The Risk Models with Non-Local Poisson Processes

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Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of
Master of Philosophy

January 2023

Acknowledgements

I would like to acknowledge Prof. Nikolai Leonenko as a fantastic mentor and supervisor, who was happy to supervise a student without any great grades or research experiences.

I would like to give a special thanks to Gordon Cheng for the helps in programming whenever I needed.

Last but most importantly, a huge thanks to my parents for choosing to give me the best education that I could possibly have.

Abstract

The Poisson process is the most commonly used point process in modelling counting phenomena [20]. Even if the counting process has non-stationary increments, it can be shown to converge to the Poisson process if observed sufficiently long after a transient period as long as it constitutes a renewal process [43]. As such, it is important to review the key characteristics of the Poisson process as it serves as the main building block of more complex models.

In the first part of this thesis, we propose two fractional risk models, where the classical risk process is time-changed by the mixture of tempered stable inverse subordinators. We characterise the risk processes by deriving the marginal distributions and establish the corresponding moments and covariance structure.

In the second part of this thesis, we study the main characteristics of these models such as ruin probability and time of ruin, and illustrate the results with Monte Carlo simulations. The data suggests that the time of ruin can be approximated by the inverse gaussian distribution and its generalisations.

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Chapter 1

Introduction

The homogenous Poisson process is used to model random occurrence of events. The modelling of events can be done in relation to time and space. However, the Poisson process can be applied only if events satisfy particular conditions, such as they are independent of each, there are no two events occurring at exactly the same time or located at exactly the same space and the probability of the event should increase with the increase in the interval of the time or space in which the event is expected. The rate of intensity of events is represented by a constant term which also represents the mean density of the events modelled in time or space. The homogeneous Poisson process assumes that the rate of occurrence is stationary [20].

Proposition 3.1.1 shows that the distribution of arrivals depends only on the length of the interval and is independent of the location of the interval. In other words, the intensity of arrivals does not change over time. However, modelling real-life phenomena including risk probabilities often requires the rate of occurrence to vary with time as stationarity can only be assumed for a short period of time. A natural generalisation of the homogeneous Poisson process that would account for non-stationarity involves treating the rate λ as a function of t rather than a constant. As λ varies over time, it can be interpreted as instantaneous intensity of the arrival flow at time t [44]. A constant intensity corresponds to the homogeneous Poisson process described in Section 3.1.1.

Generally, a subordinated process is understood as a superposition of two independent stochastic processes [23]. The main process is referred to as the parent process, or the outer process. The time of the process is replaced by an independent stochastic process called the outer process, or the subordinator. This allows for altering certain characteristics of the parent process while still keeping some of its properties. Well-known subordinators covered in the present review include Poisson process, one-sided stable process with stability index $\alpha \in (0, 1)$ (α-stable subordinator), and tempered stable subordinators [23]. Subordinators arise naturally in risk theory. In a general perturbed risk process, the cumulative claim process has to be increasing. Assuming stationary independent increments, this implies that the cumulative claim process should be modelled using subordinators [18].

Studying subordinators involves describing the first-passage time of the process. The inverse subordinator is the process obtained by considering the first-passage time of a subordinator [51]. It is possible for a Lévy subordinator to be itself a first passage time of some other Lévy process. The use of inverse subordinators that is the most relevant to the present paper is achieving a process with sub-diffusion. Sub-diffusion, or anomalous diffusion, refers to a property of the process variance growing at a rate which varies non-linearly with time [51]. While the moments of a subordinator may be infinite, the moments of the inverses of specific types of subordinators are finite.

As described in Section 2.1, subordinators have infinite moments, but the first-hitting time process of the subordinator may have finite moments. For α-stable subordinators, all moments of order α or less are infinite while the first moment of its inverse grows as t^α [51]. In general, stable distributions possess heavy tails and are infinitely divisible which provides a rich class of Lévy processes [23]. The key limitation of α-stable processes is that generally they have non-finite second-order moments, which will be addressed by considering tempered stable subordinators in Section 2.3.

The inverse α-stable subordinator is defined using the Mittag-Leffler family of functions and fractional derivatives. The link between fractional calculus and probability is reflected in the interpretation of equations describing diffusion processes [38]. Specifically, in a Brownian motion, the probability of a jump exceeding a certain length x falls off as a power law $x^{-\alpha}$ for some $\alpha \in (0, 2)$. Then the random walk limit is an α-stable Lévy process with particle traces being fractals of dimension α [1]. The particle density is described by a diffusion equation which includes a fractional derivative of order α.

The link between subordinators and risk models is in the use of inverse subordinators instead of time in Poisson processes [30]. As will be shown in Section 3.2 , the fractional Poisson process defined as a renewal process with Mittag-Leffler waiting times can be also obtained by time-changing via an inverse α-stable subordinator [37].

The Mittag-Leffler function can be regarded as an extension of the complex exponential function. Section 2.2.2 introduces a one-parameter generalisation as an infinite series. This parametrisation corresponds to the commonly used two-parameter definition with $\beta = 1$. The Mittag-Leffler distribution is a heavy-tailed distribution. As such, when used to define waiting times for a Poisson-type renewal process it implies fewer arrivals on average for sufficiently large t compared to the classical Poisson process described in Section 3.1 [9].

As noted in Section 2.2, α-stable processes do not have finite second-order moments unless they correspond to the Gaussian case [26]. This issue is addressed by the process of tempering which in essence makes the tail probabilities decay faster. The present study follows established literature on risk models and considers exponential tempering of α-stable distributions [16]. This ensures that the resulting distributions are infinitely divisible, have finite moments of all orders, and have exponentially decaying tail probabilities [23]. Exponential tempering generates a class of continuous time Lévy processes that are no longer self-similar. In line with Section 2.1, inverse tempered α-stable subordinators are defined as first-hitting times of a tempered α-stable subordinator [26]. Tempering allows the inverse subordinator to model power law waiting times while still having finite moments [1].

Mixtures of tempered α-stable subordinators generalise both tempered stable subordinators and α-stable subordinators [16]. As such, they can also be used instead of the process time in subordinated Poisson processes to allow for more flexibility in modelling deviations from Poisson processes [20].

Sections 2.1–2.4 have introduced the notions of inverse subordinators and some of the major classes of these processes. These developments tie in with risk modelling by unifying fractional and time-fractal processes. In particular, the fractional Poisson process is defined as a renewal process with waiting times that are distributed according to the Mittag-Leffler distribution. Such a process can be shown to be equivalent to an ordinary Poisson process subordinated so that the process time is replaced by an inverse α-stable subordinator [37]. One of the key differences between an ordinary Poisson process and a fractional Poisson process is that the latter does not have stationary and independent increments [43]. It follows that fractional Poisson processes are not Lévy processes or Markov processes. However, this is not necessarily a shortcoming as the light-tailed distributed waiting times of Lévy processes fail to accurately describe phenomena with long memory [20].

The key advantage of fractional processes is that they allow for modelling anomalous diffusion described by space-time fractional diffusion equations for processes whose variances vary in time according to a power law [2]. More generally, the fractional Poisson process can also be viewed as the thinning limit of counting processes associated with renewal processes with inter-arrival times that are power-law distributed [43]. The power law describing how variance changes with time is reflected in the notion of long range dependence [34] which was proved for fractional Poisson processes by Biard and Saussereau [2]. Fractional Poisson processes are particularly well-suited for risk modelling as they allow for capturing heavy-tailed distributed interarrival times [10].

Section 3.2 considers fractional extensions for both homogeneous and non-homogeneous Poisson processes. Defining a fractional non-homogeneous Poisson process as a classical renewal process is not possible since it is an additive process with an intensity function that is deterministic and time-dependent [30]. This illustrates the usefulness of the subordinator construction developed in Sections 2.1–2.4 as a general renewal process can be considered instead. This is described in Section 3.4 which defines the fractional non-homogeneous Poisson process as an ordinary non-homogeneous Poisson process time-changed by the inverse α-stable subordinator.

Section 2.4 described mixtures of tempered α-stable subordinators while Section 3.2 introduced fractional Poisson processes. Section 3.3 illustrates further applications of notions developed in Section 2.4 to the Poisson process. This yields a generalisation of both homogeneous and non-homogeneous fractional Poisson processes by time-changing via a mixture of tempered stable subordinators [16]. The resulting process is more flexible compared to a Poisson process subordinated by a single inner process while still retaining the advantages of fractional processes and tempered subordinators that allow its application to real-life risk models.

The classical risk model describes the number of claims on a time interval as a Poisson process [40]. The total claims on the interval can therefore be retrieved as a sum of claim sizes, assumed to be non-negative and *i.i.d.*, over the Poisson-modelled number of claims. The risk, or surplus, process is then understood as the sum of the initial surplus and premiums less total claims at a specific time. This characterisation, while being relatively simplistic, allows for explicitly calculating the time of ruin, the infinite-horizon survival probabilities, and the infinite-horizon ruin probabilities as a function of the initial surplus. The finite-horizon ruin probability and finite-horizon survival probability can be retrieved as a function of time and the initial surplus. Both finite- and infinite-horizon survival probabilities are non-decreasing functions of the initial surplus and a non-increasing function of time, but in general can be non-differentiable or even discontinuous [40]. The classical risk model is extended by considering a fractional Poisson process as the underlying process for the number of claims on a time interval.

The safety loading ρ is required to be strictly positive for the net profit condition to hold [24]. Non-positive values of the safety loading parameter correspond to companies that will necessarily go bankrupt. The time scale of the counting process $N_\alpha(t)$ and the payment process $Y_\alpha(t)$ should be coordinated as otherwise, the model would overestimate the number of claims or premiums and the model would violate the profit condition. Compared to a model based on the ordinary Poisson process, the fractional model implies a covariance structure that captures accumulation effects [24]. In particular, the premium associated with the simple compound Poisson process described in Definition 3.4.1 is strictly lower than the premium associated with the fractional compound Poisson process described in Definition 3.4.3.

Comparing the expressions for covariance between $R_1(t)$ and $R_2(t)$ shows that the main difference is in the safety loading factor ρ entering the expression for the covariance of $R_2(t)$. As such, the risk process of the first type exhibiting the long range dependence property implies that the risk process of the second type also exhibits the long range dependence property [19].

For fractional Poisson processes, simulation of trajectories is enabled by relating subordinators to α-stable distributions [24]. This illustrates why the description of fractional Poisson processes as a process time-changed by an inverse stable subordinator is particularly useful for applications and modelling. The present study relies on the acceptance-rejection method to generate increments from mixtures of subordinators. This is a general method for drawing from a random distribution for which the density is bounded by a density of another distribution scaled by some constant. The study fixes model parameters such as intensity rate for the homogeneous Poisson process, functional form of the intensity rate for the non-homogeneous Poisson process, tempering parameters, and subordinator mixture weights. This allows for focusing on the differences between process types given identical parameters which should help illustrate how more advanced models may be more suitable for describing real-life scenarios. The impact and choice of risk model parameters on ruin probabilities is discussed in later chapters of the study.

Chapter 2

Subordinator and Inverse subordinator

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2.1 Subordinator and Inverse subordinator

Definition 2.1.1 (Homogeneous Lévy process)**.** The process $L := \{L(t)\}_{t\geq 0}$ is said to be a homogeneous Lévy process if all the following conditions hold:

- The process has independent and stationary increments,
- $L(0) = 0$ almost surely (a.s.),
- L is stochastically continuous, i.e. for any $\varepsilon > 0$ and $t > 0$,

$$\lim_{h\to 0} \mathrm{P}\left(|L(t+h) - L(t)| > \varepsilon\right) = 0,$$

- L is cadlag a.s., also called right continuous with left limits (RCLL),
- For $0 \leq s \leq t$, $L(t) - L(s) \stackrel{d}{=} L(t-s)$.

Definition 2.1.2 (Nonhomogeneous Lévy process)**.** The process $L := \{L(t)\}_{t\geq 0}$ is said to be a nonhomogeneous Lévy process if all the following conditions hold:

- The process has independent increments,
- $L(0) = 0$ a.s.,
- L is stochastically continuous,
- L is cadlag a.s.

Examples of homogeneous Lévy processes include HPP and homogeneous Wiener process; and NPP is an example of nonhomogeneous Lévy process.

Definition 2.1.3 (Subordinators)**.** A Lévy process $L := \{L(t)\}_{t\geq 0}$ is said to be a subordinator if it is non-decreasing a.s. [27].

Definition 2.1.4 (Inverse subordinators)**.** An inverse subordinator is the first-hitting time process of the given subordinator, i.e. for a subordinator $L(t)$, the inverse subordinator is defined by [25]

$$Y(t) = \inf\{s \geq 0 : L(s) > t\}. \tag{2.1}$$

Definition 2.1.5 (Renewal function)**.** The renewal function of a process $Y(t)$ is defined by

$$U(t) = \mathbb{E}[Y(t)]. \tag{2.2}$$

Properties

Definition 2.1.6 (Laplace exponent)**.** For the following Laplace transform,

$$\mathbb{E}\left[e^{-sL(t)}\right] = e^{-t\phi(s)}, \quad s, t \geq 0, \tag{2.3}$$

the function ϕ is unique, and it is said to be the Laplace exponent.

From [50], we have Equations 2.4 and 2.5.

Let $L(t),\ t \geq 0$ be a Lévy subordinator with Laplace exponent

$$\phi(s) = \mu s + \int_{(0,\infty)} (1 - e^{-sx})\Pi(dx), \quad s \geq 0, \tag{2.4}$$

where $\mu \geq 0$ is the drift and the Lévy measure Π on $\mathbb{R}_+ \cup \{0\}$ satisfies

$$\int_0^\infty (1 \wedge x)\Pi(dx) < \infty. \tag{2.5}$$

The Lévy measure in Equation 2.4 is a measure on the σ-algebra of Borel sets of $(0, \infty)$, which satisfies the condition of Equation 2.5. However, as this is the only condition needs to be met for a Lévy measure, it is not necessarily to be finite, for example, the Lévy measure of an α-stable subordinator is not finite, as the integral of such a measure diverges at zero (see Equation 2.14).

The process $Y(t),\ t \geq 0$, is non-decreasing and its sample paths are a.s. continuous if L is strictly increasing. Also $Y(t)$ is, in general, non-Markovian with non-stationary and non-independent increments.

We have

$$\{L(u_i) < t_i, i = 1, \ldots, n\} = \{Y(t_i) > u_i, i = 1, \ldots, n\}. \tag{2.6}$$

Let

$$H_u(t) = \mathrm{P}\{Y(u) < t\}. \tag{2.7}$$

From Equation 2.4 we have

$$\int_0^\infty e^{-st} dH_u(t) = e^{-t\phi(s)}, \tag{2.8}$$

and

$$\tilde{U}(s) = \int_0^\infty U(t)e^{-st}dt = \frac{1}{s\phi(s)}, \tag{2.9}$$

where $\tilde{U}(s)$ denotes the Laplace transform of $U(s)$; thus, $\tilde{U}$ characterizes the process $Y(t)$ (since ϕ characterizes L) [50].

And we have the covariance function (see [50, Equation (17)])

$$\mathrm{Cov}(Y(t_1), Y(t_2)) = \int_0^{t_1 \wedge t_2} (U(t_1 - \tau) + U(t_2 - \tau))dU(\tau) - U(t_1)U(t_2). \tag{2.10}$$

2.2 α-stable subordinator and inverse α-stable subordinator

In this section we present some results from [24].

2.2.1 α-stable subordinator

Let $L_\alpha(t)$, $t \geq 0$, be an α-stable subordinator with $\phi(s) = s^\alpha$, $0 < \alpha < 1$ (cadlag, continuous in probability, with independent and stationary increments), with α to be the stability index, whose density $g(t,x)$ is such that $L_\alpha(1)$ has probability density function

$$\begin{aligned} g_\alpha(x) &= g(1,x) \\ &= \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\Gamma(\alpha k+1)}{k!}\frac{1}{x^{\alpha k+1}}\sin(\pi k\alpha) \\ &= \frac{1}{x}W_{-\alpha,0}(-x^{-\alpha}), \quad x > 0, \end{aligned} \tag{2.11}$$

and with Laplace transform

$$\mathbb{E}\left[e^{-sL_\alpha(t)}\right] = \exp\left(-ts^\alpha\right), \quad s \geq 0. \tag{2.12}$$

Here we use the Wright's generalised Bessel function (see, e.g., [17])

$$W_{\gamma,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k)\Gamma(\beta+\gamma k)}, \quad z \in \mathbb{C},$$

where $\gamma > -1$, and $\beta \in \mathbb{C}$ [47], and for $\beta = 0, \gamma = -\alpha \in (-1,0)$

$$W_{-\alpha,0}(z) = \sum_{k=1}^{\infty}\frac{\sin(\pi k\alpha)}{\pi}\frac{z^k\Gamma(1+\alpha k)}{k!} \tag{2.13}$$

by reciprocity relation for the $\Gamma-$function. Also

$$g_\alpha(x) \sim \frac{\left(\frac{\alpha}{x}\right)^{\frac{2-\alpha}{2(1-\alpha)}}}{\sqrt{2\pi\alpha(1-\alpha)}}\exp\left\{-(1-\alpha)\left(\frac{x}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}}\right\}, \quad x \to 0, \tag{2.14}$$

and

$$g_\alpha(x) \sim \frac{\alpha}{\Gamma(1-\alpha)x^{1+\alpha}}, \quad x \to \infty. \tag{2.15}$$

Remark.

