EXISTENCE AND UNIQUENESS OF GLOBAL WEAK SOLUTIONS TO STRAIN-LIMITING VISCOELASTICITY WITH DIRICHLET BOUNDARY DATA*

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Abstract. We consider a system of evolutionary equations that is capable of describing certain viscoelastic effects in linearized yet nonlinear models of solid mechanics. The constitutive relation, involving the Cauchy stress, the small strain tensor and the symmetric velocity gradient, is given in an implicit form. For a large class of these implicit constitutive relations, we establish the existence and uniqueness of a global-in-time large-data weak solution. Then we focus on the class of so-called limiting strain models, i.e., models for which the magnitude of the strain tensor is known to remain small a priori, regardless of the magnitude of the Cauchy stress tensor. For this class of models, a new technical difficulty arises. The Cauchy stress is only an integrable function over its domain of definition, resulting in the underlying function spaces being nonreflexive and thus the weak compactness of bounded sequences of elements of these spaces is lost. Nevertheless, even for problems of this type we are able to provide a satisfactory existence theory, as long as the initial data have finite elastic energy and the boundary data fulfil natural compatibility conditions.

Key words. nonlinear viscoelasticity, strain-limiting theory, evolutionary problem, global existence, weak solution, regularity

AMS subject classifications. 35M13, 35K99, 74D10, 74H20

1. Introduction.

This paper is devoted to the study of the following nonlinear system of partial differential equations (PDEs).We assume that $\Omega \subset \mathbb{R}^d$ is a given bounded open domain. We denote the associated parabolic cylinder by $Q := (0,T) \times \Omega$ and its spatial boundary by $\Gamma := (0,T) \times \partial\Omega$, where $T > 0$ is the length of the time interval of interest. For given data $\mathbf{G} : \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}^{d\times d}_{sym}$, $\boldsymbol{f} : Q \to \mathbb{R}^d$, $\boldsymbol{u}_I : \Omega \to \mathbb{R}^d$, $\boldsymbol{v}_0 : \Omega \to \mathbb{R}^d$, $\boldsymbol{u}_\Gamma : \Gamma \to \mathbb{R}^d$ and α, $\beta > 0$, we seek a couple $(\boldsymbol{u}, \mathbf{T}) : Q \to \mathbb{R}^d \times \mathbb{R}^{d\times d}_{sym}$ satisfying

$$\partial^2_{tt}\boldsymbol{u} - \operatorname{div}\mathbf{T} = \boldsymbol{f} \quad \text{in } Q, \tag{1.1a}$$

$$\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta\boldsymbol{\varepsilon}(\partial_t\boldsymbol{u}) = \mathbf{G}(\mathbf{T}) \quad \text{in } Q, \tag{1.1b}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_I, \quad \partial_t\boldsymbol{u}(0) = \boldsymbol{v}_0 \quad \text{in } \Omega, \tag{1.1c}$$

$$\boldsymbol{u} = \boldsymbol{u}_\Gamma \quad \text{on } \Gamma. \tag{1.1d}$$

Here, (1.1a) represents an approximation[1] of the balance of linear momentum, where $\boldsymbol{f}$ is the density of the external body forces, $\boldsymbol{u}$ is the displacement, $\mathbf{T}$ denotes the Cauchy stress tensor and the operator div denotes the divergence operator with respect to the spatial variables $x_1, \ldots, x_d$. The Cauchy stress tensor $\mathbf{T}$ is implicitly related to the small strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}})$ and to the symmetric velocity gradient $\boldsymbol{\varepsilon}(\partial_t\boldsymbol{u}) := \partial_t(\boldsymbol{\varepsilon}(\boldsymbol{u}))$ via (1.1b). The initial displacement and the initial velocity are given by (1.1c) and the Dirichlet boundary condition for the displacement

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[1] In fact, the density ϱ of the solid should also appear in (1.1a). In principle, ϱ is a function of space and should satisfy the equation for the balance of mass. Since we are dealing with small strains here, that is, the case when the displacement gradient of the solid is small, assuming that the solid is homogeneous at initial time $t = 0$, we consider the density to be equal to a constant for all times $t \in (0,T)$. We scale the density to be identically equal to one for simplicity. We refer also to the discussion in [8]. However, under suitable assumptions, we can extend the results presented herein to the case of variable density.

is represented by (1.1d). A more detailed discussion concerning the relevance of (1.1) to problems in viscoelasticity is contained in Section 1.2.

It remains to specify the form of the implicit constitutive law (1.1b). The minimal assumptions imposed on the mapping $\mathbf{G}$ throughout the paper are the following. We assume that the function $\mathbf{G} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is a continuous mapping such that, for some $p \in [1,\infty)$, some positive constants C_1 and C_2, and for all $\mathbf{T}, \mathbf{W} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, the following inequalities hold:

$$\big(\mathbf{G}(\mathbf{T}) - \mathbf{G}(\mathbf{W})\big)\cdot(\mathbf{T}-\mathbf{W}) \geq 0, \tag{A1}$$

$$\mathbf{G}(\mathbf{T})\cdot\mathbf{T} \geq C_1|\mathbf{T}|^p - C_2, \tag{A2}$$

$$|\mathbf{G}(\mathbf{T})| \leq C_2(1+|\mathbf{T}|)^{p-1}, \tag{A3}$$

where $|\cdot|$ stands for the usual Frobenius matrix norm. Assumptions (A1)–(A3) are sufficient for the existence and uniqueness of a weak solution provided that $p \in (1,\infty)$. For $p = 1$, however, we must impose a more restrictive assumption because of the lack of compactness experienced when working in $L^1(Q)$. Namely, we assume that there exists a strictly convex function $\phi \in \mathcal{C}^2(\mathbb{R}_+;\mathbb{R}_+)$ such that $\phi(0) = \phi'(0) = 0$, $|\phi''(s)| \leq C(1+s)^{-1}$ for every $s \in \mathbb{R}_+$, and for all $\mathbf{T} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ there holds

$$\mathbf{G}(\mathbf{T}) = \frac{\phi'(|\mathbf{T}|)\mathbf{T}}{|\mathbf{T}|}. \tag{A4}$$

We note that the structure of the constitutive relation (1.1b) is vital to many of the estimates in our work. In particular, we have the following memory kernel structure:

$$\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) = \mathrm{e}^{-\frac{\alpha}{\beta}t}\boldsymbol{\varepsilon}(\boldsymbol{u}(0)) + \int_0^t \frac{\mathrm{e}^{\frac{\alpha}{\beta}(\tau-t)}}{\beta}\mathbf{G}(\mathbf{T}(\tau))\,\mathrm{d}\tau.$$

This representation of the strain $\boldsymbol{\varepsilon}(\boldsymbol{u})$ allows us to obtain bounds on this term, given bounds on the initial strain $\boldsymbol{\varepsilon}(\boldsymbol{u}(0))$ and the stress tensor $\mathbf{T}$.

Concerning the initial and boundary data, we assume that we are given a function $\boldsymbol{u}_0 : Q \to \mathbb{R}^d$ fulfilling, in an appropriate sense, the initial and boundary conditions

$$\begin{aligned} \boldsymbol{u}_0(0) &= \boldsymbol{u}_I && \text{in } \Omega,\\ \partial_t\boldsymbol{u}_0(0) &= \boldsymbol{v}_0 && \text{in } \Omega,\\ \boldsymbol{u}_0 &= \boldsymbol{u}_\Gamma && \text{on } \Gamma. \end{aligned}$$

Although not the standard approach, such a joint treatment of the initial and boundary conditions simplifies the exposition here, as it avoids nonessential technical details concerning the choice of function spaces for the data and the corresponding trace theorems. We henceforth formulate all assumptions on the initial and boundary data in terms of $\boldsymbol{u}_0$, rather than $\boldsymbol{u}_I$, $\boldsymbol{v}_0$ and $\boldsymbol{u}_\Gamma$. While this choice may appear nontrivial upon first glance, the function spaces for $\boldsymbol{u}_0$ stated below are the same as those for the weak solution $\boldsymbol{u}$. Hence it is necessary that such a $\boldsymbol{u}_0$ exists. Otherwise our construction of a weak solution would not be possible.

1.1. Statement of the main results. First, we formulate our result for the case when $p > 1$. Here, p and p' are dual exponents.

Theorem 1.1. *Let $1 < p < 2d/(d-2)$, let $\mathbf{G}$ satisfy (A1), (A2) and (A3), and let α, $\beta > 0$ be arbitrary. Assume that the data satisfy the following hypotheses:*

$$\begin{aligned} &\boldsymbol{u}_0 \in W^{1,p'}(0,T;W^{1,p'}(\Omega;\mathbb{R}^d)) \cap W^{2,p}(0,T;(W^{1,p'}_0(\Omega;\mathbb{R}^d))^*) \cap \mathcal{C}^1([0,T];L^2(\Omega;\mathbb{R}^d)),\\ &\boldsymbol{f} \in L^p(0,T;(W^{1,p'}_0(\Omega;\mathbb{R}^d))^*). \end{aligned} \tag{1.2}$$

There exists a couple $(\boldsymbol{u},\mathbf{T})$ fulfilling

$$\boldsymbol{u} \in \mathcal{C}^1([0,T];L^2(\Omega;\mathbb{R}^d)) \cap W^{1,p'}(0,T;W^{1,p'}(\Omega;\mathbb{R}^d)) \cap W^{2,p}(0,T;(W^{1,p'}_0(\Omega;\mathbb{R}^d))^*), \tag{1.3}$$

$$\mathbf{T} \in L^p(0,T;L^p(\Omega;\mathbb{R}^{d\times d}_{sym})) \tag{1.4}$$

and solving (1.1) *in the following sense:*

$$\langle \partial_{tt}\boldsymbol{u}, \boldsymbol{w}\rangle + \int_\Omega \mathbf{T}\cdot\nabla\boldsymbol{w} = \langle \boldsymbol{f}, \boldsymbol{w}\rangle \qquad \forall\, \boldsymbol{w}\in W_0^{1,p'}(\Omega;\mathbb{R}^d), \quad \textit{for a.e. } t\in(0,T), \tag{1.5}$$

$$\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathbf{G}(\mathbf{T}) \qquad \textit{a.e. in } Q, \tag{1.6}$$

and

$$\boldsymbol{u}-\boldsymbol{u}_0 = \mathbf{0} \quad \textit{a.e. on } \Gamma \qquad \textit{and} \qquad \boldsymbol{u}(0)-\boldsymbol{u}_0(0) = \partial_t\boldsymbol{u}(0)-\partial_t\boldsymbol{u}_0(0) = \mathbf{0} \quad \textit{a.e. in } \Omega. \tag{1.7}$$

Furthermore, the function $\boldsymbol{u}$ *is unique. If, additionally, the mapping* $\mathbf{G}$ *is strictly monotonic, then* $\mathbf{T}$ *is also unique.*

Before proceeding, we first comment on the assertions of Theorem 1.1. The proof of Theorem 1.1 is based on the relevant a priori estimates. The function spaces considered in (1.3), (1.4) correspond to the structural assumptions imposed on $\mathbf{G}$, namely the coercivity assumption (A2) and the growth condition (A3). Since $p>1$, we have a "standard" function space setting, so the nonlinearity in (1.6) can be identified by using a modification of Minty's method. Theorem 1.1 can also be understood as an extension of the results established in [8]. In a similar way to the work presented here, the authors of [8] treat a viscoelastic solid model of generalized Kelvin–Voigt type. However, they consider a constitutive relation for the Cauchy stress of the following explicit form:

$$\mathbf{T} = \mathbf{T}_{el}(\boldsymbol{\varepsilon}(\boldsymbol{u})) + \mathbf{T}_{vis}(\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u})) \qquad \text{a.e. in } Q.$$

The regularity results for such models are available in [7]. It is remarkable that while (1.6) can be fully justified from the physical point of view via implicit constitutive theory, (see [29], [31] for example) the above explicit form $\mathbf{T} = \mathbf{T}_{el} + \mathbf{T}_{vis}$ can be justified for particular choices of $\mathbf{T}_{el}$ and $\mathbf{T}_{vis}$ only.

In contrast with the case $p>1$, almost none of the above applies in the case that $p=1$, or for the limit, as $p\to 1_+$, of the sequence of solutions constructed in Theorem 1.1. Indeed, for similar models in the purely elastic, steady setting, it was demonstrated in [3] that $\mathbf{T}$ is, in general, a Radon measure and therefore one cannot consider (1.6) pointwise in Q. Nevertheless, it was shown there that under some structural assumptions on $\mathbf{G}$ (corresponding to (A4)), $\mathbf{T}$ is integrable.

A similar situation is studied in [2] but with $p\to\infty$. In general, this leads to solutions $\boldsymbol{u}$ in the spaces of bounded variation. However, under a structural assumption related to (A4), one can again overcome such difficulties and show the existence of a solution that belongs to a Sobolev space. We expect something similar in our setting when $p=1$. Therefore, in order to avoid difficulties associated with the interpretation of $\partial_{tt}\boldsymbol{u}$ and the interpretation of the sense in which the initial data are attained, we assume here, for simplicity, that the right-hand side $\boldsymbol{f}\in L^2(Q;\mathbb{R}^d)$. We also use a variational formulation which is slightly different from (1.5). Nevertheless, we will show that (1.5) still holds locally in $(0,T)$ and, in the case of more regular initial data, we are able to show the continuity with respect to time of $\boldsymbol{u}$ and $\partial_t\boldsymbol{u}$ on the whole time interval $[0,T]$.

Inspired by [3], if $p=1$ we assume in addition to (A1)–(A3) that we have (A4). It follows from these structural assumptions that, for all $s\in\mathbb{R}_+$, we have

$$\begin{aligned} \frac{C_1 s}{2} - C_2 \le \phi(s) &\le C_2 s,\\ 0\le \phi'(s) &\le C_2. \end{aligned}$$

Since ϕ is convex, we deduce that there exists an $L>0$ such that

$$L := \lim_{s\to\infty}\phi'(s) \ge \phi'(t) \qquad \forall\, t\in\mathbb{R}. \tag{1.8}$$

The number L plays an essential role in the subsequent analysis, in particular in the assumptions on the initial and boundary data. Indeed, thanks to (A4), we see that

$$L = \lim_{|\mathbf{W}|\to\infty}|\mathbf{G}(\mathbf{W})| \ge |\mathbf{G}(\mathbf{T})| \qquad \forall\, \mathbf{T}\in\mathbb{R}^{d\times d}_{\mathrm{sym}}. \tag{1.9}$$

Hence, if (1.1b) is satisfied, we necessarily have that

$$|\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u})| \le L \qquad \text{a.e. in } Q. \tag{1.10}$$

Consequently, if such a $\boldsymbol{u}$ exists, it is natural to assume that (1.10) must also hold for the initial and boundary data. That is, we must have

$$|\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0)| \le L \qquad \text{a.e. in } Q. \tag{1.11}$$

In fact, we require in the existence analysis that (1.11) is satisfied with a strict inequality sign. We call this the *safety strain condition.*

Theorem 1.2. *For some strictly convex $\phi \in \mathcal{C}^2(\mathbb{R}_+;\mathbb{R}_+)$, let $\mathbf{G}$ satisfy (A1)–(A4) with $p = 1$. Assume that the data satisfy the following hypotheses:*

$$\begin{aligned} &\boldsymbol{u}_0 \in W^{1,\infty}(0,T;W^{1,2}(\Omega;\mathbb{R}^d)) \cap W^{2,1}(0,T;L^2(\Omega;\mathbb{R}^d)),\\ &\boldsymbol{f} \in L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \end{aligned} \tag{1.12}$$

with the safety strain condition

$$\|\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\|_{L^\infty(Q;\mathbb{R}^{d\times d}_{sym})} < L, \tag{1.13}$$

and for every $\delta > 0$ we have

$$\operatorname*{ess\,sup}_{(t,x)\in(\delta,T)\times\Omega} |\partial_{tt}\boldsymbol{\varepsilon}(\boldsymbol{u}_0(t,x))| < \infty. \tag{1.14}$$

There exists a unique couple $(\boldsymbol{u},\mathbf{T})$ fulfilling

$$\boldsymbol{u} \in W^{1,\infty}(0,T;W^{1,2}(\Omega;\mathbb{R}^d)) \cap \mathcal{C}^1([0,T];L^2(\Omega;\mathbb{R}^d)) \cap W^{2,2}(\delta,T;L^2(\Omega;\mathbb{R}^d)), \tag{1.15}$$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) \in L^\infty(Q;\mathbb{R}^{d\times d}_{sym}), \tag{1.16}$$

$$\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}) \in L^\infty(Q;\mathbb{R}^{d\times d}_{sym}), \tag{1.17}$$

$$\mathbf{T} \in L^1(0,T;L^1(\Omega;\mathbb{R}^{d\times d}_{sym})), \tag{1.18}$$

for every $\delta > 0$, and satisfying

$$\int_\Omega \partial_{tt}\boldsymbol{u}\cdot\boldsymbol{w} + \mathbf{T}\cdot\nabla\boldsymbol{w}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{w}\,\mathrm{d}x \qquad \forall\,\boldsymbol{w} \in W^{1,\infty}_0(\Omega;\mathbb{R}^d), \quad \textit{for a.e. } t \in (0,T), \tag{1.19}$$

$$\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathbf{G}(\mathbf{T}) \qquad \textit{a.e. in } Q, \tag{1.20}$$

and

$$\boldsymbol{u} - \boldsymbol{u}_0 = \mathbf{0} \quad \textit{a.e. on } \Gamma \qquad \textit{and} \qquad \boldsymbol{u}(0) - \boldsymbol{u}_0(0) = \partial_t\boldsymbol{u}(0) - \partial_t\boldsymbol{u}_0(0) = \mathbf{0} \quad \textit{a.e. in } \Omega. \tag{1.21}$$

This theorem answers the question of existence of weak solutions to the problem under the assumptions (A1)–(A4) when $p = 1$ and therefore provides an existence result for limiting strain models where the symmetric displacement gradient and symmetric velocity gradient remain bounded. In Section 1.2, we discuss the physical background and the importance of this model.

In our proof, we rely on an approximation of the strain-limiting problem where in the constitutive relation we replace $\mathbf{G}$ with $\mathbf{G}_n(\mathbf{T}) = \mathbf{G}(\mathbf{T}) + \frac{\mathbf{T}}{n}$. However, if we consider a regularisation of the form $\mathbf{G}_n(\mathbf{T}) = \mathbf{G}(\mathbf{T}) + \frac{\mathbf{T}}{n(1+|\mathbf{T}|^{1-\frac{1}{n}})}$, taking the limit $n \to \infty$ exactly corresponds to taking the limit $p \to 1_+$. Such a regularisation is considered in [9], for example. However, in order to simplify the exposition, we only consider the linear regularisation term of the form $\frac{\mathbf{T}}{n}$.

A similar existence result was established recently in [11]. However, there are certain essential differences, which make the results of the present paper much stronger. First, in [11] the authors only consider the prototypical model

$$\mathbf{G}(\mathbf{T}) := \frac{\mathbf{T}}{(1+|\mathbf{T}|^q)^{\frac{1}{q}}}, \tag{1.22}$$

while we are able to cover here a more general class of models under hypothesis (A4). The corresponding potential ϕ (whose existence is assumed in (A4)) for the model (1.22) is given by

$$\phi(s) := \int_0^s \frac{t}{(1+t^q)^{\frac{1}{q}}}\,\mathrm{d}t, \qquad s \in \mathbb{R}_+.$$

The role of the parameter q in (1.22) is indicated in Fig. 1. Furthermore, the paper [11] is

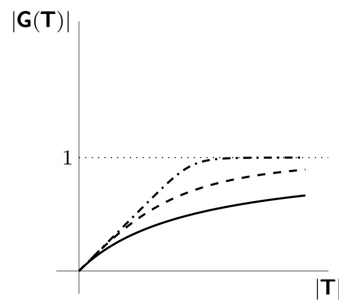

Fig. 1: Dependence of $|\mathsf{G}|$ on $|\mathsf{T}|$ for the prototype model (1.22). The three curves correspond to $q = 1$ (solid curve), $q = 2$ (dashed curve) and $q = 10$ (dash-dotted curve). Clearly, $|\mathsf{G}(\mathsf{T})|$ tends to 1 more rapidly with increasing q when $|\mathsf{G}(\mathsf{T})| > 1$.

concerned with the spatially periodic setting, which simplifies the analysis in an essential way, most notably with regards to the derivation of the relevant a priori estimates. We are not able to derive estimates of the same strength as those in [11]. This is the consequence of working in the nonperiodic setting, as well as the choice of a more general constitutive relation. However, by an application of Chacon's biting lemma and Egoroff's theorem, we are able to overcome these difficulties and obtain a complete existence result.

