# Stochastic finite strain analysis of inhomogeneous hyperelastic solids 

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#### Abstract

This thesis presents a theoretical study of static and dynamic inflation, and finite amplitude oscillatory motion of inhomogeneous spherical shells and cylindrical tubes of stochastic hyperelastic material. These bodies are deformed by radially symmetric uniform inflation, and, in the dynamic case, are subject to either a surface dead load or an impulse traction, applied uniformly in the radial direction. We consider composite shells and tubes with two concentric stochastic homogeneous neo-Hookean phases, and inhomogeneous bodies of stochastic neo-Hookean material with constitutive parameters varying continuously in the radial direction. For the homogeneous materials, we define the elastic parameters as spatially-independent random variables, while for the radially inhomogeneous bodies, we take the parameters as spatially-dependent random fields, described by Gamma probability density functions. Under radially symmetric dynamic deformation treated as quasi-equilibrated motion, we show that these bodies oscillate, i.e., their radius increases up to a point, then decreases, then increases again, and so on, and the amplitude and period of the oscillations are characterised by probability distributions, depending on the initial conditions, geometry, and the probabilistic material properties.


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## Chapter 1

## Introduction

Extensive studies of oscillatory motions of cylindrical or spherical shells of linear elastic material have been driven by a wide range of industrial applications [ $5,6,20,31]$. In contrast, time-dependent finite oscillations of cylindrical tubes and spherical shells of nonlinear elastic material have received less attention. Internally pressurised hollow cylinders and spheres are relevant in many biological and engineering structures $[3,11,30,52-54,72]$.

The first experimental observations of inflation instabilities in cylindrical and spherical balloons of rubber-like material were reported in [75]. Cylindrical tubes of homogeneous isotropic incompressible hyperelastic material subject to finite symmetric inflation and stretching were analysed for the first time in [105]. For elastic spherical shells, finite radially symmetric inflation was investigated first in [47], then in $[2,114]$. For both elastic tubes and spherical shells, in [26], it was shown that, depending on the particular material and initial geometry, the internal pressure may increase monotonically, or increase and then decrease, or increase, decrease, and then increase again. Further studies examining these deformations for different constitutive descriptions can be found in [9, 44, 94, 141]. Localised bulging in long inflated isotropic hyperelastic tubes of arbitrary thickness was modelled and analysed within the framework of nonlinear elasticity in $[34,37,38$, 55, 138].

Large amplitude oscillations of spherical shells and tubes of homogeneous isotropic incompressible nonlinear hyperelastic material were formulated as special cases of quasi-equilibrated motions in [132]. These are the class of motions for which, at every time instant, the deformed configuration is a possible static configuration under the given forces.

Free and forced axially symmetric radial oscillations of infinitely long, isotropic incompressible cylindrical tubes, with arbitrary wall thickness, were described for the first time in $[68,69]$. For spherical shells, oscillatory motions were derived analogously in $[56,70,135]$. For the combined radial-axial large amplitude oscillations of hyperelastic cylindrical tubes, in [109], the surface tractions necessary to maintain the periodic motions were discussed, and the results were applied to a tube sealed at both ends and filled with an incompressible fluid. The dynamic deformation of cylindrical tubes of Mooney-Rivlin material [92,106] in finite amplitude radial oscillation was obtained in $[109,111,112]$. Theoretical and experimental studies of cylindrical and spherical shells of rubberlike material under external pressure were presented in [137]. The finite amplitude radial oscillations of homogeneous isotropic incompressible hyperelastic spherical and cylindrical shells under a constant pressure difference between the inner and the outer surface were studied theoretically in [25].

In [60], the dynamic problem of axially symmetric oscillations of cylindrical tubes of transversely isotropic incompressible material, with radial transverse isotropy, was considered. The dynamic deformation of a longitudinally anisotropic thin-walled cylindrical tube under radial oscillations was obtained in [110]. Radial oscillations of inhomogeneous thick-walled cylindrical and spherical shells of neo-Hookean material, with a material parameter varying continuously along the radial direction, were examined in [35].

In [4], for pressurised homogeneous isotropic compressible hyperelastic tubes of arbitrary wall thickness under uniform radial dead-load traction, the stability of
the finitely deformed state and small radial vibrations about this state were treated using the theory of small deformations superposed on large elastic deformations, while the governing equations were solved numerically. In [134], the dynamic inflation of hyperelastic spherical membranes of Mooney-Rivlin material subjected to a uniform step pressure was considered, and the absence of damping in these models was discussed. It was concluded that, as the amplitude and period of oscillations are strongly influenced by the rate of internal pressure, if the pressure was suddenly imposed and the inflation process was short, then sustained oscillations due to the dominant elastic effects could be observed. For many systems under slowly increasing pressure, strong damping would typically prevent oscillatory motion [29]. The dynamic response of incompressible hyperelastic cylindrical and spherical shells subjected to periodic loading was examined in [103,104].

Radial oscillations of cylindrical tubes and spherical shells of Mooney-Rivlin and Gent hyperelastic materials were analysed in [17,18], for both thick-walled and thin-walled structures. It was found that, generally, both the amplitude and period of oscillations decrease when the stiffness of the material increases. The influence of material constitutive law on the dynamic behaviour of cylindrical and spherical shells was investigated also in $[10,12,108,140]$ where the results for Yeoh [139] and Mooney-Rivlin material models were compared. In [21], the static and dynamic behaviour of circular cylindrical shells of homogeneous isotropic incompressible hyperelastic material modelling arterial walls were considered. In [116], the nonlinear static and dynamic behaviour of a spherical membrane of neo-Hookean or Mooney-Rivlin material, subject to a uniformly distributed radial pressure on the inner surface, was analysed, and the influence of the material constants was discussed.

However, deterministic approaches, which are based on average data values, can greatly underestimate or overestimate mechanical responses, and stochastic representations accounting also for data dispersion are needed to improve assess-
ment and predictions $[40,59,67,98,100,121,129]$.
Recently, radial oscillations of cylindrical and spherical shells of hyperelastic material, treated as quasi-equilibrated motions, were reviewed and extended to stochastic hyperelastic bodies in [83, 91]. Namely, spherical and cylindrical shells of stochastic isotropic incompressible hyperelastic material were analysed in [83] where particular attention was given to the periodic (oscillatory) motion and timedependent stresses taking into account the probabilistic model parameters. In [91], the cavitation and finite amplitude oscillations under radially symmetric finite deformation of homogeneous and radially inhomogeneous spheres of stochastic hyperelastic material was studied analytically.

Stochastic hyperelastic models are described by strain-energy densities with the parameters characterised by probability density functions (see [50,51,124-128] and also $[36,88])$. These are advanced phenomenological models that rely on the finite elasticity theory and on the notion of entropy [62-64,113,122] to enable the propagation of uncertainties from input data to output quantities of interest [120]. They can be also combined with Bayesian approaches $[16,80]$ for model selection [36, 88, 98, 107].

For stochastic homogeneous incompressible hyperelastic bodies, the effect of probabilistic model parameters on predicted mechanical responses was demonstrated theoretically on the various instability problems: the static cavitation of a sphere under uniform tensile dead load [84], the inflation of pressurised spherical and cylindrical shells [82], the classic problem of the Rivlin cube [89], the rotation and perversion of stochastic incompressible anisotropic hyperelastic cylindrical tubes [90]. In [84], in addition to the well-known case of stable cavitation post-bifurcation at the critical dead load, it was shown, for the first time, the existence of unstable (snap) cavitation for some (deterministic or stochastic) isotropic incompressible materials satisfying Baker-Ericksen inequalities [14]. These problems, for which the elastic solutions are obtained explicitly, can offer significant
insight into how the uncertainties in input parameters are propagated to output quantities.

A similar stochastic methodology was developed to study instabilities in liquid crystal elastomers in $[86,87]$. To investigate the effect of probabilistic parameters in the case of more complex geometries and loading conditions, numerical approaches have been proposed in [127,128].

The scope of this thesis is to extend the analysis of [83,91] to large strain deformations and oscillatory motions of inhomogeneous cylindrical tubes and spherical shells of stochastic hyperelastic material. Chapter 2 provides a summary of the stochastic elasticity prerequisites, and introduces the notion of quasi-equilibrated motion. Chapters 3 and 4 present an analysis of radially symmetric deformations of composite cylindrical tubes and spherical shells, respectively, with thin walls consisting of two concentric homogeneous stochastic neo-Hookean phases. Chapter 5 is devoted to the analysis of finite amplitude oscillations of a composite formed from two concentric homogeneous tubes of different stochastic neo-Hookean material, and of inhomogeneous tubes with radially varying material properties. For spherical shells, a similar analysis is carried out in Chapter 6. Results are summarised and concluding remarks are drawn in the last chapter. Some of these results were published in [81].

## Chapter 2

## Prerequisites

In this chapter, we summarise fundamental elements of finite elasticity and probability theory, which are then combined to produce stochastic finite elasticity. This is the main theoretical framework for this thesis. We also include a summary of the cylindrical and spherical polar coordinate systems, which are extensively used in the following chapters. Classical texts on the theory of nonlinear elasticity include $[13,48,99,133]$. A comprehensive review on nonlinear elastic material parameters is provided in [85]. The elasticity theory is applied to biological growth in [43], and to engineering problems in [57]. A recent introductory text focusing on linear elasticity, plasticity, viscoelasticity, and coupled field theories, such as thermoelasticity, chemoelasticity, poroelasticity, and piezoelectricity, with two chapters devoted to finite elasticity is [8]. For probability theory, we refer to [49]. Relevant books on uncertainty quantification in solids are [32, 33, 121].

### 2.1 Finite elasticity

The main objective of finite elasticity is to predict reversible mechanical changes that occur in solid material bodies under internal or external forces. The following are preliminary concepts in nonlinear finite elasticity.

Definition 2.1.1 A finite elastic deformation of the body from the reference con-
figuration $\mathcal{B}_{0}$ to the current configuration $\mathcal{B}$ is defined by a one-to-one, orientationpreserving mapping

$$
\begin{equation*}
\chi: \Omega \rightarrow \mathbb{R}^{3} . \tag{2.1}
\end{equation*}
$$

We denote

$$
\begin{equation*}
\boldsymbol{x}=\chi(\boldsymbol{X}), \tag{2.2}
\end{equation*}
$$

where $\boldsymbol{X}$ is the Lagrangian variable, for the reference configuration, and $\boldsymbol{x}$ is the Eulerian variable, for the deformed configuration. The one-to-one condition on $\Omega$ guarantees that interpenetration of matter is avoided. However, self-contact on the body's surface is permitted, hence, this transformation need not be injective on $\bar{\Omega}$ (see Figure 2.1).


Figure 2.1: Schematic of an elastic body in the reference state (left) and the deformed state with self-contact (right).

Definition 2.1.2 The first derivative of the deformation $\chi$ with respect to the reference configuration, known as the gradient tensor, is defined as follows,

$$
\boldsymbol{F}=\nabla \chi=\operatorname{Grad} \chi=\left[\begin{array}{lll}
\partial \chi_{1} / \partial X_{1} & \partial \chi_{1} / \partial X_{2} & \partial \chi_{1} / \partial X_{3}  \tag{2.3}\\
\partial \chi_{2} / \partial X_{1} & \partial \chi_{2} / \partial X_{2} & \partial \chi_{2} / \partial X_{3} \\
\partial \chi_{3} / \partial X_{1} & \partial \chi_{3} / \partial X_{2} & \partial \chi_{3} / \partial X_{3}
\end{array}\right] .
$$

This has a positive determinant, i.e., $J=\operatorname{det} \boldsymbol{F}>0$ on $\Omega$. The deformation gradient $\boldsymbol{F}$ measures local changes in distance, while the Jacobian J measures volumetric changes. Changes in areas are measured by the cofactor $\operatorname{Cof} \boldsymbol{F}=J \boldsymbol{F}^{-T}$, where " $T$ " denotes the transpose and " $-T$ " the inverse of the transpose, or equivalently, the transpose of the inverse.

Definition 2.1.3 The displacement field is defined by

$$
\begin{equation*}
u(X)=x-X \tag{2.4}
\end{equation*}
$$

and its gradient is equal to

$$
\begin{equation*}
\nabla \boldsymbol{u}=\operatorname{Grad} \boldsymbol{u}=\boldsymbol{F}-\boldsymbol{I}, \tag{2.5}
\end{equation*}
$$

where $\boldsymbol{I}=\operatorname{diag}(1,1,1)$ is the identity tensor.

Definition 2.1.4 The following tensors are symmetric and positive definite by construction: the right Cauchy-Green tensor $\boldsymbol{C}=\boldsymbol{F}^{T} \boldsymbol{F}$, and the left Cauchy-Green tensor $\boldsymbol{B}=\boldsymbol{F} \boldsymbol{F}^{T}$.

Definition 2.1.5 The principal invariants satisfy

$$
\begin{gather*}
I_{1}(\boldsymbol{B})=\operatorname{tr} \boldsymbol{B}=I_{1}(\boldsymbol{C}),  \tag{2.6}\\
I_{2}(\boldsymbol{B})=\frac{1}{2}\left[(\operatorname{tr} \boldsymbol{B})^{2}-\operatorname{tr}\left(\boldsymbol{B}^{2}\right)\right]=\operatorname{tr}(\operatorname{Cof} \boldsymbol{B})=I_{2}(\boldsymbol{C}),  \tag{2.7}\\
I_{3}(\boldsymbol{B})=\operatorname{det} \boldsymbol{B}=I_{3}(\boldsymbol{C}), \tag{2.8}
\end{gather*}
$$

where $\operatorname{Cof} \boldsymbol{F}=(\operatorname{det} \boldsymbol{F}) \boldsymbol{F}^{-T}$ is the cofactor of the tensor $\boldsymbol{F}$ and $\operatorname{det} \boldsymbol{F}$ is the determinant. Equivalently, the principal invariants can be expressed in terms of the
principal stretches as follows:

$$
\begin{align*}
& I_{1}(\boldsymbol{B})=\lambda_{1}^{2}+\lambda_{2}^{2}+\lambda_{3}^{2},  \tag{2.9}\\
& I_{2}(\boldsymbol{B})=\lambda_{1}^{2} \lambda_{2}^{2}+\lambda_{2}^{2} \lambda_{3}^{2}+\lambda_{3}^{2} \lambda_{1}^{2},  \tag{2.10}\\
& I_{3}(\boldsymbol{B})=\lambda_{1}^{2} \lambda_{2}^{2} \lambda_{3}^{2} . \tag{2.11}
\end{align*}
$$

Definition 2.1.6 $A$ homogeneous material is a material for which there exists a reference configuration such that all of the material particles respond in the same way to the deformations described with respect to this configuration, i.e., the material properties are independent of position.

Definition 2.1.7 An isotropic material is a material that has the same mechanical behaviour in all directions.

Definition 2.1.8 An incompressible material is a material that can undertake only volume preserving (isochoric) deformations, for which $J=1$.

Definition 2.1.9 Hyperelastic materials, or Green elastic materials, are the class of elastic material models that are described by a strain energy density function. In particular, for homogeneous isotropic hyperelastic materials, having the same properties in all directions, the strain-energy function depends only on the deformation gradient $\boldsymbol{F}$, i.e.,

$$
\begin{equation*}
W=W(\boldsymbol{F}) . \tag{2.12}
\end{equation*}
$$

These strain-energy functions are usually assumed to be equal to zero in the reference configuration, i.e., $W(\boldsymbol{I})=0$.

Definition 2.1.10 The Cauchy stress tensor of a homogeneous isotropic incompressible hyperelastic material takes the form

$$
\begin{equation*}
\boldsymbol{T}=J^{-1} \frac{\partial W}{\partial \boldsymbol{F}} \boldsymbol{F}^{T}-p \boldsymbol{I}=-p \boldsymbol{I}+\beta_{1} \boldsymbol{B}+\beta_{-1} \boldsymbol{B}^{-1} . \tag{2.13}
\end{equation*}
$$

where $p$ is the Lagrange multiplier associated with the incompressibility condition ( $I_{3}=1$ ), and

$$
\begin{equation*}
\beta_{1}=\frac{2}{\sqrt{I_{3}}} \frac{\partial W}{\partial I_{1}}, \quad \beta_{-1}=-2 \sqrt{I_{3}} \frac{\partial W}{\partial I_{2}} \tag{2.14}
\end{equation*}
$$

are scalar functions of the principal invariants. This stress tensor is symmetric, and is co-axial (i.e., it has the same eigenvectors) with the left Cauchy-Green tensor $\boldsymbol{B}$. The physical interpretation of the Cauchy stress tensor is that it represents the internal force per unit of deformed area acting within the solid.

Definition 2.1.11 The first Piola-Kirchhoff stress tensor of a homogeneous isotropic incompressible hyperelastic material is defined by

$$
\begin{equation*}
\boldsymbol{P}=J \boldsymbol{T} \boldsymbol{F}^{-T}=\frac{\partial W}{\partial \boldsymbol{F}}-p \boldsymbol{F}^{-T} . \tag{2.15}
\end{equation*}
$$

This stress tensor is not symmetric in general, and the physical interpretation of this stress tensor is that it represents the internal force per unit of reference area acting within the deformed solid. Its transpose $\boldsymbol{P}^{T}$ is known as the nominal (or engineering) stress tensor.

Definition 2.1.12 The second Piola-Kirchhoff stress tensor of a homogeneous isotropic incompressible hyperelastic material is defined by

$$
\begin{equation*}
\boldsymbol{S}=\boldsymbol{F}^{-1} \boldsymbol{P}=J \boldsymbol{F}^{-1} \boldsymbol{T} \boldsymbol{F}^{-T}=2 \frac{\partial W}{\partial \boldsymbol{C}}-p \boldsymbol{C}^{-1} . \tag{2.16}
\end{equation*}
$$

This stress tensor is symmetric, and is coaxial with the right Cauchy-Green tensor C.

When the displacement is infinitesimally small, the Cauchy stress tensor and the two Piola-Kirchhoff stress tensors coincide.

### 2.2 Quasi-equilibrated motion

For the large strain dynamic deformation of an elastic solid, Cauchy's laws of motion (balance laws of linear and angular momentum) are governed by the following Eulerian field equations [133, p. 40],

$$
\begin{align*}
& \rho \ddot{\mathbf{x}}=\operatorname{div} \mathbf{T}+\rho \mathbf{b},  \tag{2.17}\\
& \mathbf{T}=\mathbf{T}^{T}, \tag{2.18}
\end{align*}
$$

where $\rho$ is the material density, which is assumed constant, $\mathbf{x}=\chi(\mathbf{X}, t)$ is the motion of the elastic solid, with velocity $\dot{\mathbf{x}}=\partial \chi(\mathbf{X}, t) / \partial t$ and acceleration $\ddot{\mathbf{x}}=$ $\partial^{2} \chi(\mathbf{X}, t) / \partial t^{2}, \mathbf{b}=\mathbf{b}(\mathbf{x}, t)$ is the body force, $\mathbf{T}=\mathbf{T}(\mathbf{x}, t)$ is the Cauchy stress tensor, and the superscript $T$ defines the transpose.

Definition 2.2.1 [133, p. 208] A quasi-equilibrated motion, $\boldsymbol{x}=\chi(\boldsymbol{X}, t)$, is the motion of a homogeneous incompressible elastic solid subject to a given body force, $\boldsymbol{b}=\boldsymbol{b}(\boldsymbol{x}, t)$, whereby, for each value of $t, \boldsymbol{x}=\chi(\boldsymbol{X}, t)$ defines a static deformation that satisfies the equilibrium conditions under the body force $\boldsymbol{b}=\boldsymbol{b}(\boldsymbol{x}, t)$.

Theorem 2.2.2 [133, p. 208] (see also [83] for a proof) A quasi-equilibrated motion, $\boldsymbol{x}=\chi(\boldsymbol{X}, t)$, of a homogeneous incompressible elastic solid subject to a given body force, $\boldsymbol{b}=\boldsymbol{b}(\boldsymbol{x}, t)$, is dynamically possible, subject to the same body force, if and only if the motion is circulation preserving with a single-valued acceleration potential $\xi$, i.e.,

$$
\begin{equation*}
\ddot{\boldsymbol{x}}=-\operatorname{grad} \xi . \tag{2.19}
\end{equation*}
$$

For the condition (2.19) to be satisfied, it is necessary that

$$
\begin{equation*}
\operatorname{curl} \ddot{\boldsymbol{x}}=\boldsymbol{0} \text {. } \tag{2.20}
\end{equation*}
$$

Then, the Cauchy stress tensor takes the form

$$
\begin{equation*}
\boldsymbol{T}=-\rho \xi \boldsymbol{I}+\boldsymbol{T}^{(0)}, \tag{2.21}
\end{equation*}
$$

where $\boldsymbol{T}^{(0)}$ is the Cauchy stress for the equilibrium state at time $t$ and $\boldsymbol{I}=$ $\operatorname{diag}(1,1,1)$ is the identity tensor. In this case, the stress field is determined by the present configuration alone. In particular, the shear stresses in the motion are the same as those of the equilibrium state at time $t$.

### 2.3 Polar systems of coordinates

### 2.3.1 Cylindrical Coordinates

We denote by $(r, \theta, z)$ the cylindrical polar coordinates consisting of the radius $r \in[0, \infty]$, the azimuth $\theta \in(-\pi, \pi]$, and the height $z \in(-\infty, \infty)$, and the associated basis vectors by $\left(\mathbf{e}_{r}, \mathbf{e}_{\theta}, \mathbf{e}_{z}\right)$. The cylindrical coordinates are defined with respect to a set of Cartesian (rectangular) coordinates ( $x, y, z$ ), with the basis vectors $\left(\mathbf{e}_{x}, \mathbf{e}_{y}, \mathbf{e}_{z}\right)$, and the two sets of coordinates are related as follows,

$$
x=r \cos \theta, \quad y=r \sin \theta, \quad z=z,
$$

and

$$
r=\sqrt{x^{2}+y^{2}}, \quad \theta=\arctan \frac{y}{x}, \quad z=z .
$$

Their corresponding basis vectors are related by

$$
\mathbf{e}_{x}=\mathbf{e}_{r} \cos \theta-\mathbf{e}_{\theta} \sin \theta, \quad \mathbf{e}_{y}=\mathbf{e}_{r} \sin \theta+\mathbf{e}_{\theta} \cos \theta, \quad \mathbf{e}_{z}=\mathbf{e}_{z}
$$

and

$$
\mathbf{e}_{r}=\mathbf{e}_{x} \cos \theta+\mathbf{e}_{y} \sin \theta, \quad \mathbf{e}_{\theta}=-\mathbf{e}_{x} \sin \theta+\mathbf{e}_{y} \cos \theta, \quad \mathbf{e}_{z}=\mathbf{e}_{z} .
$$

For time-dependent systems, we have $r=r(t), \theta=\theta(t)$ and $z=z(t)$, where $t$ denotes the time variable, and differentiating the cylindrical basis vectors with respect to time gives

$$
\dot{\mathbf{e}}_{r}=\dot{\theta} \mathbf{e}_{\theta}, \quad \dot{\mathbf{e}}_{\theta}=-\dot{\theta} \mathbf{e}_{r}, \quad \dot{\mathbf{e}}_{z}=0 .
$$

The position, velocity and acceleration in cylindrical coordinates are, respectively,

$$
\begin{aligned}
& \mathbf{p}=r \mathbf{e}_{r}+z \mathbf{e}_{z}, \\
& \dot{\mathbf{p}}=\dot{r} \mathbf{e}_{r}+r \dot{\theta} \mathbf{e}_{\theta}+\dot{z} \mathbf{e}_{z}, \\
& \ddot{\mathbf{p}}=\left(\ddot{r}-r \dot{\theta}^{2}\right) \mathbf{e}_{r}+(r \ddot{\theta}+2 \dot{r} \dot{\theta}) \mathbf{e}_{\theta}+\ddot{z} \mathbf{e}_{z} .
\end{aligned}
$$

The gradient of a scalar function $f$ in cylindrical polar coordinates is

$$
\nabla f=\frac{\partial f}{\partial r} \mathbf{e}_{r}+\frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_{\theta}+\frac{\partial f}{\partial z} \mathbf{e}_{z} .
$$

The divergence of a vector field $\mathbf{f}=\left(f_{r}, f_{\theta}, f_{z}\right)^{T}$ is

$$
\nabla \cdot \mathbf{f}=\frac{1}{r} \frac{\partial\left(r f_{r}\right)}{\partial r}+\frac{1}{r} \frac{\partial f_{\theta}}{\partial \theta}+\frac{\partial f_{z}}{\partial z} .
$$

The curl of a vector field $\mathbf{f}=\left(f_{r}, f_{\theta}, f_{z}\right)^{T}$ is

$$
\nabla \times \mathbf{f}=\left(\frac{1}{r} \frac{\partial f_{z}}{\partial \theta}-\frac{\partial f_{\theta}}{\partial z}\right) \mathbf{e}_{r}+\left(\frac{\partial f_{r}}{\partial z}-\frac{\partial f_{z}}{\partial r}\right) \mathbf{e}_{\theta}+\frac{1}{r}\left(\frac{\partial\left(r f_{\theta}\right)}{\partial r}-\frac{\partial f_{r}}{\partial \theta}\right) \mathbf{e}_{z}
$$

### 2.3.2 Spherical Coordinates

We denote by $(r, \theta, \phi)$ the spherical polar coordinates consisting of the radius $r \in[0, \infty]$, the azimuth $\theta \in(-\pi, \pi]$, and the inclination $\phi \in[0, \pi]$, and the associated basis vectors by $\left(\mathbf{e}_{r}, \mathbf{e}_{\theta}, \mathbf{e}_{\phi}\right)$. The spherical coordinates are defined with respect to a set of Cartesian coordinates $(x, y, z)$, with the basis vectors $\left(\mathbf{e}_{x}, \mathbf{e}_{y}, \mathbf{e}_{z}\right)$,
and the two sets of coordinates are related as follows,

$$
x=r \cos \theta \sin \phi, \quad y=r \sin \theta \sin \phi, \quad z=r \cos \phi
$$

and

$$
r=\sqrt{x^{2}+y^{2}+z^{2}}, \quad \theta=\arctan \frac{y}{x}, \quad \phi=\arccos \frac{z}{r} .
$$

Their corresponding basis vectors are related as follows,

$$
\begin{aligned}
& \mathbf{e}_{x}=\mathbf{e}_{r} \cos \theta \sin \phi-\mathbf{e}_{\theta} \sin \theta+\mathbf{e}_{\phi} \cos \theta \cos \phi, \\
& \mathbf{e}_{y}=\mathbf{e}_{r} \sin \theta \sin \phi+\mathbf{e}_{\theta} \cos \theta+\mathbf{e}_{\phi} \sin \theta \cos \phi, \\
& \mathbf{e}_{z}=\mathbf{e}_{r} \cos \phi-\mathbf{e}_{\phi} \sin \phi,
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathbf{e}_{r}=\mathbf{e}_{x} \cos \theta \sin \phi+\mathbf{e}_{y} \sin \theta \sin \phi+\mathbf{e}_{z} \cos \phi \\
& \mathbf{e}_{\theta}=-\mathbf{e}_{x} \sin \theta+\mathbf{e}_{y} \cos \theta \\
& \mathbf{e}_{\phi}=\mathbf{e}_{x} \cos \theta \cos \phi+\mathbf{e}_{y} \sin \theta \cos \phi-\mathbf{e}_{z} \sin \phi
\end{aligned}
$$

For time-dependent systems, $r=r(t), \theta=\theta(t)$ and $\phi=\phi(t)$, and differentiating the spherical basis vectors with respect to time gives

$$
\begin{aligned}
& \dot{\mathbf{e}}_{r}=(\dot{\theta} \sin \phi) \mathbf{e}_{\theta}+\dot{\phi} \mathbf{e}_{\phi}, \\
& \dot{\mathbf{e}}_{\theta}=-(\dot{\theta} \sin \phi) \mathbf{e}_{r}-(\dot{\theta} \cos \phi) \mathbf{e}_{\phi}, \\
& \dot{\mathbf{e}}_{\phi}=-\dot{\phi} \mathbf{e}_{r}+(\dot{\theta} \cos \phi) \mathbf{e}_{\theta} .
\end{aligned}
$$

The position, velocity and acceleration in spherical coordinates are, respectively,

$$
\begin{aligned}
\mathbf{p} & =r \mathbf{e}_{r} \\
\dot{\mathbf{p}} & =\dot{r} \mathbf{e}_{r}+(r \dot{\theta} \sin \phi) \mathbf{e}_{\theta}+r \dot{\phi} \mathbf{e}_{\phi} \\
\ddot{\mathbf{p}} & =\left(\ddot{r}-r \dot{\theta}^{2} \sin ^{2} \phi-r \dot{\phi}^{2}\right) \mathbf{e}_{r} \\
& +(r \ddot{\theta} \sin \phi+2 \dot{r} \dot{\theta} \sin \phi+2 r \dot{\theta} \dot{\phi} \cos \phi) \mathbf{e}_{\theta} \\
& +\left(r \ddot{\phi}+2 \dot{r} \dot{\phi}-r \dot{\theta}^{2} \sin \phi \cos \phi\right) \mathbf{e}_{\phi}
\end{aligned}
$$

The gradient of a scalar function $f$ in spherical polar coordinates is

$$
\nabla f=\frac{\partial f}{\partial r} \mathbf{e}_{r}+\frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_{\theta}+\frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \mathbf{e}_{\phi}
$$

The divergence of a vector field $\mathbf{f}=\left(f_{r}, f_{\theta}, f_{\phi}\right)^{T}$ is

$$
\nabla \cdot \mathbf{f}=\frac{1}{r^{2}} \frac{\partial\left(r^{2} f_{r}\right)}{\partial r}+\frac{1}{r \sin \theta} \frac{\partial f_{\theta}}{\partial \theta}+\frac{1}{r \sin \theta} \frac{\partial\left(f_{\phi} \sin \theta\right)}{\partial \phi} .
$$

The curl of a vector field $\mathbf{f}=\left(f_{r}, f_{\theta}, f_{\phi}\right)^{T}$ is

$$
\begin{aligned}
\nabla \times \mathbf{f} & =\frac{1}{r \sin \theta}\left[\frac{\partial\left(f_{\phi} \sin \theta\right)}{\partial \theta}-\frac{\partial f_{\theta}}{\partial \phi}\right] \mathbf{e}_{r}+\frac{1}{r}\left[\frac{1}{\sin \theta} \frac{\partial f_{r}}{\partial \phi}-\frac{\partial\left(r f_{\phi}\right)}{\partial r}\right] \mathbf{e}_{\theta} \\
& +\frac{1}{r}\left[\frac{\partial\left(r f_{\theta}\right)}{\partial r}-\frac{\partial f_{r}}{\partial \theta}\right] \mathbf{e}_{\phi} .
\end{aligned}
$$

### 2.4 Probability theory

Definition 2.4.1 A probability space is a mathematical model of an experiment characterised by a triple $(\Theta, \mathcal{F}, P)$, where $\Theta$ is the space of possible outcomes, or the sample space, with each outcome being the result of a single experimental test, $\mathcal{F}$ is a collection of events, with each event being a subset of $\Theta$, and $P: \mathcal{F} \rightarrow[0,1]$ is a function that assigns probabilities to individual events, such that events that almost certainly will not happen have probability 0 , and events that will happen almost surely have probability 1.

Definition 2.4.2 Given a probability space $(\Theta, \mathcal{F}, P)$, a random variable is a realvalued function, $X: \Theta \rightarrow \mathbb{R}$, mapping a sample space $\Theta$ into the real line $\mathbb{R}$, such that $(X \leq x)=\{\theta \in \Theta: X(\theta) \leq x\} \in \mathcal{F}$, for every real number $x \in \mathbb{R}$. For any fixed $\theta \in \Theta$, the deterministic value $X(\theta) \in \mathbb{R}$ is called a realisation, or a sample, of the random variable $X$. The corresponding (cumulative) distribution function is $F_{X}: \mathbb{R} \rightarrow[0,1]$, defined by $F_{X}(x)=P(X \leq x)$.

Definition 2.4.3 The random variable $X$ is called continuous if there exists a non-negative function $f_{X}: \mathbb{R} \rightarrow[0, \infty)$, such that the corresponding cumulative distribution function takes the form

$$
F_{X}(x)=P(X \leq x)=\int_{-\infty}^{x} f_{X}(u) \mathrm{d} u, \quad x \in \mathbb{R}
$$

The function $f_{X}$ is called the probability density function (pdf) of $X$.
Definition 2.4.4 The random variable $X$ is called discrete if the range of its possible values is a countable set $\left\{x_{1}, x_{2}, \ldots\right\} \subset \mathbb{R}$. Its distribution function takes the form

$$
F_{X}(x)=\sum_{i: x_{i} \leq x} P\left(X=x_{i}\right) .
$$

Definition 2.4.5 $A$ random vector of length $n$ is a vector $\boldsymbol{X}=\left(X_{1}, X_{2}, \ldots, X_{n}\right)$, where, $X_{i}: \Theta \rightarrow \mathbb{R}$ is a random variable, for all $i=1,2, \ldots, n$. The corresponding joint distribution function is a function $F_{\boldsymbol{X}}: \mathbb{R}^{n} \rightarrow[0,1]$, given by $F_{\boldsymbol{X}}\left(x_{1}, x_{2}, \ldots, x_{n}\right)=P\left(X_{1} \leq x_{1}, X_{2} \leq x_{2}, \ldots, X_{n} \leq x_{n}\right)$. If $\boldsymbol{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$, then the joint probability function can be written in the form $F_{\boldsymbol{X}}(\boldsymbol{x})=P(\boldsymbol{X} \leq \boldsymbol{x})$.