1. The α-stable subordinator has no finite moments.
2. The sample path of an α-stable subordinator is not continuous as it contains jumps at the points of events.

2.2.2 Inverse α-stable subordinator

Then we have the inverse α-stable subordinator

$$Y_\alpha(t) = \inf\{u \geq 0 : L_\alpha(u) > t\}, \tag{2.16}$$

which has density function of

$$f_\alpha(t,x) = \frac{t}{\alpha} x^{-1-\frac{1}{\alpha}} g_\alpha(tx^{-\frac{1}{\alpha}}), \quad x \geq 0,\ t > 0. \tag{2.17}$$

Definition 2.2.1 (Fractional Caputo-Djrbashian derivative)**.** The fractional Caputo-Djrbashian derivative ${}^C D_t^\alpha$ is given by (see [5])

$${}^C D_t^\alpha = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \frac{df(\tau)}{d\tau} \frac{d\tau}{(t-\tau)^\alpha}, \quad \alpha \in (0,1). \tag{2.18}$$

Let

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0,\ z \in \mathbb{C}$$

be the Mittag-Leffler function [13],[17].

Remark. It reduces to $E_\alpha(-z) = e^{-z}$ if $\alpha = 1$.

Asymptotic behaviour

From [35], we have the following asymptotic behaviours for $E_\alpha(-t^\alpha)$:

$$E_\alpha(-t^\alpha) \sim \exp\left(-\frac{t^\alpha}{\Gamma(1+\alpha)}\right), \quad t \to 0, \tag{2.19}$$

$$E_\alpha(-t^\alpha) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad t \to \infty. \tag{2.20}$$

Recall the following:

(i) The Laplace transform of the Mittag-Leffler function is of the form

$$\int_0^\infty e^{-st} E_\alpha(-t^\alpha)dt = \frac{s^{\alpha-1}}{1+s^\alpha}, \quad 0 < \alpha < 1,\ s \geq 0,\ t \geq 0.$$

(ii) The Mittag-Leffler function is a solution of the fractional equation with fractional Caputo-Djrbashian derivative ${}^C D_t^\alpha$

$${}^C D_t^\alpha E_\alpha(at^\alpha) = aE_\alpha(at^\alpha).$$

Proposition.

(i)

$$\mathbb{E}\left[e^{-sY_\alpha(t)}\right] = \sum_{n=0}^{\infty} \frac{(-st^\alpha)^n}{\Gamma(\alpha n+1)} = E_\alpha(-st^\alpha),\ s \geq 0,$$

(ii) both processes $L_\alpha(t), t \geq 0$ and $Y_\alpha(t)$ are self-similar

$$\frac{L_\alpha(at)}{a^{1/\alpha}} \stackrel{d}{=} L_\alpha(t),\ \frac{Y_\alpha(at)}{a^\alpha} \stackrel{d}{=} Y_\alpha(t), \quad a > 0,$$

(iii)

$$\frac{\partial \mathbb{E}(Y_\alpha(t_1)\cdots Y_\alpha(t_k))}{\partial t_1 \cdots \partial t_k} = \frac{1}{\Gamma^k(\alpha)} \frac{1}{[t_1(t_2-t_1)\cdots(t_k-t_{k-1})]^{1-\alpha}}, \quad 0 < t_1 < \cdots < t_k.$$

In particular,

(a) we have the first moment of $Y_\alpha(t)$,

$$\mathbb{E}\left[Y_\alpha(t)\right] = \frac{t^\alpha}{\Gamma(1+\alpha)};$$

(b) and we have the covariance function as

$$\begin{aligned}&\operatorname{Cov}\left(Y_\alpha(s), Y_\alpha(t)\right)\\ &= \frac{1}{\Gamma(1+\alpha)\Gamma(\alpha)} \int_0^{\min(s,t)} \left((s-\tau)^\alpha + (t-\tau)^\alpha\right) \tau^{\alpha-1} d\tau - \frac{(st)^\alpha}{\Gamma^2(1+\alpha)}.\end{aligned} \tag{2.21}$$

Comments.

1. Notice that this last property can be interpreted as long-range dependence.
2. There is a (complicated) form of all finite-dimensional distributions of $Y_\alpha(t), t \geq 0$, in the form of Laplace transforms, see [3].

Remark.

1. The inverse α-stable subordinator has finite moments.
2. Unlike α-stable subordinator, the sample path of an inverse α-stable subordinator is continuous.

Alternate form of α-stable and inverse α-stable subordinator

The density function of the α-stable subordinator, $g_\alpha(x,t)$, has the following integral form [22]

$$g_\alpha(x,t) = \frac{1}{\pi} \int_0^\infty e^{-ux - tu^\alpha \cos \alpha\pi} \sin\left(tu^\alpha \sin \alpha\pi\right) du, \quad x > 0. \tag{2.22}$$

The density function of the inverse α-stable subordinator, $f_\alpha(x,t)$, has the following integral form (see [22]):

$$f_\alpha(x,t) = \frac{1}{\pi} \int_0^\infty u^{\alpha-1} e^{-tu - xu^\alpha \cos(\alpha\pi)} \sin(\alpha\pi - xu^\alpha \sin \alpha\pi)\, du, \quad x > 0. \tag{2.23}$$

The density function of the α-stable subordinator at $t = 1$, shorten as $g_\alpha(x)$, could also be expressed as followed (see [39]):

$$g_\alpha(x) = \frac{\alpha}{1-\alpha} \frac{1}{\pi x} \int_0^\pi u e^{-u}\, d\varphi, \quad 0 < \alpha < 1, \tag{2.24}$$

where

$$u = u(\varphi) = \frac{\sin(1-\alpha)\varphi}{\sin\varphi} \left(\frac{\sin\alpha\varphi}{x \sin\varphi}\right)^{\frac{\alpha}{1-\alpha}}.$$

The density function of the inverse α-stable subordinator at $t=1$, $f_\alpha(x)$, could also be expressed as followed (see [45]):

$$f_\alpha(x) = \frac{1}{\alpha} x^{-\left(1+\frac{1}{\alpha}\right)} g_\alpha\left(x^{-\frac{1}{\alpha}}\right), \ x > 0. \tag{2.25}$$

Asymptotic behaviour

From [45], we have the following asymptotic behaviours for $f_\alpha(x)$:

$$f_\alpha(x) \to \frac{\sin(\alpha\pi)}{\alpha\pi}\Gamma(1+\alpha), \quad x \to 0^+, \tag{2.26}$$

$$f_\alpha(x) \sim \frac{K}{\alpha} x^{\frac{2\alpha-1}{2(1-\alpha)}} \exp\left(-Ax^{\frac{1}{1-\alpha}}\right), \quad x \to \infty, \tag{2.27}$$

where

$$A = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}},$$

$$K = \frac{\alpha^{\frac{1}{2(1-\alpha)}}}{\sqrt{2\pi(1-\alpha)}}.$$

From [39], we have the following asymptotic behaviors for $g_\alpha(x)$:

$$g_\alpha(x) \to Kx^{-\frac{2-\alpha}{2(1-\alpha)}} \exp\left(-Ax^{-\frac{\alpha}{1-\alpha}}\right), \quad x \to 0^+, \tag{2.28}$$

$$g_\alpha(x) \sim \frac{\sin\alpha\pi}{\pi}\Gamma(1+\alpha)x^{-(1+\alpha)}, \quad x \to \infty. \tag{2.29}$$

Thus,

$$f_\alpha(x,t) \to \frac{1}{t^\alpha}\frac{\sin(\alpha\pi)}{\alpha\pi}\Gamma(1+\alpha), \quad x \to 0^+, \tag{2.30}$$

and

$$f_\alpha(x,t) \sim \frac{Kt^{-\frac{\alpha}{2(1-\alpha)}}}{\alpha} x^{\frac{2\alpha-1}{2(1-\alpha)}} \exp\left(-\frac{Ax^{\frac{1}{1-\alpha}}}{t^{\frac{\alpha}{1-\alpha}}}\right), \quad x \to \infty \tag{2.31}$$

for any fixed t.

2.3 Tempered α-stable subordinator and inverse tempered α-stable subordinator

2.3.1 Tempered α-stable subordinator

Definition 2.3.1 (Tempered α-stable subordinator)**.** The tempered α-stable subordinator $S_{\alpha,\lambda}(t)$, with tempering parameter $\lambda > 0$ and stability index $\alpha \in (0,1)$, is a Lévy process which has density function

$$g_{\alpha,\lambda}(x,t) = e^{-\lambda x + \lambda^\alpha t} g_\alpha(x,t), \tag{2.32}$$

where $g_\alpha(x,t)$ is the density function of the α-stable subordinator; and with Laplace transform

$$\mathbb{E}\left(e^{-sS_{\alpha,\lambda}(t)}\right) = e^{-t[(s+t)^\alpha - \lambda^\alpha]}, \quad s > 0. \tag{2.33}$$

The first two moments and covariance of $S_{\alpha,\lambda}(t)$ are given by

$$\begin{aligned} \mathbb{E}\left(S_{\alpha,\lambda}(t)\right) &= \alpha\lambda^{\alpha-1}t \\ \mathbb{E}\left(S^2_{\alpha,\lambda}(t)\right) &= \alpha(1-\alpha)\lambda^{\alpha-2}t + (\alpha\lambda^{\alpha-1}t)^2 \\ \operatorname{Cov}\left(S_{\alpha,\lambda}(s), S_{\alpha,\lambda}(t)\right) &= \alpha(1-\alpha)\lambda^{\alpha-2}\min(s,t), \quad s,t \geq 0 \end{aligned} \tag{2.34}$$

as shown in [16].

2.3.2 Inverse tempered α-stable subordinator

Definition 2.3.2. We have the inverse tempered α-stable subordinator $Y_{\alpha,\lambda}(t)$, defined by

$$Y_{\alpha,\lambda}(t) = \inf\{u \geq 0 : S_{\alpha,\lambda}(u) > t\}, \quad t \geq 0. \tag{2.35}$$

Let $f_{\alpha,\lambda}(x,t)$ denotes the density function of $Y_{\alpha,\lambda}(t)$. Then $f_{\alpha,\lambda}(x,t)$ has the Laplace transform given by

$$\mathcal{L}\left(f_{\alpha,\lambda}(x,t)\right) = \frac{1}{s}\left[(s+\lambda)^\alpha - \lambda^\alpha\right] e^{-x[(s+\lambda)^\alpha - \lambda^\alpha]}; \tag{2.36}$$

and

$$f_{\alpha,\lambda}(x,t) = \frac{1}{\pi} e^{\lambda^\alpha x - \lambda t} \int_0^\infty \frac{e^{-ty - xy^\beta \cos(\beta\pi)}}{y+\lambda} \left[\lambda^\beta \sin\left(xy^\beta \sin(\beta\pi)\right) + y^\beta \sin\left(\beta\pi - xy^\beta \sin(\beta\pi)\right)\right] dy, \quad x > 0, \lambda > 0, \beta \in (0,1). \tag{2.37}$$

2.4 Mixture of tempered α-stable subordinators and inverse mixture of tempered α-stable subordinators

In this section we present some results from [16].

2.4.1 Mixture of tempered α-stable subordinators

We define a mixture of tempered stable subordinators (MTSS) as Lévy stochastic process $\{S_m(t);\ t \geq 0\}$ with the following Laplace exponent

$$\mathbb{E}[\exp\{-sS_m(t)\}] = e^{-t\phi(s)}, \quad s > 0, \tag{2.38}$$

where

$$\phi(s) = c_1\left((s+\lambda_1)^{\alpha_1} - \lambda_1^{\alpha_1}\right) + c_2\left((s+\lambda_2)^{\alpha_2} - \lambda_2^{\alpha_2}\right), \quad s > 0, \tag{2.39}$$

and $\alpha_i \in (0,1), \lambda_i \geq 0, c_i \geq 0, i = 1,2, c_1 + c_2 = 1$.

Note that for $\alpha_1 = \alpha$, $c_1 = 1$, $c_2 = 0$, $\lambda_1 = 0$ the process $S_m(t)$ reduces to the α-stable subordinator $S_\alpha(t)$ defined in Equation 2.11.

An alternative definition of the MTSS can be given as a sum of the independent tempered stable Lévy processes $S_i(t)$, $i = 1,2$ with Laplace exponents $\phi_i(s) = (s+\lambda_i)^{\alpha_i} - \lambda_i^{\alpha_i}, \quad i = 1,2$ with time scaling and the conditions $c_1 + c_2 = 1$ such that

$$S_m(t) = S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t) = S_1(c_1 t) + S_2(c_2 t), \quad t \geq 0. \tag{2.40}$$

The advantage of tempered stable subordinators over $\alpha-$stable subordinators is that they have finite moments of all orders and their density is also infinitely divisible. The first moment and covariance structure of the MTSS are given by

$$\mathbb{E}[S_m(t)] = t\left(c_1\alpha_1\lambda_1^{\alpha_1-1} + c_2\alpha_2\lambda_2^{\alpha_2-1}\right) \tag{2.41}$$

and

$$Cov(S_m(s), S_m(t)) = \min(t,s)\left[c_1\alpha_1(1-\alpha_1)\lambda_1^{\alpha_1-2} + c_2\alpha_2(1-\alpha_2)\lambda_2^{\alpha_2-2}\right]. \tag{2.42}$$

The pdf $g_m(x,t) = \frac{d}{dx}\mathbb{P}[S_m(t) \le x]$ has the following representation for $\lambda_1 \neq \lambda_2$:

$$\begin{aligned}
g_m(x,t) &= \frac{1}{\pi}\int_0^\infty \exp\left(-x\lambda_2\right)\exp(-wx)\exp\left[t\left(c_1\lambda_1^{\alpha_1} + c_2\lambda_2^{\alpha_2}\right)\right]\\
&\times \exp\left[-t\left(c_1(\lambda_1-\lambda_2)^{\alpha_1}\sum_{k=0}^{\infty}\binom{\alpha_1}{k}\frac{w^k}{(\lambda_1-\lambda_2)^k}\cos\pi k + c_2 w^{\alpha_2}\cos\pi\alpha_2\right)\right]\\
&\times \sin\left(c_1 t(\lambda_1-\lambda_2)^{\alpha_1}\sum_{k=0}^{\infty}\binom{\alpha_1}{k}\frac{w^k}{(\lambda_1-\lambda_2)^k}\sin\left(\pi k\right) + c_2 t w^{\alpha_2}\sin\left(\pi\alpha_2\right)\right)dw\\
&+ \frac{1}{\pi}\int_0^{\lambda_2-\lambda_1}\exp\left(-x\lambda_1\right)\exp(-wx)\exp\left[t\left(c_1\lambda_1^{\alpha_1} + c_2\lambda_2^{\alpha_2}\right)\right]\\
&\times \exp\left[-t\left(c_1 w^{\alpha_1}\cos(\pi\alpha_1) + c_2(\lambda_1-\lambda_2)^{\alpha_2}\sum_{k=0}^{\infty}\binom{\alpha_2}{k}\frac{w^k}{(\lambda_1-\lambda_2)^k}\cos\left(\pi k\right)\right)\right]\\
&\times \sin\left(c_1 t w^{\alpha_1}\sin\left(\pi\alpha_1\right) + c_1 t(\lambda_1-\lambda_2)^{\alpha_2}\sum_{k=0}^{\infty}\binom{\alpha_2}{k}\frac{w^k}{(\lambda_1-\lambda_2)^k}\sin\left(\pi k\right)\right)dw.
\end{aligned} \tag{2.43}$$

If $\lambda_1, \lambda_2 = \lambda$ and $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$, then

$$\begin{aligned}
g_m(x,t) = \frac{1}{\pi}\int_0^\infty &\exp\left(-x\lambda\right)\exp(-wx)\exp\left[t\left(c_1\lambda^{\alpha_1} + c_2\lambda^{\alpha_2}\right)\right]\\
&\times \exp\left[-t\left(c_1 w^{\alpha_1}\cos\left(\pi\alpha_1\right) + c_2 w^{\alpha_2}\cos\left(\pi\alpha_2\right)\right)\right]\\
&\times \sin\left(t(c_1 w^{\alpha_1}\sin\left(\pi\alpha_1\right) + c_2 w^{\alpha_2}\sin\left(\pi\alpha_2\right))\right)dw,
\end{aligned} \tag{2.44}$$

where $c_1 + c_2 = 1$ and $c_1, c_2 \ge 0$.

