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# On polynomial transformations preserving purely imaginary zeros

(In memory of Mark W. Coffey)

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## ABSTRACT

In this present work polynomial transformations are identified that preserve the property of the polynomials having all zeros lying on the imaginary axis. Existence results concerning families of polynomials whose generalised Mellin transforms have zeros all lying on the critical line  $\Re s = \frac{1}{2}$  are then derived. Inherent structures are identified from which a simple proof relating to the Gegenbauer family of orthogonal polynomials is subsequently deduced. Some discussion about the choice of generalised Mellin transform is also given.

## AMS CLASSIFICATION

30C15, 11B83, 44A20, 33C45

## KEYWORDS

orthogonal polynomials, critical zeros, pure imaginary zeros, generalised Mellin transform.

## 1. Introduction

In the paper ‘Mellin transforms with only critical zeros: Legendre functions’ [1] M. W. Coffey and the second author discuss generalised Mellin transforms [2–4] of Legendre polynomials [5–7]. They start from the fact that the Legendre polynomials  $P_n$ ,  $n = 0, 1, 2, 3, \dots$  satisfy the Legendre differential equation

$$(1 - x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0,$$

and apply the generalised Mellin transform  $P_n \mapsto M_n$  to the polynomials  $P_n$  where

$$M_n(s) = \int_0^1 \frac{x^{s-1}}{(1-x^2)^{\frac{1}{2}}} P_n(x) dx.$$

They show that the functions  $M_n$  satisfy

$$\begin{aligned} (n(n+1) - 1 - 2s(s-1)) M_n(s) + (s-n)(s+n+1) M_n(s+2) \\ + (s-1)(s-2) M_n(s-2) = 0. \end{aligned} \tag{1.1}$$

They went on to show that the generalised Mellin transform  $M_n$  may be written as a product of Gamma functions and a polynomial function  $p_n$ , where all the zeros of the polynomials  $p_n$  lie on the critical line. Their proof, which was based on (1.1), involves demonstrating that all the zeros of the polynomials  $q_n$ , defined by

$$q_n(s) = p_n(s + 1/2),$$

lie on the imaginary axis. The generalised Mellin transform they used appeared via explorative modification of results due to Bump, Choi and Ng [8,9] that gave interesting results and linked to the functional relation for the Riemann zeta function. However, the detail of this argument is omitted and it is not entirely obvious to the reader how to reconstruct the full argument.

To redress this Lettington and Coffey published a technical proof for generalised Mellin transforms of the larger family of Gegenbauer polynomials in their 2020 paper [10], that utilised continuous Hahn polynomials [11].

In this present work the authors expand upon these results by combining a first order differential-difference relation, with a result similar to one that appears in Titchmarsh [12]. The two structure theorems (below) obtained concern polynomial transformations that preserve the property of the zeros all lying on the imaginary axis  $\Re s = 0$ . Subsequently a simple proof of the Gegenbauer result detailed in [10] is then deduced. Some discussion about the choice of generalised Mellin transform is also given.

With the background and motivation outlined we now state our main results below.

**Theorem 1.1.** *Let  $P$  be a polynomial function of degree  $\ell$ , all of whose zeros lie on the imaginary axis, then the polynomial function  $Q_1$  defined by*

$$Q_1(z) = (m - z)P(z + 1) + (m + z)P(z - 1)$$

*will also have all its zeros on the imaginary axis in the case that  $m \leq 0$  or  $m \geq \ell$ .*

**Theorem 1.2.** *Let  $P$  be a polynomial function of degree  $\ell$ , all of whose zeros lie on the imaginary axis, then the polynomial function  $Q_2$  defined by*

$$Q_2(z) = (m - z)(z + 1/2)P(z + 1) + (m + z)(z - 1/2)P(z - 1)$$

*will also have all its zeros on the imaginary axis in the case that  $m \leq 0$  or  $m \geq \ell + 1/2$ .*

## 2. The differential-difference relation

Our starting point is the sequence of polynomials  $\{P_n\}_{n=0}^{\infty}$  defined by the differential-difference equation

$$(n + 1)P_{n+1}(x) = a_n x P_n(x) - (1 - x^2)P'_n(x), \quad (2.1)$$

where  $P_0(x) = 1$ ,  $a_n > n - 1$  and  $a_n \neq n$ , which is a special case of

$$P_{n+1}(x) = -\frac{1}{b_n}(a_n x P_n(x) - (1 - x^2)P'_n(x)) \quad (2.2)$$

with  $-b_n = n + 1$ . The differential-difference relation (2.2) is itself a special case of the more general differential-difference equation

$$P_{n+1}(x) = A_n(x)P'_n(x) + B_n(x)P_n(x), \quad n \geq 0 \quad P_0(x) = 1, \quad (2.3)$$

with  $A_n(x) = (x^2 - 1)/b_n$ , and  $B_n(x) = -a_n x/b_n$ . The polynomial systems (2.3) have been studied in greater generality by Dominici et al. in [13,14], where  $A_n(x)$  and  $B_n(x)$  are polynomials of degree at most 2 and 1 respectively. In [13] the authors study the zeros of polynomial solutions to equation (2.3), in which they analyze when their zeros are real and simple and whether the zeros of polynomials of adjacent degree interlace. Their results hold for general families of polynomials including sequences of classical orthogonal polynomials as well as Euler-Frobenius, Bell and other polynomials. Later on, in [14] the asymptotics of their zero counting distribution is deduced.

