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GEOMETRIC STRUCTURES ON THE ORBITS OF LOOP DIFFEOMORPHISM GROUPS AND RELATED HEAVENLY-TYPE HAMILTONIAN SYSTEMS. I

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We present a review of differential-geometric and Lie-algebraic approaches to the investigation of a broad class of nonlinear integrable differential systems of "heavenly" type associated with Hamiltonian flows on the spaces conjugate to the loop Lie algebras of vector fields on the tori. These flows are generated by the corresponding orbits of the coadjoint action of loop diffeomorphism groups and satisfy the Lax–Sato-type vector-field compatibility conditions. We analyze the corresponding hierarchies of conservation laws and their relationships with Casimir invariants. We consider typical examples of these systems and establish their complete integrability by using the developed Lie-algebraic construction. We also describe new generalizations of the integrable dispersion-free systems of heavenly type for which the corresponding generating elements of the orbits have factorized structures, which allows their extension to the multidimensional case.

1. Introduction

It is known that the investigations of integrability of complex mathematical models of contemporary natural science or the corresponding nonlinear differential equations and dynamical systems is an actual field [4, 7, 23] of mathematical research starting from the time of creation of the inverse-scattering method and the application of differential-geometric, algebraic-geometric, and operator-spectral methods [2, 13–16, 18, 19, 23] for their detailed analysis. The indicated nonlinear models are, to a certain extent, universal because they appear in numerous fields of physics (such as solid-state physics, nonlinear optics, hydrodynamics, plasma physics, etc.) in the course of both theoretical and applied investigations. At the same time, the integrability of these models is closely connected with numerous directions of modern mathematics and characterized by the presence of rich and beautiful structures encountered in these models.

In the present review, we mainly consider integrable systems of multidimensional dispersion-free dynamic flows and partial differential equations possessing a modified Lax–Sato-type representation associated with their hidden group symmetry and Hamiltonian structure. Systems of this kind appear in mechanics, in the general relativity theory, in the differential geometry, and in the general theory of integrable dynamical systems. Among these systems, we can especially mention the Boyer–Finley equation and the Plebański heavenly type equation that describe a class of self-dual 4-manifolds, the dispersion-free Kadomtsev–Petviashvili equation known as the Khokhlov–Zabolotskii equation in the nonlinear acoustics, and in the Einstein–Weyl theory of structures. The integrability of these systems was investigated by using different advanced approaches, including, in particular,

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the symmetry analysis, the differential-geometric methods, the techniques of dispersion-free $\bar{\partial}$-dressing and factorization, the Virasoro constraints, the hydrodynamic reductions, etc.

In the present paper, we present a survey of a certain class of differential-geometric and Lie-algebraic structures characterizing the classical hydrodynamic-type dynamical systems that are important both for the description and construction of their exact solutions and for the detailed analysis of the properties of mathematical objects related to them. The first examples and the corresponding Hamiltonian structures were considered in [24–28]. Later, numerous examples of systems of dispersion-free partial differential equations were analyzed in detail in [29–31, 49, 69–72]. These systems are called heavenly type systems. For the first time, this name was introduced by Plebański in [32]. Heavenly type systems were studied in numerous works (see [8–12, 20–22, 26–28, 31–42]) by using different approaches. In particular, the differential-geometric and symplectic methods of their investigation were extensively used in [26–28, 33–41]. In recent works [29, 30], the general Lie-algebraic scheme was developed for the construction of the Lax–Sato integrable heavenly type differential systems. The indicated scheme is based on the application of the Adler–Kostant–Symes classical geometric structure (AKS-theory) and the R-operator structures connected with it [16–18, 24–28, 42–45] to the loop Lie algebra $\widetilde{\operatorname{diff}}(\mathbb{T}^n)$ of the vector fields on an n-dimensional torus $\mathbb{T}^n$ and its holomorphic generalization. According to the scheme developed by the authors, these differential systems follow from the commutativity condition for Hamiltonian flows on regular spaces conjugate to the above-mentioned Lie algebras given by the R-deformed Lie–Poisson bracket and the corresponding Casimir invariants as Hamiltonians. For each of these Lie algebras, the commutativity condition on the orbits of coadjoint action is reduced to the Lax–Sato representations of the heavenly type systems.

In the cited works, it was also indicated that, in most cases, integrable systems of the heavenly type are generated by elements of a regular space conjugate to a loop Lie algebra with a special structure of total differential or a structure proportional to this structure over the ring of smooth functions on a torus. Moreover, in the space of modules [46, 47] of gauge connectedness on $\mathbb{T}^n$, for the coadjoint actions of the corresponding Casimir invariants, there exists a canonical symplectic structure, which enables us to study the geometric nature of these systems by using the cohomological approaches proposed in [46, 48] for the case of Riemannian surfaces. In addition, the authors established the relationship between the Hamiltonian flows constructed by us and the well-known Lagrange–d'Alembert principle of classical mechanics. In particular, in [10], a generalization of the Lie-algebraic scheme developed in [29, 30] was proposed for $n = 1$ in the case of a loop Lie algebra of superconformal vector fields on the supercircle $\mathbb{S}^{1|N} \simeq \mathbb{S}^1 \times \Lambda_1$, where $\Lambda := \Lambda_0 \oplus \Lambda_1$ and $\Lambda_0 \supset \mathbb{C}$ is the Grassmann algebra over the field $\mathbb{C}$. Further, new Lax–Sato integrable superanalogs of some known heavenly type systems were obtained [30, 49]. We also note that nonassociative and noncommutative algebras of flows on a torus $\mathbb{T}^m$, $m \in \mathbb{N}$, were used in [30, 50–53], for the analysis of heavenly type systems. In [26–28], the authors developed the general Lie-algebraic approach to the construction of bi-Hamiltonian heavenly type systems based on the use of a central extension of a so-called loop Lie algebra of vector fields on a circle.