For the special case $\alpha_1 = \alpha_2 = \alpha$ and $\lambda_1 = \lambda_2 = 0$ with the condition $c_1 + c_2 = 1$ the formulae (2.43) and (2.44) reduce to the known formulae for the $\alpha-$stable subordinator:

$$g_m(x,t) = \frac{1}{\alpha t x^{1+\alpha}} M_\alpha\left(\frac{t}{x^\alpha}\right),$$

where the M-Wright function $M_\alpha(z)$ is defined by

$$M_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k + (1-\alpha))}, \quad z \in \mathbb{C}. \tag{2.45}$$

2.4.2 Inverse mixture of tempered α-stable subordinators

Next, we define the inverse MTSS (IMTSS) as the right continuous stochastic process

$$Y_m(t) = \inf\{u > 0 : \; S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(u) > t\} = \inf\{u > 0 : S_m(u) > t\}, \quad t \geq 0. \tag{2.46}$$

Note that for $\alpha_1 = \alpha,\ c_1 = 1,\ c_2 = 0,$ and $\lambda_1 = 0$ the process $Y_m(t)$ reduces to the inverse α-stable subordinator $Y_\alpha(t)$ defined in Equation 2.16.

The renewal function for the IMTSS is given by

$$U_m(t) = \mathbb{E}[Y_m(t)] \tag{2.47}$$

and its Laplace transform by

$$\tilde{U}(s) = \frac{1}{\phi(s)}, \tag{2.48}$$

where $\phi(s)$ is given by (2.39). The asymptotic behavior of this renewal function is as follows:

$$U(t) \sim \begin{cases} \dfrac{t^{\alpha_1+\alpha_2-\min(\alpha_1,\alpha_2)}}{\Gamma(1+\min(\alpha_1,\alpha_2))\left(c_1 t^{\alpha_2-\min(\alpha_1,\alpha_2)} + c_2 t^{\alpha_1-\min(\alpha_1,\alpha_2)}\right)}, & \text{as } t \to 0, \\[2ex] \dfrac{t}{\left(c_1\alpha_1{\lambda_1}^{\alpha_1-1} + c_2\alpha_2{\lambda_2}^{\alpha_2-1}\right)}, \ \lambda_1, \lambda_2 > 0, & \text{as } t \to \infty, \\[2ex] \dfrac{t^{\alpha_1+\alpha_2-\min(\alpha_1,\alpha_2)}}{\Gamma(1+\min(\alpha_1,\alpha_2))\left(c_1 t^{\alpha_2-\min(\alpha_1,\alpha_2)} + c_2 t^{\alpha_1-\min(\alpha_1,\alpha_2)}\right)}, \ \lambda_1 = \lambda_2 = 0, & \text{as } t \to \infty. \end{cases}$$

The cdf of IMTSS is given by

$$\mathbb{P}[Y_m(t) \leq x] = \int_t^\infty g_m(x,u)du, \tag{2.49}$$

and the pdf by

$$h_m(x,t) = \frac{d}{dx}\mathbb{P}[Y_m(t) \leq x] = \frac{d}{dx}\left(1 - \int_t^\infty g_m(x,u)du\right). \tag{2.50}$$

The covariance structure is

$$Cov(Y_m(s), Y_m(t)) = \int_0^{s\wedge t} [U(t-\tau) - U(s-\tau)]d\tau - U(s)U(t), \quad s,t \geq 0, \tag{2.51}$$

where $U(t)$ is defined in (2.47).

Chapter 3

Count and risk processes

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3.1 Poisson processes

3.1.1 Homogeneous Poisson process

Definition 3.1.1 (Homogeneous Poisson process (HPP) (first definition))**.** The process $N^\mu = \{N^\mu(t), t \in [0, \infty)\}$ is said to be a HPP having rate μ, $\mu > 0$, if the following conditions are met.

- $N^\mu(0) = 0$.
- The process has independent and stationary increments.
- The number of arrivals P_m in any interval of length $\tau > 0$ is Poisson distributed with mean $(\mu\tau)$.

Thus, we have

$$\begin{aligned} P_m(t) &= \mathrm{P}\{N^\mu(t+u) - N^\mu(u) = m\} \\ &= \frac{e^{-[\mu(t+u)-\mu u]}\,[\mu(t+u) - \mu u]^m}{m!}, \quad m = 0, 1, 2, \ldots \end{aligned} \tag{3.1}$$

HPP could also be defined as follow.

Definition 3.1.2 (HPP (second definition))**.** The process $N^\mu = \{N^\mu(t), t \in [0,\infty)\}$ is said to be a HPP having rate $\mu, \mu > 0$, if the following conditions are met.

- $N^\mu(0) = 0$.
- The process has independent and stationary increments.
- For any $t \in [0,\infty)$, as $h \to 0$,

$$\begin{aligned} \mathrm{P}(N^\mu(t+h) = m \mid N^\mu(t) = m) &= \mathrm{P}\{N^\mu(t+h) - N^\mu(t) = 0\} \\ &= 1 - \mu th + o(h); \\ \mathrm{P}(N^\mu(t+h) = m+1 \mid N^\mu(t) = m) &= \mathrm{P}\{N^\mu(t+h) - N^\mu(t) = 1\} \\ &= \mu th + o(h); \\ \mathrm{P}(N^\mu(t+h) = m+k \mid N^\mu(t) = m) &= \mathrm{P}\{N^\mu(t+h) - N^\mu(t) = k\} \\ &= o(h), k \geq 2. \end{aligned}$$

Proposition 3.1.1. Let $N^\mu(t)$ be a HPP with rate μ. Let T_n be the time of n^{th} arrival, and Y_n be the inter-arrival time between the $(n-1)^{\text{th}}$ and the n^{th} arrival, i.e. $Y_n = T_n - T_{n-1}$. Then,

$$\begin{aligned} \mathrm{P}(Y_1 > t) &= \mathrm{P}\left(\text{no arrival in } (0,t]\right) \\ &= e^{-\mu t}; \end{aligned}$$

$$\begin{aligned} \mathrm{P}(Y_2 > t \mid Y_1 = t_1) &= \mathrm{P}\left(\text{no arrival in } (t_1, t_1 + t] \mid Y_1 = t_1\right) \\ &= \mathrm{P}\left(\text{no arrival in } (t_1, t_1 + t]\right) \ \text{(independent increments)} \\ &= e^{-\mu t}. \end{aligned}$$

$$\begin{aligned}
P(Y_{n+1} > t \mid Y_1 = t_1, \dots, Y_n = t_n) &= P\left(\text{no arrival in } (t_n, t_n + t] \mid Y_1 = t_1, \dots, Y_n = t_n\right) \\
&= P\left(\text{no arrival in } (t_n, t_n + t]\right) \\
&= e^{-\mu t}.
\end{aligned}$$

Then we have

$$F_{Y_n}^{\mathrm{HPP}}(t) = \begin{cases} 1 - e^{-\mu t}, & \text{if } t \geq 0 \\ 0, & \text{otherwise} \end{cases}, \quad n \in \mathbb{Z}^+.$$

Thus, we have $Y_n \sim \operatorname{Exp}(\mu)$.

Definition 3.1.3 (Erlang distribution)**.** Let $Y_1, Y_2, \dots, Y_n$ denotes n *i.i.d.* random variables, each of them has an exponential distribution with parameter μ. Then we have $\sum_{i=1}^{n} Y_i = T_n$, where T_n has a Erlang distribution with parameters n and μ, i.e. $T_n \sim \operatorname{Erlang}(n, \mu)$.

An Erlang distribution has the following probability density function:

$$f_{T_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \quad t \geq 0, \lambda \geq 0, n \in \mathbb{Z}^+, \tag{3.2}$$

and the cumulative distribution function:

$$F_{T_n}(t) = 1 - e^{-\lambda t} \sum_{k=0}^{n-1} \frac{1}{k!} (\lambda t)^k. \tag{3.3}$$

Remark. The Erlang distribution reduces to an exponential distribution when $n = 1$.

Proposition 3.1.2 (Arrival time)**.** Recall that $T_n = Y_1 + \ldots + Y_n$, since $Y_1, \dots, Y_n$ are *i.i.d.* exponential random variable, thus, we have $T_n \sim \operatorname{Erlang}(n, \mu)$.

Following the results from above, the Poisson process could also be defined as follow:

Definition 3.1.4 (HPP (third definition) (renewal representation))**.**

$$N^{\mu}(t) := \max\left\{n \geq 0 : Y_1 + \cdots + Y_n \leq t\right\}, \tag{3.4}$$

where $Y_1, \dots, Y_n$ are n *i.i.d* exponential random variables, and we have $N^{\mu}(t)_{t \geq 0}$ is a HPP with independent and stationary increments.

Covariance function

Let $X(t), t \geq 0$ be a HPP with rate $\mu > 0$, we have the covariance function:

$$\operatorname{Cov}(X(s), X(t)) = \mu(\min(s,t)), \quad s \geq 0, t \geq 0. \tag{3.5}$$

3.1.2 Nonhomogeneous Poisson process

Definition 3.1.5 (Nonhomogeneous Poisson process). Let $\lambda(t) : [0,\infty) \to [0,\infty)$ be a pre-specified integrable function. The process $NN^{\Lambda(t)} = \{NN^{\Lambda(t)}(t),\ t \in [0,\infty)\}$ is said to be a nonhomogeneous Poisson process (NPP) having rate $\lambda(t)$, $\lambda(t) \geq 0$, if the following conditions are met.

- $NN^{\Lambda(t)}(0) = 0$.
- The process has independent increments.
- For any $t \in [0,\infty)$, as $h \to 0$,

$$\begin{aligned}
\mathrm{P}(NN^{\Lambda(t)}(t+h) = m \mid NN^{\Lambda(t)}(t) = m) &= \mathrm{P}\{NN^{\Lambda(t)}(t+h) - NN^{\Lambda(t)}(t) = 0\} \\
&= 1 - \lambda(t)h + o(h); \\
\mathrm{P}(NN^{\Lambda(t)}(t+h) = m+1 \mid NN^{\Lambda(t)}(t) = m) &= \mathrm{P}\{NN^{\Lambda(t)}(t+h) - NN^{\Lambda(t)}(t) = 1\} \\
&= \lambda(t)h + o(h); \\
\mathrm{P}(NN^{\Lambda(t)}(t+h) = m+k \mid NN^{\Lambda(t)}(t) = m) &= \mathrm{P}\{NN^{\Lambda(t)}(t+h) - NN^{\Lambda(t)}(t) = k\} \\
&= o(h), k \geq 2.
\end{aligned}$$

Let

$$\Lambda(t) = \int_0^t \lambda(u)du.$$

Then we have

$$\begin{aligned}
P_m(t) &= \mathrm{P}\{NN^{\Lambda(t)}(t+u) - NN^{\Lambda(t)}(u) = m\} \\
&= \frac{e^{-[\Lambda(t+u)-\Lambda(u)]}\,[\Lambda(t+u) - \Lambda(u)]^m}{m!}, \quad m = 0, 1, 2, \ldots,
\end{aligned} \tag{3.6}$$

and

$$F(t) = \mathrm{P}(Y_1 \leq t) = 1 - \exp(-\Lambda(t)), \quad t \geq 0. \tag{3.7}$$

Remark. If $\lambda(t)$ is a positive constant at all t, then

$$\Lambda(t) = \lambda t, \Lambda(t+u) - \Lambda(u) = \lambda(t+u) - \lambda u = \lambda t.$$

i.e. it reduces to a HPP with rate λ.

Arrival time

T_n has the following distribution functions [31]:

$$F_{T_n}^{\text{NPP}}(t) = 1 - e^{-\Lambda(t)} \sum_{k=0}^{n-1} \frac{[\Lambda(t)]^k}{t!}, \tag{3.8}$$

and

$$f_{T_n}^{\text{NPP}}(t) = e^{-\Lambda(t)} \frac{\lambda(t)\,[\Lambda(t)]^{n-1}}{(n-1)!}. \tag{3.9}$$

Covariance function

Let $X^{\Lambda(t)}(t), t \geq 0$ be a NPP with rate $\lambda(t), \lambda(t) \geq 0$, we have the covariance function:

$$\text{Cov}\left(X^{\Lambda(t)}(s), X^{\Lambda(t)}(t)\right) = \Lambda\left(\min\left(s,t\right)\right), \quad s \geq 0, t \geq 0. \tag{3.10}$$

Similar to Definition 3.1.4, NPP could also be constructed as follow: [30]

Definition 3.1.6 (NPP (second definition) (renewal representation))**.** Let $Y_1, Y_2, \ldots$ be a sequence of *i.i.d.* non-negative random variables with distribution function

$$F(t) = \text{P}\left(Y_n \leq t\right) = 1 - e^{-\Lambda(t)}, \quad t \geq 0,$$

with

$$\Lambda(t) \to \infty, \quad t \to \infty.$$

Define

$$\zeta'_n := \max\left\{Y_1, \ldots, Y_n\right\}, \quad n = 1, 2, \ldots\ ,$$
$$\chi_n := \inf\left\{k \in \mathbb{N} : \zeta'_k > \zeta'_{n-1}\right\}, \quad n = 2, 3, \ldots$$

with $\chi_1 = 1$, let

$$\zeta_n := \zeta'_{\chi_n},$$

and

$$NN^{\Lambda(t)}(t) := \sup\{k \in \mathbb{N} : \zeta_k \le t\} = \sum_{n=0}^{\infty} n \mathbb{1}_{\{\zeta_n \le t \le \zeta_{n+1}\}}, \quad t \ge 0$$

where $\zeta_0 = 0$, and we have $NN^{\Lambda(t)}(t)_{t\ge 0}$ is a NPP with independent increments.

3.2 Fractional Poisson processes

3.2.1 Fractional homogeneous Poisson process

Definition 3.2.1 (Fractional homogeneous Poisson process (first definition))**.** The Fractional homogeneous Poisson process (FHPP) $N_\alpha(t)$, is defined by

$$N_\alpha(t) = N^1(Y_\alpha(t)),\ t \ge 0, \quad \alpha \in (0,1), \tag{3.11}$$

where $N^1(t)$ denotes a HPP as defined in subsection 3.1.1 with parameter $\mu = 1$, independent of $Y_\alpha(t)$, which is the inverse stable subordinator as defined in subsection 2.2.2.

Remark. It reduces to a HPP with rate μ if $\alpha = 1$ as shown in [28].

Definition 3.2.2 (FHPP (second definition) (renewal representation))**.** FHPP could also be defined with Mittag-Leffler distribution:

$$\begin{aligned} N_\alpha(t) &= \max(n : Y_1 + \cdots + Y_n \le t) \\ &= \sum_{j=1}^{\infty} \mathbb{1}_{\{Y_1 + \cdots + Y_j \le t\}} \\ &= \sum_{j=1}^{\infty} \mathbb{1}_{U_j \le G_\alpha(t)}, \quad t \ge 0, \end{aligned} \tag{3.12}$$

where $\{Y_j\}, j = 1, 2, \ldots,$ denote the interarrival times of FHPP, which are *i.i.d.* random variables with strictly monotone Mittag-Leffler distribution, with probability function (see [24]):

$$F_\alpha(t) = \mathrm{P}(Y_j \le t) = 1 - E_\alpha(-\mu t^\alpha), \quad t \ge 0, \alpha \in (0,1), j = 1, 2, \ldots, \tag{3.13}$$

$$G_\alpha(t) = P(Y_1 + \cdots + Y_k \le t) = \int_0^t h^{(k)}(x)\, dx,$$

where

$$\begin{aligned} h^{(k)}(x) &= \alpha\mu^k \frac{x^{k\alpha-1}}{(k-1)!} E_\alpha^{(k)}(-\mu x^\alpha) \\ &= \mu^k x^{\alpha k-1} E_{\alpha,\alpha k+1}^k(-\mu x^\alpha), \quad \alpha \in (0,1), x > 0 \end{aligned}$$

Remark.

1. Mittag-Leffler distribution has no finite moments for $\alpha \in (0,1)$. Also, it reduces to an exponential distribution if $\alpha = 1$ as shown in subsection 2.2.2.
2. The interarrival time of the HPP $Y_j^{HPP} \overset{i.i.d.}{\sim} \mathrm{Exp}\,(\mu)$, whereas the interarrival time of the FHPP are $i.i.d.$ Mittag-Leffler distributed.