Also of relevance to our results are the so called *tangent polynomials* (see §6.5 of [14]), which satisfy (2.3) with  $A_n(x) = x^2 + 1$  and  $B_n(x) = 1$  for all  $n = 0, 1, 2, \dots$ . They have all imaginary zeros and hence satisfy the conditions of Theorems 1.1 or 1.2.

Here we consider the generalized Mellin transform of sequences of polynomials  $\{P_n(x)\}_{n=0}^\infty$  defined as above. The reason for choosing the definition (2.1) is that in Section 5 we focus specifically on the Gegenbauer family of orthogonal polynomials  $\{C_n^\lambda\}_{n=0}^\infty$  which satisfy

$$(n+1)C_{n+1}^\lambda(x) = (n+2\lambda)x C_n^\lambda(x) - (1-x^2)C_n^{\lambda'}(x),$$

where  $C_0^\lambda(x) = 1$ ,  $\lambda > -1/2$ ,  $\lambda \neq 0$ . Setting  $C_n^\lambda = P_n$  and  $n+2\lambda = a_n$  we see that the above is of the form given in (2.1).

By using fundamental properties of zeros of polynomials generated as in Theorem 1.1 and Theorem 1.2, we deduce that the polynomial factors of the generalized Mellin transforms of Gegenbauer polynomials have all their zeros on the critical line  $z \in \mathbb{C}$ ,  $\Re z = 1/2$ .

Before proving Theorems (1.1) and (1.2) in Section 3, we first notice that, just like the Gegenbauer polynomials, the polynomials  $\{P_n\}_{n=0}^\infty$ , are alternately even or odd and obey the functional equation  $P_n(-x) = (-1)^n P_n(x)$ .

### 3. Proof of theorems

**Proof of Theorem 1.1.** If  $\ell = 0$  then  $P(z) = a$  then  $Q_1(z) = 2ma$  and there are no zeros to consider. If  $\ell = 1$  then  $Q_1(z) = 2a(m-1)z$  whose only zero is at 0, on the imaginary axis.

For the case  $\ell \geq 2$  we look for points on the imaginary axis which are zeros of  $Q_1$ . Consider the path  $t \mapsto z(t) = it$  which runs up the imaginary axis. The point  $it$  will be a zero of  $Q_1$  provided that

$$(m - it)P(it + 1) = -(m + it)P(it - 1), \quad (3.1)$$

which will be the case if and only if both

$$|(m - it)P(it + 1)| = |(m + it)P(it - 1)| \quad (3.2)$$

and

$$\arg((m - it)P(it + 1)) = \arg(-(m + it)P(it - 1)). \quad (3.3)$$

Let the zeros of  $P$ , which by assumption are all on the imaginary axis, be at  $\imath a_1, \imath a_2, \dots, \imath a_\ell$  where the  $a_j \in \mathbb{R}$  then

$$P(z) = c(z - \imath a_1)(z - \imath a_2) \cdots (z - \imath a_\ell),$$

where  $c \neq 0$  is a constant, so that (3.1) becomes

$$\begin{aligned} & (m - it)c(it - (\imath a_1 - 1))(it - (\imath a_2 - 1)) \cdots (it - (\imath a_\ell - 1)) \\ &= -(m + it)c(it - (\imath a_1 + 1))(it - (\imath a_2 + 1)) \cdots (it - (\imath a_\ell + 1)). \end{aligned}$$

We have

$$|m - it| = |m + it| \quad \text{and} \quad |it - (\imath a_j - 1)| = |it - (\imath a_j + 1)|$$

for all  $a_j$  and so (3.2) is satisfied. If in addition (3.3) holds then  $it$  will be a zero of  $Q_1(z)$ . Equation (3.3) is equivalent to

$$\arg \left( \frac{(it - (\imath a_1 - 1))(it - (\imath a_2 - 1)) \cdots (it - (\imath a_\ell - 1))}{(it - (\imath a_1 + 1))(it - (\imath a_2 + 1)) \cdots (it - (\imath a_\ell + 1))} \right) = \arg \left( \frac{-(m + it)}{(m - it)} \right). \quad (3.4)$$