We now briefly describe the structure of the present paper.

In the first section, we consider some main concepts and mathematical structures that form a basis of the differential-geometric Lie approach to the study of integrable differential Lax–Sato-type equations.

In the second section, we describe Lie-algebraic structures associated with these equations in the space conjugate to the Lie algebra and the associated Lie–Poisson structures and formulate an algebraic criterion for the existence of integrable Lax-type flows.

The third section is devoted to the differential-geometric analysis of a group of diffeomorphisms of a torus and the construction of the canonical Lie–Poisson structure on the space conjugate to its Lie algebra.

In the fourth section, we present the description of integrable Hamiltonian systems generated by orbits of the coadjoint action of the loop group of diffeomorphisms on the space conjugate to its Lie algebra.

Integrable multidimensional heavenly Lax–Sato-type systems and the associated conformal structures generating these equations are discussed in the fifth and sixth sections. Among them, one can find equations quite important for the contemporary research fields in physics, hydrodynamics, and especially, Riemannian geometry connected with interesting conformal structures on metric Riemann spaces, namely, with the Einstein and Einstein–Weyl metric equations, the second Plebański conformal metric equation, the Dunajski metric equation, etc. Moreover, some of them are based on the holomorphic generating elements in certain special subdomains of the complex plane and their analysis requires certain modifications of their theoretical substantiation. Furthermore, since the general differential-geometric structure of generating elements connected with some equations of conformal metrics is invariant with respect to the space dimension of the analyzed Riemann spaces, they can be analytically described in the multidimensional case. In particular, we study the Einstein–Weyl metric equation, the modified Einstein–Weyl metric equation, the system of heavenly Dunajski equations, the equations of the first and second conformal structures generating the corresponding integrable heavenly equations, the inverse first heavenly Shabat equation, the first and modified heavenly Plebański equations and their multidimensional generalizations, the heavenly Husain equation and its multidimensional generalizations, and the general Monge equation and its multidimensional generalizations.

The short seventh section is devoted to the construction of superconformal analogs of the heavenly Whitham equation. In the eighth section, we investigate geometric structures related to the one-dimensional completely integrable hydrodynamic Chaplygin system. It turns out that this system is closely connected with differential systems on a torus and with orbits of the loop group of diffeomorphisms associated with them. This geometric structure made it possible to establish an additional correlation between the generating differential forms on a torus and give an analytic description of the infinite hierarchy of new integrable hydrodynamic systems. As shown in [3], these systems are closely connected with a class of completely integrable Monge-type equations whose geometric structure has been recently comprehensively analyzed in [6] by using a somewhat different approach based on the properties of embeddings of general differential systems (defined on jet-submanifolds in the Plücker coordinates) in Grassmann manifolds. In particular, this approach poses an interesting problem of finding the relationships between different geometric approaches to the description of completely integrable dispersion-free differential systems.

In the last two sections, we develop an analog of the Lie-algebraic scheme proposed in [10, 29, 30] for central extensions of the loop Lie algebra of vector fields on an n-dimensional torus $\mathbb{T}^n$ for any $n \in \mathbb{N}$, which is the semidirect sum of the Lie algebra of vector fields on $\mathbb{T}^n$ for the corresponding conjugate regular space and a loop Lie algebra of holomorphic generalizations of the vector fields on the torus $\mathbb{T}^n$. We apply the proposed Lie-algebraic scheme for the construction of Lax–Sato integrable modified and generalized heavenly Mikhalev–Pavlov-type systems in the four-dimensional space and the modified heavenly Martínez Alonso–Shabat system in the four-dimensional space.

2. Lie Algebras, Associated Poisson Structures, and the Existence of Integrable Lax-Type Flows

Let $(\tilde{\mathcal{G}};[\cdot,\cdot])$ be a Lie algebra over the field $\mathbb{C}$ and let $\tilde{\mathcal{G}}^*$ be the natural conjugate space. Consider a tensor element $r \in \tilde{\mathcal{G}} \otimes \tilde{\mathcal{G}} \simeq \mathrm{Hom}(\tilde{\mathcal{G}}^*;\tilde{\mathcal{G}})$ with partition into the symmetric and antisymmetric parts of the form $r = k \oplus \sigma$, where the symmetric tensor $k \in \tilde{\mathcal{G}} \otimes \tilde{\mathcal{G}}$ is nondegenerate. This enables us to introduce, on the Lie algebra $\tilde{\mathcal{G}}$, a nondegenerate symmetric bilinear form $(\cdot|\cdot)\colon \tilde{\mathcal{G}} \otimes \tilde{\mathcal{G}} \to \mathbb{C}$ by using the expression $(a|b) := k^{-1}(ab)$ for any $a, b \in \tilde{\mathcal{G}}$. The composition of mappings $R := \sigma \circ k^{-1}\colon \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}$ acting according to the rule

$$\tilde{\mathcal{G}} \stackrel{k^{-1}}{\to} \tilde{\mathcal{G}}^* \stackrel{\sigma}{\to} \tilde{\mathcal{G}}$$

determines the R-operator structure

$$[a,b]_R := [Ra,b] + [a,Rb]$$

on the Lie algebra $\tilde{\mathcal{G}}$ for any $a, b \in \tilde{\mathcal{G}}$. The following theorem enables us to introduce the Poisson structure on the space conjugate to $\tilde{\mathcal{G}}$ [42, 48, 54, 55]:

Theorem 2.1. *For any $\alpha, \beta \in \tilde{\mathcal{G}}^*$, consider a bracket*

$$\{\alpha, \beta\} := ad^*_{r\alpha}\beta - ad^*_{r\beta}\alpha. \tag{2.1}$$

Bracket (2.1) is Poisson if and only if the R-operator structure on the Lie algebra $\tilde{\mathcal{G}}$ specifies the Lie structure on $\tilde{\mathcal{G}}$, i.e., for any $a, b \in \tilde{\mathcal{G}}$, the Yang–Baxter equality

$$[Ra, Rb] - R[a, b]_R = -[a, b]$$

is true.