Note that, FHPP has the following probability mass function [24], [28]:

$$\begin{aligned} P_\alpha(t,k) &= \mathrm{P}(N_\alpha(t) = k) \\ &= \frac{(\mu t^\alpha)^k}{k!} \sum_{j=1}^{\infty} \frac{(j+k)!}{j!} \frac{(-\mu t^\alpha)^j}{\Gamma\left(\alpha\left(j+k\right)+1\right)} \\ &= \frac{(\mu t^\alpha)^k}{k!} E_\alpha^{(k)}(-\mu t^\alpha), \\ &= (\mu t^\alpha)^k E_{\alpha,\alpha k+1}^{k+1}(-\mu t^\alpha), \quad k = 0,1,2,\ldots,\ t \ge 0,\ \alpha \in (0,1) \end{aligned} \tag{3.14}$$

where $E_\alpha(z)$ is the Mittag-Leffler function, $E_\alpha^{(k)}(z)$ is the k-th derivative of $E_\alpha(z)$, and $E_{\alpha,\beta}^\gamma(z)$ is the three-parametric generalised Mittag-Leffler function defined by

$$E_{\alpha,\beta}^\gamma(z) = \sum_{j=0}^{\infty} \frac{\gamma^{(j)}\, z^j}{j!\,\Gamma(\alpha j + \beta)}, \quad \alpha > 0, \beta > 0, \gamma > 0, z \in \mathbb{C}, \tag{3.15}$$

where $\gamma^{(j)}$ is the rising factorial, defined by

$$\gamma^{(j)} := \begin{cases} 1, & \text{if } j = 0 \\ \dfrac{\Gamma(\gamma + j)}{\Gamma(\gamma)}, & \text{if } j = 1, 2, \ldots \end{cases}$$

The probability function, $P_\alpha(t,k)$, follows the following fractional Kolmogorov equation:

$$ {}^C D_t^\alpha P(N_\alpha(t)=k) = \mu(P_\alpha(t,k-1) - P_\alpha(t,k)), \quad 0<\alpha<1, \tag{3.16} $$

with initial condition

$$ P_\alpha(0,k) = \begin{cases} 0, & \text{if } k \neq 0, \\ 1, & \text{if } k = 0, \end{cases} \quad \alpha \in (0,1), $$

and we defined that $P_\alpha(k,-1) \equiv 0$.

Moments and covariance

We have

$$ \mathbb{E}\left(N_\alpha(t)\right) = \frac{\mu t^\alpha}{\Gamma(1+\alpha)} \tag{3.17} $$

$$ \operatorname{Var}\left(N_\alpha(t)\right) = \frac{\mu^2 t^{2\alpha}}{\Gamma^2(1+\alpha)}\left[\frac{\alpha\Gamma(\alpha)}{\Gamma(2\alpha)} - 1\right] + \frac{\mu t^\alpha}{\Gamma(1+\alpha)}, \ t \geq 0, \tag{3.18} $$

$$ \operatorname{Cov}\left(N_\alpha(s), N_\alpha(t)\right) = \frac{\mu(\min(s,t))^\alpha}{\Gamma(1+\alpha)} + \mu^2 \operatorname{Cov}\left(Y_\alpha(s), Y_\alpha(t)\right), \tag{3.19} $$

where $\operatorname{Cov}\left(Y_\alpha(s), Y_\alpha(t)\right)$ is given in Equation 2.21 (see [24]).

Arrival time

The arrival time T_n of FHPP has the following distribution function [31]:

$$ \begin{aligned} F_{T_n}^{\text{FHPP}}(t) &= \int_0^\infty f_\alpha(t,u) F_{T_n}^{\text{HPP}}(u)\, du \\ &= 1 - \sum_{k=0}^{n-1} \frac{(\mu t^\alpha)^k}{k!} E_\alpha(-\mu t^\alpha), \end{aligned} \tag{3.20} $$

where f_α is given in Equation 2.17, and $E_\alpha(z)$ is the one parameter Mittag-Leffler function.

3.2.2 Fractional nonhomogeneous Poisson process

Definition 3.2.3 (Fractional nonhomogeneous Poisson process (first definition)). The fractional nonhomogeneous Poisson process (FNPP) $NN_\alpha(t)$, is defined by [24]:

$$NN_\alpha(t) := N^1\left(\Lambda\left(Y_\alpha(t)\right)\right), \ t \geq 0, \ \alpha \in (0,1), \tag{3.21}$$
$$\Lambda(t) := \int_0^t \lambda(s)\, ds, \quad t \geq 0,$$

with

$$\Lambda(t) \to \infty, t \to \infty,$$

where $N^1 = \{N^1(t), t \geq 0\}$ denotes a HPP with $\mu = 1$, and $Y_\alpha(t)$ denotes an inverse α-stable subordinator, which is independent of N^1.

Following Definition 3.1.6, FNPP could also be constructed as follow: [30]

Definition 3.2.4 (FNPP (second definition) (renewal representation)).

$$NN_\alpha(t) = \sum_{n=0}^{\infty} n \mathbb{1}_{\{\zeta_n \leq Y_\alpha(t) \leq \zeta_{n+1}\}} \overset{a.s.}{=} \sum_{n=0}^{\infty} n \mathbb{1}_{\{L_\alpha(\zeta_n) \leq t \leq L_\alpha(\zeta_{n+1})\}},$$

where ζ_n is defined in Definition 3.1.6, and where $L_\alpha(Y_\alpha(t)) = t$ iff t is not a jump time of L_α.

Remark. The renewal representations of HPP and FHPP are based on the interarrival time Y_j, whereas the renewal representations of the nonhomogeneous counterpart are based on $\zeta_j = \zeta'_{\chi_j} = \max\left(Y_1, \ldots, Y_{\chi_j}\right)$, i.e. the maximum value of the interarrival time.

The probability mass function is given by [24]

$$\mathrm{P}\left(NN_\alpha(t)\right) = \int_0^\infty e^{-\Lambda(u)} \frac{\Lambda(u)^k}{k!} f_\alpha(t,u)\, du, \quad k = 0, 1, 2, \ldots, \tag{3.22}$$

where $f_\alpha(t,u)$ is given by Equation 2.17.

Arrival time

The arrival time T_n has the following distribution function [31]:

$$F_{T_n}^{\text{FNPP}}(t) = \int_0^{\infty} f_\alpha(t,u) F_{T_n}^{\text{NPP}}(u)\, du. \tag{3.23}$$

3.3 Mixed tempered fractional Poisson processes

3.3.1 Mixed tempered fractional homogeneous Poisson process

Definition 3.3.1 (Mixed tempered fractional homogeneous Poisson process)**.** We introduce the mixed tempered fractional homogeneous Poisson process (MTFHPP) as the counting process

$$\{Z_m(t) = N^1(Y_m(t));\ t \geq 0\}, \tag{3.24}$$

where $N^1 = \{N^1(t); t \geq 0\}$ is a homogeneous Poisson process with intensity $\mu = 1$, independent of IMTSS $Y_m(t)$ defined by Equation 2.46.

The marginal distribution is as follows:

$$f_m(x) = \mathbb{P}[Z_m^h(t) = x] = \frac{1}{x!} \int_0^{\infty} e^{\mu u} [\mu u]^x h_m(t,u) du, \quad x \in \mathbb{N}_0, \tag{3.25}$$

where $h_m(t,u)$ is defined by Equation 2.50. It satisfies the governing equations with the same fractional operators (see ([16]) Proposition 9).

Employing the results of Leonenko at al. (2014) [29] Proposition 9, one can show that

$$\text{Cov}(Z_m(s), Z_m(t)) = \frac{\mu(\min(t,s))^\alpha}{\Gamma(1-\alpha)} + \mu^2 \text{Cov}(Y_m(s), Y_m(t)), \quad 0 \leq s \leq t, \tag{3.26}$$

where $\text{Cov}(Y_m(s), Y_m(t))$ is given by Equation 2.51.

3.3.2 Mixed tempered fractional nonhomogeneous Poisson process

Definition 3.3.2 (Mixed tempered fractional non-homogeneous Poisson process)**.** We introduce the mixed tempered fractional nonhomogeneous Poisson process (MTFNPP) as the counting process

$$\{NZ_m(t) = N^1(\Lambda(Y_m(t)));\ t \geq 0\}, \tag{3.27}$$

where $N^1 = \{N^1(t); t \geq 0\}$ is a homogeneous Poisson process with intensity $\mu = 1$, independent of IMTSS $Y_m(t)$, and $\Lambda : [0, \infty) \to [0, \infty)$ is a right continuous, non-decreasing function such that

$$\Lambda(0) = 0, \quad \Lambda(t) - \Lambda(t-) \leq 1, \quad \Lambda(\infty) = \infty, \quad t > 0$$
$$\Lambda(t) = \int_0^t \lambda(u)du < \infty,$$

where $\lambda(u),\ u \geq 0$ is an intensity function.

The marginal distribution are as follows:

$$f_m(x) = \mathbb{P}[NZ_m(t) = x] = \frac{1}{x!}\int_0^\infty e^{\Lambda(u)}[\Lambda(u)]^x h_m(t,u)du, \quad x \in \mathbb{N}_0, \tag{3.28}$$

where $h_m(t, u)$ is defined by Equation 2.50. Equations 3.25 and 3.28 satisfy the governing equations with the same fractional operators (see ([16]) Proposition 9). This fact justifies our terminology.

Again, by employing the results of Leonenko at al. (2014) [29] Proposition 9, we can show that

$$\mathrm{Cov}(Z_m(s), Z_m(t)) = \frac{\Lambda((\min(t,s))^\alpha)}{\Gamma(1-\alpha)} + \mu^2\mathrm{Cov}(Y_m(s), Y_m(t)), \quad 0 \leq s \leq t, \tag{3.29}$$

where $\mathrm{Cov}(Y_m(s), Y_m(t))$ is given by Equation 2.51.

Following Leonenko et al. (2019) [30] one can approximate the distributions of the stochastic process $Z_m^h(t)$ as follows:

$$\frac{\left[Z_m^h(t) - \mu Y_m(t)\right]}{\sqrt{\mu}} \xrightarrow{J_1} B(Y_m(t)), \tag{3.30}$$

where $\{B(t); t \geq 0\}$ is a standard Brownian Motion independent of $Y_m(t)$ and convergence holds in J_1 Skorokhod topology. For other approximations see again [30].

3.4 Fractional risk processes

3.4.1 Classical risk models

Definition 3.4.1 (Compound Poisson process)**.** Suppose that $N^\mu(t)$ is a HPP, and $X_1, X_2, \ldots = X$ are *i.i.d.* random variables which independent of $N^\mu(t)$. Then the sum of $X_1, X_2, \ldots, X_{N^\mu(t)}$, is said to be a compound Poisson process, i.e.

$$S(t) = \sum_{k=1}^{N^\mu(t)} X_k \tag{3.31}$$

is said to be a compound Poisson process (see [11], [12], [49]).

We have

$$\mathbb{E}\left(S(t)\right) = \mu t \mathbb{E}\left(X\right) \tag{3.32}$$

$$\operatorname{Var}\left(S(t)\right) = \mu t \mathbb{E}\left(X^2\right) \tag{3.33}$$

$$\operatorname{Cov}\left(S(s), S(t)\right) = \mu \min(s,t) \mathbb{E}\left(X^2\right) \tag{3.34}$$

Definition 3.4.2 (Classical risk process)**.** In classical Cramer-Lundberg risk process (also known as classical compound Poisson risk model), the risk process $R(t), t \geq 0$, is defined by:

$$R(t) := \ u + ct - \sum_{k=1}^{N^\mu(t)} X_k, \quad t \geq 0, \tag{3.35}$$

where u is the initial surplus, c is the gross risk premium rate, $X_k : k \geq 1$ are *i.i.d.* random variables denoting the claim sizes, with $X_k > 0 : k \geq 1$, and $N^{\mu}(t)$ denotes the number of claims during the interval $(0, t]$ [48].

We consider a generalisation of the classical risk model, by replacing $N^{\mu}(t)$ with desired processes.

Definition 3.4.3 (Fractional compound Poisson process)**.** Suppose that $N_{\alpha}(t)$ is a FHPP, and $X_1, X_2, \ldots = X$ are *i.i.d.* random variables which independent of $N_{\alpha}(t)$. Then the sum of $X_1, X_2, \ldots, X_{N_{\alpha}(t)}$, is said to be a fractional compound Poisson process, i.e.

$$S_{\alpha}(t) = \sum_{k=1}^{N_{\alpha}(t)} X_k \tag{3.36}$$

is said to be a fractional compound Poisson process, other processes could also be computed similarly by replacing $N_{\alpha}(t)$.

Definition 3.4.4 (Fractional homogeneous risk process)**.** The classical risk process was extended by Kumar, et al. (see [24]), consider

$$R(t) := \; u + m\mu(1 + \rho)Y_{\alpha}(t) - \sum_{k=1}^{N_{\alpha}(t)} X_k, \quad t \geq 0, \tag{3.37}$$

where u is the initial surplus, $X_k : k \geq 1$ are *i.i.d.* random variables denoting the claim sizes, with $X_k > 0 : k \geq 1$ with mean $m > 0$, ρ is the safety loading, $Y_{\alpha}(t)$ is an inverse α-stable subordinator, and $N_{\alpha}(t)$ is a FHPP.

For nonhomogeneous version, the process is defined by

Definition 3.4.5 (Fractional nonhomogeneous risk process)**.**

$$R(t) := \; u + m(1 + \rho)\Lambda(Y_{\alpha}(t)) - \sum_{k=1}^{NN_{\alpha}(t)} X_k, \quad t \geq 0, \tag{3.38}$$

where u is the initial surplus, $X_k : k \geq 1$ are *i.i.d.* random variables denoting the claim sizes, with $X_k > 0 : k \geq 1$, ρ is the safety loading, $Y_{\alpha}(t)$ is an inverse α-stable subordinator, and $NN_{\alpha}(t)$ is a FNPP.

3.4.2 Mixed risk model

To extend the usage of classical Cramer-Lundberg risk model for mixed risk model, we consider a generalisation of it, namely we consider the risk process of the first type as

$$R_1(t) = u + ct - \sum_{i=1}^{NZ_m(t)} X_i, \tag{3.39}$$

where u is the initial capital of the insurance company, c is the constant premium rate, and $\{X_i\}_{i\geq 1}$ are positive $i.i.d.$ random variables independent of the fractional counting process $Z_m(t)$ defined by (Equation 3.24). We interpret $\{X_i\}_{i\geq 1}$ as the claim sizes of the counting process $NZ_m(t)$ with inter-arrival times $\{W_i\}_{i\geq 1}$. The net profit condition $c\mathbb{E}[W_i] > \mathbb{E}[X_i]$ is imposed to ensure that ruin does not happen with certainty. We also assume that random variables X_i have two moments and employ the following notation: $m = \mathbb{E}[X_i]$.

Note that for $\lambda_1 = \lambda_2 = \lambda$ the process (Equation 3.39) was introduced and studied in Kataria and Khandakar (2021) [21]. Observe that for the homogeneous Poisson process (i.e. $\Lambda(t) = \mu t$) we get

$$\mathbb{E}[R_1(t)] = u + m\rho\mu U(t),$$

$$Cov(R_1(s), R_1(t)) = \mathbb{E}[X_1^2]\mathbb{E}[NZ_m(s)] + \mu^2(\mathbb{E}[X_1])^2 Cov(NZ_m(s), NZ_m(t)), \quad 0 < s \leq t,$$

where $U(t)$ is given by (Equation 3.26).

Following Kumar et. al (2019) [24], we also introduce the risk process of the second type

$$R_2(t) = u + m(1+\rho)\Lambda(Y_m(t)) - \sum_{i=1}^{NZ_m(t)} X_i, \tag{3.40}$$

where ρ is the safety loading factor defined by

$$\rho = \frac{c\mathbb{E}[W_1] - \mathbb{E}[X_1]}{\mathbb{E}[X_1]}. \tag{3.41}$$

Note that

$$\mathbb{E}[R_2(t)] = u + m\rho\mu U(t),$$
$$Cov(R_2(s), R_2(t)) = \mathbb{E}[X_1^2]\mathbb{E}[NZ_m(s)] + m^2\mu^2\rho^2 Cov(NZ_m(s), NZ_m(t)),$$

where $Cov(NZ_m(s), NZ_m(t))$ is given by (Equation 3.26).

Observe that the homogeneous versions of models (Equation 3.39), (Equation 3.40) are obtained by setting the intensity function $\lambda(t) = \mu t$. This yields

$$R_3(t) = u + ct - \sum_{i=1}^{Z_m(t)} X_i, \tag{3.42}$$

and

$$R_4(t) = u + m(1+\rho)Y_m(t) - \sum_{i=1}^{Z_m(t)} X_i. \tag{3.43}$$

For all risk processes $R_i(t), i = 1, 2, 3, 4$ we introduce the ruin probabilities with finite horizon (finite-time ruin probabilities) [49]

$$\psi_i(u, T) = P[\inf_{0 \le t \le T} R_i(t) < 0 \mid R_i(0) = u] \quad i = 1, 2, \quad T > 0 \tag{3.44}$$

and the time of ruin

$$T_i = \inf\{t > 0 : \; R_i(t) < 0\}, \quad i = 1, 2. \tag{3.45}$$

These characteristics will be investigated numerically in Section 4.2. It is worth noting that the ruin probability and numerical approximations of the first passage were intensively investigated in various generalizations of classical risk models (see [9], [36], [50]).