Finally, in [11] the initial data are assumed to be quite regular. They are supposed to belong to the Sobolev space $W^{k,2}(\Omega;\mathbb{R}^d)$ with $k > \frac{d}{2}$. This is related to the choice of the method used to prove the existence of a weak solution. In this paper we do not require such strong regularity of the initial data, although in the current setting it is difficult to describe the correct space-time trace spaces, because we are dealing with L^∞-type spaces and symmetric gradients. Since we want to state the result in its full generality, and, in particular, to be able to admit time-dependent boundary data, we assume a certain compatibility condition via an *a priori* prescribed space-time function $\boldsymbol{u}_0$ that we use in order to impose the initial and boundary conditions. This further justifies our choice of working with a function $\boldsymbol{u}_0$ incorporating both the boundary and the initial data.

The existence of $\boldsymbol{u}_0$ satisfying the safety strain condition (1.13) is necessary for the existence of a solution and is used when deriving appropriate *a priori* estimates. The assumption $(1.12)_1$ concerning the temporal regularity of $\boldsymbol{u}_0$ is required in order to ensure that $\boldsymbol{u}_0$ and $\partial_t \boldsymbol{u}_0$ have meaningful traces at time $t = 0$. Finally, the assumption (1.14) prescribes the required temporal smoothness of the boundary data. It only involves $t \in (\delta, T)$ for $\delta > 0$. Hence it does not affect the regularity of the initial condition or the compatibility between the boundary and initial data. We give several examples for simplified settings regarding the boundary conditions in the following remark.

Remark 1.3. We discuss two cases of boundary and initial data from (1.1c)–(1.1d) for which it is easy to construct a function $\boldsymbol{u}_0$ that satisfies the assumptions (1.12)–(1.14).

Boundary data independent of time. Suppose that $\boldsymbol{u}_\Gamma$ is independent of time and $\boldsymbol{u}_I \in W^{1,2}(\Omega;\mathbb{R}^d)$ satisfies the compatibility condition $\boldsymbol{u}_I|_{\partial\Omega} = \boldsymbol{u}_\Gamma$. The boundary data are independent of time so it is natural to assume that $\boldsymbol{v}_0 \in W_0^{1,2}(\Omega;\mathbb{R}^d)$, where

$$\|\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_I) + \beta\boldsymbol{\varepsilon}(\boldsymbol{v}_0)\|_{L^\infty(\Omega;\mathbb{R}^{d\times d}_{sym})} < L. \tag{1.23}$$

We set

$$\boldsymbol{u}_0(t,x) := \mathrm{e}^{-\frac{\alpha t}{\beta}}\boldsymbol{u}_I(x) + \frac{\alpha\boldsymbol{u}_I(x)+\beta\boldsymbol{v}_0(x)}{\alpha}(1-\mathrm{e}^{-\frac{\alpha t}{\beta}}).$$

A direct computation yields that

$$\partial_t\boldsymbol{u}_0(t,x) = \boldsymbol{v}_0(x)\,\mathrm{e}^{-\frac{\alpha t}{\beta}},$$

and thus $\boldsymbol{u}_0(0,x) = \boldsymbol{u}_I(x)$, $\partial_t\boldsymbol{u}_0(0,x) = \boldsymbol{v}_0(x)$ for $x \in \Omega$ and $\boldsymbol{u}_0|_\Gamma = \boldsymbol{u}_\Gamma$. Moreover,

$$\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0) = \alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_I) + \beta\boldsymbol{\varepsilon}(\boldsymbol{v}_0).$$

Consequently, $\boldsymbol{u}_0$ satisfies (1.13) provided (1.23) holds. The validity of (1.14) is obvious.

Time-dependent boundary data. In this setting, we assume the existence of a function $\tilde{\boldsymbol{u}}$ such that $\tilde{\boldsymbol{u}}(0,x) = \boldsymbol{u}_I(x)$ for $x \in \Omega$ and $\tilde{\boldsymbol{u}}|_\Gamma = \boldsymbol{u}_\Gamma$. In addition, we assume the natural compatibility condition $\boldsymbol{v}_0(\cdot) = \partial_t\boldsymbol{u}_\Gamma(0,\cdot)$ on $\partial\Omega$. We adopt the following assumption on $\tilde{\boldsymbol{u}}$ and $\boldsymbol{v}_0$:

$$\|\alpha\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}) + \beta(\partial_t\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}) - \partial_t\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}(0,\cdot)) + \boldsymbol{\varepsilon}(\boldsymbol{v}_0(\cdot)))\|_{L^\infty(Q;\mathbb{R}^{d\times d}_{sym})} < L. \tag{1.24}$$

We define

$$\boldsymbol{u}_0(t,x) := \tilde{\boldsymbol{u}}(t,x) + \frac{\beta(\boldsymbol{v}_0(x) - \partial_t\tilde{\boldsymbol{u}}(0,x))}{\alpha}(1-\mathrm{e}^{-\frac{\alpha t}{\beta}}).$$

Clearly, $\boldsymbol{u}_0(0,x) = \tilde{\boldsymbol{u}}(0,x) = \boldsymbol{u}_I(x)$ for $x \in \Omega$ and $\boldsymbol{u}_0 = \boldsymbol{u}_\Gamma$ on Γ. The time derivative of $\boldsymbol{u}_0$ is

$$\partial_t\boldsymbol{u}_0(t,x) = \partial_t\tilde{\boldsymbol{u}}(t,x) + (\boldsymbol{v}_0(x) - \partial_t\tilde{\boldsymbol{u}}(0,x))\,\mathrm{e}^{-\frac{\alpha t}{\beta}}.$$

Thus $\partial_t\boldsymbol{u}_0(0,x) = \boldsymbol{v}_0(x)$ for $x \in \Omega$. In addition, since

$$\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0) = \alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_I) + \beta(\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_I) - \partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_I(0)) + \boldsymbol{\varepsilon}(\boldsymbol{v}_0)),$$

we see that (1.13) is equivalent to (1.24). The assumption (1.14) is only related to our extension of the boundary data inside of Ω and the temporal regularity of the boundary data.

1.2. Relevance to the modelling of viscoelastic solids. With these results in mind, we now discuss the importance of such problems. We often encounter materials exhibiting viscoelastic response. By definition, viscoelasticity involves the material response of both elastic solids and viscous fluids, which can be modelled linearly or nonlinearly. We refer to [13] for an extensive overview. On the other hand, it is well-known that implicit constitutive theories allow for a more general structure in modelling than explicit ones (cf. [29], [30]), where the strain can be given as a function of the stress. Indeed, this is the case in our constitutive relation (1.1b) in system (1.1). Rajagopal's main contribution [31] to the theory was to show that a nonlinear relationship between the stress and the strain can be obtained after linearizing the strain. The relation (1.1b) is obtained by Erbay and Şengül in [18] as a result of application of the linearization procedure introduced by Rajagopal (see e.g., [33] for details) to the relation between the stress and the strain tensors under the assumption that the magnitude of the strain is small. For models of this type it is possible that once the magnitude of the strain has reached a certain limiting value (as is the case in Theorem 1.2), any further increase of the magnitude of the stress causes no changes in the strain. These models are called *strain-limiting* and/or *strain-locking* models and such behaviour has been observed in numerous experiments (see [15] and references therein). For a further discussion of such models in the purely elastic setting or in the setting of the generalized Kelvin–Voigt model we refer to [8], and in the viscoelastic setting to [15, 18, 14, 12].

We note that the term *ideal-locking material* was introduced by Prager [28] (see also [27]). In the extreme cases, the strain (resp. stress) can increase arbitrarily without any further increase in the stress (resp. strain). However, in his study Prager neglects the elastic stresses in comparison to the much larger stresses that can be supported in the locked state. This is a more limited setting than that given by Rajagopal's framework of implicit constitutive theory.

A potential application of strain-limiting models is in the context of fracture mechanics and crack propagation. Under a linear relationship between the stress and strain, in the anti-plane setting, the stress and the strain behave like $r^{-\frac{1}{2}}$, where r is the distance to the crack tip [32]. In particular, both the stress and the strain experience a singularity at the crack tip. However, this contradicts the standing assumption in the derivation of the model, namely, that one is in the small-strain regime. A better model for studying fracture in brittle materials might ensure that the magnitude of the strain tensor remains bounded *a priori* even in the presence of a stress singularity, as is the case for the model considered here.

There has been some analysis in the literature of strain-limiting models of fracture, particularly in the time-independent setting from a computational point of view. In [24, 25], the authors consider a strain-limiting model in the anti-plane strain setting, studying a plate with a V-notch. The one-dimensional setting allows the reduction of the problem by use of the Airy stress function. Studying the problem numerically, the stress is shown to concentrate around the tip of the V-notch. We notice that this contradicts the asymptotic analysis performed in [35], where the stress is shown to vanish in the vicinity of the crack tip. This conflict is likely due to the fact that solutions of nonlinear PDEs can exhibit very different behaviour to what is suggested by formal asymptotic analysis. We mention the similar studies in [26, 10, 21], considering different geometric settings.

Furthermore, there has been recent study of a finite-element discretisation of problems based on strain-limiting elasticity in [37]. The authors study the time-independent problem in three different crack geometries in the anti-plane setting. The numerical results presented in [37] indicate that the linearised strain remains bounded *a priori* below a fixed value, while the value of the stress is able to be very high. Indeed, near the crack tip, the stress grows significantly faster than the strain. The strain does not exhibit a singularity near the crack tip, in contrast to the linear model, which is also studied in [37] for comparison.

The aforementioned literature all deal with time-independent problems. Here, we only study the time-dependent problem. Furthermore, we only consider viscoelastic solids. However, the study of implicitly constituted fluids is a very rich, active area of current research. We refer to [29, 30] for the modelling background on these fluids, of which strain-limiting fluids are a special subclass. For the corresponding mathematical analysis, we point the reader to [4] for the steady case and [5] for the unsteady case; however, we note that those studies do not cover a strain-limiting problem analogous to the one explored here. We refer to [6] for the analysis of a related parabolic type problem with the bounded gradient.

Strain-limiting problems have also been considered in the quasi-static setting, that is, with the term $\partial^2_{tt}\boldsymbol{u}$ is neglected from the balance of momentum equation. In [22], the authors consider the quasi-static system in a domain with a fixed crack set. Under certain conditions on the constitutive relation, they show that a weak solution of the problem exists. However, they are only able to show that a weak solution exists in the space of measures. In particular, the stress tensor is shown to be in the space $C([0,T];\mathcal{M}(\overline{\Omega})^{d\times d})$, where $\mathcal{M}(\overline{\Omega})$ is the space of Radon measures on $\overline{\Omega}$. We mention also [23] for a similar problem.

A similar problem is studied in [16] but in an abstract setting. The authors consider

$$\partial^2_{tt}u + A\partial_t u + Bu = f,$$

where u is scalar-valued. Assuming that A, B are operators on 'nice' function spaces and by considering a sequence of approximating problems based on temporal discretization, the authors prove the existence of a weak solution to this doubly nonlinear problem. We also mention the related work [17], where the authors consider

$$\partial^2_{tt}u - \operatorname{div}(F(\nabla\partial_t u) + \nabla u) = f,$$

supplemented with a Dirichlet boundary condition. The function F satisfies a suitable growth condition; namely, F is assumed to be a continuous, monotone function such that there exists an N-function (see [1, p. 228] for the definition) φ for which

$$F(\boldsymbol{v})\cdot\boldsymbol{v} \geq c\big(\varphi(\boldsymbol{v}) + \varphi^*(F(\boldsymbol{v}))\big),$$

where φ^* is the convex conjugate of φ. The existence of such a φ ensures that one is not in any kind of strain-limiting setting. In particular, it is not the case that ∇u is *a priori* uniformly bounded on its domain of definition.

Finally, we note the analysis in [36]. There, the author considers the system of equations

$$\partial_{tt}^2 \boldsymbol{u} - \operatorname{div}\left(\mathsf{G}(\nabla\partial_t \boldsymbol{u}, \nabla \boldsymbol{u})\right) = \boldsymbol{f}.$$

The restrictions on G are however such that any physically realistic constitutive relation is excluded. In particular, the uniform strict monotonicity assumption eliminates the strain-limiting case. However, the author suggests that the methods employed in the paper could be used in order to extend the results to physically more realistic cases. We note also that in [36] the full gradient is considered, rather than the symmetric gradient as is discussed here. One should refer to the review [13] for more related work on classical nonlinear viscoelasticity.

Now we introduce some basic kinematics in order to discuss these limiting strain models from a mathematical perspective. We denote by $\boldsymbol{u}(\mathbf{X},t) := \boldsymbol{x}(\mathbf{X},t) - \mathbf{X}$ the displacement of a given body at a space-time point $(\mathbf{X},t)$, where $\mathbf{X}$ is the position vector in the reference configuration and $\boldsymbol{x}(\mathbf{X},t)$ is the position vector in the current configuration. We denote the deformation of the body, which is assumed to be stress-free initially, by $\boldsymbol{\chi}(\mathbf{X},t)$. The deformation gradient is defined as $\mathbf{F} = \partial\boldsymbol{\chi}/\partial\mathbf{X}$. We define the *left Cauchy–Green deformation tensor* as $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$, the velocity as $\boldsymbol{v} = \partial\boldsymbol{\chi}/\partial t$ and denote by D the symmetric part of the gradient of the velocity field $\mathsf{L} = \nabla_{\mathbf{x}}\boldsymbol{v}$. Under the small displacement gradient assumption, that is,

$$\|\nabla_{\mathbf{X}}\boldsymbol{u}\|_{L^\infty(Q;\mathbb{R}^{d\times d})} = O(\delta), \qquad 0<\delta\ll 1, \tag{1.25}$$

one can consider the linearized strain defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\left[\nabla_{\mathbf{X}}\boldsymbol{u} + (\nabla_{\mathbf{X}}\boldsymbol{u})^{\mathrm{T}}\right]. \tag{1.26}$$

We consider a general constitutive relation between the Cauchy stress tensor T, the deformation $\mathbf{B}$ and the symmetric velocity gradient $\mathbf{D}$. Noticing that $\mathbf{B} = \mathbf{I} + 2\boldsymbol{\varepsilon} + (\nabla_{\mathbf{X}}\boldsymbol{u})(\nabla_{\mathbf{X}}\boldsymbol{u})^{\mathrm{T}}$ and linearising under the assumptions (1.25), we obtain a relationship between the Cauchy stress, the linearised strain and the strain rate $\boldsymbol{\varepsilon}(\partial_t\boldsymbol{u})$. In particular, we obtain (1.1b).

As is explained in [15], in the purely elastic setting, starting from the following constitutive relation between the stress and the strain

$$\mathsf{G}(\mathsf{T},\mathsf{B}) = \mathbf{0}, \tag{1.27}$$

for frame-indifferent and isotropic bodies, one can obtain the representation

$$\begin{aligned}\mathsf{G}(\mathsf{T},\mathsf{B}) = {}& \chi_0\mathsf{I} + \chi_1\mathsf{T} + \chi_2\mathsf{T} + \chi_3\mathsf{T}^2 + \chi_4\mathsf{B}^2 + \chi_5(\mathsf{T}\mathsf{B}+\mathsf{B}\mathsf{T})\\ &+ \chi_6(\mathsf{T}^2\mathsf{B}+\mathsf{B}\mathsf{T}^2) + \chi_7(\mathsf{B}^2\mathsf{T}+\mathsf{T}\mathsf{B}^2) + \chi_8(\mathsf{T}^2\mathsf{B}^2+\mathsf{B}^2\mathsf{T}^2),\end{aligned} \tag{1.28}$$

where the functions χ_i, $i=0,\dots,8$, depend only on the scalar invariants of T and B, which can be expressed in terms of

$$\operatorname{tr}\mathsf{T}, \operatorname{tr}\mathsf{B}, \operatorname{tr}\mathsf{T}^2, \operatorname{tr}\mathsf{B}^2, \operatorname{tr}\mathsf{T}^3, \operatorname{tr}\mathsf{B}^3, \operatorname{tr}\mathsf{T}\mathsf{B}, \operatorname{tr}\mathsf{T}^2\mathsf{B}, \operatorname{tr}\mathsf{T}\mathsf{B}^2, \operatorname{tr}\mathsf{T}^2\mathsf{B}^2.$$

Under the smallness assumption (1.25), we have that $|\mathsf{B} - (\mathsf{I}+\boldsymbol{\varepsilon})| = O(\delta^2)$, with $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u})$. Thus, at the end of the linearization process, (1.28) gives a nonlinear relationship between T and $\boldsymbol{\varepsilon}$. In many studies a simpler subclass of constitutive relations than (1.28) is considered, namely

$$\mathsf{B} = \tilde{\chi}_0\mathsf{I} + \tilde{\chi}_1\mathsf{T} + \tilde{\chi}_2\mathsf{T}^2. \tag{1.29}$$

Under the assumption (1.25), the equality (1.29) becomes

$$\boldsymbol{\varepsilon} = \bar{\chi}_0\mathsf{I} + \bar{\chi}_1\mathsf{T} + \bar{\chi}_2\mathsf{T}^2, \tag{1.30}$$

with some invariant-dependent coefficients $\bar{\chi}_i$, $i = 0, 1, 2$. The analysis of a limiting strain problem with a constitutive relation of the form $\boldsymbol{\varepsilon} = \mathbf{G}(\mathbf{T})$, which is a more general version of (1.30), with a bounded mapping $\mathbf{G}$, as those considered here, was also studied in [9], [3], where the authors highlight the analytical difficulties associated with such models, most notably the lack of weak compactness of approximations to the stress tensor in $L^1(\Omega; \mathbb{R}^{d\times d}_{sym})$. We rely on methods developed in [3] in order to show that (1.19) holds for our proposed solution of the problem. The additional time-dependence here presents further difficulties in the analysis. In particular, we must develop suitable space-time estimates.

As is discussed in [34], we can consider a general implicit constitutive relation of the form

$$\mathbf{G}(\mathbf{T}, \mathbf{B}, \mathbf{D}) = \mathbf{0}. \tag{1.31}$$

Motivated by the constitutive equation for the classical Kelvin-Voigt model and considering the simplification of (1.31) under the assumption of frame-indifference and isotropy, we obtain the following subclass of such implicit models:

$$\alpha\mathbf{B} + \beta\mathbf{D} = \gamma_0\mathbf{I} + \gamma_1\mathbf{T} + \gamma_2\mathbf{T}^2, \tag{1.32}$$

where $\gamma_i = \gamma_i(I_1, I_2, I_3)$, $i = 0, 1, 2$, $I_1 = \mathrm{tr}\mathbf{T}, I_2 = \frac{1}{2}\mathrm{tr}\mathbf{T}^2, I_3 = \frac{1}{3}\mathrm{tr}\mathbf{T}^3$, for nonnegative constants α and β. We note that under assumption (1.25), we can interchange derivatives with respect to $\boldsymbol{x}$ and $\mathbf{X}$. In particular, also assuming a similar smallness assumption for $\|\nabla_{\mathbf{X}}\boldsymbol{v}\|_{L^\infty(Q;\mathbb{R}^{d\times d})}$, the linearized counterpart of $\mathbf{D}$ can be identified with $\partial_t\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\partial_t\boldsymbol{u})$. Therefore, assuming (1.25) and writing the right-hand side of (1.32) more generally as a nonlinear function of $\mathbf{T}$, one obtains (1.1b), as required.