At the point  $it$

$$\frac{it - (\imath a_j - 1)}{it - (\imath a_j + 1)} = -\frac{(1 + \imath(t - a_j))^2}{1 + (t - a_j)^2},$$

so

$$\arg \left( \frac{it - (\imath a_j - 1)}{it - (\imath a_j + 1)} \right) = \arg(-1) + 2 \arg(1 + \imath(t - a_j)),$$

and for  $1 \leq j \leq \ell$ , the continuous function  $A_j$ , defined by

$$A_j : t \mapsto \pi + 2 \arctan(t - a_j)$$

gives an argument for

$$\frac{it - (\imath a_j - 1)}{it - (\imath a_j + 1)}$$

at the point  $it$  on the imaginary axis with

$$A_j(t) \rightarrow \begin{cases} 0 & \text{as } t \rightarrow -\infty, \\ 2\pi & \text{as } t \rightarrow \infty, \end{cases}$$

and  $A = \sum_{j=1}^{\ell} A_j$ , gives an argument for

$$\left( \frac{(it - (ia_1 - 1))(it - (ia_2 - 1)) \cdots (it - (ia_{\ell} - 1))}{(it - (ia_1 + 1))(it - (ia_2 + 1)) \cdots (it - (ia_{\ell} + 1))} \right)$$

at the point  $it$  on the imaginary axis with

$$A(t) \rightarrow \begin{cases} 0 & \text{as } t \rightarrow -\infty, \\ 2\ell\pi & \text{as } t \rightarrow \infty. \end{cases}$$

The functions  $A_j$  and  $A$  are increasing functions on  $(-\infty, \infty)$ . Similarly, because

$$\left( -\frac{m + it}{m - it} \right) = \left( -\frac{(m + it)^2}{m^2 + t^2} \right),$$

we have

$$\arg \left( -\frac{m + it}{m - it} \right) = \arg(-1) + 2 \arg(m + it),$$

and so the continuous function  $B : t \mapsto B(t) = \pi + 2 \arctan(t/m)$ , gives an argument for

$$\left( -\frac{m + it}{m - it} \right)$$

at the point  $it$  on the imaginary axis. For  $m > 0$ ,  $B$  is increasing on  $(-\infty, \infty)$  and decreasing for  $m < 0$  with respectively,

$$B(t) \rightarrow \begin{cases} 0 & \text{as } t \rightarrow -\infty, \\ 2\pi & \text{as } t \rightarrow +\infty, \end{cases} \quad \text{and} \quad B(t) \rightarrow \begin{cases} 2\pi & \text{as } t \rightarrow -\infty, \\ 0 & \text{as } t \rightarrow +\infty. \end{cases}$$

Define the  $\ell - 1$  points  $t_j$ ,  $j = 1, 2, \dots, \ell - 1$  where  $A(t_j) = 2j\pi$ , and the  $\ell$  intervals such that

$$I_1 = (-\infty, t_1], I_j = [t_{j-1}, t_j], \text{ for } 2 \leq j \leq \ell - 1, I_{\ell} = [t_{\ell-1}, \infty).$$

In the interval  $I_j$  the function  $A$  increases from  $2(j-1)\pi$  to  $2j\pi$ , for  $1 \leq j \leq \ell$ .

We consider the behaviour of the the functions  $A$  and  $B$  for different values of  $m$  as follows.

**The case  $m < 0$ .** We first note that  $Q_1$  has the same degree,  $\ell$ , as  $P$  because for

$$P(z) = c_{\ell} z^{\ell} + \cdots,$$

where the dots indicate powers of  $z$  of order  $\ell - 1$  or less, we have

$$Q_1(z) = (m - z)P(z + 1) + (m + z)P(z - 1) = 2(m - \ell)c_{\ell}z^{\ell} + \cdots, \quad (3.5)$$

which is of order  $\ell$  because  $m - \ell \neq 0$

We therefore seek to locate all  $\ell$  zeros of  $Q_1$ . In the interval  $I_j$  the function  $A$  increases from  $2(j-1)\pi$  to  $2j\pi$ , for  $1 \leq j \leq \ell$  and, because  $m < 0$ , the function  $B$  is decreasing taking values between  $2\pi$  and  $0$ . It follows that there will be at least one point  $\tilde{t}_j$  in  $I_j$  where

$$A(\tilde{t}_j) = 2(j-1)\pi + B(\tilde{t}_j).$$

This means that each of the  $\ell$  points  $i\tilde{t}_j$ , for  $1 \leq j \leq \ell$ , satisfies equation (3.2) and we have located all  $\ell$  zeros of  $Q_1$  on the imaginary axis.

**The case  $m > \ell$ .** Again  $Q_1$  has the same degree,  $\ell$ , as  $P$  but this time because  $m > 0$ , the function  $B(t)$  is increasing. As above we find at least one point  $\tilde{t}_j$  in  $I_j$ , for  $2 \leq j \leq \ell-1$ , where

$$A(\tilde{t}_j) = 2(j-1)\pi + B(\tilde{t}_j).$$

This means that each of the  $\ell-2$  points  $i\tilde{t}_j$ , for  $2 \leq j \leq \ell-1$ , satisfies equation (3.2) and we have located at least  $\ell-2$  zeros of  $Q_1$  on the imaginary axis. We next locate two more zeros which will account for all  $\ell$  zeros. To do this consider the interval  $I_\ell = [t_{\ell-1}, \infty)$  and use

$$\arctan(x) = \frac{\pi}{2} - \arctan\left(\frac{1}{x}\right).$$

We have

$$A_j(t) = \pi + 2 \arctan(t - a_j) = 2\pi - 2 \arctan\left(\frac{1}{t - a_j}\right)$$

and therefore

$$A(t) = 2\ell\pi - 2 \sum_{j=1}^{\ell} \arctan\left(\frac{1}{t - a_j}\right).$$

Also

$$B(t) = \pi + 2 \arctan(t/m) = 2\pi - 2 \arctan\left(\frac{m}{t}\right),$$

then at  $t_{\ell-1}$

$$A(t_{\ell-1}) = 2(\ell-1)\pi \quad \text{and} \quad B(t_{\ell-1}) > 0$$

and so

$$B(t_{\ell-1}) + 2(\ell-1)\pi - A(t_{\ell-1}) > 0.$$

As  $t \rightarrow \infty$

$$A(t) \rightarrow 2\ell\pi - \frac{2\ell}{t} \quad \text{and} \quad B(t) \rightarrow 2\pi - \frac{2m}{t}.$$