By using this theorem, we can construct Hamiltonian flows of the Lax type on the conjugate space $\tilde{\mathcal{G}}^*$ in the case where there exists a Killing-type functional $Tr(\cdot)$ generating, on $\tilde{\mathcal{G}}$, the symmetric and ad-invariant product

$$Tr(ab) := (a|b), \quad (a|[b, c]) = (([a, b]|, c)$$

for any a, b and $c \in \tilde{\mathcal{G}}$. Then, for any element $a \in \tilde{\mathcal{G}}$, the Hamiltonian flow has the standard Lax form

$$da/dt = [\mathrm{grad}(h), a],$$

where the element $\mathrm{grad}(h) \in \tilde{\mathcal{G}}$ is associated with a certain functional $h \in \mathcal{D}(\tilde{\mathcal{G}})$.

For the loop Lie algebra

$$\tilde{\mathcal{G}} := \widetilde{\mathrm{diff}}(\mathbb{T}^n)$$

on the torus $\mathbb{T}^n$, it is known that a Tr-type functional on $\tilde{\mathcal{G}}$ does not exist. As a result, it becomes necessary to study Hamiltonian flows on the conjugate loop space

$$\tilde{\mathcal{G}}^* \simeq \tilde{\Lambda}^1(\mathbb{T}^n)$$

of meromorphic differential forms on the torus $\mathbb{T}^n$ and to obtain integrable dispersion-free equations as compatibility conditions for the corresponding vector fields generated by the Casimir invariants on $\tilde{\mathcal{G}}^*$. This procedure is more complicated than the standard procedure and requires the use of a larger number of geometric instruments and the properties of the structure of coadjoint orbits for elements generating the hierarchy of integrable Hamiltonian flows. In particular, it is necessary to study the reducing properties of this hierarchy guaranteeing the existence of nontrivial Casimir invariants on these coadjoint orbits.

The application of the indicated ideas to the central extensions of Lie algebras enables one to construct new classes of commuting Hamiltonian flows on the extended conjugate space

$$\bar{\mathcal{G}}^* := \tilde{\mathcal{G}}^* \oplus \mathbb{C}.$$

These Hamiltonian flows are generated by the elements $(\tilde{a} \ltimes \tilde{l}; \alpha) \in \bar{\mathcal{G}}^*$ and Casimir invariants constructed on the orbits in $\tilde{\mathcal{G}}^*$.

In most cases, generating elements can be obtained as specially factorized differential objects whose geometric nature is studied quite poorly. With the help of the Lie-algebraic approach, it was established that the corresponding commutativity condition for the constructed factorized Hamiltonian flows is the compatibility condition for a system of three linear vector-field Lax–Sato-type equations. As examples illustrating these mathematical structures, we obtain the generalizations of the dispersion-free Mikhalev–Pavlov and Martínez Alonso–Shabat systems for which generating elements have special factorized structures with the help of which they can be extended to the multidimensional case.

3. Group of Diffeomorphisms $\mathrm{Diff}(\mathbb{T}^n)$ and the Associated Differential-Geometric Structures

We consider an n-dimensional torus $\mathbb{T}^n$ and points $X \in \mathbb{T}^n$ as Lagrange variables for the configuration $\eta \in \mathrm{Diff}(\mathbb{T}^n)$. The manifold $\mathbb{T}^n$ defined as a target configuration space $\eta \in \mathrm{Diff}(\mathbb{T}^n)$ is called space or Euler configuration and its points are called space or Euler points. We denote these points by small letters $x \in \mathbb{T}^n$. Then any one-parameter configuration $\mathrm{Diff}(\mathbb{T}^n)$ is a time-dependent, $t \in \mathbb{R}$, family of diffeomorphisms [56–60], which can be represented in the form

$$\mathbb{T}^n \ni x = \eta(X,t) := \eta_t(X) \in \mathbb{T}^n$$

for any initial configuration $X \in \mathbb{T}^n$ and some mappings $\eta_t \in \mathrm{Diff}(\mathbb{T}^n),\ t \in \mathbb{R}$.

To study the flows in the space of Lagrange configurations $\eta \in \mathrm{Diff}(\mathbb{T}^n)$ with respect to the time variable $t \in \mathbb{R}$ generated by the group of diffeomorphisms $\eta_t \in \mathrm{Diff}(\mathbb{T}^n),\ t \in \mathbb{R}$, we describe the structures of the tangent $(T_{\eta_t}(\mathrm{Diff}(\mathbb{T}^n)))$ and cotangent $(T^*_{\eta_t}(\mathrm{Diff}(\mathbb{T}^n)))$ spaces to the group of diffeomorphisms $\mathrm{Diff}(\mathbb{T}^n)$ at the points $\eta_t \in \mathrm{Diff}(\mathbb{T}^n)$ for any $t \in \mathbb{R}$.

