defined on $\mathbb{R}^N$, $c_{\alpha\beta} = c_{\beta\alpha}$ such that

$$\sum_{|\alpha|=|\beta|=m} c_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq 0,$$

for all $x \in \mathbb{R}^N$, $(\xi_\alpha)_{|\alpha|=m} \in \mathbb{R}^m$. For all measurable open sets $\Omega \in \mathbb{R}^N$ we define

$$Q_\Omega(u,v) = \int_\Omega \left(c_{\alpha\beta} D^\alpha u D^\beta v + uv \right) dx \tag{4.1}$$

Let $V(\Omega)$ be a linear subspace of $H^m(\Omega)$ containing $H_0^m(\Omega)$. Assume that $V(\Omega)$ endowed with the norm $Q_\Omega(\cdot)^{1/2}$ is complete. Then there exists a unique self-adjoint operator $A_{V(\Omega)}$ such that

$$Q_\Omega(u,v) = (A_{V(\Omega)}^{1/2} u, A_{V(\Omega)}^{1/2} v) \tag{4.2}$$

for all $u, v \in V(\Omega)$.

For $\epsilon \geq 0$, let Ω_ϵ be a bounded domain of $\mathbb{R}^N$. In this setting we can give the following

Definition 6. *Let $W(\Omega_0)$ be a linear subspace of $H^m(\Omega_0)$ containing $H_0^m(\Omega_0)$. Assume that $W(\Omega_0)$ endowed with the norm $Q_{\Omega_0}^{1/2}$ is complete. The sequence of operators $(A_{V(\Omega_\epsilon)})_{\epsilon>0} \cup \{A_{W(\Omega_0)}\}$, defined as in (4.2) with Ω replaced by Ω_ϵ, and Q_{Ω_ϵ} as in (4.1) for all $\epsilon > 0$, is said to be spectrally stable if $(A_{V(\Omega_\epsilon)})_{\epsilon>0}$ is a spectrally exact approximation of $A_{W(\Omega_0)}$ and $W(\Omega_0) = V(\Omega_0)$.*

With Definition 6, [2, Theorem 3.5] can be rephrased as:

Theorem 3. *Assume that Condition (C), see* [2, Definition 3.1]*, is satisfied by the sequence of operators $A_{V(\Omega_\epsilon)}$, $A_{V(\Omega)}$ associated with the quadratic forms Q_{Ω_ϵ}, Q_Ω. Then the sequence of operators $(A_{V(\Omega_\epsilon)})_{\epsilon>0}$ is spectrally stable.*

5. Proof of Theorem 1(iii), (iv)

To prove *(iii)* and *(iv)* in Theorem 1, we will show that Condition (C), see [2, Definition 3.1], holds for the operators A_{Ω_ϵ} associated to (1.2). An application of Theorem 3 will then prove the claims.

Establishing Condition (C) will require several lemmata. We first establish a general lemma concerning the limiting boundary behaviour of sequences $(u_\epsilon)_\epsilon$ such that $u_\epsilon \in H^3(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon)$ and $\|u_\epsilon\|_{H^3(\Omega_\epsilon)} < \infty$, for all $\epsilon > 0$.

For $\epsilon > 0$, we define

$$\Omega_\epsilon^\sharp = \{(\bar{x}, x_N) \in \mathbb{R}^N : \bar{x} \in \overline{W}, -1 \leq x_N < g_\epsilon(\bar{x})\}, \quad \Omega^\sharp = \overline{W} \times [-1, 0),$$

and for any $l \in \mathbb{N}$, $\epsilon > 0$ we set

$$H^l_{0,*}(\Omega_\epsilon) = \overline{C^\infty_c(\Omega^\sharp_\epsilon)}^{H^l(\Omega_\epsilon)}, \quad H^l_{0,*}(\Omega) = \overline{C^\infty_c(\Omega^\sharp)}^{H^l(\Omega)}.$$

In the case of sequence of functions in $(u_\epsilon)_{\epsilon>0}$, $u_\epsilon \in H^3(\Omega_\epsilon) \cap H^1_0(\Omega_\epsilon)$, we have the following result

Lemma 4. *Let $Y = [-1/2, 1/2]^{N-1}$, $\alpha \in \mathbb{R}$, $\alpha > 0$. Let $\Omega = W \times (-1, 0)$, where $W \subset \mathbb{R}^{N-1}$ is bounded domain of class C^3. Let Ω_ϵ be as in (1.1). Let $(u_\epsilon)_{\epsilon>0}$ be such that $H^3(\Omega_\epsilon) \cap H^1_{0,*}(\Omega_\epsilon)$ for all $\epsilon > 0$ and $u_\epsilon|_\Omega \to u$ weakly in $H^3(\Omega)$. Let also $\hat{u} \in L^2(W, H^3(Y \times (-1, 0)))$ be defined by (5.18). Then:*

(i) If $\alpha > 5/2$ then $u \in H^3(\Omega) \cap H^1_{0,*}(\Omega)$;
(ii) If $\alpha = 5/2$ then $u \in H^3(\Omega) \cap H^1_{0,*}(\Omega)$ and for $i, j \in \{1, \dots, N-1\}$,

$$\frac{\partial^2 \hat{u}}{\partial y_i \partial y_j}(\bar{x}, \bar{y}, 0) = -\frac{\partial u}{\partial x_N}(\bar{x}, \bar{y}, 0) \frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j}. \tag{5.1}$$

(iii) If $0 < \alpha < 5/2$ then $u \in H^3(\Omega) \cap H^2_{0,*}(\Omega)$;
(iv) If $0 < \alpha \leq 1$ then $u \in H^3_{0,*}(\Omega)$

Proof. Fix $0 < \epsilon < 1$. We find convenient to treat first the case $\alpha \geq 3/2$. Since $u_\epsilon \in H^1_0(\Omega_\epsilon)$

$$u_\epsilon(\bar{x}, g_\epsilon(\bar{x})) = 0, \quad \text{for a.e. } \bar{x} \in W. \tag{5.2}$$

Note that the function $u_\epsilon(\cdot, g_\epsilon(\cdot)) \in H^{5/2}(W) \subset H^2(W)$. Differentiation (5.2) with respect to x_i and then with respect to x_j, $i, j \in \{1, \dots, N-1\}$ gives

$$\begin{aligned} &\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_j}(\bar{x}, g_\epsilon(\bar{x})) + \frac{\partial^2 u_\epsilon}{\partial x_i \partial x_N}(\bar{x}, g_\epsilon(\bar{x})) \frac{\partial g_\epsilon(\bar{x})}{\partial x_j} + \frac{\partial^2 u_\epsilon}{\partial x_j \partial x_N}(\bar{x}, g_\epsilon(\bar{x})) \frac{\partial g_\epsilon(\bar{x})}{\partial x_i} \\ &+ \frac{\partial^2 u_\epsilon}{\partial x_N^2}(\bar{x}, g_\epsilon(\bar{x})) \frac{\partial g_\epsilon(\bar{x})}{\partial x_i} \frac{\partial g_\epsilon(\bar{x})}{\partial x_j} + \frac{\partial u_\epsilon}{\partial x_N}(\bar{x}, g_\epsilon(\bar{x})) \frac{\partial^2 g_\epsilon(\bar{x})}{\partial x_i \partial x_j} = 0, \end{aligned} \tag{5.3}$$

for a.e. $\bar{x} \in W$. For $v \in H^1(\Omega_\epsilon)$, let $\hat{v}(\bar{x}, y)$ for all $\bar{x} \in \widehat{W}_\epsilon$, $\bar{y} \in Y$, $y_N \in (-1/\epsilon, \epsilon^{\alpha-1} b(\bar{y}))$ be as in Definition 1. It is understood that $\hat{v}$ is set to be zero for all $\bar{x} \in W \setminus \widehat{W}_\epsilon$.
To shorten the notation, define $y_\epsilon := \epsilon^{\alpha-1} b(\bar{y})$, $\epsilon > 0$, $\bar{y} \in W$, and note that by periodicity of b, $b(\bar{y}) = b([\bar{x}/\epsilon] + \bar{y}) = \epsilon^{-\alpha} \widehat{g_\epsilon}(\bar{x}, \bar{y})$ for all $(\bar{x}, \bar{y}) \in C^k_\epsilon \times Y$.

An application of the unfolding operator to equality (5.3), with the help of Lemma 2 gives

$$\begin{aligned} &\frac{1}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x}, \bar{y}, y_\epsilon) + \frac{\epsilon^{\alpha-1}}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, y_\epsilon) \frac{\partial b(\bar{y})}{\partial y_j} + \frac{\epsilon^{\alpha-1}}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_j \partial y_N}(\bar{x}, \bar{y}, y_\epsilon) \frac{\partial b(\bar{y})}{\partial y_i} \\ &+ \frac{\epsilon^{2\alpha-2}}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x}, \bar{y}, y_\epsilon) \frac{\partial b(\bar{y})}{\partial y_i} \frac{\partial b(\bar{y})}{\partial y_j} + \frac{\epsilon^{\alpha-2}}{\epsilon} \frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, y_\epsilon) \frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} = 0, \end{aligned}$$

for a.e. $\bar{x} \in W$, for a.e. $\bar{y} \in Y$. Define

$$\begin{aligned} \hat{\Psi}_\epsilon(\bar{x}, y) = &\frac{1}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x}, y) + \frac{\epsilon^{\alpha-1}}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, y) \frac{\partial b(\bar{y})}{\partial y_j} + \frac{\epsilon^{\alpha-1}}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_j \partial y_N}(\bar{x}, y) \frac{\partial b(\bar{y})}{\partial y_i} \\ &+ \frac{\epsilon^{2\alpha-2}}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x}, y) \frac{\partial b(\bar{y})}{\partial y_i} \frac{\partial b(\bar{y})}{\partial y_j} + \frac{\epsilon^{\alpha-2}}{\epsilon} \frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, y) \frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j}, \end{aligned}$$

for a.e. $\bar{x} \in W$, for a.e. $\bar{y} \in Y$. Let also $\hat{Y} := \{y \in \mathbb{R}^N : \bar{y} \in Y, -1 < y_N < \epsilon^{\alpha-1} b(\bar{y})\}$. Then $\hat{\Psi}_\epsilon \in L^2(W, H^1(\hat{Y}))$.