Therefore

$$B(t) + 2(\ell - 1)\pi - A(t) \rightarrow \frac{2(\ell - m)}{t}$$

and, because in this case  $m > \ell$ ,

$$B(t) + 2(\ell - 1)\pi - A(t) \text{ becomes negative as } t \rightarrow \infty.$$

It follows that there will be a point  $\tilde{t}_\ell$  where

$$B(\tilde{t}_\ell) + 2(\ell - 1)\pi - A(\tilde{t}_\ell) = 0,$$

giving another zero of  $Q_1$  on the imaginary axis. Similar considerations for the interval  $I_1 = (-\infty, t_1]$  give another zero of  $Q_1$  on the imaginary axis showing that all  $\ell$  zeros of  $Q_1$  lie on the imaginary axis.

**The case  $m = 0$ .** If  $m = 0$  then  $Q_1(z) = -zP(z + 1) + zP(z - 1)$ , and the zeros of  $Q_1$  are 0 together with the zeros of  $P(z + 1) - P(z - 1)$ . This will have a zero at the point  $it$  on the imaginary axis provided both  $|P(it + 1)| = |P(it - 1)|$ , and  $\arg P(it + 1) = \arg P(it - 1)$ . The first of these will be satisfied as above and the second when

$$\arg \left( \frac{P(it + 1)}{P(it - 1)} \right) = 1$$

which is the case at the  $\ell - 1$  points  $t_j$ ,  $j = 1, 2, \dots, \ell - 1$  where  $A(t_j) = 2j\pi$  as defined above.

**The case  $m = \ell$ .** From (3.5) we see

$$Q_1(z) = (m - z)P(z + 1) + (m + z)P(z - 1) = 2c_{\ell-1}z^{\ell-1} + \dots$$

and so the degree of  $Q_1$  is at most  $\ell - 1$ . Considerations as above show that there are still  $\ell - 2$  zeros on the imaginary axis. If  $2c_{\ell-1} = 0$  this accounts for all of them but if  $2c_{\ell-1} \neq 0$  there is one more zero,  $z_0$  say, to be located. Let the zeros of  $P$ , which by assumption are all on the imaginary axis, be at  $ia_1, ia_1, \dots, ia_\ell$ , where the  $a_j \in \mathbb{R}$ . Then we have that

$$P(z) = c(z - ia_1)(z - ia_2) \dots (z - ia_\ell),$$

where  $c \neq 0$  is a constant. Because  $z_0$  is a zero it is a solution of (3.1), and so satisfies after cancellation with  $c$ ,

$$\begin{aligned} & (m - z_0)(z_0 - (ia_1 - 1))(z_0 - (ia_2 - 1)) \dots (z_0 - (ia_\ell - 1)) \\ &= -(m + z_0)(z_0 - (ia_1 + 1))(z_0 - (ia_2 + 1)) \dots (z_0 - (ia_\ell + 1)). \end{aligned}$$

Taking the complex conjugate of this equation and putting  $z_1 = -\bar{z}_0$  gives

$$\begin{aligned} & (m + z_1)(z_1 - (ia_1 + 1))(z_1 - (ia_2 + 1)) \dots (z_1 - (ia_\ell + 1)) \\ &= -(m - z_1)(z_1 - (ia_1 - 1))(z_1 - (ia_2 - 1)) \dots (z_1 - (ia_\ell - 1)), \end{aligned}$$



and so  $z_1$  is also a zero of equation (3.1). As there was only one more zero to be located we must have  $z_1 = z_0$  therefore  $z_0 = -\bar{z}_0$  and so  $z_0$  lies on the imaginary axis as required.  $\square$

**Proof of Theorem 1.2.** The proof of Theorem 1.2 is as above except that for the extra terms  $(z + 1/2)$  and  $(z - 1/2)$  an additional argument function,  $A_{\ell+1}$ , is required. Because

$$\arg \left( \frac{(it + 1/2)}{(it - 1/2)} \right) = \arg(-1) + 2 \arg(1 + 2it) ,$$

and the continuous function defined by  $A_{\ell+1} : t \mapsto \pi + 2 \arctan(2t)$ , gives an argument for

$$\frac{(it + 1/2)}{(it - 1/2)}$$

at the point  $it$  on the imaginary axis with

$$A_{\ell+1}(t) \rightarrow \begin{cases} 0 & \text{as } t \rightarrow -\infty, \\ 2\pi & \text{as } t \rightarrow \infty. \end{cases}$$