We first describe the space $T_{\eta_t}(\mathrm{Diff}(\mathbb{T}^n))$ tangent to the group of diffeomorphisms $\mathrm{Diff}(\mathbb{T}^n)$ at the point $\eta \in \mathrm{Diff}(\mathbb{T}^n)$ by using the structure developed in [56, 57, 61]. In particular, we consider the Lagrange configuration $\eta \in \mathrm{Diff}(\mathbb{T}^n)$ and define the space $T_\eta(\mathrm{Diff}(\mathbb{T}^n))$ tangent to $\eta \in \mathrm{Diff}(\mathbb{T}^n)$ as a collection of vectors

$$\xi_\eta := d\eta_\tau/d\tau|_{\tau=0},$$

where

$$\mathbb{R} \ni \tau \mapsto \eta_\tau \in \mathrm{Diff}(\mathbb{T}^n), \qquad \eta_\tau|_{\tau=0} = \eta,$$

is a smooth curve on $\mathrm{Diff}(\mathbb{T}^n)$ and, for any point $X \in \mathbb{T}^n$, the equality

$$\xi_\eta(X) = d\eta_\tau(X)/d\tau|_{\tau=0}$$

is true. The last relation means that the vectors $\xi_\eta(X) \in T_{\eta(X)}(\mathbb{T}^n),\ X \in \mathbb{T}^n$, specify the vector field

$$\xi : \mathbb{T}^n \to T(\mathbb{T}^n)$$

on $\mathbb{T}^n$ for any $\eta \in \mathrm{Diff}(\mathbb{T}^n)$. Hence, the tangent space $T_\eta(\mathrm{Diff}(\mathbb{T}^n))$ coincides with a set of vector fields on $\mathbb{T}^n$:

$$T_\eta(\mathrm{Diff}(\mathbb{T}^n)) \simeq \{\xi_\eta \in \Gamma(T(\mathbb{T}^n)) \colon \xi_\eta(X) \in T_{\xi(X)}(\mathbb{T}^n)\}.$$

Similarly, the cotangent space $T^*_\eta(\mathrm{Diff}(\mathbb{T}^n))$ is formed by all linear functionals on $\mathbb{T}^n$ over $\eta \in \mathrm{Diff}(\mathbb{T}^n)$:

$$T^*_\eta(\mathrm{Diff}(\mathbb{T}^n)) = \left\{\alpha_\eta \in \Lambda^1(\mathbb{T}^n) \otimes \Lambda^3(\mathbb{T}^n) \colon \alpha_\eta(X) \in T^*_{\eta(X)}(\mathbb{T}^n) \otimes |\Lambda^3(\mathbb{T}^n)|\right\}$$

with respect to the ordinary nondegenerate convolution $(\cdot|\cdot)_c$ on $T^*_\eta(\mathrm{Diff}(\mathbb{T}^n)) \times T_\eta(\mathrm{Diff}(\mathbb{T}^n))$:

If $\alpha_\eta \in T^*_\eta(\mathrm{Diff}(\mathbb{T}^n))$ and $\xi_\eta \in T_\eta(\mathrm{Diff}(\mathbb{T}^n))$, where

$$\alpha_\eta|_X = \langle \alpha_\eta(X)|dx\rangle \otimes d^3X \qquad \text{and} \qquad \xi_\eta|_X = \langle \xi_\eta(X)|\partial/\partial x\rangle,$$

then

$$(\alpha_\eta|\xi_\eta)_c := \int\limits_{\mathbb{T}^n} \langle \alpha_\eta(X)|\xi_\eta(X)\rangle d^3X.$$

This structure enables one to identify the cotangent bundle $T^*_\eta(\mathrm{Diff}(\mathbb{T}^n))$ for a fixed Lagrange configuration $\eta \in \mathrm{Diff}(\mathbb{T}^n)$ with the tangent space $T_\eta(\mathrm{Diff}(\mathbb{T}^n))$ because the tangent space $T(\mathbb{T}^n)$ is endowed with a natural intrinsic metric $\langle\cdot|\cdot\rangle$ at the point $\eta(X) \in \mathbb{T}^n$, which enables one to identify the spaces $T(\mathbb{T}^n)$ and $T^*(\mathbb{T}^n)$ by using the corresponding metric isomorphism $\sharp\colon T^*(\mathbb{T}^n) \to T(\mathbb{T}^n)$. We can construct an extension of this isomorphism to $T^*_\eta(\mathrm{Diff}(\mathbb{T}^n))$ for $\eta \in \mathrm{Diff}(\mathbb{T}^n)$ as follows: for any elements $\alpha_\eta, \beta_\eta \in T^*_\eta(\mathrm{Diff}(\mathbb{T}^n))$,

$$\alpha_\eta|_X = \langle \alpha_\eta(X)|dx\rangle \otimes d^3X \qquad \text{and} \qquad \beta_\eta|_X = \langle \beta_\eta(X)|dx\rangle \otimes d^3X \in T^*_\eta(\mathrm{Diff}(\mathbb{T}^n)),$$

we define a metric

$$(\alpha_\eta|\beta_\eta) := \int\limits_{\mathbb{T}^n} \langle \alpha^\sharp_\eta(X)|\beta^\sharp_\eta(X)\rangle d^3X,$$

where, by definition,

$$\alpha^\sharp_\eta(X) := \sharp\langle \alpha_\eta(X)|dx\rangle \qquad \text{and} \qquad \beta^\sharp_\eta(X) := \sharp\langle \beta_\eta(X)|dx\rangle \in T_{\eta(X)}(\mathbb{T}^n)$$

for any $X \in \mathbb{T}^n$.