Since $\hat{\Psi}_\epsilon(\bar{x},y,y_\epsilon)=0$ we have that $|\hat{\Psi}_\epsilon(\bar{x},\bar{y},0)|\leq\int_0^{y_\epsilon}|\partial_{y_N}\hat{\Psi}_\epsilon(\bar{x},\bar{y},t)|\,dt$ for a.e. $\bar{x}\in W$, $\bar{y}\in Y$, from which we deduce

$$
\begin{aligned}
|\hat{\Psi}_\epsilon(\bar{x},\bar{y},0)| &\leq \left(\epsilon^{\alpha-1}\|b\|_\infty\right)^{1/2}\Bigg[\frac{1}{\epsilon^2}\left\|\frac{\partial^3\hat{u}_\epsilon}{\partial y_i\partial y_j\partial y_N}(\bar{x},\bar{y},\cdot)\right\|_{L^2(0,y_\epsilon)}\\
&+\frac{\epsilon^{\alpha-1}}{\epsilon^2}\|\nabla b\|_\infty\left\|\frac{\partial^3\hat{u}_\epsilon}{\partial y_i\partial y_N^2}(\bar{x},\bar{y},\cdot)\right\|_{L^2(0,y_\epsilon)}+\frac{\epsilon^{\alpha-1}}{\epsilon^2}\|\nabla b\|_\infty\left\|\frac{\partial^3\hat{u}_\epsilon}{\partial y_j\partial y_N^2}(\bar{x},\bar{y},\cdot)\right\|_{L^2(0,y_\epsilon)}\\
&+\frac{\epsilon^{2\alpha-2}}{\epsilon^2}\|\nabla b\|_\infty^2\left\|\frac{\partial^3\hat{u}_\epsilon}{\partial y_N^3}(\bar{x},\bar{y},\cdot)\right\|_{L^2(0,y_\epsilon)}+\frac{\epsilon^{\alpha-2}}{\epsilon}\|D^2b\|_\infty\left\|\frac{\partial^2\hat{u}_\epsilon}{\partial y_N^2}(\bar{x},\bar{y},\cdot)\right\|_{L^2(0,y_\epsilon)}\Bigg],
\end{aligned}
\tag{5.4}
$$

Let us define $\hat{Y}_{>0}:=\hat{Y}\cap\{y_N\in\mathbb{R}:y_N>0\}$. We square both hand sides of (5.4) and integrate over $W\times Y$ to get

$$
\begin{aligned}
\int_W\int_Y|\hat{\Psi}_\epsilon(\bar{x},\bar{y},0)|^2d\bar{y}d\bar{x}&\leq C(\|b\|^2_{C^2(Y)}+\|\nabla b\|^4_\infty)\epsilon^{\alpha-1}\Bigg[\frac{1}{\epsilon^4}\|D_y^3\hat{u}_\epsilon\|^2_{L^2(W\times\hat{Y}_{>0})}\\
+\frac{\epsilon^{2\alpha-2}}{\epsilon^4}\|D_y^3\hat{u}_\epsilon\|^2_{L^2(W\times\hat{Y}_{>0})}&+\frac{\epsilon^{4\alpha-4}}{\epsilon^4}\|D_y^3\hat{u}_\epsilon\|^2_{L^2(W\times\hat{Y}_{>0})}+\frac{\epsilon^{2\alpha-4}}{\epsilon^2}\left\|\frac{\partial^2\hat{u}_\epsilon}{\partial y_N^2}\right\|^2_{L^2(W\times\hat{Y}_{>0})}\Bigg],
\end{aligned}
\tag{5.5}
$$

Due to (3.4) and some basic estimates, (5.5) implies that

$$
\begin{aligned}
&\|\hat{\Psi}_\epsilon(\bar{x},\bar{y},0)\|^2_{L^2(W\times Y)}\\
&\qquad\leq C\|D^3u_\epsilon\|^2_{L^2(\Omega_\epsilon)}(\epsilon^\alpha+\epsilon^{3\alpha-2}+\epsilon^{5\alpha-4})+C\epsilon^{3\alpha-4}\left\|\frac{\partial^2u_\epsilon}{\partial x_N^2}\right\|^2_{L^2(\Omega_\epsilon\setminus\Omega)}\\
&\qquad\leq C(\epsilon^\alpha+\epsilon^{4\alpha-4})+o(\epsilon^\alpha)
\end{aligned}
\tag{5.6}
$$

where in the last inequality we used the classical one-dimensional Sobolev estimate $\|\partial^2_{x_N^2}u_\epsilon\|^2_{L^2(\Omega_\epsilon\setminus\Omega)}\leq C|\Omega_\epsilon\setminus\Omega|\|\partial^2_{x_N^2}u_\epsilon\|^2_{W^{1,2}(\Omega_\epsilon)}$, for some $C>0$, which holds since $\partial^2_{x_N^2}u_\epsilon$ is in $H^1(\Omega_\epsilon)$, $\epsilon>0$, with uniformly bounded norm. Note that since $\alpha\geq 3/2>1$, (5.6) implies

$$
\int_W\int_Y\epsilon^{-1}\left|\hat{\Psi}_\epsilon(\bar{x},\bar{y},0)-\int_Y\hat{\Psi}_\epsilon(\bar{x},\bar{z},0)d\bar{z}\right|^2d\bar{y}d\bar{x}=O(\epsilon^{\alpha-1})\to 0,
\tag{5.7}
$$

as $\epsilon\to 0$. We can rewrite (5.7) as

$$
\int_W\int_Y\left|T_1+\cdots+T_5\right|^2d\bar{y}d\bar{x}\to 0,\quad\text{as }\epsilon\to 0,
\tag{5.8}
$$

where

$$T_1 = \frac{1}{\epsilon^{5/2}}\left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x},\bar{y},0) - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x},\bar{z},0)d\bar{z}\right);$$

$$T_2 = \frac{\epsilon^{\alpha-1}}{\epsilon^{5/2}}\left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x},\bar{y},0)\frac{\partial b(\bar{y})}{\partial y_j} - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x},\bar{z},0)\frac{\partial b(\bar{z})}{\partial y_j}d\bar{z}\right);$$

$$T_3 = \frac{\epsilon^{\alpha-1}}{\epsilon^{5/2}}\left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_j \partial y_N}(\bar{x},\bar{y},0)\frac{\partial b(\bar{y})}{\partial y_i} - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_j \partial y_N}(\bar{x},\bar{z},0)\frac{\partial b(\bar{z})}{\partial y_i}d\bar{z}\right);$$

$$T_4 = \frac{\epsilon^{2\alpha-2}}{\epsilon^{5/2}}\left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x},\bar{y},0)\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j} - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x},\bar{z},0)\frac{\partial b(\bar{z})}{\partial y_i}\frac{\partial b(\bar{z})}{\partial y_j}d\bar{z}\right);$$

$$T_5 = \frac{\epsilon^{\alpha-2}}{\epsilon^{3/2}}\left(\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{y},0)\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} - \int_Y \frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{z},0)\frac{\partial^2 b(\bar{z})}{\partial y_i \partial y_j}d\bar{z}\right).$$

Recall that the function U_ϵ defined by

$$U_\epsilon(\bar{x},y) = \hat{u}_\epsilon(\bar{x},y) - \int_Y \left(\hat{u}_\epsilon(\bar{x},\bar{\zeta},0) - \sum_{|\eta|=2}\int_Y D_y^\eta \hat{u}_\epsilon(\bar{x},\bar{\zeta},0)\,\mathrm{d}\bar{\zeta}\right)\frac{\bar{\zeta}^\eta}{\eta!}\,\mathrm{d}\bar{\zeta}$$
$$- \int_Y \nabla_y \hat{u}_\epsilon(\bar{x},\bar{\zeta},0)\,\mathrm{d}\bar{\zeta}\cdot y - \sum_{|\eta|=2}\int_Y D_y^\eta \hat{u}_\epsilon(\bar{x},\bar{\zeta},0)\,\mathrm{d}\bar{\zeta}\,\frac{y^\eta}{\eta!},$$

is such that the sequence $(\epsilon^{-5/2}U_\epsilon)$ is uniformly bounded in $L^2(W,H^3(Y\times(d,0))$, for any $d<0$, see Lemma 3. Note also that $D_y^\eta U_\epsilon = D_y^\eta \hat{u}_\epsilon - \int_Y D_y^\eta \hat{u}_\epsilon(\cdot,\bar{z},\cdot)d\bar{z}$ for any $|\eta|=2$. Using these facts we deduce that

$$\int_W\int_Y |T_1|^2 d\bar{y}d\bar{x} = \int_W\int_Y \left|\epsilon^{-5/2}\frac{\partial^2 U_\epsilon}{\partial y_i\partial y_j}(\bar{x},\bar{y},0)\right|^2 d\bar{y}d\bar{x}$$
$$\le C\left\|\epsilon^{-5/2}\frac{\partial^2 U_\epsilon}{\partial y_i\partial y_j}\right\|^2_{L^2(W,H^1(Y\times(-1,0))}$$
$$\le C\|\epsilon^{-5/2}D_y^3\hat{u}_\epsilon\|^2_{L^2(W\times Y\times(-1,0))} \le C\|D^3 u_\epsilon\|^2_{L^2(\Omega)},$$

where we have used a trace inequality, the Poincaré-Wirtinger inequality, and the exact integration formula (3.4). Hence T_1 is bounded in $L^2(W\times Y)$, uniformly in $\epsilon>0$.

Consider now T_2. Note that the function $\frac{\partial b}{\partial y_j}$ has null average over Y because of periodicity. Hence,

$$\int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i\partial y_N}(\bar{x},\bar{z},0)\frac{\partial b(\bar{z})}{\partial y_j}d\bar{z} = \int_Y \frac{\partial b(\bar{z})}{\partial y_j}\left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i\partial y_N}(\bar{x},\bar{z},0) - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i\partial y_N}(\bar{x},\bar{t},0)d\bar{t}\right)d\bar{z}$$

and

$$\int_W\int_Y \epsilon^{2\alpha-2-5}\left|\int_Y \frac{\partial b(\bar{z})}{\partial y_j}\left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i\partial y_N}(\bar{x},\bar{z},0) - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i\partial y_N}(\bar{x},\bar{t},0)d\bar{t}\right)d\bar{z}\right|^2 d\bar{y}d\bar{x}$$
$$\le C\frac{\epsilon^{2\alpha}}{\epsilon^2}\int_W\int_Y\int_Y \left|\epsilon^{-5/2}\left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i\partial y_N}(\bar{x},\bar{z},0) - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i\partial y_N}(\bar{x},\bar{t},0)d\bar{t}\right)\right|^2 d\bar{z}d\bar{y}d\bar{x} \qquad (5.9)$$
$$\le C\epsilon^{2\alpha-2}\|\epsilon^{-5/2}\partial^2_{y_iy_N}U_\epsilon(\cdot,\cdot,0)\|^2_{L^2(W\times Y)}$$
$$\le C\epsilon^{2\alpha-2}\|\epsilon^{-5/2}D_y^3\hat{u}_\epsilon\|^2_{L^2(W\times Y\times(-1,0))} \to 0,$$

as $\epsilon \to 0$, for all $\alpha > 1$. We deduce that

$$\begin{aligned}\int_W \int_Y |T_2|^2 d\bar{y} d\bar{x} &\leq C \int_W \int_Y \left| \frac{\epsilon^{\alpha-1}}{\epsilon^{5/2}} \left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, 0) \frac{\partial b(\bar{y})}{\partial y_j} \right) \right|^2 d\bar{y} d\bar{x} \\ &+ C \int_W \int_Y \left| \int_Y \epsilon^{\alpha-1-5/2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{z}, 0) \frac{\partial b(\bar{z})}{\partial y_j} d\bar{z} \right|^2 d\bar{y} d\bar{x} \\ &\leq C \int_W \int_Y \left| \frac{\epsilon^{\alpha}}{\epsilon^{3/2}} \left(\frac{1}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, 0) \frac{\partial b(\bar{y})}{\partial y_j} \right) \right|^2 d\bar{y} d\bar{x} + o(1),\end{aligned} \tag{5.10}$$

as $\epsilon \to 0$. We claim that

$$\frac{1}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, 0) \frac{\partial b(\bar{y})}{\partial y_j} \to \frac{\partial^2 u}{\partial x_i \partial x_N}(\bar{x}, 0) \frac{\partial b(\bar{y})}{\partial y_j}, \tag{5.11}$$

in $L^2(W \times Y)$ as $\epsilon \to 0$. Since $u_\epsilon|_\Omega \to u$ weakly in $H^3(\Omega)$, by the compactness of the trace operator we have that

$$\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_N}(\bar{x}, 0) \to \frac{\partial^2 u}{\partial x_i \partial x_N}(\bar{x}, 0), \tag{5.12}$$

in $L^2(W)$, as $\epsilon \to 0$. Now define

$$\overline{\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_N}}(\bar{x}) := \frac{1}{\epsilon^{N-1}} \int_{C_\epsilon(\bar{x})} \frac{\partial^2 u_\epsilon}{\partial x_i \partial x_N}(\bar{t}, 0)\, d\bar{t},$$

where $C_\epsilon(\bar{x})$ is as in (3.3). Note that, by a change of variable,

$$\overline{\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_N}}(\bar{x}) = \int_Y \widehat{\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_N}}(\bar{x}, \bar{z}, 0) d\bar{z} = \frac{1}{\epsilon^2} \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{z}, 0)\, d\bar{z}.$$

By (5.12) we deduce that

$$\overline{\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_N}} \to \frac{\partial^2 u}{\partial x_i \partial x_N}(\cdot, 0),$$

strongly in $L^2(W)$ as $\epsilon \to 0$. Here, we have used the fact that if a sequence of functions v_ϵ converges strongly in L^2 to v then $\overline{v_\epsilon}$ converges strongly in L^2 to v. We give a proof of this in Lemma 8 in Appendix (B). Since $\epsilon^{-5/2} \partial_{y_i y_N} U_\epsilon$ is uniformly bounded in $L^2(W \times Y)$, for all $\epsilon > 0$ due to Lemma 2, it follows that

$$\frac{1}{\epsilon^2} \left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\cdot, \cdot, 0) - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\cdot, \bar{z}, 0) d\bar{z} \right) \to 0,$$

in $L^2(W \times Y)$ as $\epsilon \to 0$. Hence, $\frac{1}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, 0) \to \frac{\partial^2 u}{\partial x_i \partial x_N}(\bar{x}, 0)$ in $L^2(W \times Y)$ as $\epsilon \to 0$, which proves the claim. Since $\alpha > 3/2$, by recalling (5.10) we then deduce that T_2 vanishes in $L^2(W \times Y)$ as $\epsilon \to 0$.