Definition 2.4.6 The random variables $X_{1}, X_{2}, \ldots, X_{n}$ are called jointly continuous if there exists a joint non-negative function $f_{\boldsymbol{X}}(\boldsymbol{x})$, with $\boldsymbol{X}=\left(X_{1}, X_{2}, \ldots, X_{n}\right)$ and $\boldsymbol{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$, such that the corresponding distribution function takes the form

$$
F_{\boldsymbol{X}}(\boldsymbol{x})=\int_{-\infty}^{x_{n}} \ldots \int_{-\infty}^{x_{2}} \int_{-\infty}^{x_{1}} f_{\boldsymbol{X}}\left(u_{1}, u_{1}, \ldots, u_{n}\right) \mathrm{d} u_{1} \mathrm{~d} u_{2} \ldots \mathrm{~d} u_{n}, \quad \boldsymbol{x} \in \mathbb{R}^{n}
$$

The function $f_{\boldsymbol{X}}$ is called the joint probability density function of $X_{1}, X_{2}, \ldots, X_{n}$.

Definition 2.4.7 The random variables $X_{1}, X_{2}, \ldots, X_{n}$ are called jointly discrete if the range of its possible values of the random vector $\boldsymbol{X}=\left(X_{1}, X_{2}, \ldots, X_{n}\right)$ is a countable subset of $\mathbb{R}^{n}$. Its joint distribution function takes the form

$$
F_{\boldsymbol{X}}(\boldsymbol{x})=\sum_{i: \boldsymbol{x}^{(i)} \leq \boldsymbol{x}} P\left(\boldsymbol{X}=\boldsymbol{x}^{(i)}\right) .
$$

Definition 2.4.8 The random variables $X_{1}, X_{2}, \ldots, X_{n}$ are independent if the events $\left(X_{i}=x_{i}\right)$ are independent for all $i=1,2, \ldots, n$, i.e.,

$$
P\left(X_{1}=x_{1}, X_{2}=x_{2}, \ldots, X_{n}=x_{n}\right)=P\left(X_{1}=x_{1}\right) P\left(X_{2}=x_{2}\right) \ldots P\left(X_{n}=x_{n}\right),
$$

for all sets $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\} \in \mathbb{R}$ and all their finite subsets.

Definition 2.4.9 The mean value, or expected value, or mathematical expectation of a continuous random variable $X$ with probability function $f_{X}$ is

$$
E[X]=\int_{-\infty}^{\infty} x f_{X}(x) \mathrm{d} x
$$

provided that

$$
\int_{-\infty}^{\infty}|x| f_{X}(x) \mathrm{d} x<\infty
$$

Definition 2.4.10 The moment generating function of a random variable $X$ is the function $M_{X}: \mathbb{R} \rightarrow[0, \infty)$, given by $M_{X}(t)=E\left[e^{t X}\right]$. This function may be approximated by the Taylor's expansion

$$
M_{X}(t)=\sum_{k=0}^{\infty} E\left[X^{k}\right] \frac{t^{k}}{t!},
$$

where $E\left[X^{k}\right], k>1$, is called the $k$ th moment. $X$ is a $q$-order random variable if $E\left[X^{q}\right]<\infty$.

Definition 2.4.11 For a continuous random variable $X$, the $k$ th central moment is $(X-E[X])^{k}, k>1$, and its expected value is given by

$$
E\left[(X-E[X])^{k}\right]=\int_{-\infty}^{\infty}(x-E[X])^{k} f_{X}(x) \mathrm{d} x
$$

if

$$
\int_{-\infty}^{\infty}\left|(x-E[X])^{k}\right| f_{X}(x) \mathrm{d} x<\infty
$$

Definition 2.4.12 For a random variable $X$, the $2 n d$ central moment is the variance, defined by

$$
\operatorname{Var}[X]=E\left[(X-E[X])^{2}\right]=E\left[X^{2}\right]-E[X]^{2}=\sum_{i \geq 1}\left(x_{i}-E[X]\right)^{2} P\left(X_{i}=x_{i}\right)
$$

The square root of the variance, $\sqrt{\operatorname{Var}[X]}$, is called the standard deviation of $X$.

Definition 2.4.13 The mean value, or expected value, or mathematical expectation of a discrete random variable $X$, with possible values in a countable set $\left\{x_{1}, x_{2}, \ldots\right\} \in \mathbb{R}$, is

$$
E[X]=\sum_{i \geq 1} x_{i} P\left(X_{i}=x_{i}\right)
$$

provided that

$$
\sum_{i \geq 1}\left|x_{i}\right| P\left(X_{i}=x_{i}\right)<\infty
$$

Definition 2.4.14 In general, for a discrete random variable $X$, the expected value of the $k$ th central moment $(X-E[X])^{k}, k>1$, is

$$
E\left[(X-E[X])^{k}\right]=\sum_{i \geq 1}\left(x_{i}-E[X]\right)^{k} P\left(X_{i}=x_{i}\right)
$$

if

$$
\sum_{i \geq 1}\left|\left(x_{i}-E[X]\right)^{k}\right| P\left(X_{i}=x_{i}\right)<\infty .
$$

# 2.5. STOCHASTIC STRAIN-ENERGY FUNCTIONS FOR HYPERELASTIC MATERIALS 

Definition 2.4.15 The covariance of two random variables $X_{1}$ and $X_{2}$ is defined by

$$
\operatorname{Cov}\left[X_{1}, X_{2}\right]=E\left[\left(X_{1}-E\left[X_{1}\right]\right)\left(X_{2}-E\left[X_{2}\right]\right)\right]=E\left[X_{1} X_{2}\right]-E\left[X_{1}\right] E\left[X_{2}\right] .
$$

Definition 2.4.16 $A$ random field $U(\boldsymbol{x})$ is an uncountable family of random variables depending on a deterministic variable $\boldsymbol{x}=\left(x_{1}, \ldots, x_{d}\right)^{T} \in \Omega \subset \mathbb{R}^{d}$. When $d=1$ and the deterministic variable is time, $x=t$, the random field is a random function of time, $U(t)$, known as a stochastic process.

Definition 2.4.17 The mean function of a random field $U(\boldsymbol{x})$ is a real-valued function depending on a deterministic variable $\boldsymbol{x} \in \Omega \subset \mathbb{R}^{d}, E[U(\boldsymbol{x})]: \Omega \rightarrow \mathbb{R}$, where $E[\cdot]$ denotes the mathematical expectation (mean value).

### 2.5 Stochastic strain-energy functions for hyperelastic materials

For our general definition of stochastic isotropic incompressible hyperelastic materials, we rely on the following assumptions [36, 82-84, 88, 89, 91]:
(A1) Material objectivity [43, 99, 133]: the constitutive equations must be invariant under changes of frame of reference. This requires that the scalar strainenergy function, $W=W(\mathbf{F})$, depending only on the deformation gradient $\mathbf{F}$, with respect to the reference configuration, is unaffected by a superimposed rigid-body transformation (involving a change of position) after deformation, i.e., $W\left(\mathbf{R}^{T} \mathbf{F}\right)=W(\mathbf{F})$, where $\mathbf{R} \in S O(3)$ is a proper orthogonal tensor (rotation). Material objectivity is guaranteed by defining strain-energy functions in terms of the scalar invariants.
(A2) Material isotropy [43, 99, 133]: the strain-energy function is unaffected by a superimposed rigid-body transformation prior to deformation, i.e., $W(\mathbf{F R})=$

### 2.5. STOCHASTIC STRAIN-ENERGY FUNCTIONS FOR HYPERELASTIC MATERIALS

$W(\mathbf{F})$, where $\mathbf{R} \in S O(3)$. For isotropic materials, the strain-energy function is a symmetric function of the principal stretches $\left\{\lambda_{i}\right\}_{i=1,2,3}$ of $\mathbf{F}$, i.e., $W(\mathbf{F})=\mathcal{W}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)$.
(A3) Baker-Ericksen (BE) inequalities [14,77]: the greater principal Cauchy stress occurs in the direction of the greater principal stretch, or equivalently,

$$
\begin{equation*}
\left(\lambda_{i} \frac{\partial \mathcal{W}}{\partial \lambda_{i}}-\lambda_{j} \frac{\partial \mathcal{W}}{\partial \lambda_{j}}\right)\left(\lambda_{i}-\lambda_{j}\right)>0 \quad \text { if } \quad \lambda_{i} \neq \lambda_{j}, \quad i, j=1,2,3 \tag{2.22}
\end{equation*}
$$

When any two principal stretches are equal, the strict inequality ">" in (2.22) is replaced by " $\geq$ ".
(A4) For any given finite deformation, at any point in the material, the shear modulus, $\mu$, and its inverse, $1 / \mu$, are second-order random variables, i.e., they have finite mean value and finite variance [124-128].

Assumptions (A1)-(A3) are inherited from the finite elasticity theory [43, 85, 99, 133]. In particular, (A3) guarantees that the shear modulus is positive [85]. Assumption (A4) places random variables at the foundation of stochastic hyperelastic models [88,124-126]. A random variable is usually described in terms of its mean value and its variance, which contains information about the range of values about the mean value $[22,27,59,79,95]$.

In order to derive analytical results for the instability problems presented in this thesis, we confine our attention to a class of stochastic hyperelastic models defined by the strain-energy density

$$
\begin{equation*}
\mathcal{W}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)=\frac{\mu}{2}\left(\lambda_{1}^{2}+\lambda_{2}^{2}+\lambda_{3}^{2}-3\right), \tag{2.2.2}
\end{equation*}
$$

where the shear modulus, $\mu=\mu(R)>0$, is a random field depending on the radius $R$ in a system of cylindrical or spherical polar coordinates, and $\lambda_{1}, \lambda_{2}$, and $\lambda_{3}$ are the principal stretch ratios. When $\mu$ is independent of $R$, the material model

### 2.5. STOCHASTIC STRAIN-ENERGY FUNCTIONS FOR HYPERELASTIC MATERIALS

(2.23) reduces to the stochastic (homogeneous) neo-Hookean model [88, 124].

For the shear modulus, $\mu=\mu(R)$, at any fixed $R$, we rely on the following available information, which guarantees that assumption (A4) holds [117-119,121],

$$
\left\{\begin{array}{l}
E[\mu]=\underline{\mu}>0,  \tag{2.24}\\
E[\log \mu]=\nu, \quad \text { such that }|\nu|<+\infty .
\end{array}\right.
$$

By the principle of maximum entropy [62-64], for any fixed $R, \mu$ follows a Gamma probability distribution with the shape and scale parameters $\rho_{1}=\rho_{1}(R)>0$ and $\rho_{2}=\rho_{2}(R)>0$, respectively. Hence,

$$
\begin{equation*}
\underline{\mu}=\rho_{1} \rho_{2}, \quad \operatorname{Var}[\mu]=\rho_{1} \rho_{2}^{2}, \tag{2.25}
\end{equation*}
$$

where $\underline{\mu}$ is the mean value, $\operatorname{Var}[\mu]=\|\mu\|^{2}$ is the variance, and $\|\mu\|$ is the standard deviation of $\mu$. The corresponding probability density function takes the form [1, 49, 65]

$$
\begin{equation*}
g\left(\mu ; \rho_{1}, \rho_{2}\right)=\frac{\mu^{\rho_{1}-1} e^{-\mu / \rho_{2}}}{\rho_{2}^{\rho_{1}} \Gamma\left(\rho_{1}\right)}, \quad \text { for } \mu>0 \text { and } \rho_{1}, \rho_{2}>0 \tag{2.26}
\end{equation*}
$$

where $\Gamma: \mathbb{R}_{+}^{*} \rightarrow \mathbb{R}$ is the complete Gamma function

$$
\begin{equation*}
\Gamma(z)=\int_{0}^{+\infty} t^{z-1} e^{-t} \mathrm{~d} t \tag{2.27}
\end{equation*}
$$

The word 'hyperparameters' is also used for the parameters $\rho_{1}$ and $\rho_{2}$ of the Gamma distribution, to distinguish them from the material parameter $\mu$ and other material constants [121, p. 8].

In the limiting case when $\rho_{1} \rightarrow \infty$, the Gamma distribution can be approximated by the Gaussian (normal) distribution with mean value $\rho_{1} \rho_{2}$ and standard deviation $\sqrt{\rho_{1}} \rho_{2}[36,82]$ (see Appendix A for a proof).

When $\mu=\mu_{1}+\mu_{2}$, setting a fixed constant value $b>-\infty$, such that $\mu_{i}>b$,

# 2.5. STOCHASTIC STRAIN-ENERGY FUNCTIONS FOR HYPERELASTIC MATERIALS 

$i=1,2$ (e.g., $b=0$ if $\mu_{1}>0$ and $\mu_{2}>0$, although $b$ is not unique in general), we define the auxiliary random variable

$$
\begin{equation*}
R_{1}=\frac{\mu_{1}-b}{\mu-2 b} \tag{2.28}
\end{equation*}
$$

such that $0<R_{1}<1$. Then, the random model parameters can be expressed equivalently as follows,

$$
\begin{equation*}
\mu_{1}=R_{1}(\mu-2 b)+b, \quad \mu_{2}=\mu-\mu_{1}=\left(1-R_{1}\right)(\mu-2 b)+b . \tag{2.29}
\end{equation*}
$$

It is reasonable to assume

$$
\begin{cases}E\left[\log R_{1}\right]=\nu_{1}, & \text { such that }\left|\nu_{1}\right|<+\infty  \tag{2.30}\\ E\left[\log \left(1-R_{1}\right)\right]=\nu_{2}, & \text { such that }\left|\nu_{2}\right|<+\infty\end{cases}
$$

in which case, the random variable $R_{1}$ follows a standard Beta distribution, with hyperparameters $\xi_{1}>0$ and $\xi_{2}>0$ satisfying

$$
\begin{equation*}
\underline{R}_{1}=\frac{\xi_{1}}{\xi_{1}+\xi_{2}}, \quad \operatorname{Var}\left[R_{1}\right]=\frac{\xi_{1} \xi_{2}}{\left(\xi_{1}+\xi_{2}\right)^{2}\left(\xi_{1}+\xi_{2}+1\right)}, \tag{2.31}
\end{equation*}
$$

where $\underline{R}_{1}$ is the mean value and $\operatorname{Var}\left[R_{1}\right]$ is the variance of $R_{1}$. The associated probability density function is

$$
\begin{equation*}
\beta\left(r ; \xi_{1}, \xi_{2}\right)=\frac{r^{\xi_{1}-1}(1-r)^{\xi_{2}-1}}{B\left(\xi_{1}, \xi_{2}\right)}, \quad \text { for } r \in(0,1) \text { and } \xi_{1}, \xi_{2}>0 \tag{2.32}
\end{equation*}
$$

where $B: \mathbb{R}_{+}^{*} \times \mathbb{R}_{+}^{*} \rightarrow \mathbb{R}$ is the Beta function

$$
\begin{equation*}
B(x, y)=\int_{0}^{1} t^{x-1}(1-t)^{y-1} \mathrm{~d} t \tag{2.33}
\end{equation*}
$$

Then, for the random coefficients given by (2.29), the corresponding mean values
are

$$
\begin{equation*}
\underline{\mu}_{1}=\underline{R}_{1}(\underline{\mu}-2 b)+b, \quad \underline{\mu}_{2}=\underline{\mu}^{-} \underline{\mu}_{1}=\left(1-\underline{R}_{1}\right)(\underline{\mu}-2 b)+b, \tag{2.34}
\end{equation*}
$$

and the variances and covariance take the form, respectively,

$$
\begin{align*}
& \operatorname{Var}\left[\mu_{1}\right]=(\underline{\mu}-2 b)^{2} \operatorname{Var}\left[R_{1}\right]+\left(\underline{R}_{1}\right)^{2} \operatorname{Var}[\mu]+\operatorname{Var}[\mu] \operatorname{Var}\left[R_{1}\right],  \tag{2.35}\\
& \operatorname{Var}\left[\mu_{2}\right]=(\underline{\mu}-2 b)^{2} \operatorname{Var}\left[R_{1}\right]+\left(1-\underline{R}_{1}\right)^{2} \operatorname{Var}[\mu]+\operatorname{Var}[\mu] \operatorname{Var}\left[R_{1}\right],  \tag{2.36}\\
& \operatorname{Cov}\left[\mu_{1}, \mu_{2}\right]=\frac{1}{2}\left(\operatorname{Var}[\mu]-\operatorname{Var}\left[\mu_{1}\right]-\operatorname{Var}\left[\mu_{2}\right]\right) . \tag{2.37}
\end{align*}
$$

Throughout this thesis, numerical computations were carried out in Matlab, where we made specific use of inbuilt functions for random number generation. The particular values for the stochastic parameters were taken from $[82,83]$.

## Chapter 3

## Radially symmetric inflation of cylindrical tubes

In this chapter, we study uniform inflation instabilities under radially symmetric finite deformation in two concentric thin-walled homogeneous cylindrical tubes of stochastic neo-Hookean material, which are continuously attached to each other throughout the deformation. We find that, in order to increase the chance of stable inflation, the outer tube must be sufficiently softer than the inner tube. First, we review the problem of radially symmetric inflation instability in homogeneous incompressible hyperelastic tubes presented in [82], then extend the analysis to two concentric homogeneous tubes.

### 3.1 Homogeneous tube

We consider a circular cylindrical tube of homogeneous isotropic incompressible hyperelastic material described by the strain-energy function [82]

$$
\begin{equation*}
\mathcal{W}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)=\frac{\mu_{1}}{2 m^{2}}\left(\lambda_{1}^{2 m}+\lambda_{2}^{2 m}+\lambda_{3}^{2 m}-3\right)+\frac{\mu_{2}}{2 n^{2}}\left(\lambda_{1}^{2 n}+\lambda_{2}^{2 n}+\lambda_{3}^{2 n}-3\right), \tag{3.1}
\end{equation*}
$$

where $m$ and $n$ are deterministic constants, and $\mu_{1}$ and $\mu_{2}$ are random variables defined by probability distributions.

The tube is deformed through the combined effects of inflation and extension (see Figure 3.1)

$$
\begin{equation*}
r=f(R) R, \quad \theta=\Theta, \quad z=\alpha Z, \tag{3.2}
\end{equation*}
$$

where $(R, \Theta, Z)$ and $(r, \theta, z)$ are the cylindrical polar coordinates in the reference and current configurations, respectively, such that $A \leq R \leq B$, and $\alpha>0$, with $A, B$ and $\alpha$ given deterministic constants, and $f(R)>0$ a given deterministic function.


Figure 3.1: Schematic of a cylindrical tube, with undeformed inner and outer radii $A$ and $B$, respectively, showing the reference state (left), and the deformed state, with inner and outer radii $a$ and $b$, respectively (right).

The deformation gradient is $\mathbf{F}=\operatorname{diag}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)$, where

$$
\begin{equation*}
\lambda_{1}=f(R)+R \frac{\mathrm{~d} f}{\mathrm{~d} R}=\lambda^{-1} \alpha^{-1}, \quad \lambda_{2}=f(R)=\lambda, \quad \lambda_{3}=\alpha, \tag{3.3}
\end{equation*}
$$

with $\lambda_{1}, \lambda_{2}$ and $\lambda_{3}$ representing the radial, tangential and longitudinal stretch ratio, respectively.

For the cylindrical tube, the equilibrium equation in the reference configuration
reads [82]

$$
\begin{equation*}
\frac{\mathrm{d} P_{11}}{\mathrm{~d} \lambda} \lambda^{-1} \alpha^{-1}+\frac{P_{11}-P_{22}}{1-\lambda^{2} \alpha}=0 \tag{3.4}
\end{equation*}
$$

where $\mathbf{P}=\left(P_{i j}\right)_{i, j=1,2,3}$ denotes the first Piola-Kirchhoff stress tensor. For an incompressible material,

$$
\begin{equation*}
P_{11}=\frac{\partial \mathcal{W}}{\partial \lambda_{1}}-\frac{p}{\lambda_{1}}, \quad P_{22}=\frac{\partial \mathcal{W}}{\partial \lambda_{2}}-\frac{p}{\lambda_{2}}, \tag{3.5}
\end{equation*}
$$

where $p$ is the Lagrange multiplier for the incompressibility constraint $(\operatorname{det} \mathbf{F}=1)$.
We denote

$$
\begin{equation*}
W(\lambda)=\mathcal{W}\left(\lambda^{-1} \alpha^{-1}, \lambda, \alpha\right), \tag{3.6}
\end{equation*}
$$

where $\lambda=r / R$ and $\alpha=z / Z$. Then, by (3.1),

$$
\begin{equation*}
W(\lambda)=\frac{\mu_{1}}{2 m^{2}}\left(\lambda^{-2 m} \alpha^{-2 m}+\lambda^{2 m}+\alpha^{2 m}-3\right)+\frac{\mu_{2}}{2 n^{2}}\left(\lambda^{-2 n} \alpha^{-2 n}+\lambda^{2 n}+\alpha^{2 n}-3\right) . \tag{3.7}
\end{equation*}
$$

By (3.5), since

$$
\begin{align*}
\frac{\partial \mathcal{W}}{\partial \lambda_{1}} & =\frac{\mu_{1}}{m} \lambda_{1}^{2 m-1}+\frac{\mu_{2}}{n} \lambda_{1}^{2 n-1}  \tag{3.8}\\
& =\frac{\mu_{1}}{m} \lambda^{-2 m+1} \alpha^{-2 m+1}+\frac{\mu_{2}}{n} \lambda^{-2 n+1} \alpha^{-2 n+1}
\end{align*}
$$

and

$$
\begin{align*}
\frac{\partial \mathcal{W}}{\partial \lambda_{2}} & =\frac{\mu_{1}}{m} \lambda_{2}^{2 m-1}+\frac{\mu_{2}}{n} \lambda_{2}^{2 n-1}  \tag{3.9}\\
& =\frac{\mu_{1}}{m} \lambda^{2 m-1}+\frac{\mu_{2}}{n} \lambda^{2 n-1}
\end{align*}
$$

we obtain

$$
\begin{align*}
\frac{\mathrm{d} W}{\mathrm{~d} \lambda} & =\frac{\partial W}{\partial \lambda_{1}} \frac{\partial \lambda_{1}}{\partial \lambda}+\frac{\partial W}{\partial \lambda_{2}} \frac{\partial \lambda_{2}}{\partial \lambda} \\
& =-\lambda^{-2} \alpha^{-1} \frac{\partial \mathcal{W}}{\partial \lambda_{1}}+\frac{\partial \mathcal{W}}{\partial \lambda_{2}} \\
& =-\frac{1}{\lambda^{2} \alpha}\left(P_{11}+\frac{p}{\lambda_{1}}\right)+P_{22}+\frac{p}{\lambda_{2}}  \tag{3.10}\\
& =-\frac{1}{\lambda^{2} \alpha}\left(P_{11}+p \lambda \alpha\right)+P_{22}+\frac{p}{\lambda} \\
& =-\frac{1}{\lambda^{2} \alpha} P_{11}-\frac{p}{\lambda}+P_{22}+\frac{p}{\lambda}
\end{align*}
$$

Hence,

$$
\begin{equation*}
\frac{\mathrm{d} W}{\mathrm{~d} \lambda}=-\frac{P_{11}}{\lambda^{2} \alpha}+P_{22} . \tag{3.11}
\end{equation*}
$$

Setting the external pressure, at $R=B$, equal to zero, by (3.4) and (3.11), the internal pressure expressed in terms of the Cauchy stress, at $R=A$, is equal to

$$
\begin{align*}
T & =-\left.\frac{P_{11}}{\lambda \alpha}\right|_{\lambda=\lambda_{a}} \\
& =-\int_{\lambda_{a}}^{\lambda_{b}} \frac{P_{11}}{\lambda^{2} \alpha} d \lambda+\int_{\lambda_{a}}^{\lambda_{b}} \frac{\mathrm{~d} P_{11}}{\mathrm{~d} \lambda} \lambda^{-1} \alpha^{-1} \mathrm{~d} \lambda  \tag{3.12}\\
& =-\int_{\lambda_{a}}^{\lambda_{b}} \frac{P_{11}}{\lambda^{2} \alpha} d \lambda-\int_{\lambda_{a}}^{\lambda_{b}} \frac{P_{11}-P_{22}}{1-\lambda^{2} \alpha} \mathrm{~d} \lambda \\
& =\int_{\lambda_{a}}^{\lambda_{b}} \frac{\mathrm{~d} W}{\mathrm{~d} \lambda} \frac{\mathrm{~d} \lambda}{1-\lambda^{2} \alpha},
\end{align*}
$$

where $\lambda_{a}=a / A$ and $\lambda_{b}=b / B$ are the stretches for the inner and outer radii. By the incompressibility constraint, the material volume in the cylindrical tube is conserved,, that is, $\pi \alpha\left(b^{2}-a^{2}\right)=\pi\left(B^{2}-A^{2}\right)$, or equivalently, $a=A \lambda_{a}$ and $b=B \lambda_{b}$. Hence,

$$
\begin{equation*}
\lambda_{b}^{2}=\left(\lambda_{a}^{2}-\frac{1}{\alpha}\right)\left(\frac{A}{B}\right)^{2}+\frac{1}{\alpha} . \tag{3.13}
\end{equation*}
$$

Therefore, the internal pressure can be described in terms of the inner stretch ratio, $\lambda_{a}$, only.

For the cylindrical tube, if there is a change in the monotonicity of $T$, then a
limit-point instability occurs. Assuming that the tube is thin, so that

$$
\begin{equation*}
0<\epsilon=\frac{B-A}{A} \ll 1 \tag{3.14}
\end{equation*}
$$

the internal pressure, $T$, is a function of inner stretch ratio, $\lambda_{a}$, and it can be approximated by

$$
\begin{equation*}
T(\lambda)=\frac{\epsilon}{\lambda \alpha} \frac{\mathrm{d} W}{\mathrm{~d} \lambda} \tag{3.15}
\end{equation*}
$$

That is,

$$
\begin{equation*}
T(\lambda)=\frac{\epsilon}{\alpha}\left[\frac{\mu_{1}}{m}\left(\lambda^{2 m-2}+\lambda^{-2 m-2} \alpha^{-2 m}\right)+\frac{\mu_{2}}{n}\left(\lambda^{2 n-2}-\lambda^{-2 n-2} \alpha^{-2 n}\right)\right] . \tag{3.16}
\end{equation*}
$$

To find the critical value of $\lambda$ where a limit-point of instability occurs, we solve for $\lambda>1$, the equation

$$
\begin{equation*}
\frac{\mathrm{d} T}{\mathrm{~d} \lambda}=0 \tag{3.17}
\end{equation*}
$$

### 3.2 Two concentric tubes

Next, we take a composite formed from two concentric tubes (see Figure 3.2), with strain-energy density

$$
\mathcal{W}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)= \begin{cases}\mathcal{W}^{(1)}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right), & \text { if } C<R<A  \tag{3.18}\\ \mathcal{W}^{(2)}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right), & \text { if } A<R<B\end{cases}
$$

where

$$
\begin{align*}
& \mathcal{W}^{(1)}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)=\frac{\mu^{(1)}}{2 m^{2}}\left(\lambda_{1}^{2 m}+\lambda_{2}^{2 m}+\lambda_{3}^{2 m}-3\right),  \tag{3.19}\\
& \mathcal{W}^{(2)}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)=\frac{\mu^{(2)}}{2 n^{2}}\left(\lambda_{1}^{2 n}+\lambda_{2}^{2 n}+\lambda_{3}^{2 n}-3\right) \tag{3.20}
\end{align*}
$$

with $m, n$ deterministic constants and $\mu^{(1)}, \mu^{(2)}$ random variables. For the composite tube, we assume continuity of the displacement across the interface between
the two concentric tubes.


Figure 3.2: Schematic of a composite cylindrical tube made of two concentric homogeneous tubes, with undeformed outer radii $A$ and $B$, respectively, showing the reference state (left), and the deformed state, with outer radii $a$ and $b$ of the concentric tubes, respectively (right).

We denote

$$
W(\lambda)= \begin{cases}W^{(1)}\left(\lambda^{-1} \alpha^{-1}, \lambda, \alpha\right), & \text { if } C<R<A  \tag{3.21}\\ W^{(2)}\left(\lambda^{-1} \alpha^{-1}, \lambda, \alpha\right), & \text { if } A<R<B\end{cases}
$$

where

$$
\begin{align*}
W^{(1)}(\lambda) & =\frac{\mu^{(1)}}{2 m^{2}}\left(\lambda^{-2 m} \alpha^{-2 m}+\lambda^{2 m}+\alpha^{2 m}-3\right),  \tag{3.22}\\
W^{(2)}(\lambda) & =\frac{\mu^{(2)}}{2 n^{2}}\left(\lambda^{-2 n} \alpha^{-2 n}+\lambda^{2 n}+\alpha^{2 n}-3\right) \tag{3.23}
\end{align*}
$$

In this case, the internal pressure is equal to

$$
\begin{equation*}
T=\int_{\lambda_{c}}^{\lambda_{a}} \frac{\mathrm{~d} W^{(1)}}{\mathrm{d} \lambda} \frac{\mathrm{~d} \lambda}{1-\lambda^{2} \alpha}+\int_{\lambda_{a}}^{\lambda_{b}} \frac{\mathrm{~d} W^{(2)}}{\mathrm{d} \lambda} \frac{\mathrm{~d} \lambda}{1-\lambda^{2} \alpha}, \tag{3.24}
\end{equation*}
$$

where $\lambda_{c}=c / C, \lambda_{a}=a / A$, and $\lambda_{b}=b / B$.

When the composite tube is thin, so that

$$
\begin{equation*}
0<\frac{\epsilon}{2}=\frac{B-A}{A} \approx \frac{A-C}{C} \ll 1 \tag{3.25}
\end{equation*}
$$

the internal pressure can be approximated by

$$
\begin{align*}
T(\lambda) & =\frac{\epsilon}{2 \lambda \alpha} \frac{\mathrm{~d} W^{(1)}}{\mathrm{d} \lambda}+\frac{\epsilon}{2 \lambda \alpha} \frac{\mathrm{~d} W^{(2)}}{\mathrm{d} \lambda} \\
& =\frac{\epsilon}{2 \lambda \alpha}\left[\frac{\mu^{(1)}}{m}\left(-\lambda^{-2 m-1} \alpha^{-2 m}+\lambda^{2 m-1}\right)+\frac{\mu^{(2)}}{n}\left(-\lambda^{-2 n-1} \alpha^{-2 n}+\lambda^{2 n-1}\right)\right] \\
& =\frac{\epsilon}{2 \alpha}\left[\frac{\mu^{(1)}}{m}\left(-\lambda^{-2 m-2} \alpha^{-2 m}+\lambda^{2 m-2}\right)+\frac{\mu^{(2)}}{n}\left(-\lambda^{-2 n-2} \alpha^{-2 n}+\lambda^{2 n-2}\right)\right] . \tag{3.26}
\end{align*}
$$

Thus

$$
\begin{align*}
\frac{\mathrm{d} T}{\mathrm{~d} \lambda}= & \frac{\epsilon}{2 \lambda \alpha} \frac{\mu^{(1)}}{m}\left[(2 m+2) \lambda^{-2 m-3} \alpha^{-2 m}+(2 m-2) \lambda^{2 m-3}\right]  \tag{3.27}\\
& +\frac{\epsilon}{2 \lambda \alpha} \frac{\mu^{(2)}}{n}\left[(2 n+2) \lambda^{-2 n-3} \alpha^{-2 n}+(2 n-2) \lambda^{2 n-3}\right]
\end{align*}
$$

and equation

$$
\begin{equation*}
\frac{\mathrm{d} T}{\mathrm{~d} \lambda}=0 \tag{3.28}
\end{equation*}
$$

is equivalent

$$
\begin{align*}
& \frac{\mu^{(1)}}{m}\left[(2 m+2) \lambda^{-2 m-3} \alpha^{-2 m}+(2 m-2) \lambda^{2 m-3}\right] \\
& +\frac{\mu^{(2)}}{n}\left[(2 n+2) \lambda^{-2 n-3} \alpha^{-2 n}+(2 n-2) \lambda^{2 n-3}\right]=0 . \tag{3.29}
\end{align*}
$$

Next, similarly to [82], we specialise to the case when $m=1 / 2, n=-3 / 2$, and $\alpha=1$, and obtain

$$
\begin{equation*}
\frac{\mu^{(1)}}{\mu^{(1)}+\mu^{(2)}}=\eta(\lambda)=\frac{1+5 \lambda^{-6}}{1+5 \lambda^{-6}+3 \lambda^{-2}-9 \lambda^{-4}} . \tag{3.30}
\end{equation*}
$$

Instability occurs if the equation (3.30) has a solution.