By using the structure described above, we can construct smooth functionals on the cotangent bundle

$$T^*(\mathrm{Diff}(\mathbb{T}^n))$$

that are invariant under the coadjoint action of the diffeomorphism group $\mathrm{Diff}(\mathbb{T}^n)$. In addition, the cotangent bundle $T^*(\mathrm{Diff}(\mathbb{T}^n))$ is *a priori* equipped with a canonical symplectic structure [42, 43, 45, 56, 57, 60, 62–65], which is equivalent to the bracket of smooth functionals on $T^*(\mathrm{Diff}(\mathbb{T}^n))$. This enables us to study the corresponding Hamiltonian flows, their hidden symmetries, and integrability.

Further, we consider the cotangent bundle $T^*(\mathrm{Diff}(\mathbb{T}^n))$ as a smooth manifold with canonical symplectic structure [56, 62], which is equivalent to the canonical Poisson bracket on the space of smooth functionals given on this manifold.

Since the cotangent space $T^*_\eta(\mathrm{Diff}(\mathbb{T}^n))$ for $\eta \in \mathrm{Diff}(\mathbb{T}^n)$ is shifted under the $R_{\eta^{-1}}$-action toward the space $T^*_{Id}(\mathrm{Diff}(\mathbb{T}^n))$, $Id \in \mathrm{Diff}(\mathbb{T}^n)$, is diffeomorphic to the space $\mathrm{Diff}^*(\mathbb{T}^n)$ conjugate to the Lie algebra $\mathrm{Diff}(\mathbb{T}^n) \simeq \Gamma(T(\mathbb{T}^n))$ of the vector fields on $\mathbb{T}^n$ {this was shown by S. Lie as early as in 1887 (see, e.g., [60, 66–68])}, this canonical Poisson bracket on $T^*_\eta(\mathrm{Diff}(\mathbb{T}^n))$ turns into the classical Lie-Poisson bracket on the conjugate space $\mathcal{G}^*$ [57, 60, 62, 64, 66, 67]. In addition, the orbits of the group of diffeomorphisms $\mathrm{Diff}(\mathbb{T}^n)$ on $T^*(\mathrm{Diff}(\mathbb{T}^n))$ are mapped into coadjoint orbits on the conjugate space $\mathcal{G}^*$ generated by the corresponding elements of the Lie algebra $\mathcal{G}$. The following lemma enables one to construct this Lie–Poisson bracket:

Lemma 3.1. *The Lie algebra* $\mathrm{diff}(\mathbb{T}^n) \simeq \Gamma(T(\mathbb{T}^n))$ *is given by the commutation relation*

$$[a_1, a_2] = \langle a_1|\nabla\rangle a_2 - \langle a_2|\nabla\rangle a_1 \tag{3.1}$$

for any vector fields $a_1, a_2 \in \Gamma(T(\mathbb{T}^n))$ *on the manifold* $\mathbb{T}^n$.

Proof. The commutation relation (3.1) follows from the definition of the group operation of multiplication

$$(\varphi_{1,t} \circ \varphi_{2,t})(X) = \varphi_{2,t}(\varphi_{1,t}(X))$$

for any group diffeomorphisms $\varphi_{1,t}, \varphi_{2,t} \in \mathrm{Diff}(\mathbb{T}^n),\ t \in \mathbb{R}$, and $X \in \mathbb{T}^n$ under the condition that $a_j(X) := d\varphi_{j,t}(X)/dt|_{t=0}$ and $\varphi_{j,t}|_{t=0} = Id \in \mathrm{Diff}(\mathbb{T}^n),\ j = \overline{1,2}$.

To find the Poisson bracket on the cotangent space $T^*_\eta(\mathrm{Diff}(\mathbb{T}^n))$ for any $\eta \in \mathrm{Diff}(\mathbb{T}^n)$, we consider the cotangent space $T^*_\eta(\mathrm{Diff}(\mathbb{T}^n)) \simeq \mathrm{Diff}^*(\mathbb{T}^n)$, i.e., the space conjugate to the tangent space $T_\eta(\mathrm{Diff}(\mathbb{T}^n))$ of the left-invariant vector fields $\mathrm{Diff}(\mathbb{T}^n)$ for any $\eta \in \mathrm{Diff}(\mathbb{T}^n)$, and a canonical symplectic structure on $T^*_\eta(\mathrm{Diff}(\mathbb{T}^n))$ of the form $\omega^{(2)}(\mu, \eta) := \delta\alpha(\mu, \eta)$. Moreover, the canonical Liouville form

$$\alpha(\mu, \eta) := (\mu|\delta\eta)_c \in \Lambda^1_{(\mu,\eta)}(T^*_\eta(\mathrm{Diff}(\mathbb{T}^n)))$$

is *a priori* defined at the point $(\mu, \eta) \in T^*_\eta(\mathrm{Diff}(\mathbb{T}^n))$ of the tangent space $T_\eta(\mathrm{Diff}(\mathbb{T}^n)) \simeq \Gamma(T(M))$ of right-invariant vector fields on the torus $\mathbb{T}^n$. If we find the corresponding Poisson bracket for smooth functions $(\mu|a)_c, (\mu|b)_c \in C^\infty(T^*_\eta(\mathrm{Diff}(\mathbb{T}^n)); \mathbb{R})$ on $T^*_\eta(\mathrm{Diff}(\mathbb{T}^n)) \simeq \mathrm{Diff}^*(\mathbb{T}^n),\ \eta \in \mathrm{Diff}(\mathbb{T}^n)$, then we can formulate the following theorem:

Theorem 3.1. *The Lie–Poisson bracket on the conjugate space* $T^*_\eta(\mathrm{Diff}(\mathbb{T}^n)) \simeq \mathrm{diff}^*(\mathbb{T}^n),\ \eta \in M$, *is given by the expression*

$$\{f, g\}(\mu) = (\mu|[\delta g(\mu)/\delta\mu, \delta f(\mu)/\delta\mu])_c \tag{3.2}$$

for any smooth functionals $f, g \in C^\infty(\mathcal{G}^*; \mathbb{R})$.