Chapter 4

Simulations

This chapter is based on the 'Risk process with mixture of tempered stable inverse subordinators: analysis and synthesis' coauthor with Tetyana Kadankova accepted on 28th October, 2021 by 'Random Operators and Stochastic Equations' and has been reproduced here with the permission of the copyright holder.

4.1 Simulations methodologies

The python code for simulations could be found on https://github.com/VinventN/Processes.

4.1.1 Simulation of processes

In this section we provide simulations of the sample paths of the two traditional types of Poisson processes: homogeneous Poisson process $N(t)$ (HPP) defined in Section 3.1.1, and non-homogeneous Poisson process $NN(t)$ (NPP) defined in Section 3.1.2; along with four types of fractional Poisson processes: fractional homogeneous Poisson process $N_\alpha(t)$ (FHPP) defined by Section 3.2.1, and its non-homogeneous version $NN_\alpha(t)$ (FNPP) defined by Section 3.2.2, the mixed tempered stable fractional homogeneous Poisson process $Z_m(t)$ (MTFHPP) defined by Definition 3.3.1, and its non-homogeneous version $NZ_m(t)$ (MTFNPP) defined by Definition 3.3.2. These processes serve as counting processes of the claims in the models defined in Equations (3.39), (3.40), (3.42) and (3.43). We use the following base parameterisation:

- the homogeneous Poisson process $N^{\mu}(t)$ has the intensity $\mu = 4$;
- the intensity function of the non-homogeneous Poisson process is

$$\lambda(t) = \frac{\gamma}{\beta}\left(\frac{t}{\beta}\right)^{\gamma-1}, \quad \gamma = 0.9,\ \beta = 0.2;$$

- the parameters of the mixture tempered stable inverse process are set as follows:

$$\alpha_1 = 0.9, \alpha_2 = 0.5, \lambda_1 = 0.3, \lambda_2 = 0.7, c_1 = 0.75, c_2 = 0.25.$$

and the results of the simulations could be found in Figure 4.1.

Homogeneous Poisson process

From Proposition 3.1.1, it was shown that the waiting time of HPP follows an exponential distribution with parameter λ, and a HPP could be simulated using Algorithm 1. The results of the simulation could be found in Figure 4.1a.

Nonhomogeneous Poisson process

Since NPP has a time dependent rate $\lambda(t)$, Algorithm 1 is not suitable for such simulation. However, the thinning algorithm could be used for NPP simulation (see [8], [32], [41]), a proof of that could be found in [8, Theorem 4.2]. An NPP could be simulated using Algorithm 2, and the results could be found in Figure 4.1b.

α-stable subordinator

α-stable subordinator $S_\alpha(t)$ could be simulated by using $S_\alpha(t)$ using $S_\alpha(t+dt) - S_\alpha(t) \overset{d}{=} S_\alpha(dt) \overset{d}{=} (dt)^{1/\alpha}$ [16], where

$$S_\alpha(1) \overset{d}{=} \frac{\sin(\alpha\pi U_1)\left[\sin(1-\alpha)\pi U_1\right]^{1/\alpha-1}}{\left[\sin(\pi U_1)\right]^{1/\alpha}\left|\ln U_2\right|^{1/\alpha-1}}.$$

It could also be simulated by using the α-stable distribution (see [24]), where

$$L_\alpha(t) \sim S(\alpha, \beta, \sigma, \mu),$$

which has the characteristics function

$$\mathbb{E}\left[e^{itX}\right] = \begin{cases} \exp\left[-\sigma^\alpha |t|^\alpha \left(1 - i\beta \operatorname{sgn}(t) \tan\left(\frac{\pi\alpha}{2}\right)\right) + it\mu\right], & \alpha \neq 1 \\ \exp\left[-\sigma|t| \left(1 + i\beta \frac{2}{\pi} \operatorname{sgn}(t) \log|t|\right) + it\mu\right], & \alpha = 1, \end{cases} \tag{4.1}$$

with

$$\beta = 1,\ \sigma = \left(t \cos\frac{\pi\alpha}{2}\right)^{\frac{1}{\alpha}},\ \mu = 0.$$

Inverse α-stable subordinator

An inverse α-stable subordinator could be simulated by swapping the axes of an α-stable subordinator due to the nature of an inverse function.

Fractional homogeneous Poisson process

Simulating of the paths of the fractional homogeneous Poisson process (FHPP) is based on the following fact: the inter-arrival times $\{T_i;\ i \geq 1\}$ of FHPP have the Mittag-Leffler distribution. We employ the approach proposed in [4], where Cahoy, Uchaikin and Woyczynski used the following representation for $\{T_i;\ i \geq 1\}$:

$$T_i \stackrel{d}{=} T = \frac{|\ln U_1|^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}}} L_\alpha(1), \tag{4.2}$$

where $U_1 \stackrel{i.i.d.}{\sim} \mathrm{U}(0,1)$, and $L_\alpha(1)$ is an α-stable subordinator evaluated at $t = 1$. The following result from [6] provides an algorithm for simulation of the inter-arrival times of FHPP:

$$L_\alpha(1) \stackrel{d}{=} \frac{\sin(\alpha\pi U_2)\,[\sin((1-\alpha)\pi U_2)]^{\frac{1}{\alpha}-1}}{[\sin(\pi U_2)]\,|\ln U_3|^{\frac{1}{\alpha}-1}}, \tag{4.3}$$

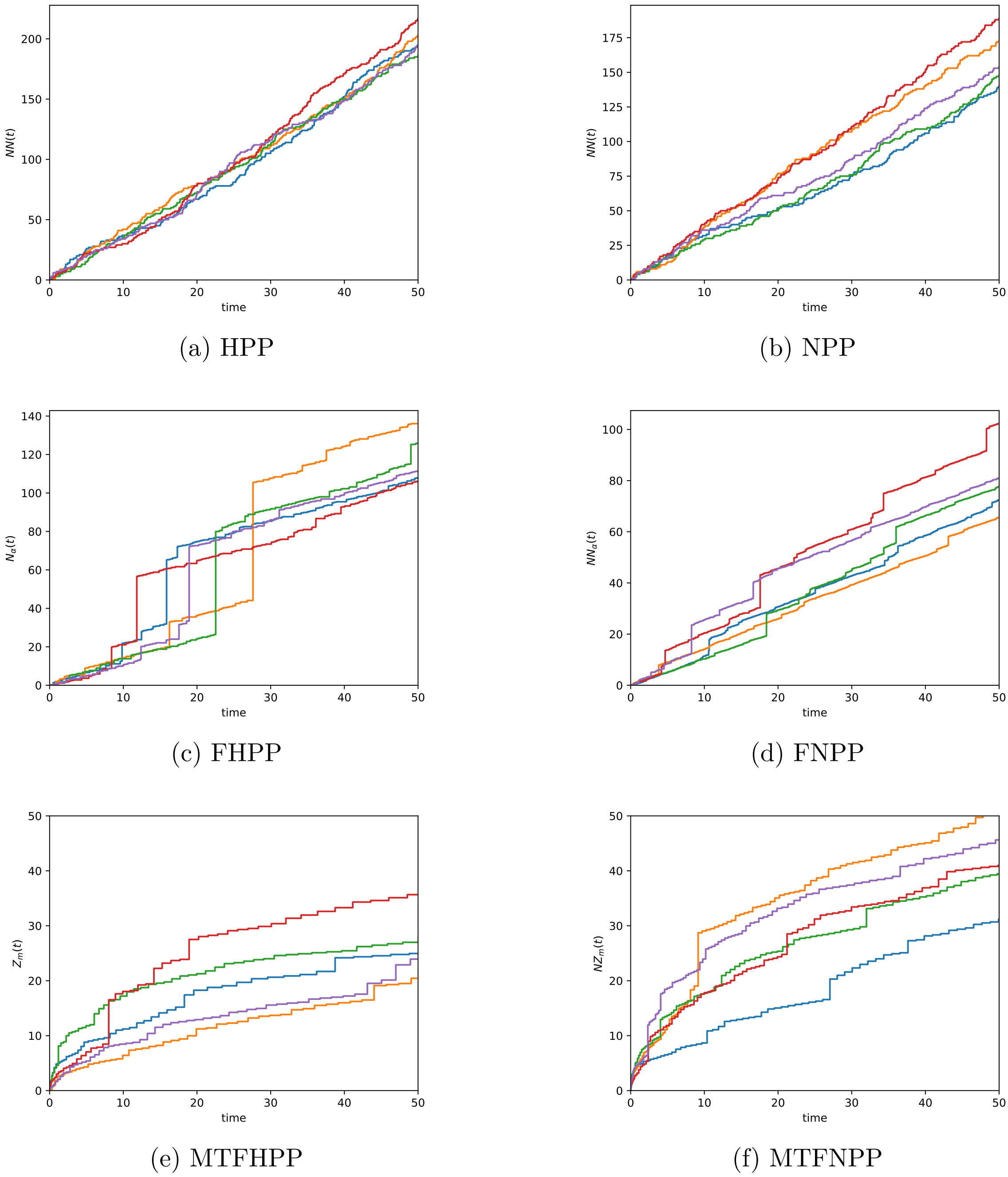
where $U_2, U_3 \stackrel{i.i.d.}{\sim} \mathrm{U}(0,1)$.

Fractional nonhomogeneous Poisson process

The simulation of FNPP could be done by using the same thinning algorithm to a FHPP, as shown in Algorithm 4, the results is shown in Figure 4.1d.

Mixed tempered fractional Poisson process

The simulations of the mixed tempered processes (MTFHPP and MTFNPP) could be done with the 'acceptance-rejection method' as shown in [16], and the algorithms could be found in Algorithms 5 and 6.



(a) HPP

(b) NPP

(c) FHPP

(d) FNPP

(e) MTFHPP

(f) MTFNPP

Fig. 4.1 Simulations of the processes

4.1.2 Simulation of ruin probability

Definition 4.1.1 (Inverse Gaussian distribution)**.** The probability density function of an inverse Gaussian (IG) distribution is given by (see [46])

$$f(x;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi x^3}}\exp\left[-\frac{\lambda}{2\mu^2}\frac{(x-\mu^2)}{x}\right], \quad x>0, \mu>0, \lambda>0, \tag{4.4}$$

with cumulative distribution function:

$$F(x;\mu,\lambda) = \Phi\left(\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}-1\right)\right) + \exp\left(\frac{2\lambda}{\mu}\right)\Phi\left(-\frac{\lambda}{x}\left(\frac{x}{\mu}+1\right)\right). \tag{4.5}$$

The Laplace transform of an inverse Gaussian distribution is given by (see [46])

$$\mathbb{E}\left[e^{-sX}\right] = \exp\left\{\frac{\lambda}{\mu}\left[1-\left(1+\frac{2\mu^2 s}{\lambda}\right)^{\frac{1}{2}}\right]\right\}. \tag{4.6}$$

The first two moments of an inverse Gaussian distribution is given by

$$\mathbb{E}\left[X\right] = \mu$$
$$\mathbb{E}\left[X^2\right] = \frac{\mu^2(\lambda+\mu)}{\lambda}$$

Definition 4.1.2 (Generalised Inverse Gaussian distribution)**.** The probability density function of a generalised inverse Gaussian (GIG) distribution is given by (see [14])

$$f(x;a,b,p) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})}x^{p-1}e^{-\frac{ax+b/x}{2}}, \ x>0, a>0, b>0, p\in\mathbb{R}, \tag{4.7}$$

where K_p is the modified Bessel function of the second kind, which is given by (see [52])

$$K_p(x) = \int_0^\infty e^{-x\cosh u}\cosh(pu)du, \ ,\Re(x)>0.$$

with cumulative distribution function:

$$F(x;a,b,p) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})}\left(\frac{2}{a}\right)^p\gamma\left(p,\frac{ax}{2},\frac{ab}{4}\right) \tag{4.8}$$

where $\gamma(\alpha, x, b)$ is the generalised lower incomplete gamma function, given by (see [7])

$$\gamma(\alpha, x, b) = \int_0^x t^{\alpha-1} e^{-t-\frac{b}{t}} dt.$$

The first two moments of a generalised inverse Gaussian distribution is given by

$$\mathbb{E}[X] = \frac{\sqrt{b}K_{p+1}(\sqrt{ab})}{\sqrt{a}K_p(\sqrt{ab})}$$
$$\mathbb{E}\left[X^2\right] = \frac{bK_{p+2}(\sqrt{ab})}{aK_p(\sqrt{ab})}.$$

Definition 4.1.3 (Exponentially modified Gaussian distribution)**.** The probability density function of a exponentially modified Gaussian distribution (exGaussian) (EMG) distribution is given by (see [15])

$$f(x; \mu, \sigma, \lambda) = \frac{\lambda}{2} \exp\left(2\mu + \lambda\sigma^2 - 2x\right) \operatorname{erfc}\left(\frac{\mu + \lambda\sigma^2 - x}{\sqrt{2}\sigma}\right), \quad x \in \mathbb{R}, \mu \in \mathbb{R}, \sigma > 0, \lambda > 0, \tag{4.9}$$

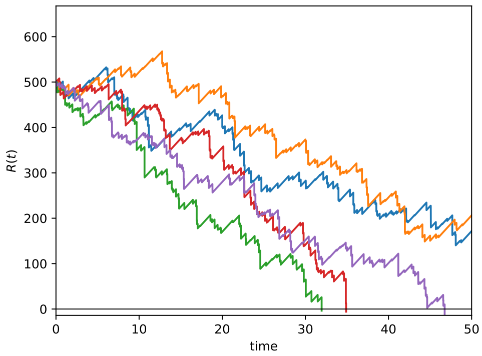
where $\operatorname{erf}(x)$ is the error function defined by $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, and $\operatorname{erfc}(x)$ is the complementary error function defined by $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$; with cumulative distribution function:

$$F(x; \mu, \sigma, \lambda) = \Phi(u, 0, v) - \exp\left(-\frac{u + v^2}{2 + \log(\Phi(u, v^2, v))}\right) \tag{4.10}$$

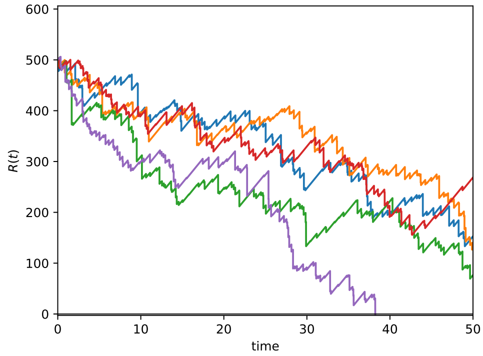
where $u = \lambda(x - \mu), v = \lambda\sigma$.