T_3 is exactly T_2 with swapped indexes i and j, hence also T_3 vanishes in $L^2(W \times Y)$ as $\epsilon \to 0$.

We then consider T_4. By arguing as in (5.11) we deduce that

$$\frac{1}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x}, \bar{y}, 0) \frac{\partial b(\bar{y})}{\partial y_i} \frac{\partial b(\bar{y})}{\partial y_j} \to \frac{\partial^2 u}{\partial y_N^2}(\bar{x}, 0) \frac{\partial b(\bar{y})}{\partial y_i} \frac{\partial b(\bar{y})}{\partial y_j}, \tag{5.13}$$

in $L^2(W\times Y)$ as $\epsilon\to 0$, so the integral in Y of the left-hand side of (5.13) is convergent. Thus,

$$T_4=\frac{\epsilon^{2\alpha}}{\epsilon^{5/2}}\left(\frac{1}{\epsilon^2}\frac{\partial^2\hat{u}_\epsilon}{\partial y_N^2}(\bar{x},\bar{y},0)\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j}-\int_Y\frac{1}{\epsilon^2}\frac{\partial^2\hat{u}_\epsilon}{\partial y_N^2}(\bar{x},\bar{z},0)\frac{\partial b(\bar{z})}{\partial y_i}\frac{\partial b(\bar{z})}{\partial y_j}d\bar{z}\right)\to 0, \tag{5.14}$$

in $L^2(W\times Y)$ as $\epsilon\to 0$ for all $\alpha>5/4$, hence in particular for any $\alpha\geq 3/2$. Finally, we consider T_5. Arguing as in the proof of Claim (5.11) we can prove that

$$\frac{1}{\epsilon}\frac{\partial\hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{y},0)\frac{\partial^2 b(\bar{y})}{\partial y_i\partial y_j}\to\frac{\partial u}{\partial y_N}(\bar{x},0)\frac{\partial^2 b(\bar{y})}{\partial y_i\partial y_j}, \tag{5.15}$$

in $L^2(W\times Y)$ as $\epsilon\to 0$ and

$$\int_Y\frac{1}{\epsilon}\frac{\partial\hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{z},0)\frac{\partial^2 b(\bar{z})}{\partial y_i\partial y_j}d\bar{z}\to\frac{\partial u}{\partial y_N}(\bar{x},0)\int_Y\frac{\partial^2 b(\bar{z})}{\partial y_i\partial y_j}d\bar{z}=0, \tag{5.16}$$

in $L^2(W\times Y)$ as $\epsilon\to 0$, where the right-hand side of (5.16) is zero due to periodicity of b. We now consider different cases according to the value of α.

Case $3/2<\alpha<5/2$**.** In this case, by summarising the previous results we have that T_1 is uniformly bounded in $L^2(W\times Y)$ as $\epsilon\to 0$, whereas T_2,T_3,T_4 tend to zero in $L^2(W\times Y)$ as $\epsilon\to 0$. Then (5.8) implies that there exists a constant $M>0$ such that

$$\left(\int_W\int_Y|T_5|^2d\bar{y}d\bar{x}\right)^{1/2}\leq\left(\int_W\int_Y|T_1+T_2+T_3+T_4|^2d\bar{y}d\bar{x}\right)^{1/2}+o(1)\leq M,$$

as $\epsilon\to 0$. Thus,

$$\left\|\frac{1}{\epsilon}\frac{\partial\hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{y},0)\frac{\partial^2 b(\bar{y})}{\partial y_i\partial y_j}-\int_Y\frac{1}{\epsilon}\frac{\partial\hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{z},0)\frac{\partial^2 b(\bar{z})}{\partial y_i\partial y_j}d\bar{z}\right\|_{L^2(W\times Y)}=O(\epsilon^{5/2-\alpha}),$$

as $\epsilon\to 0$. By letting $\epsilon\to 0$ and recalling (5.15) and (5.16) we deduce that $\frac{\partial u}{\partial y_N}(\bar{x},0)\frac{\partial^2 b(\bar{y})}{\partial y_i\partial y_j}=0$, for a.e. $\bar{x}\in W$, for a.e. $\bar{y}\in Y$, and since b is not affine we deduce that

$$\frac{\partial u}{\partial x_N}(\bar{x},0)=0, \tag{5.17}$$

for a.e. $\bar{x}\in W$. We conclude that $u\in H^3(\Omega)\cap H^2_{0,*}$.

Case $\alpha=5/2$**.** In this case, we have the estimate

$$\left(\int_W\int_Y|T_1+T_5|^2d\bar{y}d\bar{x}\right)^{1/2}\leq\left(\int_W\int_Y|T_2+T_3+T_4|^2d\bar{y}d\bar{x}\right)^{1/2}+o(1)=o(1),$$

as $\epsilon\to 0$. Thus,

$$\frac{1}{\epsilon^{5/2}}\left(\frac{\partial^2\hat{u}_\epsilon}{\partial y_i\partial y_j}(\bar{x},\bar{y},0)-\int_Y\frac{\partial^2\hat{u}_\epsilon}{\partial y_i\partial y_j}(\bar{x},\bar{z},0)d\bar{z}\right)+\frac{1}{\epsilon}\frac{\partial\hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{y},0)\frac{\partial^2 b(\bar{y})}{\partial y_i\partial y_j}\to 0,$$

as $\epsilon\to 0$. Now since $(\epsilon^{-5/2}U_\epsilon)$ is uniformly bounded in $L^2(W\times Y\times(d,0))$, there exists a subsequence of $(\epsilon^{-5/2}U_\epsilon)$ and a function $\hat{u}\in L^2(W,H^3(Y\times(d,0)))$ such that

$$\epsilon^{-5/2}U_\epsilon\rightharpoonup\hat{u}, \tag{5.18}$$

in $L^2(W,H^3(Y\times(d,0)))$. (5.18) implies that

$$\frac{1}{\epsilon^{5/2}}\left(\frac{\partial^2\hat{u}_\epsilon}{\partial y_i\partial y_j}(\bar{x},\bar{y},0)-\int_Y\frac{\partial^2\hat{u}_\epsilon}{\partial y_i\partial y_j}(\bar{x},\bar{z},0)d\bar{z}\right)\to\frac{\partial^2\hat{u}}{\partial y_i\partial y_j}(\bar{x},\bar{y},0),$$

strongly in $L^2(W\times Y)$ as $\epsilon\to 0$. Moreover, according to (5.15) we deduce that

$$\frac{\partial^2\hat{u}}{\partial y_i\partial y_j}(\bar{x},\bar{y},0)=-\frac{\partial u}{\partial y_N}(\bar{x},0)\frac{\partial^2 b(\bar{y})}{\partial y_i\partial y_j}, \tag{5.19}$$

for a.e. $\bar{x}\in W$, a.e. $\bar{y}\in Y$, which is (5.1).

Case $\alpha\leq 1$. In this case we give a more direct proof based on a different definition of the unfolding operator. We define

$$\hat{Y}=\{(\bar{y},y_N):\bar{y}\in Y,-1<y_N<b(\bar{y})\}, \tag{5.20}$$

and

$$\hat{u}_\epsilon(\bar{x},\bar{y},y_N):=u_\epsilon\left(\epsilon\left[\frac{\bar{x}}{\epsilon}\right]+\epsilon\bar{y},\epsilon^\alpha y_N\right), \tag{5.21}$$

for all $(\bar{x},y)\in W\times\hat{Y}$, for all $u_\epsilon\in H^3(\Omega_\epsilon)$. Note that $\hat{u}_\epsilon$, $\epsilon\in(0,1]$, are defined on a fixed domain of $\mathbb{R}^N$. Then, starting from the identity (5.2) we deduce the analogous of (5.3), which namely reads

$$\begin{aligned}&\frac{1}{\epsilon^2}\frac{\partial^2\hat{u}_\epsilon}{\partial y_i\partial y_j}(\bar{x},\bar{y},b(\bar{y}))+\frac{\epsilon^{\alpha-1}}{\epsilon^{\alpha+1}}\frac{\partial^2\hat{u}_\epsilon}{\partial y_i\partial y_N}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_j}\\&+\frac{\epsilon^{\alpha-1}}{\epsilon^{\alpha+1}}\frac{\partial^2\hat{u}_\epsilon}{\partial y_j\partial y_N}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_i}+\frac{\epsilon^{2\alpha-2}}{\epsilon^{2\alpha}}\frac{\partial^2\hat{u}_\epsilon}{\partial y_N^2}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j}\\&+\frac{\epsilon^{\alpha-2}}{\epsilon^\alpha}\frac{\partial\hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial^2 b(\bar{y})}{\partial y_i\partial y_j}=0.\end{aligned} \tag{5.22}$$

If $\alpha=1$, by arguing as in (8.4) below, we have

$$\frac{1}{\epsilon^2}\frac{\partial^2\hat{u}_\epsilon}{\partial y_i\partial y_j}(\bar{x},\bar{y},b(\bar{y}))\to\frac{\partial^2 u}{\partial x_i\partial x_j}(\bar{x},0),$$

$$\frac{1}{\epsilon^2}\frac{\partial^2\hat{u}_\epsilon}{\partial y_i\partial y_N}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_j}\to\frac{\partial^2 u}{\partial x_i\partial x_N}(\bar{x},0)\frac{\partial b(\bar{y})}{\partial y_j},$$

$$\frac{1}{\epsilon^2}\frac{\partial^2\hat{u}_\epsilon}{\partial y_N^2}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j}\to\frac{\partial^2 u}{\partial x_N^2}(\bar{x},0)\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j},$$

as $\epsilon\to 0$, where the limits are taken in $L^2(W\times Y)$. According to (5.22), we immediately discover that

$$\left\|\frac{1}{\epsilon}\frac{\partial\hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{y},b(\bar{y}))\right\|_{L^2(W\times\hat{Y})}\leq C\epsilon, \tag{5.23}$$

for all $\epsilon>0$. By (5.23) we deduce that

$$\frac{\partial u}{\partial x_N}(\bar{x},0)=0, \tag{5.24}$$

and that there exists a function $\zeta\in L^2(W)$ such that, up to a subsequence,

$$\frac{1}{\epsilon^2}\frac{\partial\hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{y},b(\bar{y}))\rightharpoonup\zeta(\bar{x}),$$

in $L^2(W\times Y)$ as $\epsilon\to 0$. The fact that ζ does not depend on $\bar{y}$ is an easy consequence of the following argument. Let $\varphi\in C_c^\infty(W\times Y)$. Then

$$\int_{W\times Y}\frac{1}{\epsilon^2}\frac{\partial\hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial\varphi}{\partial y_i}\,d\bar{x}d\bar{y}=-\int_{W\times Y}\frac{1}{\epsilon^2}\frac{\partial^2\hat{u}_\epsilon}{\partial y_N\partial y_i}(\bar{x},\bar{y},b(\bar{y}))\varphi\,d\bar{x}d\bar{y},$$

and passing to the limit as $\epsilon \to 0$ we deduce that

$$\int_{W\times Y} \zeta \frac{\partial \varphi}{\partial y_i}\, d\bar{x}d\bar{y} = -\int_{W\times Y} \frac{\partial^2 u}{\partial x_N \partial x_i}(\bar{x},0)\varphi\, d\bar{x}d\bar{y} = 0, \tag{5.25}$$

where we have used that $\frac{\partial^2 u}{\partial x_N \partial x_i}(\bar{x},0) = 0$ because of (5.24). Equation(5.25) implies that ζ is weakly differentiable in y_i and that $\frac{\partial \zeta}{\partial y_i} = 0$.
Taking the limit as $\epsilon \to 0$ in $L^2(W\times Y)$ in (5.22) we deduce that

$$\begin{aligned} &\frac{\partial^2 u}{\partial x_i \partial x_j}(\bar{x},0) + \frac{\partial^2 u}{\partial x_i \partial x_N}(\bar{x},0)\frac{\partial b(\bar{y})}{\partial y_j} + \frac{\partial^2 u}{\partial x_j \partial x_N}(\bar{x},0)\frac{\partial b(\bar{y})}{\partial y_i} \\ &+ \frac{\partial^2 u}{\partial x_N^2}(\bar{x},0)\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j} + \zeta(\bar{x})\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} = 0. \end{aligned} \tag{5.26}$$