The important consequence of this additional term is that as  $t \rightarrow \infty$

$$A(t) \rightarrow 2(\ell + 1)\pi - \frac{2l}{t} - \frac{1}{t} .$$

Therefore

$$B(t) + 2\ell\pi - A(t) \rightarrow \frac{2(\ell + 1/2 - m)}{t}$$

and, in order to obtain a point  $\tilde{t}_{\ell+1}$  in  $I_{\ell+1} = [t_{\ell+1}, \infty)$  where

$$B(\tilde{t}_{\ell+1}) + 2l\pi - A(\tilde{t}_{\ell+1}) = 0 ,$$

a point which is needed to locate a further zero of  $Q_2$  on the imaginary axis, we require  $m > l + 1/2$ .  $\square$

**Remark 3.1.** Theorems 1.1 and 1.2 apply more generally than we will need in that they apply to any polynomials having all their zeros on the imaginary axis whereas we will only be concerned with odd or even polynomials with real coefficients. They do not tell us what happens in Theorem 1.1 if  $0 < m < l$  and in Theorem 1.2 if  $0 < m < l + 1/2$ . In some cases the polynomials  $Q_1$  and  $Q_2$  may still have all zeros on the imaginary axis. For example for the polynomial  $P(z) = z - \imath$ , whose only zero is on the imaginary axis, we have  $Q_1(z) = -2m\imath + 2(m - 1)z$  and so  $Q_1$  has a zero on the imaginary axis, at  $\imath/(m - 1)$ , for all values of  $m$  except  $m = 1$  in which case  $Q_1$  has no zeros.

**Remark 3.2.** We also note that for  $m < 0$ , both Theorems 1.1 and 1.2 may be proved without using the argument equality (3.3). If  $P(z) = c(z - \imath a_1)(z - \imath a_2) \cdots (z - \imath a_\ell)$ ,

is a polynomial with all zeros on the imaginary axis, where  $c \neq 0$  is a constant, and if  $z_0$  is a zero of

$$Q_1(z) = (m - z)P(z + 1) + (m + z)P(z - 1),$$

then

$$\begin{aligned} & |(m - z_0)| |(z_0 - (\imath a_1 - 1))| |(z_0 - (\imath a_2 - 1))| \cdots |(z_0 - (\imath a_\ell - 1))| \\ &= |(m + z_0)| |(z_0 - (\imath a_1 + 1))| |(z_0 - (\imath a_2 + 1))| \cdots |(z_0 - (\imath a_\ell + 1))|. \end{aligned}$$

Now if  $m < 0$  then all the points  $m, (\imath a_1 - 1), (\imath a_2 - 1), \dots, (\imath a_\ell - 1)$  lie to the left of the imaginary axis and  $-m, (\imath a_1 + 1), (\imath a_2 + 1), \dots, (\imath a_\ell + 1)$  to the right so that unless  $z_0$  is on the imaginary axis the above equality of the moduli cannot be satisfied. This is similar to the Lemma in Section 10.23 of Titchmarsh [12]. The same argument will not work for all  $m$  because for example if  $P(z) = z^2 + 1$ , where both zeros are imaginary, then for the case  $m = 1$

$$Q_1(z) = (m - z)P(z + 1) + (m + z)P(z - 1) = 4 - 2z^2,$$

which has zeros at  $z = \pm\sqrt{2}$ , which are not on the imaginary axis. Whilst for  $m = 3$ , we have  $Q_1(z) = 12 + 2z^2$ , whose zeros are  $\pm\imath\sqrt{6}$ , in accordance with Theorem 1.1.

In the following section we consider the choice of generalised Mellin transform, before applying the theory developed to the Gegenbauer family of orthogonal polynomials in Section 5.

#### 4. The choice of generalised Mellin transform

We introduce a generalised Mellin transform of the form

$$\mathcal{M}(f)(s) = \int_a^b m(x)x^{s-1}f(x) \, dx, \quad (4.1)$$

where the function  $m$  and the limits of integration will be chosen to provide a simple functional recurrence relation for the transformed functions. Put

$$M_n(s) = \int_a^b m(x)x^{s-1}P_n(x) \, dx.$$

Applying the transform (4.1) to equation (2.1), we obtain, after integration by parts,

$$\begin{aligned} (n+1)M_{n+1}(s) &= -(-a_n + s + 1)M_n(s+1) + (s-1)M_n(s-1) \\ &\quad - \int_a^b \frac{d}{dx} (m(x)x^{s-1}(1-x^2)P_n(x)) \, dx + \int_a^b m'(x)(1-x^2)x^{s-1}P_n(x) \, dx. \end{aligned} \quad (4.2)$$

We choose the limits of integration  $a = 0$ ,  $b = 1$  so that the first integral vanishes. There are already terms  $M_n(s+1)$  and  $M_n(s-1)$  on the right hand side of equation

(4.2) and the second integral will provide similar terms if we choose  $m$  so that

$$m'(x)(1-x^2) = \left(\alpha x + \frac{\beta}{x}\right)m(x), \quad (4.3)$$

where  $\alpha$  and  $\beta$  are constants. With this choice the second integral becomes

$$\int_0^1 \left(\alpha x + \frac{\beta}{x}\right)m(x)x^{s-1}P_n(x) dx = \alpha M_n(s+1) + \beta M_n(s-1)$$

and equation (4.2) becomes

$$(n+1)M_{n+1}(s) = (\alpha + a_n - s - 1)M_n(s+1) + (s-1+\beta)M_n(s-1). \quad (4.4)$$