The first two moments of a exponentially modified Gaussian distribution is given by

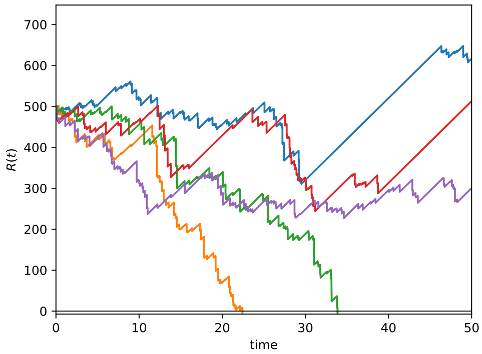
$$\mathbb{E}[X] = \mu + \frac{1}{\lambda}$$
$$\mathbb{E}\left[X^2\right] = \sigma^2 + \mu^2 + \frac{2(1 + \lambda\mu)}{\lambda^2}.$$



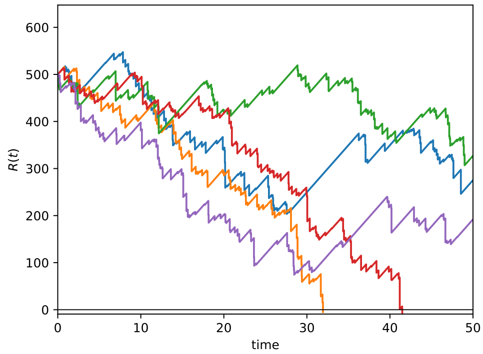
(a) HPP



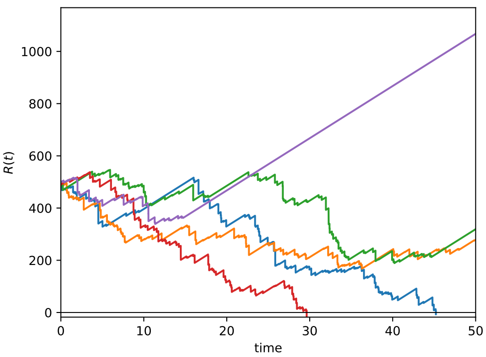
(b) NPP



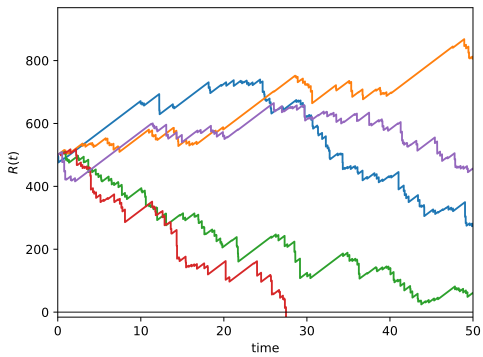
(c) FHPP



(d) FNPP



(e) MTFHPP



(f) MTFNPP

Fig. 4.2 Simulations of risk processes

4.2 Numerical investigation of the ruin probabilities and ruin times for the mixed risk model

4.2.1 Setups

In this section we adhere to the following assumptions (unless specified otherwise):

- the initial capital of the insurance company is set to $u = 500$;
- claim sizes $\{X_i\}_{i\geq 0}$ are *i.i.d.* exponentially distributed, with expected claim size equal to 10;
- the gross risk premium rate is set to $c = 20$, the safety loading factor is $\rho = 0.1$;
- the claims arrive according to the homogeneous Poisson process with intensity $\mu = 4$;
- for the non-homogeneous Poisson process we assume that the intensity function is

$$\lambda(t) = \frac{\gamma}{\beta}\left(\frac{t}{\beta}\right)^{\gamma-1} \qquad \gamma = 0.9,\ \beta = 0.2;$$

- the parameters of the mixture tempered stable inverse process are set as

$$\alpha_1 = 0.9, \alpha_2 = 0.5, \lambda_1 = 0.3, \lambda_2 = 0.7, c_1 = 0.75, c_2 = 0.25.$$

For the sake of simplicity we adhere to the following notation: the risk models (3.39) - (3.43) will be denoted by Model 1, Model 2, Model 3 and by Model 4 respectively. We first simulated 200 000 realisations of sample paths of each risk process $R_i(t)$, $i = 1, \dots 4$. Based on these simulations we created samples of ruin times and then we fitted different distributions to these samples. All available distributions in `scipy` package from python were used for distribution fitting using `scipy.stats.rv_continuous.fit`. The best fit is produced by (*i*) IG; (*ii*) GIG and (*iii*) the EMG with the following parameterisations:

(i) for the IG:

$$f(x;\delta,\eta) = \sqrt{\frac{\eta}{2\pi x^3}}\exp\left[-\frac{\eta}{2\delta^2}\frac{(x-\delta^2)}{x}\right],\ x > 0, \delta > 0, \eta > 0; \qquad (4.11)$$

(ii) GIG has the density:

$$f(x;a,b,p) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} e^{-\frac{ax+b/x}{2}}, \; x>0, a>0, b>0, p \in \mathbb{R}, \tag{4.12}$$

(iii) and for the EMG:

$$f(x;\kappa,\sigma,q_0) = \frac{q_0}{2} \exp\left(2\kappa + q_0\sigma^2 - 2x\right) \operatorname{erfc}\left(\frac{\kappa + q_0\sigma^2 - x}{\sqrt{2}\sigma}\right), \quad x \in \mathbb{R}, \kappa \in \mathbb{R}, \sigma > 0, q_0 > 0, \tag{4.13}$$

where $\operatorname{erf}(x)$ is the error function defined by $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, and $\operatorname{erfc}(x)$ is the complementary error function defined by $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$.

Table 4.1 contains the maximum likelihood estimators of the fitted distributions. Table 4.2 presents the errors and the goodness of fit measures such as the sum of squared error (SSE), Akaike information criterion (AIC), Bayesian information criterion (BIC), and the log-likelihood (LLH). As shown in Table 4.2, the GIG and IG have the lowest value of SSE, and therefore they will be used for further analyses in this paper.

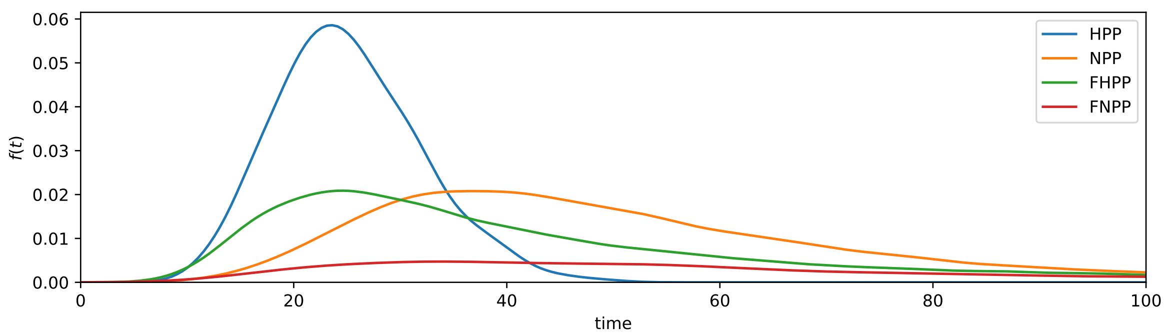


Fig. 4.3 Distributions of the time of ruin for HPP, NPP, FHPP, and FNPP

Further, we compared the behaviour of the finite-time ruin probabilities $\psi(u,t)$ as a function of some model parameters. To do this, we simulated the finite-time ruin probabilities obtained from Models 1-4.

Table 4.3 shows that under Model 2 and Model 4 the ruin probabilities are remarkably higher than those under Model 1 and Model 3. Simulations suggest that the ruin probabilities drop more rapidly as the initial capital u increases for Model 1 and Model

Table 4.1 Parameters of the fitted distributions using `scipy.stats.rv_continuous.fit`

Distribution	Parameters
Model 1	
IG	$\delta = 94.752, \eta = 164.627$
GIG	$a = 151.685, b = 0.020, p = -0.294$
EMG	$\kappa = 22.543, \sigma = 7.268, q_0 = 0.003$
Model 2	
IG	$\delta = 26.391, \eta = 38.051$
GIG	$a = 99.070, b = 2.960 \times 10^{-8}, p = -3.179$
EMG	$\kappa = 10.171, \sigma = 0.004, q_0 = 0.073$
Model 3	
IG	$\delta = 93.504, \eta = 158.924$
GIG	$a = 135.072, b = 0.022, p = -0.108$
EMG	$\kappa = 23.644, \sigma = 8.051, q_0 = 0.014$
Model 4	
IG	$\delta = 26.216, \eta = 38.680$
GIG	$a = 99.493, b = 3.888 \times 10^{-5}, p = -3.183$
EMG	$\kappa = 7.587, \sigma = 1.192, q_0 = 0.054$

Table 4.2 Statistics for fitted distributions of the time of ruin

Distribution	SSE	AIC	BIC
Model 1			
IG	3.592×10^{-6}	1888.216	-103635.007
GIG	$3.086 \times 10^{-6*}$	1867.221*	-104374.570^*
EMG	4.972×10^{-6}	1942.557	-102033.463
Model 2			
IG	$3.80 \times 10^{-5*}$	5600.448	$-8.987 \times 10^{5*}$
GIG	3.75×10^{-4}	2890.382*	-7.995×10^{5}
EMG	5.10×10^{-5}	10403.993	-8.852×10^{5}
Model 3			
IG	2.033×10^{-6}	1833.934*	-105738.071
GIG	$1.980 \times 10^{-6*}$	1840.089	-105859.845^*
EMG	6.098×10^{-6}	1870.819	-100360.953
Model 4			
IG	$3.60 \times 10^{-5*}$	5546.795	$-8.904 \times 10^{5*}$
GIG	3.40×10^{-4}	2887.251*	-7.953×10^{5}
EMG	4.70×10^{-5}	10502.987	-8.800×10^{5}

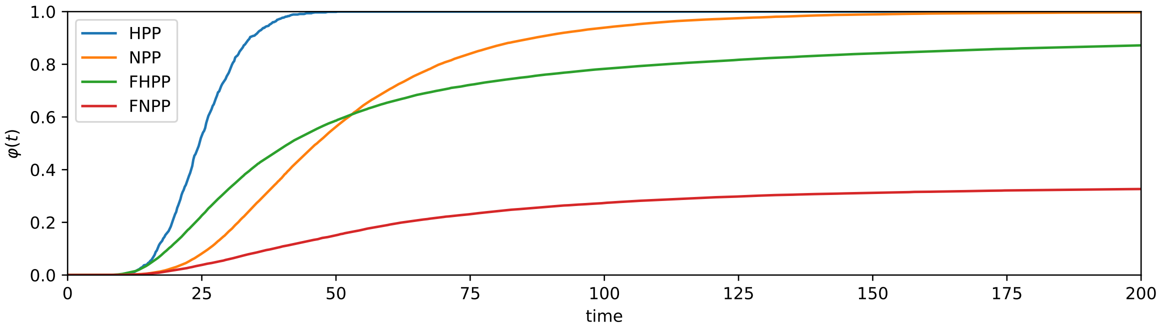
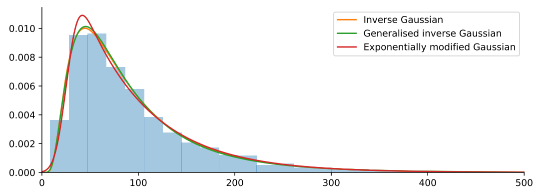
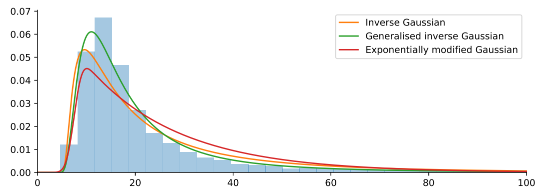


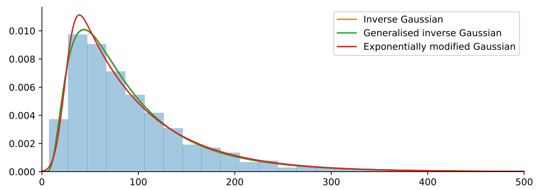
Fig. 4.4 Probability of ruin $\phi(t)$ for HPP, NPP, FHPP, and FNPP



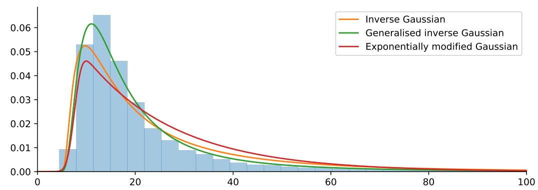
(a) Model 1



(b) Model 2



(c) Model 3



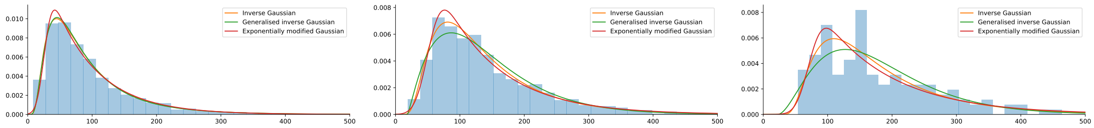
(d) Model 4

Fig. 4.5 Fitted distributions of simulations of time of ruin

3. By comparing $\psi(u, 2000)$ and $\psi(u, 100)$, we established that Model 1 and Model 3 have lower proportions of ruins occurred before time $t = 100$, which indicates that their distributions have heavier tails compared to those of Model 2 and Model 4.

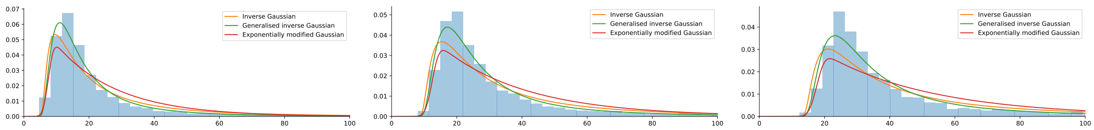
Figures 4.3 and 4.4 show the time of ruin and probabilities of ruin of HPP, NPP, FHPP, FNPP. Figure 4.5 shows the probability density functions of the ruin probability with their fitted distributions. Figures 4.6–4.9 contain the ruin probabilities of Model 1 (MTFNPP) and Model 3 (MTFHPP) with different initial capitals u. Figures 4.10–4.13 show the ruin probabilities of MTFHPP and MTFNPP with different values of α_1 and u under Models 1 - 4.

From Figures 4.10b, 4.11b, 4.12b and 4.13b, we observe the following behavior as u increases:



(a) $u = 500$ (b) $u = 750$ (c) $u = 1000$

Fig. 4.6 Fitted distributions of simulations of time of ruin for Model 1 with different initial capital values



(a) $u = 500$ (b) $u = 750$ (c) $u = 1000$

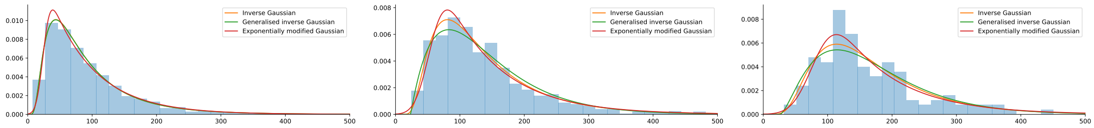
Fig. 4.7 Fitted distributions of simulations of time of ruin for Model 2 with different initial capital values

- the probabilities of ruin reduce;
- the distributions of time of ruin are less left skewed;
- the distributions of time of ruin have heavier tails.

Complete numerical results can be found in Appendix B.

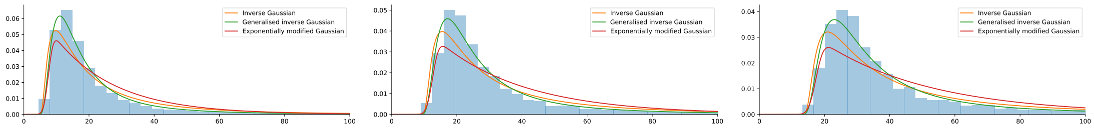
Table 4.3 Finite-time ruin probabilities $\psi(u, t)$

	Model 1	Model 2	Model 3	Model 4
$\psi(500, 2000)$	0.024635	0.0448	0.02448	0.044195
$\psi(500, 100)$	0.016425	0.04362	0.01626	0.04301
$\psi(750, 2000)$	0.00422	0.02429	0.00411	0.02441
$\psi(750, 100)$	0.00166	0.023135	0.001585	0.0232
$\psi(1000, 2000)$	0.00072	0.01591	0.00066	0.01649
$\psi(1000, 100)$	0.000165	0.01472	0.00013	0.015245



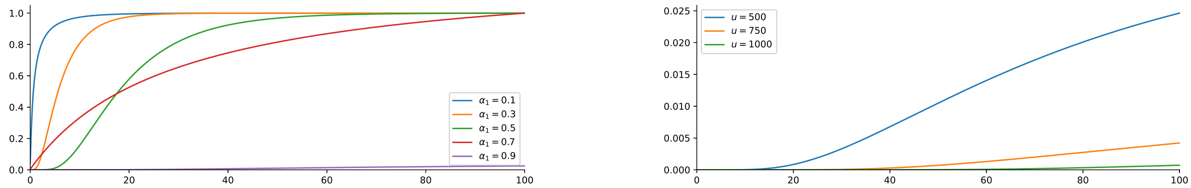
(a) $u = 500$ (b) $u = 750$ (c) $u = 1000$

Fig. 4.8 Fitted distributions of simulations of time of ruin for Model 3 for different initial capital values



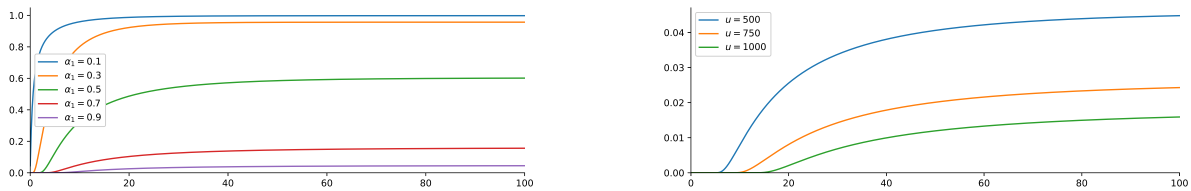
(a) $u = 500$ (b) $u = 750$ (c) $u = 1000$

Fig. 4.9 Fitted distributions of simulations of time of ruin for Model 4 for different initial capital values



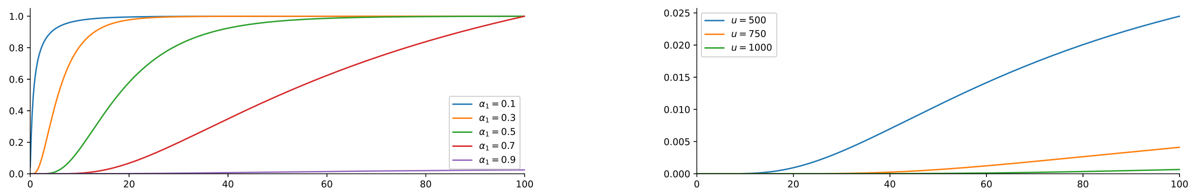
(a) α_1 (b) u

Fig. 4.10 Probability of ruin of Model 1 under different parameterisations



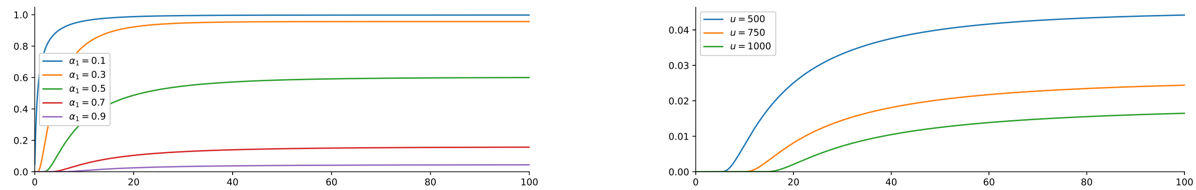
(a) α_1 (b) u

Fig. 4.11 Probability of ruin of Model 2 under different parameterisations



(a) α_1 (b) u

Fig. 4.12 Probability of ruin of model 3 under different parameterisations



(a) α_1 (b) u

Fig. 4.13 Probability of ruin of Model 4 under different parameterisations

Chapter 5

Conclusions

In this thesis, we introduce the mixed tempered fractional Poisson process, which would be more flexible in real-life modelling due to the extra parameters.