Because of (5.24) the first three summands in (5.26) are zero. Hence, (5.26) implies that

$$\frac{\partial^2 u}{\partial x_N^2}(\bar{x},0)\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j} + \zeta(\bar{x})\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} = 0. \tag{5.27}$$

Recall now that since b is Y-periodic, its derivatives are periodic and with null average on Y. An integration in Y in (5.27) yields

$$\frac{\partial^2 u}{\partial x_N^2}(\bar{x},0)\int_Y \frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j} d\bar{y} = 0,$$

for almost all $\bar{x} \in W$. Since this holds for all $i,j = 1,\dots,N-1$ we can in particular choose $i = j$ so that $\frac{\partial^2 u}{\partial x_N^2}(\bar{x},0)\int_Y |\nabla b|^2 d\bar{y} = 0$, and since b is non constant it must be $\frac{\partial^2 u}{\partial x_N^2}(\bar{x},0) = 0$ for almost all $\bar{x} \in W$.

If $\alpha < 1$ we can argue in a similar way. Namely, we multiply each side of (5.22) by $\epsilon^{2-2\alpha}$ in order to obtain

$$\begin{aligned} &\frac{1}{\epsilon^{2\alpha}}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x},\bar{y},b(\bar{y})) + \frac{\epsilon^{1-\alpha}}{\epsilon^{\alpha+1}}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_j} \\ &+ \frac{\epsilon^{1-\alpha}}{\epsilon^{\alpha+1}}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_j \partial y_N}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_i} + \frac{1}{\epsilon^{2\alpha}}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j} \\ &+ \frac{1}{\epsilon^{2\alpha}}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} = 0. \end{aligned} \tag{5.28}$$

Since $u(\bar{x},0) = 0$, a.a $x \in W$, the first three summands in (5.28) are vanishing as $\epsilon \to 0$. Then we deduce that

$$\frac{\partial^2 u}{\partial x_N^2}(\bar{x},0)\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j} + \lim_{\epsilon\to 0}\frac{1}{\epsilon^{2\alpha}}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} = 0.$$

This first implies that

$$\left\| \frac{1}{\epsilon^\alpha}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial^2 b}{\partial y_i \partial y_j} \right\|_{L^2(W\times Y)} \le C\epsilon^\alpha,$$

hence $\frac{\partial u}{\partial x_N}(\bar{x},0) = 0$. Moreover, we deduce that up to a subsequence there exists $\zeta \in L^2(W)$ such that $\frac{1}{\epsilon^{2\alpha}}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{y},b(\bar{y})) \rightharpoonup \zeta(\bar{x})$ in $L^2(W\times Y)$ as $\epsilon \to 0$. Then arguing as in the case $\alpha = 1$ we deduce that $\frac{\partial^2 u}{\partial x_N^2}(\bar{x},0) = 0$. □

Proof of Theorem 1(iii),(iv). We first prove Claim (*iii*). We will show that the Condition (C), defined in [2, Def. 3.1] holds with $V(\Omega_\epsilon) = H^3(\Omega_\epsilon)\cap H_0^1(\Omega_\epsilon)$ and $V(\Omega) = H^3(\Omega)\cap H_0^1(\Omega)\cap H^2_{0,*}(\Omega)$. In [2, Def. 3.1] we choose

$$K_\epsilon = \{x \in \Omega : x_N < -\epsilon\},$$

$\epsilon \in (0,1]$, $T_\epsilon : V(\Omega) \to V(\Omega_\epsilon)$ as in (3.2) and $E_\epsilon : V(\Omega_\epsilon) \to H^m(\Omega)$ as the restriction operator $E_\epsilon u_\epsilon = u_\epsilon|_\Omega$, $\epsilon \in (0,1]$. With this choices it is not difficult to verify that conditions (C1), (C2)(i), (C2)(iii), (C3)(i) and (C3)(ii) hold true. Then it is sufficient to prove the validity of conditions $(C2)(ii)$ and $(C3)(iii)$.

In order to show that $(C2)(ii)$ holds it is sufficient to use Lemma 4(iii) and its proof. Indeed, if $\alpha > 3/2$ then $\lim_{\epsilon\to 0}\|T_\epsilon\varphi\|_{H^3(\Omega_\epsilon\setminus K_\epsilon)} = 0$ for all $\varphi \in V(\Omega)$.

Condition $(C3)(iii)$ now follows directly from Lemma 4(iii), since we have proved that if $u_\epsilon \in V(\Omega_\epsilon)$ is such that $u_\epsilon|_\Omega \rightharpoonup u$ and $3/2 < \alpha < 5/2$, then $u \in V(\Omega)$.

Hence Condition (C) holds and [2, Thm 3.5] now yields the claim.
The proof of Claim (iv) is similar. We show that Condition (C) holds with $V(\Omega_\epsilon) = H^3(\Omega_\epsilon) \cap H^1_0(\Omega_\epsilon)$, $\epsilon \in (0,1]$, $V(\Omega) = H^3(\Omega) \cap H^1_0(\Omega) \cap H^3_{0,*}(\Omega)$, T_ϵ the extension-by-zero operator, and E_ϵ the restriction operator defined above. Then conditions (C1)-(C3) hold true. Note that Condition (C3)(iii) follows directly from Lemma 4(iv).

□

6. Proof of Theorem 1(ii)

In this section, we shall consider the case $\alpha = 5/2$ of Theorem 1. We refer to Section 3 for the notation about Φ_ϵ, h_ϵ, T_ϵ, C^k_ϵ, $\hat{u}$, $w^{3,2}_{Per_Y}(Y \times (-\infty, 0))$. We divide the proof in two subsections. Since the proof follows the same strategy as [21], [2], we will only sketch the proofs and refer to [2] for further details in the case of the biharmonic operator with SBC.

6.1. **Macroscopic limit.** Let $f_\epsilon \in L^2(\Omega_\epsilon)$ and $f \in L^2(\Omega)$ be such that $f_\epsilon \rightharpoonup f$ in $L^2(\mathbb{R}^N)$ as $\epsilon \to 0$, with the understanding that the functions are extended by zero outside their natural domains. Let $v_\epsilon \in V(\Omega_\epsilon) = H^3(\Omega_\epsilon) \cap H^1_0(\Omega_\epsilon)$ be such that

$$A_{\Omega_\epsilon} v_\epsilon = f_\epsilon, \tag{6.1}$$

for all $\epsilon > 0$ small enough. Then $\|v_\epsilon\|_{H^3(\Omega_\epsilon)} \le M$ for all $\epsilon > 0$ sufficiently small, hence, possibly passing to a subsequence there exists $v \in H^3(\Omega) \cap H^1_0(\Omega)$ such that $v_\epsilon \rightharpoonup v$ in $H^3(\Omega)$ and $v_\epsilon \to v$ in $L^2(\Omega)$.
Let $\varphi \in V(\Omega) = H^3(\Omega) \cap H^1_0(\Omega)$ be a fixed test function. Since $T_\epsilon\varphi \in V(\Omega_\epsilon)$, by (6.1) we get

$$\int_{\Omega_\epsilon} D^3 v_\epsilon : D^3(T_\epsilon\varphi)\,\mathrm{d}x + \int_{\Omega_\epsilon} v_\epsilon T_\epsilon\varphi\,\mathrm{d}x = \int_{\Omega_\epsilon} f_\epsilon T_\epsilon\varphi\,\mathrm{d}x, \tag{6.2}$$

and passing to the limit as $\epsilon \to 0$ we have that

$$\int_{\Omega_\epsilon} v_\epsilon T_\epsilon\varphi\,\mathrm{d}x \to \int_\Omega v\varphi\,\mathrm{d}x, \quad \int_{\Omega_\epsilon} f_\epsilon T_\epsilon\varphi\,\mathrm{d}x \to \int_\Omega f\varphi\,\mathrm{d}x.$$

Now consider the first integral in the right-hand side of (6.2). Let us define $K_\epsilon = W \times (-1, -\epsilon)$. By splitting the integral in three terms corresponding to $\Omega_\epsilon \setminus \Omega$, $\Omega \setminus K_\epsilon$ and K_ϵ and by arguing as in [2, Section 8.3] one can show that

$$\int_{K_\epsilon} D^3 v_\epsilon : D^3(T_\epsilon\varphi)\,\mathrm{d}x \to \int_\Omega D^3 v : D^3\varphi\,\mathrm{d}x, \quad \int_{\Omega_\epsilon\setminus\Omega} D^3 v_\epsilon : D^3(T_\epsilon\varphi)\,\mathrm{d}x \to 0,$$

as $\epsilon \to 0$. Let $Q_\epsilon = \widehat{W}_\epsilon \times (-\epsilon, 0)$. We split again the remaining integral in two summands,

$$\int_{\Omega_\epsilon\setminus K_\epsilon} D^3 v_\epsilon : D^3(T_\epsilon\varphi)\,\mathrm{d}x = \int_{\Omega_\epsilon\setminus(K_\epsilon\cup Q_\epsilon)} D^3 v_\epsilon : D^3(T_\epsilon\varphi)\,\mathrm{d}x + \int_{Q_\epsilon} D^3 v_\epsilon : D^3(T_\epsilon\varphi)\,\mathrm{d}x. \tag{6.3}$$

Again, by arguing as in [2, Section 8.3] it is possible to prove that

$$\int_{\Omega_\epsilon\setminus(K_\epsilon\cup Q_\epsilon)} D^3 v_\epsilon : D^3(T_\epsilon\varphi)\,\mathrm{d}x \to 0,$$

as $\epsilon\to 0$. We now require two technical lemmata.

Lemma 5. *For all $y\in Y\times(-1,0)$ and $i,j,k=1,\dots,N$ the functions $\hat{h}_\epsilon(\bar{x},y)$, $\widehat{\frac{\partial h_\epsilon}{\partial x_i}}(\bar{x},y)$, $\widehat{\frac{\partial^2 h_\epsilon}{\partial x_i\partial x_j}}(\bar{x},y)$ and $\widehat{\frac{\partial^3 h_\epsilon}{\partial x_i\partial x_j\partial x_k}}(\bar{x},y)$ are independent of $\bar{x}$. Moreover,*

$$\hat{h}_\epsilon(\bar{x},y)=O(\epsilon^{5/2}),\ \widehat{\frac{\partial h_\epsilon}{\partial x_i}}(\bar{x},y)=O(\epsilon^{3/2}),\ \widehat{\frac{\partial^2 h_\epsilon}{\partial x_i\partial x_j}}(\bar{x},y)=O(\epsilon^{1/2}),$$

as $\epsilon\to 0$, for all $i,j=1,\dots,N$, uniformly in $y\in Y\times(-1,0)$, and

$$\epsilon^{1/2}\widehat{\frac{\partial^3 h_\epsilon}{\partial x_i\partial x_j\partial x_k}}(\bar{x},y)\to\frac{\partial^3(b(\bar{y})(y_N+1)^4)}{\partial y_i\partial y_j\partial y_k},$$

as $\epsilon\to 0$, for all $i,j,k=1,\dots,N$, uniformly in $y\in Y\times(-1,0)$.