The minimum value of the function

$$
\begin{equation*}
\eta(\lambda)=\frac{\lambda^{6}+5}{\lambda^{6}+5+3 \lambda^{4}-9 \lambda^{2}}, \quad \lambda>1, \tag{3.31}
\end{equation*}
$$

is $\eta_{\min }=\min \eta(\lambda)=0.8035$, attained at $\lambda=2.4895$. This minimum was derived numerically,

Hence, instability occurs if

$$
\begin{equation*}
\frac{\mu^{(1)}}{\mu^{(1)}+\mu^{(2)}} \geq \eta_{\min }=0.8035 \tag{3.32}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
\frac{\mu^{(1)}}{\mu^{(2)}} \geq \frac{\eta_{\min }}{1-\eta_{\min }}=\frac{0.8035}{1-0.8035}=4.0890 . \tag{3.33}
\end{equation*}
$$

Therefore, the condition for the existence of limit-point instability is that the outer tube is softer than the inner tube.

### 3.3 Stochastic elastic tubes

For a cylindrical tube of stochastic hyperelastic material described by the strainenergy function (3.18) with $m=1 / 2$ and $n=-3 / 2$, by (3.33), the probability distribution of stable inflation, such that the internal pressure always increases as the radial stretch increases, is equal to

$$
\begin{equation*}
P_{1}\left(\mu^{(1)}<4.0890 \mu^{(2)}\right)=\int_{0}^{4.0890 \mu^{(2)}} p_{1}(u) \mathrm{d} u \tag{3.34}
\end{equation*}
$$

and the probability distribution of inflation instability occurring is

$$
\begin{equation*}
P_{2}\left(\mu^{(1)}>4.0890 \mu^{(2)}\right)=1-P_{1}\left(\mu^{(1)}<4.0890 \mu^{(2)}\right)=1-\int_{0}^{4.0890 \mu^{(2)}} p_{1}(u) \mathrm{d} u \tag{3.35}
\end{equation*}
$$

where $p_{1}(u)=g\left(u ; \rho_{1}, \rho_{2}\right)$ is defined by (2.26) if the random variable follows a Gamma distribution, or $p_{1}=f(u ; \underline{u},\|u\|)$ if the random variable follows a normal
distribution with mean value $\underline{u}$ and standard deviation $\|u\|$.


Figure 3.3: Probability distributions (3.34)-(3.35) of whether instability can occur or not for a composite with two concentric cylindrical tubes of stochastic material described by (3.21) with $m=1 / 2$ and $n=-3 / 2$, and the shear modulus, $\mu^{(1)}$, following either a Gamma distribution (2.26) with shape and scale parameters $\rho_{1}^{(1)}=405.0214$ and $\rho_{2}^{(1)}=0.0101$, respectively (continuous lines), or a normal distribution with mean value $\underline{\mu}^{(1)}=4.0907$ and standard deviation $\left\|\mu^{(1)}\right\|=0.2302$ (dashed lines). Darker colours represent analytically derived solutions, given by equation (3.34)-(3.35), and lighter ones show stochastically generated data. The vertical line at the critical value, $\mu^{(2)}=4.0907 / 4.0890=1.0004$, separates the expected regions based only on the mean value $\underline{\mu}^{(1)}=4.0907$.

The probability distributions given by equations (3.34)-(3.35) are illustrated in Figure 3.3, where blue lines are for $P_{1}$ and red lines for $P_{2}$. For the numerical approximation, ( $0, \underline{\mu}^{(1)} / 2$ ) was divided into 100 steps, then for each value of $\mu^{(2)}$, 100 random values of $\mu^{(1)}$ were numerically generated from a specified Gamma (or normal) distribution and compared with the inequalities defining the two intervals for values of $\mu^{(1)}$. For the deterministic case, which is based on the mean value of the shear modulus, $\underline{\mu}^{(1)}=\rho_{1}^{(1)} \rho_{2}^{(1)}=4.0907$, the critical value of $\mu^{(2)}=1.0004$ strictly separates the cases where inflation instability can occur or not. For the stochastic problem, for the same critical value, there is, by definition, exactly $50 \%$ chance of stable inflation (blue solid or dashed line if the shear modulus is Gamma or normal distributed, respectively), and $50 \%$ chance of a limit-point instability (red solid or dashed line). To increase the probability of stable inflation ( $P_{1} \approx 1$ ), one must consider sufficiently large values of $\mu^{(2)}$, above the expected critical value, whereas a limit-point instability has better chance to occur $\left(P_{2} \approx 1\right)$ if $\mu^{(2)}$ is small enough. However, there will always be competition between the two cases.


Figure 3.4: The normalised internal pressure, $T(\lambda)$, for the inflation of cylindrical tubes of hyperelastic materials, defined by (3.18), with $m=1 / 2$ and $n=-3 / 2$. In this deterministic case, inflation instability occurs if $\mu^{(1)} / \mu^{(2)} \geq 4.0890$.


Figure 3.5: Probability distribution of the normalised internal pressure, $T(\lambda)$ defined by (3.26), for the inflation of a cylindrical tube of stochastic hyperelastic material, given by (3.18) with $m=1 / 2$ and $n=-3 / 2$, when $\mu^{(1)}$ follows a Gamma distribution with $\rho_{1}^{(1)}=405.0214, \rho_{2}^{(1)}=0.0101$ and $\mu^{(2)}$ follows a Gamma distribution with $\rho_{1}^{(2)}=\rho_{1}^{(1)}, \rho_{2}^{(2)}=\rho_{2}^{(1)} / 4$. The dashed black line corresponds to the expected pressure based only on mean parameter values.

Figure 3.5 shows the probability distribution of the normalised internal pressure, $T(\lambda)$ defined by (3.26), for the inflation of a cylindrical tube of stochastic hyperelastic material, given by (3.18) with $m=1 / 2$ and $n=-3 / 2$, when $\mu^{(1)}$ follows a Gamma distribution with $\rho_{1}^{(1)}=405.0214, \rho_{2}^{(1)}=0.0101$ and $\mu^{(2)}$ follows a Gamma distribution with $\rho_{1}^{(2)}=\rho_{1}^{(1)}, \rho_{2}^{(2)}=\rho_{2}^{(1)} / 4$. The dashed black line corresponds to the expected pressure based only on mean parameter values.

Note that, in Figures 3.4 and 3.5, there is more variability in the responses for smaller values of $\lambda$ and less variability for large values of $\lambda$. This is the exact opposite behaviour compared to that obtained in [82], where a single homogeneous tube was treated, and it was found that there was less variability in the responses for smaller values of $\lambda$ and more variability for larger values of $\lambda$. The different stochastic behaviours of the two cases is due to the different probability distributions characterising the respective material parameters. Namely, in [82], the a homogeneous stochastic Mooney-Rivlin material was treated, where the two material components followed Beta distributions, while here, two concentric stochastic neo-Hookean are considered, with the respective shear moduli following Gamma distributions.

## Chapter 4

## Radially symmetric inflation of spherical shells

In this chapter, we study uniform inflation instabilities under radially symmetric finite deformation in two concentric thin-walled homogeneous spherical shells of stochastic neo-Hookean material, which are continuously attached to each other throughout the deformation. For these shells also, we find that, in order to increase the probability of stable inflation, the outer shell must be sufficiently softer than the inner shell. First, we review the problem of radially symmetric inflation instability in homogeneous incompressible hyperelastic shells presented in [82], then extend the analysis to two concentric homogeneous shells.

### 4.1 Homogeneous spherical shell

We consider a spherical shell of homogeneous isotropic incompressible hyperelastic material described by the strain-energy function defined by (3.1), subject to the following radially symmetric deformation (see Figure 4.1)

$$
\begin{equation*}
r=f(R) R, \quad \theta=\Theta, \quad \phi=\Phi \tag{4.1}
\end{equation*}
$$

where $(R, \Theta, \Phi)$ and $(r, \theta, \phi)$ are the spherical polar coordinates in the reference and current configurations, respectively, such that $A \leq R \leq B$, with $A$ and $B$ given deterministic constants and $f(R)>0$ a given deterministic function.


Figure 4.1: Schematic of a spherical shell with undeformed inner and outer radii $A$ and $B$, respectively, showing the reference state (left), and the deformed state, with inner and outer radii $a$ and $b$, respectively (right).

The deformation gradient is $\mathbf{F}=\operatorname{diag}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)$, where

$$
\begin{equation*}
\lambda_{1}=f(R)+R \frac{\mathrm{~d} f}{\mathrm{~d} R}=\lambda^{-2}, \quad \lambda_{2}=\lambda_{3}=f(R)=\lambda, \tag{4.2}
\end{equation*}
$$

where $\lambda_{1}, \lambda_{2}$ and $\lambda_{3}$ representing the radial, tangential and longitudinal stretch ratio, respectively, and $\mathrm{d} f / \mathrm{d} R$ denotes differentiation of $f$ with respect to $R$.

For the spherical shell, the equilibrium equation in the reference configuration is [82]

$$
\begin{equation*}
\frac{\mathrm{d} P_{11}}{\mathrm{~d} \lambda} \frac{1}{\lambda^{2}}-2 \frac{P_{11}-P_{22}}{1-\lambda^{3}}=0 \tag{4.3}
\end{equation*}
$$

where $\mathbf{P}=\left(P_{i j}\right)_{i, j=1,2,3}$ is the first Piola-Kirchhoff stress tensor. For an incompressible material,

$$
\begin{equation*}
P_{11}=\frac{\partial \mathcal{W}}{\partial \lambda_{1}}-\frac{p}{\lambda_{1}}, \quad P_{22}=\frac{\partial \mathcal{W}}{\partial \lambda_{2}}-\frac{p}{\lambda_{2}} \tag{4.4}
\end{equation*}
$$

where $p$ is the Lagrange multiplier for the incompressibility constraint ( $\operatorname{det} \mathbf{F}=1$ ).

We denote

$$
\begin{equation*}
W(\lambda)=\mathcal{W}\left(\lambda^{-2}, \lambda, \lambda\right) \tag{4.5}
\end{equation*}
$$

where $\lambda=r / R>1$. Then, by (3.1),

$$
\begin{equation*}
W(\lambda)=\frac{\mu_{1}}{2 m^{2}}\left(\lambda^{-4 m}+2 \lambda^{2 m}-3\right)+\frac{\mu_{2}}{2 n^{2}}\left(\lambda^{-4 n}+2 \lambda^{2 n}-3\right) . \tag{4.6}
\end{equation*}
$$

By (3.5), since

$$
\begin{align*}
\frac{\partial \mathcal{W}}{\partial \lambda_{1}} & =\frac{\mu_{1}}{m} \lambda_{1}^{2 m-1}+\frac{\mu_{2}}{n} \lambda_{1}^{2 n-1}  \tag{4.7}\\
& =\frac{\mu_{1}}{m} \lambda^{-4 m+2}+\frac{\mu_{2}}{n} \lambda^{-4 n+2}
\end{align*}
$$

and

$$
\begin{align*}
\frac{\partial \mathcal{W}}{\partial \lambda_{2}} & =\frac{\mu_{1}}{m} \lambda_{2}^{2 m-1}+\frac{\mu_{2}}{n} \lambda_{2}^{2 n-1}  \tag{4.8}\\
& =\frac{\mu_{1}}{m} \lambda^{2 m-1}+\frac{\mu_{2}}{n} \lambda^{2 n-1}
\end{align*}
$$

we obtain

$$
\begin{align*}
\frac{\mathrm{d} W}{\mathrm{~d} \lambda} & =\frac{\partial W}{\partial \lambda_{1}} \frac{\partial \lambda_{1}}{\partial \lambda}+\frac{\partial W}{\partial \lambda_{2}} \frac{\partial \lambda_{2}}{\partial \lambda} \\
& =-2 \lambda^{-3} \frac{\partial \mathcal{W}}{\partial \lambda_{1}}+\frac{\partial \mathcal{W}}{\partial \lambda_{2}} \\
& =-\frac{2}{\lambda^{3}}\left(P_{11}+\frac{p}{\lambda_{1}}\right)+2 P_{22}+2 \frac{p}{\lambda_{2}}  \tag{4.9}\\
& =-\frac{2}{\lambda^{3}}\left(P_{11}+\lambda^{2} p\right)+2 P_{22}+\frac{2 p}{\lambda} \\
& =-\frac{2}{\lambda^{3}} P_{11}-\frac{2 p}{\lambda}+2 P_{22}+\frac{2 p}{\lambda} .
\end{align*}
$$

Hence,

$$
\begin{equation*}
\frac{\mathrm{d} W}{\mathrm{~d} \lambda}=-\frac{2 P_{11}}{\lambda^{3}}+2 P_{22} . \tag{4.10}
\end{equation*}
$$

Setting the external pressure, at $R=B$, equal to zero, by (4.3) and (4.10), the
internal pressure expressed in terms of the Cauchy stress, at $R=A$, is equal to

$$
\begin{align*}
T & =-\left.\frac{P_{11}}{\lambda^{2}}\right|_{\lambda=\lambda_{a}} \\
& =-2 \int_{\lambda_{a}}^{\lambda_{b}} \frac{P_{11}}{\lambda^{3}} d \lambda+\int_{\lambda_{a}}^{\lambda_{b}} \frac{\mathrm{~d} P_{11}}{\mathrm{~d} \lambda} \lambda^{-2} \mathrm{~d} \lambda \\
& =-2 \int_{\lambda_{a}}^{\lambda_{b}} \frac{P_{11}}{\lambda^{3}} d \lambda+2 \int_{\lambda_{a}}^{\lambda_{b}} \frac{P_{11}-P_{22}}{1-\lambda^{3}} \mathrm{~d} \lambda \\
& =-2 \int_{\lambda_{a}}^{\lambda_{b}} \frac{P_{11}\left(1-\lambda^{3}\right)+P_{11} \lambda^{3}-P_{22} \lambda^{3}}{\lambda^{3}\left(1-\lambda^{3}\right)} \mathrm{d} \lambda  \tag{4.11}\\
& =-2 \int_{\lambda_{a}}^{\lambda_{b}} \frac{P_{11}-p_{22} \lambda^{3}}{\lambda^{3}\left(1-\lambda^{3}\right)} \mathrm{d} \lambda \\
& =\int_{\lambda_{a}}^{\lambda_{b}}\left(\frac{-2 P_{11}}{\lambda^{3}}+2 p_{22}\right) \frac{\mathrm{d} \lambda}{1-\lambda^{3}} \\
& =\int_{\lambda_{a}}^{\lambda_{b}} \frac{\mathrm{~d} W}{\mathrm{~d} \lambda} \frac{\mathrm{~d} \lambda}{1-\lambda^{3}},
\end{align*}
$$

where $\lambda_{a}=a / A$ and $\lambda_{b}=b / B$ are the stretches ratios for the inner and outer radii, respectively. By the material incompressibility condition, the material volume in the spherical shell is conserved, that is, $4 \pi\left(b^{3}-a^{3}\right)=4 \pi\left(B^{3}-A^{3}\right)$, or equivalently, $a=A \lambda_{a}$ and $b=B \lambda_{b}$. Hence,

$$
\begin{equation*}
\lambda_{b}^{3}=\left(\lambda_{a}^{3}-1\right)\left(\frac{A}{B}\right)^{3}+1 \tag{4.12}
\end{equation*}
$$

Therefore, the internal pressure can be described in terms of the inner stretch ratio, $\lambda_{a}$, only.

For the spherical shell, limit-point instability occurs if there is a change in the monotonicity of $T$. When the spherical shell is thin, so that

$$
\begin{equation*}
0<\epsilon=\frac{B-A}{A} \ll 1, \tag{4.13}
\end{equation*}
$$

the internal pressure, $T$, is a function of inner stretch ratio, $\lambda_{a}$, and it can be approximated as follows

$$
\begin{equation*}
T(\lambda)=\frac{\epsilon}{\lambda^{2}} \frac{\mathrm{~d} W}{\mathrm{~d} \lambda} \tag{4.14}
\end{equation*}
$$

That is,

$$
\begin{equation*}
T(\lambda)=\frac{\epsilon}{\lambda^{2}}\left[\frac{-2 \mu^{(1)}}{m}\left(\lambda^{-4 m-1}-\lambda^{2 m-1}\right)-\frac{4 \mu^{(2)}}{n}\left(\lambda^{-4 n-1}-2 \lambda^{2 n-1}\right)\right] . \tag{4.15}
\end{equation*}
$$

To find the critical value of $\lambda$ where a limit-point of instability occurs, we solve for $\lambda>1$, the equation

$$
\begin{equation*}
\frac{\mathrm{d} T}{\mathrm{~d} \lambda}=0 \tag{4.16}
\end{equation*}
$$

### 4.2 Two concentric spherical shells

We also consider a composite formed from two concentric spheres (see Figure 4.2), with the strain-energy density given by equations (3.18)- (3.20). For the composite sphere, we assume continuity of the displacement across the interface between the two concentric spheres.


Figure 4.2: Schematic of a composite spherical shell made of two concentric homogeneous shells, with undeformed outer radii $A$ and $B$, respectively, showing the reference state (left), and the deformed state, with outer radii $a$ and $b$ of the concentric shells, respectively (right).

We denote

$$
W(\lambda)= \begin{cases}W^{(1)}\left(\lambda^{-2}, \lambda, \lambda\right), & \text { if } C<R<A  \tag{4.17}\\ W^{(2)}\left(\lambda^{-2}, \lambda, \lambda\right), & \text { if } A<R<B\end{cases}
$$

where

$$
\begin{align*}
W^{(1)}(\lambda) & =\frac{\mu^{(1)}}{2 m^{2}}\left(\lambda^{-4 m}+2 \lambda^{2 m}-3\right),  \tag{4.18}\\
W^{(2)}(\lambda) & =\frac{\mu^{(2)}}{2 n^{2}}\left(\lambda^{-4 n}+2 \lambda^{2 n}-3\right) . \tag{4.19}
\end{align*}
$$

In this case, the internal pressure is equal to

$$
\begin{equation*}
T=\int_{\lambda_{c}}^{\lambda_{a}} \frac{\mathrm{~d} W^{(1)}}{\mathrm{d} \lambda} \frac{\mathrm{~d} \lambda}{1-\lambda^{3}}+\int_{\lambda_{a}}^{\lambda_{b}} \frac{\mathrm{~d} W^{(2)}}{\mathrm{d} \lambda} \frac{\mathrm{~d} \lambda}{1-\lambda^{3}}, \tag{4.20}
\end{equation*}
$$

where $\lambda_{c}=c / C, \lambda_{a}=a / A$, and $\lambda_{b}=b / B$.
When the sphere is thin, so that

$$
\begin{equation*}
0<\frac{\epsilon}{2}=\frac{B-A}{A} \approx \frac{A-C}{C} \ll 1, \tag{4.21}
\end{equation*}
$$

the internal pressure can be approximated by

$$
\begin{align*}
T(\lambda) & =\frac{\epsilon}{\lambda^{2}} \frac{\mathrm{~d} W^{(1)}}{\mathrm{d} \lambda}+\frac{\epsilon}{\lambda^{2}} \frac{\mathrm{~d} W^{(2)}}{\mathrm{d} \lambda} \\
& =\frac{\epsilon}{\lambda^{2}}\left[\frac{\mu^{(1)}}{m}\left(-2 \lambda^{-4 m-1}+2 \lambda^{2 m-1}\right)+\frac{\mu^{(2)}}{n}\left(-2 \lambda^{-4 n-1}+2 \lambda^{2 n-1}\right)\right]  \tag{4.22}\\
& =\epsilon\left[\frac{\mu^{(1)}}{m}\left(-\lambda^{-4 m-3}+\lambda^{2 m-3}\right)+\frac{\mu^{(2)}}{n}\left(-\lambda^{-4 n-3}+\lambda^{2 n-3}\right)\right] .
\end{align*}
$$

Thus

$$
\begin{align*}
\frac{\mathrm{d} T}{\mathrm{~d} \lambda}= & \epsilon \frac{\mu^{(1)}}{m}\left[(4 m+3) \lambda^{-4 m-4}+(2 m-3) \lambda^{2 m-4}\right]  \tag{4.2}\\
& +\epsilon \frac{\mu^{(2)}}{n}\left[(4 n+3) \lambda^{-4 n-4}+(2 n-3) \lambda^{2 n-4}\right],
\end{align*}
$$

and equation

$$
\begin{equation*}
\frac{\mathrm{d} T}{\mathrm{~d} \lambda}=0 \tag{4.24}
\end{equation*}
$$

is equivalent to

$$
\begin{align*}
& \frac{\mu^{(1)}}{m}\left[(4 m+3) \lambda^{-4 m-4}+(2 m-3) \lambda^{2 m-4}\right]  \tag{4.25}\\
& +\frac{\mu^{(2)}}{n}\left[(4 n+3) \lambda^{-4 n-4}+(2 n-3) \lambda^{2 n-4}\right]=0
\end{align*}
$$

Next, we specialise to the case when $m=1$ and $n=-1$, and obtain

$$
\begin{equation*}
\frac{\mu^{(1)}}{\mu^{(1)}+\mu^{(2)}}=\frac{\lambda^{8}+5 \lambda^{2}}{\lambda^{8}+5 \lambda^{2}+\lambda^{6}-7} . \tag{4.26}
\end{equation*}
$$

Instability occurs if the equation (4.26) has a solution.
The minimum value of $\mu^{(1)} /\left(\mu^{(1)}+\mu^{(2)}\right)$ such that instability occurs is the minimum of the function

$$
\begin{equation*}
\eta(\lambda)=\frac{\lambda^{8}+5 \lambda^{2}}{\lambda^{8}+5 \lambda^{2}+\lambda^{6}-7}, \quad \lambda>1 \tag{4.27}
\end{equation*}
$$

which is $\eta_{\min }=\min \eta(\lambda)=0.8234$. This minimum was obtained numerically.
Hence, instability occurs if

$$
\begin{equation*}
\frac{\mu^{(1)}}{\mu^{(1)}+\mu^{(2)}} \geq \eta_{\min }=0.8234 \tag{4.28}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
\frac{\mu^{(1)}}{\mu^{(2)}} \geq \frac{\eta_{\min }}{1-\eta_{\min }}=\frac{0.8234}{1-0.8234}=4.6625 . \tag{4.29}
\end{equation*}
$$

So the condition for the existence of limit-point instability is that the outer sphere is softer than the inner sphere.

### 4.3 Stochastic spherical shells

For a spherical shell of stochastic hyperelastic material described by the strainenergy function (3.18) with $m=1$ and $n=-1$ (see also [82]), by (4.29), the probability of stable inflation, such that the internal pressure always increases as
the radial stretch increases, is equal to

$$
\begin{equation*}
P_{1}\left(\mu^{(1)}<4.6625 \mu^{(2)}\right)=\int_{0}^{4.6625 \mu^{(2)}} p_{1}(u) \mathrm{d} u \tag{4.30}
\end{equation*}
$$

and the probability of inflation instability occurring is

$$
\begin{equation*}
P_{2}\left(\mu^{(1)}>4.6625 \mu^{(2)}\right)=1-P_{1}\left(\mu^{(1)}<4.6625 \mu^{(2)}\right)=1-\int_{0}^{4.6625 \mu^{(2)}} p_{1}(u) \mathrm{d} u \tag{4.31}
\end{equation*}
$$

where $p_{1}(u)=g\left(u ; \rho_{1}, \rho_{2}\right)$ is defined by (2.26) if the random variable follows a Gamma distribution, or $p_{1}=f(u ; \underline{u},\|u\|)$ if the random variable follows a normal distribution with mean value $\underline{u}$ and standard deviation $\|u\|$.


Figure 4.3: Probability distributions (4.30)-(4.31) of whether instability can occur or not for a composite with two concentric spherical shells of stochastic material described by (4.17) with $m=1$ and $n=-1$, and the shear modulus $\mu^{(1)}$ following either a Gamma distribution (2.26) with shape and scale parameters $\rho_{1}^{(1)}=405.0214$ and $\rho_{2}^{(1)}=0.0101$, respectively (continuous lines), or a normal distribution with mean value $\underline{\mu}^{(1)}=4.0907$ and standard deviation $\left\|\mu^{(1)}\right\|=0.2302$ (dashed lines). Darker colours represent analytically derived solutions, given by equation (4.30)-(4.31), whereas lighter colour show stochastically generated data. The vertical line at the critical value, $\mu^{(2)}=4.0907 / 4.6625=0.8774$, separates the expected regions based only on the mean value $\underline{\mu}^{(1)}=4.0907$.

The probability distributions given by equations (4.30)-(4.31) are illustrated in Figure 4.3, where blue lines are for $P_{1}$ and red lines for $P_{2}$. For the numerical approximation, ( $0, \underline{\mu}^{(1)} / 2$ ) was divided into 100 steps, then for each value of $\mu^{(2)}$, 100 random values of $\mu^{(1)}$ were numerically generated from a specified Gamma (or normal) distribution and compared with the inequalities defining the two intervals for values of $\mu^{(1)}$.


Figure 4.4: The normalised internal pressure, $T(\lambda)$, for the inflation of spherical shells of hyperelastic materials, defined by (4.17), with $m=1$ and $n=-1$. In this deterministic case, inflation instability occurs if $\mu^{(1)} / \mu^{(2)} \geq 4.6625$.


Figure 4.5: Probability distribution of the normalised internal pressure, $T(\lambda)$, for the inflation of a spherical shells of stochastic hyperelastic material, given by (4.17) with $m=1$ and $n=-1$, when $\mu^{(1)}$ follows a Gamma distribution with $\rho_{1}^{(1)}=405.0214, \rho_{2}^{(1)}=0.0101$ and $\mu^{(2)}$ follows a Gamma distribution with $\rho_{1}^{(2)}=\rho_{1}^{(1)}, \rho_{2}^{(2)}=\rho_{2}^{(1)} / 4$. The dashed black line corresponds to the expected pressure based only on mean parameter values.

For the deterministic case, which is based on the mean value of the shear modulus, $\underline{\mu}^{(1)}=\rho_{1}^{(1)} \rho_{2}^{(1)}=4.0907$, the critical value of $\mu^{(2)}=0.8774$ strictly separates the cases where inflation instability can occur or not. For the stochastic problem, for the same critical value, there is, by definition, exactly $50 \%$ chance of stable inflation (blue solid or dashed line if the shear modulus is Gamma or normal distributed, respectively), and $50 \%$ chance of a limit-point instability (red solid or dashed line). To increase the probability of stable inflation ( $P_{1} \approx 1$ ), one must consider sufficiently large values of $\mu^{(2)}$, above the expected critical value, whereas a limit-point instability has better chance to occur $\left(P_{2} \approx 1\right)$ if $\mu^{(2)}$ is small enough. However, there will always be competition between the two cases.

Figure 4.5 shows the probability distribution of the normalised internal pressure, $T(\lambda)$, for the inflation of a spherical shells of stochastic hyperelastic material, given by (4.17) with $m=1$ and $n=-1$, when $\mu^{(1)}$ follows a Gamma distribution with $\rho_{1}^{(1)}=405.0214, \rho_{2}^{(1)}=0.0101$ and $\mu^{(2)}$ follows a Gamma distribution with $\rho_{1}^{(2)}=\rho_{1}^{(1)}, \rho_{2}^{(2)}=\rho_{2}^{(1)} / 4$. The dashed black line corresponds to the expected pressure based only on mean parameter values.

Similarly to the case of concentric tubes, in Figures (4.4) and (4.5), there is more variability in the responses for smaller values of $\lambda$ and less variability for larger values of $\lambda$. This is the exact opposite behaviour compared to that obtained in [82], where a single homogeneous shell was treated, for which there was less variability in the responses for smaller values of $\lambda$ and more variability for larger values of $\lambda$. The different stochastic behaviours of the two cases is due to the different probability distributions characterising the respective material parameters.

## Chapter 5

## Quasi-equilibrated radial-axial

## motion of stochastic hyperlastic

## tubes

In this chapter, we examine the behaviour under quasi-equilibrated radial motion of two concentric homogeneous tubes, which are continuously attached to each other throughout the deformation, and of a radially inhomogeneous tube of stochastic incompressible hyperelastic material. These tubes are deformed by radially symmetric inflation when subject to either a surface dead load (which is constant in the reference configuration) or an impulse traction (which is kept constant in the current configuration), applied uniformly in the radial direction. For these tubes, we demonstrate that the amplitude and period of the oscillations are characterised by probability distributions, depending on the initial conditions and the probabilistic material properties. Our analysis confirms and extends the analysis for stochastic homogeneous incompressible hyperelastic tubes presented in [83].

### 5.1 Radial-axial motion of stochastic hyperelastic cylindrical tubes

For a circular cylindrical tube, the combined radial and axial motion is described by

$$
\begin{equation*}
r^{2}=c^{2}+\frac{R^{2}-C^{2}}{\alpha}, \quad \theta=\Theta, \quad z=\alpha Z \tag{5.1}
\end{equation*}
$$

where $(R, \Theta, Z)$ and $(r, \theta, z)=(r(t), \theta(t), z(t))$, with $t$ the time variable, are the cylindrical polar coordinates in the reference and current configuration, respectively, such that $C \leq R \leq B, C$ and $B$ are the inner and outer radii in the undeformed state, respectively, $c=c(t)$ and $b=b(t)=\sqrt{c^{2}+\left(B^{2}-C^{2}\right) / \alpha}$ are the inner and outer radius at time $t$, respectively, and $\alpha>0$ is a given constant. When $\alpha=1$, the time-dependent deformation (5.1) simplifies to that studied also in $[17,68,69]$. The case when $\alpha$ is time-dependent was considered in [109].