This thesis had the aim of exploring the relationship between ruin probabilities and the newly introduced mixed tempered fractional risk models.

5.1 Summaries

The results in Section 4.2 suggest that the inverse Gaussian distribution fits well for the risk models under mixed tempered fractional Poisson process, especially model 1 and 3, but also the other processes as they are special cases within MTFPP.

Furthermore, there are noteworthy drops in ruin probabilities of model 1 and 3 as the initial capital increase in comparison to the other models.

Although the exact formula of the risk process remains a complex open problem, we propose a simple but practical method for time of ruin and finite-time ruin probability approximation in Chapter 4, with the use of the inverse Gaussian and generalised inverse Gaussian distribution, it is far less complex and time consuming than full Monte Carlo simulation.

5.2 Limitations

Insurance companies are often not received the premium payments continuously and uniformly throughout the year in real life, e.g. travel insurance tends to have higher number of claims and more policies are bought during summer, i.e. more claims and higher premium rate. Also, the use of exponential distribution for claim sizes X_i in Section 4.1.2 might not be suitable for some insurance providers.

However, for simplicity, and also to reduce the need of computational power, the Monte Carlo simulations in Section 4.2 are done in the assumption that the premium payments are received continuously and uniformly with exponentially distributed claim sizes.

5.3 Future work

We believe that there are still many open problems for further research. We suggest that further research may be carried out in the following areas:

1. For insurance providers which have higher proportion of seasonal policies, we could extend the model that was discussed in Equation (3.39),

$$R(t) = u + c(t) - \sum_{i=1}^{NZ_m(t)} X_i, \tag{5.1}$$

   where $c(t)$ is the premium rate function over time. Also, a periodic function could be useful for $\lambda(t)$ as the distribution of claim sizes and the number of claims over each calender year are usually very similar every year.

2. For insurance companies which have high proportion of lower claim sizes, lighter-tailed distributions would be more suitable, e.g. generalised rectified Gaussian distribution [42].

3. The obtained results in this paper do not include the effects of reinsurance or copayment which are common in some types of policies.

4. Further research may be carried out in the relationship between the collective risk model used in this paper, and the individual risk model [53].

5. Mixture of more than 2 TSSs could be introduced in further studies.

6. The theory for ruin probability and time of ruin for the risk models with some particular non-local Poisson processes.

7. Translated gamma approximation discussed in [12] could also be further studied for non-local processes.

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Appendix A

Simulations' algorithms

Algorithm 1: Simulation of Homogeneous Poisson process with rate μ, on $[0, T]$ [8]

Input: μ, T

Step 1: generate increments of $Y_n \sim \text{Exp}(\mu)$ (see Propositions 3.1.1 and 3.1.2);

Step 2: sum up the increments to give $N^\mu(t)$;

Output: $N^\mu(t)$

Algorithm 2: Simulation of Nonhomogeneous Poisson process with time dependent rate $\lambda(t)$, on $[0, T]$ [8]

Input: $\lambda(t), T$

Step 1: generate increments of $Y_n \sim \text{Exp}(\mu)$;

Step 2: generate $NN^\mu(t)$ using the following 'thinning algorithm':

(i) generate Y_n;

(ii) generate $U \overset{i.i.d.}{\sim} \text{U}(0, 1)$;

(iii) if $U \leq \frac{\lambda(t)}{\sup_{0 \leq t \leq T} \lambda(t)}$, accept Y_n;

(iv) goto *(i)*;

Step 3: sum up the increments to give $NN^\mu(t)$;

Output: $NN^\mu(t)$

Algorithm 3: Simulation of Fractional Homogeneous Poisson process with rate λ, and stability index α, on $[0, T]$ [4], [33]

Input: μ, T, α

Step 1: generate $U_1, U_2, U_3, \overset{i.i.d.}{\sim} \mathrm{U}(0,1)$;

Step 2: generate increments of Y_n, where

$$Y_n = \frac{|\ln U_1|^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}}} \frac{\sin(\alpha\pi U_2)\,[\sin((1-\alpha)\pi U_2)]^{\frac{1}{\alpha}-1}}{[\sin(\pi U_2)]\,|\ln U_3|^{\frac{1}{\alpha}-1}};$$

Step 3: sum up the increments to give $N_\alpha(t)$;

Output: $N_\alpha(t)$

Algorithm 4: Simulation of Fractional Nonhomogeneous Poisson process with time dependent rate $\lambda(t)$, and stability index α, on $[0, T]$

Input: $\lambda(t), T, \alpha$

Step 1: generate $U_1, U_2, U_3, \overset{i.i.d.}{\sim} \mathrm{U}(0,1)$;

Step 2: generate increments of Y_n, where

$$Y_n = \frac{|\ln U_1|^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}}} \frac{\sin(\alpha\pi U_2)\,[\sin((1-\alpha)\pi U_2)]^{\frac{1}{\alpha}-1}}{[\sin(\pi U_2)]\,|\ln U_3|^{\frac{1}{\alpha}-1}};$$

Step 3: apply the thinning algorithm (see Algorithm 2);

Step 4: sum up the increments to give $N_\alpha(t)$;

Output: $N_\alpha(t)$

Algorithm 5: Simulation of mixture tempered fractional Poisson processes with rate μ or $\Lambda(t)$, and fractional indices α_1, α_2, tempered indices λ_1, λ_2, and mixture indices c_1, c_2, on $[0, T]$ [16]

Input: $\mu, T, \alpha_1, \alpha_2, \lambda_1, \lambda_2, c_1, c_2$

Step 1: generate $U_1, U_2 \overset{i.i.d.}{\sim} \mathrm{U}\,(0,1)$;

Step 2: generate increments of the α-stable subordinator $S_\alpha(t)$ using $S_\alpha(t+dt) - S_\alpha(t) \overset{d}{=} S_\alpha(dt) \overset{d}{=} (dt)^{1/\alpha}$, where

$$S_\alpha(1) \overset{d}{=} \frac{\sin\left(\alpha\pi U_1\right)\left[\sin\left(1-\alpha\right)\pi U_1\right]^{1/\alpha-1}}{\left[\sin\left(\pi U_1\right)\right]^{1/\alpha}\left|\ln U_2\right|^{1/\alpha-1}};$$

Step 3: generate $S_1(c_1 t)$ and $S_2(c_2 t)$ using the following 'accept-reject algorithm':

(i) generate $S_\alpha(dt)$;

(ii) generate $U_3 \overset{i.i.d.}{\sim} \mathrm{U}\,(0,1)$, if $U_3 \leq e^{-\lambda S_\alpha(dt)}$, accept $S_\alpha(dt)$; otherwise, go back to *(i)*;

(iii) sum up the increments to get $S_1(c_1 t)$ and repeat to get $S_2(c_2 t)$;

Step 4: generate MTSS $S_m(t)$ by summing up $S_1(c_1 t)$ and $S_2(c_2 t)$ (see Equation 2.40);

Step 5: generate IMTSS $Y_m(t)$ by inverting the axes;

Step 6: generate MTFHPP $NZ_m(t)$ using Equation 3.24;

Output: $Z_m(t)$

Algorithm 6: Simulation of mixture tempered fractional Poisson processes with rate μ or $\Lambda(t)$, and fractional indices α_1, α_2, tempered indices λ_1, λ_2, and mixture indices c_1, c_2, on $[0, T]$ [16]

Input: $\lambda(t), T, \alpha_1, \alpha_2, \lambda_1, \lambda_2, c_1, c_2$

Step 1: generate $U_1, U_2 \overset{i.i.d.}{\sim} \mathrm{U}\,(0, 1)$;

Step 2: generate increments of the α-stable subordinator $S_\alpha(t)$ using $S_\alpha(t + dt) - S_\alpha(t) \overset{d}{=} S_\alpha(dt) \overset{d}{=} (dt)^{1/\alpha}$, where

$$S_\alpha(1) \overset{d}{=} \frac{\sin(\alpha\pi U_1)\left[\sin(1-\alpha)\pi U_1\right]^{1/\alpha-1}}{\left[\sin(\pi U_1)\right]^{1/\alpha}\left|\ln U_2\right|^{1/\alpha-1}};$$

Step 3: apply the thinning algorithm (see Algorithm 2);

Step 4: generate $S_1(c_1t)$ and $S_2(c_2t)$ using the following 'accept-reject algorithm':

(i) generate $S_\alpha(dt)$;

(ii) generate $U_3 \overset{i.i.d.}{\sim} \mathrm{U}\,(0, 1)$, if $U_3 \leq e^{-\lambda S_\alpha(dt)}$, accept $S_\alpha(dt)$; otherwise, go back to *(i)*;

(iii) sum up the increments to get $S_1(c_1t)$ and repeat to get $S_2(c_2t)$;

Step 5: generate MTSS $S_m(t)$ by summing up $S_1(c_1t)$ and $S_2(c_2t)$ (see Equation 2.40);

Step 6: generate IMTSS $Y_m(t)$ by inverting the axes;

Step 7: generate MTFNPP $NZ_m(t)$ using Equation 3.27;

Output: $NZ_m(t)$

Appendix B

Full numerical results

Table B.1 Ruin probabilities with substituted α_1 and α_2

	α_1					α_2				
	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
model 1										
$\psi(500, 2000)$	1	1	1	1	0.024635	0.024515	0.02424	0.024635	0.025235	0.02528
$\psi(500, 100)$	1	1	0.999245	0.62831	0.016425	0.016135	0.015885	0.016425	0.016585	0.016975
$\psi(750, 2000)$	1	1	1	0.99999	0.00422	0.00423	0.004125	0.00422	0.00408	0.004265
$\psi(750, 100)$	1	1	0.997115	0.38287	0.00166	0.00175	0.001595	0.00166	0.001545	0.0016
$\psi(1000, 2000)$	1	1	1	0.99999	0.00072	0.000875	0.00072	0.00072	0.000645	0.00077
$\psi(1000, 100)$	1	1	0.991185	0.189275	0.000165	0.00013	0.000125	0.000165	0.0001	0.00013
model 2										
$\psi(500, 2000)$	0.9979	0.95665	0.60129	0.15593	0.0448	0.044855	0.044575	0.0448	0.04438	0.044
$\psi(500, 100)$	0.9969	0.953405	0.594195	0.152305	0.04362	0.0435	0.04339	0.04362	0.043075	0.04281
$\psi(750, 2000)$	0.997685	0.93539	0.429315	0.083135	0.02429	0.02464	0.024705	0.02429	0.02444	0.024175
$\psi(750, 100)$	0.996535	0.930115	0.41913	0.079155	0.023135	0.02331	0.02349	0.023135	0.0231	0.02291
$\psi(1000, 2000)$	0.997485	0.91031	0.30683	0.055145	0.01591	0.016625	0.016445	0.01591	0.01667	0.016335
$\psi(1000, 100)$	0.99621	0.902835	0.294325	0.05099	0.01472	0.015285	0.01522	0.01472	0.01531	0.015055
model 3										
$\psi(500, 2000)$	1	1	1	1	0.02448	0.02521	0.02502	0.02448	0.02522	0.024315
$\psi(500, 100)$	1	1	0.99925	0.628715	0.01626	0.01672	0.016395	0.01626	0.01651	0.01593
$\psi(750, 2000)$	1	1	1	1	0.00411	0.00429	0.004245	0.00411	0.004105	0.00394
$\psi(750, 100)$	1	1	0.99733	0.38232	0.001585	0.00173	0.00174	0.001585	0.001585	0.001545
$\psi(1000, 2000)$	1	1	1	1	0.00066	0.000685	0.000845	0.00066	0.00068	0.00064
$\psi(1000, 100)$	1	1	0.991465	0.18908	0.00013	0.000125	0.00016	0.00013	0.000145	0.00009
model 4										
$\psi(500, 2000)$	0.997895	0.95672	0.59985	0.156675	0.044195	0.04464	0.044095	0.044195	0.043765	0.044225
$\psi(500, 100)$	0.996875	0.953735	0.59346	0.153315	0.04301	0.04343	0.04294	0.04301	0.0425	0.043015
$\psi(750, 2000)$	0.99763	0.93537	0.42681	0.08326	0.02441	0.02466	0.02465	0.02441	0.02403	0.024275
$\psi(750, 100)$	0.99649	0.93065	0.417205	0.07968	0.0232	0.023405	0.02346	0.0232	0.02275	0.02302
$\psi(1000, 2000)$	0.99744	0.910655	0.306885	0.05536	0.01649	0.016655	0.016605	0.01649	0.016035	0.01594
$\psi(1000, 100)$	0.996145	0.90372	0.295105	0.051675	0.015245	0.01539	0.015385	0.015245	0.01473	0.014625

Table B.2 Ruin probabilities with substituted λ_1 and λ_2

	α_1					α_2				
	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
model 1										
$\psi(500, 2000)$	0.00187	0.024635	0.218995	0.90633	0.99998	0.003025	0.00672	0.01302	0.024635	0.04448
$\psi(500, 100)$	0.001755	0.016425	0.074645	0.21114	0.42669	0.002695	0.00545	0.00987	0.016425	0.025825
$\psi(750, 2000)$	0.00009	0.00422	0.10491	0.848245	0.99995	0.000185	0.000555	0.00158	0.00422	0.00968
$\psi(750, 100)$	0.00007	0.00166	0.01323	0.06067	0.17547	0.00014	0.000305	0.000765	0.00166	0.003035
$\psi(1000, 2000)$	0.00001	0.00072	0.049985	0.7837	0.99986	0.000005	0.000035	0.000215	0.00072	0.002115
$\psi(1000, 100)$	0.000005	0.000165	0.001725	0.011755	0.052075	0	0.000005	0.000045	0.000165	0.000255
model 2										
$\psi(500, 2000)$	0.05172	0.0448	0.04338	0.045505	0.04938	0.04145	0.040745	0.04289	0.0448	0.04666
$\psi(500, 100)$	0.04914	0.04362	0.042545	0.04477	0.048795	0.04014	0.03965	0.04176	0.04362	0.045475
$\psi(750, 2000)$	0.031885	0.02429	0.02247	0.02245	0.02301	0.022725	0.022355	0.023845	0.02429	0.02575
$\psi(750, 100)$	0.02923	0.023135	0.02163	0.02171	0.02235	0.0214	0.02127	0.022695	0.023135	0.024535
$\psi(1000, 2000)$	0.022705	0.01591	0.0145	0.01403	0.01428	0.01539	0.01477	0.015815	0.01591	0.01734
$\psi(1000, 100)$	0.020015	0.01472	0.01366	0.013285	0.01361	0.014025	0.013585	0.014565	0.01472	0.016075
model 3										
$\psi(500, 2000)$	0.00203	0.02448	0.219865	0.905655	0.99994	0.002905	0.00709	0.013225	0.02448	0.04468
$\psi(500, 100)$	0.001885	0.01626	0.07502	0.21098	0.426665	0.00267	0.00583	0.00995	0.01626	0.02597
$\psi(750, 2000)$	0.00011	0.00411	0.104535	0.849115	0.999885	0.000165	0.000585	0.00163	0.00411	0.00992
$\psi(750, 100)$	0.000075	0.001585	0.013425	0.060395	0.176805	0.000135	0.000385	0.000765	0.001585	0.003105
$\psi(1000, 2000)$	0.00001	0.00066	0.04992	0.784365	0.999815	0.000005	0.0004	0.0002	0.00066	0.00225
$\psi(1000, 100)$	0.000005	0.00013	0.001625	0.012005	0.0516	0	0.000015	0.000035	0.00013	0.00026
model 4										
$\psi(500, 2000)$	0.053195	0.044195	0.04431	0.04629	0.049345	0.040565	0.04081	0.04348	0.044195	0.046595
$\psi(500, 100)$	0.05054	0.04301	0.043355	0.04556	0.048705	0.03937	0.039655	0.042325	0.04301	0.04545
$\psi(750, 2000)$	0.032735	0.02441	0.022945	0.02244	0.02319	0.02214	0.022745	0.02417	0.02441	0.025885
$\psi(750, 100)$	0.03007	0.0232	0.021975	0.02169	0.02254	0.020945	0.02156	0.02298	0.0232	0.024695
$\psi(1000, 2000)$	0.023515	0.01649	0.014675	0.014005	0.01427	0.01503	0.01517	0.01584	0.01649	0.017565
$\psi(1000, 100)$	0.02072	0.015245	0.013725	0.01325	0.0136	0.0138	0.013875	0.01467	0.015245	0.01637