Proof. We refer to [21, Lemma 4] and [2, Lemma 8.27], where similar computations were carried out in the case of strong intermediate boundary conditions. □

Lemma 6. *Let $v_\epsilon\in V(\Omega_\epsilon)=H^3(\Omega_\epsilon)\cap H^1_0(\Omega_\epsilon)$ be such that $\|v_\epsilon\|_{H^3(\Omega_\epsilon)}\le M$ for all $\epsilon>0$. Assume that up to a subsequence $v_\epsilon|_\Omega\rightharpoonup v$ in $H^3(\Omega)$. Let φ be a fixed function in $V(\Omega)$. Let $\hat{v}\in L^2(W,w^{3,2}_{\mathrm{Per}_Y}(Y\times(-\infty,0)))$ be as in Lemma 3. Then*

$$\begin{aligned}&\int_{Q_\epsilon} D^3 v_\epsilon : D^3(T_\epsilon\varphi)\,\mathrm{d}x\to\\ &-\int_W\int_{Y\times(-1,0)}(D^3_y(\hat{v}) : D^3(b(\bar{y})(1+y_N)^4)\,\mathrm{d}y\,\frac{\partial\varphi}{\partial x_N}(\bar{x},0)\mathrm{d}\bar{x}.\end{aligned}\tag{6.4}$$

Proof. In the following calculations we use the index notation and we drop the summation symbols. We calculate

$$\begin{aligned}&\int_{Q_\epsilon} D^3 v_\epsilon : D^3(T_\epsilon\varphi)\,\mathrm{d}x=\int_{Q_\epsilon}\frac{\partial^3 v_\epsilon}{\partial x_i\partial x_j\partial x_h}\frac{\partial^3(\varphi\circ\Phi_\epsilon)}{\partial x_i\partial x_j\partial x_h}\,\mathrm{d}x\\ &=\int_{Q_\epsilon}\frac{\partial^3 v_\epsilon}{\partial x_i\partial x_j\partial x_h}\frac{\partial^3\varphi}{\partial x_k\partial x_l\partial x_m}(\Phi_\epsilon(x))\frac{\partial\Phi^{(k)}_\epsilon}{\partial x_i}\frac{\partial\Phi^{(l)}_\epsilon}{\partial x_j}\frac{\partial\Phi^{(m)}_\epsilon}{\partial x_h}\,\mathrm{d}x\\ &+\int_{Q_\epsilon}\frac{\partial^3 v_\epsilon}{\partial x_i\partial x_j\partial x_h}\frac{\partial^2\varphi}{\partial x_k\partial x_l}(\Phi_\epsilon(x))\Big[\frac{\partial\Phi^{(k)}_\epsilon}{\partial x_i}\frac{\partial^2\Phi^{(l)}_\epsilon}{\partial x_j\partial x_h}+\frac{\partial\Phi^{(k)}_\epsilon}{\partial x_j}\frac{\partial^2\Phi^{(l)}_\epsilon}{\partial x_i\partial x_h}+\frac{\partial\Phi^{(k)}_\epsilon}{\partial x_h}\frac{\partial^2\Phi^{(l)}_\epsilon}{\partial x_i\partial x_j}\Big]\,\mathrm{d}x,\\ &+\int_{Q_\epsilon}\frac{\partial^3 v_\epsilon}{\partial x_i\partial x_j\partial x_h}\frac{\partial\varphi}{\partial x_k}(\Phi_\epsilon(x))\frac{\partial^3\Phi^{(k)}_\epsilon}{\partial x_i\partial x_j\partial x_h}\,\mathrm{d}x.\end{aligned}\tag{6.5}$$

It is not difficult to prove that the first integral in the right-hand side of (6.5) vanishes as $\epsilon\to 0$, see the proof of [21, Proposition 2]. We then consider the second integral in the right hand side of (6.5). Note that all the terms with $l\neq N$ vanish. Thus, without loss of generality we set $l=N$. Consider separately the case $k\neq N$, and $k=N$.

Case $k\neq N$: by the exact integration formula (3.4) we obtain

$$\begin{aligned}&\left|\int_{Q_\epsilon}\frac{\partial^3 v_\epsilon}{\partial x_i\partial x_j\partial x_h}\frac{\partial^2\varphi}{\partial x_k\partial x_N}(\Phi_\epsilon(x))\,\delta_{ki}\frac{\partial^2\Phi^{(N)}_\epsilon}{\partial x_j\partial x_h}\,\mathrm{d}x\right|\\ &\qquad\le C\epsilon^{1/2}\|\epsilon^{-5/2}\hat{v}_\epsilon\|_{W^{3,2}(\widehat{W}_\epsilon\times Y\times(-1,0))}\left\|\frac{\partial\varphi}{\partial x_k\partial x_N}\right\|_{L^2(Q_\epsilon)}\to 0,\end{aligned}$$

as $\epsilon \to 0$.
Case $k = N$: in this case (3.4) applied to (6.5) gives

$$\int_{Q_\epsilon} D^3 v_\epsilon : D^3(T_\epsilon \varphi)\,\mathrm{d}x = \epsilon^{-5} \int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \frac{\partial^3 \hat{v_\epsilon}}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 \varphi}{\partial x_N^2}(\widehat{\Phi_\epsilon}(y)) \cdot \\ \cdot \left[\frac{\partial \widehat{\Phi_\epsilon}^{(N)}}{\partial y_i} \frac{\partial^2 \widehat{\Phi_\epsilon}^{(N)}}{\partial y_j \partial y_h} + \frac{\partial \widehat{\Phi_\epsilon}^{(N)}}{\partial y_j} \frac{\partial^2 \widehat{\Phi_\epsilon}^{(N)}}{\partial y_i \partial y_h} + \frac{\partial \widehat{\Phi_\epsilon}^{(N)}}{\partial y_h} \frac{\partial^2 \widehat{\Phi_\epsilon}^{(N)}}{\partial y_i \partial y_j} \right] \mathrm{d}\bar{x}\mathrm{d}y, \tag{6.6}$$

and since we are summing on the indexes $i, j, h \in 1, \dots, N$, (6.6) equals

$$3\epsilon^{-5} \int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \frac{\partial^3 \hat{v_\epsilon}}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 (\varphi(\widehat{\Phi_\epsilon}(y)))}{\partial x_N^2} \frac{\partial \widehat{\Phi_\epsilon}^{(N)}}{\partial y_i} \frac{\partial^2 \widehat{\Phi_\epsilon}^{(N)}}{\partial y_j \partial y_h}\,\mathrm{d}\bar{x}\mathrm{d}y.$$

Note now that

$$\frac{\partial \widehat{\Phi_\epsilon}^{(k)}}{\partial y_i} = \begin{cases} \epsilon\delta_{ki}, & \text{if } k \neq N, \\ \epsilon\delta_{Ni} - \epsilon\widehat{\frac{\partial h_\epsilon}{\partial x_i}}, & \text{if } k = N. \end{cases} \qquad \frac{\partial^2 \widehat{\Phi_\epsilon}^{(k)}}{\partial y_i \partial y_j} = \begin{cases} 0, & \text{if } k \neq N, \\ -\epsilon^2 \widehat{\frac{\partial^2 h_\epsilon}{\partial x_i \partial x_j}}, & \text{if } k = N. \end{cases}$$

Thus, we have

$$3\epsilon^{-5} \int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \frac{\partial^3 \hat{v_\epsilon}}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 (\varphi(\widehat{\Phi_\epsilon(y)}))}{\partial x_N^2} \frac{\partial \widehat{\Phi_\epsilon}^{(N)}}{\partial y_i} \frac{\partial^2 \widehat{\Phi_\epsilon}^{(N)}}{\partial y_j \partial y_h}\,\mathrm{d}\bar{x}\mathrm{d}y \\ = -3\epsilon^{-2} \int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \frac{\partial^3 \hat{v_\epsilon}}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 (\varphi(\widehat{\Phi_\epsilon}(y)))}{\partial x_N^2} \left(\delta_{Ni} - \widehat{\frac{\partial h_\epsilon}{\partial x_i}} \right) \widehat{\frac{\partial^2 h_\epsilon}{\partial x_j \partial x_h}}\,\mathrm{d}\bar{x}\mathrm{d}y. \tag{6.7}$$

It is not difficult to see that the right-hand side of (6.7) vanishes as $\epsilon \to 0$, due to (3.4) and Lemma 5. It remains to treat only the third integral in the right hand side of (6.5). We apply the exact integration formula (3.4) in order to obtain

$$\epsilon \int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \widehat{\frac{\partial^3 v_\epsilon}{\partial x_i \partial x_j \partial x_h}} \frac{\partial \varphi}{\partial x_N}(\widehat{\Phi_\epsilon}(y)) \widehat{\frac{\partial^3 \Phi_\epsilon^{(N)}}{\partial x_i \partial x_j \partial x_h}}\,\mathrm{d}\bar{x}\mathrm{d}y \\ = -\int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \left[\epsilon^{-5/2} \frac{\partial^3 \hat{v_\epsilon}}{\partial y_i \partial y_j \partial y_h} \right] \left[\frac{\partial \varphi}{\partial x_N}(\widehat{\Phi_\epsilon}(y)) \right] \left[\epsilon^{1/2} \widehat{\frac{\partial^3 h_\epsilon}{\partial x_i \partial x_j \partial x_h}} \right] \mathrm{d}\bar{x}\mathrm{d}y.$$

By Lemma 3 it is clear that $\epsilon^{-5/2} \frac{\partial^3 \hat{v_\epsilon}}{\partial y_i \partial y_j \partial y_h} \to \frac{\partial^3 \hat{v}}{\partial y_i \partial y_j \partial y_h}$, weakly in $L^2(W \times Y \times (-\infty, 0))$ as $\epsilon \to 0$. Moreover, by Lemma 5 $\epsilon^{1/2} \widehat{\frac{\partial^3 \Phi_\epsilon^{(N)}}{\partial x_i \partial x_j \partial x_h}} \to -\frac{\partial^3 (b(\bar{y})(1+y_N)^4)}{\partial y_i \partial y_j \partial y_h}$, uniformly in $W \times Y \times (-1, 0)$ as $\epsilon \to 0$. Hence,

$$\int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \left[\epsilon^{-5/2} \frac{\partial^3 \hat{v_\epsilon}}{\partial y_i \partial y_j \partial y_h} \right] \left[\frac{\partial \varphi}{\partial x_N}(\widehat{\Phi_\epsilon}(y)) \right] \left[\epsilon^{1/2} \widehat{\frac{\partial^3 \Phi_\epsilon^{(N)}}{\partial x_i \partial x_j \partial x_h}} \right] \mathrm{d}\bar{x}\mathrm{d}y \\ \to -\int_{W\times Y\times(-1,0)} \frac{\partial^3 \hat{v}}{\partial y_i \partial y_j \partial y_h} \frac{\partial \varphi}{\partial x_N}(\bar{x}, 0) \frac{\partial^3 (b(\bar{y})(1+y_N)^4)}{\partial y_i \partial y_j \partial y_h}\,\mathrm{d}\bar{x}\mathrm{d}y.$$

as $\epsilon \to 0$. □

The previous discussion yields the following

Theorem 4. *Let $f_\epsilon \in L^2(\Omega_\epsilon)$ and $f \in L^2(\Omega)$ be such that $f_\epsilon|_\Omega \rightharpoonup f$ in $L^2(\Omega)$. Let $v_\epsilon \in H^3(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon)$ be the solutions to $A_{\Omega_\epsilon} v_\epsilon = f_\epsilon$. Then, possibly passing to a subsequence, there exists $v \in H^3(\Omega) \cap H_0^1(\Omega)$ and $\hat{v} \in L^2(W, w^{3,2}_{Per_Y}(Y \times (\infty, 0)))$*

such that $v_\epsilon|_\Omega \rightharpoonup v$ *in* $H^3(\Omega)$, $v_\epsilon|_\Omega \to v$ *in* $L^2(\Omega)$ *and such that statements* (a) *and* (b) *in Lemma 3 hold. Moreover,*

$$-\int_W \int_{Y\times(-1,0)} (D_y^3(\hat v) : D^3(b(\bar y)(1+y_N)^4)\,\mathrm{d}y\,\frac{\partial\varphi}{\partial x_N}(\bar x,0)\mathrm{d}\bar x + \int_\Omega D^3 v : D^3\varphi + u\varphi\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x,$$

for all $\varphi \in H^3(\Omega)\cap H_0^1(\Omega)$.