Proof. By using the definition of the Poisson bracket for smooth functions

$$(\mu|a)_c,\ (\mu|b)_c \in C^\infty(T^*_\eta(\mathrm{Diff}(\mathbb{T}^n)); \mathbb{R})$$

on the symplectic space $T^*_\eta(\mathrm{Diff}(\mathbb{T}^n))$ [56, 62], we get

$$\begin{aligned}\{\mu(a), \mu(b)\} &:= \delta\alpha(X_a, X_b)\\ &= X_a(\alpha|X_b)_c - X_b(\alpha|X_a)_c - (\alpha|[X_a, X_b])_c,\end{aligned} \tag{3.3}$$

where

$$X_a := \delta(\mu|a)_c/\delta\mu = a \in \mathrm{diff}(\mathbb{T}^n) \qquad \text{and} \qquad X_b := \delta(\mu|b)_c/\delta\mu = b \in \mathrm{diff}(\mathbb{T}^n).$$

In view of the right invariance of the vector fields $X_a, X_b \in T_\eta(\mathrm{Diff}(\mathbb{T}^n)),\ X_a(\alpha|X_b)_c = 0$ and $X_b(\alpha|X_a)_c = 0$, the Poisson bracket (3.3) turns into

$$\{(\mu|a)_c, (\mu|b)_c\} = -(\alpha|[X_a, X_b])_c$$

$$= (\mu|[b,a])_c = (\mu|[\delta(\mu|b)_c/\delta\mu, \delta(\mu|a)_c/\delta\mu])_c$$

for all $(\mu,\eta) \in T^*_\eta(\mathrm{Diff}(\mathbb{T}^n)) \simeq \mathrm{Diff}^*(\mathbb{T}^n),\ \eta \in \mathrm{Diff}(\mathbb{T}^n)$, and any $a, b \in \mathrm{diff}(\mathbb{T}^n)$. The Poisson bracket (3.3) can be easily generalized to

$$\{f,g\}(\mu) = (\mu|[\delta g(\mu)/\delta\mu, \delta f(\mu)/\delta\mu])_c$$

for any smooth functionals $f, g \in C^\infty(\mathcal{G}^*;\mathbb{R})$, which completes the proof of the theorem.

By using the Lie–Poisson bracket (3.2), it is possible to construct Hamiltonian flows on the conjugate space $\mathrm{diff}^*(\mathbb{T}^n)$ in the form $\partial l/\partial t = -ad^*_{\mathrm{grad}\, h(l)} l$ for any element $l \in \mathrm{diff}^*(\mathbb{T}^n),\ t \in \mathbb{R}$. Here, by definition,

$$\frac{d}{d\varepsilon} h(l+\varepsilon m)|_{\varepsilon=0} := (m|\,\mathrm{grad}\, h(l))_c$$

for some smooth Hamiltonian function $h \in C^\infty(\mathrm{Diff}^*(\mathbb{T}^n);\mathbb{R})$.

If a system has, in addition to the Hamiltonian function, sufficiently many additional global invariants, then it is possible to expect that the procedure of reduction transforms the system into its complete integrable differential form.

4. Vector Fields on a Torus and Their Lie-Algebraic Properties

Consider the Lie loop group $\tilde{G} := \widetilde{\mathrm{diff}}(\mathbb{T}^n)$, i.e., the set of smooth mappings $\{\mathbb{C}^1 \supset \mathbb{S}^1 \longrightarrow G := \mathrm{Diff}(\mathbb{T}^n)\}$ holomorphically extended, respectively, from the circle $\mathbb{S}^1 \subset \mathbb{C}^1$ to the set $\mathbb{D}^1_+$ of interior points of the circle $\mathbb{S}^1$ and to the set $\mathbb{D}^1_-$ of its exterior points $\lambda \in \mathbb{C}\backslash\overline{\mathbb{D}}^1_+$. The corresponding Lie algebra admits the following splitting: $\tilde{\mathcal{G}} := \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$, where

$$\tilde{\mathcal{G}}_+ := \widetilde{\mathrm{diff}}(\mathbb{T}^n)_+ \subset \Gamma\big(\mathbb{D}^1_+ \times \mathbb{T}^n; T(\mathbb{D}^1_+ \times \mathbb{T}^n)\big)$$

is the Lie subalgebra containing the vector fields on the manifold $\mathbb{S}^1 \times \mathbb{T}^n$ holomorphically extended to the disk $\mathbb{D}^1_+$ and

$$\tilde{\mathcal{G}}_- := \widetilde{\mathrm{diff}}(\mathbb{T}^n)_- \subset \Gamma\big(\mathbb{D}^1_- \times \mathbb{T}^n; T(\mathbb{D}^1_- \times \mathbb{T}^n)\big)$$

is the Lie subalgebra containing the vector fields on the manifold $\mathbb{C} \times \mathbb{T}^n$ holomorphic on the set $\mathbb{D}^1_-$. The conjugate space $\tilde{\mathcal{G}}^* := \tilde{\mathcal{G}}^*_+ \oplus \tilde{\mathcal{G}}^*_-$, where the space