Models of the type (1.32) were considered in [34] in order to describe viscoelastic solid bodies. The model is a generalization of the classical (linear) Kelvin–Voigt model, which in one space dimension involves the constitutive relation

$$\sigma = E\epsilon + \eta\epsilon_t, \tag{1.33}$$

where σ denotes the scalar stress, ϵ the scalar strain, and E, η are constants signifying the modulus of elasticity and the viscosity, respectively. As mentioned previously, it is worth noting that similar models have been considered in [8, 7], where the authors assumed that the stress $\mathbf{T}$ was a sum of the elastic $\mathbf{T}_{el}$ and viscous $\mathbf{T}_{vis}$ parts. Considering implicit relations for each component separately, they obtained $\mathbf{T}_{el} = \mathbf{H}(\boldsymbol{\varepsilon})$, $\mathbf{T}_{vis} = \mathbf{G}(\boldsymbol{\varepsilon}_t)$ for nonlinear mappings $\mathbf{H}$, $\mathbf{G}$. However, the assumptions that were made there on $\mathbf{H}$ and $\mathbf{G}$ result in a problem that is not of strain-limiting type. This, together with the additive decomposition of the stress considered there, gives an analysis that is very different from the one performed here.

There is some analysis, albeit limited, available in the literature for problem (1.1). In particular, studies of the one-dimensional case have been performed. In [18], the authors derive the equation

$$\sigma_{xx} + \beta\sigma_{xxt} = g(\sigma)_{tt}, \tag{1.34}$$

using the equation of motion (1.1a) together with the constitutive relation (1.1b) and setting $\alpha = 1$, with σ denoting the scalar stress. In (1.34), the nonlinearity g corresponds to $\mathbf{G}$ in problem (1.1). The authors investigate conditions on the function g under which travelling wave solutions exist. Furthermore, in [20] the authors prove the local-in-time existence of solutions for equation (1.34). In this work, we cannot proceed in the same way and derive a single equation, on account of the fact that we are not working in one spatial dimension. In particular, the symmetric gradient does not reduce to a classical gradient operator as in the one-dimensional case, a property that is exploited in [18] and [20].

A related problem is studied in [19] where the authors look at a stress-rate problem rather than a strain-rate one. In the one-dimensional setting, this results in the equation

$$\sigma_{xx} + \gamma\sigma_{ttt} = h(\sigma)_{tt}. \tag{1.35}$$

The constitutive law for the study is $\epsilon+\gamma\sigma_t = h(\sigma)$. We note that the travelling wave solutions of equations (1.34) and (1.35) coincide. However, we do not attempt to treat the stress-rate problem in higher dimensions in this work.

We close this section with a thermodynamical justification of the model (1.1). In particular, we show that an energy-dissipation balance holds and that the sum of the kinetic energy and the elastic energy is a decreasing function of time. We suppose that the constitutive relation can be written as

$$\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta\boldsymbol{\varepsilon}(\partial_t\boldsymbol{u}) = \frac{\partial\varphi}{\partial\mathbf{T}}(\mathbf{T}) =: \mathbf{G}(\mathbf{T})$$

where φ is a function from $\mathbb{R}^{d\times d}$ to $\mathbb{R}_+$ defined by $\varphi(\mathbf{T}) = \phi(|\mathbf{T}|)$. We suppose that $\phi(0) = \phi'(0) = 0$ and $\phi \in \mathcal{C}^2(\mathbb{R}_+;\mathbb{R}_+)$ is strictly convex. Clearly this is the case if (A4) holds. Under these assumptions, φ is also strictly convex, noting that ϕ is strictly increasing on $[0,\infty)$. Furthermore, $\mathbf{G}$ is monotone. We define the convex conjugate φ^* by

$$\varphi^*(\boldsymbol{\varepsilon}) = \sup_{\mathbf{T}\in\mathbb{R}^{d\times d}_{sym}} \big(\boldsymbol{\varepsilon}\cdot\mathbf{T} - \varphi(\mathbf{T})\big).$$

We note that φ^* is also convex and, for any $\mathbf{T}\in\mathbb{R}^{d\times d}_{sym}$, the following identity holds:

$$\varphi^*(\mathbf{G}(\mathbf{T})) + \varphi(\mathbf{T}) = \mathbf{G}(\mathbf{T})\cdot\mathbf{T}. \tag{1.36}$$

Thus, the function $\mathbf{G}^{-1} = \frac{\partial\varphi^*}{\partial\mathbf{T}}$ is also monotone. With these facts in mind, formally testing (1.1a) against $\partial_t\boldsymbol{u}$ and assuming the absence of body forces, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\partial_t\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega \mathbf{T}\cdot\boldsymbol{\varepsilon}(\partial_t\boldsymbol{u})\,\mathrm{d}x = 0. \tag{1.37}$$

However, the integrand in the second term on the right-hand side can be rewritten as

$$\begin{aligned}
\mathbf{T}\cdot\boldsymbol{\varepsilon}(\partial_t\boldsymbol{u}) &= \frac{\partial\varphi^*}{\partial\mathbf{T}}(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}))\cdot\boldsymbol{\varepsilon}(\partial_t\boldsymbol{u}) + \left(\mathbf{T} - \frac{\partial\varphi^*}{\partial\mathbf{T}}(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}))\right)\cdot\boldsymbol{\varepsilon}(\partial_t\boldsymbol{u}) \\
&= \frac{1}{\alpha}\partial_t(\varphi^*(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}))) + \frac{1}{\beta}\left(\mathbf{T} - \frac{\partial\varphi^*}{\partial\mathbf{T}}(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}))\right)\cdot(\mathbf{G}(\mathbf{T}) - \alpha\boldsymbol{\varepsilon}(\boldsymbol{u})) \\
&= \frac{1}{\alpha}\partial_t(\varphi^*(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}))) + \frac{1}{\beta}\big(\mathbf{T} - \mathbf{G}^{-1}(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}))\big)\cdot(\mathbf{G}(\mathbf{T}) - \alpha\boldsymbol{\varepsilon}(\boldsymbol{u})).
\end{aligned}$$

Substituting this back into (1.37) and defining $\mathbf{T}_0 := \mathbf{G}^{-1}(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}))$, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \frac{1}{2}|\partial_t\boldsymbol{u}|^2 + \frac{\varphi^*(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}))}{\alpha}\,\mathrm{d}x\right) + \frac{1}{\beta}\int_\Omega (\mathbf{T}-\mathbf{T}_0)\cdot(\mathbf{G}(\mathbf{T}) - \mathbf{G}(\mathbf{T}_0))\,\mathrm{d}x = 0. \tag{1.38}$$

Recalling that $\mathbf{G}$ is monotone, we deduce that

$$\sup_{t\in(0,T)}\left(\int_\Omega \frac{1}{2}|\partial_t\boldsymbol{u}|^2 + \frac{\varphi^*(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}))}{\alpha}\,\mathrm{d}x\right) \le \int_\Omega \frac{1}{2}|\partial_t\boldsymbol{u}_0(0)|^2 + \frac{\varphi^*(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0(0)))}{\alpha}\,\mathrm{d}x.$$

Consequently, the sum of the kinetic energy and elastic energy is decreasing. The extra term that appears in (1.38) corresponds to the dissipation. In particular, we have an energy-dissipation balance that holds in accordance with the laws of thermodynamics.

The structure of the remainder of the paper is as follows. In Section 2 we prove Theorem 1.1. We structure the proof in the following way. First, in Section 2.1 we use a Galerkin method and find a weak solution to an approximate problem. In Section 2.2, we obtain uniform bounds on the sequence of Galerkin solutions, and use these in Section 2.3 in order to take the limit as $n\to\infty$. Finally, we show that the limit is the correct one in Section 2.4. We prove uniqueness in Section 2.5. In Section 3 we obtain further temporal and spatial regularity estimates for these solutions. Finally, in Section 4 we consider the case $p = 1$ and give the proof of Theorem 1.2.

2. Proof of Theorem 1.1. To prove the existence of a weak solution, we use a compactness argument based on a sequence of Galerkin approximations. Since G is not invertible in general, we introduce the following regularization:

$$\mathsf{G}_n(\mathsf{T}) := \mathsf{G}(\mathsf{T}) + n^{-1}|\mathsf{T}|^{p-2}\,\mathsf{T}.$$

For all $n \in \mathbb{N}$, the regularized mapping still satisfies (A1)–(A3), with C_2 replaced by (C_2+1). However, additionally, the inequality (A1) is strict whenever $\mathsf{T} \neq \mathsf{W}$. Therefore, it directly follows from the theory of monotone operators that there exists a continuous inverse $\mathsf{G}_n^{-1} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$.

2.1. Galerkin approximation. Let $\{\boldsymbol{\omega}_j\}_{j=1}^\infty$ be a basis[2] of $W_0^{m^*,2}(\Omega;\mathbb{R}^d)$, which is orthonormal in $L^2(\Omega;\mathbb{R}^d)$ for an arbitrary $m^* > \frac{d}{2}+1$. We denote by P^n the projection of $W_0^{m^*,2}(\Omega;\mathbb{R}^d)$ onto the linear hull of $\{\boldsymbol{\omega}_j\}_{j=1}^n$. This is a continuous linear operator by standard properties of Hilbert projections. The choice of m^* guarantees that we have the continuous embedding $W^{m^*,2}(\Omega;\mathbb{R}^d) \subset C^1(\overline{\Omega};\mathbb{R}^d)$. In particular, the sequence of projections $(P^n\boldsymbol{w})_n$ is bounded in $W^{1,p'}(\Omega;\mathbb{R}^d)$, for every $\boldsymbol{w} \in W_0^{m^*,2}(\Omega;\mathbb{R}^d)$, a fact that we use in later estimates.

We look for a function $\boldsymbol{u}^n$ of the form

$$\boldsymbol{u}^n(t,x) = \boldsymbol{u}_0(t,x) + \sum_{i=1}^n C_i^n(t)\boldsymbol{\omega}_i(x),$$

such that for all $j = 1,2,\dots,n$ and almost all $t \in (0,T)$ it solves the following problem:

$$\int_\Omega \partial_{tt}^2\boldsymbol{u}^n \cdot \boldsymbol{\omega}_j + \mathsf{G}_n^{-1}\left(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n)\right) \cdot \nabla\boldsymbol{\omega}_j \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{\omega}_j\rangle, \tag{2.1a}$$

$$\boldsymbol{u}^n(0) = \boldsymbol{u}_0(0), \tag{2.1b}$$

$$\partial_t\boldsymbol{u}^n(0) = \partial_t\boldsymbol{u}_0(0). \tag{2.1c}$$

We denote by $\boldsymbol{C}^n$ the vector of coefficients $(C_i^n)_{i=1}^n$. It follows that (2.1b) and (2.1c) are equivalent to $\boldsymbol{C}^n(0) = \mathbf{0}$ and $\partial_t\boldsymbol{C}^n(0) = \mathbf{0}$, respectively. Since G_n^{-1} is continuous and the basis functions $\{\boldsymbol{\omega}_j\}_{j=1}^\infty$ are orthonormal in $L^2(\Omega;\mathbb{R}^d)$, equation (2.1a) reduces to

$$\partial_{tt}C_i^n(t) = F_i(t, \boldsymbol{C}^n(t), \partial_t\boldsymbol{C}^n(t)),$$

where F_i is a Carathéodory mapping for every $i = 1,2,\dots,n$. Hence, using standard Carathéodory theory for systems of ordinary differential equations, we deduce that there exists a solution on some maximal time interval $(0,T^*)$. Furthermore, either we must have $|\boldsymbol{C}^n(t)| + |\partial_t\boldsymbol{C}^n(t)| \to \infty$ as $t \to T^*_-$ or we can extend the solution to the whole interval $(0,T)$. We next show that the latter is true by establishing uniform bounds on the sequence of Galerkin approximations.

2.2. Uniform bounds. First, let us define

$$\mathsf{T}^n := \mathsf{G}_n^{-1}\left(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n)\right),$$

which is clearly equivalent to

$$\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n) = \mathsf{G}(\mathsf{T}^n) + n^{-1}|\mathsf{T}^n|^{p-2}\mathsf{T}^n. \tag{2.2}$$

We multiply (2.1a) by $\partial_t C_j^n + \frac{\alpha}{\beta}C_j^n$ and sum the resulting identities with respect to the indices $j = 1,\dots,n$ to obtain

$$\begin{aligned}&\int_\Omega \partial_{tt}\boldsymbol{u}^n \cdot \left[\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0) + \frac{\alpha}{\beta}(\boldsymbol{u}^n - \boldsymbol{u}_0)\right] + \mathsf{T}^n : \Big(\frac{\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n)}{\beta} - \frac{\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0)}{\beta}\Big)\,\mathrm{d}x \\ &= \langle \boldsymbol{f}, \partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0) + \frac{\alpha}{\beta}(\boldsymbol{u}^n - \boldsymbol{u}_0)\rangle.\end{aligned} \tag{2.3}$$

[2]Such a basis can be found by looking for eigenfunctions $\boldsymbol{\omega}_j \in W_0^{m^*,2}(\Omega;\mathbb{R}^d)$ of the problem

$$-\Delta^{m^*}\boldsymbol{\omega}_j = \lambda_j\boldsymbol{\omega}_j \qquad \text{on } \Omega.$$

It follows from (2.2) that

$$\mathbf{T}^n \cdot \left(\frac{\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n)}{\beta}\right) = \frac{1}{\beta}\left(\mathbf{G}(\mathbf{T}^n)\cdot\mathbf{T}^n + n^{-1}|\mathbf{T}^n|^p\right).$$

Also, we can write

$$\int_\Omega \partial_{tt}(\boldsymbol{u}^n - \boldsymbol{u}_0)\cdot(\boldsymbol{u}^n - \boldsymbol{u}_0)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)\cdot(\boldsymbol{u}^n - \boldsymbol{u}_0)\,\mathrm{d}x - \int_\Omega |\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)|^2\,\mathrm{d}x.$$

Using these two identities in (2.3), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)|^2 + \frac{2\alpha}{\beta}\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)\cdot(\boldsymbol{u}^n - \boldsymbol{u}_0)\,\mathrm{d}x + \frac{1}{\beta}\int_\Omega \mathbf{G}(\mathbf{T}^n)\cdot\mathbf{T}^n + n^{-1}|\mathbf{T}^n|^p\,\mathrm{d}x\\
&= \langle \boldsymbol{f}, \partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)\rangle + \int_\Omega \mathbf{T}^n\cdot\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \partial_{tt}\boldsymbol{u}_0\cdot\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)\,\mathrm{d}x\\
&\quad + \frac{\alpha}{\beta}\int_\Omega |\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)|^2 - \partial_{tt}\boldsymbol{u}_0\cdot(\boldsymbol{u}^n - \boldsymbol{u}_0) + \mathbf{T}^n\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\,\mathrm{d}x + \langle \boldsymbol{f}, (\boldsymbol{u}^n - \boldsymbol{u}_0)\rangle.
\end{aligned} \tag{2.4}$$

We define on $[0, T]$ the function

$$Y^n := \frac{1}{4}\int_\Omega |\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)|^2 + |\boldsymbol{u}^n - \boldsymbol{u}_0|^2 + \left|\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0) + \frac{2\alpha}{\beta}(\boldsymbol{u}^n - \boldsymbol{u}_0)\right|^2\,\mathrm{d}x.$$

Using this, we rewrite the first term on the left-hand side of (2.4) as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)|^2 + \frac{2\alpha}{\beta}\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)\cdot(\boldsymbol{u}^n - \boldsymbol{u}_0)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}Y^n - \left(\frac{\alpha^2}{\beta^2} + \frac{1}{4}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\boldsymbol{u}^n - \boldsymbol{u}_0|^2\,\mathrm{d}x.$$

Consequently, utilising this identity in (2.4), using (A2) to deal with the second term on the left-hand side, and applying the Hölder inequality to the terms on the right-hand side together with the Poincaré and Korn inequalities, it follows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}Y^n + \frac{C_1}{\beta}\int_\Omega |\mathbf{T}^n|^p\,\mathrm{d}x - \left(\frac{\alpha^2}{\beta^2} + \frac{1}{4}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\boldsymbol{u}^n - \boldsymbol{u}_0|^2\,\mathrm{d}x\\
&\le C(\|\boldsymbol{\varepsilon}(\boldsymbol{u}^n)\|_{p'} + \|\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n)\|_{p'} + \|\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\|_{p'} + \|\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\|_{p'})(\|\boldsymbol{f}\|_{(W_0^{1,p'})^*} + \|\partial_{tt}\boldsymbol{u}_0\|_{(W_0^{1,p'})^*})\\
&\quad + C(\|\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\|_{p'} + \|\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\|_{p'})\|\mathbf{T}^n\|_p + C\,(1 + Y^n)\,,
\end{aligned} \tag{2.5}$$

where C is a generic constant that is independent of n. To bound the right-hand side, we use (2.2) to observe that

$$\partial_t\left(\mathrm{e}^{\frac{\alpha}{\beta}t}\boldsymbol{\varepsilon}(\boldsymbol{u}^n)\right) = \frac{\mathrm{e}^{\frac{\alpha}{\beta}t}}{\beta}(\mathbf{G}(\mathbf{T}^n) + n^{-1}|\mathbf{T}^n|^{p-2}\mathbf{T}^n).$$

After integration with respect to time, this yields

$$\boldsymbol{\varepsilon}(\boldsymbol{u}^n(t)) = \mathrm{e}^{-\frac{\alpha}{\beta}t}\boldsymbol{\varepsilon}(\boldsymbol{u}_0(0)) + \mathrm{e}^{-\frac{\alpha}{\beta}t}\int_0^t \frac{\mathrm{e}^{\frac{\alpha}{\beta}\tau}}{\beta}(\mathbf{G}(\mathbf{T}^n(\tau) + n^{-1}|\mathbf{T}^n(\tau)|^{p-2}\mathbf{T}^n(\tau))\,\mathrm{d}\tau.$$

As discussed previously, this memory property follows from the specific structure of the constitutive relation. Namely, the elasticity and viscosity tensors are each a positive scalar multiple of the identity tensor. Using properties of the Bochner integral, it follows that

$$\begin{aligned}
\|\boldsymbol{\varepsilon}(\boldsymbol{u}^n(t))\|_{p'}^{p'} &\le C\left(\int_0^t \|\mathbf{G}(\mathbf{T}^n) + n^{-1}|\mathbf{T}^n|^{p-2}\mathbf{T}^n\|_{p'}^{p'}\,\mathrm{d}\tau + \|\boldsymbol{u}_0(0)\|_{1,p'}^{p'}\right)\\
&\le C\left(\int_0^t \|\mathbf{T}^n\|_p^p\,\mathrm{d}\tau + \|\boldsymbol{u}_0(0)\|_{1,p'}^{p'} + 1\right),
\end{aligned} \tag{2.6}$$

where for the second inequality we have used (A3). Consequently, using (2.6) and (2.2), we have the following bound:

$$\|\partial_t \boldsymbol{\varepsilon}(\boldsymbol{u}^n(t))\|_{p'}^{p'} \leq C\left(1 + \|\boldsymbol{u}_0(0)\|_{1,p'}^{p'} + \|\mathbf{T}^n(t)\|_p^p + \int_0^t \|\mathbf{T}^n\|_p^p \,\mathrm{d}\tau\right). \tag{2.7}$$

To bound the final term on the left-hand side of (2.5), we notice that performing differentiation in the time variable yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\boldsymbol{u}^n - \boldsymbol{u}_0|^2 \,\mathrm{d}x &= \int_\Omega 2\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)\cdot(\boldsymbol{u}^n - \boldsymbol{u}_0)\,\mathrm{d}x \\ &\leq \int_\Omega |\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)|^2 + |\boldsymbol{u}^n - \boldsymbol{u}_0|^2 \,\mathrm{d}x \\ &\leq 4Y^n. \end{aligned} \tag{2.8}$$