6.2. **Microscopic limit.** Let $\psi \in C^\infty(\overline{W}\times\overline{Y}\times]-\infty,0])$ be such that $\operatorname{supp}\psi \subset C\times\overline{Y}\times[d,0]$ for some compact set $C\subset W$, $d\in]-\infty,0[$, and assume that $\psi(\bar x,\bar y,0)=0$ for all $(\bar x,\bar y)\in W\times Y$. Let ψ be Y-periodic in the variable $\bar y$. We set

$$\psi_\epsilon(x) = \epsilon^{\frac{5}{2}}\psi\Big(\bar x, \frac{\bar x}{\epsilon}, \frac{x_N}{\epsilon}\Big),$$

for all $\epsilon>0$, $x\in W\times]-\infty,0]$. Then $T_\epsilon\psi_\epsilon \in V(\Omega_\epsilon)$ for sufficiently small ϵ, hence we can plug it in the weak formulation of the problem in Ω_ϵ in order to get

$$\int_{\Omega_\epsilon} D^3 v_\epsilon : D^3(T_\epsilon\psi_\epsilon)\,\mathrm{d}x + \int_{\Omega_\epsilon} v_\epsilon T_\epsilon\psi_\epsilon\,\mathrm{d}x = \int_{\Omega_\epsilon} f_\epsilon T_\epsilon\psi_\epsilon\,\mathrm{d}x. \tag{6.8}$$

It is not difficult to prove that

$$\int_{\Omega_\epsilon} v_\epsilon T_\epsilon\psi_\epsilon\,dx \to 0, \qquad \int_{\Omega_\epsilon} f_\epsilon T_\epsilon\psi\epsilon\,dx \to 0, \tag{6.9}$$

as $\epsilon\to 0$, and by arguing as in [2, Eq. (8.20), p. 29] we deduce that

$$\int_{\Omega_\epsilon\setminus\Omega} D^3 v_\epsilon : D^3(T_\epsilon\psi_\epsilon)\,dx \to 0, \tag{6.10}$$

as $\epsilon\to 0$. Moreover, by arguing as in [2, Lemma 8.47] it is possible to prove that

$$\int_\Omega D^3 v_\epsilon : D^3(T_\epsilon\psi_\epsilon)\,\mathrm{d}x \to \int_{W\times Y\times(-\infty,0)} D_y^3\hat v(\bar x,y) : D_y^3\psi(\bar x,y)\,\mathrm{d}\bar x\mathrm{d}y, \tag{6.11}$$

as $\epsilon\to 0$. Then we have the following

Theorem 5. *Let* $\hat v \in L^2(W, w^{3,2}_{Per_Y}(Y\times(-\infty,0)))$ *be the function from Theorem 4. Then*

$$\int_{W\times Y\times(-\infty,0)} D_y^3\hat v(\bar x,y) : D_y^3\psi(\bar x,y)\mathrm{d}\bar x\mathrm{d}y = 0,$$

for all $\psi \in L^2(W, w^{3,2}_{Per_Y}(Y\times(-\infty,0)))$ *such that* $\psi(\bar x,\bar y,0)=0$ *on* $W\times Y$. *Moreover, for any* $i,j=1,\dots,N-1$, *we have*

$$\frac{\partial^2\hat v}{\partial y_i\partial y_j}(\bar x,\bar y,0) = -\frac{\partial^2 b}{\partial y_i\partial y_j}(\bar y)\frac{\partial v}{\partial x_N}(\bar x,0) \qquad \text{on } W\times Y, \tag{6.12}$$

Proof. We need only to prove (6.12) since the first part of the statement follows from (6.8), (6.9), (6.10), (6.11) (see also the proof of [2, Theorem 8.53]). By applying Lemma 4, case $\alpha=5/2$ to $v_\epsilon\in H^3(\Omega_\epsilon)\cap H_0^1(\Omega_\epsilon)$ we deduce the validity of (6.12). □

Lemma 7. *There exists* $V\in w^{3,2}_{Per_Y}(Y\times(-\infty,0))$ *satisfying the equation*

$$\int_{Y\times(-\infty,0)} D^3 V : D^3\psi\,\mathrm{d}y = 0, \tag{6.13}$$

for all $\psi\in w^{3,2}_{Per_Y}(Y\times(-\infty,0))$ *such that* $\psi(\bar y,0)=0$ *on* Y, *and the boundary condition*

$$V(\bar y,0) = b(\bar y), \qquad \text{on } Y.$$

Function V is unique up to a sum of a monomial in y_N of the form ay_N^2. Moreover $V \in W^{6,2}_{Per_Y}(Y \times (d,0))$ for any $d<0$ and it satisfies the equation

$$\Delta^3 V = 0, \qquad \text{in } Y \times (d,0),$$

subject to the boundary conditions

$$\begin{cases} \frac{\partial^2(\Delta V)}{\partial y_N^2} + 2\frac{\partial^2}{\partial y_N^2}(\Delta_{N-1}V) = 0, & \text{on } Y, \\ \frac{\partial^3 V}{\partial y_N^3}(\bar{y},0) = 0, & \text{on } Y. \end{cases}$$

Proof. Existence, uniqueness and regularity of V follows as in [2, Lemma 8.60]. Note that in order to find the boundary conditions satisfied by V on Y we need to use the Triharmonic Green Formula (7.4) with V in place of f and ψ in place of φ. We choose test functions ψ as in the statement with bounded support in the y_N-direction. We then deduce that

$$\begin{aligned} \int_{Y\times(-\infty,0)} D^3V : D^3\psi \,\mathrm{d}y = &- \int_{Y\times(-\infty,0)} \Delta^3 V \psi \,\mathrm{d}y + \int_Y \frac{\partial^3 V}{\partial y_N^3}\frac{\partial^2\psi}{\partial y_N^2}\,\mathrm{d}\bar{y} \\ &- \int_Y \left(\frac{\partial^2(\Delta V)}{\partial y_N^2} + 2\Delta_{N-1}\left(\frac{\partial^2 V}{\partial y_N^2}\right)\right)\frac{\partial\psi}{\partial y_N}\,\mathrm{d}\bar{y}, \end{aligned}$$

hence V is triharmonic and satisfies the boundary conditions in the statement. □

Theorem 6 (Characterisation of the strange term)**.** *Let V be the function defined in Lemma 7. Let $v, \hat{v}$ be as in Theorem 4. Then*

$$\hat{v}(\bar{x},y) = -V(y)\frac{\partial v}{\partial x_N}(\bar{x},0) + a(\bar{x})y_N^2.$$

for some function $a \in L^2(W)$. Moreover we have the following equalities:

$$\begin{aligned} \int_{Y\times(-\infty,0)} |D^3V|^2 dy &= \int_{Y\times(-\infty,0)} D^3V : D^3(b(\bar{y})(1+y_N^4))dy \\ &= \int_Y \left(\frac{\partial(\Delta^2 V)}{\partial y_N} + \Delta_{N-1}\left(\frac{\partial\Delta V}{\partial y_N}\right) + \Delta_{N-1}^2\left(\frac{\partial V}{\partial y_N}\right)\right) b(\bar{y})d\bar{y}. \quad (6.14) \end{aligned}$$

Proof. Let ϕ be the real-valued function defined on $Y\times]-\infty,0]$ by

$$\phi(y) = \begin{cases} b(\bar{y})(1+y_N)^4, & \text{if } -1 \le y_N \le 0, \\ 0, & \text{if } y_N < -1. \end{cases}$$

Then $\phi \in H^3(Y\times(-\infty,0))$, and $\phi(\bar{y},0) = 0$ for all $\bar{y} \in Y$. Now note that the function $\psi = V - \phi$ is a suitable test-function in equation (6.13); by plugging it in we get

$$\int_{Y\times(-\infty,0)} |D^3V|^2 \,dy = \int_{Y\times(-\infty,0)} D^3V : D^3(b(\bar{y})(1+y_N)^4)\,dy$$

By applying (7.4) on the right-hand side of the former equation, and by keeping in account that V is as in Lemma 7, so $\Delta^3 V = 0$ in $Y\times(d,0)$ for all $d<0$, we deduce that

$$\begin{aligned} &\int_{Y\times(-\infty,0)} D^3V : D^3(b(\bar{y})(1+y_N)^4)\,dy = \\ &\qquad \int_Y \left(\frac{\partial(\Delta^2 V)}{\partial y_N} + \Delta_{N-1}\left(\frac{\partial(\Delta V)}{\partial y_N}\right) + \Delta_{N-1}^2\left(\frac{\partial V}{\partial y_N}\right)\right) b(\bar{y})\,d\bar{y}. \end{aligned}$$

□

By Lemma 7 and Theorem 6 it is now easy to deduce (iii) of Theorem 1.

Proof of Theorem 1(iii). Note that the function v of Theorem 4 satisfies

$$-\int_W \int_{Y\times(-1,0)} (D_y^3(\hat{v}) : D^3(b(\bar{y})(1+y_N)^4)\,\mathrm{d}y\,\frac{\partial\varphi}{\partial x_N}(\bar{x},0)\mathrm{d}\bar{x} + \int_\Omega D^3 v : D^3\varphi + u\varphi\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x, \quad (6.15)$$

for all $\varphi \in H^3(\Omega)\cap H_0^1(\Omega)$. By Theorem 6 the first integral in the left-hand side of (6.15) can be equivalently rewritten as

$$\int_W \left(\int_{Y\times(-\infty,0)} |D^3V|^2\,\mathrm{d}y\right)\frac{\partial v}{\partial x_N}(\bar{x},0)\frac{\partial\varphi}{\partial x_N}(\bar{x},0)\mathrm{d}\bar{x},$$

where V is defined in Lemma 7. By (7.4)

$$\begin{aligned}&\int_\Omega D^3 v : D^3\varphi\,dx = -\int_\Omega \Delta^3 v\,\varphi\,dx + \int_{\partial\Omega}\frac{\partial^3 f}{\partial n^3}\frac{\partial^2\varphi}{\partial n^2}\,dS\\ &+\int_{\partial\Omega}\left((n^T D^3 v)_{\partial\Omega} : D_{\partial\Omega} n\right) - \frac{\partial^2(\Delta v)}{\partial n^2} - 2\,\mathrm{div}_{\partial\Omega}(D^3 v[n\otimes n])_{\partial\Omega}\Big)\frac{\partial\varphi}{\partial n}\,dS,\end{aligned} \quad (6.16)$$

for all $\varphi \in H^3(\Omega)\cap H_0^1(\Omega)$. In particular, we deduce that on $W\times\{0\}$ we have the following boundary integral

$$\int_W \left(-\frac{\partial^2(\Delta v)}{\partial x_N^2}(\bar{x},0) - 2\Delta_{N-1}\left(\frac{\partial^2 v}{\partial x_N^2}\right)(\bar{x},0) + K_1\frac{\partial v}{\partial x_N}(\bar{x},0)\right)\frac{\partial\varphi}{\partial x_N}(\bar{x},0)\mathrm{d}\bar{x}, \quad (6.17)$$

where $K_1 = \int_{Y\times(-\infty,0)} |D^3V|^2$. Then, by (6.15), (6.16), (6.17) and the arbitrariness of φ we deduce the statement of Theorem 1, part (iii). □