$$\tilde{\mathcal{G}}^*_+ \subset \Gamma(\mathbb{D}^1_+ \times \mathbb{T}^n; T^*(\mathbb{D}^1_+ \times \mathbb{T}^n))$$

contains differential forms on the manifold $\mathbb{S}^1 \times \mathbb{T}^n$ holomorphically extended to the set $\mathbb{C}\backslash\overline{\mathbb{D}}^1_+$ and the conjugate space

$$\tilde{\mathcal{G}}^*_- \subset \Gamma\big(\mathbb{D}^1_- \times \mathbb{T}^n; T^*(\mathbb{D}^1_- \times \mathbb{T}^n)\big)$$

contains differential forms on the manifold $\mathbb{S}^1 \times \mathbb{T}^n$ holomorphically extended to the set $\mathbb{D}^1_+$ so that the space $\tilde{\mathcal{G}}^*_+$ is dual to $\tilde{\mathcal{G}}_+$ and the space $\tilde{\mathcal{G}}^*_-$ is dual to $\tilde{\mathcal{G}}_-$ with respect to the convolution on the product $\tilde{\mathcal{G}}^* \times \tilde{\mathcal{G}}$:

$$(\tilde{l}|\tilde{a}) := \mathrm{res}_\lambda \int\limits_{\mathbb{T}^n} \langle l, a\rangle dx \tag{4.1}$$

for any vector field

$$\tilde{a} := \left\langle a(\mathrm{x}), \frac{\partial}{\partial \mathrm{x}} \right\rangle \in \tilde{\mathcal{G}}$$

and the differential form $\tilde{l} := \langle l(\mathrm{x}), d\mathrm{x} \rangle \in \tilde{\mathcal{G}}^*$ on $\mathbb{C} \times \mathbb{T}^n$ that depends on the coordinate

$$\mathrm{x} := (\lambda; x) \in \mathbb{C} \times \mathbb{T}^n.$$

Here, $\langle \cdot, \cdot \rangle$ is the ordinary scalar product in the Euclidean space $\mathbb{E}^{n+1}$ and

$$\frac{\partial}{\partial \mathrm{x}} := \left(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right)^{\top}$$

is the ordinary gradient vector. Splitting the Lie algebra $\tilde{\mathcal{G}}$ into the direct sum

$$\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-, \tag{4.2}$$

we get the following splitting into the direct sum:

$$\tilde{\mathcal{G}}^* = \tilde{\mathcal{G}}^*_+ \oplus \tilde{\mathcal{G}}^*_-$$

with respect to convolution (4.1). If we determine the set of smooth invariant Casimir functionals $h\colon \tilde{\mathcal{G}}^* \to \mathbb{R}$ on the conjugate space $\tilde{\mathcal{G}}^*$ by the action of the coadjoint Lie algebra $\tilde{\mathcal{G}}$

$$ad^*_{\nabla h(\tilde{l})} \tilde{l} = 0 \tag{4.3}$$

on the generating (so-called "seed") element $\tilde{l} \in \tilde{\mathcal{G}}^*$, then we can explicitly construct a broad class of multidimensional completely integrable dispersion-free (so-called "heavenly") nonlinear commuting Hamiltonian systems by using the classical Adler–Kostant–Symes scheme [26–28, 30]:

$$d\tilde{l}/dt := -ad^*_{\nabla h_+(\tilde{l})} \tilde{l} \tag{4.4}$$

for all $h \in \mathrm{I}(\tilde{\mathcal{G}}^*)$, $\nabla h(\tilde{l}) := \nabla h_+(\tilde{l}) \oplus \nabla h_-(\tilde{l}) \in \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$, on the corresponding functional manifold. Moreover, the commuting flows (4.4) can be represented as compatible systems of the vector-field Lax–Sato-type equations [30] on the functional manifold of generating element $C^2(\mathbb{C} \times \mathbb{T}^n; \mathbb{C})$, which generate the complete set of first integrals on this manifold.

5. Lie-Algebraic Structures and Integrable Hamiltonian Systems

Consider the loop Lie algebra $\tilde{\mathcal{G}}$ defined above. Elements of this algebra can be represented in the form

$$a(x;\lambda) := \left\langle a(x;\lambda), \frac{\partial}{\partial \mathrm{x}} \right\rangle = \sum_{j=1}^{n} a_j(x;\lambda) \frac{\partial}{\partial x_j} + a_0(x;\lambda) \frac{\partial}{\partial \lambda} \in \tilde{\mathcal{G}}$$

for some vectors $a(x;\lambda) \in \mathbb{E} \times \mathbb{E}^n$ holomorphic in $\lambda \in \mathbb{D}^1_{\pm}$ for all $x \in \mathbb{T}^n$, where

$$\frac{\partial}{\partial \mathrm{x}} := \left(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)^{\top}$$

is the generalized gradient vector with respect to the variable $\mathrm{x} := (\lambda, x) \in \mathbb{C} \times \mathbb{T}^n$. The Lie algebra $\tilde{\mathcal{G}}$ as the direct sum of subalgebras (4.2) enables us to introduce the classical $\mathcal{R}$-structure as follows:

$$[\tilde{a}, \tilde{b}]_{\mathcal{R}} := [\mathcal{R}\tilde{a}, \tilde{b}] + [\tilde{a}, \mathcal{R}\tilde{b}]$$

for any $\tilde{a}, \tilde{b} \in \tilde{\mathcal{G}}$, where $\mathcal{R} := (P_+ - P_-)/2$ and $P_{\pm}\tilde{\mathcal{G}} := \tilde{\mathcal{G}}_{\pm} \subset \tilde{\mathcal{G}}$.