Hence, using (2.6) and (2.7) for the terms appearing on the right-hand side of (2.5), using (2.8) for the last term on the left-hand side, and applying Young's inequality to the resulting right-hand side, we deduce that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(Y^n + \frac{C_1}{4\beta}\int_0^t \|\mathbf{T}^n\|_p^p \,\mathrm{d}\tau\right) + \frac{C_1}{4\beta}\|\mathbf{T}^n\|_p^p \\ &\leq C\left(Y^n + \frac{C_1}{4\beta}\int_0^t \|\mathbf{T}^n\|_p^p \,\mathrm{d}\tau\right) + C\sup_{t\in[0,T]}\|\boldsymbol{u}_0(t)\|_{1,p'}^{p'} \\ &\quad + C\left(\|\partial_t \boldsymbol{\varepsilon}(\boldsymbol{u}_0)\|_{p'}^{p'} + \|\boldsymbol{f}\|_{(W_0^{1,p'})^*}^p + \|\partial_{tt}\boldsymbol{u}_0\|_{(W_0^{1,p'})^*}^p\right). \end{aligned} \tag{2.9}$$

Using Grönwall's lemma and the assumptions on the data, we get that

$$\sup_{t\in(0,T)} Y^n(t) + \int_0^T \|\mathbf{T}^n\|_p^p \,\mathrm{d}\tau \leq C(\boldsymbol{u}_0, \boldsymbol{f}) + Y^n(0) = C(\boldsymbol{u}_0, \boldsymbol{f}). \tag{2.10}$$

From the definition of Y^n, the bounds (2.6), (2.7), and Korn's inequality, we deduce that

$$\sup_{t\in(0,T)}\left(\|\partial_t \boldsymbol{u}^n\|_2^2 + \|\boldsymbol{u}^n\|_2^2 + \|\boldsymbol{u}^n\|_{1,p'}^{p'}\right) + \int_0^T \|\mathbf{T}^n\|_p^p + \|\partial_t \boldsymbol{u}^n\|_{1,p'}^{p'} \,\mathrm{d}t \leq C(\boldsymbol{u}_0, \boldsymbol{f}). \tag{2.11}$$

It remains to provide a bound on $\partial_{tt}\boldsymbol{u}^n$. We define $\mathcal{V} := \{\boldsymbol{w} \in W_0^{m^*,2}(\Omega;\mathbb{R}^d), \|\boldsymbol{w}\|_{m^*,2} = 1\}$. Using the orthonormality of the basis and the continuity of P^n as a linear operator on $W_0^{m^*,2}(\Omega;\mathbb{R}^d)$, we deduce from (2.1a) that

$$\begin{aligned} \|\partial_{tt}\boldsymbol{u}^n(t)\|_{(W_0^{m^*,2}(\Omega;\mathbb{R}^d))^*} &= \sup_{\boldsymbol{w}\in\mathcal{V}}\int_\Omega \partial_{tt}\boldsymbol{u}^n(t)\cdot\boldsymbol{w}\,\mathrm{d}x \\ &= \sup_{\boldsymbol{w}\in\mathcal{V}}\int_\Omega \partial_{tt}\boldsymbol{u}^n(t)\cdot P^n\boldsymbol{w}\,\mathrm{d}x \\ &= \sup_{\boldsymbol{w}\in\mathcal{V}}\left(\langle \boldsymbol{f}, P^n\boldsymbol{w}\rangle - \int_\Omega \mathbf{T}^n(t)\cdot\nabla(P^n\boldsymbol{w})\,\mathrm{d}x\right) \\ &\leq \sup_{\boldsymbol{w}\in\mathcal{V}}\left((\|\boldsymbol{f}(t)\|_{(W_0^{1,p'}(\Omega;\mathbb{R}^d))^*} + \|\mathbf{T}^n(t)\|_p)\|P^n\boldsymbol{w}\|_{1,p'}\right) \\ &\leq C\sup_{\boldsymbol{w}\in\mathcal{V}}\left((\|\boldsymbol{f}(t)\|_{(W_0^{1,p'}(\Omega;\mathbb{R}^d))^*} + \|\mathbf{T}^n(t)\|_p)\|P^n\boldsymbol{w}\|_{m^*,2}\right) \\ &\leq C(\|\boldsymbol{f}(t)\|_{(W_0^{1,p'}(\Omega;\mathbb{R}^d))^*} + \|\mathbf{T}^n(t)\|_p), \end{aligned}$$

where we have used the fact that $W^{m^*,2}(\Omega;\mathbb{R}^d)$ is continuously embedded into $W^{1,p'}(\Omega;\mathbb{R}^d)$. Therefore, it follows from (2.11) that

$$\int_0^T \|\partial_{tt}\boldsymbol{u}^n\|_{(W_0^{m^*,2}(\Omega;\mathbb{R}^d))^*}^p \,\mathrm{d}t \leq C\int_0^T \|\boldsymbol{f}\|_{(W_0^{1,p'}(\Omega;\mathbb{R}^d))^*}^p + \|\mathbf{T}^n\|_p^p \,\mathrm{d}t \leq C(\boldsymbol{u}_0, \boldsymbol{f}). \tag{2.12}$$

2.3. Limit $n \to \infty$. Using the bounds from Section 2.2 in conjunction with the reflexivity and separability of the underlying spaces, we can find a subsequence, that we do not relabel, such that

$$\begin{aligned} \mathsf{G}(\mathsf{T}^n) &\rightharpoonup \bar{\mathsf{G}} && \text{weakly in } L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})), \\ \boldsymbol{u}^n &\overset{*}{\rightharpoonup} \boldsymbol{u} && \text{weakly}^* \text{ in } W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)), \\ \boldsymbol{u}^n &\rightharpoonup \boldsymbol{u} && \text{weakly in } W^{1,p'}(0,T;W^{1,p'}(\Omega;\mathbb{R}^d)), \\ \mathsf{T}^n &\rightharpoonup \mathsf{T} && \text{weakly in } L^{p}(0,T;L^{p}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})), \\ \partial_{tt}\boldsymbol{u}^n &\rightharpoonup \partial_{tt}\boldsymbol{u} && \text{weakly in } L^{p}(0,T;(W^{m^*,2}_0(\Omega;\mathbb{R}^d))^*). \end{aligned} \tag{2.13}$$

Hence, we see that T fulfils (1.4) and $\boldsymbol{u}$ belongs to the first two spaces indicated in (1.3). In addition, thanks to the fact that $W^{1,p'}(\Omega;\mathbb{R}^d)$ is compactly embedded into $L^2(\Omega;\mathbb{R}^d)$, using the Aubin–Lions lemma, up to a further subsequence that we do not relabel, we have that

$$\begin{aligned} \boldsymbol{u}^n &\to \boldsymbol{u} && \text{strongly in } \mathcal{C}([0,T];L^2(\Omega;\mathbb{R}^d)), \\ \partial_t\boldsymbol{u}^n &\to \partial_t\boldsymbol{u} && \text{strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)) \cap \mathcal{C}([0,T];(W^{m^*,2}_0(\Omega;\mathbb{R}^d))^*). \end{aligned} \tag{2.14}$$

It follows directly from the fact that $\boldsymbol{u}^n(0) = \boldsymbol{u}_0(0)$ and $\partial_t\boldsymbol{u}^n(0) = \partial_t\boldsymbol{u}_0(0)$ and the convergence result (2.14) that we have

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{and} \quad \partial_t\boldsymbol{u}(0) = \partial_t\boldsymbol{u}_0(0).$$

Next, we let $n \to \infty$ in (2.1a). Let $\phi \in \mathcal{C}^\infty([0,T])$ be arbitrary. We multiply (2.1a) by ϕ and integrate the result over $(0,T)$ to get

$$\int_0^T \langle \partial_{tt}\boldsymbol{u}^n, \boldsymbol{\omega}_j\rangle \phi \,\mathrm{d}t + \int_0^T \int_\Omega \mathsf{T}^n \cdot \nabla(\boldsymbol{\omega}_j\phi)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle \boldsymbol{f}, \boldsymbol{\omega}_j\rangle \phi\,\mathrm{d}t,$$

for every $j \in \{1,\dots,n\}$. Thus, for a fixed j, we can let $n \to \infty$. Using the weak convergence result (2.13), we deduce that

$$\int_0^T \langle \partial_{tt}\boldsymbol{u}, \boldsymbol{\omega}_j\rangle \phi \,\mathrm{d}t + \int_0^T \int_\Omega \mathsf{T} \cdot \nabla(\boldsymbol{\omega}_j\phi)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle \boldsymbol{f}, \boldsymbol{\omega}_j\rangle \phi\,\mathrm{d}t.$$

Since j and ϕ are arbitrary, and recalling that $\{\boldsymbol{\omega}_j\}_{j=1}^\infty$ forms a basis of $W^{m^*,2}_0(\Omega;\mathbb{R}^d)$, it follows that

$$\langle \partial_{tt}\boldsymbol{u}, \boldsymbol{w}\rangle + \int_\Omega \mathsf{T}\cdot\nabla\boldsymbol{w}\,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{w}\rangle \qquad \forall\, \boldsymbol{w} \in W^{m^*,2}_0(\Omega;\mathbb{R}^d), \quad \text{for a.e. } t \in (0,T). \tag{2.15}$$

Consequently, by the density of $W^{m^*,2}_0(\Omega;\mathbb{R}^d)$ in $W^{1,p'}_0(\Omega;\mathbb{R}^d)$, we see that, for almost all $t \in (0,T)$, we have $\partial_{tt}\boldsymbol{u} \in (W^{1,p'}_0(\Omega;\mathbb{R}^d))^*$. Furthermore, we have

$$\|\partial_{tt}\boldsymbol{u}^n(t)\|_{(W^{1,p'}_0(\Omega;\mathbb{R}^d))^*} = \sup_{\boldsymbol{w}\in W^{1,p'}_0(\Omega;\mathbb{R}^d),\, \|\boldsymbol{w}\|_{1,p'}=1} \left[-\int_\Omega \mathsf{T}^n(t)\cdot\nabla\boldsymbol{w}\,\mathrm{d}x + \langle \boldsymbol{f}(t), \boldsymbol{w}\rangle \right].$$

Using (2.11) and (2.13), it follows that

$$\int_0^T \|\partial_{tt}\boldsymbol{u}^n\|^p_{(W^{1,p'}_0(\Omega;\mathbb{R}^d))^*}\,\mathrm{d}t \le C\int_0^T \|\mathsf{T}^n\|_p^p + \|\boldsymbol{f}\|^p_{(W^{1,p'}_0(\Omega;\mathbb{R}^d))^*}\,\mathrm{d}t \le C(\boldsymbol{u}_0, \boldsymbol{f}). \tag{2.16}$$

Hence, (2.15) can be strengthened so that (1.5) holds. In addition, by standard parabolic interpolation and the fact that $\partial_t\boldsymbol{u}_0 \in \mathcal{C}([0,T];L^2(\Omega;\mathbb{R}^d))$, we see that $\boldsymbol{u}$ satisfies (1.3).

Finally, letting $n \to \infty$ in (2.2) and using (2.13), we see that

$$\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}) = \overline{\mathsf{G}} \quad \text{a.e. in } Q. \tag{2.17}$$

Hence, in order to show (1.6) and deduce the existence of a weak solution, it remains to show that $\overline{\mathsf{G}} = \mathsf{G}(\mathsf{T})$ a.e. in Q.

2.4. Identification of the nonlinearity. In order to identify the nonlinearity, we use monotone operator theory. Let $\phi \in \mathcal{C}_0^1((0,T))$ be an arbitrary nonnegative function. We multiply (2.3) by ϕ and integrate the result over $(0,T)$. With the help of integration by parts, and the fact that $\boldsymbol{u}^n(0) = \boldsymbol{u}_0(0)$ and $\phi(0) = \phi(T) = 0$, we observe that

$$\begin{aligned}
&\int_0^T \int_\Omega \mathbf{T}^n \cdot \big(\partial_t \boldsymbol{\varepsilon}(\boldsymbol{u}^n) + \frac{\alpha}{\beta}\boldsymbol{\varepsilon}(\boldsymbol{u}^n)\big)\phi \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_\Omega \left(\frac{|\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)|^2}{2} + \frac{\alpha}{\beta}\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)\cdot(\boldsymbol{u}^n - \boldsymbol{u}_0)\right)\phi' \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{\alpha}{\beta}\int_0^T \int_\Omega |\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)|^2 \phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \mathbf{T}^n \cdot \big(\partial_t \boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \frac{\alpha}{\beta}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\big)\phi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \langle \boldsymbol{f} - \partial_{tt}\boldsymbol{u}_0, \partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0) + \frac{\alpha}{\beta}(\boldsymbol{u}^n - \boldsymbol{u}_0)\rangle \phi \,\mathrm{d}t.
\end{aligned} \tag{2.18}$$

Next, we use the weak convergence results (2.13) and the strong convergence results (2.14) to identify the limits on the right-hand side of (2.18). In particular, we see that

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T \int_\Omega \mathbf{T}^n \cdot \big(\partial_t \boldsymbol{\varepsilon}(\boldsymbol{u}^n) + \frac{\alpha}{\beta}\boldsymbol{\varepsilon}(\boldsymbol{u}^n)\big)\phi \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_\Omega \left(\frac{|\partial_t(\boldsymbol{u} - \boldsymbol{u}_0)|^2}{2} + \frac{\alpha}{\beta}\partial_t(\boldsymbol{u} - \boldsymbol{u}_0)\cdot(\boldsymbol{u} - \boldsymbol{u}_0)\right)\phi' \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{\alpha}{\beta}\int_0^T \int_\Omega |\partial_t(\boldsymbol{u} - \boldsymbol{u}_0)|^2 \phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \mathbf{T} \cdot \big(\partial_t \boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \frac{\alpha}{\beta}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\big)\phi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \langle \boldsymbol{f} - \partial_{tt}\boldsymbol{u}_0, \partial_t(\boldsymbol{u} - \boldsymbol{u}_0) + \frac{\alpha}{\beta}(\boldsymbol{u} - \boldsymbol{u}_0)\rangle \phi \,\mathrm{d}t.
\end{aligned} \tag{2.19}$$

Next, we use (1.5) to evaluate the terms on the right-hand side of (2.19). We note that, as a result of the regularity of $\boldsymbol{u}$, both $\boldsymbol{u} - \boldsymbol{u}_0$ and $\partial_t(\boldsymbol{u} - \boldsymbol{u}_0)$ are admissible test functions in (1.5). Using these two choices as the test function $\boldsymbol{w}$, multiplying the resulting equalities by ϕ and integrating over $(0,T)$, we can apply integration by parts in order to obtain the following identity:

$$\begin{aligned}
&\int_0^T \int_\Omega \mathbf{T} \cdot \big(\partial_t \boldsymbol{\varepsilon}(\boldsymbol{u}^n) + \frac{\alpha}{\beta}\boldsymbol{\varepsilon}(\boldsymbol{u}^n)\big)\phi \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_\Omega \left(\frac{|\partial_t(\boldsymbol{u} - \boldsymbol{u}_0)|^2}{2} + \frac{\alpha}{\beta}\partial_t(\boldsymbol{u} - \boldsymbol{u}_0)\cdot(\boldsymbol{u} - \boldsymbol{u}_0)\right)\phi' \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{\alpha}{\beta}\int_0^T \int_\Omega |\partial_t(\boldsymbol{u} - \boldsymbol{u}_0)|^2 \phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \mathbf{T} \cdot \big(\partial_t \boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \frac{\alpha}{\beta}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\big)\phi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \langle \boldsymbol{f} - \partial_{tt}\boldsymbol{u}_0, \partial_t(\boldsymbol{u} - \boldsymbol{u}_0) + \frac{\alpha}{\beta}(\boldsymbol{u} - \boldsymbol{u}_0)\rangle \phi \,\mathrm{d}t.
\end{aligned} \tag{2.20}$$

Comparing (2.19) with (2.20), we see that

$$\limsup_{n\to\infty} \int_Q \phi \mathbf{T}^n \cdot (\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}^n) + \beta \partial_t \boldsymbol{\varepsilon}(\boldsymbol{u}^n))\,\mathrm{d}x\,\mathrm{d}t = \int_Q \phi \mathbf{T} \cdot (\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta \partial_t \boldsymbol{\varepsilon}(\boldsymbol{u}))\,\mathrm{d}x\,\mathrm{d}t. \tag{2.21}$$

Therefore, using the nonnegativity of ϕ, we observe that

$$
\begin{aligned}
\limsup_{n\to\infty}\int_Q \phi \mathbf{G}(\mathbf{T}^n)\cdot\mathbf{T}^n \,\mathrm{d}x\,\mathrm{d}t &\le \limsup_{n\to\infty}\int_Q \phi(\mathbf{G}(\mathbf{T}^n)+n^{-1}|\mathbf{T}^n|^{p-2}\mathbf{T}^n)\cdot\mathbf{T}^n \,\mathrm{d}x\,\mathrm{d}t\\
&\overset{(2.2)}{=} \limsup_{n\to\infty}\int_Q \phi\mathbf{T}^n\cdot(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n)+\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n))\,\mathrm{d}x\,\mathrm{d}t\\
&\overset{(2.21)}{=} \int_Q \phi\mathbf{T}\cdot(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u})+\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}))\,\mathrm{d}x\,\mathrm{d}t\\
&\overset{(2.17)}{=} \int_Q \phi\mathbf{T}\cdot\overline{\mathbf{G}}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{2.22}
$$

The inequality (2.22) is the key to identifying the nonlinearity. Let $\mathbf{W}\in L^p(Q,\mathbb{R}^{d\times d}_{\mathrm{sym}})$ be arbitrary. Using the monotonicity assumption (A1), the weak convergence results (2.13), the bound (2.22) and the nonnegativity of ϕ, we obtain

$$0\le \limsup_{n\to\infty}\int_Q \phi\,(\mathbf{G}(\mathbf{T}^n)-\mathbf{G}(\mathbf{W}))\cdot(\mathbf{T}^n-\mathbf{W})\,\mathrm{d}x\,\mathrm{d}t \le \int_Q \phi\left(\overline{\mathbf{G}}-\mathbf{G}(\mathbf{W})\right)\cdot(\mathbf{T}-\mathbf{W})\,\mathrm{d}x\,\mathrm{d}t.$$

Setting $\mathbf{W}=\mathbf{T}-\kappa\mathbf{B}$ for an arbitrary $\mathbf{B}\in L^{p'}(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $\kappa>0$, we divide through by κ to deduce that

$$0\le\int_Q\phi\left(\overline{\mathbf{G}}-\mathbf{G}(\mathbf{T}-\kappa\mathbf{B})\right)\cdot\mathbf{B}\,\mathrm{d}x\,\mathrm{d}t.$$

Hence, since $\mathbf{G}$ is continuous, we let $\kappa\to 0_+$ and deduce that

$$0\le\int_Q\phi\left(\overline{\mathbf{G}}-\mathbf{G}(\mathbf{T})\right)\cdot\mathbf{B}\,\mathrm{d}x\,\mathrm{d}t.$$

As $\mathbf{B}$ and ϕ are arbitrary, we conclude that

$$\overline{\mathbf{G}}=\mathbf{G}(\mathbf{T})\quad\text{a.e. in } Q.$$

Thus we have proved the existence of a weak solution.