7. Appendix (A)

We give here a proof of the Triharmonic Green Formula. We refer to Section 3 for the tangential calculus notation and related results. We first note that by using tangential calculus it is possible to prove that

$$D^2 f(x) = \left(D^2_{\partial\Omega} f(x) + \frac{\partial}{\partial n}\left(\nabla_{\partial\Omega} f(x)\right)\otimes n(x) + n(x)\otimes\nabla_{\partial\Omega}\left(\frac{\partial f(x)}{\partial n}\right) + \frac{\partial^2 f(x)}{\partial n^2} n(x)\otimes n(x)\right) + \frac{\partial f(x)}{\partial n} D_{\partial\Omega} n(x), \quad (7.1)$$

for all $x\in\partial\Omega$. Then formula (7.1) can be equivalently rewritten as

$$D^2 f(x) = \left(D^2_{\partial\Omega} f(x) + \nabla_{\partial\Omega}\left(\frac{\partial f(x)}{\partial n}\right)\otimes n(x) + n(x)\otimes\nabla_{\partial\Omega}\left(\frac{\partial f(x)}{\partial n}\right) + \frac{\partial^2 f(x)}{\partial n^2} n(x)\otimes n(x)\right) - (D_{\partial\Omega} n(x))(\nabla_{\partial\Omega} f(x))\otimes n(x) + \frac{\partial f(x)}{\partial n} D_{\partial\Omega} n(x), \quad (7.2)$$

for all $x\in\partial\Omega$. Finally, note that if we take the trace on both hand sides of (7.2) we recover the classical decomposition formula for the Laplacian at the boundary

$$\Delta f(x) = \Delta_{\partial\Omega} f(x) + \frac{\partial^2 f(x)}{\partial n^2} + \mathcal{H}(x)\frac{\partial f(x)}{\partial n},$$

for all $x\in\partial\Omega$, where $\mathcal{H}$ is the curvature of $\partial\Omega$.

Theorem 7 (Triharmonic Green Formula - general domain)**.** *Let Ω be a bounded domain of $\mathbb{R}^N$ of class $C^{0,1}$. Let $f \in C^6(\overline{\Omega})$, $\varphi \in C^3(\overline{\Omega})$. Then*

$$\int_\Omega D^3 f : D^3\varphi \, dx = -\int_\Omega \Delta^3 f \, \varphi \, dx + \int_{\partial\Omega} (n^T D^3 f) : D^2\varphi \, dS$$
$$- \int_{\partial\Omega} (n^T D^2(\Delta f))_{\partial\Omega} \cdot \nabla_{\partial\Omega}\varphi \, dS - \int_{\partial\Omega} \frac{\partial^2(\Delta f)}{\partial n^2} \frac{\partial\varphi}{\partial n} \, dS + \int_{\partial\Omega} \frac{\partial(\Delta^2 f)}{\partial n}\varphi \, dS. \quad (7.3)$$

If moreover Ω is of class C^3 then

$$\begin{aligned}\int_\Omega D^3 f : D^3\varphi \, dx &= -\int_\Omega \Delta^3 f \, \varphi \, dx + \int_{\partial\Omega} \frac{\partial^3 f}{\partial n^3} \frac{\partial^2\varphi}{\partial n^2} \, dS \\ &+ \int_{\partial\Omega} \left(\big((n^T D^3 f)_{\partial\Omega} : D_{\partial\Omega} n\big) - \frac{\partial^2(\Delta f)}{\partial n^2} - 2\,\mathrm{div}_{\partial\Omega}(D^3 f[n\otimes n])_{\partial\Omega} \right) \frac{\partial\varphi}{\partial n} \, dS \\ &+ \int_{\partial\Omega} \Bigg(\mathrm{div}^2_{\partial\Omega}\big((n^T D^3 f)_{\partial\Omega}\big) + \mathrm{div}_{\partial\Omega}\left(D_{\partial\Omega} n (D^3 f[n\otimes n])_{\partial\Omega}\right) \\ &\qquad + \frac{\partial(\Delta^2 f)}{\partial n} + \mathrm{div}_{\partial\Omega}\left(n^T D^2(\Delta f)\right)_{\partial\Omega} \Bigg)\varphi \, dS.\end{aligned} \quad (7.4)$$

Proof. Repeated integrations by parts establish that

$$\begin{aligned}\int_\Omega D^3 f : D^3\varphi \, dx &= \int_\Omega \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} \frac{\partial^3\varphi}{\partial x_i \partial x_j \partial x_k} \, dx \\ &= -\int_\Omega \Delta^3 f \varphi \, dx + \int_{\partial\Omega} (n^T D^3 f) : D^2\varphi \, dS - \int_{\partial\Omega} (n^T D^2(\Delta f)) \cdot \nabla\varphi \, dS + \int_{\partial\Omega} \frac{\partial(\Delta^2 f)}{\partial n}\varphi \, dS,\end{aligned} \quad (7.5)$$

where summation symbols on i, j, k from 1 to N have been dropped. Then (7.3) follows from (7.5) by decomposing the gradient appearing in the third integral on the right-hand side of (7.5) in tangential and normal components, see Definition 2. In order to prove (7.4) we need first to decompose the hessian matrix appearing in the first boundary integral on the right-hand side of (7.3). By using formula (7.2) on $D^2\varphi$ we deduce that

$$\begin{aligned}\int_{\partial\Omega} (n^T D^3 f) : D^2\varphi \, dS &= \int_{\partial\Omega} (n^T D^3 f)_{\partial\Omega} : D^2_{\partial\Omega}\varphi \, dS \\ &+ 2\int_{\partial\Omega} (D^3 f[n\otimes n])_{\partial\Omega} \cdot \nabla_{\partial\Omega}\left(\frac{\partial\varphi}{\partial n}\right) dS \\ &- \int_{\partial\Omega} \left(D_{\partial\Omega} n (D^3 f[n\otimes n])_{\partial\Omega}\right) \cdot \nabla_{\partial\Omega}\varphi \, dS \\ &+ \int_{\partial\Omega} \big((n^T D^3 f)_{\partial\Omega} : D_{\partial\Omega} n\big) \frac{\partial\varphi}{\partial n} \, dS + \int_{\partial\Omega} \frac{\partial^3 f}{\partial n^3} \frac{\partial^2\varphi}{\partial n^2} \, dS.\end{aligned} \quad (7.6)$$

In (7.6) the symbol $D^3 f[n\otimes n]$ stands for the vector having as i-th component $\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} n_j n_k$, where sums over j and k are understood. Note also that the third integral on the right-hand side of (7.6) is deduced from

$$-\int_{\partial\Omega} (n^T D^3 f) : \big(D_{\partial\Omega} n (\nabla_{\partial\Omega}\varphi) \otimes n\big) \, dS,$$

by using the following equalities

$$\begin{aligned}(n^TD^3f) : \big(D_{\partial\Omega}n(\nabla_{\partial\Omega}\varphi)\otimes n\big) &= \big(D_{\partial\Omega}n(\nabla_{\partial\Omega}\varphi)\big)^T(n^TD^3f)n\\ &= (\nabla_{\partial\Omega}\varphi)^T\big((D_{\partial\Omega}n)^T(D^3f[n\otimes n])_{\partial\Omega}\big)\\ &= \big((D_{\partial\Omega}n)(D^3f[n\otimes n])_{\partial\Omega}\big)\cdot\nabla_{\partial\Omega}\varphi.\end{aligned}$$

In the third equality we have used the fact that $D_{\partial\Omega}n$ is a symmetric matrix. Now, since Ω is of class C^2, we plan to apply the Tangential Divergence theorem (see Theorem 2) to the first, the second, and the third integral in the right-hand side of (7.6). We consider separately the first integral. Let us note that for every matrix $A=(a_{ij}(x))_{ij}$ with coefficients $a_{ij}\in C^2(\overline{\Omega})$ and for every function $\psi\in C^2(\overline{\Omega})$, we have

$$\int_{\partial\Omega}\mathrm{div}_{\partial\Omega}\big((A)_{\partial\Omega}(\nabla_{\partial\Omega}\psi)\big)\,dS=0$$

by (3.8). Here $((A)_{\partial\Omega})_{ij}=(a_{ij}\circ p)|_{\partial\Omega}$, where p is defined in Section 3. Hence,

$$\int_{\partial\Omega}(\mathrm{div}_{\partial\Omega}(A)_{\partial\Omega})\cdot\nabla_{\partial\Omega}\psi+(A)_{\partial\Omega}:D^2_{\partial\Omega}\psi\,dS=0. \tag{7.7}$$

Finally, a further application of the Tangential Green formula (see (3.9)) on the first summand on the right-hand side of (7.7) yields

$$\int_{\partial\Omega}(\mathrm{div}^2_{\partial\Omega}(A)_{\partial\Omega})\psi\,dS=\int_{\partial\Omega}(A)_{\partial\Omega}:D^2_{\partial\Omega}\psi\,dS \tag{7.8}$$

for all matrix $A\in C^2(\overline{\Omega})^{N\times N}$, for every function $\psi\in C^2(\overline{\Omega})$. Then, by applying Formula (7.8) to the first integral in the right-hand side of (7.6) with $A=(n^TD^3f)$ and $\psi=f$, and by using (3.9) on the second and third integral in the right-hand side of (7.6) we deduce that

$$\begin{aligned}&\int_{\partial\Omega}(n^TD^3f):D^2\varphi\,dS=\int_{\partial\Omega}\mathrm{div}^2_{\partial\Omega}\big((n^TD^3f)_{\partial\Omega}\big)\varphi\,dS\\ &-2\int_{\partial\Omega}\mathrm{div}_{\partial\Omega}\big((D^3f[n\otimes n])_{\partial\Omega}\big)\frac{\partial\varphi}{\partial n}\,dS+\int_{\partial\Omega}\mathrm{div}_{\partial\Omega}\Big(D_{\partial\Omega}n(D^3f[n\otimes n])_{\partial\Omega}\Big)\varphi\,dS\\ &+\int_{\partial\Omega}\big((n^TD^3f)_{\partial\Omega}:D_{\partial\Omega}n\big)\frac{\partial\varphi}{\partial n}\,dS+\int_{\partial\Omega}\frac{\partial^3f}{\partial n^3}\frac{\partial^2\varphi}{\partial n^2}\,dS,\end{aligned} \tag{7.9}$$

where we have denoted with $(V)_{\partial\Omega}$ the projection of V on the tangent plane to $\partial\Omega$, as defined in §3. By applying the Tangential Divergence Theorem to the second boundary integral on the right-hand side of (7.3) we finally deduce that

$$-\int_{\partial\Omega}(n^TD^2(\Delta f))_{\partial\Omega}\cdot\nabla_{\partial\Omega}\varphi\,dS=\int_{\partial\Omega}\mathrm{div}_{\partial\Omega}\big(n^TD^2(\Delta f)\big)_{\partial\Omega}\,\varphi\,dS. \tag{7.10}$$

By (7.9) and (7.10) we get (7.4), concluding the proof. □

8. Appendix (B)

Proposition 1. *Let $u\in H^3(\Omega)\cap H^1_0(\Omega)$ be the function defined in the statement of Lemma (4). If $1<\alpha<2$ then $\frac{\partial u}{\partial x_N}(\bar{x},0)=0$ for almost all $\bar{x}\in W$.*