The radial-axial motion (5.1) of the cylindrical tube is fully determined by the inner radius $c$ at time $t$, which in turn is obtained from the initial conditions. Thus, the acceleration $\ddot{r}$ can be computed in terms of the acceleration $\ddot{c}$ on the inner surface. By the governing equations (5.1), condition (2.20) is valid for $\mathbf{x}=$ $(r, \theta, z)^{T}$, since

$$
\ddot{\mathbf{x}}=\left(\ddot{r}-r \dot{\theta}^{2}\right) \mathbf{e}_{r}+(r \ddot{\theta}+2 \dot{r} \dot{\theta}) \mathbf{e}_{\theta}+\ddot{z} \mathbf{e}_{z}
$$

and

$$
\mathbf{0}=\operatorname{curl} \ddot{\mathbf{x}}=\left(\frac{1}{r} \frac{\partial \ddot{x}_{z}}{\partial \theta}-\frac{\partial \ddot{x}_{\theta}}{\partial z}\right) \mathbf{e}_{r}+\left(\frac{\partial \ddot{x}_{r}}{\partial z}-\frac{\partial \ddot{x}_{z}}{\partial r}\right) \mathbf{e}_{\theta}+\frac{1}{r}\left(\frac{\partial\left(r \ddot{x}_{\theta}\right)}{\partial r}-\frac{\partial \ddot{x}_{r}}{\partial \theta}\right) \mathbf{e}_{z},
$$

where $\ddot{\mathbf{x}}=\left(\ddot{x}_{r}, \ddot{x}_{\theta}, \ddot{x}_{z}\right)^{T}$. Hence, (5.1) describes na quasi-equilibrated motion, such that

$$
\begin{equation*}
-\frac{\partial \xi}{\partial r}=\ddot{r}=\frac{\dot{c}^{2}}{r}+\frac{c \ddot{c}}{r}-\frac{c^{2} \dot{c}^{2}}{r^{3}}, \tag{5.2}
\end{equation*}
$$

where $\xi$ is the acceleration potential satisfying (2.19), i.e., the energy or potential

### 5.1. RADIAL-AXIAL MOTION OF STOCHASTIC HYPERELASTIC CYLINDRICAL TUBES

applied in the system to increase the energy and accelerate the radial speed.
Integrating (5.2) gives [133, p. 215]

$$
\begin{equation*}
-\xi=\dot{c}^{2} \log r+c \ddot{c} \log r+\frac{c^{2} \dot{c}^{2}}{2 r^{2}}=\dot{r}^{2} \log r+r \ddot{r} \log r+\frac{1}{2} \dot{r}^{2} . \tag{5.3}
\end{equation*}
$$

The deformation gradient of (5.1), with respect to the polar coordinates $(R, \Theta, Z)$, is equal to

$$
\begin{equation*}
\mathbf{F}=\operatorname{diag}\left(\frac{R}{\alpha r}, \frac{r}{R}, \alpha\right), \tag{5.4}
\end{equation*}
$$

the Cauchy-Green deformation tensor is

$$
\begin{equation*}
\mathbf{B}=\mathbf{F}^{2}=\operatorname{diag}\left(\frac{R^{2}}{\alpha^{2} r^{2}}, \frac{r^{2}}{R^{2}}, \alpha^{2}\right), \tag{5.5}
\end{equation*}
$$

and the principal invariants take the form

$$
\begin{align*}
& I_{1}=\operatorname{tr}(\mathbf{B})=\frac{R^{2}}{\alpha^{2} r^{2}}+\frac{r^{2}}{R^{2}}+\alpha^{2} \\
& I_{2}=\frac{1}{2}\left[(\operatorname{tr} \mathbf{B})^{2}-\operatorname{tr}\left(\mathbf{B}^{2}\right)\right]=\frac{\alpha^{2} r^{2}}{R^{2}}+\frac{R^{2}}{r^{2}}+\frac{1}{\alpha^{2}},  \tag{5.6}\\
& I_{3}=\operatorname{det} \mathbf{B}=1 .
\end{align*}
$$

The principal components of the equilibrium Cauchy stress tensor at time $t$ are

$$
\begin{align*}
& T_{r r}^{(0)}=-p^{(0)}+\beta_{1} \frac{R^{2}}{\alpha^{2} r^{2}}+\beta_{-1} \frac{\alpha^{2} r^{2}}{R^{2}}, \\
& T_{\theta \theta}^{(0)}=T_{r r}^{(0)}+\left(\beta_{1}-\beta_{-1} \alpha^{2}\right)\left(\frac{r^{2}}{R^{2}}-\frac{R^{2}}{\alpha^{2} r^{2}}\right),  \tag{5.7}\\
& T_{z z}^{(0)}=T_{r r}^{(0)}+\left(\beta_{1}-\beta_{-1} \frac{r^{2}}{R^{2}}\right)\left(\alpha^{2}-\frac{R^{2}}{\alpha^{2} r^{2}}\right),
\end{align*}
$$

where $p^{(0)}$ is the Lagrangian multiplier for the incompressibility constraint ( $I_{3}=$ $1)$, and

$$
\begin{equation*}
\beta_{1}=2 \frac{\partial W}{\partial I_{1}}, \quad \beta_{-1}=-2 \frac{\partial W}{\partial I_{2}} \tag{5.8}
\end{equation*}
$$

are the nonlinear material parameters, with $I_{1}$ and $I_{2}$ given by (5.6). For a cylin-
drical tube made of a homogeneous incompressible neo-Hookean material, $\beta_{-1}=0$.
As the stress components depend only on the radius $r$, the system of equilibrium equations reduces to

$$
\begin{equation*}
\frac{\partial T_{r r}^{(0)}}{\partial r}=\frac{T_{\theta \theta}^{(0)}-T_{r r}^{(0)}}{r} . \tag{5.9}
\end{equation*}
$$

Hence, by (5.7) and (5.9), the radial Cauchy stress for the equilibrium state at time $t$ is equal to

$$
\begin{equation*}
T_{r r}^{(0)}(r, t)=\int\left(\beta_{1}-\beta_{-1} \alpha^{2}\right)\left(\frac{r^{2}}{R^{2}}-\frac{R^{2}}{\alpha^{2} r^{2}}\right) \frac{\mathrm{d} r}{r}+\psi(t), \tag{5.10}
\end{equation*}
$$

where $\psi=\psi(t)$ is an arbitrary function of time. Substitution of (5.3) and (5.10) into (2.21) then gives the principal Cauchy stress components at time $t$ as follows,

$$
\begin{align*}
& T_{r r}(r, t)=\rho\left(a \ddot{a} \log r+\dot{a}^{2} \log r+\frac{a^{2} \dot{a}^{2}}{2 r^{2}}\right)+\int\left(\beta_{1}-\beta_{-1} \alpha^{2}\right)\left(\frac{r^{2}}{R^{2}}-\frac{R^{2}}{\alpha^{2} r^{2}}\right) \frac{\mathrm{d} r}{r}+\psi(t), \\
& T_{\theta \theta}(r, t)=T_{r r}(r, t)+\left(\beta_{1}-\beta_{-1} \alpha^{2}\right)\left(\frac{r^{2}}{R^{2}}-\frac{R^{2}}{\alpha^{2} r^{2}}\right) \\
& T_{z z}(r, t)=T_{r r}(r, t)+\left(\beta_{1}-\beta_{-1} \frac{r^{2}}{R^{2}}\right)\left(\alpha^{2}-\frac{R^{2}}{\alpha^{2} r^{2}}\right) . \tag{5.11}
\end{align*}
$$

### 5.2 Oscillatory motion of a composite with two concentric homogeneous cylindrical tubes

To study explicitly the behaviour under the quasi-equilibrated radial motion of a composite formed from two concentric homogeneous cylindrical tubes (see Figure 3.2), we focus on composite tubes with two stochastic neo-Hookean phases. In this case, we define the following strain-energy function,

$$
\mathcal{W}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)=\left\{\begin{array}{cl}
\frac{\mu^{(1)}}{2}\left(\lambda_{1}^{2}+\lambda_{2}^{2}+\lambda_{3}^{2}-3\right), & C<R<A  \tag{5.12}\\
\frac{\mu^{(2)}}{2}\left(\lambda_{1}^{2}+\lambda_{2}^{2}+\lambda_{3}^{2}-3\right), & A<R<B
\end{array}\right.
$$

### 5.2. OSCILLATORY MOTION OF A COMPOSITE WITH TWO CONCENTRIC HOMOGENEOUS CYLINDRICAL TUBES

where $C<R<A$ and $A<R<B$ denote the radii of the inner and outer tube in the reference configuration, respectively, and the corresponding shear moduli $\mu^{(1)}$ and $\mu^{(2)}$ are (spatially-independent) random variables characterised by the Gamma distributions $g\left(u ; \rho_{1}^{(1)}, \rho_{2}^{(1)}\right)$ and $g\left(u ; \rho_{1}^{(2)}, \rho_{2}^{(2)}\right)$, defined by (2.26).

For the composite tube, we assume continuity of the displacement across the interface between the two concentric tubes. We denote the radial pressures acting on the curvilinear surfaces $r=c(t)$ and $r=b(t)$ at time $t$ as $T_{1}(t)$ and $T_{2}(t)$, respectively, and impose the continuity condition for the stress components across their interface, $r=a(t)$. Evaluating $T_{1}(t)=-T_{r r}(c, t)$ and $T_{2}(t)=-T_{r r}(b, t)$ at $r=c$ and $r=b$, respectively, and subtracting the results, gives

$$
\begin{align*}
T_{1}(t)-T_{2}(t)= & \frac{\rho}{2}\left[c \ddot{c} \log \frac{b^{2}}{c^{2}}+\dot{c}^{2} \log \frac{b^{2}}{c^{2}}+\dot{c}^{2}\left(\frac{c^{2}}{b^{2}}-1\right)\right] \\
& +\int_{c}^{a} \mu^{(1)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{2}}{\alpha^{2} r^{2}}\right) \frac{\mathrm{d} r}{r}+\int_{a}^{b} \mu^{(2)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{2}}{\alpha^{2} r^{2}}\right) \frac{\mathrm{d} r}{r} . \\
= & \frac{\rho C^{2}}{2}\left[\left(\frac{c}{C} \frac{\ddot{c}}{C}+\frac{\dot{c}^{2}}{C^{2}}\right) \log \frac{b^{2}}{c^{2}}+\frac{\dot{c}^{2}}{C^{2}}\left(\frac{c^{2}}{b^{2}}-1\right)\right]  \tag{5.13}\\
& +\int_{c}^{a} \mu^{(1)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{2}}{\alpha^{2} r^{2}}\right) \frac{\mathrm{d} r}{r}+\int_{a}^{b} \mu^{(2)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{2}}{\alpha^{2} r^{2}}\right) \frac{\mathrm{d} r}{r} .
\end{align*}
$$

We set the notation

$$
\begin{equation*}
u=\frac{r^{2}}{R^{2}}=\frac{r^{2}}{\alpha\left(r^{2}-c^{2}\right)+C^{2}}, \quad x=\frac{c}{C}, \quad \gamma=\frac{B^{2}}{C^{2}}-1, \tag{5.14}
\end{equation*}
$$

and rewrite

$$
\begin{align*}
\left(\frac{c}{C} \frac{\ddot{c}}{C}+\frac{\dot{c}^{2}}{C^{2}}\right) \log \frac{b^{2}}{c^{2}}+\frac{\dot{c}^{2}}{C^{2}}\left(\frac{c^{2}}{b^{2}}-1\right) & =\left(\ddot{x} x+\dot{x}^{2}\right) \log \left(1+\frac{\gamma}{\alpha x^{2}}\right)-\dot{x}^{2} \frac{\frac{\gamma}{\alpha x^{2}}}{1+\frac{\gamma}{\alpha x^{2}}} \\
& =\frac{1}{2 x} \frac{\mathrm{~d}}{\mathrm{~d} x}\left[\dot{x}^{2} x^{2} \log \left(1+\frac{\gamma}{\alpha x^{2}}\right)\right] . \tag{5.15}
\end{align*}
$$

By (5.1) and (5.14), we obtain

$$
\begin{align*}
& \int_{c}^{a} \mu^{(1)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{2}}{\alpha^{2} r^{2}}\right) \frac{\mathrm{d} r}{r}+\int_{a}^{b} \mu^{(2)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{2}}{\alpha^{2} r^{2}}\right) \frac{\mathrm{d} r}{r} \\
= & \int_{c}^{a} \mu^{(1)}\left[\frac{r^{2}}{\alpha\left(r^{2}-c^{2}\right)+C^{2}}-\frac{\alpha\left(r^{2}-c^{2}\right)+C^{2}}{\alpha^{2} r^{2}}\right] \frac{\mathrm{d} r}{r} \\
& +\int_{a}^{b} \mu^{(2)}\left[\frac{r^{2}}{\alpha\left(r^{2}-c^{2}\right)+C^{2}}-\frac{\alpha\left(r^{2}-c^{2}\right)+C^{2}}{\alpha^{2} r^{2}}\right] \frac{\mathrm{d} r}{r} \\
= & -\frac{1}{2} \int_{c^{2} / C^{2}}^{a^{2} / A^{2}} \mu^{(1)} \frac{1+\alpha u}{\alpha^{2} u^{2}} \mathrm{~d} u-\frac{1}{2} \int_{a^{2} / A^{2}}^{b^{2} / B^{2}} \mu^{(2)} \frac{1+\alpha u}{\alpha^{2} u^{2}} \mathrm{~d} u  \tag{5.16}\\
= & -\frac{1}{2} \int_{x^{2}}^{x^{2} \frac{C^{2}}{A^{2}}+\frac{1}{\alpha}\left(1-\frac{C^{2}}{A^{2}}\right)} \mu^{(1)} \frac{1+\alpha u}{\alpha^{2} u^{2}} \mathrm{~d} u-\frac{1}{2} \int_{x^{2} \frac{C^{2}}{A^{2}}+\frac{1}{\alpha}\left(1-\frac{C^{2}}{A^{2}}\right)}^{\frac{\gamma}{\alpha}} \\
= & \frac{1}{2} \int_{x^{2} \frac{C^{2}}{A^{2}}+\frac{1}{\alpha}\left(1-\frac{C^{2}}{A^{2}}\right)}^{x^{2}} \mu^{(1)} \frac{1+\alpha u}{\alpha^{2} u^{2}} \mathrm{~d} u+\frac{1}{2} \int_{\frac{x^{2}+\frac{\gamma}{\alpha}}{x^{2}} \frac{C^{2}}{A^{2}}+\frac{1}{\alpha}\left(1-\frac{C^{2}}{A^{2}}\right)}^{\mathrm{d}} u \\
\mu^{(2)} & \frac{1+\alpha u}{\alpha^{2} u^{2}} \mathrm{~d} u .
\end{align*}
$$

In the above calculations, we used the following relations,

$$
\begin{align*}
& r=\left[\frac{u\left(C^{2}-\alpha c^{2}\right)}{1-\alpha u}\right]^{1 / 2}  \tag{5.17}\\
& \frac{\mathrm{~d} r}{\mathrm{~d} u}=\frac{C^{2}-\alpha c^{2}}{2(1-\alpha u)^{2}}\left[\frac{u\left(C^{2}-\alpha c^{2}\right)}{1-\alpha u}\right]^{-1 / 2}=\frac{r}{2 u(1-\alpha u)}  \tag{5.18}\\
& \left(u-\frac{1}{\alpha^{2} u}\right) \frac{1}{2 u(1-\alpha u)}=\frac{\alpha^{2} u^{2}-1}{2 \alpha^{2} u^{2}(1-\alpha u)}=-\frac{1+\alpha u}{2 \alpha^{2} u^{2}} . \tag{5.19}
\end{align*}
$$

We now express equation (5.13) equivalently as follows,

$$
\begin{align*}
2 x \frac{T_{1}(t)-T_{2}(t)}{\rho C^{2}} & =\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} x}\left[\dot{x}^{2} x^{2} \log \left(1+\frac{\gamma}{\alpha x^{2}}\right)\right] \\
& +\frac{x}{\rho C^{2}} \int_{x^{2} \frac{C^{2}}{A^{2}}+\frac{1}{\alpha}\left(1-\frac{C^{2}}{A^{2}}\right)}^{x^{2}} \mu^{(1)} \frac{1+\alpha u}{\alpha^{2} u^{2}} \mathrm{~d} u  \tag{5.20}\\
& +\frac{x}{\rho C^{2}} \int_{\frac{x^{2}+\frac{\alpha}{\alpha}}{1+\gamma}}^{x^{2} C^{2}} \frac{1}{\alpha}\left(1-\frac{C^{2}}{A^{2}}\right)
\end{align*} \mu^{(2)} \frac{1+\alpha u}{\alpha^{2} u^{2}} \mathrm{~d} u .
$$

In the static case, where $\dot{c}=0$ and $\ddot{c}=0$, (5.13) becomes

$$
\begin{equation*}
T_{1}(t)-T_{2}(t)=\int_{c}^{a} \mu^{(1)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{2}}{\alpha^{2} r^{2}}\right) \frac{\mathrm{d} r}{r}+\int_{a}^{b} \mu^{(2)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{2}}{\alpha^{2} r^{2}}\right) \frac{\mathrm{d} r}{r} \tag{5.21}
\end{equation*}
$$

and (5.20) reduces to

$$
\begin{align*}
2 \frac{T_{1}(t)-T_{2}(t)}{\rho C^{2}} & =\frac{1}{\rho C^{2}} \int_{x^{2} \frac{C^{2}}{A^{2}}+\frac{1}{\alpha}\left(1-\frac{C^{2}}{A^{2}}\right)}^{x^{2}} \mu^{(1)} \frac{1+\alpha u}{\alpha^{2} u^{2}} \mathrm{~d} u  \tag{5.22}\\
& +\frac{1}{\rho C^{2}} \int_{\frac{x^{2}+\frac{\gamma}{\alpha}}{1+\gamma}}^{x^{2} \frac{C^{2}}{\alpha}+\frac{1}{\alpha}\left(1-\frac{C^{2}}{A^{2}}\right)} \mu^{(2)} \frac{1+\alpha u}{\alpha^{2} u^{2}} \mathrm{~d} u .
\end{align*}
$$

For the dynamic tube, we define

$$
\begin{align*}
G(x, \gamma) & =\frac{1}{\rho C^{2}} \int_{\frac{1}{\sqrt{\alpha}}}^{x} \zeta\left[\int_{\zeta^{2} \frac{C^{2}}{A^{2}}+\frac{1}{\alpha}\left(1-\frac{C^{2}}{A^{2}}\right)}^{\zeta^{2}} \mu^{(1)} \frac{1+\alpha u}{\alpha^{2} u^{2}} \mathrm{~d} u\right] \mathrm{d} \zeta \\
& +\frac{1}{\rho C^{2}} \int_{\frac{1}{\sqrt{\alpha}}}^{x} \zeta\left[\int_{\frac{\zeta^{2}+\frac{\gamma}{\alpha}}{\zeta^{2} \frac{C^{2}}{\alpha}}+\frac{1}{\alpha}\left(1-\frac{C^{2}}{A^{2}}\right)}^{l} \mu^{(2)} \frac{1+\alpha u}{\alpha^{2} u^{2}} \mathrm{~d} u\right] \mathrm{d} \zeta . \tag{5.23}
\end{align*}
$$

This function will be used to establish whether the radial motion of the tube is oscillatory or not.

### 5.2.1 Composite with two concentric homogeneous tubes subject to impulse traction

We set the pressure impulse (suddenly applied pressure difference)

$$
2 \alpha \frac{T_{1}(t)-T_{2}(t)}{\rho C^{2}}= \begin{cases}0 & \text { if } t \leq 0  \tag{5.24}\\ p_{0} & \text { if } t>0\end{cases}
$$

with $p_{0}$ constant in time. Integrating (5.20) implies

$$
\begin{equation*}
\frac{1}{2} \dot{x}^{2} x^{2} \log \left(1+\frac{\gamma}{\alpha x^{2}}\right)+G(x, \gamma)=\frac{p_{0}}{2 \alpha}\left(x^{2}-\frac{1}{\alpha}\right)+C_{0} \tag{5.25}
\end{equation*}
$$

where $G(x, \gamma)$ is defined by (5.23) and

$$
\begin{equation*}
C_{0}=\frac{1}{2} \dot{x}_{0}^{2} x_{0}^{2} \log \left(1+\frac{\gamma}{\alpha x_{0}^{2}}\right)+G\left(x_{0}, \gamma\right)-\frac{p_{0}}{2 \alpha}\left(x_{0}^{2}-\frac{1}{\alpha}\right) \tag{5.26}
\end{equation*}
$$

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with the initial conditions $x(0)=x_{0}$ and $\dot{x}(0)=\dot{x}_{0}$. By (5.25), the velocity is equal to

$$
\begin{equation*}
\dot{x}= \pm \sqrt{\frac{\frac{p_{0}}{\alpha}\left(x^{2}-\frac{1}{\alpha}\right)+2 C_{0}-2 G(x, \gamma)}{x^{2} \log \left(1+\frac{\gamma}{\alpha x^{2}}\right)}} . \tag{5.27}
\end{equation*}
$$

This nonlinear elastic system is analogous to the motion of a point mass with energy

$$
\begin{equation*}
E=\frac{1}{2} m(x) \dot{x}^{2}+V(x) \tag{5.28}
\end{equation*}
$$

where the energy is $E=C_{0}$, the potential is given by $V(x)=G(x, \gamma)-\frac{p_{0}}{2 \alpha}\left(x^{2}-\frac{1}{\alpha}\right)$ and the position-dependent mass is $m(x)=x^{2} \log \left(1+\frac{\gamma}{\alpha x^{2}}\right)$. The solutions of interest are either static or periodic solutions.

The radial motion is periodic if and only if the frequency equation

$$
\begin{equation*}
G(x, \gamma)=\frac{p_{0}}{2 \alpha}\left(x^{2}-\frac{1}{\alpha}\right)+C_{0} \tag{5.29}
\end{equation*}
$$

has exactly two distinct solutions, representing the amplitudes of the oscillation, $x=x_{1}$ and $x=x_{2}$, such that $0<x_{1}<x_{2}<\infty$. Then, by (5.14), the minimum and maximum radii of the inner surface in the oscillation are equal to $x_{1} C$ and $x_{2} C$, respectively, and by (5.27), the period of oscillation is equal to

$$
\begin{equation*}
T=2\left|\int_{x_{1}}^{x_{2}} \frac{\mathrm{~d} x}{\dot{x}}\right|=2\left|\int_{x_{1}}^{x_{2}} \sqrt{\frac{x^{2} \log \left(1+\frac{\gamma}{\alpha x^{2}}\right)}{\frac{p_{0}}{\alpha}\left(x^{2}-\frac{1}{\alpha}\right)+2 C_{0}-2 G(x, \gamma)}} \mathrm{d} x\right| . \tag{5.30}
\end{equation*}
$$

For the stochastic composite tube, the amplitude and period of the oscillation are random variables described by probability distributions.

To examine $G(x, \gamma)$ defined by (5.23), we rewrite this function equivalently as

$$
\begin{equation*}
G(x, \gamma)=G_{1}(x, \gamma)+G_{2}(x, \gamma) \tag{5.31}
\end{equation*}
$$

where

$$
\begin{equation*}
G_{1}(x, \gamma)=\frac{1}{\rho C^{2}} \int_{\frac{1}{\sqrt{\alpha}}}^{x} \zeta\left[\int_{\zeta^{2} \frac{C^{2}}{A^{2}}+\frac{1}{\alpha}\left(1-\frac{C^{2}}{A^{2}}\right)}^{\zeta^{2}} \mu^{(1)} \frac{1+\alpha u}{\alpha^{2} u^{2}} \mathrm{~d} u\right] \mathrm{d} \zeta \tag{5.32}
\end{equation*}
$$

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and

$$
\begin{equation*}
G_{2}(x, \gamma)=\frac{1}{\rho C^{2}} \int_{\frac{1}{\sqrt{\alpha}}}^{x} \zeta\left[\int_{\zeta^{2}}^{\zeta^{2} \frac{C^{2}}{A^{2}}+\frac{1}{\alpha}\left(1-\frac{C^{2}}{A^{2}}\right)} \mu^{(2)} \frac{1+\alpha u}{\alpha^{2} u^{2}} \mathrm{~d} u+\int_{\frac{\zeta^{2}+\frac{\gamma}{\alpha}}{1+\gamma}}^{\zeta^{2}} \mu^{(2)} \frac{1+\alpha u}{\alpha^{2} u^{2}} \mathrm{~d} u\right] \mathrm{d} \zeta \tag{5.33}
\end{equation*}
$$

Proceeding as in [83], we obtain

$$
\begin{equation*}
G_{1}(x, \gamma)=\frac{\mu^{(1)}}{2 \alpha \rho C^{2}}\left(\frac{1}{\alpha}-x^{2}\right) \log \left[\frac{C^{2}}{A^{2}}+\frac{1}{\alpha x^{2}}\left(1-\frac{C^{2}}{A^{2}}\right)\right] \tag{5.34}
\end{equation*}
$$

and

$$
\begin{equation*}
G_{2}(x, \gamma)=\frac{\mu^{(2)}}{2 \alpha \rho C^{2}}\left(x^{2}-\frac{1}{\alpha}\right)\left\{\log \left[\frac{C^{2}}{A^{2}}+\frac{1}{\alpha x^{2}}\left(1-\frac{C^{2}}{A^{2}}\right)\right]+\log \frac{1+\gamma}{1+\frac{\gamma}{\alpha x^{2}}}\right\} . \tag{5.35}
\end{equation*}
$$

By (5.34) and (5.35), the function $G(x, \gamma)$ defined by (5.31) takes the form

$$
\begin{align*}
G(x, \gamma) & =\frac{1}{2 \alpha \rho C^{2}}\left(x^{2}-\frac{1}{\alpha}\right)\left\{\left(\mu^{(2)}-\mu^{(1)}\right) \log \left[\frac{C^{2}}{A^{2}}+\frac{1}{\alpha x^{2}}\left(1-\frac{C^{2}}{A^{2}}\right)\right]\right.  \tag{5.36}\\
& \left.+\mu^{(2)} \log \frac{1+\gamma}{1+\frac{\gamma}{\alpha x^{2}}}\right\} .
\end{align*}
$$

This function is monotonically decreasing for $0<x<1 / \sqrt{\alpha}$ and increasing for $x>1 / \sqrt{\alpha}$. In particular, when $\mu^{(1)}=\mu^{(2)}$, the function corresponding to the homogeneous tube is recovered [83].

Assuming that the shear moduli $\mu^{(1)}$ and $\mu^{(2)}$ have a lower bound

$$
\begin{equation*}
\mu^{(1)}>\eta, \quad \mu^{(2)}>\eta, \tag{5.37}
\end{equation*}
$$

for some constant $\eta>0$, it follows that

$$
\begin{equation*}
\lim _{x \rightarrow 0} G(x, \gamma)=\lim _{x \rightarrow \infty} G(x, \gamma)=\infty \tag{5.38}
\end{equation*}
$$

We examine the following two cases:

### 5.2. OSCILLATORY MOTION OF A COMPOSITE WITH TWO CONCENTRIC HOMOGENEOUS CYLINDRICAL TUBES

(i) If $p_{0}=0$ and $C_{0}>0$, then equation (5.29) has exactly two solutions, $x=x_{1}$ and $x=x_{2}$, satisfying $0<x_{1}<1 / \sqrt{\alpha}<x_{2}<\infty$, for any positive constant $C_{0}$. Note that these oscillations are not 'free' in general, since, by (5.11), if $T_{r r}(r, t)=0$ at $r=c$ and $r=b$, so that $T_{1}(t)=T_{2}(t)=0$, then $T_{\theta \theta}(r, t) \neq 0$ and $T_{z z}(r, t) \neq 0$ at $r=c$ and $r=b$, unless $\alpha \rightarrow 1$ and $r^{2} / R^{2} \rightarrow 1$ [109]. In Figure 5.1, we represent the stochastic function $G(x, \gamma)$, defined by (5.36), intersecting the line $C_{0}=7$, and the associated velocity, given by (5.27), when $\alpha=1, \rho=1$, and the shear modulus of the inner phase, $\mu^{(1)}$, follows a Gamma distribution with $\rho_{1}^{(1)}=405$ and $\rho_{2}^{(1)}=4.05 / \rho_{1}^{(1)}=0.01$, while the shear modulus of the outer phase, $\mu^{(2)}$, is drawn from a Gamma distribution with $\rho_{1}^{(2)}=405$ and $\rho_{2}^{(2)}=4.2 / \rho_{1}^{(2)}$.
(ii) When $p_{0} \neq 0$ and $C_{0} \geq 0$, substitution of (5.36) in (5.29) gives

$$
\begin{equation*}
p_{0}=\frac{2 \alpha\left(G-C_{0}\right)}{x^{2}-\frac{1}{\alpha}} \tag{5.39}
\end{equation*}
$$

The right-hand side of the above equation is a monotonically increasing function of $x$, implying that there exists a unique positive $x$ satisfying (5.39) if and only if the following condition holds,

$$
\begin{equation*}
\lim _{x \rightarrow 0} \frac{2 \alpha\left(G-C_{0}\right)}{x^{2}-\frac{1}{\alpha}}<p_{0}<\lim _{x \rightarrow \infty} \frac{2 \alpha\left(G-C_{0}\right)}{x^{2}-\frac{1}{\alpha}} . \tag{5.40}
\end{equation*}
$$

By (5.14) and (5.40), the necessary and sufficient condition that oscillatory motions occur is that

$$
\begin{equation*}
-\infty<p_{0}<\frac{\mu^{(1)}}{\rho C^{2}} \log \frac{A^{2}}{C^{2}}+\frac{\mu^{(2)}}{\rho C^{2}} \log \frac{B^{2}}{A^{2}} \tag{5.41}
\end{equation*}
$$

where $\mu^{(1)}$ and $\mu^{(2)}$ are described by the Gamma probability density functions $g\left(u ; \rho_{1}^{(1)}, \rho_{2}^{(1)}\right)$ and $g\left(u ; \rho_{1}^{(2)}, \rho_{2}^{(2)}\right)$, respectively. After rescaling, $\mu^{(1)} \log \frac{A^{2}}{C^{2}}$ follows the Gamma distribution with shape parameter $\rho_{1}^{(1)}$ and scale parameter $\rho_{2}^{(1)} \log \frac{A^{2}}{C^{2}}$ and $\mu^{(2)} \log \frac{B^{2}}{A^{2}}$ follows the Gamma distribution with shape parameter $\rho_{1}^{(2)}$ and


Figure 5.1: The function $G(x, \gamma)$, defined by (5.36), intersecting the (dashed red) line $C_{0}=7$ (left), and the associated velocity, given by (5.27) (right), for a dynamic composite tube with two concentric stochastic neo-Hookean phases, with inner radii $A=1$ and $C=1 / 2$, respectively, assuming that $\alpha=1, \rho=1$, and $\mu^{(1)}$ follows a Gamma distribution with $\rho_{1}^{(1)}=405$ and $\rho_{2}^{(1)}=4.05 / \rho_{1}^{(1)}=0.01$, while $\mu^{(2)}$ is drawn from a Gamma distribution with $\rho_{1}^{(2)}=405$ and $\rho_{2}^{(2)}=4.2 / \rho_{1}^{(2)}$. The dashed black lines correspond to the expected values based only on mean values, $\underline{\mu}^{(1)}=4.05$ and $\underline{\mu}^{(2)}=4.2$. Each distribution was calculated from the average of 1000 stochastic simulations.


Figure 5.2: The function $G(x, \gamma)$, defined by (5.36), intersecting the (dashed red) line $\frac{p_{0}}{2 \alpha}\left(x^{2}-\frac{1}{\alpha}\right)+C_{0}$ when $p_{0}=1$ and $C_{0}=2$ (left), and the associated velocity, given by (5.27) (right), for a dynamic composite tube with two concentric stochastic neo-Hookean phases, with inner radii $A=1$ and $C=1 / 2$, respectively, under impulse traction, assuming that $\alpha=1, \rho=1$, and $\mu^{(1)}$ follows a Gamma distribution with $\rho_{1}^{(1)}=405$ and $\rho_{2}^{(1)}=4.05 / \rho_{1}^{(1)}=0.01$, while $\mu^{(2)}$ is drawn from a Gamma distribution with $\rho_{1}^{(2)}=405$ and $\rho_{2}^{(2)}=4.2 / \rho_{1}^{(2)}$. The dashed black lines correspond to the expected values based only on mean values, $\underline{\mu}^{(1)}=4.05$ and $\underline{\mu}^{(2)}=4.2$. Each distribution was calculated from the average of 1000 stochastic simulations.
scale parameter $\rho_{2}^{(2)} \log \frac{B^{2}}{A^{2}}$. Then, $\mu^{(1)} \log \frac{A^{2}}{C^{2}}+\mu^{(2)} \log \frac{B^{2}}{A^{2}}$ is a random variable characterised by the sum of the two rescaled Gamma distributions [78, 93] (see Appendix B). An example is shown in Figure 5.2, where $p_{0}=1$ and $C_{0}=2$, while the geometric and material parameters for the composite tube are as in the previous case.

Thin-walled tube. In particular, when the tube wall is thin, such that $0<$ $\gamma \ll 1$ and $\alpha=1$, if we assume that $A^{2} / C^{2}=B^{2} / A^{2}=\gamma / 2+1$, then (5.36) becomes

$$
\begin{equation*}
G(x, \gamma)=\frac{\mu^{(1)}+\mu^{(2)}}{4 \rho C^{2}}\left(x^{2}-1\right) \log \frac{1+\gamma}{1+\frac{\gamma}{x^{2}}} . \tag{5.42}
\end{equation*}
$$

In this case, the problem reduces to that of a thin-walled homogeneous cylindrical tube with shear modulus $\left(\mu^{(1)}+\mu^{(2)}\right) / 2$ [83].

### 5.2.2 Composite with two concentric homogeneous tubes subject to dead-load traction

We now assume that the outer circular surface of the composite tube is free, such that $T_{2}(t)=0$, while the inner surface is subject to a dead-load pressure $P_{1}(t)$ satisfying

$$
2 \frac{P_{1}(t)}{\rho C^{2}}=2 \alpha x \frac{T_{1}(t)}{\rho C^{2}}=\left\{\begin{align*}
0 & \text { if } t \leq 0  \tag{5.43}\\
p_{0} & \text { if } t>0
\end{align*}\right.
$$

with $p_{0}$ constant in time. Integrating (5.20) implies

$$
\begin{equation*}
\frac{1}{2} \dot{x}^{2} x^{2} \log \left(1+\frac{\gamma}{\alpha x^{2}}\right)+G(x, \gamma)=\frac{p_{0}}{\alpha}\left(x-\frac{1}{\sqrt{\alpha}}\right)+C_{0} \tag{5.44}
\end{equation*}
$$

where $G(x, \gamma)$ is defined by (5.23) and

$$
\begin{equation*}
C_{0}=\frac{1}{2} \dot{x}_{0}^{2} x_{0}^{2} \log \left(1+\frac{\gamma}{\alpha x_{0}^{2}}\right)+G\left(x_{0}, \gamma\right)-\frac{p_{0}}{\alpha}\left(x_{0}-\frac{1}{\sqrt{\alpha}}\right), \tag{5.45}
\end{equation*}
$$

with the initial conditions $x(0)=x_{0}$ and $\dot{x}(0)=\dot{x}_{0}$. By (5.25), the velocity is

$$
\begin{equation*}
\dot{x}= \pm \sqrt{\frac{\frac{2 p_{0}}{\alpha}\left(x-\frac{1}{\sqrt{\alpha}}\right)+2 C_{0}-2 G(x, \gamma)}{x^{2} \log \left(1+\frac{\gamma}{\alpha x^{2}}\right)}} \tag{5.46}
\end{equation*}
$$

The radial motion is periodic if and only if the frequency equation

$$
\begin{equation*}
G(x, \gamma)=\frac{p_{0}}{\alpha}\left(x-\frac{1}{\sqrt{\alpha}}\right)+C_{0} \tag{5.47}
\end{equation*}
$$

has exactly two distinct solutions, representing the amplitudes of the oscillation, $x=x_{1}$ and $x=x_{2}$, such that $0<x_{1}<x_{2}<\infty$. By (5.14), the minimum and maximum radii of the inner surface in the oscillation are equal to $x_{1} C$ and $x_{2} C$, respectively, and by (5.27), the period of oscillation is equal to

$$
\begin{equation*}
T=2\left|\int_{x_{1}}^{x_{2}} \frac{\mathrm{~d} x}{\dot{x}}\right|=2\left|\int_{x_{1}}^{x_{2}} \sqrt{\frac{x^{2} \log \left(1+\frac{\gamma}{\alpha x^{2}}\right)}{\frac{2 p_{0}}{\alpha}\left(x-\frac{1}{\sqrt{\alpha}}\right)+2 C_{0}-2 G(x, \gamma)}} \mathrm{d} x\right| \tag{5.48}
\end{equation*}
$$

For the stochastic composite tube, the amplitude and period of the oscillation are random variables described by probability distributions.