2.5. Uniqueness of solutions. To complete the proof of Theorem 1.1, it remains to show uniqueness of the weak solution. To this end, let $(\boldsymbol{u}_1,\mathbf{T}_1)$ and $(\boldsymbol{u}_2,\mathbf{T}_2)$ be two weak solutions of (1.1) emanating from the same data. We denote $\boldsymbol{u}:=\boldsymbol{u}_1-\boldsymbol{u}_2$. Then, using (1.5), we see that

$$\langle\partial_{tt}\boldsymbol{u},\boldsymbol{w}\rangle+\int_\Omega(\mathbf{T}_1-\mathbf{T}_2)\cdot\boldsymbol{\varepsilon}(\boldsymbol{w})\,\mathrm{d}x=0\quad\forall\,\boldsymbol{w}\in W^{1,p'}_0(\Omega;\mathbb{R}^d)\text{ and a.e. } t\in(0,T).$$

We have that $\boldsymbol{u}$ and $\partial_t\boldsymbol{u}$ belong to $W^{1,p'}_0(\Omega;\mathbb{R}^d)$ for almost all $t\in(0,T)$. Hence we can set $\boldsymbol{w}=\beta\partial_t\boldsymbol{u}+\alpha\boldsymbol{u}$ in the above to deduce that, for almost all $t\in(0,T)$, the following holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega\frac{\beta}{2}|\partial_t\boldsymbol{u}|^2+\alpha\partial_t\boldsymbol{u}\cdot\boldsymbol{u}\,\mathrm{d}x\right)+\int_\Omega(\mathbf{T}_1-\mathbf{T}_2)\cdot(\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u})+\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}))\,\mathrm{d}x=\int_\Omega\alpha|\partial_t\boldsymbol{u}|^2\,\mathrm{d}x.$$

Following the same procedure that is used to derive the previous *a priori* estimates and using the constitutive relation (1.6), we obtain

$$
\begin{aligned}
&\frac14\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\beta|\partial_t\boldsymbol{u}|^2+\beta|\boldsymbol{u}|^2+\beta\left|\partial_t\boldsymbol{u}+\frac{2\alpha}{\beta}\boldsymbol{u}\right|^2\,\mathrm{d}x+\int_\Omega(\mathbf{G}(\mathbf{T}_1)-\mathbf{G}(\mathbf{T}_2))\cdot(\mathbf{T}_1-\mathbf{T}_2)\,\mathrm{d}x\\
&=\int_\Omega\alpha|\partial_t\boldsymbol{u}|^2+\left(\beta+\frac{\alpha^2}{\beta}\right)|\boldsymbol{u}|^2\,\mathrm{d}x\\
&\le C(\alpha,\beta)\int_\Omega\beta|\partial_t\boldsymbol{u}|^2+\beta|\boldsymbol{u}|^2+\beta\left|\partial_t\boldsymbol{u}+\frac{2\alpha}{\beta}\boldsymbol{u}\right|^2\,\mathrm{d}x.
\end{aligned}
$$

The second term on the left-hand side is nonnegative thanks to (A1) so we can apply Grönwall's inequality. Since $\boldsymbol{u}(0)=\partial_t\boldsymbol{u}(0)=\mathbf{0}$, we deduce that $\boldsymbol{u}=\mathbf{0}$ a.e. in Q. In addition, by monotonicity, we also obtain that $\big(\mathbf{G}(\mathbf{T}_1)-\mathbf{G}(\mathbf{T}_2)\big)\cdot(\mathbf{T}_1-\mathbf{T}_2)=0$ a.e. in Q. This proves that $\boldsymbol{u}_1=\boldsymbol{u}_2$ a.e. in Q. If $\mathbf{G}$ is strictly monotone then also $\mathbf{T}_1=\mathbf{T}_2$.

3. Regularity estimates. In this section we prove the higher regularity estimates for the solution constructed in Theorem 1.1. We note that this is an essential part in the proof of the existence of a solution for the limiting strain model, that is, the case $p = 1$. Indeed, as the focus turns to the limiting strain model, in this part we assume that there exists a strictly convex $\mathcal{C}^2$-function $F : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ such that, for all $\mathbf{T} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$,

$$\frac{\partial F(\mathbf{T})}{\partial \mathbf{T}} = \mathbf{G}(\mathbf{T}). \tag{3.1}$$

In this case, $\mathbf{G}$ is strongly monotone. In order to simplify the subsequent notation, for an arbitrary $\mathbf{T} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, we denote

$$\mathcal{A}(\mathbf{T}) := \frac{\partial^2 F(\mathbf{T})}{\partial \mathbf{T}\partial \mathbf{T}} = \frac{\partial \mathbf{G}(\mathbf{T})}{\partial \mathbf{T}}, \qquad \mathcal{A}^{ij}_{kl}(\mathbf{T}) := \frac{\partial \mathbf{G}_{ij}(\mathbf{T})}{\partial \mathbf{T}_{kl}}.$$

We define a $\mathbf{T}$-dependent scalar product on $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ by

$$(\mathbf{V}, \mathbf{W})_{\mathcal{A}(\mathbf{T})} := \mathcal{A}(\mathbf{T})\mathbf{V} \cdot \mathbf{W} = \sum_{i,j,k,l=1}^{d} \frac{\partial \mathbf{G}_{ij}(\mathbf{T})}{\partial \mathbf{T}_{kl}} \mathbf{V}_{ij}\mathbf{W}_{kl}. \tag{3.2}$$

The fact that (3.2) does indeed define a scalar product follows from the fact that $\mathbf{G}$ has a potential F. In particular, we know that for all $\mathbf{T} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ there holds $\frac{\partial \mathbf{G}_{ij}(\mathbf{T})}{\partial \mathbf{T}_{kl}} = \frac{\partial \mathbf{G}_{kl}(\mathbf{T})}{\partial \mathbf{T}_{ij}}$, that is, symmetry. Furthermore, $\mathcal{A}(\mathbf{T})$ is positive definite as a result of the convexity assumption.

In what follows, we split the regularity estimates. First, we focus on time regularity. Then we consider regularity with respect to the spatial variable. We provide only a formal proof of the results. Nevertheless, the time regularity proof is fully rigorous since it can be deduced at the level of Galerkin approximations. The spatial regularity proof is only formal, but can be justified by using a standard difference quotient technique. We emphasise that we do not impose any coercivity and growth assumptions on $\mathcal{A}$ here because, in the case $p = 1$, we lose such information.

We note that, if $p \in (1, \infty)$, one can usually assume that

$$|(\mathbf{V}, \mathbf{W})_{\mathcal{A}(\mathbf{T})}| \le C_3(1 + |\mathbf{T}|)^{p-2} |\mathbf{V}|\, |\mathbf{W}|, \qquad (\mathbf{W}, \mathbf{W})_{\mathcal{A}(\mathbf{T})} \ge C_4(1 + |\mathbf{T}|)^{p-2} |\mathbf{W}|^2. \tag{3.3}$$

Under assumption (3.3), the regularity estimates can be deduced in an easier way. However, they are not included here as the more challenging case of $p = 1$ is our primary interest. Also, it is worth observing that our prototype models (1.22) do not satisfy $(3.3)_2$ and in general, the assumption $(3.3)_2$ is not satisfied when $p = 1$.

Defining the convex conjugate F^* of F as in Section 1.2, we recall that, from the definition of $\mathbf{G}$, we have that

$$F(\mathbf{T}) + F^*(\mathbf{G}(\mathbf{T})) = \mathbf{G}(\mathbf{T}) \cdot \mathbf{T}. \tag{3.4}$$

3.1. Time regularity. Here, we improve the bound on the time derivative. This bound is used in the existence proof for the limiting strain model in order to pass to the limit in the term $\partial_{tt}\boldsymbol{u}$ in the weak formulation. We formulate the following lemma locally in time in order to keep the initial data as general as possible.

LEMMA 3.1. *Let $p \in (1, \infty)$ and suppose that (3.1) holds with $\mathbf{G}$ fulfilling (A1)–(A3). Assume that $\boldsymbol{f} \in L^2(0, T; L^2(\Omega; \mathbb{R}^d))$ and $\boldsymbol{u}_0 \in W^{2,p'}(\delta, T; W^{1,p'}(\Omega; \mathbb{R}^d))$ for every $\delta > 0$. For any weak solution to (1.1) and for every $\delta > 0$, the following bound holds:*

$$\begin{aligned} &\sup_{t\in(\delta,T)} \int_\Omega F^*(\mathbf{G}(\mathbf{T}))\,\mathrm{d}x + \int_\delta^T \|\partial_{tt}\boldsymbol{u}\|_2^2\,\mathrm{d}t \\ &\le C(\alpha,\beta)\left(\int_{\frac{\delta}{2}}^T \int_\Omega |\boldsymbol{f}|_2^2 + |\partial_t\boldsymbol{u}|_2^2 + |\partial_{tt}\boldsymbol{u}_0|_2^2 + |\partial_t\boldsymbol{u}_0|_2^2 + |\mathbf{T}\cdot\partial_t(\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0))|\,\mathrm{d}x\,\mathrm{d}t\right) \\ &\quad + \frac{C(\alpha,\beta)}{\delta}\int_0^\delta \int_\Omega F^*(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}(\tau)) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}(\tau))) + |\partial_t\boldsymbol{u}(\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{3.5}$$

If additionally $\boldsymbol{u}_0 \in W^{2,p'}(0,T;W^{1,p'}(\Omega;\mathbb{R}^d))$, *we have the following global-in-time bound:*

$$
\begin{aligned}
&\sup_{t\in(0,T)} \int_\Omega F^*(\mathsf{G}(\mathsf{T}))\,\mathrm{d}x + \int_0^T \|\partial_{tt}\boldsymbol{u}\|_2^2\,\mathrm{d}t \\
&\le C(\alpha,\beta)\left(\int_Q |\boldsymbol{f}|_2^2 + |\partial_t \boldsymbol{u}|_2^2 + |\partial_{tt}\boldsymbol{u}_0|_2^2 + |\partial_t \boldsymbol{u}_0|_2^2 + |\mathsf{T}\cdot\partial_t(\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0)+\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0))|\,\mathrm{d}x\,\mathrm{d}t\right) \\
&\quad + C(\alpha,\beta)\int_\Omega F^*(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0(0)) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0(0))) + |\partial_t\boldsymbol{u}_0(0)|^2\,\mathrm{d}x.
\end{aligned} \tag{3.6}
$$

Proof. Recalling that $\boldsymbol{f} \in L^2(0,T;L^2(\Omega,\mathbb{R}^d))$, we set $\boldsymbol{w} := \beta\partial_{tt}(\boldsymbol{u}-\boldsymbol{u}_0) + \alpha\partial_t(\boldsymbol{u}-\boldsymbol{u}_0)$ in (1.5) to observe that, for almost all $t \in (0,T)$,

$$
\begin{aligned}
&\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\boldsymbol{u}\|_2^2 + \int_\Omega \beta|\partial_{tt}\boldsymbol{u}|^2 + \mathsf{T}\cdot\big(\alpha\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta\partial_{tt}\boldsymbol{\varepsilon}(\boldsymbol{u})\big)\,\mathrm{d}x \\
&= \int_\Omega \boldsymbol{f}\cdot\big(\alpha\partial_t(\boldsymbol{u}-\boldsymbol{u}_0) + \beta\partial_{tt}(\boldsymbol{u}-\boldsymbol{u}_0)\big) + \partial_{tt}\boldsymbol{u}\cdot\big(\alpha\partial_t\boldsymbol{u}_0 + \beta\partial_{tt}\boldsymbol{u}_0\big) \\
&\quad + \mathsf{T}\cdot\big(\alpha\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \beta\partial_{tt}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\big)\,\mathrm{d}x.
\end{aligned} \tag{3.7}
$$

For the third term on the left-hand side of (3.7), using (1.1b), we see that

$$
\begin{aligned}
\int_\Omega \mathsf{T}\cdot(\alpha\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta\partial_{tt}\boldsymbol{\varepsilon}(\boldsymbol{u}))\,\mathrm{d}x &= \int_\Omega \mathsf{G}^{-1}(\mathsf{G}(\mathsf{T})) : \partial_t\mathsf{G}(\mathsf{T})\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F^*(\mathsf{G}(\mathsf{T}))\,\mathrm{d}x,
\end{aligned}
$$

recalling that $\mathsf{G}^{-1}(\mathsf{T}) = \frac{\partial F^*}{\partial \mathsf{T}}(\mathsf{T})$. Thus, using this in (3.7) and applying Young's inequality, we obtain the following bound:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega F^*(\mathsf{G}(\mathsf{T})) + \frac{\alpha}{2}|\partial_t\boldsymbol{u}|^2\,\mathrm{d}x\right) + \frac{\beta}{2}\|\partial_{tt}\boldsymbol{u}\|_2^2 \\
&\le C(\alpha,\beta)(\|\boldsymbol{f}\|_2^2 + \|\partial_t\boldsymbol{u}\|_2^2 + \|\partial_{tt}\boldsymbol{u}_0\|_2^2 + \|\partial_t\boldsymbol{u}_0\|_2^2) + \int_\Omega \mathsf{T}\cdot\partial_t(\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0)).
\end{aligned} \tag{3.8}
$$

Integrating (3.8) over $(0,T)$ and using the fact that

$$
F^*(\mathsf{G}(\mathsf{T}(0))) = F^*(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0)),
$$

we deduce (3.6). Similarly, integrating (3.8) over (τ,t) where $\delta/2 \le \tau \le \delta \le t \le T$ are arbitrary, we deduce that

$$
\begin{aligned}
&\sup_{t\in(\delta,T)}\left(\int_\Omega F^*(\mathsf{G}(\mathsf{T})) + \frac{\alpha|\partial_t\boldsymbol{u}|^2}{2}\,\mathrm{d}x\right) + \int_\delta^T \frac{\beta}{2}\|\partial_{tt}\boldsymbol{u}\|_2^2\,\mathrm{d}t \\
&\le C(\alpha,\beta)\int_{\frac{\delta}{2}}^T \int_\Omega |\boldsymbol{f}|^2 + |\partial_t\boldsymbol{u}|^2 + |\partial_{tt}\boldsymbol{u}_0|^2 + |\partial_t\boldsymbol{u}_0|^2 + |\mathsf{T}\cdot\partial_t(\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0))|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C(\alpha,\beta)\int_\Omega F^*(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}(\tau)) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}(\tau))) + |\partial_t\boldsymbol{u}(\tau)|^2\,\mathrm{d}x.
\end{aligned} \tag{3.9}
$$

Integrating with respect to $\tau \in (\delta/2,\delta)$ and dividing by δ, we directly obtain (3.5). ∎

3.2. Spatial regularity. Here, we improve the spatial regularity of the weak solution. In particular, we prove a weighted bound on $\nabla\mathsf{T}$, which is a key tool for obtaining the existence of a weak solution for the limiting strain model, that is, in the case $p = 1$.

LEMMA 3.2. *Let the assumptions of Lemma 3.1 be satisfied. Also, assume that* $\partial_t \boldsymbol{u}_0(0) \in W^{1,2}(\Omega;\mathbb{R}^d)$ *and*

$$\int_0^T \int_\Omega |\mathcal{A}(\mathbf{T})||\mathbf{T}|^2 + |\mathcal{A}(\mathbf{T})||\boldsymbol{f}|^2 \,\mathrm{d}x\,\mathrm{d}t < \infty.$$

Then, for an arbitrary open set $\Omega' \subset \overline{\Omega'} \subset \Omega$, *for any* $\delta > 0$, *we have the following bound:*

$$\begin{aligned} &\sup_{t\in(\delta,T)} \|\partial_t \nabla \boldsymbol{u}\|_{L^2(\Omega')} + \sum_{k=1}^d \int_\delta^T \int_{\Omega'} (\partial_k \mathbf{T}, \partial_k \mathbf{T})_{\mathcal{A}(\mathbf{T})} \,\mathrm{d}x\,\mathrm{d}t \\ &\le C(\Omega',\delta) \int_0^T \int_\Omega |\mathbf{T}||\mathbf{G}(\mathbf{T})| + |\mathcal{A}(\mathbf{T})||\mathbf{T}|^2 + |\boldsymbol{f}|^2 + |\nabla \boldsymbol{u}|^2 + |\partial_t \nabla \boldsymbol{u}|^2 + |\mathcal{A}(\mathbf{T})||\boldsymbol{f}|^2 \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.10}$$

If, additionally, $\boldsymbol{u}_0 \in \mathcal{C}^1([0,T];W^{1,2}(\Omega;\mathbb{R}^d))$, *then we also have*

$$\begin{aligned} &\sup_{t\in(0,T)} \|\partial_t \nabla \boldsymbol{u}\|_{L^2(\Omega')} + \sum_{k=1}^d \int_0^T \int_{\Omega'} (\partial_k \mathbf{T}, \partial_k \mathbf{T})_{\mathcal{A}(\mathbf{T})} \,\mathrm{d}x\,\mathrm{d}t \\ &\le C(\Omega') \int_0^T \int_\Omega |\mathbf{T}||\mathbf{G}(\mathbf{T})| + |\mathcal{A}(\mathbf{T})||\mathbf{T}|^2 + |\boldsymbol{f}|^2 + |\nabla \boldsymbol{u}|^2 + |\partial_t \nabla \boldsymbol{u}|^2 + |\mathcal{A}(\mathbf{T})||\boldsymbol{f}|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + C\|\partial_t \nabla \boldsymbol{u}_0(0)\|_2^2. \end{aligned} \tag{3.11}$$

Proof. Fix an arbitrary nonnegative smooth compactly supported $\varphi \in \mathcal{C}_0^\infty(\Omega)$. For the test function in (1.5), we choose $\boldsymbol{w} := -\operatorname{div}(\varphi^2 \nabla(\alpha \boldsymbol{u} + \beta \partial_t \boldsymbol{u}))$. Then we integrate by parts to deduce the following identity:

$$\begin{aligned} &\frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\partial_t \nabla \boldsymbol{u}\varphi|^2 \,\mathrm{d}x + \alpha \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \partial_t \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u}\varphi^2 \,\mathrm{d}x \\ &\quad + \int_\Omega \sum_{i,j,k=1}^d \partial_k \mathbf{T}_{ij} \partial_j \big(\varphi^2(\alpha \partial_k \boldsymbol{u}_i + \beta \partial_t \partial_k \boldsymbol{u}_i)\big) \,\mathrm{d}x \\ &= -\int_\Omega \boldsymbol{f} \cdot \operatorname{div}\big(\varphi^2 \nabla(\alpha \boldsymbol{u} + \beta \partial_t \boldsymbol{u})\big) \,\mathrm{d}x + \alpha \int_\Omega |\partial_t \nabla \boldsymbol{u}\varphi|^2 \,\mathrm{d}x. \end{aligned} \tag{3.12}$$

This can be rewritten in the more useful form

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\beta}{4}|\partial_t \nabla \boldsymbol{u}\varphi|^2 + \frac{1}{2\beta}\Big|\alpha \nabla \boldsymbol{u}\varphi + \beta \partial_t \nabla \boldsymbol{u}\varphi\Big|^2 \,\mathrm{d}x + \int_\Omega \sum_{i,j,k=1}^d \partial_k \mathbf{T}_{ij} \partial_j \big(\varphi^2(\alpha \partial_k \boldsymbol{u}_i + \beta \partial_t \partial_k \boldsymbol{u}_i)\big) \,\mathrm{d}x \\ &= -\int_\Omega \boldsymbol{f} \cdot \operatorname{div}\big(\varphi^2 \nabla(\alpha \boldsymbol{u} + \beta \partial_t \boldsymbol{u})\big) \,\mathrm{d}x + \alpha \int_\Omega |\partial_t \nabla \boldsymbol{u}\varphi|^2 \,\mathrm{d}x + \frac{\alpha^2}{2\beta^2}\int_\Omega \partial_t \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u}\varphi^2 \,\mathrm{d}x. \end{aligned} \tag{3.13}$$