Proof. In this proof we use the definition of $\hat{Y}$ and $\hat{u}$ introduced in (5.20) and (5.21). Note that

$$\epsilon^\alpha\int_W\int_{\hat{Y}}\frac{1}{\epsilon^2}|\nabla_{\bar{y}}\hat{u}_\epsilon|^2+\frac{1}{\epsilon^{2\alpha}}\left|\frac{\partial\hat{u}_\epsilon}{\partial y_N}\right|^2d\bar{x}dy=\int_{\Omega_\epsilon}|\nabla u_\epsilon|^2\,dx, \tag{8.1}$$

where we have used formula (3.4). Since $\alpha<2$ we deduce that $\nabla_y\hat{u}_\epsilon\to0$ in $L^2(W\times\hat{Y})^N$. In a similar way one proves that $D^\beta_y\hat{u}_\epsilon\to0$ for all the multiindexes β

such that $1 \leq |\beta| \leq 3$. Now note that $\int_W \int_Y |\hat{u}_\epsilon(\bar{x}, \bar{y}, 0)|^2 d\bar{y} d\bar{x} = \int_W |u_\epsilon(\bar{x}, 0)|^2 d\bar{x} \leq C$, uniformly in $\epsilon > 0$. Thus,

$$\begin{aligned}
&\int_W \int_{\hat{Y}} |\hat{u}_\epsilon(\bar{x}, y)|^2 dy d\bar{x} \\
&\leq 2 \int_W \int_{\hat{Y}} |\hat{u}_\epsilon(\bar{x}, y) - \hat{u}_\epsilon(\bar{x}, \bar{y}, 0)|^2 dy d\bar{x} + 2(b(\bar{y}) + 1) \int_W \int_Y |\hat{u}_\epsilon(\bar{x}, \bar{y}, 0)|^2 d\bar{y} d\bar{x} \\
&\leq 2 \int_W \int_{\hat{Y}} \int_0^{y_N} \left| \frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, t) \right|^2 dt |y_N| dy d\bar{x} + C \leq C',
\end{aligned}$$

hence $\hat{u}_\epsilon$ is uniformly bounded in $L^2(W, H^3(\hat{Y}))$ and up to a subsequence $\hat{u}_\epsilon \rightharpoonup \hat{u}$ in $L^2(W, H^3(\hat{Y}))$, for some function $\hat{u} \in L^2(W, H^3(\hat{Y}))$. Actually $\hat{u}$ does not depend on y; indeed $\nabla_y \hat{u}_\epsilon \to 0$ in $L^2(W \times \hat{Y})^N$ implies that $\nabla_y \hat{u} = 0$. Since $u_\epsilon \to u$ weakly in $H^3(\Omega)$, by the Trace Theorem, $u_\epsilon(\bar{x}, 0) \to u(\bar{x}, 0)$ strongly in $L^2(W)$. By Lemma 8 below, we deduce that

$$\overline{u_\epsilon}(\bar{x}) = \frac{1}{\epsilon^{N-1}} \int_{C_\epsilon(\bar{x})} u_\epsilon(\bar{t}, 0) d\bar{t} \to u(\bar{x}, 0), \tag{8.2}$$

strongly in $L^2(W)$ as $\epsilon \to 0$. By a change of variable it is easy to see that $\overline{u_\epsilon}(\bar{x}) = \int_Y \hat{u}_\epsilon(\bar{x}, \bar{z}, 0)\, d\bar{z}$ for almost all $\bar{x} \in W$. By Poincaré inequality it is also easy to prove that

$$\left\| \hat{u}_\epsilon - \int_Y \hat{u}_\epsilon(\cdot, \bar{z}, 0) d\bar{z} \right\|_{L^2(W \times \hat{Y})} \leq C \|\nabla_{\bar{y}} \hat{u}_\epsilon\|_{L^2(W \times \hat{Y})} \to 0, \tag{8.3}$$

as $\epsilon \to 0$, according to (8.1). Then, by (8.2) and (8.3) we have

$$\begin{aligned}
&\|\hat{u}_\epsilon - u(\bar{x}, 0)\|_{L^2(W \times \hat{Y})} \\
&\leq \left\| \hat{u}_\epsilon - \int_Y \hat{u}_\epsilon(\cdot, \bar{z}, 0) d\bar{z} \right\|_{L^2(W \times \hat{Y})} + \left\| \int_Y \hat{u}_\epsilon(\cdot, \bar{z}, 0) d\bar{z} - u(\bar{x}, 0) \right\|_{L^2(W \times \hat{Y})} \to 0,
\end{aligned} \tag{8.4}$$

as $\epsilon \to 0$, which implies that $\hat{u}(\bar{x}) = u(\bar{x}, 0)$ for almost all $\bar{x} \in W$. Now we unfold the following identity

$$\begin{aligned}
&\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_j}(\bar{x}, g_\epsilon(\bar{x})) + \frac{\partial^2 u_\epsilon}{\partial x_i \partial x_N}(\bar{x}, g_\epsilon(\bar{x})) \frac{\partial g_\epsilon(\bar{x})}{\partial x_j} + \frac{\partial^2 u_\epsilon}{\partial x_j \partial x_N}(\bar{x}, g_\epsilon(\bar{x})) \frac{\partial g_\epsilon(\bar{x})}{\partial x_i} \\
&+ \frac{\partial^2 u_\epsilon}{\partial x_N^2}(\bar{x}, g_\epsilon(\bar{x})) \frac{\partial g_\epsilon(\bar{x})}{\partial x_i} \frac{\partial g_\epsilon(\bar{x})}{\partial x_j} + \frac{\partial u_\epsilon}{\partial x_N}(\bar{x}, g_\epsilon(\bar{x})) \frac{\partial^2 g_\epsilon(\bar{x})}{\partial x_i \partial x_j} = 0,
\end{aligned}$$

in order to obtain

$$\begin{aligned}
&\frac{1}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x}, \bar{y}, b(\bar{y})) + \frac{\epsilon^{\alpha-1}}{\epsilon^{\alpha+1}} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, b(\bar{y})) \frac{\partial b(\bar{y})}{\partial y_j} + \frac{\epsilon^{\alpha-1}}{\epsilon^{\alpha+1}} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_j \partial y_N}(\bar{x}, \bar{y}, b(\bar{y})) \frac{\partial b(\bar{y})}{\partial y_i} \\
&+ \frac{\epsilon^{2\alpha-2}}{\epsilon^{2\alpha}} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x}, \bar{y}, b(\bar{y})) \frac{\partial b(\bar{y})}{\partial y_i} \frac{\partial b(\bar{y})}{\partial y_j} + \frac{\epsilon^{\alpha-2}}{\epsilon^\alpha} \frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, b(\bar{y})) \frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} = 0.
\end{aligned} \tag{8.5}$$

Note that $\frac{1}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x}, \bar{y}, b(\bar{y})) \to \frac{\partial^2 u}{\partial x_i \partial x_j}(\bar{x}, 0) = 0$, and $\frac{1}{\epsilon^{\alpha+1}} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, b(\bar{y})) \to \frac{\partial^2 u}{\partial x_i \partial x_N}(\bar{x}, 0)$, $\frac{1}{\epsilon^{2\alpha}} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x}, \bar{y}, b(\bar{y})) \to \frac{\partial^2 u}{\partial x_N^2}(\bar{x}, 0)$ as $\epsilon \to 0$, where the limits are in $L^2(W \times Y)$. Hence, if $1 < \alpha < 2$ we deduce that all the summands in (8.5) are vanishing in $L^2(W \times Y)$ with the possible exception of $\frac{\epsilon^{\alpha-2}}{\epsilon^\alpha} \frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, b(\bar{y})) \frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j}$. Since equality (8.5) must hold, this implies that also this last summand is bounded; hence,

$$\frac{1}{\epsilon^\alpha} \frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, b(\bar{y})) \frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} \to 0,$$

in $L^2(W\times Y)$ as $\epsilon\to 0$, and consequently $\frac{\partial u}{\partial x_N}(\bar{x},0)\frac{\partial^2 b(\bar{y})}{\partial y_i\partial y_j}=0$ for almost all $(\bar{x},\bar{y})\in(W\times Y)$. Then $\frac{\partial u}{\partial x_N}(\bar{x},0)=0$ for almost all $\bar{x}\in W$, concluding the proof. □

Lemma 8. *Let $(v_\epsilon)_\epsilon$ be a sequence of functions in $L^2(\Theta)$, for a given bounded open set $\Theta\subset\mathbb{R}^N$. Let $v\in L^2(\Theta)$, and assume that $v_\epsilon\to v$ in $L^2(\Theta)$. For all $\epsilon>0$ let $C_\epsilon(x)=\{y\in\mathbb{R}^N:|x-y|<\epsilon\}$ and we define*

$$\overline{v_\epsilon}(x)=\frac{1}{\epsilon^N}\int_{C_\epsilon(x)}v_\epsilon(y)\,dy,$$

for almost all $x\in\Theta$. Then $\overline{v_\epsilon}\to v$ in $L^2(\Theta)$ as $\epsilon\to 0$.

Proof. We claim that

$$\overline{v}(x):=\frac{1}{\epsilon^N}\int_{C_\epsilon(x)}v(y)dy\to v(x), \tag{8.6}$$

strongly in $L^2(\Theta)$ as $\epsilon\to 0$. Let $\delta>0$ be fixed and let $w\in C^1(\Theta)\cap L^2(\Theta)$ such that $\|v-w\|_{L^2(\Theta)}\le\delta$. Then

$$\begin{aligned}\overline{v}(x)-v(x)&=\frac{1}{\epsilon^N}\int_{C_\epsilon(x)}(v(y)-v(x))\,dy\\&=\frac{1}{\epsilon^N}\int_{C_\epsilon(x)}(v(y)-w(y))\,dy+(w(x)-v(x))+\frac{1}{\epsilon^N}\int_{C_\epsilon(x)}(w(y)-w(x))\,dy.\end{aligned}$$

Let us define $\Theta^\epsilon=\{x\in\Theta:\operatorname{dist}(x,\partial\Theta)>\epsilon\}$. Note that

$$\begin{aligned}\int_{\Theta^\epsilon}\left|\frac{1}{\epsilon^N}\int_{C_\epsilon(x)}(v(y)-w(y))dy\right|^2dx&\le\int_{\Theta^\epsilon}\frac{1}{\epsilon^N}\int_{C_\epsilon(x)}|v(y)-w(y)|^2\,dydx\\&=\int_{\Theta^\epsilon}|v(y)-w(y)|^2\left(\frac{1}{\epsilon^N}\int_{C_\epsilon(y)}dx\right)dy\le C\delta^2\end{aligned}$$

where we have used Jensen's inequality and Tonelli Theorem. Moreover, it is clear that

$$\left\|\frac{1}{\epsilon^N}\int_{C_\epsilon(x)}(w(y)-w(x))dy\right\|_{L^2(\Theta)}\le C\epsilon.$$

Hence, $\|\overline{v}-v\|_{L^2(\Theta^\epsilon)}\le C(\delta+\epsilon)\le C'\delta$, concluding the proof of claim (8.6). Now note that

$$\|\overline{v_\epsilon}-\overline{v}\|_{L^2(\Theta^\epsilon)}\le\left(\int_{\Theta^\epsilon}\left(\frac{1}{\epsilon^N}\int_{C_\epsilon(x)}|v_\epsilon(y)-v(y)|^2dy\right)dx\right)^{1/2}.$$

By Tonelli Theorem we can exchange the order of the integrals in order to obtain

$$\int_{\Theta^\epsilon}\left(\frac{1}{\epsilon^N}\int_{C_\epsilon(x)}|v_\epsilon(y)-v(y)|^2dy\right)dx\le\|v_\epsilon-v\|^2_{L^2(\Theta)}\frac{1}{\epsilon^N}\int_{C_\epsilon(y)}dx=\|v_\epsilon-v\|^2_{L^2(\Theta)}.$$

Hence, $\|\overline{v_\epsilon}-\overline{v}\|_{L^2(\Theta^\epsilon)}\le\|v_\epsilon-v\|_{L^2(\Theta)}$; consequently,

$$\|\overline{v_\epsilon}-v\|_{L^2(\Theta^\epsilon)}\le\|\overline{v_\epsilon}-\overline{v}\|_{L^2(\Theta^\epsilon)}+\|\overline{v}-v\|_{L^2(\Theta^\epsilon)}\le\|v_\epsilon-v\|_{L^2(\Theta)}+\|\overline{v}-v\|_{L^2(\Theta^\epsilon)},$$

and the right-hand side tends to zero as $\epsilon\to 0$. □

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