The case with $p_{0}=0$ is similar to that when an impulse traction was assumed. For $p_{0} \neq 0$ and $C_{0} \geq 0$, substitution of (5.36) in (5.47) implies

$$
\begin{equation*}
p_{0}=\frac{\alpha\left(G-C_{0}\right)}{x-\frac{1}{\sqrt{\alpha}}} . \tag{5.49}
\end{equation*}
$$

The right-hand side of the above equation is a monotonically increasing function of $x$, implying that there exists a unique positive $x$ satisfying (5.49) if and only if the following condition holds,

$$
\begin{equation*}
-\infty=\lim _{x \rightarrow 0} \frac{\alpha\left(G-C_{0}\right)}{x-\frac{1}{\sqrt{\alpha}}}<p_{0}<\lim _{x \rightarrow \infty} \frac{\alpha\left(G-C_{0}\right)}{x-\frac{1}{\sqrt{\alpha}}}=\infty \tag{5.50}
\end{equation*}
$$

An example is shown in Figure 5.3, where $p_{0}=5$ and $C_{0}=0$, and the geometric


Figure 5.3: The function $G(x, \gamma)$, defined by (5.36), intersecting the (dashed red) line $\frac{p_{0}}{\alpha}\left(x-\frac{1}{\sqrt{\alpha}}\right)+C_{0}$ when $p_{0}=5$ and $C_{0}=0$ (left), and the associated velocity, given by (5.27) (right), for a dynamic composite tube with two concentric stochastic neo-Hookean phases, with inner radii $A=1$ and $C=1 / 2$, respectively, under dead-load traction, assuming that $\alpha=1, \rho=1$, and $\mu^{(1)}$ follows a Gamma distribution with $\rho_{1}^{(1)}=405$ and $\rho_{2}^{(1)}=4.05 / \rho_{1}^{(1)}=0.01$, while $\mu^{(2)}$ is drawn from a Gamma distribution with $\rho_{1}^{(2)}=405$ and $\rho_{2}^{(2)}=4.2 / \rho_{1}^{(2)}$. The dashed black lines correspond to the expected values based only on mean values, $\underline{\mu}^{(1)}=4.05$ and $\underline{\mu}^{(2)}=4.2$. Each distribution was calculated from the average of 1000 stochastic simulations.
and material parameters for the composite tube are as in the previous case.

### 5.3 Oscillatory motion of a radially inhomogeneous cylindrical tube

We further consider the inflation of a radially inhomogeneous tube of stochastic neo-Hookean-like hyperelastic material characterised by the strain-energy function (2.23). Intuitively, one can regard the radially inhomogeneous tube as an extension of the composite with two concentric phases to the case with infinitely many concentric layers and continuous inhomogeneity. Our inhomogeneous model is similar to those proposed in [35] where the dynamic inflation of cylindrical tubes and spherical shells was treated explicitly. Clearly, different models will lead to different results. However, the purpose here is to illustrate our stochastic approach in a mathematically transparent manner by combining it with analytical

### 5.3. OSCILLATORY MOTION OF A RADIALLY INHOMOGENEOUS CYLINDRICAL TUBE

approaches for the mechanical solution whenever possible.


Figure 5.4: Examples of Gamma distribution, with hyperparameters $\rho_{1}=405$. $B^{4} / R^{4}$ and $\rho_{2}=0.01 \cdot R^{4} / B^{4}$, for the nonlinear shear modulus $\mu(R)$ given by (5.51).

Specifically, we assume a class of stochastic inhomogeneous hyperelastic models defined by(2.23), where $\mu=\mu(R)$ takes the form

$$
\begin{equation*}
\mu(R)=C_{1}+C_{2} \frac{R^{2}}{C^{2}} \tag{5.51}
\end{equation*}
$$

such that $\mu(R)>0$, for all $C \leq R \leq B, C_{1}>0$ is a single-valued (deterministic) constant, and $C_{2}$ is a random value defined by a given probability distribution. This function form was chosen for convenience, so that the problem of oscillatory motion for the cylindrical tube could be treated explicitly.

When the mean value of the shear modulus $\mu(R)$ described by (5.51) does not depend on $R$, as $C_{1}, R$ and $C$ are deterministic and $C_{2}$ is probabilistic, we have

$$
\begin{equation*}
\underline{\mu}=C_{1}, \quad \operatorname{Var}[\mu]=\operatorname{Var}\left[C_{2}\right] \frac{R^{4}}{C^{4}} \tag{5.52}
\end{equation*}
$$

where $\operatorname{Var}\left[C_{2}\right]$ is the variance of $C_{2}$, while the mean value of $C_{2}$ is $\underline{C}_{2}=0$.
By (2.25) and (5.52), the hyperparameters of the corresponding Gamma dis-
tribution, defined by (2.26), take the form

$$
\begin{equation*}
\rho_{1}=\frac{C_{1}}{\rho_{2}}, \quad \rho_{2}=\frac{\operatorname{Var}[\mu]}{C_{1}}=\frac{\operatorname{Var}\left[C_{2}\right]}{C_{1}} \frac{R^{4}}{C^{4}} \tag{5.53}
\end{equation*}
$$

For example, we can choose two constant values, $C_{0}>0$ and $C_{1}>0$, and set the hyperparameters for the Gamma distribution at any given $R$ as follows,

$$
\begin{equation*}
\rho_{1}=\frac{C_{1}}{C_{0}} \frac{C^{4}}{R^{4}}, \quad \rho_{2}=C_{0} \frac{R^{4}}{C^{4}} \tag{5.54}
\end{equation*}
$$

By (5.51), $C_{2}=\left(\mu(R)-C_{1}\right) C^{2} / R^{2}$ is the shifted Gamma-distributed random variable with mean value $\underline{C}_{2}=0$ and variance $\operatorname{Var}\left[C_{2}\right]=\rho_{1} \rho_{2}^{2} C^{4} / R^{4}=C_{0} C_{1}$.

In Figure 5.4, we show Gamma distributions with $\rho_{1}=405 \cdot B^{4} / R^{4}$ and $\rho_{2}=$ $0.01 \cdot R^{4} / B^{4}$. By (5.51) and (5.54), $C_{0}=0.01 \cdot C^{4} / B^{4}, C_{1}=\underline{\mu}=\rho_{1} \rho_{2}=4.05$ and $C_{2}=\mu(C)-C_{1}$. In particular, for a tube with infinitely thick wall, as $R$ decreases to $C, \rho_{1}$ increases, while $\rho_{2}$ decreases, and the Gamma distribution converges to a normal distribution $[36,82]$.

The shear modulus given by (5.51) takes the equivalent form

$$
\begin{equation*}
\mu(u)=C_{1}+C_{2} \frac{x^{2}-\frac{1}{\alpha}}{u-\frac{1}{\alpha}}, \tag{5.55}
\end{equation*}
$$

where $u=r^{2} / R^{2}$ and $x=c / C$, as denoted in (5.14).
Next, writing the invariants given by (5.6) in the equivalent form

$$
\begin{equation*}
I_{1}=\frac{1}{\alpha^{2} u}+u+\alpha^{2}, \quad I_{2}=\alpha^{2} u+\frac{1}{u}+\frac{1}{\alpha^{2}}, \quad I_{3}=1 \tag{5.56}
\end{equation*}
$$

and substituting these in (5.8) gives

$$
\begin{align*}
& \beta_{1}=2 \frac{\partial W}{\partial I_{1}}=\mu+\frac{d \mu}{\mathrm{~d} u} \frac{\mathrm{~d} u}{\mathrm{~d} I_{1}}\left(I_{1}-3\right)  \tag{5.57}\\
& \beta_{-1}=-2 \frac{\partial W}{\partial I_{2}}=-\frac{\mathrm{d} \mu}{\mathrm{~d} u} \frac{\mathrm{~d} u}{\mathrm{~d} I_{2}}\left(I_{1}-3\right)
\end{align*}
$$

where $\mu$ is defined by (5.55). Therefore,

$$
\begin{align*}
& \beta_{1}=C_{1}+C_{2} \frac{x^{2}-\frac{1}{\alpha}}{u-\frac{1}{\alpha}}\left[1-\frac{u^{3}+u^{2}\left(\alpha^{2}-3\right)+\frac{u}{\alpha^{2}}}{\left(u-\frac{1}{\alpha}\right)^{2}\left(u+\frac{1}{\alpha}\right)}\right],  \tag{5.58}\\
& \beta_{-1}=C_{2} \frac{x^{2}-\frac{1}{\alpha}}{u-\frac{1}{\alpha}} \frac{u^{3}}{\alpha^{2}}+u^{2}\left(1-\frac{3}{\alpha^{2}}\right)+\frac{u}{\alpha^{4}} \\
&\left(u-\frac{1}{\alpha}\right)^{2}\left(u+\frac{1}{\alpha}\right)
\end{align*}
$$

and

$$
\begin{equation*}
\beta_{1}-\beta_{-1} \alpha^{2}=C_{1}+2 C_{2} \frac{x^{2}-\frac{1}{\alpha}}{u-\frac{1}{\alpha}}\left[\frac{1}{2}-\frac{u^{3}+u^{2}\left(\alpha^{2}-3\right)+\frac{u}{\alpha^{2}}}{\left(u-\frac{1}{\alpha}\right)^{2}\left(u+\frac{1}{\alpha}\right)}\right] . \tag{5.59}
\end{equation*}
$$

Recalling that the stress components are described by (5.11), and following a similar procedure as in the previous section, we set the pressure impulse as in (5.24). Then, similarly to (5.23), using (5.59), we define the function

$$
\begin{align*}
G(x, \gamma) & =\frac{C_{1}}{\rho C^{2}} \int_{\frac{1}{\sqrt{\alpha}}}^{x}\left(\zeta \int_{\frac{\zeta^{2}+\frac{\gamma}{\alpha}}{1+\gamma}}^{\zeta^{2}} \frac{1+\alpha u}{\alpha^{2} u^{2}} \mathrm{~d} u\right) \mathrm{d} \zeta \\
& +\frac{2 C_{2}}{\rho C^{2}} \int_{\frac{1}{\sqrt{\alpha}}}^{x}\left\{\left(\zeta^{3}-\frac{\zeta}{\alpha}\right) \int_{\frac{\zeta^{2}+\frac{\gamma}{\alpha}}{1+\gamma}}^{\zeta^{2}} \frac{1+\alpha u}{\alpha^{2} u^{2}\left(u-\frac{1}{\alpha}\right)}\left[\frac{1}{2}-\frac{u^{3}+u^{2}\left(\alpha^{2}-3\right)+\frac{u}{\alpha^{2}}}{\left(u-\frac{1}{\alpha}\right)^{2}\left(u+\frac{1}{\alpha}\right)}\right] \mathrm{d} u\right\} \mathrm{d} \zeta \\
& =\frac{C_{1}}{2 \alpha \rho C^{2}}\left(x^{2}-\frac{1}{\alpha}\right) \log \frac{1+\gamma}{1+\frac{\gamma}{\alpha x^{2}}} \\
& +\frac{C_{2}}{\rho C^{2}} \int_{\frac{1}{\sqrt{\alpha}}}^{x}\left\{\left(\zeta^{3}-\frac{\zeta}{\alpha}\right)\left[\frac{1}{\alpha \zeta^{2}}-\frac{1+\gamma}{\alpha \zeta^{2}+\gamma}-\frac{\alpha^{3}-3 \alpha+2}{\left(\alpha \zeta^{2}-1\right)^{2}}+\frac{\alpha^{3}-3 \alpha+2}{\left(\frac{\alpha \zeta^{2}+\gamma}{1+\gamma}-1\right)^{2}}\right]\right\} \mathrm{d} \zeta . \tag{5.60}
\end{align*}
$$

We restrict our attention on the following limiting cases:

Thick-walled tube. If the tube has an infinitely thick wall, such that $\gamma \rightarrow \infty$ and $\alpha=1$, then (5.60) takes the form

$$
\begin{align*}
G(x) & =\frac{C_{1}}{\rho C^{2}}\left(x^{2}-1\right) \log x-\frac{C_{2}}{\rho C^{2}} \int_{1}^{x} \frac{\left(\zeta^{2}-1\right)^{2}}{\zeta} \mathrm{~d} \zeta  \tag{5.61}\\
& =\frac{C_{1}}{\rho C^{2}}\left(x^{2}-1\right) \log x-\frac{C_{2}}{4 \rho C^{2}}\left(x^{4}-4 x^{2}+4 \log x+3\right) .
\end{align*}
$$

# 5.3. OSCILLATORY MOTION OF A RADIALLY INHOMOGENEOUS CYLINDRICAL TUBE 



Figure 5.5: The function $G(x, \gamma)$, defined by (5.61), intersecting the (dashed red) line $C_{0}=2$ (left), and the associated velocity, given by (5.27) (right), for a dynamic radially inhomogeneous tube with infinitely thick wall having inner radius $C=1$, assuming that $\alpha=1, \rho=1$, and $\mu$ follows a Gamma distribution with $\rho_{1}=405 / R^{4}$ and $\rho_{2}=0.01 \cdot R^{4}$. The dashed black lines correspond to the expected values based only on mean values. Each distribution was calculated from the average of 1000 stochastic simulations.


Figure 5.6: The function $G(x, \gamma)$, defined by (5.61), intersecting the (dashed red) line $\frac{p_{0}}{2 \alpha}\left(x^{2}-\frac{1}{\alpha}\right)+C_{0}$ when $p_{0}=1$ and $C_{0}=1$ (left), and the associated velocity, given by (5.27) (right), for a dynamic radially inhomogeneous tube with infinitely thick wall having inner radius $C=1$ under impulse traction, assuming that $\alpha=1$, $\rho=1$, and $\mu$ follows a Gamma distribution with $\rho_{1}=405 / R^{4}$ and $\rho_{2}=0.01$. $R^{4}$. The dashed black lines correspond to the expected values based only on mean values. Each distribution was calculated from the average of 1000 stochastic simulations.


Figure 5.7: The function $G(x, \gamma)$, defined by (5.61), intersecting the (dashed red) line $\frac{p_{0}}{\alpha}\left(x-\frac{1}{\sqrt{\alpha}}\right)+C_{0}$ when $p_{0}=5$ and $C_{0}=0$ (left), and the associated velocity, given by (5.27) (right), for a dynamic radially inhomogeneous tube with infinitely thick wall having inner radius $C=1$ under dead-load traction, assuming that $\alpha=1, \rho=1$, and $\mu$ follows a Gamma distribution with $\rho_{1}=405 / R^{4}$ and $\rho_{2}=$ $0.01 \cdot R^{4}$. The dashed black lines correspond to the expected values based only on mean values. Each distribution was calculated from the average of 1000 stochastic simulations.

Examples are presented in Figure 5.5 for the case with no impulse or dead-load traction, in Figure 5.6 for the case when the pressure impulse is given by (5.24), and in Figure 5.7 for the case when the dead-load traction is given by (5.43). The problem reduces to that of a homogeneous tube if $C_{2}=0$ (see [83]). For both homogeneous and inhomogeneous tubes, the amplitude and period of the oscillations depend on the initial conditions and the probabilistic material properties. However, a less detailed explicit analysis is possible for the inhomogeneous case.

Thin-walled tube. If the tube wall is thin, such that $0<\gamma \ll 1$ and $\alpha=1$, then the shear modulus defined by (5.51) takes the form $\mu=C_{1}+C_{2}$, and the problem reduces to that of a homogeneous tube with thin wall (see [83]).

## Chapter 6

## Quasi-equilibrated radial-axial <br> motion of stochastic hyperelastic

## spherical shells

In this chapter, we analyse the behaviour under quasi-equilibrated radial motion of two concentric homogeneous spherical shells, which are continuously attached to each other throughout the deformation, and of radially inhomogeneous shells of stochastic incompressible hyperelastic material. These spherical shells are also deformed by radially symmetric inflation when subject to either a surface dead load (which is constant in the reference configuration) or an impulse traction (which is kept constant in the current configuration), applied uniformly in the radial direction. For these shells as well, we find that the amplitude and period of the oscillations are characterised by probability distributions, depending on the initial conditions and the probabilistic material properties. Our results confirm and extend the results for stochastic homogeneous incompressible hyperelastic spherical shells presented in [83] where a similar analytical approach is employed.

### 6.1 Radial-axial motion of stochastic hyperelastic spherical shells

For a spherical shell, the radial motion is described by [15, 18, 56, 70]

$$
\begin{equation*}
r^{3}=c^{3}+R^{3}-C^{3}, \quad \theta=\Theta, \quad \phi=\Phi, \tag{6.1}
\end{equation*}
$$

where $(R, \Theta, \Phi)$ and $(r, \theta, \phi)=(r(t), \theta(t), \phi(t))$, with $t$ the time variable, are the spherical polar coordinates in the reference and current configuration, respectively, such that $C \leq R \leq B, C$ and $B$ are the inner and outer radii in the undeformed state, and $c=c(t)$ and $b=b(t)=\sqrt[3]{c^{3}+B^{3}-C^{3}}$ are the inner and outer radii at time $t$, respectively.

As for the cylindrical tube, the radial motion (6.1) of the spherical shell is determined entirely by the inner radius $c$ at time $t$. By the governing equations (6.1), condition (2.20) is valid for $\mathbf{x}=(r, \theta, \phi)^{T}$, since

$$
\begin{aligned}
\ddot{\mathbf{x}} & =\left(\ddot{r}-r \dot{\theta}^{2} \sin ^{2} \phi-r \dot{\phi}^{2}\right) \mathbf{e}_{r} \\
& +(r \ddot{\theta} \sin \phi+2 \dot{r} \dot{\theta} \sin \phi+2 r \dot{\theta} \dot{\phi} \cos \phi) \mathbf{e}_{\theta} \\
& +\left(r \ddot{\phi}+2 \dot{r} \dot{\phi}-r \dot{\theta}^{2} \sin \phi \cos \phi\right) \mathbf{e}_{\phi}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{0}=\operatorname{curl} \ddot{\mathbf{x}} & =\frac{1}{r \sin \theta}\left[\frac{\partial\left(\ddot{x}_{\phi} \sin \theta\right)}{\partial \theta}-\frac{\partial \ddot{x}_{\theta}}{\partial \phi}\right] \mathbf{e}_{r} \\
& +\frac{1}{r}\left[\frac{1}{\sin \theta} \frac{\partial \ddot{x}_{r}}{\partial \phi}-\frac{\partial\left(r \ddot{x}_{\phi}\right)}{\partial r}\right] \mathbf{e}_{\theta} \\
& +\frac{1}{r}\left[\frac{\partial\left(r \ddot{x}_{\theta}\right)}{\partial r}-\frac{\partial \ddot{x}_{r}}{\partial \theta}\right] \mathbf{e}_{\phi},
\end{aligned}
$$

where $\ddot{\mathbf{x}}=\left(\ddot{x}_{r}, \ddot{x}_{\theta}, \ddot{x}_{\phi}\right)^{T}$. Hence, (6.1) describes a quasi-equilibrated motion, such that

$$
\begin{equation*}
-\frac{\partial \xi}{\partial r}=\ddot{r}=\frac{2 c \dot{c}^{2}+c^{2} \ddot{c}}{r^{2}}-\frac{2 c^{4} \dot{c}^{2}}{r^{5}} \tag{6.2}
\end{equation*}
$$

### 6.1. RADIAL-AXIAL MOTION OF STOCHASTIC HYPERELASTIC SPHERICAL SHELLS

where $\xi$ is the acceleration potential satisfying (2.19). Integrating (6.2) gives [133, p. 217]

$$
\begin{equation*}
-\xi=-\frac{2 c \dot{c}^{2}+c^{2} \ddot{c}}{r}+\frac{c^{4} \dot{c}^{2}}{2 r^{4}}=-r \ddot{r}-\frac{3}{2} \dot{r}^{2} . \tag{6.3}
\end{equation*}
$$

For the deformation (6.1), the gradient tensor with respect to the polar coordinates $(R, \Theta, \Phi)$ takes the form

$$
\begin{equation*}
\mathbf{F}=\operatorname{diag}\left(\frac{R^{2}}{r^{2}}, \frac{r}{R}, \frac{r}{R}\right), \tag{6.4}
\end{equation*}
$$

the Cauchy-Green tensor is equal to

$$
\begin{equation*}
\mathbf{B}=\mathbf{F}^{2}=\operatorname{diag}\left(\frac{R^{4}}{r^{4}}, \frac{r^{2}}{R^{2}}, \frac{r^{2}}{R^{2}}\right), \tag{6.5}
\end{equation*}
$$

and the corresponding principal invariants are

$$
\begin{align*}
& I_{1}=\operatorname{tr}(\mathbf{B})=\frac{R^{4}}{r^{4}}+2 \frac{r^{2}}{R^{2}}, \\
& I_{2}=\frac{1}{2}\left[(\operatorname{tr} \mathbf{B})^{2}-\operatorname{tr}\left(\mathbf{B}^{2}\right)\right]=\frac{r^{4}}{R^{4}}+2 \frac{R^{2}}{r^{2}},  \tag{6.6}\\
& I_{3}=\operatorname{det} \mathbf{B}=1 .
\end{align*}
$$

Then, the principal components of the equilibrium Cauchy stress at time $t$ are

$$
\begin{align*}
& T_{r r}^{(0)}=-p^{(0)}+\beta_{1} \frac{R^{4}}{r^{4}}+\beta_{-1} \frac{r^{4}}{R^{4}}, \\
& T_{\theta \theta}^{(0)}=T_{r r}^{(0)}+\left(\beta_{1}-\beta_{-1} \frac{r^{2}}{R^{2}}\right)\left(\frac{r^{2}}{R^{2}}-\frac{R^{4}}{r^{4}}\right),  \tag{6.7}\\
& T_{\phi \phi}^{(0)}=T_{\theta \theta}^{(0)},
\end{align*}
$$

where $p^{(0)}$ is the Lagrangian multiplier for the incompressibility constraint ( $I_{3}=$ $1)$, and

$$
\begin{equation*}
\beta_{1}=2 \frac{\partial W}{\partial I_{1}}, \quad \beta_{-1}=-2 \frac{\partial W}{\partial I_{2}} \tag{6.8}
\end{equation*}
$$

with $I_{1}$ and $I_{2}$ given by (6.6). For a spherical shell made of a homogeneous incompressible neo-Hookean material, $\beta_{-1}=0$.

As the stress components depend only on the radius $r$, the system of equilib-
rium equations reduces to

$$
\begin{equation*}
\frac{\partial T_{r r}^{(0)}}{\partial r}=2 \frac{T_{\theta \theta}^{(0)}-T_{r r}^{(0)}}{r} \tag{6.9}
\end{equation*}
$$

Hence, by (6.7) and (6.9), the radial Cauchy stress for the equilibrium state at $t$ is equal to

$$
\begin{equation*}
T_{r r}^{(0)}(r, t)=2 \int\left(\beta_{1}-\beta_{-1} \frac{r^{2}}{R^{2}}\right)\left(\frac{r^{2}}{R^{2}}-\frac{R^{4}}{r^{4}}\right) \frac{\mathrm{d} r}{r}+\psi(t) \tag{6.10}
\end{equation*}
$$

where $\psi=\psi(t)$ is an arbitrary function of time. Substitution of (6.3) and (6.10) into (2.21) gives the following principal Cauchy stresses at time $t$,

$$
\begin{align*}
T_{r r}(r, t) & =-\rho\left(\frac{c^{2} \ddot{c}+2 c \dot{c}^{2}}{r}-\frac{c^{4} \dot{c}^{2}}{2 r^{4}}\right) \\
& +2 \int\left(\beta_{1}-\beta_{-1} \frac{r^{2}}{R^{2}}\right)\left(\frac{r^{2}}{R^{2}}-\frac{R^{4}}{r^{4}}\right) \frac{\mathrm{d} r}{r}+\psi(t),  \tag{6.11}\\
T_{\theta \theta}(r, t) & =T_{r r}(r, t)+\left(\beta_{1}-\beta_{-1} \frac{r^{2}}{R^{2}}\right)\left(\frac{r^{2}}{R^{2}}-\frac{R^{4}}{r^{4}}\right), \\
T_{\phi \phi}(r, t) & =T_{\theta \theta}(r, t) .
\end{align*}
$$

### 6.2 Oscillatory motion of a composite with two concentric homogeneous spherical shells

We investigate the behaviour under quasi-equilibrated radial motion of a composite formed from two stochastic neo-Hookean phases, similar to those containing two concentric spheres of different neo-Hookean material treated deterministically in [58] and [115], and stochastically in [91] (see Figure 4.2). We define the following strain-energy function,

$$
\mathcal{W}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)=\left\{\begin{array}{cl}
\frac{\mu^{(1)}}{2}\left(\lambda_{1}^{2}+\lambda_{2}^{2}+\lambda_{3}^{2}-3\right), & C<R<A  \tag{6.12}\\
\frac{\mu^{(2)}}{2}\left(\lambda_{1}^{2}+\lambda_{2}^{2}+\lambda_{3}^{2}-3\right), & A<R<B
\end{array}\right.
$$

### 6.2. OSCILLATORY MOTION OF A COMPOSITE WITH TWO CONCENTRIC HOMOGENEOUS SPHERICAL SHELLS

where $C<R<A$ and $A<R<B$ denote the radii of the inner and outer sphere in the reference configuration, and the corresponding shear moduli $\mu^{(1)}$ and $\mu^{(2)}$ are (spatially-independent) random variables characterised by the Gamma distributions $g\left(u ; \rho_{1}^{(1)}, \rho_{2}^{(1)}\right)$ and $g\left(u ; \rho_{1}^{(2)}, \rho_{2}^{(2)}\right)$, respectively.

For the composite spherical shell deforming by (6.1), we assume continuity of the displacement across the interface between the two concentric spheres. We set the inner and outer radial pressures acting on the curvilinear surfaces, $r=c(t)$ and $r=b(t)$ at time $t$, as $T_{1}(t)$ and $T_{2}(t)$, respectively [133, pp. 217-219]. Then, evaluating $T_{1}(t)=-T_{r r}(c, t)$ and $T_{2}(t)=-T_{r r}(b, t)$, using (6.11), with $r=c$ and $r=b$, respectively, and subtracting the results, gives

$$
\begin{align*}
T_{1}(t)-T_{2}(t) & =\rho\left[\left(c^{2} \ddot{c}+2 c \dot{c}^{2}\right)\left(\frac{1}{c}-\frac{1}{b}\right)-\frac{c^{4} \dot{c}^{2}}{2}\left(\frac{1}{c^{4}}-\frac{1}{b^{4}}\right)\right] \\
& +2 \int_{c}^{a} \mu^{(1)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{4}}{r^{4}}\right) \frac{\mathrm{d} r}{r}+2 \int_{a}^{b} \mu^{(2)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{4}}{r^{4}}\right) \frac{\mathrm{d} r}{r} \\
& =\rho\left[\left(c \ddot{c}+2 \dot{c}^{2}\right)\left(1-\frac{c}{b}\right)-\frac{\dot{c}^{2}}{2}\left(1-\frac{c^{4}}{b^{4}}\right)\right] \\
& +2 \int_{c}^{a} \mu^{(1)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{4}}{r^{4}}\right) \frac{\mathrm{d} r}{r}+2 \int_{a}^{b} \mu^{(2)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{4}}{r^{4}}\right) \frac{\mathrm{d} r}{r}  \tag{6.13}\\
& =\rho C^{2}\left[\left(\frac{c}{C} \frac{\ddot{c}}{C}+2 \frac{\dot{c}^{2}}{C^{2}}\right)\left(1-\frac{c}{b}\right)-\frac{\dot{c}^{2}}{2 C^{2}}\left(1-\frac{c^{4}}{b^{4}}\right)\right] \\
& +2 \int_{c}^{a} \mu^{(1)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{4}}{r^{4}}\right) \frac{\mathrm{d} r}{r}+2 \int_{a}^{b} \mu^{(2)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{4}}{r^{4}}\right) \frac{\mathrm{d} r}{r} .
\end{align*}
$$

Setting the notation

$$
\begin{equation*}
u=\frac{r^{3}}{R^{3}}=\frac{r^{3}}{r^{3}-c^{3}+C^{3}}, \quad x=\frac{c}{C}, \quad \gamma=\frac{B^{3}}{C^{3}}-1, \tag{6.14}
\end{equation*}
$$

we rewrite

$$
\begin{aligned}
\left(\frac{c}{C} \frac{\ddot{c}}{C}+2 \frac{\dot{c}^{2}}{C^{2}}\right)\left(1-\frac{c}{b}\right) & -\frac{\dot{c}^{2}}{2 C^{2}}\left(1-\frac{c^{4}}{b^{4}}\right) \\
& =\left(\ddot{x} x+2 \dot{x}^{2}\right)\left[1-\left(1+\frac{\gamma}{x^{3}}\right)^{-1 / 3}\right]-\frac{\dot{x}^{2}}{2}\left[1-\left(1+\frac{\gamma}{x^{3}}\right)^{-4 / 3}\right] \\
& =\left(\ddot{x} x+\frac{3}{2} \dot{x}^{2}\right)\left[1-\left(1+\frac{\gamma}{x^{3}}\right)^{-1 / 3}\right]-\frac{\dot{x}^{2}}{2} \frac{\gamma}{x^{3}}\left(1+\frac{\gamma}{x^{3}}\right)^{-4 / 3} \\
& =\frac{1}{2 x^{2}} \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{\dot{x}^{2} x^{3}\left[1-\left(1+\frac{\gamma}{x^{3}}\right)^{-1 / 3}\right]\right\} .
\end{aligned}
$$

By (6.1) and (6.14), we obtain

$$
\begin{aligned}
& \int_{c}^{a} \mu^{(1)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{4}}{r^{4}}\right) \frac{\mathrm{d} r}{r}+\int_{a}^{b} \mu^{(2)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{4}}{r^{4}}\right) \frac{\mathrm{d} r}{r} \\
& =\int_{c}^{a} \mu^{(1)}\left[\left(\frac{r^{3}}{r^{3}-c^{3}+C^{3}}\right)^{2 / 3}-\left(\frac{r^{3}-c^{3}+C^{3}}{r^{3}}\right)^{4 / 3}\right] \frac{\mathrm{d} r}{r} \\
& +\int_{a}^{b} \mu^{(2)}\left[\left(\frac{r^{3}}{r^{3}-c^{3}+C^{3}}\right)^{2 / 3}-\left(\frac{r^{3}-c^{3}+C^{3}}{r^{3}}\right)^{4 / 3}\right] \frac{\mathrm{d} r}{r} \\
& =-\frac{1}{3} \int_{c^{3} / C^{3}}^{a^{3} / A^{3}} \mu^{(1)} \frac{1+u}{u^{7 / 3}} \mathrm{~d} u-\frac{1}{3} \int_{a^{3} / A^{3}}^{b^{3} / B^{3}} \mu^{(2)} \frac{1+u}{u^{7 / 3}} \mathrm{~d} u \\
& =-\frac{1}{3} \int_{x^{3}}^{\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1} \mu^{(1)} \frac{1+u}{u^{7 / 3}} \mathrm{~d} u-\frac{1}{3} \int_{\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1}^{\frac{x^{3}+\gamma}{1+\gamma}} \mu^{(2)} \frac{1+u}{u^{7 / 3}} \mathrm{~d} u \\
& =\frac{1}{3} \int_{\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1}^{x^{3}} \mu^{(1)} \frac{1+u}{u^{7 / 3}} \mathrm{~d} u+\frac{1}{3} \int_{\frac{x^{3}+\gamma}{A^{3}+\gamma}}^{\left.\frac{x^{3}}{1+\gamma}-1\right)+1} \mu^{(2)} \frac{1+u}{u^{7 / 3}} \mathrm{~d} u .
\end{aligned}
$$

In the above calculations, we used the following relations,

$$
\begin{align*}
& r=\left[\frac{u\left(C^{3}-c^{3}\right)}{1-u}\right]^{1 / 3}  \tag{6.15}\\
& \frac{\mathrm{~d} r}{\mathrm{~d} u}=\frac{C^{3}-c^{3}}{3(1-u)^{2}}\left[\frac{u\left(C^{3}-c^{3}\right)}{1-u}\right]^{-2 / 3}=\frac{r}{3 u(1-u)}  \tag{6.16}\\
& \left(u^{2 / 3}-\frac{1}{u^{4 / 3}}\right) \frac{1}{3 u(1-u)}=\frac{u^{2}-1}{3 u^{7 / 3}(1-u)}=-\frac{1+u}{3 u^{7 / 3}} . \tag{6.17}
\end{align*}
$$

We then express (6.13) equivalently as follows

$$
\begin{align*}
2 x^{2} \frac{T_{1}(t)-T_{2}(t)}{\rho C^{2}} & =\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\dot{x}^{2} x^{3}\left[1-\left(1+\frac{\gamma}{x^{3}}\right)^{-1 / 3}\right]\right\} \\
& +\frac{4 x^{2}}{3 \rho C^{2}} \int_{\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1}^{x^{3}} \mu^{(1)} \frac{1+u}{u^{7 / 3}} \mathrm{~d} u  \tag{6.18}\\
& +\frac{4 x^{2}}{3 \rho C^{2}} \int_{\frac{x^{3}+\gamma}{1+\gamma}}^{\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1} \mu^{(2)} \frac{1+u}{u^{7 / 3}} \mathrm{~d} u .
\end{align*}
$$

In the static case, where $\dot{c}=0$ and $\ddot{c}=0$, (6.13) reduces to

$$
\begin{equation*}
T_{1}(t)-T_{2}(t)=2 \int_{c}^{a} \mu^{(1)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{4}}{r^{4}}\right) \frac{\mathrm{d} r}{r}+2 \int_{a}^{b} \mu^{(2)}\left(\frac{r^{2}}{R^{2}}-\frac{R^{4}}{r^{4}}\right) \frac{\mathrm{d} r}{r} \tag{6.19}
\end{equation*}
$$

and (6.18) becomes

$$
\begin{equation*}
2 \frac{T_{1}(t)-T_{2}(t)}{\rho A^{2}}=\frac{4}{3 \rho C^{2}}\left[\int_{\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1}^{x^{3}} \mu^{(1)} \frac{1+u}{u^{7 / 3}} \mathrm{~d} u+\int_{\frac{x^{3}+\gamma}{1+\gamma}}^{\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1} \mu^{(2)} \frac{1+u}{u^{7 / 3}} \mathrm{~d} u\right] \tag{6.20}
\end{equation*}
$$

For the dynamic shell, we define

$$
\begin{equation*}
\left.H(x, \gamma)=\frac{4}{3 \rho C^{2}} \int_{1}^{x} \zeta^{2}\left[\int_{\frac{C^{3}}{A^{3}}\left(\zeta^{3}-1\right)+1}^{\zeta^{3}} \mu^{(1)} \frac{1+u}{u^{7 / 3}} \mathrm{~d} u+\int_{\frac{\zeta^{3}+\gamma}{1+\gamma}}^{\frac{C^{3}}{A^{3}}} \zeta^{3}-1\right)+1 \quad \mu^{(2)} \frac{1+u}{u^{7 / 3}} \mathrm{~d} u\right] \mathrm{d} \zeta \tag{6.21}
\end{equation*}
$$

This function will be used to establish whether the radial motion of the composite shell is oscillatory or not.