Next, we show that the second integral on the left-hand side is the key source of information. We

use (1.1b), integration by parts and the symmetry of T to observe that
(3.14)

$$
\begin{aligned}
&\int_\Omega \sum_{i,j,k=1}^{d} \partial_k \mathsf{T}_{ij} \partial_j(\varphi^2(\alpha\partial_k \boldsymbol{u}_i + \beta\partial_t\partial_k \boldsymbol{u}_i))\,\mathrm{d}x \\
&= \sum_{i,j,k=1}^{d} \int_\Omega \partial_k \mathsf{T}_{ij}(\varphi^2(\alpha\partial_k\partial_j \boldsymbol{u}_i + \beta\partial_t\partial_k\partial_j \boldsymbol{u}_i)) + 2\partial_k \mathsf{T}_{ij}\varphi\partial_j\varphi(\alpha\partial_k \boldsymbol{u}_i + \beta\partial_t\partial_k \boldsymbol{u}_i)\,\mathrm{d}x \\
&= \sum_{i,j,k=1}^{d} \int_\Omega \partial_k \mathsf{T}_{ij}\varphi^2\partial_k(\alpha\boldsymbol{\varepsilon}_{ij}(\boldsymbol{u}) + \beta\partial_t\boldsymbol{\varepsilon}_{ij}(\boldsymbol{u})) + 4\partial_k \mathsf{T}_{ij}\varphi\partial_j\varphi(\alpha\boldsymbol{\varepsilon}_{ik}(\boldsymbol{u}) + \beta\partial_t\boldsymbol{\varepsilon}_{ik}(\boldsymbol{u}))\,\mathrm{d}x \\
&\quad - 2\sum_{i,j,k=1}^{d} \int_\Omega \partial_k \mathsf{T}_{ij}\varphi\partial_j\varphi(\alpha\partial_i \boldsymbol{u}_k + \beta\partial_t\partial_i \boldsymbol{u}_k)\,\mathrm{d}x \\
&= \sum_{i,j,k=1}^{d} \int_\Omega \partial_k \mathsf{T}_{ij}\varphi^2\partial_k \mathsf{G}_{ij}(\mathsf{T}) - 4\mathsf{T}_{ij}\partial_k(\varphi\partial_j\varphi)\mathsf{G}_{ik}(\mathsf{T}) - 4\mathsf{T}_{ij}\varphi\partial_j\varphi\partial_k \mathsf{G}_{ik}(\mathsf{T})\,\mathrm{d}x \\
&\quad + \sum_{i,j,k=1}^{d} \int_\Omega \mathsf{T}_{ij}\partial_{kj}(\varphi^2)\partial_i(\alpha\boldsymbol{u}_k + \beta\partial_t\boldsymbol{u}_k)\,\mathrm{d}x + 2\sum_{i,j,k=1}^{d} \int_\Omega \mathsf{T}_{ij}\varphi\partial_j\varphi\partial_i(\alpha\partial_k \boldsymbol{u}_k + \beta\partial_t\partial_k \boldsymbol{u}_k)\,\mathrm{d}x \\
&= \int_\Omega \sum_{k=1}^{d} (\partial_k \mathsf{T}\varphi, \partial_k \mathsf{T}\varphi)_{\mathcal{A}(\mathsf{T})} - 4\sum_{i,j,k=1}^{d} \mathsf{T}_{ij}\partial_k(\varphi\partial_j\varphi)\mathsf{G}_{ik}(\mathsf{T}) - 4\sum_{i,j,k=1}^{d} \mathsf{T}_{ij}\varphi\partial_j\varphi\partial_k \mathsf{G}_{ik}(\mathsf{T})\,\mathrm{d}x \\
&\quad - \sum_{i,j,k=1}^{d} \int_\Omega \partial_j \mathsf{T}_{ij}\partial_k(\varphi^2)\partial_i(\alpha\boldsymbol{u}_k + \beta\partial_t\boldsymbol{u}_k)\,\mathrm{d}x - \sum_{i,j,k=1}^{d} \int_\Omega \mathsf{T}_{ij}\partial_k(\varphi^2)\partial_{ij}(\alpha\boldsymbol{u}_k + \beta\partial_t\boldsymbol{u}_k)\,\mathrm{d}x \\
&\quad + 2\sum_{i,j,k=1}^{d} \int_\Omega \mathsf{T}_{ij}\varphi\partial_j\varphi\partial_i \mathsf{G}_{kk}(\mathsf{T})\,\mathrm{d}x \\
&=: \sum_{m=1}^{6} I_m.
\end{aligned}
$$

We need to determine what bounds can be deduced from (3.14). In particular, we show that the terms $I_2, \dots, I_6$ can be bounded in terms of I_1 and the data. The simplest bound is for I_2. In particular, it directly follows that

$$
|I_2| \le C(\varphi)\int_\Omega |\mathsf{T}|\,|\mathsf{G}(\mathsf{T})|\,\mathrm{d}x.
$$

Letting δ_{nk} denote the Kronecker delta, in order to bound I_3 we first rewrite it as

$$
\begin{aligned}
\sum_{i,j,k=1}^{d} \mathsf{T}_{ij}\varphi\partial_j\varphi\partial_k \mathsf{G}_{ik}(\mathsf{T}) &= \sum_{i,j,k,l,m,n=1}^{d} \delta_{nk}\mathsf{T}_{ij}\varphi\partial_j\varphi\mathcal{A}^{ik}_{lm}(\mathsf{T})\partial_n \mathsf{T}_{lm} \\
&= \sum_{j,n=1}^{d}\left(\sum_{i,k,l,m=1}^{d} \mathcal{A}^{ik}_{lm}(\mathsf{T})\partial_n \mathsf{T}_{lm}\delta_{nk}\mathsf{T}_{ij}\varphi\partial_j\varphi\right).
\end{aligned}
$$

Using the Cauchy–Schwarz inequality and the fact that $\mathcal{A}$ generates a scalar product, applying

Young's inequality we find that

$$
\begin{aligned}
|I_3| &\leq C \int_\Omega \left| \sum_{j,n=1}^{d} \left(\sum_{i,k,l,m=1}^{d} \mathcal{A}_{lm}^{ik}(\mathbf{T}) \partial_n \mathbf{T}_{lm} \delta_{nk} \mathbf{T}_{ij} \varphi \partial_j \varphi \right) \right| \mathrm{d}x \\
&\leq C \int_\Omega \left| \sum_{j,n=1}^{d} \left(\sum_{i,k,l,m=1}^{d} \mathcal{A}_{lm}^{ik}(\mathbf{T}) \partial_n \mathbf{T}_{lm} \varphi \partial_n \mathbf{T}_{ik} \varphi \right)^{\frac{1}{2}} \right. \\
&\qquad \left. \cdot \left(\sum_{i,k,l,m=1}^{d} \mathcal{A}_{lm}^{ik}(\mathbf{T}) \delta_{nm} \mathbf{T}_{lj} \partial_j \varphi \delta_{nk} \mathbf{T}_{ij} \partial_j \varphi \right)^{\frac{1}{2}} \right| \mathrm{d}x \\
&\leq \frac{I_1}{8} + C(\varphi) \int_\Omega |\mathcal{A}(\mathbf{T})| |\mathbf{T}|^2 \, \mathrm{d}x.
\end{aligned}
$$

The term I_6 can be bounded in a very similar way. In particular, we have

$$
|I_6| \leq \frac{I_1}{8} + C(\varphi) \int_\Omega |\mathcal{A}(\mathbf{T})| |\mathbf{T}|^2 \, \mathrm{d}x.
$$

For I_4, we use the equation (1.1a) and Young's inequality to obtain

$$
\begin{aligned}
|I_4| &= \left| \sum_{i,k=1}^{d} \int_\Omega (\boldsymbol{f}_i - \partial_{tt} \boldsymbol{u}_i) \partial_k(\varphi^2) \partial_i (\alpha \boldsymbol{u}_k + \beta \partial_t \boldsymbol{u}_k) \, \mathrm{d}x \right| \\
&\leq C(\varphi) \int_\Omega |\boldsymbol{f}|^2 + |\partial_{tt} \boldsymbol{u}|^2 + |\partial_t \nabla \boldsymbol{u}|^2 + |\nabla \boldsymbol{u}|^2 \, \mathrm{d}x.
\end{aligned}
$$

Finally, to evaluate I_5, we first recall the following identity

$$
\begin{aligned}
&\partial_{ij}(\alpha \boldsymbol{u}_k + \beta \partial_t \boldsymbol{u}_k) \\
&= \partial_i(\alpha \boldsymbol{\varepsilon}_{jk}(\boldsymbol{u}) + \beta \partial_t \boldsymbol{\varepsilon}_{jk}(\boldsymbol{u})) + \partial_j(\alpha \boldsymbol{\varepsilon}_{ik}(\boldsymbol{u}) + \beta \partial_t \boldsymbol{\varepsilon}_{ik}(\boldsymbol{u})) - \partial_k(\alpha \boldsymbol{\varepsilon}_{ij}(\boldsymbol{u}) + \beta \partial_t \boldsymbol{\varepsilon}_{ij}(\boldsymbol{u})).
\end{aligned} \tag{3.15}
$$

Then, we rewrite I_5 with the help of (1.1b) to find that

$$
I_5 = - \sum_{i,j,k=1}^{d} \int_\Omega \mathbf{T}_{ij} \partial_k(\varphi^2) \left(\partial_i \mathbf{G}_{jk}(\mathbf{T}) + \partial_j \mathbf{G}_{ik}(\mathbf{T}) - \partial_k \mathbf{G}_{ij}(\mathbf{T}) \right) \mathrm{d}x.
$$

Hence, we see that we are in the same situation as with the term I_3 and we deduce that

$$
|I_5| \leq \frac{I_1}{8} + C(\varphi) \int_\Omega |\mathcal{A}(\mathbf{T})| |\mathbf{T}|^2 \, \mathrm{d}x.
$$

Thus we have suitable bounds on the left-hand side of (3.13). We rewrite the first term on the right-hand side of (3.13) in the following way:

$$
\begin{aligned}
&\int_\Omega \boldsymbol{f} \cdot \operatorname{div}(\varphi^2 (\alpha \nabla \boldsymbol{u} + \beta \partial_t \nabla \boldsymbol{u})) \, \mathrm{d}x \\
&= \sum_{i,j=1}^{d} \int_\Omega \boldsymbol{f}_i (\partial_j(\varphi^2)(\alpha \partial_j \boldsymbol{u}_i + \beta \partial_t \partial_j \boldsymbol{u}_i) + \varphi^2 (\alpha \partial_{jj} \boldsymbol{u}_i + \beta \partial_t \partial_{jj} \boldsymbol{u}_i)) \, \mathrm{d}x \\
&= \sum_{i,j=1}^{d} \int_\Omega \boldsymbol{f}_i (\partial_j(\varphi^2)(\alpha \partial_j \boldsymbol{u}_i + \beta \partial_t \partial_j \boldsymbol{u}_i) + \varphi^2 (2 \partial_j \mathbf{G}_{ij}(\mathbf{T}) - \partial_i \mathbf{G}_{jj}(\mathbf{T})) \, \mathrm{d}x.
\end{aligned}
$$

Using Young's inequality on the first term and a procedure similar to the one used for I_3 for the second, we get

$$\left|\int_\Omega \boldsymbol{f}\cdot \operatorname{div}(\varphi^2(\alpha\nabla \boldsymbol{u}+\beta\partial_t\nabla\boldsymbol{u}))\,\mathrm{d}x\right| \le \frac{I_1}{8}+C(\varphi)\int_\Omega |\boldsymbol{f}|^2+|\nabla\boldsymbol{u}|^2+|\partial_t\nabla\boldsymbol{u}|^2+|\mathcal{A}(\mathbf{T})||\boldsymbol{f}|^2\,\mathrm{d}x. \tag{3.16}$$

Substituting the above bounds into (3.13) and using a similar procedure to the one used in the proof of Lemma 3.1, we deduce (3.11) and (3.10). ∎

4. Limiting strain - Proof of Theorem 1.2. As in the proof of Theorem 1.1, in order to prove Theorem 1.2 we first introduce an approximate problem. However, we are able to make use of the knowledge obtained from Theorem 1.1. Indeed, we define a function on $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ by

$$\mathbf{G}^n(\mathbf{T}) := \mathbf{G}(\mathbf{T}) + n^{-1}\mathbf{T}. \tag{4.1}$$

Since $\mathbf{G}$ satisfies (A1)–(A3) with $p=1$, it is evident that $\mathbf{G}^n$ satisfies (A1)–(A3) with $p=2$. Therefore, as a result of Theorem 1.1, there exists a couple $(\boldsymbol{u}^n,\mathbf{T}^n)$, fulfilling[3]

$$\boldsymbol{u}^n \in \mathcal{C}^1([0,T];L^2(\Omega;\mathbb{R}^d))\cap W^{1,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^d))\cap W^{2,2}(0,T;(W^{1,2}_0(\Omega;\mathbb{R}^d))^*), \tag{4.2}$$

$$\mathbf{T}^n \in L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})) \tag{4.3}$$

and satisfying

$$\langle \partial_{tt}\boldsymbol{u}^n,\boldsymbol{w}\rangle + \int_\Omega \mathbf{T}^n\cdot\nabla\boldsymbol{w}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{w}\,\mathrm{d}x \qquad \forall\,\boldsymbol{w}\in W^{1,2}_0(\Omega;\mathbb{R}^d) \quad \text{for a.e. } t\in(0,T), \tag{4.4}$$

and

$$\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n)+\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n) = \mathbf{G}^n(\mathbf{T}^n) = \mathbf{G}(\mathbf{T}^n)+n^{-1}\mathbf{T}^n \quad \text{a.e. in } Q. \tag{4.5}$$

We note that we can replace the duality pairing by the integral over Ω in the term containing $\boldsymbol{f}$ thanks to the assumed regularity of $\boldsymbol{f}$. Moreover, we know that[4]

$$\boldsymbol{u}^n=\boldsymbol{u}_0 \quad \text{on } \Gamma\cup(\{0\}\times\Omega), \qquad \partial_t\boldsymbol{u}^n=\partial_t\boldsymbol{u}_0 \quad \text{on } \{0\}\times\Omega.$$

We want to consider the limit as $n\to\infty$ in order to prove the existence of a solution to the limiting strain problem in the sense of Theorem 1.2.

4.1. A priori n-independent bounds. We start with bounds that are independent of the order of approximation. For this purpose, we use and mimic some of the steps from the preceding sections. We start with the first uniform bound. Setting $\boldsymbol{w}:=\beta\partial_t(\boldsymbol{u}^n-\boldsymbol{u}_0)+\alpha(\boldsymbol{u}-\boldsymbol{u}_0)$ in (4.4), applying the same algebraic manipulations as those used for (2.4), we deduce that

$$\begin{aligned}
&\frac{\beta}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\partial_t(\boldsymbol{u}^n-\boldsymbol{u}_0)|^2+\left|\partial_t(\boldsymbol{u}^n-\boldsymbol{u}_0)+\frac{2\alpha}{\beta}(\boldsymbol{u}^n-\boldsymbol{u}_0)\right|^2\mathrm{d}x+\int_\Omega \mathbf{G}^n(\mathbf{T}^n)\cdot\mathbf{T}^n\,\mathrm{d}x\\
&=\int_\Omega \mathbf{T}^n\cdot(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0)+\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0))\,\mathrm{d}x+\alpha\int_\Omega|\partial_t(\boldsymbol{u}^n-\boldsymbol{u}_0)|^2\,\mathrm{d}x\\
&\quad+\int_\Omega(\boldsymbol{f}-\partial_{tt}\boldsymbol{u}_0)\cdot(\alpha(\boldsymbol{u}^n-\boldsymbol{u}_0)+\beta\partial_t(\boldsymbol{u}^n-\boldsymbol{u}_0))\,\mathrm{d}x+\frac{2\alpha^2}{\beta}\int_\Omega\partial_t(\boldsymbol{u}^n-\boldsymbol{u}_0)\cdot(\boldsymbol{u}^n-\boldsymbol{u}_0)\,\mathrm{d}x.
\end{aligned}\tag{4.6}$$

[3] We assume a slightly different restriction on $\boldsymbol{u}_0$ than in Theorem 1.1. However, the proof of Theorem 1.1 can be easily adapted to this case.

[4] In case that Ω is not a Lipschitz domain, the identity below is not understood in the sense of traces but in the sense that $\boldsymbol{u}-\boldsymbol{u}_0\in W^{1,1}_0(\Omega;\mathbb{R}^d)$ for almost all $t\in(0,T)$, where $W^{1,1}_0(\Omega;\mathbb{R}^d)$ defined as the closure of $C^\infty_0(\Omega;\mathbb{R}^d)$ in the norm of $W^{1,1}(\Omega;\mathbb{R}^d)$.

In order to obtain the required a priori estimate, we need to use the safety strain condition. In particular, it follows from (1.13) that there exists a $\delta > 0$ such that

$$|\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0)| \le L - 2\delta \qquad \text{a.e. in } Q, \tag{4.7}$$

where L is defined as in (1.9). Defining $F(\mathbf{T}) := \phi(|\mathbf{T}|)$, it follows from the convexity of ϕ that, for any $\tilde{\delta} > 0$, there exists a $C_{\tilde{\delta}}$ such that, for all $\mathbf{T} \in \mathbb{R}^{d\times d}_{sym}$,

$$F(\mathbf{T}) \ge (L - \tilde{\delta})|\mathbf{T}| - C_{\tilde{\delta}}. \tag{4.8}$$

We choose $\tilde{\delta} = \delta$ as in (4.7) and let C_δ be the corresponding constant from (4.8). Since δ depends in principle on $\boldsymbol{u}_0$ and F, we do not trace the dependence of C on δ in what follows. Consequently, for the second term on the left-hand side of (4.6), we can use (3.4) and (4.5) to deduce that

$$\mathbf{G}^n(\mathbf{T}^n) \cdot \mathbf{T}^n = n^{-1}|\mathbf{T}^n|^2 + F(\mathbf{T}^n) + F^*(\mathbf{G}(\mathbf{T}^n)) \ge (L - \delta)|\mathbf{T}^n| + n^{-1}|\mathbf{T}^n|^2 - C.$$

Furthermore, the first term on the right-hand side of (4.6) can be bounded by using (4.7) in the following way:

$$\int_\Omega \mathbf{T}^n \cdot (\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0))\,\mathrm{d}x \le (L - 2\delta)\|\mathbf{T}^n\|_1.$$

Therefore, it follows from (4.6), the above bounds and Hölder's inequality that

$$\begin{aligned} &\frac{\beta}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)|^2 + \left|\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0) + \frac{2\alpha}{\beta}(\boldsymbol{u}^n - \boldsymbol{u}_0)\right|^2 \mathrm{d}x + \delta\|\mathbf{T}^n\|_1 + n^{-1}\|\mathbf{T}^n\|_2^2 \\ &\le C\left(\int_\Omega \beta|\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)|^2 + \beta\left|\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0) + \frac{2\alpha}{\beta}(\boldsymbol{u}^n - \boldsymbol{u}_0)\right|^2 \mathrm{d}x + \|\boldsymbol{f}\|_2^2 + \|\partial_{tt}\boldsymbol{u}_0\|_2^2 + 1\right). \end{aligned} \tag{4.9}$$

An application of Grönwall's lemma yields

$$\sup_{t\in(0,T)}\left(\|\partial_t\boldsymbol{u}^n(t)\|_2^2 + \|\boldsymbol{u}^n(t)\|_2^2\right) + \int_0^T \|\mathbf{T}^n\|_1 + n^{-1}\|\mathbf{T}^n\|_2^2\,\mathrm{d}t \le C(\boldsymbol{f}, \boldsymbol{u}_0), \tag{4.10}$$

where we use assumption (1.12) regarding the data. It follows from (1.6) and the above bound that

$$\int_Q |\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n)|^2\,\mathrm{d}x\,\mathrm{d}t \le \int_Q (L + n^{-1}|\mathbf{T}^n|)^2\,\mathrm{d}x\,\mathrm{d}t \le C(\boldsymbol{f}, \boldsymbol{u}_0). \tag{4.11}$$

However, we know that

$$|\mathbf{G}^n(\mathbf{T}^n) \cdot \mathbf{T}^n| \le L|\mathbf{T}^n| + \frac{|\mathbf{T}^n|^2}{2}.$$

Hence, as a result of (4.10), we have that

$$\int_Q |\mathbf{G}^n(\mathbf{T}^n) \cdot \mathbf{T}^n|\,\mathrm{d}x\,\mathrm{d}t \le C(\boldsymbol{f}, \boldsymbol{u}_0). \tag{4.12}$$

Furthermore, arguing as with (2.6) and making use of (4.10), (4.11), we deduce that

$$\sup_{t\in(0,T)} \|\boldsymbol{u}^n\|_{1,2} + \int_0^T \|\partial_t\boldsymbol{u}^n\|_{1,2}^2\,\mathrm{d}t \le C(\boldsymbol{f}, \boldsymbol{u}_0). \tag{4.13}$$