The solution of the differential equation (4.3) for  $m$  is

$$m(x) = \frac{Cx^\beta}{(1-x^2)^{\frac{\alpha+\beta}{2}}},$$

where  $C$  is a constant and our Mellin transform is

$$M_n(s) = \int_0^1 \frac{Cx^{\beta+s-1}}{(1-x^2)^{\frac{\alpha+\beta}{2}}} P_n(x) dx.$$

The constant  $C$  may be set to 1, as it appears in every term and may be cancelled, and  $\beta$  to 0 as the effect of  $\beta$  is simply to translate  $s$  to  $s+\beta$ . We will look at the choice of  $\alpha$  in Section 5. With these choices we have

$$M_n(s) = \int_0^1 \frac{x^{s-1}}{(1-x^2)^{\frac{\alpha}{2}}} P_n(x) dx,$$

and the recurrence relation for the Mellin transforms given in (4.4) becomes

$$(n+1)M_{n+1}(s) = (\alpha + a_n - s - 1)M_n(s+1) + (s-1)M_n(s-1). \quad (4.5)$$

Next consider the generalised Mellin transforms of the  $\{P\}_{n=0}^\infty$ , which as already remarked at the end of section 2 are alternatively even or odd, taking the case of even  $n$  with  $n = 2m$  first. Here we have

$$P_{2m}(x) = a_{2m,2m}x^{2m} + a_{2m-2,2m}x^{2m-2} + \cdots + a_{2j,2m}x^{2j} + \cdots + a_{0,2m}.$$

The generalised Mellin transform is linear so we look at the transforms of powers of  $x$ . Using

$$\int_0^1 x^{2a}(1-x^2)^b dx = \frac{\Gamma(a+1/2)\Gamma(b+1)}{2\Gamma(a+b+3/2)}$$

we obtain the transform of  $x^{2j}$

$$\int_0^1 \frac{x^{s-1}x^{2j}}{(1-x^2)^{\alpha/2}} dx = \frac{\Gamma(s/2+j)\Gamma(-\alpha/2+1)}{2\Gamma((s-\alpha)/2+j+1)}.$$

We now wish to express the value of this integral as a product of a polynomial in  $s$  multiplied by Gamma functions of  $s$  that depends only on  $\alpha$  and  $m = n/2$ . We use the functional equation for the Gamma function  $\Gamma(s+1) = s\Gamma(s)$  to obtain

$$\Gamma(s/2 + j) = f_{j,1}(s)\Gamma(s/2),$$

where  $f_{j,1}(s) = (s/2 + j - 1)(s/2 + j - 2) \cdots (s/2)$  and

$$\Gamma((s - \alpha)/2 + j + 1) = \frac{\Gamma((s - \alpha)/2 + m + 1)}{f_{j,2}(s)},$$

where  $f_{j,2}(s) = ((s - \alpha)/2 + j + 1)((s - \alpha)/2 + j + 2) \cdots ((s - \alpha)/2 + m)$ . Hence the generalised Mellin transform of  $x^{2j}$  is of the form

$$\int_0^1 \frac{x^{s-1}x^{2j}}{(1-x^2)^{\alpha/2}} dx = \frac{f_{j,1}(s)f_{j,2}(s)\Gamma(s/2)\Gamma(-\alpha/2 + 1)}{2\Gamma((s - \alpha)/2 + m + 1)}.$$

This holds for each term in the generalised Mellin transform of  $P_n$  and so this may be written as

$$M_n(s) = \int_0^1 \frac{x^{s-1}P_n(x)}{(1-x^2)^{\alpha/2}} dx = \frac{p_n(s)\Gamma(s/2)\Gamma(-\alpha/2 + 1)}{2\Gamma((s + n - \alpha)/2 + 1)}, \quad (4.6)$$

where  $p_n(s)$  is a polynomial in  $s$  obtained from the  $f_{j,1}(s)$ ,  $f_{j,2}(s)$  and the coefficients  $a_{2j,2m}$ , where  $j = 0, \dots, m$

For the case of odd  $n = 2m + 1$  we have

$$P_{2m+1}(x) = a_{2m+1,2m+1}x^{2m+1} + a_{2m-1,2m+1}x^{2m-1} + \cdots + a_{2j+1,2m+1}x^{2j+1} + \cdots + a_{1,2m+1}x$$

and it may similarly be shown

$$M_n(s) = \frac{p_n(s)\Gamma((s+1)/2)\Gamma(-\alpha/2 + 1)}{2\Gamma((s + n - \alpha)/2 + 1)}. \quad (4.7)$$

We take equations (4.6) and (4.7) as the definitions of the polynomial family  $\{p_n\}_{n=0}^\infty$ .