### 6.2.1 Composite with two concentric homogeneous spherical shells subject to impulse traction

We set the pressure impulse

$$
2 \frac{T_{1}(t)-T_{2}(t)}{\rho C^{2}}=\left\{\begin{array}{cl}
0 & \text { if } t \leq 0  \tag{6.22}\\
p_{0} & \text { if } t>0
\end{array}\right.
$$

with $p_{0}$ constant in time. Integrating (6.18) implies

$$
\begin{equation*}
\dot{x}^{2} x^{3}\left[1-\left(1+\frac{\gamma}{x^{3}}\right)^{-1 / 3}\right]+H(x, \gamma)=\frac{p_{0}}{3}\left(x^{3}-1\right)+C_{0} \tag{6.23}
\end{equation*}
$$

where $H(x, \gamma)$ is given by (6.21), and

$$
\begin{equation*}
C_{0}=\dot{x}_{0}^{2} x_{0}^{3}\left[1-\left(1+\frac{\gamma}{x_{0}^{3}}\right)^{-1 / 3}\right]+H\left(x_{0}, \gamma\right)-\frac{p_{0}}{3}\left(x_{0}^{3}-1\right) \tag{6.24}
\end{equation*}
$$

with the initial conditions $x(0)=x_{0}$ and $\dot{x}(0)=\dot{x}_{0}$. By (6.23), the velocity is equal to

$$
\begin{equation*}
\dot{x}= \pm \sqrt{\frac{\frac{p_{0}}{3}\left(x^{3}-1\right)+C_{0}-H(x, \gamma)}{x^{3}\left[1-\left(1+\frac{\gamma}{x^{3}}\right)^{-1 / 3}\right]}} . \tag{6.25}
\end{equation*}
$$

This system is analogous to the motion of a point mass with energy

$$
\begin{equation*}
E=\frac{1}{2} m(x) \dot{x}^{2}+V(x) \tag{6.26}
\end{equation*}
$$

where the energy is $E=C_{0}$, the potential is given by $V(x)=H(x, \gamma)-\frac{p_{0}}{3}\left(x^{3}-1\right)$ and the position-dependent mass is $m(x)=2 x^{3}\left[1-\left(1+\frac{\gamma}{x^{3}}\right)^{-1 / 3}\right]$. The solutions of interest are either static or periodic solutions.

Oscillatory motion of the composite spherical shell occurs if and only if the frequency equation

$$
\begin{equation*}
H(x, \gamma)=\frac{p_{0}}{3}\left(x^{3}-1\right)+C_{0} \tag{6.27}
\end{equation*}
$$

has exactly two distinct solutions, representing the amplitudes of the oscillation, $x=x_{1}$ and $x=x_{2}$, such that $0<x_{1}<x_{2}<\infty$. In this case, the minimum and maximum radii of the inner surface in the oscillation are given by $x_{1} A$ and $x_{2} A$, respectively, and the period of oscillation is equal to

$$
\begin{equation*}
T=2\left|\int_{x_{1}}^{x_{2}} \frac{\mathrm{~d} x}{\dot{x}}\right|=2\left|\int_{x_{1}}^{x_{2}} \sqrt{\frac{x^{3}\left[1-\left(1+\frac{\gamma}{x^{3}}\right)^{-1 / 3}\right]}{\frac{p_{0}}{3}\left(x^{3}-1\right)+C_{0}-H(x, \gamma)}} \mathrm{d} x\right| . \tag{6.28}
\end{equation*}
$$

### 6.2. OSCILLATORY MOTION OF A COMPOSITE WITH TWO CONCENTRIC HOMOGENEOUS SPHERICAL SHELLS

For the stochastic composite shell, the amplitude and period of the oscillation are random variables characterised by probability distributions.

To examine $H(x, \gamma)$ defined by (6.21), we rewrite this function in the equivalent form

$$
\begin{equation*}
H(x, \gamma)=H_{1}(x, \gamma)+H_{2}(x, \gamma), \tag{6.29}
\end{equation*}
$$

where

$$
\begin{equation*}
H_{1}(x, \gamma)=\frac{4 \mu^{(1)}}{3 \rho C^{2}} \int_{1}^{x} \zeta^{2}\left[\int_{\frac{C^{3}}{A^{3}}\left(\zeta^{3}-1\right)+1}^{\zeta^{3}} \frac{1+u}{u^{7 / 3}} \mathrm{~d} u\right] \mathrm{d} \zeta \tag{6.30}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{2}(x, \gamma)=\frac{4 \mu^{(2)}}{3 \rho C^{2}} \int_{1}^{x} \zeta^{2}\left[\int_{\zeta^{3}}^{\frac{C^{3}}{A^{3}}\left(\zeta^{3}-1\right)+1} \frac{1+u}{u^{7 / 3}} \mathrm{~d} u+\int_{\frac{\zeta^{3}+\gamma}{1+\gamma}}^{\zeta^{3}} \frac{1+u}{u^{7 / 3}} \mathrm{~d} u\right] \mathrm{d} \zeta . \tag{6.31}
\end{equation*}
$$

Proceeding as in [83], we obtain

$$
\begin{align*}
H_{1}(x, \gamma) & =\frac{\mu^{(1)}}{\rho C^{2}}\left(x^{3}-1\right) \frac{2 x^{3}-1}{x^{3}+x^{2}+x} \\
& -\frac{\mu^{(1)}}{\rho C^{2}}\left(x^{3}-1\right) \frac{2\left[\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1\right]-1}{\left[\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1\right]+\left[\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1\right]^{2 / 3}+\left[\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1\right]^{1 / 3}} \tag{6.32}
\end{align*}
$$

and

$$
\begin{align*}
H_{2}(x, \gamma) & =\frac{\mu^{(2)}}{\rho C^{2}}\left(x^{3}-1\right) \frac{2\left[\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1\right]-1}{\left[\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1\right]+\left[\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1\right]^{2 / 3}+\left[\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1\right]^{1 / 3}} \\
& -\frac{\mu^{(2)}}{\rho C^{2}}\left(x^{3}-1\right) \frac{2 \frac{x^{3}+\gamma}{1+\gamma}-1}{\frac{x^{3}+\gamma}{1+\gamma}+\left(\frac{x^{3}+\gamma}{1+\gamma}\right)^{2 / 3}+\left(\frac{x^{3}+\gamma}{1+\gamma}\right)^{1 / 3}} . \tag{6.3}
\end{align*}
$$

By (6.32) and (6.33), the function $H(x, \gamma)$ defined by (6.29) takes the form

$$
\begin{align*}
& H(x, \gamma)=\frac{\mu^{(1)}}{\rho C^{2}}\left(x^{3}-1\right) \frac{2 x^{3}-1}{x^{3}+x^{2}+x} \\
& -\frac{\mu^{(1)}}{\rho C^{2}}\left(x^{3}-1\right) \frac{2\left[\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1\right]-1}{\left[\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1\right]+\left[\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1\right]^{2 / 3}+\left[\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1\right]^{1 / 3}} \\
& +\frac{2\left[\frac{\mu^{3}}{A^{3}}\left(x^{3}-1\right)+1\right]-1}{\rho C^{2}}\left(x^{3}-1\right) \frac{C^{3}}{\left[\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1\right]+\left[\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1\right]^{2 / 3}+\left[\frac{C^{3}}{A^{3}}\left(x^{3}-1\right)+1\right]^{1 / 3}} \\
& -\frac{\mu^{(2)}}{\rho C^{2}}\left(x^{3}-1\right) \frac{x^{3}+\gamma}{1+\gamma}-1  \tag{6.34}\\
& \frac{x^{3}+\gamma}{1+\gamma}+\left(\frac{x^{3}+\gamma}{1+\gamma}\right)^{2 / 3}+\left(\frac{x^{3}+\gamma}{1+\gamma}\right)^{1 / 3} .
\end{align*}
$$

The above function is monotonically decreasing for $0<x<1$ and increasing for $x>1$. In particular, when $\mu^{(1)}=\mu^{(2)}$, the function corresponding to homogeneous spherical shell is obtained [83].

We consider the following two cases:
(i) If $p_{0}=0$ and $C_{0}>0$, then equation (6.27) has exactly two solutions, $x=x_{1}$ and $x=x_{2}$, satisfying $0<x_{1}<1<x_{2}<\infty$, for any positive constant $C_{0}$. An example is shown in Figure 6.1, where $C_{0}=7$. These oscillations are not 'free' in general, since, by (6.11), if $T_{r r}(r, t)=0$ at $r=c$ and $r=b$, so that $T_{1}(t)=T_{2}(t)=0$, then $T_{\theta \theta}(r, t)=T_{\phi \phi}(r, t) \neq 0$ at $r=c$ and $r=b$, unless $r^{2} / R^{2} \rightarrow 1$.
(ii) When $p_{0} \neq 0$ and $C_{0} \geq 0$, substitution of (6.34) in (6.27) implies that the necessary and sufficient condition for the motion to be oscillatory is that $p_{0}$ satisfies

$$
\begin{equation*}
-\infty=\lim _{x \rightarrow 0} \frac{3\left(H(x, \gamma)-C_{0}\right)}{x^{3}-1}<p_{0}<\sup _{0<x<\infty} \frac{3\left(H(x, \gamma)-C_{0}\right)}{x^{3}-1} \tag{6.35}
\end{equation*}
$$



Figure 6.1: The function $H(x, \gamma)$, defined by (6.34), intersecting the (dashed red) line $C_{0}=7$ (left), and the associated velocity, given by (6.25) (right), for a dynamic composite tube with two concentric stochastic neo-Hookean phases, with inner radii $A=1$ and $C=1 / 2$, respectively, assuming that $\rho=1$ and $\mu^{(1)}$ follows a Gamma distribution with $\rho_{1}^{(1)}=405$ and $\rho_{2}^{(1)}=4.05 / \rho_{1}^{(1)}=0.01$, while $\mu^{(2)}$ is drawn from a Gamma distribution with $\rho_{1}^{(2)}=405$ and $\rho_{2}^{(2)}=4.2 / \rho_{1}^{(2)}$. The dashed black lines correspond to the expected values based only on mean values, $\underline{\mu}^{(1)}=4.05$ and $\underline{\mu}^{(2)}=4.2$. Each distribution was calculated from the average of 1000 stochastic simulations.


Figure 6.2: The function $H(x, \gamma)$, defined by (6.34), intersecting the (dashed red) line $\frac{p_{0}}{3}\left(x^{3}-1\right)+C_{0}$ when $p_{0}=1$ and $C_{0}=1$ (left), and the associated velocity, given by (6.25) (right), for a dynamic composite tube with two concentric stochastic neo-Hookean phases, with inner radii $A=1$ and $C=1 / 2$, respectively, under impulse traction, assuming that $\rho=1$ and $\mu^{(1)}$ follows a Gamma distribution with $\rho_{1}^{(1)}=405$ and $\rho_{2}^{(1)}=4.05 / \rho_{1}^{(1)}=0.01$, while $\mu^{(2)}$ is drawn from a Gamma distribution with $\rho_{1}^{(2)}=405$ and $\rho_{2}^{(2)}=4.2 / \rho_{1}^{(2)}$. The dashed black lines correspond to the expected values based only on mean values, $\underline{\mu}^{(1)}=4.05$ and $\underline{\mu}^{(2)}=4.2$. Each distribution was calculated from the average of $\overline{0} 00$ stochastic simulations.
where "sup" denotes supremum. This case is exemplified in Figure 6.2 where $p_{0}=1$ and $C_{0}=1$.

Thin-walled shell. In particular, when the shell wall is thin, such that $0<$ $\gamma \ll 1$, if we assume that $A^{3} / C^{3}=B^{3} / A^{3}=\gamma / 2+1$, then (6.34) becomes

$$
\begin{equation*}
H(x, \gamma)=\frac{\mu^{(1)}+\mu^{(2)}}{2 \rho C^{2}}\left(x^{3}-1\right)\left[\frac{2 x^{3}-1}{x^{3}+x^{2}+x}-\frac{2 \frac{x^{3}+\gamma}{1+\gamma}-1}{\frac{x^{3}+\gamma}{1+\gamma}+\left(\frac{x^{3}+\gamma}{1+\gamma}\right)^{2 / 3}+\left(\frac{x^{3}+\gamma}{1+\gamma}\right)^{1 / 3}}\right] . \tag{6.36}
\end{equation*}
$$

In this case, the problem reduces to that of a thin-walled homogeneous spherical shell with shear modulus $\left(\mu^{(1)}+\mu^{(2)}\right) / 2$ [83].

### 6.2.2 Composite with two concentric homogeneous spherical shells subject to dead-load traction

We further assume that the outer circular surface of the composite sphere is free, such that $T_{2}(t)=0$, while the inner surface is subject to a dead-load pressure $P_{1}(t)$ satisfying

$$
2 \frac{P_{1}(t)}{\rho C^{2}}=2 x^{2} \frac{T_{1}(t)}{\rho C^{2}}=\left\{\begin{align*}
0 & \text { if } t \leq 0  \tag{6.37}\\
p_{0} & \text { if } t>0
\end{align*}\right.
$$

with $p_{0}$ constant in time. Integrating (6.18) implies

$$
\begin{equation*}
\dot{x}^{2} x^{3}\left[1-\left(1+\frac{\gamma}{x^{3}}\right)^{-1 / 3}\right]+H(x, \gamma)=p_{0}(x-1)+C_{0} \tag{6.38}
\end{equation*}
$$

where $H(x, \gamma)$ is given by (6.21), and

$$
\begin{equation*}
C_{0}=\dot{x}_{0}^{2} x_{0}^{3}\left[1-\left(1+\frac{\gamma}{x_{0}^{3}}\right)^{-1 / 3}\right]+H\left(x_{0}, \gamma\right)-p_{0}\left(x_{0}-1\right), \tag{6.39}
\end{equation*}
$$

with the initial conditions $x(0)=x_{0}$ and $\dot{x}(0)=\dot{x}_{0}$. By (6.23), the velocity is

$$
\begin{equation*}
\dot{x}= \pm \sqrt{\frac{p_{0}(x-1)+C_{0}-H(x, \gamma)}{x^{3}\left[1-\left(1+\frac{\gamma}{x^{3}}\right)^{-1 / 3}\right]}} . \tag{6.40}
\end{equation*}
$$

Oscillatory motion of the composite spherical shell occurs if and only if the frequency equation

$$
\begin{equation*}
H(x, \gamma)=p_{0}(x-1)+C_{0} \tag{6.41}
\end{equation*}
$$

has exactly two distinct solutions, representing the amplitudes of the oscillation, $x=x_{1}$ and $x=x_{2}$, such that $0<x_{1}<x_{2}<\infty$. In this case, the minimum and maximum radii of the inner surface in the oscillation are given by $x_{1} A$ and $x_{2} A$, respectively, and the period of oscillation is equal to

$$
\begin{equation*}
T=2\left|\int_{x_{1}}^{x_{2}} \frac{\mathrm{~d} x}{\dot{x}}\right|=2\left|\int_{x_{1}}^{x_{2}} \sqrt{\frac{x^{3}\left[1-\left(1+\frac{\gamma}{x^{3}}\right)^{-1 / 3}\right]}{p_{0}(x-1)+C_{0}-H(x, \gamma)}} \mathrm{d} x\right| \tag{6.42}
\end{equation*}
$$



Figure 6.3: The function $H(x, \gamma)$, defined by (6.34), intersecting the (dashed red) line $p_{0}(x-1)+C_{0}$ when $p_{0}=10$ and $C_{0}=0$ (left), and the associated velocity, given by (6.25) (right), for a dynamic composite tube with two concentric stochastic neo-Hookean phases, with inner radii $A=1$ and $C=1 / 2$, respectively, under dead-load traction, assuming that $\rho=1$ and $\mu^{(1)}$ follows a Gamma distribution with $\rho_{1}^{(1)}=405$ and $\rho_{2}^{(1)}=4.05 / \rho_{1}^{(1)}=0.01$, while $\mu^{(2)}$ is drawn from a Gamma distribution with $\rho_{1}^{(2)}=405$ and $\rho_{2}^{(2)}=4.2 / \rho_{1}^{(2)}$. The dashed black lines correspond to the expected values based only on mean values, $\underline{\mu}^{(1)}=4.05$ and $\underline{\mu}^{(2)}=4.2$. Each distribution was calculated from the average of $\overline{1} 000$ stochastic simulations.

The case with $p_{0}=0$ is similar to that when an impulse traction was assume. For $p_{0} \neq 0$ and $C_{0} \geq 0$, substitution of (6.34) in (6.27) implies that the necessary and sufficient condition for the motion to be oscillatory is that $p_{0}$ satisfies

$$
\begin{equation*}
-\infty=\lim _{x \rightarrow 0} \frac{\left(H(x, \gamma)-C_{0}\right)}{x-1}<p_{0}<\sup _{0<x<\infty} \frac{\left(H(x, \gamma)-C_{0}\right)}{x-1}=+\infty \tag{6.43}
\end{equation*}
$$

An example is shown in Figure 6.3, where $p_{0}=5$ and $C_{0}=0$, and the geometric and material parameters for the composite tube are as in the previous case.

### 6.3 Oscillatory motion of a radially inhomogeneous spherical shell

We also examine the oscillatory motion of radially inhomogeneous incompressible spherical shells of stochastic hyperelastic material described by a neo-Hookean-like strain-energy function, with the constitutive parameter varying continuously along the radial direction. Similarly to the case of a radially inhomogeneous tube, the radially inhomogeneous sphere can be regarded as an extension of the composite with two concentric phases to the case with infinitely many concentric layers and continuous inhomogeneity. Our inhomogeneous model is similar to those proposed in [91], where the cavitation and radial oscillatory motion of a stochastic sphere was treated explicitly.

As in [91] where the cavitation of radially inhomogeneous spheres was treated analytically, here, we define the class of stochastic inhomogeneous hyperelastic models (2.23) with the shear modulus taking the form (see also [35])

$$
\begin{equation*}
\mu(R)=C_{1}+C_{2} \frac{R^{3}}{C^{3}} \tag{6.44}
\end{equation*}
$$

where $\mu(R)>0$, for all $C \leq R \leq B, C_{1}>0$ is a single-valued (deterministic) constant, and $C_{2}$ is a random value defined by a given probability distribution.


Figure 6.4: Examples of Gamma distribution, with hyperparameters $\rho_{1}=405$. $B^{6} / R^{6}$ and $\rho_{2}=0.01 \cdot R^{6} / B^{6}$, for the shear modulus $\mu(R)$ given by (6.44).

This function form was chosen so that the problem of oscillatory motion for the spherical shell could be treated explicitly. We note that $C_{1}$ could also be chosen to be probabilistic, but this case would be more involved.

When the mean value of the shear modulus $\mu(R)$, described by (6.44), does not depend on $R$, as $C_{1}, R$ and $C$ are deterministic and $C_{2}$ is probabilistic, we have

$$
\begin{equation*}
\underline{\mu}=C_{1}, \quad \operatorname{Var}[\mu]=\operatorname{Var}\left[C_{2}\right] \frac{R^{6}}{C^{6}} \tag{6.45}
\end{equation*}
$$

where $\operatorname{Var}\left[C_{2}\right]$ is the variance of $C_{2}$, while the mean value of $C_{2}$ is $\underline{C}_{2}=0$.
By (2.25) and (6.45), the hyperparameters of the corresponding Gamma distribution, defined by (2.26), take the form

$$
\begin{equation*}
\rho_{1}=\frac{C_{1}}{\rho_{2}}, \quad \rho_{2}=\frac{\operatorname{Var}[\mu]}{C_{1}}=\frac{\operatorname{Var}\left[C_{2}\right]}{C_{1}} \frac{R^{6}}{C^{6}} . \tag{6.46}
\end{equation*}
$$

For example, we can choose two constant values, $C_{0}>0$ and $C_{1}>0$, and set the hyperparameters for the Gamma distribution at any given $R$ as follows,

$$
\begin{equation*}
\rho_{1}=\frac{C_{1}}{C_{0}} \frac{C^{6}}{R^{6}}, \quad \rho_{2}=C_{0} \frac{R^{6}}{C^{6}} . \tag{6.47}
\end{equation*}
$$

By (6.44), $C_{2}=\left(\mu(R)-C_{1}\right) C^{3} / R^{3}$ is the shifted Gamma-distributed random
variable with mean value $\underline{C}_{2}=0$ and variance $\operatorname{Var}\left[C_{2}\right]=\rho_{1} \rho_{2}^{2} C^{6} / R^{6}=C_{0} C_{1}$.
In Figure 6.4, we show Gamma distributions with $\rho_{1}=405 \cdot B^{6} / R^{6}$ and $\rho_{2}=$ $0.01 \cdot R^{6} / B^{6}$. By (6.44) and (6.47), $C_{0}=0.01 \cdot C^{6} / B^{6}, C_{1}=\mu=\rho_{1} \rho_{2}=4.05$ and $C_{2}=\mu(C)-C_{1}$. In particular, for a shell with infinitely thick wall, as $R$ decreases to $C, \rho_{1}$ increases, while $\rho_{2}$ decreases, and the Gamma distribution converges to a normal distribution $[36,82]$.

The shear modulus defined by (6.44) can be expressed equivalently as

$$
\begin{equation*}
\mu(u)=C_{1}+C_{2} \frac{x^{3}-1}{u-1} \tag{6.48}
\end{equation*}
$$

where $u=r^{3} / R^{3}$ and $x=c / C$, as denoted in (6.14).
Next, writing the invariants given by (6.6) in the equivalent form

$$
\begin{equation*}
I_{1}=u^{-4 / 3}+2 u^{2 / 3}, \quad I_{2}=u^{4 / 3}+2 u^{-2 / 3}, \quad I_{3}=1, \tag{6.49}
\end{equation*}
$$

and substituting these in (6.8) gives

$$
\begin{align*}
& \beta_{1}=2 \frac{\partial W}{\partial I_{1}}=\mu+\frac{d \mu}{\mathrm{~d} u} \frac{\mathrm{~d} u}{\mathrm{~d} I_{1}}\left(I_{1}-3\right) \\
& \beta_{-1}=-2 \frac{\partial W}{\partial I_{2}}=-\frac{\mathrm{d} \mu}{\mathrm{~d} u} \frac{\mathrm{~d} u}{\mathrm{~d} I_{2}}\left(I_{1}-3\right) \tag{6.50}
\end{align*}
$$

where $\mu$ is defined by (6.48). Therefore,

$$
\begin{align*}
\beta_{1} & =C_{1}+\frac{3 C_{2}}{4} \frac{x^{3}-1}{u-1}\left[\frac{4}{3}-\frac{2 u^{3}-3 u^{7 / 3}+u}{(u-1)^{2}(u+1)}\right] \\
\beta_{-1} & =\frac{3 C_{2}}{4} \frac{x^{3}-1}{u-1} \frac{2 u^{7 / 3}-3 u^{5 / 3}+u^{1 / 3}}{(u-1)^{2}(u+1)}, \tag{6.51}
\end{align*}
$$

and

$$
\begin{equation*}
\beta_{1}-\beta_{-1} \frac{r^{2}}{R^{2}}=C_{1}+\frac{3 C_{2}}{2} \frac{x^{3}-1}{u-1}\left[\frac{2}{3}-\frac{2 u^{3}-3 u^{7 / 3}+u}{(u-1)^{2}(u+1)}\right] . \tag{6.52}
\end{equation*}
$$

Recalling the stress components described by (6.11), and following a similar approach as in the previous section, we set the pressure impulse as in (6.22). Then,


Figure 6.5: The function $H(x, \gamma)$, defined by (6.54), intersecting the (dashed red) line $C_{0}=3$ (left), and the associated velocity, given by (6.25) (right), for a dynamic radially inhomogeneous shell with infinitely thick wall having inner radius $C=1$, assuming that $\rho=1$ and $\mu$ follows a Gamma distribution with $\rho_{1}=405 / R^{6}$ and $\rho_{2}=0.01 \cdot R^{6}$. The dashed black lines correspond to the expected values based only on mean values. Each distribution was calculated from the average of 1000 stochastic simulations.


Figure 6.6: The function $H(x, \gamma)$, defined by (6.54), intersecting the (dashed red) line $\frac{p_{0}}{3}\left(x^{3}-1\right)+C_{0}$ when $p_{0}=1$ and $C_{0}=1$ (left), and the associated velocity, given by (6.25) (right), for a dynamic radially inhomogeneous shell with infinitely thick wall having inner radius $C=1$, under impulse traction, assuming that $\rho=1$ and $\mu$ follows a Gamma distribution with $\rho_{1}=405 / R^{6}$ and $\rho_{2}=0.01 \cdot R^{6}$. The dashed black lines correspond to the expected values based only on mean values. Each distribution was calculated from the average of 1000 stochastic simulations.

### 6.3. OSCILLATORY MOTION OF A RADIALLY INHOMOGENEOUS SPHERICAL SHELL



Figure 6.7: The function $H(x, \gamma)$, defined by (6.54), intersecting the (dashed red) line $p_{0}(x-1)+C_{0}$ when $p_{0}=5$ and $C_{0}=0$ (left), and the associated velocity, given by (6.25) (right), for a dynamic radially inhomogeneous shell with infinitely thick wall having inner radius $C=1$, under dead-load traction, assuming that $\rho=1$ and $\mu$ follows a Gamma distribution with $\rho_{1}=405 / R^{6}$ and $\rho_{2}=0.01 \cdot R^{6}$. The dashed black lines correspond to the expected values based only on mean values. Each distribution was calculated from the average of 1000 stochastic simulations.
similarly to (6.21), using (6.52), we define the function

$$
\begin{align*}
H(x, \gamma) & =\frac{4 C_{1}}{3 \rho C^{2}} \int_{1}^{x}\left(\zeta^{2} \int_{\frac{\zeta^{3}+\gamma}{1+\gamma}}^{\zeta^{3}} \frac{1+u}{u^{7 / 3}} \mathrm{~d} u\right) \mathrm{d} \zeta \\
& +\frac{2 C_{2}}{\rho C^{2}} \int_{1}^{x}\left\{\left(\zeta^{5}-\zeta^{2}\right) \int_{\frac{\zeta^{3}+\gamma}{1+\gamma}}^{\zeta^{3}} \frac{1+u}{u^{7 / 3}(u-1)}\left[\frac{2}{3}-\frac{2 u^{3}-3 u^{7 / 3}+u}{(u-1)^{2}(u+1)}\right] \mathrm{d} u\right\} \mathrm{d} \zeta \\
& =\frac{C_{1}}{\rho C^{2}}\left(x^{3}-1\right)\left[\frac{2 x^{3}-1}{x^{3}+x^{2}+x}-\frac{2 \frac{x^{3}+\gamma}{1+\gamma}-1}{\frac{x^{3}+\gamma}{1+\gamma}+\left(\frac{x^{3}+\gamma}{1+\gamma}\right)^{2 / 3}+\left(\frac{x^{3}+\gamma}{1+\gamma}\right)^{1 / 3}}\right] \\
& +\frac{C_{2}}{\rho C^{2}} \int_{1}^{x}\left(\zeta^{5}-\zeta^{2}\right)\left[\frac{2 \zeta^{6}-3 \zeta^{4}+1}{\zeta^{4}\left(\zeta^{3}-1\right)^{2}}-\frac{2\left(\frac{\zeta^{3}+\gamma}{1+\gamma}\right)^{2}-3\left(\frac{\zeta^{3}+\gamma}{1+\gamma}\right)^{4 / 3}+1}{\left(\frac{\zeta^{3}+\gamma}{1+\gamma}\right)^{4 / 3}\left(\frac{\zeta^{3}+\gamma}{1+\gamma}-1\right)^{2}}\right] \mathrm{d} \zeta . \tag{6.53}
\end{align*}
$$

We focus our attention on the following limiting cases:

Thick-walled shell. If the shell has an infinitely thick wall, such that $\gamma \rightarrow \infty$, then (6.53) takes the form

$$
\begin{align*}
H(x) & =\frac{C_{1}}{\rho C^{2}}\left(x^{3}-1\right)\left(\frac{2 x^{3}-1}{x^{3}+x^{2}+x}-\frac{1}{3}\right) \\
& +\frac{C_{2}}{\rho C^{2}} \int_{1}^{x} \frac{2 \zeta^{6}-3 \zeta^{4}+1}{\zeta^{2}\left(\zeta^{3}-1\right)} \mathrm{d} \zeta \\
& =\frac{C_{1}}{\rho C^{2}}\left(x^{3}-1\right)\left(\frac{2 x^{3}-1}{x^{3}+x^{2}+x}-\frac{1}{3}\right) \\
& +\frac{C_{2}}{\rho C^{2}}\left[x^{2}+\frac{1}{x}-\frac{3}{2} \log \left(x^{2}+x+1\right)+\sqrt{3} \arctan \frac{2 x+1}{\sqrt{3}}-2+3 \log \sqrt{3}-\frac{\pi}{\sqrt{3}}\right] \tag{6.54}
\end{align*}
$$

Examples are presented in Figure 6.5 for the case without impulse or dead-load traction, in Figure 6.6 for the case when the pressure impulse is given by (6.22), and in Figure 6.7 for the case when the dead-load traction is given by (6.37). The problem reduces to that of a homogeneous sphere if $C_{2}=0$ (see [83]). For both homogeneous and inhomogeneous spheres, the amplitude and period of the oscillations depend on the initial conditions and the probabilistic material properties. However, a less detailed explicit analysis is possible for the inhomogeneous case.

Thin-walled shell. If the shell wall is thin, such that $0<\gamma \ll 1$, then the shear modulus defined by (6.44) takes the form $\mu=C_{1}+C_{2}$, and the problem reduces to that of a homogeneous shell with thin wall (see [83]).

## Chapter 7

## Conclusion

In this thesis, we studied analytically the static and dynamic finite deformation and oscillatory motion for: (1) composite cylindrical tubes and spherical shells with two concentric stochastic homogeneous neo-Hookean phases, and (2) inhomogeneous tubes and shells of neo-Hookean-like material with the constitutive parameter varying continuously in the radial direction. For the homogeneous materials, the shear moduli are spatially-independent random variables, while for the radially inhomogeneous bodies, the shear moduli are spatially-dependent random fields. Under the physically realistic assumptions that, for any given finite deformation, at any point in the material, the shear modulus and its inverse are positive and have finite mean value and finite variance, by the principle of maximum entropy, this modulus follows a Gamma probability density function.