4.2. Regularity via n-independent bounds. The bounds (4.10), (4.12) and (4.13) are not sufficient to pass to the limit $n \to \infty$, since we only have a priori control on $\mathbf{T}^n$ in a nonreflexive space $L^1(Q; \mathbb{R}^{d\times d})$. In particular, at best we have that the weak star limit of $\mathbf{T}^n$ is a measure. Therefore, the pointwise relation (1.20) is neither meaningful nor likely to be valid in this case. Instead, we improve our information by using the regularity technique introduced in Section 3. Namely, we use Lemma 3.1 and Lemma 3.2. First, we define an approximation F_n of the potential F by

$$F_n(\mathbf{T}) := F(\mathbf{T}) + \frac{|\mathbf{T}|^2}{2n}.$$

We have that

$$\frac{\partial F^n(\mathbf{T})}{\partial \mathbf{T}} = \mathbf{G}_n(\mathbf{T}) = \mathbf{G}(\mathbf{T}) + n^{-1}\mathbf{T}.$$

We now apply the results from Section 3 with $p = 2$, replacing $(\boldsymbol{u}, F, \mathbf{G})$ with the triple $(\boldsymbol{u}^n, F_n, \mathbf{G}_n)$. Using the definition of $\mathbf{G}_n$, we define $\mathcal{A}_n$ in an analogous way to $\mathcal{A}$. In particular, we write

$$\begin{aligned}(\mathcal{A}_n(\mathbf{T}^n))_{ijkl} &:= \frac{\partial}{\partial \mathbf{T}^n_{kl}}\left(\frac{\phi'(|\mathbf{T}^n|)}{|\mathbf{T}^n|}\mathbf{T}^n_{ij} + n^{-1}\mathbf{T}^n_{ij}\right)\\ &= \delta_{ik}\delta_{jl}\left(n^{-1} + \frac{\phi'(|\mathbf{T}^n|)}{|\mathbf{T}^n|}\right) + \left(\frac{\phi''(|\mathbf{T}^n|)|\mathbf{T}^n| - \phi'(|\mathbf{T}^n|)}{|\mathbf{T}^n|}\right)\frac{\mathbf{T}^n_{ij}\mathbf{T}^n_{kl}}{|\mathbf{T}^n|^2}.\end{aligned}$$

Consequently, using the fact that $\phi'(0) = 0$ and $\phi''(s) \le C(1+s)^{-1}$, we see that

$$|\mathcal{A}_n(\mathbf{T}^n)| \le Cn^{-1} + \frac{C}{1+|\mathbf{T}^n|}. \tag{4.14}$$

With this in mind, we first discuss regularity with respect to time. We see that all assumptions of Lemma 3.1 are satisfied. Therefore we have, for every $\delta > 0$, the following inequality:

(4.15)
$$\begin{aligned}&\sup_{t\in(\delta,T)}\int_\Omega F_n^*(\mathbf{G}_n(\mathbf{T}^n))\,\mathrm{d}x + \int_\delta^T \|\partial_{tt}\boldsymbol{u}^n\|_2^2\,\mathrm{d}t\\ &\le C(\alpha,\beta)\left(\int_{\frac{\delta}{2}}^T\int_\Omega |\boldsymbol{f}|_2^2 + |\partial_t\boldsymbol{u}^n|_2^2 + |\partial_{tt}\boldsymbol{u}_0|_2^2 + |\partial_t\boldsymbol{u}_0|_2^2 + |\mathbf{T}^n\cdot\partial_t(\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0))|\,\mathrm{d}x\,\mathrm{d}t\right)\\ &\quad + \frac{C(\alpha,\beta)}{\delta}\int_0^\delta\int_\Omega F_n^*(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n(\tau)) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n(\tau))) + |\partial_t\boldsymbol{u}^n(\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

We focus on the right-hand side. For the second integral on the right-hand side, it follows from the properties of the convex conjugate function and the uniform bounds (4.10), (4.12), (4.13) that we have

$$\begin{aligned}\int_0^\delta\int_\Omega F_n^*(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n)) + |\partial_t\boldsymbol{u}^n|^2\,\mathrm{d}x\,\mathrm{d}\tau &= \int_0^\delta\int_\Omega F_n^*(\mathbf{G}_n(\mathbf{T}^n)) + |\partial_t\boldsymbol{u}^n|^2\,\mathrm{d}x\,\mathrm{d}\tau\\ &\le \int_0^\delta\int_\Omega \big(F_n^*(\mathbf{G}_n(\mathbf{T}^n)) + F_n(\mathbf{T}^n)\big) + |\partial_t\boldsymbol{u}^n|^2\,\mathrm{d}x\,\mathrm{d}\tau\\ &= \int_Q \mathbf{G}_n(\mathbf{T}^n)\cdot\mathbf{T}^n + |\partial_t\boldsymbol{u}^n|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le C(\boldsymbol{u}_0, \boldsymbol{f}),\end{aligned}$$

using property (1.36) with $(F, \mathbf{G})$ replaced by $(F_n, \mathbf{G}_n)$ in order to deduce the second inequality. For the first term on the right-hand side of (4.15), we use Hölder's inequality, the assumptions on the data (1.12), (1.13), (1.14) and the uniform bound (4.10) in order to deduce that

$$\begin{aligned}\int_{\frac{\delta}{2}}^T \int_\Omega |\boldsymbol{f}|_2^2 + |\partial_t \boldsymbol{u}^n|_2^2 + |\partial_{tt}\boldsymbol{u}_0|_2^2 + |\partial_t \boldsymbol{u}_0|_2^2 + |\mathbf{T}^n \cdot \partial_t(\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0))| \,\mathrm{d}x\,\mathrm{d}t \\ \le C(\boldsymbol{u}_0, \boldsymbol{f}) + \| |\partial_{tt}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)| + |\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0)| \|_{L^\infty((\frac{\delta}{2},T)\times\Omega)} \int_0^T \int_\Omega |\mathbf{T}^n| \,\mathrm{d}x\,\mathrm{d}t \\ \le C(\boldsymbol{u}_0, \boldsymbol{f}).\end{aligned}$$

It follows from the above bounds and (4.15) that, for every $\delta > 0$, we have

$$\sup_{t\in(\delta,T)} \int_\Omega F_n^*(\mathbf{G}_n(\mathbf{T}^n)) \,\mathrm{d}x + \int_\delta^T \|\partial_{tt}\boldsymbol{u}^n\|_2^2 \,\mathrm{d}t \le C(\boldsymbol{f}, \boldsymbol{u}_0). \tag{4.16}$$

Similarly, in case that (1.14) holds for $\delta = 0$, we use (3.6). By an analogous computation to the above, we deduce that

$$\begin{aligned}&\sup_{t\in(0,T)} \int_\Omega F_n^*(\mathbf{G}_n(\mathbf{T}^n)) \,\mathrm{d}x + \int_0^T \|\partial_{tt}\boldsymbol{u}^n\|_2^2 \,\mathrm{d}t \\ &\le C(\boldsymbol{f}, \boldsymbol{u}_0) + C\int_\Omega F_n^*(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0(0)) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0(0))) \,\mathrm{d}x \\ &\le C(\boldsymbol{f}, \boldsymbol{u}_0) + C\int_\Omega F^*(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0(0)) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0(0))) \,\mathrm{d}x \\ &\le C(\boldsymbol{f}, \boldsymbol{u}_0),\end{aligned} \tag{4.17}$$

using the fact that $F_n^* \le F^*$ and assumptions (1.13), (1.14) with $\delta = 0$.

Next, we consider the spatial regularity estimates. For an arbitrary open set $\Omega' \subset \overline{\Omega'} \subset \Omega$ and for any $\delta > 0$, it follows from (3.10) that

$$\begin{aligned}&\sup_{t\in(\delta,T)} \|\partial_t\nabla\boldsymbol{u}^n\|_{L^2(\Omega')} + \sum_{k=1}^d \int_\delta^T \int_{\Omega'} (\partial_k\mathbf{T}^n, \partial_k\mathbf{T}^n)_{\mathcal{A}_n(\mathbf{T}^n)} \,\mathrm{d}x\,\mathrm{d}t \\ &\le C(\Omega', \delta)\int_Q |\mathbf{T}^n||\mathbf{G}_n(\mathbf{T}^n)| + |\mathcal{A}_n(\mathbf{T}^n)||\mathbf{T}^n|^2 + |\boldsymbol{f}|^2 + |\nabla\boldsymbol{u}^n|^2 + |\partial_t\nabla\boldsymbol{u}^n|^2 + |\mathcal{A}_n(\mathbf{T}^n)||\boldsymbol{f}|^2 \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.18}$$

Since $|\mathbf{T}^n||\mathbf{G}_n(\mathbf{T}^n)| = |\mathbf{T}^n \cdot \mathbf{G}_n(\mathbf{T}^n)|$, we can use (4.10), (4.12) and (4.13) to deduce that

$$\int_Q |\mathbf{T}^n||\mathbf{G}_n(\mathbf{T}^n)| + |\boldsymbol{f}|^2 + |\nabla\boldsymbol{u}^n|^2 + |\partial_t\nabla\boldsymbol{u}^n|^2 \,\mathrm{d}x\,\mathrm{d}t \le C(\boldsymbol{u}_0, \boldsymbol{f}).$$

It only remains to bound the terms involving $\mathcal{A}_n$ on the right-hand side of (4.18). To this end, we note that

$$\int_Q |\mathcal{A}_n(\mathbf{T}^n)||\mathbf{T}^n|^2 + |\mathcal{A}_n(\mathbf{T}^n)||\boldsymbol{f}|^2 \,\mathrm{d}x\,\mathrm{d}t \le C\int_Q n^{-1}|\mathbf{T}^n|^2 + |\mathbf{T}^n| + |\boldsymbol{f}|^2 \le C(\boldsymbol{u}_0, \boldsymbol{f}),$$

where the last inequality follows from (4.10) and the assumptions on $\boldsymbol{f}$. Using these inequalities for the terms appearing on the right-hand side of (4.18), we immediately deduce that

$$\sup_{t\in(\delta,T)} \|\partial_t\nabla\boldsymbol{u}^n\|_{L^2(\Omega')} + \sum_{k=1}^d \int_\delta^T \int_{\Omega'} (\partial_k\mathbf{T}^n, \partial_k\mathbf{T}^n)_{\mathcal{A}_n(\mathbf{T}^n)} \,\mathrm{d}x\,\mathrm{d}t \le C(\boldsymbol{u}_0, \boldsymbol{f}, \Omega'). \tag{4.19}$$

Similarly, if $\boldsymbol{u}_0 \in \mathcal{C}^1([0,T]; W^{1,2}(\Omega;\mathbb{R}^d))$ we can use (3.11) and perform similar computations to find that

$$\sup_{t\in(0,T)} \|\partial_t\nabla\boldsymbol{u}^n\|_{L^2(\Omega')} + \sum_{k=1}^d \int_0^T \int_{\Omega'} (\partial_k\mathbf{T}^n, \partial_k\mathbf{T}^n)_{\mathcal{A}_n(\mathbf{T}^n)} \,\mathrm{d}x\,\mathrm{d}t \le C(\Omega', \boldsymbol{u}_0, \boldsymbol{f}). \tag{4.20}$$

Next, we focus on the bounds on the second order spatial derivatives of $\partial_t \boldsymbol{u}^n$ and $\boldsymbol{u}^n$. It follows from (4.5) and the Cauchy–Schwarz inequality that

$$
\begin{aligned}
&|\partial_k(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n)+\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n))|^2\\
&=(\partial_k(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n)+\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n)))\cdot\partial_k\mathbf{G}_n(\mathbf{T}^n)\\
&=(\partial_k(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n)+\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n)),\partial_k\mathbf{T}^n)_{\mathcal{A}_n(\mathbf{T}^n)}\\
&\le(\partial_k(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n)+\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n)),\partial_k(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n)+\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n)))^{\frac12}_{\mathcal{A}_n(\mathbf{T}^n)}(\partial_k\mathbf{T}^n,\partial_k\mathbf{T}^n)^{\frac12}_{\mathcal{A}_n(\mathbf{T}^n)}\\
&\le C|\partial_k(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n)+\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n))|(\partial_k\mathbf{T}^n,\partial_k\mathbf{T}^n)^{\frac12}_{\mathcal{A}_n(\mathbf{T}^n)}.
\end{aligned}
$$

Therefore,

$$
|\partial_k(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n)+\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n))|^2\le C(\partial_k\mathbf{T}^n,\partial_k\mathbf{T}^n)_{\mathcal{A}_n(\mathbf{T}^n)}.
$$

Using this and (4.19), simple algebraic manipulations imply that

$$
\int_\delta^T\int_{\Omega'}|\nabla(\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n)+\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n))|^2\,\mathrm{d}x\,\mathrm{d}t\le C(\boldsymbol{u}_0,\boldsymbol{f},\Omega'). \tag{4.21}
$$

4.3. Convergence results as $n\to\infty$ based on uniform bounds. From the uniform bounds (4.10), (4.12) and (4.13), we see that we can find a subsequence, not relabelled, such that

$$
\begin{aligned}
&\boldsymbol{u}^n\rightharpoonup\boldsymbol{u} &&\text{weakly in } W^{1,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^d)), &(4.22)\\
&\boldsymbol{u}^n\overset{*}{\rightharpoonup}\boldsymbol{u} &&\text{weakly}^* \text{ in } W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)), &(4.23)\\
&n^{-1}\mathbf{T}^n\to\mathbf{0} &&\text{strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})). &(4.24)
\end{aligned}
$$

In addition, using the regularity estimates (4.16), (4.21), as well as the Aubin–Lions lemma, we deduce that, for every $\delta>0$,

$$
\begin{aligned}
&\boldsymbol{u}^n\rightharpoonup\boldsymbol{u} &&\text{weakly in } W^{2,2}(\delta,T;L^2(\Omega;\mathbb{R}^d)), &(4.25)\\
&\boldsymbol{u}^n\rightharpoonup\boldsymbol{u} &&\text{weakly in } W^{1,2}(\delta,T;W^{2,2}_{loc}(\Omega;\mathbb{R}^d)), &(4.26)\\
&\boldsymbol{u}^n\to\boldsymbol{u} &&\text{strongly in } W^{1,2}(\delta,T;W^{1,2}_{loc}(\Omega;\mathbb{R}^d)). &(4.27)
\end{aligned}
$$

Next, we focus on taking the limit in the constitutive relation (4.5). The mapping $\mathbf{G}$ is bounded so we have that

$$
\mathbf{G}(\mathbf{T}^n)\overset{*}{\rightharpoonup}\overline{\mathbf{G}}\qquad\text{weakly}^*\text{ in } L^\infty(Q;\mathbb{R}^{d\times d}). \tag{4.28}
$$

We need to identify $\overline{\mathbf{G}}$. We note that from (4.5), (4.23) and (4.24), we must have

$$
\overline{\mathbf{G}}=\alpha\boldsymbol{\varepsilon}(\boldsymbol{u})+\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u})\qquad\text{a.e. in } Q. \tag{4.29}
$$

Next, we show that there exists a $\mathbf{T}$ such that $\overline{\mathbf{G}}=\mathbf{G}(\mathbf{T})$. To do so, we appeal to Chacon's biting lemma and deduce from (4.10) that there exists a $\mathbf{T}\in L^1(Q;\mathbb{R}^{d\times d})$ and a nondecreasing sequence of sets $Q_1\subset Q_2\subset\cdots$, with $|Q\setminus Q_i|\to 0$ as $i\to\infty$, such that, for each $i\in\mathbb{N}$,

$$
\mathbf{T}^n\rightharpoonup\mathbf{T}\qquad\text{weakly in } L^1(Q_i;\mathbb{R}^{d\times d}). \tag{4.30}
$$

However, thanks to (4.27), (4.29) and Egoroff's theorem, we know that for every $\varepsilon>0$ and every $i\in\mathbb{N}$ there exists a $Q_{i,\varepsilon}\subset Q_i$, with $|Q_i\setminus Q_{i,\varepsilon}|\le\varepsilon$, such that

$$
\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n)+\beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n)\to\overline{\mathbf{G}}\qquad\text{strongly in } L^\infty(Q_{i,\varepsilon};\mathbb{R}^{d\times d}).
$$

Therefore, using the monotonicity of $\mathbf{G}$ and the above convergence result, we deduce, for an arbitrary $\mathbf{W} \in L^1(Q;\mathbb{R}^{d\times d})$, that

$$\begin{aligned}
0 &\le \lim_{n\to\infty} \int_{Q_{i,\varepsilon}} (\mathbf{G}(\mathbf{T}^n) - \mathbf{G}(\mathbf{W}))\cdot(\mathbf{T}^n - \mathbf{W})\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{Q_{i,\varepsilon}} \mathbf{G}(\mathbf{W})\cdot(\mathbf{W}-\mathbf{T}) - \overline{\mathbf{G}}\cdot\mathbf{W}\,\mathrm{d}x\,\mathrm{d}t + \lim_{n\to\infty}\int_{Q_{i,\varepsilon}} \mathbf{G}(\mathbf{T}^n)\cdot\mathbf{T}^n\,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_{Q_{i,\varepsilon}} \mathbf{G}(\mathbf{W})\cdot(\mathbf{W}-\mathbf{T}) - \overline{\mathbf{G}}\cdot\mathbf{W}\,\mathrm{d}x\,\mathrm{d}t + \lim_{n\to\infty}\int_{Q_{i,\varepsilon}} \mathbf{G}_n(\mathbf{T}^n)\cdot\mathbf{T}^n\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{Q_{i,\varepsilon}} \mathbf{G}(\mathbf{W})\cdot(\mathbf{W}-\mathbf{T}) - \overline{\mathbf{G}}\cdot\mathbf{W}\,\mathrm{d}x\,\mathrm{d}t + \lim_{n\to\infty}\int_{Q_{i,\varepsilon}} (\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^n) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}^n))\cdot\mathbf{T}^n\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{Q_{i,\varepsilon}} (\overline{\mathbf{G}} - \mathbf{G}(\mathbf{W}))\cdot(\mathbf{T}-\mathbf{W})\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Since $\mathbf{G}$ is a monotone mapping and $\mathbf{W}$ is arbitrary, we use Minty's method to see that

$$\overline{\mathbf{G}} = \mathbf{G}(\mathbf{T}) \qquad \text{a.e. in } Q_{i,\varepsilon}.$$

Recalling that $\varepsilon > 0$ and $i \in \mathbb{N}$ are arbitrary, (1.20) follows, using (4.29) and the above identity. Additionally, setting $\mathbf{W} := \mathbf{T}$ in the above and using the fact that $\overline{\mathbf{G}} = \mathbf{G}(\mathbf{T})$, we see that

$$\lim_{n\to\infty}\int_{Q_{i,\varepsilon}} |(\mathbf{G}(\mathbf{T}^n) - \mathbf{G}(\mathbf{T}))\cdot(\mathbf{T}^n-\mathbf{T})|\,\mathrm{d}x\,\mathrm{d}t = \lim_{n\to\infty}\int_{Q_{i,\varepsilon}} (\mathbf{G}(\mathbf{T}^n) - \mathbf{G}(\mathbf{T}))\cdot(\mathbf{T}^n-\mathbf{T})\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Consequently, we must have that

$$\mathbf{T}^n \to \mathbf{T} \qquad \text{a.e. in } Q_{i,\varepsilon},$$

as a result of the strict monotonicity of $\mathbf{G}$. However, as before, since $\varepsilon > 0$ and $i \in \mathbb{N}$ are arbitrary, we deduce that

$$\mathbf{T}^n \to \mathbf{T} \qquad \text{a.e. in } Q. \tag{4.31}$$

Using (4.10), (4.31) and Fatou's lemma, it follows that

$$\int_Q |\mathbf{T}|\,\mathrm{d}x\,\mathrm{d}t \le C(\boldsymbol{u}_0, \boldsymbol{f}). \tag{4.32}$$