Table B.3 Ruin probabilities with substituted c_1 and c_2

	c_1=0.25	c_1=0.5	c_1=0.75
model 1			
$\psi(500, 2000)$	0.373705	0.095175	0.024635
$\psi(500, 100)$	0.103525	0.044965	0.016425
$\psi(750, 2000)$	0.22971	0.03029	0.00422
$\psi(750, 100)$	0.021495	0.00635	0.00166
$\psi(1000, 2000)$	0.139765	0.009415	0.00072
$\psi(1000, 100)$	0.00311	0.000675	0.000165
model 2			
$\psi(500, 2000)$	0.042295	0.02448	0.0448
$\psi(500, 100)$	0.041625	0.02397	0.04362
$\psi(750, 2000)$	0.02097	0.013115	0.02429
$\psi(750, 100)$	0.020265	0.01258	0.023135
$\psi(1000, 2000)$	0.013345	0.00876	0.01591
$\psi(1000, 100)$	0.01261	0.008215	0.01472
model 3			
$\psi(500, 2000)$	0.373185	0.095175	0.02448
$\psi(500, 100)$	0.10449	0.04349	0.01626
$\psi(750, 2000)$	0.22966	0.03084	0.00411
$\psi(750, 100)$	0.02149	0.006105	0.001585
$\psi(1000, 2000)$	0.14103	0.01006	0.00066
$\psi(1000, 100)$	0.003365	0.00066	0.00013
model 4			
$\psi(500, 2000)$	0.04241	0.02421	0.044195
$\psi(500, 100)$	0.041645	0.023715	0.04301
$\psi(750, 2000)$	0.02069	0.01325	0.02441
$\psi(750, 100)$	0.019905	0.01277	0.0232
$\psi(1000, 2000)$	0.013355	0.00852	0.01649
$\psi(1000, 100)$	0.01252	0.00805	0.015245

Table B.4 Parameters of fitted distributions with default parameters (model 1)

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
$u = 500$		
α_1=0.1	1.729972	0.163872
α_1=0.3	6.948907	12.860763
α_1=0.5	20.908181	56.789506
α_1=0.7	107.757513	119.490737
α_1=0.9	94.751570	164.626525
α_2=0.1	96.186092	165.567190
α_2=0.3	94.831148	159.865012
α_2=0.5	94.751570	164.626525
α_2=0.7	95.839577	157.859758
α_2=0.9	93.979272	159.765785
λ_1=0.1	50.867421	177.220949
λ_1=0.3	94.751570	164.626525
λ_1=0.5	238.836382	178.097994
λ_1=0.7	401.791533	207.650535
λ_1=0.9	164.386916	183.951069
λ_2=0.1	56.438122	152.149749
λ_2=0.3	68.948354	147.452805
λ_2=0.5	79.918747	162.167089
λ_2=0.7	94.751570	164.626525
λ_2=0.9	113.424835	165.079768
c_1=0.25	325.011451	188.617942
c_1=0.5	151.226695	170.527887
c_1=0.75	94.751570	164.626525
$u = 750$		
α_1=0.1	2.555041	0.626731
α_1=0.3	10.234961	28.882778
α_1=0.5	30.934851	124.515495
α_1=0.7	159.780635	263.694484
α_1=0.9	141.474429	360.521813

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Table B.4 – continued from previous page

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
α_2=0.1	141.098453	359.667019
α_2=0.3	138.528194	335.361465
α_2=0.5	141.474429	360.521813
α_2=0.7	139.645979	364.252405
α_2=0.9	141.006311	341.317003
λ_1=0.1	73.460819	383.213809
λ_1=0.3	141.474429	360.521813
λ_1=0.5	351.620543	399.055337
λ_1=0.7	563.954678	487.120537
λ_1=0.9	243.765766	409.853490
λ_2=0.1	85.764071	419.487770
λ_2=0.3	104.737569	386.064342
λ_2=0.5	120.719256	376.950985
λ_2=0.7	141.474429	360.521813
λ_2=0.9	167.716522	361.646135
c_1=0.25	467.946740	430.665416
c_1=0.5	225.595726	376.663208
c_1=0.75	141.474429	360.521813
$u = 1000$		
α_1=0.1	3.366629	1.378773
α_1=0.3	13.466059	50.584546
α_1=0.5	40.814873	216.095932
α_1=0.7	211.638625	460.340812
α_1=0.9	180.856516	600.071880
α_2=0.1	179.325641	650.471084
α_2=0.3	189.367231	631.319072
α_2=0.5	180.856516	600.071880
α_2=0.7	189.052168	658.643196
α_2=0.9	193.197130	594.478922
λ_1=0.1	87.580711	3433.771089
λ_1=0.3	180.856516	600.071880

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Table B.4 – continued from previous page

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
λ_1=0.5	460.306563	712.559314
λ_1=0.7	710.746326	924.525727
λ_1=0.9	322.437436	722.582631
λ_2=0.1	unknown	unknown
λ_2=0.3	143.349094	1904.818292
λ_2=0.5	153.228757	835.820697
λ_2=0.7	180.856516	600.071880
λ_2=0.9	225.601339	605.589345
c_1=0.25	597.427959	798.859329
c_1=0.5	297.381151	647.705621
c_1=0.75	180.856516	600.071880

Table B.5 Parameters of fitted distributions with default parameters (model 2)

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
$u = 500$		
α_1=0.1	1.978713	0.145692
α_1=0.3	6.155867	6.869668
α_1=0.5	13.901867	15.415544
α_1=0.7	21.761345	24.802214
α_1=0.9	26.390501	38.051303
α_2=0.1	27.677103	35.185516
α_2=0.3	26.425973	38.546059
α_2=0.5	26.390501	38.051303
α_2=0.7	26.838277	37.254953
α_2=0.9	26.315400	38.194973
λ_1=0.1	36.105480	39.119173
λ_1=0.3	26.390501	38.051303
λ_1=0.5	22.030985	38.554697

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Table B.5 – continued from previous page

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
λ_1=0.7	20.230347	36.222607
λ_1=0.9	18.416626	35.598887
λ_2=0.1	28.552110	40.264927
λ_2=0.3	26.528826	41.710645
λ_2=0.5	26.468013	39.962175
λ_2=0.7	26.390501	38.051303
λ_2=0.9	26.104646	36.623078
c_1=0.25	21.184601	37.175090
c_1=0.5	23.076361	38.783114
c_1=0.75	26.390501	38.051303
u = 750		
α_1=0.1	2.765155	0.563039
α_1=0.3	9.443699	12.538340
α_1=0.5	22.688377	24.977842
α_1=0.7	35.393897	41.164240
α_1=0.9	38.984329	62.818654
α_2=0.1	41.886873	57.418830
α_2=0.3	39.313055	62.882382
α_2=0.5	38.984329	62.818654
α_2=0.7	40.316835	61.512244
α_2=0.9	39.496764	63.473851
λ_1=0.1	51.955311	65.473105
λ_1=0.3	38.984329	62.818654
λ_1=0.5	33.495066	64.645329
λ_1=0.7	31.481985	59.652981
λ_1=0.9	29.439696	55.865323
λ_2=0.1	42.790942	66.681207
λ_2=0.3	39.328498	69.104184
λ_2=0.5	39.266339	66.539661
λ_2=0.7	38.984329	62.818654
λ_2=0.9	39.505285	59.313581

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Table B.5 – continued from previous page

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
c_1=0.25	32.973702	60.764216
c_1=0.5	34.741395	65.117889
c_1=0.75	38.984329	62.818654
$u = 1000$		
α_1=0.1	3.505921	1.228717
α_1=0.3	12.764805	18.618484
α_1=0.5	32.791577	34.315365
α_1=0.7	47.985459	57.920695
α_1=0.9	51.342390	87.608275
α_2=0.1	54.698405	77.982701
α_2=0.3	51.369249	85.674501
α_2=0.5	51.342390	87.608275
α_2=0.7	52.533037	85.193807
α_2=0.9	51.024759	88.726060
λ_1=0.1	66.537056	89.540016
λ_1=0.3	51.342390	87.608275
λ_1=0.5	43.687266	90.030634
λ_1=0.7	41.959928	81.237019
λ_1=0.9	39.751174	76.091345
λ_2=0.1	56.054164	92.048740
λ_2=0.3	51.704456	94.146151
λ_2=0.5	51.593545	90.694492
λ_2=0.7	51.342390	87.608275
λ_2=0.9	51.322267	81.175297
c_1=0.25	44.087969	82.388442
c_1=0.5	45.467919	91.174670
c_1=0.75	51.342390	87.608275

Table B.6 Parameters of fitted distributions with default parameters (model 3)

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
$u = 500$		
α_1=0.1	1.736544	0.161875
α_1=0.3	6.931490	12.907127
α_1=0.5	20.863197	56.630527
α_1=0.7	108.422033	119.023242
α_1=0.9	93.504016	158.924275
α_2=0.1	93.743205	160.274760
α_2=0.3	95.504299	163.632697
α_2=0.5	93.504016	158.924275
α_2=0.7	94.601584	168.793540
α_2=0.9	94.984285	156.547996
λ_1=0.1	49.585078	141.541559
λ_1=0.3	93.504016	158.924275
λ_1=0.5	241.039832	176.352383
λ_1=0.7	401.690447	208.811837
λ_1=0.9	164.058703	184.016630
λ_2=0.1	55.934370	170.507499
λ_2=0.3	67.717129	152.813741
λ_2=0.5	79.435996	162.824444
λ_2=0.7	93.504016	158.924275
λ_2=0.9	111.593350	167.446835
c_1=0.25	322.703787	189.122655
c_1=0.5	154.198241	171.124131
c_1=0.75	93.504016	158.924275
$u = 750$		
α_1=0.1	2.570388	0.624654
α_1=0.3	10.224518	28.897045
α_1=0.5	30.852025	124.012570
α_1=0.7	160.407647	262.474519
α_1=0.9	139.559746	371.002633

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Table B.6 – continued from previous page

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
α_2=0.1	137.459643	358.439063
α_2=0.3	137.008760	327.395095
α_2=0.5	139.559746	371.002633
α_2=0.7	138.697805	347.513061
α_2=0.9	139.231662	386.475193
λ_1=0.1	87.289769	473.240187
λ_1=0.3	139.559746	371.002633
λ_1=0.5	353.492403	395.139156
λ_1=0.7	564.995102	489.344836
λ_1=0.9	243.063599	409.232935
λ_2=0.1	76.616405	359.947592
λ_2=0.3	97.898509	364.438426
λ_2=0.5	120.447093	362.036079
λ_2=0.7	139.559746	371.002633
λ_2=0.9	164.149611	366.846586
c_1=0.25	466.120151	435.295913
c_1=0.5	228.767093	382.226798
c_1=0.75	139.559746	371.002633
u = 1000		
α_1=0.1	3.376175	1.378184
α_1=0.3	13.467178	50.379486
α_1=0.5	40.767683	216.125647
α_1=0.7	212.245295	458.294992
α_1=0.9	169.613602	560.698531
α_2=0.1	189.397460	560.073063
α_2=0.3	177.511707	603.975238
α_2=0.5	169.613602	560.698531
α_2=0.7	178.709830	586.213447
α_2=0.9	188.181705	806.270969
λ_1=0.1	103.545402	806.642560
λ_1=0.3	169.613602	560.698531

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Table B.6 – continued from previous page

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
λ_1=0.5	468.260981	722.099034
λ_1=0.7	709.528981	925.983689
λ_1=0.9	322.052542	724.029683
λ_2=0.1	unknown	unknown
λ_2=0.3	134.181191	613.916658
λ_2=0.5	166.415997	918.439923
λ_2=0.7	169.613602	560.698531
λ_2=0.9	215.795323	691.987799
c_1=0.25	597.753722	798.033270
c_1=0.5	302.456751	666.933403
c_1=0.75	169.613602	560.698531

Table B.7 Parameters of fitted distributions with default parameters (model 4)

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
$u = 500$		
α_1=0.1	1.944541	0.146251
α_1=0.3	6.133941	6.918411
α_1=0.5	13.740798	15.685142
α_1=0.7	20.856469	25.904670
α_1=0.9	26.215863	38.679535
α_2=0.1	26.111083	39.258574
α_2=0.3	26.352391	37.955265
α_2=0.5	26.215863	38.679535
α_2=0.7	28.287866	35.505485
α_2=0.9	26.916467	36.809816
λ_1=0.1	35.309804	39.931642
λ_1=0.3	26.215863	38.679535
λ_1=0.5	23.632937	35.058761

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Table B.7 – continued from previous page

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
λ_1=0.7	20.542332	35.101576
λ_1=0.9	18.006967	37.356901
λ_2=0.1	28.459905	39.901391
λ_2=0.3	27.771790	39.490623
λ_2=0.5	26.232487	40.310410
λ_2=0.7	26.215863	38.679535
λ_2=0.9	25.222393	38.945344
c_1=0.25	21.422327	36.541342
c_1=0.5	22.949237	39.657906
c_1=0.75	26.215863	38.679535
$u = 750$		
α_1=0.1	2.721104	0.561131
α_1=0.3	9.351144	12.734376
α_1=0.5	22.528616	25.196908
α_1=0.7	33.453887	43.636532
α_1=0.9	39.336248	63.806251
α_2=0.1	38.726392	65.461947
α_2=0.3	39.322319	61.442361
α_2=0.5	39.336248	63.806251
α_2=0.7	42.828310	57.437312
α_2=0.9	40.539829	59.525870
λ_1=0.1	50.050510	67.815398
λ_1=0.3	39.336248	63.806251
λ_1=0.5	36.319014	56.589791
λ_1=0.7	32.056950	55.632292
λ_1=0.9	27.857572	60.514024
λ_2=0.1	42.499186	66.146091
λ_2=0.3	41.214940	64.978841
λ_2=0.5	38.938678	67.359193
λ_2=0.7	39.336248	63.806251
λ_2=0.9	37.363169	64.452708

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	Parameters for fitted Inverse Gaussian distribution	
	δ	η
c_1=0.25	33.537896	57.131860
c_1=0.5	33.760255	66.421843
c_1=0.75	39.336248	63.806251
u = 1000		
α_1=0.1	3.467921	1.245402
α_1=0.3	12.739262	18.612081
α_1=0.5	32.172029	34.896776
α_1=0.7	45.065155	61.314441
α_1=0.9	51.010255	88.889889
α_2=0.1	50.120573	91.626150
α_2=0.3	51.538577	84.753615
α_2=0.5	51.010255	88.889889
α_2=0.7	56.274942	77.453973
α_2=0.9	53.073602	80.582050
λ_1=0.1	63.822064	94.957389
λ_1=0.3	51.010255	88.889889
λ_1=0.5	48.441919	75.612427
λ_1=0.7	42.904147	75.474593
λ_1=0.9	37.171731	82.916679
λ_2=0.1	55.077654	90.996280
λ_2=0.3	54.552877	87.805984
λ_2=0.5	50.445184	94.294598
λ_2=0.7	51.010255	88.889889
λ_2=0.9	48.315660	88.532556
c_1=0.25	45.023931	77.565666
c_1=0.5	44.091545	90.966499
c_1=0.75	51.010255	88.889889