These results extend previous theoretical findings for cylindrical tubes and spherical shells of stochastic hyperelastic material with homogeneous, i.e., spatiallyindependent, elastic parameters [82, 83]. Similarly to the homogeneous case, albeit after more involved calculations, we obtain that, under radially-symmetric dynamic deformation, treated as quasi-equilibrated motion, the bodies oscillate (i.e., their radius increases up to a point, then decreases, then increases again, and so on), and the amplitude and period of the oscillations are characterised by probability distributions, depending on the initial conditions, the geometry, and
the probabilistic material properties.
The particular values for the stochastic parameters in our numerical calculations are for illustrative purposes only, and chosen similar to those in $[82,83]$. Other stochastic hyperelastic models, such as those defined in [88], can also be used instead of the neo-Hookean model. However, for compressible materials, the theorem on quasi-equilibrated dynamics is not applicable [133, p. 209]. As our analytical approach relies on the notion of quasi-equilibrated motion for incompressible cylindrical tubes and spherical shells, the same approach cannot be used for the compressible case. Nevertheless, standard elastodynamic problems can be formulated, which can then be treated numerically. Extensions to more realistic models of stochastic heterogeneous bodies also require computational tools.

The present study (see also [81] where some of our results have been published) is a continuation of the explicit investigation of how the elastic solutions of fundamental nonlinear elasticity problems can be extended to stochastic hyperelastic materials where the parameters follow probability distributions, as developed in [82-84, 89-91]. For these problems, which include random variables as basic concepts along with mechanical stresses and strains, the propagation of stochastic variation from input material parameters to output mechanical behaviour is mathematically tractable, offering valuable insights into how probabilistic approaches can be incorporated into nonlinear elasticity theory.

Our stochastic analysis can further be extended, for example, to:

- Localised bulging in inflated circular cylindrical tubes, which is likely to occur for all isotropic material models when the axial stretch is fixed and below a certain threshold value that is dependent on the material mode;
- Asymmetric inflations of spheres leading to aspherical configurations (see [99, Sec. 6.3.4]);
- Inflation instabilities in cylindrical tubes or spheres of anisotropic material, such as fibre-reinforced composites.

Potential applications of this theoretical work include, for example, the study of plant stems and blood vessels, and the design of biomedical and engineering devices, such as medical or whether balloons.

## Appendix A

## Normal distribution as the limit of a Gamma distribution

Theorem A.0.1 The limiting distribution of the Gamma distribution with shape and scale parameters $\rho_{1}$ and $\rho_{2}$, respectively, such that $\rho_{1} \rightarrow \infty$, is the normal (Gaussian) distribution with mean value $\rho_{1} \rho_{2}$ and standard deviation $\sqrt{\rho_{1}} \rho_{2}$.

Proof: If $X$ is a random variable following a Gamma probability distribution with shape parameter $\rho_{1}>0$ and scale parameter $\rho_{2}>0$, then its mean value and standard deviation, respectively, are equal to

$$
\underline{X}=\rho_{1} \rho_{2}, \quad\|X\|=\sqrt{\rho_{1}} \rho_{2} .
$$

Its probability density function takes the form

$$
\begin{equation*}
g_{X}\left(x ; \rho_{1}, \rho_{2}\right)=\frac{x^{\rho_{1}-1} e^{-\frac{x}{\rho_{2}}}}{\rho_{2}^{\rho_{1}} \Gamma\left(\rho_{1}\right)}, \quad x \geq 0 \tag{A.1}
\end{equation*}
$$

where $\Gamma: \mathbb{R}_{+}^{*} \rightarrow \mathbb{R}$ is the complete Gamma function

$$
\begin{equation*}
\Gamma(z)=\int_{0}^{\infty} t^{z-1} e^{-t} \mathrm{~d} t \tag{A.2}
\end{equation*}
$$

The corresponding cumulative distribution function is equal to

$$
\begin{equation*}
G_{X}(x)=\int_{0}^{x} g_{X}\left(u ; \rho_{1}, \rho_{2}\right) \mathrm{d} u \tag{A.3}
\end{equation*}
$$

The moment generating function of $X$ is defined by

$$
M_{X}(t)=E\left[e^{t X}\right]=\left(1-\rho_{2} t\right)^{-\rho_{1}}, \quad t<\frac{1}{\rho_{2}} .
$$

Subtracting the mean value $\underline{X}$ from $X$ and dividing the result by the standard deviation $\|X\|$ gives the following one-to-one transformation,

$$
Y=\frac{X-\underline{X}}{\|X\|}
$$

or equivalently,

$$
Y=\frac{X}{\rho_{2} \sqrt{\rho_{1}}}-\sqrt{\rho_{1}} .
$$

The moment generating function of $Y$ is then

$$
M_{Y}(t)=E\left[e^{t Y}\right]=e^{-t \sqrt{\rho_{1}}} E\left[e^{t \frac{X}{\rho_{2} \sqrt{\rho_{1}}}}\right]=e^{-t \sqrt{\rho_{1}}}\left(1-\frac{t}{\sqrt{\rho_{1}}}\right)^{-\rho_{1}}, \quad t<\sqrt{\rho_{1}} .
$$

Thus, the limiting moment generating function of $Y$ when $\rho_{1} \rightarrow \infty$ takes the form

$$
\lim _{\rho_{1} \rightarrow \infty} M_{Y}(t)=\lim _{\rho_{1} \rightarrow \infty} e^{-t \sqrt{\rho_{1}}}\left(1-\frac{t}{\sqrt{\rho_{1}}}\right)^{-\rho_{1}}, \quad-\infty<t<\infty .
$$

The above limit can be calculated as follows,

$$
\lim _{\rho_{1} \rightarrow \infty} e^{-t \sqrt{\rho_{1}}}\left(1-\frac{t}{\sqrt{\rho_{1}}}\right)^{-\rho_{1}}=\lim _{y=1 / \sqrt{\rho_{1}} \rightarrow 0_{+}} e^{-(t / y)}(1-t y)^{-1 / y^{2}}=e^{\lim _{y \rightarrow 0_{+}} \frac{-t y-\ln (1-t y)}{y^{2}}},
$$

where

$$
\lim _{y \rightarrow 0_{+}} \frac{-t y-\ln (1-t y)}{y^{2}}=\lim _{y \rightarrow 0} \frac{-t+t /(1-t y)}{2 y}=\lim _{y \rightarrow 0_{+}} \frac{t^{2}}{2(1-t y)}=\frac{t^{2}}{2} .
$$

Therefore,

$$
\lim _{\rho_{1} \rightarrow \infty} M_{Y}(t)=e^{\frac{t^{2}}{2}}, \quad-\infty<t<\infty
$$

which is the moment generating function of a normally-distributed random variable.

For a normally-distributed random variable $Y \in(-\infty, \infty)$ with mean value $\underline{Y}=0$ and standard deviation $\|Y\|=1$, the probability density function takes the form

$$
\begin{equation*}
f_{Y}(y ; 0,1)=\frac{1}{\sqrt{2 \pi}} e^{-\frac{y^{2}}{2}}, \quad-\infty<y<\infty . \tag{A.4}
\end{equation*}
$$

The associated normal cumulative distribution function is

$$
\begin{align*}
F_{Y}(y) & =\int_{-\infty}^{y} f_{Y}(u ; 0,1) \mathrm{d} u \\
& =\frac{1}{2}+\frac{1}{2} \operatorname{erf}\left(\frac{y}{\sqrt{2}}\right), \tag{A.5}
\end{align*}
$$

where

$$
\begin{align*}
\operatorname{erf}(y) & =2 F_{Y}(y \sqrt{2})-1 \\
& =\frac{2}{\sqrt{\pi}} \int_{0}^{y} e^{-t^{2}} \mathrm{~d} t \tag{A.6}
\end{align*}
$$

is the error function, giving the probability that a random variable with normal distribution of mean 0 and variance $1 / 2$ falls in the range $[-y, y]$. The error function has the following properties

$$
\begin{equation*}
\operatorname{erf}(0)=0, \quad \operatorname{erf}(\infty)=1, \quad \operatorname{erf}(-\infty)=-1, \quad \operatorname{erf}(-y)=-\operatorname{erf}(y) \tag{A.7}
\end{equation*}
$$

For $-1<z<1$, there exists a unique $\operatorname{erf}^{-1}(z)$ satisfying $\operatorname{erf}\left(\operatorname{erf}^{-1}(z)\right)=z$. Thus, the limiting distribution of a Gamma distribution with shape parameter $\rho_{1} \rightarrow \infty$ is the normal distribution.

## Appendix B

## Linear combination of

## independent Gamma

## distributions

In this appendix, we state without proof a result concerning the summation of two independent Gamma-distributed variables (see also Appendix of [90] for the result stated in the general case of $n$ independent Gamma-distributed variables, and Theorem 1 of [93] for a proof). The following theorem is applicable also to linear combinations of independent Gamma-distributed random variables by rescaling.

Theorem B.0.1 If $\left\{R_{1}, R_{2}\right\}$ are mutually independent Gamma-distributed random variables, with the corresponding shape and scale hyperparameters, $\rho_{1}^{(i)}$ and $\rho_{2}^{(i)}, i=1,2$, respectively, such that $\rho_{2}^{(1)} \leq \rho_{2}^{(2)}$, then the density of $R=R_{1}+R_{2}$ can be expressed as follows,

$$
\begin{equation*}
g(R)=C \sum_{k=0}^{\infty} \frac{\delta_{k} R^{\rho+k-1} e^{-R / \beta_{1}}}{\beta_{1}^{\rho+k} \Gamma(\rho+k)}, \quad R>0 \tag{B.1}
\end{equation*}
$$

where

$$
\begin{align*}
\rho & =\rho_{1}^{(1)}+\rho_{1}^{(2)},  \tag{B.2}\\
C & =\left(\frac{\rho_{2}^{(1)}}{\rho_{2}^{(2)}}\right)^{\rho_{1}^{(2)}}, \tag{B.3}
\end{align*}
$$

and $\delta_{k}$ satisfies

$$
\begin{equation*}
e^{\sum_{k=1}^{\infty} \gamma_{k}\left(1-\rho_{1}^{(1)} t\right)^{-k}}=\sum_{k=0}^{\infty} \delta_{k}\left(1-\rho_{1}^{(1)} t\right)^{-k} \tag{B.4}
\end{equation*}
$$

with

$$
\begin{equation*}
\gamma_{k}=\frac{1}{k} \rho_{1}^{(2)}\left(1-\frac{\rho_{2}^{(1)}}{\rho_{2}^{(2)}}\right)^{k}, \quad k=1,2, \ldots \tag{B.5}
\end{equation*}
$$

## Appendix C

## Poster for BMC-BAMC 2021



Stochastic Finite Strain Analysis of Inhomogeneous Hyperelastic Bodies

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## Appendix D

## Matlab codes

```
inflation instability of two-layer cylindrical tube
M=1e3;
lambda=linspace (1.001,12,M);
datac1=2*[1.77; 1.89; 2.01; 1.68; 1.76];
datac2=2*[0.2; 0.23; 0.21; 0.29; 0.21];
data=datac1+datac2;
rho=gamfit(data);
ho1=rho(1); rho2=rho(2);
N=1e2;
mul=gamrnd(rho1,rho2,N,1);
Mean_mu1=rho1*rho2
mu2=gamrnd(rho1,rho2/4,N,1);
Mean_mu2=Mean_mu1/4;
T=linspace (0,5,50).
f=mu1./mu2*(1./lambda-1./lambda.^3) +1/3*(lambda-1./lambda.^5);
f=mul./mu2*(1./lambda-1./lam
for i=1:M
end
pcolor(lambda,T(1:end-1)+diff(T)/2,h)
shading interp
colorbar
old on
Mean_f=1/Mean_mu2*Mean_mu1*(1./lambda-1./lambda.^3) +1/3*(lambda-1./lambda.^5);
plot(lambda,Mean_f,'--k','linewidth',1
xlabel('$\lambda$','interpreter','latex')
xlabel(($\lambdas$')
set(gca,'fontsize',20)
hcb=colorbar;
caxis([0 1])
set(get(hcb,'Title'),'String','Probability','interpreter','latex')
```

```
inflation and stretching of two-layer cylindrical tube
```

ambda=1.01:0.01:12
for $\begin{aligned} i=1: 5 \\ R=1+i\end{aligned}$
f2 $=1 / 3 *$ (lambda-1./lambda. $\wedge 5$ )
f1=R* (1./lambda-1./lambda.^3);
$\mathrm{f}=\mathrm{f} 1+\mathrm{f} 2$;
plot(lambda, f,'linewidth', 2)
hold on
end
label('s\1ambads , interpreter , latex
label ('\$T(\lambda)/(\epsilon \mu^\{(2)\})\$','interpreter','latex')
set (gca,'fontsize', 20)
=legend (S,'location', 'nw'
set (L, 'fontsize',14)
inflation instability for two-layer cylindrical tube
ho1=405.0214; rho2=0.0101; \% Rivlin \& Saunder
\% mean value and standard deviation of shear modulus
mu0 $=$ rho1*rho2;
stdo $=$ sqrt (rho1)*rho2;
std0=0. 2302; \% Rivlin \& Saunders
$\mathrm{N}=1 \mathrm{e} 2$;
critical value for inflation instability
$\mathrm{cr}=1 / 4.0890$;
c1mean=mu0*vcr
$\mathrm{L}=[135,206,250] / 255 ;$
figure('units','normalized','position',[0 $1 / 3$ 1 1/3])
C1=linspace ( $0, \mathrm{mu} 0 / 2$ );
plot (C1, gamcdf (C1/vcr, rho1, rho2), 'b','linewidth',2)
hold on
plot (C1,1-gamcdf (C1/vcr, rho1, rho2), 'r','linewidth', 2)
plot (C1, normcdf (C1/vcr, mu0, std0), '--b','linewidth', 2)
plot (C1,1-normedf (C1/vcr, mu0, std0),'--r','linewidth', 2)
numerical solution
l=1;
for $t=C 1$
$1=$ normrnd (mu0, std0, $\mathrm{N}, 1$ )
G2=gamrnd(rho1, rho2,N,1);
Amnorm $(:, 1)=1 *($ G1<t/vcr $) ; ~$
Amgamm (: , 1) $=1$ * (G2<t/vcr)
$1=1+1$;
end
$A(1,:)=\operatorname{sum}($ Hmgamm $==1)$;
( $2,:$ ) $=\operatorname{sum}($ Hmgamm==0);
plot (C1, A (1,:)/N,'-','color',Lb,'linewidth', 2)
hold on
plot(C1,A (2,:)/N,'-','color',Lr,'linewidth',2)
( $1,:$ ) $=\operatorname{sum}($ Hmnorm==1) ;
A $(2,:$ ) $=\operatorname{sum}($ Hmnorm==0);
plot (C1, A (1,: $) / \mathrm{N}, '--'$ ' color', Lb, 'linewidth', 2
hold on
plot(C1,A(2,:)/N,'--','color',Lr,'linewidth',2)
(
axis ([0, mu0/2,0,1])
xticks([0 c1mean mu0/2])
xlabel('\$\mu^\{(2)\}\$','interpreter','latex')
ylabel('Probability','interpreter','latex'
set (gca,'fontsize', 20)
=legend('Analytical probability of stable tubes inflation (gamma distribution)',...
Analytical probability of unstable tubes inflation (gamma distribution)',
Analytical probability of stable tubes inflation (normal distribution),$\cdots$
'Simulated probability of stable tubes inflation (gamma distribution)'
Simulated probability of unstable tubes inflation (gamma distribution)'
Simulated probability of stable tubes inflation (normal distribution)', '.
Simulated probability of unstable tubes inflation (normal distribution)',...
Deterministic critical value for tubes inflation instability',...
et (L, 'fontsize', 12)

```
inflation of two-layer stochastic spherical shel
M=1e3;
lambda=linspace (1.001, 10,M);
datac1=2*[1.77; 1.89; 2.01; 1.68; 1.76];
datac2=2* =2*[0.2; 0.23; 0.21; 0.29; 0.21];
data=datac1+datac2;
ho=gamfit(data);
rho1=rho(1); rho2=rho(2);
N=1e2;
u1=gamrnd(rho1,rho2,N,1);
Mean_mu1=rho1*rho2;
mu2=gamrnd(rho1,rho2/4.7,N,1);
Mean_mu2=Mean_mu1/4.7;
=1\mathrm{ inspace (0,12,60);}
=mu1./mu2*(1./lambda-1./lambda.^7) +(lambda-1./lambda.^5);
lot(lambda,f,'linewidth',2)
or i=1:M
    h(:,i)=histcounts(f(:,i),T,'normalization','probability');
end
pcolor(lambda,T(1:end-1)+diff(T)/2,h
shading interp
colorbar
colorbar
Mean f=Mean mu1/Mean_mu2*(1./lambda-1./lambda.^7) +(lambda-1./lambda.^5);
plot(lambda,Mean_f,'--k','linewidth',1)
xlabel('$\lambda$','interpreter','latex')
ylabel('$T(\lambda)/(\epsilon\mu^{(2)})$','interpreter','latex')
set(gca,'fontsize',20)
hcb=colorbar
caxis([0 1])
et(get(hcb,'Title'),'String','Probability','interpreter','latex')
```

```
inflation and stretching of two-layer spherical shell
ambda=1.01:0.01:10,
for i=1:5
    f1=R*(1./lambda-1./lambda.^7)
    2=(lambda-1./lambda.^5);
    £2= (1 ambda
    plot(lambda,f,'linewidth',
    hold on
nd
label('$\lambda$','interpreter', latex')
label('$T(\lambda)/(\epsilon\mu^{(2)})$','interpreter','latex')
et (gca,'fontsize',20)
=legend(S,'location', 'nw'
et(I,'fontsize',14)
```

inflation instability for spherical shell
ho1=405.0214; rho2=0.0101; \% Rivlin \& Saunders
mean value and standard deviation of shear modulus
mu0 $=$ rho1*rho2;
stdo $=$ sqrt (rho1) *rho2;
td0 $=0$. 2302; \% Rivlin \& Saunders
$\mathrm{N}=1 \mathrm{e}$ 2;
critical value for inflation instability
cr=1/4.6625;
1mean=mu0*vcr
$\mathrm{L}=[135,206,250] / 255$
figure('units','normalized','position',[0 1/3 1 1/3])
C1=linspace ( $0, \mathrm{mu} 0 / 2$ );
plot (C1, gamcdf (C1/vcr, rho1, rho2), 'b','linewidth',2)
hold on
plot (C1,1-gamcdf(C1/vcr,rho1,rho2),'r','linewidth', 2
plot (C1, normcdf (C1/vcr, mu0, std0), '--b','linewidth', 2)
plot (C1,1-normcdf(C1/vcr, mu0,std0),'--r','linewidth',2)
numerical solution
l=1;
for $t=C 1$
$1=$ normrnd (mu0, std0, N, 1)
G2=gamrnd(rho1, rho2,N,1);
Amnorm $(:, 1)=1 *($ G1<t/vcr $) ; ~$
$\operatorname{Hmgamm}(:, 1)=1 *($ G2<t/vcr $) ;$
$1=1+1$;
end
(1,:) $=\operatorname{sum}($ Hmgamm==1),
$\mathrm{A}(2,:)=\operatorname{sum}($ Hmgamm= $=0)$;
plot (C1, A (1,:)/N,'-','color',Lb,'linewidth', 2)
hold on
plot(C1,A (2,:)/N,'-','color',Lr,'linewidth',2)
( $1,:$ ) $=\operatorname{sum}($ Hmnorm==1) ;
A $(2,:$ ) $=\operatorname{sum}($ Hmnorm==0);
plot (C1, A (1,: $) / \mathrm{N}, '--'$ ' color', Lb, 'linewidth', 2
hold on
plot(C1,A(2,:)/N,'--','color',Lr,'linewidth',2)
(c1mean ones $(1,100)$, inspace $(0,1)$, k--', 'linewidth',2)
axis ([0, mu0/2,0,1])
xticks([0 c1mean mu0/2])
xlabel('\$\mu^\{(2)\}\$','interpreter','latex')
ylabel('Probability','interpreter','latex'
set (gca,'fontsize', 20)
=legend('Analytical probability of stable shells inflation (gamma distribution)',... 'Analytical probability of unstable shells inflation (gamma distribution)', Analytical probability of unstable shells inflation (normal diribution) ${ }^{\prime} \cdot \cdots$ Simulated probability of stable shells inflation (gamma distribution)' Simulated probability of unstable shells inflation (qamma distribution)', Simulated probability of stable shells inflation (normal distribution), , Simulated probability of unstable shells inflation (normal distribution)',. Deterministic critical value for shells inflation instability',...
'location','w');
\% gamma probability density function inhomogeneous tube
figure (2)
for $R=0.25: 0.25: 0.75$
fho1=405./R.^4; rho2=0.01*R^4;
=11nspace ( $3.5,4.5$ )
lot (L, gampdf (L, rho1, rho2), 'LineWidth', 2)
end
set (gca, 'fontsize',15)
$\mathrm{L}=$ legend (' $\$ \mathrm{R} / \mathrm{B}=0.25 \mathrm{~s}^{\prime}, \ldots$
' $\$ \mathrm{R} / \mathrm{B}=0.50 \$ 1, \ldots$
'location', 'nw');
set (L,'interpreter','latex','fontsize',20)
hold on
axis ([3.5 $4.5 \quad 0 \quad 35])$
xlabel('\$\mu(R) \$','interpreter','latex','fontsize',24)
label('Gamma probability distribution','interpreter','latex','fontsize',24)
\% gamma probability density function inhomogeneous sphere
igure (3)
for $R=0.25: 0.25: 0.75$
cho1=405./R.^6; rho2=0.01*R^6;
=1inspace(3.5,4.5)
lot (L, gampdf (L, rho1, rho2), 'LineWidth', 2)
end
set(gca,'fontsize',15)
$\mathrm{L}=$ legend (' $\$ \mathrm{R} / \mathrm{B}=0.25 \mathrm{~s}^{\prime}, \ldots$
$\$ \mathrm{R} / \mathrm{B}=0.50 \$ 1, \ldots$
location', 'nw');
set (L,'interpreter','latex','fontsize',20)
hold on

xlabel('\$\mu(R)\$','interpreter','latex','fontsize',24)
label('Gamma probability distribution','interpreter','latex','fontsize',24)
free radial oscillations of radially inhomogeneous cylindrical tubes of stochastic NH material
fs=20; rho=1; gamma=1; alpha=1; C=1;
$=1$ e3;
ho1=405; rho2=0.01;
mu=gamrnd (rho1,rho2, N, 1);
ean_mu=rho1*rho2;
C1 $=$ Mean_mu;
Mean $\mathrm{C} 1=\mathrm{Cl}$
$\mathrm{C} 2=\mathrm{mu}-\mathrm{C} 1$;
Mean C2 $=0$;
$\mathrm{M}=1 \mathrm{e} \overline{3}$;
$=1$ inspace ( $0.001,4, \mathrm{M}$ ) ;
$\mathrm{G}=\mathrm{C} 1 / \mathrm{C}^{\wedge} 2 \star(\mathrm{x} . \wedge 2-1) . \star \log (\mathrm{x})$;
$G=G-C 2 / 4 / C^{\wedge} 2 *(x \cdot \wedge 4-4 * x \cdot \wedge 2+3+4 * \log (x))$;
_mean=Mean_C1/C $2^{*}(x .)^{2-1) . * l o g(x) ; ~}$
_mean=G_mean-Mean_C2/4/C^2*(x.^4-4*x.^2+3+4*log(x))

- FREE OSCILLATIONS
$\mathrm{C}=2$;
figure (1)
plot(x, G,'linewidth', 2
$\mathrm{P}=1$ inspace
for $\mathrm{i}=1$ :
h(:,i)=histcounts (G(:,i), P, 'normalization','probability');
end
color (x, P(1:end-1) +diff (P) $/ 2, h$ )
hading interp
hcb=colorbar
set (get (hcb, 'Title'),'String','Probability','interpreter','latex')
hold on
plot (x, G_mean, , linewidth', 2)
xicks (0-4)
yticks (0:10:30)
xlabel('\$x\$','interpreter','latex','fontsize',fs
label('\$G\$','interpreter','latex','fontsize',fs
axis([0 $4-130])$
lot (linspace (m
lot (linspace (min (x), max (x)), C0*ones (1, 100), 'r--','linewidth', 2)
figure(2)
vx=sqrt ( $(2 \star$ C $0-2 * G) . / x . \wedge 2 . / \log (1+$ gamma./(alpha*x.^2)));
vx=real (vx);
plot ([x(110:386), x(110:386)],[vx(:,110:386),-vx(:,110:386)],'linewidth', 2)
Pv=linspace $(-3.5,3.5,50)$;
for $i=1: 277$
hv (:,1)=histcounts ([vx(:,i+109),-vx(:,1+109)],Pv,'normalization','probability')
end
pcolor ( $\mathrm{x}(110: 386$ ) , $\operatorname{Pv}(1:$ end- 1$)+\mathrm{diff}(\mathrm{Pv}) / 2, \mathrm{hv})$
shading interp
cb=colorbar
axis([0 1])
et (get (hcb, 'Title'), 'String','Probability','interpreter','latex')
hold on
x_mean=sqrt ((2*C0-2*G_mean)./x.^2./log (1+gamma./(alpha*x.^2))),
$x_{-m e a n=r e a l}(v x$ mean) ;
fvn=fvp (end:- $-1: 1$ );
prmx_mean $=\left[0, v x_{-}\right.$mean ( $f v p$ ) , -vx_mean (fvn) , 0$]$;
plot ([min(x(fvp)),x(fvp),x(fvn),min(x(fvp))],pmvx_mean,'k--','linewidth',2)
xlabel('\$x\$','interpreter','latex','fontsize',fs)
label( $\$ \backslash d o t\{x\}$ ', interpreter', latex', $10 n t s i z e$, fs)
plot(linspace $(0,3.5,100)$, zeros $(1,100), ' k ', ' l i n e w i d t h ', 1)$
et (gca,'fontsize',fs)
ticks $(0: 0.5: 2.5)$
ticks (-3:2:3)

```
forced radial oscillations radially inhomogeneous cylindrical tubes of stochastic NH materia
s=20; rho=1; gamma=1; alpha=1; C=1;
=1e3;
ho1=405; rho2=0.01;
mu=gamrnd (rho1,rho2,N,1)
ean_mu=rho1*rho2;
C1=Mean mu;
=m-c1.
Mean C2=0;
M=1e}\overline{3}\mathrm{ ;
=1inspace(0.001,5,M);
G=C1/C^2*(x.^2-1).*log(x);
G=G-C2/4/C^2*(x.^4-4*x.^2+3+4*log(x));
mean=Mean C1/c 2*(x. 2-1). log(x);
_mean=G_mean-Mean_C2/4/C^2*(x.^4-4*x.^2+3+4*log(x))
- FORCED OSCILIATIONS
0=1; C0=1;
figure (1)
plot(x,G,'linewidth',2)
=1\mathrm{ inspace (-2,12,50).}
or i=1:M
h(:,i)=histcounts(G(:,i),P,'normalization','probability');
end
color(x, P(1:end-1)+diff (P)/2,h)
shading interp
hcb=colorbar
set(get(hcb,'Title'),'String','Probability','interpreter','latex')
hold on
plot(x,G_mean, k--, linewidth',2)
ticks(0:3)
yticks(0:2:10)
label('$x$','interpreter','latex','fontsize',fs
label('$G$','interpreter','latex','fontsize',fs
xis([0 3 -1 10])
plot(x,p0/2*(x.^2-1/alpha)+C0,'r--','linewidth',2)
figure(2)
vx=sqrt((p0*(x.^2-1/alpha) +2*C0-2*G) ./x.^2./log(1+gamma./(alpha*x.^2)));
vx=real (vx); 
plot([x(130:300),x(130:300)],[vx(:,130:300),-vx(:,130:300)],'linewidth',2)
Pv=linspace (-2.5,2.5,30);
for i=1:171
    hv(:,i)=histcounts([vx(:,i+129),-vx(:,i+129)],Pv,'normalization','probability')
end
pcolor(x (130:300), Pv (1:end-1)+diff(Pv)/2,hv)
shading interp
hcb=colorbar
caxis([0 1])
et(get(hcb,'Title'),'String','Probability','interpreter','latex')
x mean=sqrt((p0*(x.^2-1/alpha) +2*C0-2*G mean)./x.^2./log(1+gamma./(alpha*x.^2)));
x_mean=real (vx_mean);
vn=fvp(end:-1:1);
pmvx_mean=[0,vx_mean (fvp),-vx_mean(fvn),0];
plot([min(x(fvp)),x(fvp),x(fvn),min(x(fvp))],pmvx_mean,'k--','linewidth',2)
xlabel('$x$','interpreter','latex','fontsize',fs)
ylabel('$\dot{x}$','interpreter','latex','fontsize',fs
plot(linspace(0,5,100),zeros(1,100), K',linewidth',1)
set(gca,'fontsize',fs)
ticks(0:0.2:1.6)
ticks(-2:1:2)
```

```
forced radial oscillations radially inhomogeneous cylindrical tubes of stochastic NH materia
s=20; rho=1; gamma=1; alpha=1; C=1;
=1e3;
ho1=405; rho2=0.01;
mu=gamrnd (rho1,rho2,N,1)
ean_mu=rho1*rho2;
C1=Mean mu;
=m-c1.
Mean C2=0;
M=1e}\overline{3}\mathrm{ ;
=1inspace(0.001,5,M);
G=C1/C^2*(x.^2-1).*log(x);
G=G-C2/4/C^2*(x.^4-4*x.^2+3+4*log(x));
mean=Mean_C1/C 2* (x. 2-1). log(x);
_mean=G_mean-Mean_C2/4/C^2*(x.^4-4*x.^2+3+4*log(x))
-mORCED-OSCILLATIONS
0=5; C0=0;
figure (1)
lot(x,G,'linewidth',2
=linspace ( }-2,12,50
or i=1:M
h(:,i)=histcounts(G(:,i),P,'normalization','probability');
end
color(x, P(1:end-1)+diff (P)/2,h)
shading interp
hcb=colorbar
set(get(hcb,'Title'),'String','Probability','interpreter','latex')
hold on
plot(x,G_mean, k---, linewidth',2)
xticks(0:0.5:3)
ticks(0:2:10)
xlabel('$x$','interpreter','latex','fontsize',fs
label('$G$','interpreter','latex','fontsize',fs
xis([0 2.5 -1 10])
plot(x,p0*(x-1/sqrt (alpha)) +C0,'r--','linewidth',2
% velocity
figure (2)
x=sqrt((2*p0*(x-1/sqrt(alpha))+2*C0-2*G)./x.^2./log(1+gamma./(alpha*x.^2)))
vx=real (vx); (190:330)],[vx(:, 190:330),-vx(:, 190:330)],']inewidth', 2)
plot([x(190:330),x(190:330)],[vx(:,190:330),-vx(:,190:330)],'linewidth',2)
Pv=linspace (-2,2,30);
for i=1:141
    hv(:,1)=histcounts([vx(:,1+189),-vx(:,1+189)],Pv,'normalization','probability'),
end
pcolor(x(190:330), Pv (1:end-1)+diff(Pv)/2,hv)
shading interp
hcb=colorbar
caxis([0 1])
et(get(hcb,'Title'),'String','Probability','interpreter','latex')
x mean=sqrt((2*p0*(x-1/sqrt(alpha)) +2*C0-2*G mean)./x.^2./log(1+gamma./(alpha*x.^2)))
x_mean=real (vx_mean);
vp={find(vx_mean>0)
pmvx_mean=[0,vx_mean(fvp),-vx_mean(fvn),0];
plot([min(x(fvp)),x(fvp),x(fvn),min(x(fvp))],pmvx_mean,'k--','linewidth',2)
xlabel('$x$','interpreter','latex','fontsize',fs)
label('$\dot{x)$','interpreter','latex','fontsize',fs
plot(linspace(0,5,100),zeros(1,100),'k','linewidth',1)
set(gca,'fontsize',fs)
xticks(0:0.2:1.6)
ticks(-2:0.5:2)
```