We now return to the functional recurrence relation given in (4.5), and substitute the above expressions for  $M_n$ . For the case of even  $n$ , so  $n+1$  is odd, we have

$$\begin{aligned} M_{n+1}(s) &= \frac{p_{n+1}(s)\Gamma((s+1)/2)\Gamma(-\alpha/2 + 1)}{2\Gamma((s + n + 1 - \alpha)/2 + 1)}, \\ M_n(s+1) &= \frac{p_n(s+1)\Gamma((s+1)/2)\Gamma(-\alpha/2 + 1)}{2\Gamma((s + n + 1 - \alpha)/2 + 1)}, \\ M_n(s-1) &= \frac{p_n(s-1)\Gamma((s-1)/2)\Gamma(-\alpha/2 + 1)}{2\Gamma((s + n - 1 - \alpha)/2 + 1)}. \end{aligned}$$

Using  $\Gamma((s-1)/2) = \frac{\Gamma((s+1)/2)}{(s-1)/2}$ , and  $\Gamma((s + n - 1 - \alpha)/2 + 1) = \frac{\Gamma((s+n+1-\alpha)/2+1)}{(s+n-1-\alpha)/2+1}$ , we

can rewrite  $M_n(s-1)$  as

$$M_n(s-1) = \frac{p_n(s-1)((s+n+1-\alpha))\Gamma((s+1)/2)\Gamma(-\alpha/2+1)}{2(s-1)\Gamma((s+n+1-\alpha)/2+1)}. \quad (4.8)$$

All three terms  $M_{n+1}(s)$ ,  $M_n(s+1)$ ,  $M_n(s-1)$  now have the same Gamma function factors and so, after substituting into (4.5) and cancellation, for the case of  $n$  even we find that

$$(n+1)p_{n+1}(s) = (\alpha + a_n - s - 1)p_n(s+1) + (s+n+1-\alpha)p_n(s-1). \quad (4.9)$$

For the case of odd  $n$ , so  $n+1$  is even, we have

$$\begin{aligned} M_{n+1}(s) &= \frac{p_{n+1}(s)\Gamma(s/2)\Gamma(-\alpha/2+1)}{2\Gamma((s+n+1-\alpha)/2+1)}, \\ M_n(s+1) &= \frac{p_n(s+1)\Gamma(s/2+1)\Gamma(-\alpha/2+1)}{2\Gamma((s+n+1-\alpha)/2+1)}, \\ M_n(s-1) &= \frac{p_n(s-1)\Gamma(s/2)\Gamma(-\alpha/2+1)}{2\Gamma((s+n-1-\alpha)/2+1)}. \end{aligned}$$

As above we use the functional equation for the Gamma function to give all three terms  $M_{n+1}(s)$ ,  $M_n(s+1)$ ,  $M_n(s-1)$  the same Gamma function factors and so after substituting into (4.5) and cancellation, for the case  $n$  odd we have that

$$(n+1)p_{n+1}(s) = \frac{s}{2}(\alpha + a_n - s - 1)p_n(s+1) + \frac{(s-1)}{2}(s+n+1-\alpha)p_n(s-1). \quad (4.10)$$

We are interested in the case where these polynomials have zeros on the critical line. For simplicity we will look at the polynomials  $q_n$  defined by

$$q_n(s) = p_n\left(s + \frac{1}{2}\right)$$

which, for  $n$  even, satisfy

$$(n+1)q_{n+1}(s) = \left(\alpha + a_n - \frac{3}{2} - s\right)q_n(s+1) + \left(-\alpha + n + \frac{3}{2} + s\right)q_n(s-1) \quad (4.11)$$

and for  $n$  odd  $(n+1)q_{n+1}(s) =$

$$\left(\alpha + a_n - \frac{3}{2} - s\right)\left(\frac{2s+1}{4}\right)q_n(s+1) + \left(-\alpha + n + \frac{3}{2} + s\right)\left(\frac{2s-1}{4}\right)q_n(s-1). \quad (4.12)$$

## 5. Transforms of Gegenbauer Polynomials

The Gegenbauer polynomials [2,5,10,15] obey the differential-difference equation

$$(n+1)C_{n+1}^\lambda(x) = (n+2\lambda)x C_n^\lambda(x) - (1-x^2)C_n^{\lambda'}(x), \quad (5.1)$$

where  $C_0^\lambda(x) = 1$ ,  $\lambda > -1/2$ ,  $\lambda \neq 0$ . This is (2.1) with  $a_n = n+2\lambda$ . Using the generalised Mellin transform of Section 3 we have

$$M_0(s) = \int_0^1 \frac{x^{s-1}}{(1-x^2)^{\frac{\alpha}{2}}} C_0^\lambda dx = \frac{\Gamma(s/2)\Gamma(-\alpha/2+1)}{2\Gamma((s-\alpha)/2+1)},$$

and from (4.6) we see that  $p_0$  is also the constant polynomial  $p_0(s) = 1$ . Using (4.9) and (4.10), we obtain the functional recurrence relations for the polynomial factors  $p_n(s)$ . For  $n$  even we have

$$(n+1)p_{n+1}(s) = (n+2\lambda+\alpha-s-1)p_n(s+1) + (n-\alpha+s+1)p_n(s-1), \quad (5.2)$$

and for  $n$  odd

$$(n+1)p_{n+1}(s) = (n+2\lambda+\alpha-s-1)\left(\frac{s}{2}\right)p_n(s+1) + (n-\alpha+s+1)\left(\frac{s-1}{2}\right)p_n(s-1). \quad (5.3)$$

These give  $p_1(s) = 2\lambda$ , and  $p_2(s) = \frac{\lambda}{2}(s(2\lambda+1) + \alpha - 2)$ . Because  $\lambda > -1/2$  we have  $1+2\lambda \neq 0$ , and so  $p_2$  has one zero at  $\frac{2-\alpha}{1+2\lambda}$ , which will be on the critical line provided that  $\alpha = 3/2 - \lambda$ .