Next, we focus on the boundary and initial conditions for $\boldsymbol{u}$. It is evident from the convergence result (4.22), combined with the fact that $\boldsymbol{u}^n = \boldsymbol{u}_0$ on Γ and $\boldsymbol{u}^n(0) = \boldsymbol{u}_0(0)$ on Ω, that we must have $\boldsymbol{u} = \boldsymbol{u}_0$ on Γ as well. Furthermore, it follows that

$$\|\boldsymbol{u}(t) - \boldsymbol{u}_0(0)\|_{1,2} \to 0 \qquad \text{as } t \to 0_+.$$

Concerning the attainment of the initial condition for $\partial_t\boldsymbol{u}(0)$ we need to proceed slightly differently since we only have control on $\partial_{tt}\boldsymbol{u}$ locally in $(0,T)$. We integrate (4.6) over a time interval $(0,t)$, where $0 < t < T$, and since we know that for each n the initial datum is attained we deduce that

(4.33)

$$\begin{aligned}
&\frac{1}{4}\int_\Omega \beta|\partial_t(\boldsymbol{u}^n - \boldsymbol{u}_0)(t)|^2 + \beta\left|\partial_t(\boldsymbol{u}^n-\boldsymbol{u}_0)(t) + \frac{2\alpha}{\beta}(\boldsymbol{u}^n-\boldsymbol{u}_0)(t)\right|^2\,\mathrm{d}x \\
&= \int_0^t\int_\Omega \mathbf{T}^n\cdot((\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0)) - \mathbf{G}_n(\mathbf{T}^n)) + \alpha|\partial_t(\boldsymbol{u}^n-\boldsymbol{u}_0)|^2\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \int_0^t\int_\Omega (\boldsymbol{f} - \partial_{tt}\boldsymbol{u}_0)\cdot(\alpha(\boldsymbol{u}^n-\boldsymbol{u}_0) + \beta\partial_t(\boldsymbol{u}^n-\boldsymbol{u}_0)) + \frac{2\alpha^2}{\beta}\partial_t(\boldsymbol{u}^n-\boldsymbol{u}_0)\cdot(\boldsymbol{u}^n-\boldsymbol{u}_0)\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

Our goal is to let $n \to \infty$. Since $t > 0$, we can use the "local" convergence result (4.25) to let $n \to \infty$ in the left-hand side of (4.33). To bound also the right-hand side, we first use the safety strain condition (1.13), which implies that there exists a $\mathbf{T}_0 \in L^1(Q;\mathbb{R}^{d\times d})$ such that

$$\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0) = \mathbf{G}(\mathbf{T}_0) \qquad \text{a.e. in } Q.$$

Using the monotonicity of $\mathbf{G}$, we see that

$$\mathbf{T}^n \cdot ((\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0)) - \mathbf{G}_n(\mathbf{T}^n)) \le \mathbf{T}^n \cdot (\mathbf{G}(\mathbf{T}_0) - \mathbf{G}(\mathbf{T}^n)) \le \mathbf{T}_0 \cdot (\mathbf{G}(\mathbf{T}_0) - \mathbf{G}(\mathbf{T}^n)).$$

Using the convergence results (4.22)–(4.29) applied to all terms in (4.33) with the above inequality yields the following:

$$\begin{aligned}
&\frac{1}{4}\int_\Omega \beta|\partial_t(\boldsymbol{u}-\boldsymbol{u}_0)(t)|^2 + \beta\left|\partial_t(\boldsymbol{u}-\boldsymbol{u}_0)(t) + \frac{2\alpha}{\beta}(\boldsymbol{u}-\boldsymbol{u}_0)(t)\right|^2 \mathrm{d}x\\
&\le \int_0^t\int_\Omega \mathbf{T}_0\cdot((\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \beta\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u}_0)) - \mathbf{G}(\mathbf{T})) + \alpha|\partial_t(\boldsymbol{u}-\boldsymbol{u}_0)|^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad + \int_0^t\int_\Omega (\boldsymbol{f} - \partial_{tt}\boldsymbol{u}_0)\cdot(\alpha(\boldsymbol{u}-\boldsymbol{u}_0) + \beta\partial_t(\boldsymbol{u}-\boldsymbol{u}_0)) + \frac{2\alpha^2}{\beta}\partial_t(\boldsymbol{u}-\boldsymbol{u}_0)\cdot(\boldsymbol{u}-\boldsymbol{u}_0)\,\mathrm{d}x\,\mathrm{d}\tau\\
&\le C\int_0^t \|\mathbf{T}_0\|_1 + \|\boldsymbol{f}\|_2 + \|\partial_{tt}\boldsymbol{u}_0\|_2 + 1\,\mathrm{d}\tau.
\end{aligned} \tag{4.34}$$

Letting $t \to 0_+$, we see that

$$\lim_{t\to 0_+}(\|\boldsymbol{u}(t) - \boldsymbol{u}_0(0)\|_2^2 + \|\partial_t\boldsymbol{u}(t) - \partial_t\boldsymbol{u}_0(0)\|_2^2) = 0.$$

In addition, it also follows from (4.25) that $\boldsymbol{u} \in \mathcal{C}^1([\delta,T];L^2(\Omega;\mathbb{R}^d))$ for every $\delta > 0$, which combined with the above result gives that $\boldsymbol{u} \in \mathcal{C}^1([0,T];L^2(\Omega;\mathbb{R}^d))$.

4.4. Validity of the equation in the limit. To summarize the results so far, we have found a couple $(\boldsymbol{u},\mathbf{T})$ that satisfies (1.3)–(1.18) and (1.20), (1.21). It remains to show (1.19). To do so, we use the method developed in [3]. Let g be a smooth nonnegative nonincreasing function satisfying

$$g(s) = \begin{cases} 1, & \text{for } s \in [0,1],\\ 0, & \text{for } s > 2.\end{cases}$$

For each $k \in \mathbb{N}$, let us define

$$g_k(s) := g(s/k).$$

It is clear that $g_k \nearrow 1$. Next let $\boldsymbol{v} \in \mathcal{C}_0^\infty(Q;\mathbb{R}^d)$ be arbitrary but fixed. In particular, there exist a compact subset $\Omega' \Subset \Omega$ and a $\delta > 0$ such that $\mathrm{supp}(\boldsymbol{v}) \subset [\delta, T-\delta]\times\Omega'$. Thanks to (4.25) and (4.31), all terms in (1.19) are well-defined for almost all $t \in (0,T)$ and we just need to check that the equality holds.

We fix $\delta > 0$. Using the properties of g_k, we have

$$\begin{aligned}
I &:= \int_Q \partial_{tt}\boldsymbol{u}\cdot\boldsymbol{v} + \mathbf{T}\cdot\nabla\boldsymbol{v} - \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t\\
&= \lim_{k\to\infty}\int_Q \partial_{tt}\boldsymbol{u}\cdot\boldsymbol{v}g_k(|\mathbf{T}|) + \mathbf{T}\cdot\nabla\boldsymbol{v}g_k(|\mathbf{T}|) - \boldsymbol{f}\cdot\boldsymbol{v}g_k(|\mathbf{T}|)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.35}$$

Using (4.25), (4.30), the fact that $\mathbf{T}^n \in L^2(\delta,T;W^{1,2}_{loc}(\Omega;\mathbb{R}^{d\times d}))$ for every $\delta > 0$, which follows from (4.19), and the fact that $g_k(|\mathbf{T}^n|)$ is supported only in the set where $|\mathbf{T}^n| \le 2k$, we can rewrite the right-hand side of (4.35) in the following way:

$$\begin{aligned} I &= \lim_{k\to\infty}\lim_{n\to\infty}\int_Q \partial_{tt}\boldsymbol{u}^n\cdot\boldsymbol{v}g_k(|\mathbf{T}^n|)+\mathbf{T}^n\cdot\nabla\boldsymbol{v}g_k(|\mathbf{T}^n|)-\boldsymbol{f}\cdot\boldsymbol{v}g_k(|\mathbf{T}^n|)\,\mathrm{d}x\,\mathrm{d}t\\ &= \lim_{k\to\infty}\lim_{n\to\infty}\int_Q \partial_{tt}\boldsymbol{u}^n\cdot\boldsymbol{v}g_k(|\mathbf{T}^n|)+\mathbf{T}^n\cdot\nabla(\boldsymbol{v}g_k(|\mathbf{T}^n|))-\boldsymbol{f}\cdot\boldsymbol{v}g_k(|\mathbf{T}^n|)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad-\lim_{k\to\infty}\lim_{n\to\infty}\int_Q \mathbf{T}^n\cdot(\nabla g_k(|\mathbf{T}^n|)\otimes\boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t\\ &= -\lim_{k\to\infty}\lim_{n\to\infty}\int_Q \mathbf{T}^n\cdot(\nabla g_k(|\mathbf{T}^n|)\otimes\boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}\tag{4.36}$$

where for the last equality we have used (4.4) with $\boldsymbol{w} := \boldsymbol{v}g_k(|\mathbf{T}^n|)$. This is a justified choice of test function by the following reasoning. We have $\mathbf{T}^n \in L^2(\delta,T;W^{1,2}_{loc}(\Omega;\mathbb{R}^{d\times d}))$. Hence, using the chain rule for weak derivatives, it follows that $g_k(|\mathbf{T}^n|) \in L^2(\delta,T;W^{1,2}_{loc}(\Omega;\mathbb{R}^{d\times d}))$. By the compact support property of $\boldsymbol{v}$, we deduce that $\boldsymbol{v}g_k(|\mathbf{T}^n|) \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^d))$ with support contained in $[\delta, T-\delta]\times\Omega'$.

It remains to show that the right-hand side of (4.36) vanishes. We define

$$M_{k,n}(s) := \int_0^s \frac{g_k'(t)}{\frac{\phi'(t)}{t}+n^{-1}}\,\mathrm{d}t \le \int_0^s \frac{t g_k'(t)}{\phi'(t)}\,\mathrm{d}t =: M_k(s).$$

Then, using that $|g_k'(s)| \le Cs^{-1}\chi_{\{s\in(k,2k)\}}$, we see that

$$M_k(s)\begin{cases} \le C\min\{s,k\} & \text{for all } s\ge 0,\\ = 0 & \text{for } s\le k. \end{cases}\tag{4.37}$$

Next, we use the structural assumption (A4) to rewrite the term under the limit in (4.36) as

$$\begin{aligned} &-\int_Q \mathbf{T}^n\cdot(\nabla g_k(|\mathbf{T}^n|)\otimes\boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad= -\int_Q \mathbf{G}_n(\mathbf{T}^n)\cdot(\nabla|\mathbf{T}^n|\otimes\boldsymbol{v})\frac{g_k'(|\mathbf{T}^n|)}{\frac{\phi'(|\mathbf{T}^n|)}{|\mathbf{T}^n|}+n^{-1}}\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad= -\int_Q \mathbf{G}_n(\mathbf{T}^n)\cdot(\nabla M_{k,n}(|\mathbf{T}^n|)\otimes\boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad= \int_Q \operatorname{div}\mathbf{G}_n(\mathbf{T}^n)\cdot\boldsymbol{v}M_{k,n}(|\mathbf{T}^n|)\,\mathrm{d}x\,\mathrm{d}t+\int_Q \mathbf{G}_n(\mathbf{T}^n)\cdot\nabla\boldsymbol{v}M_{k,n}(|\mathbf{T}^n|)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}\tag{4.38}$$

For the first term on the right-hand side of (4.38), we use the definition of $\mathcal{A}_n$ alongside the Cauchy–Schwarz inequality to obtain

$$\begin{aligned} &|\operatorname{div}\mathbf{G}_n(\mathbf{T}^n)\cdot\boldsymbol{v}M_{k,n}(|\mathbf{T}^n|)| = \left|\sum_{i,j,a,b=1}^d (\mathcal{A}_n(\mathbf{T}^n))^{ij}_{ab}\partial_j\mathbf{T}^n_{ab}\boldsymbol{v}_iM_{k,n}(|\mathbf{T}^n|)\right|\\ &= \left|\sum_{m=1}^d\sum_{i,j,a,b=1}^d (\mathcal{A}_n(\mathbf{T}^n))^{ij}_{ab}\partial_m\mathbf{T}^n_{ab}\delta_{mj}\boldsymbol{v}_iM_{k,n}(|\mathbf{T}^n|)\right|\\ &\le \left|\sum_{m=1}^d(\partial_m\mathbf{T}^n,\partial_m\mathbf{T}^n)^{\frac12}_{\mathcal{A}_n(\mathbf{T}^n)}\left(\sum_{i,j,a,b=1}^d(\mathcal{A}_n(\mathbf{T}^n))^{ij}_{ab}\delta_{mj}\boldsymbol{v}_i\delta_{ma}\boldsymbol{v}_bM^2_{k,n}(|\mathbf{T}^n|)\right)^{\frac12}\right|\\ &\le \left|\sum_{m=1}^d(\partial_m\mathbf{T}^n,\partial_m\mathbf{T}^n)^{\frac12}_{\mathcal{A}_n(\mathbf{T}^n)}\left((n^{-1}+\frac{C}{1+|\mathbf{T}^n|})|\boldsymbol{v}|^2M^2_{k,n}(|\mathbf{T}^n|)\right)^{\frac12}\right|. \end{aligned}$$

Using this bound in (4.38) and then in (4.36), recalling the fact that $\boldsymbol{v}$ is compactly supported, we deduce with the help of Hölder's inequality and the uniform bound (4.18) that

(4.39)
$$\begin{aligned}|I| &\le \lim_{k\to\infty}\lim_{n\to\infty}\int_Q\left|\sum_{m=1}^d(\partial_m\mathbf{T}^n,\partial_m\mathbf{T}^n)^{\frac12}_{\mathcal{A}_n(\mathbf{T}^n)}\left(\left(n^{-1}+\frac{C}{1+|\mathbf{T}^n|}\right)|\boldsymbol{v}|^2M^2_{k,n}(|\mathbf{T}^n|)\right)^{\frac12}\right|\,\mathrm{d}x\,\mathrm{d}t\\ &\le C(\boldsymbol{v})\lim_{k\to\infty}\lim_{n\to\infty}\left(\int_Q\left(n^{-1}+\frac{C}{1+|\mathbf{T}^n|}\right)M^2_{k,n}(|\mathbf{T}^n|)\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12}\\ &= C(\boldsymbol{v})\lim_{k\to\infty}\left(\int_Q\frac{M_k^2(|\mathbf{T}|)}{|\mathbf{T}|}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12},\end{aligned}$$

where for the last equality we use (4.31) and the boundedness of M_k. Consequently, using that $\mathbf{T}\in L^1(Q;\mathbb{R}^{d\times d})$ and the structure of M_k (4.37), we deduce that

$$|I|\le C(\boldsymbol{v})\lim_{k\to\infty}\left(\int_Q\frac{M_k^2(|\mathbf{T}|)}{|\mathbf{T}|}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12}\le C(\boldsymbol{v})\lim_{k\to\infty}\left(\int_{Q\cap\{|\mathbf{T}|>k\}}|\mathbf{T}|\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12}=0.$$

Since $\boldsymbol{v}$ is arbitrary, we see that (1.19) holds for almost all $t\in(0,T)$ and all smooth compactly supported $\boldsymbol{w}$. Finally, using a weak* density argument based on [3, Lemma A.3] we deduce that (1.19) holds for an arbitrary $\boldsymbol{w}\in W_0^{1,2}(\Omega,\mathbb{R}^d)$ fulfilling $\boldsymbol{\varepsilon}(\boldsymbol{w})\in L^\infty(Q;\mathbb{R}^{d\times d})$. This concludes the proof of the existence of a solution as asserted in Theorem 1.2.

4.5. Uniqueness of solutions. It remains to prove the uniqueness of such weak solutions. Let $(\boldsymbol{u}_1,\mathbf{T}_1)$ and $(\boldsymbol{u}_2,\mathbf{T}_2)$ be two solutions emanating from the same data and denote $\boldsymbol{u}:=\boldsymbol{u}_1-\boldsymbol{u}_2$. Then it follows from (1.19) that, for almost all $t\in(0,T)$ and for every $\boldsymbol{w}\in W_0^{1,\infty}(\Omega;\mathbb{R}^d)$,

$$\int_\Omega\partial_{tt}\boldsymbol{u}\cdot\boldsymbol{w}+(\mathbf{T}_1-\mathbf{T}_2)\cdot\boldsymbol{\varepsilon}(\boldsymbol{w})\,\mathrm{d}x=0. \tag{4.40}$$

Since $\partial_t\boldsymbol{\varepsilon}(\boldsymbol{u})$ and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ belong to $L^\infty(\Omega;\mathbb{R}^{d\times d})$ for almost all $t\in(0,T)$, we can again use the weak* density argument as in the previous section to deduce that (4.40) holds with $\boldsymbol{w}:=\alpha\boldsymbol{u}+\beta\partial_t\boldsymbol{u}$. Consequently, since we have

$$\alpha\boldsymbol{u}+\beta\partial_t\boldsymbol{u}=\mathbf{G}(\mathbf{T}_1)-\mathbf{G}(\mathbf{T}_2),$$

we can use the monotonicity of $\mathbf{G}$ and integration over (t_0,t), with $0<t_0<t<T$, to deduce from (4.40) that

$$\begin{aligned}0&\ge 2\int_{t_0}^t\int_\Omega\partial_{tt}\boldsymbol{u}\cdot(\alpha\boldsymbol{u}+\beta\partial_t\boldsymbol{u})\,\mathrm{d}x\,\mathrm{d}\tau\\ &=\beta\int_\Omega|\partial_t\boldsymbol{u}(t)|^2-|\partial_t\boldsymbol{u}(t_0)|^2+2\alpha\partial_t\boldsymbol{u}(t)\cdot\boldsymbol{v}(t)-2\alpha\boldsymbol{u}(t_0)\cdot\boldsymbol{u}(t_0)\,\mathrm{d}x-2\alpha\int_{t_0}^t\int_\Omega|\partial_t\boldsymbol{u}|^2\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

We note that this procedure is rigorous for every such $t_0>0$ thanks to the regularity of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ asserted in (1.15). Since $\boldsymbol{u}\in\mathcal{C}^1([0,T];L^2(\Omega;\mathbb{R}^d))$ as a result of (1.15), we can use (1.21) and let $t_0\to0_+$ in the above inequality to deduce that

$$\begin{aligned}0&\ge\beta\int_\Omega|\partial_t\boldsymbol{u}(t)|^2+2\alpha\partial_t\boldsymbol{u}(t)\cdot\boldsymbol{u}(t)\,\mathrm{d}x-2\alpha\int_0^t\int_\Omega|\partial_t\boldsymbol{u}|^2\,\mathrm{d}x\,\mathrm{d}\tau\\ &=\beta\int_\Omega|\partial_t\boldsymbol{u}(t)|^2+2\alpha\partial_t\boldsymbol{u}(t)\cdot\left(\int_0^t\partial_t\boldsymbol{u}(\tau)\,\mathrm{d}\tau\right)\mathrm{d}x-2\alpha\int_0^t\int_\Omega|\partial_t\boldsymbol{u}|^2\,\mathrm{d}x\,\mathrm{d}\tau\\ &\ge\frac{\beta}{2}\left(\|\partial_t\boldsymbol{u}(t)\|_2^2-C(\alpha,\beta,T)\int_0^t\|\partial_t\boldsymbol{u}(\tau)\|_2^2\,\mathrm{d}\tau\right)\\ &=\mathrm{e}^{-tC(\alpha,\beta,T)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-tC(\alpha,\beta,T)}\int_0^t\|\partial_t\boldsymbol{u}(\tau)\|_2^2\,\mathrm{d}\tau\right).\end{aligned}$$

Simple integration with respect to t then gives that $\partial_t \boldsymbol{u} \equiv 0$ almost everywhere in Q and consequently $\boldsymbol{u}_1 = \boldsymbol{u}_2$. By strict monotonicity, we necessarily also have that $\mathbf{T}_1 = \mathbf{T}_2$ almost everywhere in Q. Hence, uniqueness follows.

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