free radial oscillations of radially inhomogeneous cylindrical tubes of stochastic NH material fs=20; rho=1; gamma=1; C=1
$\mathrm{N}=1 \mathrm{e} 3$
ho1=405; rho2=0.01;
mu=gamrnd (rho1, rho2, $\mathrm{N}, 1$ );
ean_mu=rho1*rho2;
Mean $\mathrm{C} 1=\mathrm{C} 1$;
$\mathrm{C} 2=\mathrm{mu}-\mathrm{C} 1$;
Mean C2 $=0$;
$\mathrm{M}=1 \mathrm{e} \overline{3}$;
$\mathrm{x}=1$ inspace ( $0.001,5, \mathrm{M}$ ) ;
$\mathrm{H}=\mathrm{C} 1 / \mathrm{C}^{\wedge} 2^{\star}(\mathrm{x} \cdot \wedge 3-1) .{ }^{\star}((2 * \mathrm{x} \cdot \wedge 3-1) \cdot /(\mathrm{x} \cdot \wedge 3+\mathrm{x} \cdot \wedge 2+\mathrm{x})-1 / 3)$;
$=\mathrm{H}+\mathrm{C} 2 / \mathrm{C}^{\wedge} 2^{*}\left(3 / 2^{*} \log (3)-\mathrm{pi} /\right.$ sqrt (3) $\left.-2-3 / 2^{\star} \log (x .2+\mathrm{x}+1)+\operatorname{sqrt}(3) * \operatorname{atan}\left(\left(2^{*} \mathrm{x}+1\right) / \operatorname{sqrt}(3)\right)+\mathrm{x} . \wedge 2+1 . / \mathrm{x}\right)$,
H_mean=Mean_C1/C^2* $(x \cdot \wedge 3-1) \cdot *\left(\left(2^{*} x \cdot \wedge 3-1\right) \cdot /\left(x \cdot{ }^{\wedge} 3+x \cdot \wedge 2+x\right)-1 / 3\right)$
mean $=\mathrm{H}$ _mean + Mean_C2/C^2* $\left(3 / 2^{\star} \log (3)-\right.$ pi $/$ sqrt (3) -2
$\left.3 / 2^{\star} \log (\bar{x} . \wedge 2+x+1)+\bar{s} q r t(3) \star \tan ((2 * x+1) / \operatorname{sqrt}(3))+x . \wedge 2+1 . / x\right)$;
FREE OSCILLATIONS
$\mathrm{C} 0=3$;
igure (1
plot(x, H,'linewidth', 2)
$=$ linspace $(-5,35,50)$
$\mathrm{i}=1: \mathrm{M}$
( (: ) =histcounts (: i), P, 'normalization','probability');
color (x, P (1:end-1) +diff (P)/2,h)
shading interp
caxis([0 1])
et (get (hcb, 1itle ), string , Probability , interpreter , latex')
hold on
lot (x, H_mean, ${ }^{---', ~ ' l i n e w i d t h ', ~} 2$ )
xticks (0:3)
yticks (0:10:30
xlabel('\$x\$','interpreter','latex','fontsize', fs
label('\$H\$','interpreter','latex','fontsize',fs
( (0, 2.5 ,
plot(linspace (min (x), max (x)), C0*ones (1,100),'r--','linewidth', 2)
$\frac{\%}{\circ}$ velocity
figure (2)
$\mathrm{x}=$ sqrt ( (C0-H)./x.^3./(1-(1+gamma./x.^3).^(-1/3)))
vx=real (vx);
plot([x(90:306),x(90:306)],[vx(:,90:306),-vx(:,90:306)],'linewidth',2)
v=linspace ( $-4.5,4.5,50$ )
for $i=1: 217$
hv(:,i)=histcounts(lvx(:,i+89),-vx(:,i+89)],Pv, normalization', probability'),
pcolor ( $\mathrm{x}(90: 306$ ) , Pv (1: end-1) $+\operatorname{diff}(\operatorname{Pv}) / 2, h v$
shading interp
hcb=colorbar
et (get (hab 'Title'), 'String', 'Probability', 'interpreter', 'latex')
old on
x mean=sqrt ((C0-H mean)./x.^3./(1-(1+gamma./x.^3).^(-1/3)))
vx mean=real (vx_mean);
fvn=fvp (end:- $-1: 1$ );
mvx_mean=[0,vx_mean (fvp), -vx_mean (fvn), 0];

xlabel('\$x\$','interpreter','latex','fontsize',fs)
ylabel('\$\dot\{x\}\$','interpreter','latex','fontsize',fs)
plot (linspace $(0,4.5,100)$, zeros $(1,100), \mathrm{k}^{\prime}$, 'linewidth', 1)
et (gca,'fontsize', fs)
ticks (0:0.5:2.
yticks(-4:2:4)
forced radial oscillations radially inhomogeneous cylindrical tubes of stochastic NH material fs=20; rho=1; gamma=1; C=1;
$=1$ e3;
ho1=405; rho2=0.01;
mu=gamrnd (rho1,rho2, $\mathrm{N}, 1$ );
ean_mu=rho1*rho2;
C1 $=$ Mean_mu;
Mean $\mathrm{C} 1=\mathrm{Cl}$
C2=mu-C1;
Mean C2 $=0$;
$\mathrm{M}=1 \mathrm{e} \overline{3}$;
$\mathrm{x}=$ linspace ( $0.001,5, \mathrm{M}$ ) ;
$\mathrm{H}=\mathrm{C} 1 / \mathrm{C}^{\wedge} 2^{\star}(\mathrm{x} \cdot \wedge 3-1) . *\left(\left(2^{\star} \mathrm{x} \cdot \wedge 3-1\right) . /(\mathrm{x} \cdot \wedge 3+\mathrm{x} \cdot \wedge 2+\mathrm{x})-1 / 3\right)$;
$=H+C 2 / C^{\wedge} 2^{*}\left(3 / 2^{*} \log (3)-\mathrm{pi} /\right.$ sqrt (3) $\left.-2-3 / 2^{*} \log (x . \wedge 2+x+1)+\operatorname{sqrt}(3) * \tan ((2 * x+1) / \operatorname{sqrt}(3))+\mathrm{x} . \wedge 2+1 . / \mathrm{x}\right)$ )
H_mean=Mean_C1/C^2* $(x \cdot \wedge 3-1) \cdot *((2 * x \cdot \wedge 3-1) . /(x \cdot \wedge 3+x \cdot \wedge 2+x)-1 / 3)$;
Hean $=\mathrm{H}$ _mean + Mean_C2/C^2* $\left(3 / 2^{*} \log (3)-\mathrm{pi} /\right.$ sqrt (3) -2
$\overline{3} 2 \star \log (\bar{x} . \wedge 2+x+1)+\bar{s} q r t(3) * \operatorname{atan}((2 * x+1) / \operatorname{sqrt}(3))+x . \wedge 2+1 . / x)$
FORCED OSCILLATIONS
$0=1 ; C 0=1$;
figure (1)
plot( $\mathrm{x}, \mathrm{H}$, 'linewidth', 2 )
$=$ linspace $(-2,12,50)$
or $i=1: M$
(H) (:, i) , P, 'normalization', 'probability');
color (x, P (1:end-1) +diff (P)/2,h)
shading interp
caxis([0 1])
set (get (hcb, 1itie), string , Probability , interpreter , latex')
hold on
plot(x, H_mean, 'k--','linewidth', 2)
xticks ( $0: 2$ )
yticks ( $0: 2: 10$ )
xlabel('\$x\$','interpreter','latex','fontsize', fs)
label('\$H\$','interpreter','latex','fontsize',fs
axis([0 $2-1$ 10])

velocity
figure (2)
$\mathrm{x}=$ sqrt $\left(\left(\mathrm{p} 0 / 3^{\star}(\mathrm{x} . \wedge 3-1)+\mathrm{C} 0-\mathrm{H}\right) . / \mathrm{x} . \wedge 3 . /(1-(1+\right.$ gamma./x.^3).^(-1/3)));
$\mathrm{vx}=\mathrm{real}$ (vx);
plot([x(144:270),x(144:270)],[vx(:,144:270),-vx(:,144:270)],'linewidth',2)
Pv=linspace $(-3,3,30)$;
for $i=1: 127$
hv(:,i) =histcounts([vx(:,1+143),-vx(:,i+143)],Pv, 'normalization', probability');
end
pcolor (x (144:270), Pv (1:end-1) +diff(Pv)/2,hv)
shading interp
cb=colorbar
1])
('Title'),'String','Probability','interpreter','latex')
vx_mean=sqrt ((p0/3*(x.^3-1) +C0-H_mean)./x.^3./(1-(1+gamma./x.^3) .^(-1/3)))
$x^{\text {mean }}=$ real (vx_mean);
fvn=fvp (end:-1:1);
pmvx_mean=[0,vx_mean(fvp),-vx_mean(fvn), 0];

xlabel( $\$ x \${ }^{\prime}$, interpreter', latex', fontsize',fs)
ylabel('\$\dot\{x\}s', interpreter', latex', Iontsize',fs)
plot (linspace $(0,5,100)$, zeros $(1,100), ' k ', ' l i n e w i d t h ', 1)$
set (gca, 'fontsize',fs)
ticks (0:0.2:1.
yticks(-3:1:3)
forced radial oscillations radially inhomogeneous cylindrical tubes of stochastic NH material fs=20; rho=1; gamma=1; C=1;
$=1$ e3;
ho1=405; rho2=0.01;
mu=gamrnd (rho1,rho2, $\mathrm{N}, 1$ );
ean_mu=rho1*rho2;
C1 Mean
Mean Cl
$=\mathrm{C} 1$
$\mathrm{C} 2=\mathrm{mu}-\mathrm{C} 1$;
Mean C2 $=0$;
$\mathrm{M}=1 \mathrm{e} \overline{3}$;
$\mathrm{x}=$ linspace ( $0.001,5, \mathrm{M}$ ) ;
$\mathrm{H}=\mathrm{C} 1 / \mathrm{C}^{\wedge} 2^{\star}(\mathrm{x} \cdot \wedge 3-1) . *\left(\left(2^{\star} \mathrm{x} \cdot \wedge 3-1\right) . /(\mathrm{x} \cdot \wedge 3+\mathrm{x} \cdot \wedge 2+\mathrm{x})-1 / 3\right)$;
$=H+C 2 / C^{\wedge} 2^{*}\left(3 / 2^{*} \log (3)-\mathrm{pi} /\right.$ sqrt (3) $\left.-2-3 / 2^{*} \log (x . \wedge 2+x+1)+\operatorname{sqrt}(3) * a \tan ((2 * x+1) / \operatorname{sqrt}(3))+x . \wedge 2+1 . / x\right)$;
H_mean=Mean_C1/C^2* $(x \cdot \wedge 3-1) \cdot *((2 * x \cdot \wedge 3-1) . /(x \cdot \wedge 3+x \cdot \wedge 2+x)-1 / 3)$;
$H_{-}$mean $=H_{-}$mean+Mean_C2/C^2* $\left(3 / 2^{*} \log (3)-\mathrm{pi} /\right.$ sqrt (3) -2
$3 / 2 \star \log (\bar{x} . \wedge 2+x+1)+\bar{s} q r t(3) * \operatorname{atan}((2 * x+1) / \operatorname{sqrt}(3))+x . \wedge 2+1 . / x)$
\% FORCED OSCILLATIONS
$0=5$; $C 0=0$;
figure (1)
plot (x, H, 'linewidth', 2)
$=$ linspace $(-2,12,50)$
or $i=1$ :M
(H(:,i), P, 'normalization', 'probability');
color (x, P (1: end-1) +diff (P)/2,h)
shading interp
caxis([0 1])
set (get (hcb, 'Title'), string', 'Probability','interpreter', 'latex')
hold on
plot(x,H_mean, 'k--','linewidth', 2 )
ticks $10: 0.5: 2$
ticks ( $0: 2: 10$ )
xlabel('\$x\$','interpreter','latex','fontsize',fs)
label('\$H\$','interpreter','latex','fontsize',fs
axis([0 $2-1$ 10])
plot(x,p0/3*(x.^3-1)+C0,'r--','linewidth', 2)
\% velocity
igure(2) (p0* $(x-1)+C 0-H) . / x . \wedge 3 . /(1-(1+$ gamma./x.^3) .^(-1/3)));
$\mathrm{vx}=\mathrm{sqrt}\left(\mathrm{p} 0^{*}\right.$
$\mathrm{vx}=\mathrm{real}(\mathrm{vx})$;
plot([x(195:265),x(195:265)],[vx(:,195:265),-vx(:,195:265)],'linewidth',2)
Pv=linspace $(-2,2,30)$;
for $i=1: 71$
end
pcolor(x (195:265), Pv (1:end-1)+diff(Pv)/2,hv)
shading interp
cb=colorbar
axis([0 1])
(hab, 'Title'), 'String','Probability','interpreter', 'latex')
old on
vx mean=sqrt ((p0* $(x-1)+$ C0-H_mean)./x.^3./(1-(1+gamma./x.^3).^(-1/3)));
x mean=real (vx_mean);
fvn=fvp (end:-1:1);
pmvx_mean=[0,vx_mean(fvp),-vx_mean(fvn), 0];

xlabel('\$x\$', interpreter', 'latex', $10 n t s i z e ', f s)$
ylabel ('\$\dot $\{x\}$ ', interpreter','latex','fontsize', fs
plot (linspace $(0,5,100)$, zeros $(1,100), ' k ', ' l i n e w i d t h ', 1)$
set (gca,'fontsize',fs)
ticks(0:0.2:1.6)
ticks(-2:0.5:2)

```
free radial oscillations of concentric cylndrical tubes of stochastic NH material
S=20; rho=1; A=1; C=1/2; gamma=1; alpha=1
=1e3;
hol=405
u1=gamrnd(rho1, 4.05/rho1,N,1)
lean mu1=4.05;
u2=gamrnd(rho1,4.2/rho1,N,1);
Mean mu2=4.2
M=1e3;
=1inspace (0.001,5,M);
G=(mu2-mu1)* log(C^2/A^2+1./(alpha*x.^2)*(1-C^2/A^2))
G_mean=(Mean_mu2-Mean_mu1)* log(C^2/A^2+1./(alpha*x.^2)* (1-C^2/A^2));
_mean=Mean_mu2* (log(1+gamma) - log(1+gamma./(alpha*x.^2)));
G=G/ (2*C^2) * * (x.^2-1/alpha);
_mean=G_mean/(2*C^2).*(x.^2-1/alpha);
FREE OSCILLATIONS
C0=7;
igure(1)
lot(x, G,'linewidth',2
=1inspace (-5
        h(:,i)=histcounts(G(:,i),P,'normalization','probability');
end
color(x,P(1:end-1)+diff(P)/2,h)
shading interp
cb=colorbar
set(get(hcb,'Title'),'String','Probability','interpreter','latex')
hold on
plot(x,G_mean, , linewidth',2)
ticks(0:3)
yticks(0:10:30)
xlabel('$x$','interpreter','latex','fontsize',fs
ylabel('$G$','interpreter','latex','fontsize',fs
xis([0 3 -1 30])
plot(linspace(min(x),max(x)),C0*ones(1,100),'r--','linewidth', 2)
figure(2)
x=sqrt((2*C0-2*G)./x.^2./log(1+gamma./(alpha*x.^2)));
x=real (vx);
plot([x(82:386),x(82:386)],[vx(:,82:386),-vx(:,82:386)],'linewidth',2)
pv=linspace (-6.5,6.5,100);
for i=1:305
    Nv(:,1)=histcounts([vx(:,i+81),-vx(:,1+81)],Pv, normailzation', probability')
nd
polor(x (82:386), Pv (1:end-1)+diff(Pv)/2,hv)
shading interp
hcb=colorbar
caxis([0 1])
et(get(hcb,'Title'),'String','Probability','interpreter','latex')
x mean=sqrt ((2*C0-2*G mean)./x.^2./log(1+gamma./(alpha*x.^2))),
x_mean=real (vx_mean);
fn=fvp(end:-1:1);
pmvx_mean=[0,vx_mean (fvp),-vx_mean(fvn),0];
plot([min(x(fvp)),x(fvp),x(fvn),min(x(fvp))],pmvx_mean,'k--','linewidth',2)
xlabel('$x$','interpreter','latex','fontsize',fs)
label('$\dot{x}$','interpreter','latex','fontsize',fs)
plot(linspace (0,3.5,100),zeros(1,100),'k','linewidth',1)
et(gca,'fontsize',fs)
ticks(0:0.5:2.5)
ticks(-6:2:6)
```

```
forced radial oscillations of concentric cylindrical tubes of stochastic NH material
s=20; rho=1; A=1; C=1/2; gamma=1; alpha=1
=1e3;
hol=405
u1=gamrnd(rho1, 4.05/rho1,N,1)
Mean_mu1=4.05;
u2=gamrnd(rho1,4.2/rho1,N,1);
Me的;
=1inspace(0.001,5,M);
G=(mu2-mu1)*log (C^2/A^2+1./ (alpha*x.^2)* (1-C^2/A^2))
G=G+mu2.*(log(1+gamma) - log(1+gamma./(alpha*x.^2)));
G_mean=(Mean_mu2-Mean_mu1)* log(C^2/A^2+1./(alpha*x.^2)* (1-C^2/A^2));
mean=Mean_mu2*(log(1+gamma)-log(1+gamma./(alpha*x.^2)));
G=G/(2*C^2) .* (x.^2-1/alpha);
mean=G_mean/(2*\mp@subsup{C}{}{\wedge}2).*(x.^2-1/alpha);
- FORCED OSCILLATIONS
0=1; C0=2;
igure (1
lot(x, G,'linewidth',2
=linspace (-2,12,100)
r i=1:M
    h(:,i)=histcounts(G(:,i),P,'normalization','probability');
end
color(x, P (1: end-1)+diff (P)/2,h)
hading interp
cb=colorba
set(get(hcb,'Title'),'String','Probability','interpreter','latex')
hold on
plot(x,G_mean, k--', linewidth',2)
ticks(0:3)
yticks(0:2:10)
xlabel('$x$','interpreter','latex','fontsize',fs
label('$G$','interpreter','latex','fontsize',fs
axis([0 3 -1 10]),
plot(x,p0/2* (x.^2-1/alpha)+C0,'r--','linewidth', 2)
figure(2)
x=sqrt((p0*(x.^2-1/alpha) +2*C0-2*G) ./x.^2./log(1+gamma./(alpha*x.^2)));
```



```
plot([x(135:300),x(135:300)],[vx(:,135:300),-vx(:,135:300)],'linewidth',2)
v=linspace(-3,3,100);
for i=1:166
    hv(:,1)=histcounts([vx(:,i+134),-vx(:,i+134)],Pv, normalization', probability');
end
pcolor(x (135:300), Pv (1:end-1)+diff(Pv)/2,hv)
shading interp
hcb=colorbar
caxis([0 1])
et(get(hcb,'Title'),'String','Probability','interpreter','latex')
x mean=sqrt((p0*(x.^2-1/alpha) +2*C0-2*G mean)./x.^2./log(1+gamma./(alpha*x.^2)));
x_mean=real (vx_mean);
vn=fvp(end:-1:1);
pmvx_mean=[0,vx_mean (fvp),-vx_mean (fvn),0];
plot([min(x(fvp)),x(fvp),x(fvn),min(x(fvp))],pmvx mean,'k--','linewidth',2)
xlabel('$x$','interpreter','latex','fontsize',fs)
label('$\dot{x}$','interpreter','latex','fontsize',fs
plot(linspace(0,5,100),zeros(1,100), K',linewidth',1)
set(gca,'fontsize',fs)
ticks(0:0.2:1.4)
ticks(-3:3)
```

```
forced radial oscillations of concentric cylindrical tubes of stochastic NH material
fs=20; rho=1; A=1; C=1/2; gamma=1; alpha=1
=1e3;
hol=405
u1=gamrnd(rho1, 4.05/rho1,N,1)
Mean_mu1=4.05;
(rho1,4.2/rho1,N,1);
Me的;
=1inspace (0.001,5,M);
G=(mu2-mu1)*log (C^2/A^2+1./(alpha*x.^2)*(1-C^2/A^2))
G=G+mu2.*(log(1+gamma) - log(1+gamma./(alpha*x.^2)));
G_mean=(Mean_mu2-Mean_mu1)* log(C^2/A^2+1./(alpha*x.^2)*(1-C^2/A^2));
_mean=Mean_mu2*(log(1+gamma)-log(1+gamma./(alpha*x.^2)));
=G/ (2*C^2) .* (x.^2-1)alpha);
_mean=G_mean/(2* 2).*(x.^2-1/alpha);
- FORCED OSCILLATIONS
0=5; C0=0;
figure (1
lot(x, G,'linewidth',2
=linspace (-2,12,100)
or i=1:M
    (:,i)=histcounts(G(:,i),P,'normalization','probability');
end
poolor(x,P(1:end-1)+diff (P)/2,h)
shading interp
cb=colorbar
set(get(hcb,'Title'),'String','Probability','interpreter','latex')
hold on
plot(x,G_mean, k-- , linewidath , 2)
xticks(0-0.5:2)
ticks(0:2:10)
xlabel('$x$','interpreter','latex','fontsize',fs
label('$G$','interpreter','latex','fontsize',fs
xis([0 2 -1 10])
plot(x,p0*(x-1/sqrt (alpha)) +C0,'r--','linewidth', 2
% velocity
*)
vx=real(vx); 
v=linspace (-1.5,1.5,100);
for i=1:91
    hv(:,i)=histcounts([vx(:,i+194),-vx(:,i+194)],Pv,'normalization','probability');
end
pcolor(x(195:285),Pv(1:end-1)+diff(Pv)/2,hv)
shading interp
cb=colorbar
axis([001])
et(get(hcb,'Title'),'String','Probability','interpreter','latex')
x mean=sqrt ((2*p0*(x-1/sqrt(alpha)) +2*C0-2*G mean)./x.^2./log(1+gamma./(alpha*x.^2)))
x_mean=real (vx_mean);
fvn=fvp (end:-1:1);
pmvx_mean=[0,vx_mean (fvp),-vx_mean(fvn),0];
plot([min(x(fvp)),x(fvp),x(fvn),min(x(fvp))],pmvx_mean,'k--','linewidth',2)
xlabel('$x$','interpreter','latex','fontsize',fs)
label($\dot{x}$', interpreter',latex', fontsize, fs
plot(linspace(0,5,100),zeros(1,100), K','linewidth',1)
set(gca,'fontsize',fs)
xticks(0:0.2:1.4)
ticks(-1:0.5:1)
```

```
free radial oscillations of concentric cylindrical shells of stochastic NH material
fs=20; rho=1; A=1; C=1/2; gamma=1;
N=1e3;
ho1=405
u1=gamrnd(rho1,4.05/rho1,N,1)
Mean_mu1=4.05
(1,4.2/rho1,N,1);
Me的;
x=linspace (0.001,5,M);
H=mu1/C^2*(x.^3-1).*(2*x.^3-1)./(x.^3+x.^2+x);
H=H+(mu2-mu1) /C^2* (x.^3-1) .* (2* (C^3/A^3* (x.^3-1) +1)-1) ./ ( (C^3/A^3* (x.^3-1) +1) +(C^3/A^3* (x.^3-
1)+1).^(2/3) +(C^3/A^3* (x.^3-1)+1).^(1/3));
H}=\textrm{H}-\textrm{mu}2/\mp@subsup{C}{}{\wedge}\mp@subsup{2}{}{*}(\textrm{x}\cdot^3-1) .* (2* (x.^3+gamma)/(1+gamma)-- 
./((x. 3+gamma)/(1+gamma) +(x.^3+gamma). (2/3)/(1+gamma) .^(2/3)+(x.^3+gamma).^(1/3)/(1+gamma) .^
1/3));
H_mean=Mean_mu1/C^2* (x.^3-1).* (2*x.^3-1)./(x.^3+x.^^2+x);
-mean=H_mean+(Mean_mu2-Mean_mu1)/C^2* (x.^3-1).* (2* (C^3/A^3* (x.^3-1) +1) -1)./((C^3/A^3* (x.^3-
1)}+1)+(\mp@subsup{C}{}{\wedge}3/\mp@subsup{A}{}{\wedge}\mp@subsup{3}{}{*}(x.^\overline{3}-1)+1)\cdot^-(2/3)+(C^3/A^3* (x.^3-1)+1)\cdot^(1/3))
1)./((x.^^3+gamma)/(1+gamma)+(x.^3+gamma).^(2/3)/(1+gamma).^(2/3)+(x.^3+gamma).^(1/3)/(1+gamma).^(
1/3));
FREE OSCILLATION
C0=7;
figure(1)
plot(x,H,'linewidth',2
=1\mathrm{ inspace(-5, 35,100);}
for i=1:M
    h(:,i)=histcounts(H(:,i),P,'normalization','probability');
color(x,P(1:end-1)+diff(P)/2,h)
shading interp
caxis([0 1])
et(get(hcb,'Title'),'String','Probability','interpreter','latex')
hold on
k--','linewidth', 2)
xticks(0:3)
xlabel('$x$','interpreter','latex','fontsize',fs)
label('$H$','interpreter','latex','fontsize', fs)
axis([0 3-1 30])
set(gca,'fontsize',fs)
plot(linspace(min(x),max(x)),C0*ones(1,100),'r--','linewidth',2)
velocity
figure(2)
v=sqrt ((C0-H)./x.^3./(1-(1+gamma./x.^3).^(-1/3)))
x=real (vx);
lot([x(115:340),x(115:340)],[vx(:,115:340),-vx(:,115:340)],'linewidth',2)
Pv=linspace(-8.5,8.5,100);
    hv(:,i)=histcounts([vx(:,i+114),-vx(:,i+114)],Pv,'normalization','probability');
end
color(x(115:340),Pv(1:end-1)+diff(Pv)/2,hv)
shading interp
cb=colorbar
et(get(hcb,'Title'),'String','Probability','interpreter','latex')
hold on
x_mean=sqrt((C0-H_mean)./x.^3./(1-(1+gamma./x.^3).^(-1/3)))
x_mean=real (vx_mean);
vp=[find(vx_mean>0)];
fvn=fvp (end:-1:1);
pmvx_mean=[0,vx_mean(fvp),-vx_mean(fvn),0]
plot([min(x(fvp)),x(fvp),x(fvn}),min(x(fvp))],pmvx_mean,'k--',''linewidth',2
label('$x$','interpreter','latex','fontsize',fs)
lof
linewidth',1)
et(gca,'fontsize',fs)
ticks(0:0.5:2.5)
yticks(-8:2:8)
```

```
forced radial oscillations of concentric cylindrical tubes of stochastic NH material
fs=20; rho=1; A=1; C=1/2; gamma=1;
N=1e3;
ho1=405
u1=gamrnd(rho1, 4.05/rho1,N,1)
lean_mu1=4.05;
u2=gamrna(rho1,4.2/rho1,N,1)
=1e\overline{3}
=linspace (0.001,5,M);
H=mu1/C^2* (x.^3-1).*(2*x.^3-1)./(x.^3+x.^2+x);
H=H+(mu2-mu1) /C^2* (x.^3-1) .* (2* (C^3/A^3* (x.^3-1) +1)-1) ./ ( (C^3/A^3* (x.^3-1) +1) +(C^3/A^3* (x.^3-
1)+1).^(2/3)+(C^3/A^3* (x.^3-1)+1).^(1/3));
H}=\textrm{H}-\textrm{mu}2/\mp@subsup{C}{}{\wedge}\mp@subsup{2}{}{*}(x.^3-1) .* (2* (x.^3+gamma)/(1+gamma)- 
|/((x.3+gamma)/(1+gamma) +(x. 3+gamma) . (2/3)/(1+gamma) . (2/3) +(x.^3+gamma) .^(1/3)/(1+gamma) .^
/3));
-mean=H_mean+(Mean_mu2-Mean_mu1)/C^2* (x.^3-1).* (2* (C^3/A^3* (x.^3-1) +1) -1)./((C^3/A^3* (x.^3-
1)}+1)+(\mp@subsup{C}{}{\wedge}3/\mp@subsup{A}{}{\wedge}\mp@subsup{3}{}{*}(x.^\overline{3}-1)+1)\cdot^-(2/3)+(C^3/A^3* (x.^3-1)+1)\cdot^(1/3))
1)./((x.^^3+gamma)/(1+gamma)+(x.^3+gamma).^(2/3)/(1+gamma).^(2/3)+(x.^3+gamma).^(1/3)/(1+gamma).^(
1/3));
FORCED OSCILLATIONS
0=1; C0=1;
lot(x,H,'linewidth', 2)
=1\mathrm{ inspace (-2,12,100).}
for i=1:M
    h(:,i)=histcounts(H(:,i),P,'normalization','probability');
pcolor(x,P(1:end-1)+diff(P)/2,h)
shading interp
cb=colorbar
caxis([0 1])
et(get(hcb,'Title'),'String','Probability','interpreter','latex')
hold on
lot(x,H_mean, ---','linewidth', 2)
ticks(0:0.5:2
xlabel('$x$','interpreter','latex','fontsize',fs)
label('$H$','interpreter','latex','fontsize',fs)
axis([0 2 -1 10])
set(gca,'fontsize',fs)
plot(x,p0/3*(x.^3-1)+C0,'r--','linewidth',2)
velocity
figure(2)
v=sqrt ((p0/3* (x.^3-1)+C0-H)./x.^3./(1-(1+gamma./x.^3).^(-1/3))),
x=real (vx);
plot([x(160:255),x(160:255)],[vx(:,160:255),-vx(:,160:255)],'linewidth',2
v=linspace (-2.5, 2. 5,100);
for i=1:96
    hv(:,i)=histcounts([vx(:,i+159),-vx(:,i+159)],Pv,'normalization','probability'),
end
color(x(160:255), Pv(1:end-1)+diff(Pv)/2,hv)
shading interp
cb=colorbar
set(get(hcb,'Title'),'String','Probability','interpreter','latex')
hold on
x_mean=sqrt((p0/3*(x.^3-1) +C0-H_mean)./x.^3./(1-(1+gamma./x.^3).^(-1/3)));
x_mean=real (vx_mean);
vp=[find(vx_mean>0)];
vn=fvp (end:-1:1);
mvx_mean=[0,vx_mean(fvp),-vx_mean(fvn),0]
lot([min(x(fvp)),x(fvp),x(fvn),min(x(fvp))],pmvx_mean,'k--','linewidth',2)
label('$x$','interpreter','latex','fontsize',fs)
lotlinpac(0,5,100),zeros(1,100),'k','1inewidh',1)
plot(linspace(0,5,100),zeros(1,100),'b','1inewidth',1)
et(gca,'fontsize',fs)
xticks(0:0.2:2)
yticks(-2:2)
```

```
forced radial oscillations of concentric cylindrical tubes of stochastic NH material
fs=20; rho=1; A=1; C=1/2; gamma=1;
N=1e3;
ho1=405
u1=gamrnd(rho1, 4.05/rho1,N,1)
Mean_mu1=4.05;
ho1,4.2/rho1,N,1)
Me的;
=linspace (0.001,5,M);
H=mu1/C^2* (x.^3-1).*(2*x.^3-1)./(x.^3+x.^2+x);
H=H+(mu2-mu1) /C^2* (x.^3-1) .* (2* (C^3/A^3* (x.^3-1) +1)-1) ./ ( (C^3/A^3* (x.^3-1) +1) +(C^3/A^3* (x.^3-
1)+1).^(2/3)+(C^3/A^3* (x.^3-1)+1).^(1/3));
H=H-mu2/C^2* (x.^3-1) .* (2* (x.^3+gamma)/(1+gamma)-
|/((x. 3+gamma)/(1+gamma) +(x. 3+gamma) . (2/3)/(1+gamma) . (2/3) +(x.^3+gamma) . (1/3)/(1+gamma) .^
1/3)); ;
-mean=H_mean+(Mean_mu2-Mean_mu1)/C^2* (x.^3-1).* (2* (C^3/A^3* (x.^3-1) +1) -1)./((C^3/A^3* (x.^3-
1)}+1)+(\mp@subsup{C}{}{\wedge}3/\mp@subsup{A}{}{\wedge}\mp@subsup{3}{}{*}(x.^\overline{3}-1)+1)\cdot^-(2/3)+(C^3/A^3* (x.^3-1)+1)\cdot^(1/3))
1)./((x.^ ^ 3+gamma)/(1+gamma)+(x.^3+gamma).^(2/3)/(1+gamma).^(2/3)+(x.^3+gamma).^(1/3)/(1+gamma).^(
1/3));
Forced osctitattons
p0=10; C0=0;
figure(1)
plot(x,H,'linewidth',2
=1\mathrm{ inspace (-2,12,100).}
for i=1:M
    h(:,i)=histcounts(H(:,i),P,'normalization','probability');
end
color(x,P(1:end-1)+diff(P)/2,h)
shading interp
cb=colorbar
caxis([0 1])
et(get(hcb,'Title'),'String','Probability','interpreter','latex')
hold on
lot(x,H_mean,'--','linewidth', 2)
ticks (0:0.5:2
xlabel('$x$','interpreter','latex','fontsize',fs)
label('$H$','interpreter','latex','fontsize',fs)
axis([0 2 -1 10])
set(gca,'fontsize',fs)
plot(x,p0*(x-1)+C0,'r--','linewidth',2)
velocity
figure(2)
v=sqrt((po*(x-1)+C0-H)./x.^3./(1-(1+gamma./x.^3).^(-1/3))),
x=real (vx);
plot([x(190:300),x(190:300)],[vx(:,190:300),-vx(:,190:300)],'linewidth',2
v=linspace (-2.5, 2.5,100);
for i=1:111
    hv(:,i)=histcounts([vx(:,i+189),-vx(:,i+189)],Pv,'normalization','probability');
end
color(x(190:300), Pv(1:end-1)+diff(Pv)/2,hv)
shading interp
cb=colorbar
set(get(hcb,'Title'),'String','Probability','interpreter','latex')
hold on
x_mean=sqrt ((p0* (x-1)+C0-H_mean)./x.^3./(1-(1+gamma./x.^3).^(-1/3)));
x_mean=real (vx_mean);
vp=[find(vx_mean>0)];
fvn=fvp (end:-1:1);
mvx_mean=[0,vx_mean(fvp),-vx_mean(fvn),0]
plot([min(x(fvp)),x(fvp),x(fvn),min(x(fvp))],pmvx_mean,'k--','linewidth',2)
label('$x$','interpreter','latex','fontsize',fs)
(0)
plot(linspace(0,5,100),zeros(1,100),'k' 'linewidth',1)
et(gca,'fontsize',fs)
xticks(0:0.2:2)
yticks(-2:2)
```


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