This completes the definition of our generalised Mellin transform which now becomes

$$M_n(s) = \int_0^1 \frac{x^{s-1}}{(1-x^2)^{3/4-\lambda/2}} P_n(x) dx.$$

The integral in the definition of the generalised Mellin transform converges because we have  $\lambda > -1/2$ .

We may now use Theorems 1.1 and 1.2 to prove that all the zeros of all  $p_n$  will also lie on the critical line. We do this using the simpler polynomials  $q_n(z) = p_n(z + \frac{1}{2})$ , which obey, (4.11) for  $n$  even and (4.12) for  $n$  odd. Setting  $a_n = n+2\lambda$ ,  $\alpha = 3/2 - \lambda$  and  $m_n = n + \lambda$  in (4.11) and (4.12) we obtain for  $n$  even,

$$(n+1)q_{n+1}(z) = (m_n - z)q_n(z+1) + (m_n + z)q_n(z-1), \quad (5.4)$$

and for  $n$  odd

$$(n+1)q_{n+1}(z) = (m_n - z)\left(\frac{2z+1}{4}\right)q_n(z+1) + (m_n + z)\left(\frac{2z-1}{4}\right)q_n(z-1), \quad (5.5)$$

which exhibit the structure of the forms used in Theorems 1.1 and 1.2.

We need to show that the degree of  $q_n$  is  $n/2$  when  $n$  is even and  $(n-1)/2$  when  $n$  is odd. The first four  $q_n$  are

$$q_0(z) = 1, \quad q_1(z) = 2\lambda \quad q_2(z) = \frac{\lambda}{2}(2\lambda+1)z, \quad q_3(z) = \frac{\lambda}{3}(\lambda+1)(2\lambda+1)z,$$

and have degree  $0, 0, 1, 1$ , as given above. If  $k$  is even assume  $q_k$  has degree  $k/2$  then, as shown in the course of the proof of Theorem 1.1,  $q_{k+1}$  will also have degree  $k/2$  provided  $m_k - k/2 \neq 0$ . Similarly if  $k$  is odd assume  $q_k$  has degree  $(k-1)/2$  then  $q_{k+1}$  will have degree  $(k-1)/2 + 1$  provided  $m_k - (k-1)/2 - 1/2 \neq 0$ , which is again

$m_k - k/2 \neq 0$ . This condition is satisfied for all  $k > 1$  because  $m_k = k + \lambda$  and  $\lambda > -1/2$ .

All the zeros of the above four polynomials are on the imaginary axis. For the case  $k$  even assume  $q_k$  has all its zeros on the imaginary axis, then, from Theorem 1.1, so will  $q_{k+1}$  provided that  $m_k > k/2$  which is satisfied for all  $k \geq 1$ . For the case  $k$  odd assume  $q_k$  has all its zeros on the imaginary axis, then, from Theorem 1.2, so will  $q_{k+1}$  provided that  $m_k > (k - 1)/2 + 1/2$  which is satisfied for all  $k \geq 1$ . Thus we have proved

**Theorem 5.1.** *The polynomial factors of the generalised Mellin transforms of the Gegenbauer polynomials have all their zeros on the critical line.*

## 6. Conclusion and further work

In the above we showed that the polynomials  $\{p_n(s)\}_{n=0}^{\infty}$ , arising from generalised Mellin transforms of the Gegenbauer polynomials have the interesting property that all their zeros lie on the critical line. We have observed that these zeros appear to obey an interlacing property similar to that of the real zeros of orthogonal polynomials with regard to their positions on the critical line (a possible area for future investigation).

The interplay between the three term recurrence relation satisfied by families of orthogonal polynomials (see Favard's Theorem, p21 [6]) and our first order differential-difference relation (2.1) can be viewed as reciprocal operators. Here (2.1) represents the raising operator of the form  $C_n = (1 - x^2)D - a_n x I$  (see [16]), whereas the three term recurrence relation yields a lowering operator. Consequently it can be shown that these are semiclassical orthogonal polynomials, studied in [17,18], among others, from a structural point of view and in [19,20], from the point of view of holonomic equations.

In this work our fundamental motivation has been to understand polynomial transformations in terms of their zeros, such as those given in Theorems 1.1 and 1.2, and also the generalised Mellin transform considered here. An area of interest for future investigation is that between the Mellin transformed polynomials having critical zeros and the coefficients of the orthogonal polynomial families in the three-term recurrence relation.

## Disclosure statement

No potential conflict of interest was reported by the author(s).

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