The space $\tilde{\mathcal{G}}^* \simeq \tilde{\Lambda}^1(\mathbb{C}\times\mathbb{T}^n)$ conjugate to the algebra $\tilde{\mathcal{G}}$ of holomorphic vector fields on $\mathbb{C}\times\mathbb{T}^n$ is functionally identified with $\tilde{\mathcal{G}}$ with respect to metric (4.1). Thus, for any $f, g \in \mathrm{D}(\tilde{\mathcal{G}}^*)$, we can define two Lie–Poisson brackets

$$\{f, g\} := \big(\tilde{l}, [\nabla f(\tilde{l}), \nabla g(\tilde{l})]\big)$$

and

$$\{f, g\}_{\mathcal{R}} := \big(\tilde{l}, [\nabla f(\tilde{l}), \nabla g(\tilde{l})]_{\mathcal{R}}\big), \tag{5.1}$$

where, for any seed-element $\bar{l} \in \tilde{\mathcal{G}}^*$, the gradient vectors $\nabla f(\tilde{l})$ and $\nabla g(\tilde{l}) \in \tilde{\mathcal{G}}$ are computed in metric (4.1).

Assume that a smooth function $\gamma \in \mathrm{I}(\tilde{\mathcal{G}}^*)$ is a Casimir invariant, i.e.,

$$ad^*_{\nabla\gamma(\tilde{l})}\tilde{l} = 0 \tag{5.2}$$

for the chosen seed-element $\tilde{l} \in \tilde{\mathcal{G}}^*$. The coadjoint mapping $ad^*_{\nabla f(\tilde{l})} : \tilde{\mathcal{G}}^* \to \tilde{\mathcal{G}}^*$ for any $f \in \mathrm{D}(\tilde{\mathcal{G}}^*)$ can be represented as follows:

$$ad^*_{\nabla f(\tilde{l})}(\tilde{l}) = \left\langle \frac{\partial}{\partial \mathrm{x}}, \circ \nabla f(l) \right\rangle \bar{l} + \sum_{j=1}^{n} \left\langle \left\langle l, \frac{\partial}{\partial \mathrm{x}} \nabla f(l) \right\rangle, d\mathrm{x} \right\rangle,$$

where, by definition,

$$\nabla f(\tilde{l}) := \left\langle \nabla f(l), \frac{\partial}{\partial \mathrm{x}} \right\rangle.$$

Thus, for the Casimir function $\gamma \in \mathrm{D}(\tilde{\mathcal{G}}^*)$, condition (5.2) is equivalent to the equation

$$l \left\langle \frac{\partial}{\partial \mathrm{x}}, \nabla\gamma(l) \right\rangle + \left\langle \nabla\gamma(l), \frac{\partial}{\partial \mathrm{x}} \right\rangle l + \left\langle l, \left(\frac{\partial}{\partial \mathrm{x}} \nabla\gamma(l)\right) \right\rangle = 0. \tag{5.3}$$

For applications, this equation must be solved in the analytic form. In the case where an element $\tilde{l} \in \tilde{\mathcal{G}}^*$ is singular as $|\lambda| \to \infty$, we can consider the general asymptotic decomposition

$$\nabla\gamma := \nabla\gamma^{(p)} \sim \lambda^p \sum_{j \in \mathbb{Z}_+} \nabla\gamma_j^{(p)} \lambda^{-j} \tag{5.4}$$

for a properly chosen $p \in \mathbb{Z}_+$. Thus, substituting (5.4) in Eq. (5.3), we recursively solve this equation.

Let $h^{(y)}, h^{(t)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$ be a Casimir function for which the generators of the Hamiltonian vector field

$$\nabla h_+^{(y)}(l) := \left(\nabla \gamma^{(p_y)}(l)\right)|_+, \qquad \nabla h_+^{(t)}(l) := \left(\nabla h^{(p_t)}(l)\right)|_+ \tag{5.5}$$

are defined for some integer quantities $p_y, p_t \in \mathbb{Z}_+$. These two invariants generate the following commuting Hamiltonian flows:

$$\partial l/\partial t = -\left\langle \frac{\partial}{\partial \mathrm{x}}, \circ \nabla h_+^{(t)}(l) \right\rangle l - \left\langle l, \left(\frac{\partial}{\partial \mathrm{x}} \nabla h_+^{(t)}(l)\right)\right\rangle \tag{5.6}$$

and

$$\partial l/\partial y = -\left\langle \frac{\partial}{\partial \mathrm{x}}, \circ \nabla h_+^{(y)}(l) \right\rangle l - \left\langle l, \left(\frac{\partial}{\partial \mathrm{x}} \nabla h_+^{(y)}(l)\right)\right\rangle \tag{5.7}$$

with respect to the Lie–Poisson bracket (5.1), where $y, t \in \mathbb{R}$ are the corresponding evolutionary parameters. Since the invariants $h^{(y)}, h^{(t)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$ commute with respect to bracket (5.1), flows (5.6) and (5.7) are also commuting. As a result, the corresponding generators of the Hamiltonian vector fields

$$\nabla h_+^{(t)}(\tilde{l}) := \left\langle \nabla h_+^{(t)}(l), \frac{\partial}{\partial \, \mathrm{x}} \right\rangle, \qquad \nabla h_+^{(y)}(\tilde{l}) := \left\langle \nabla h_+^{(y)}(l), \frac{\partial}{\partial \, \mathrm{x}} \right\rangle \tag{5.8}$$

satisfy the Lax compatibility condition

$$\frac{\partial}{\partial y} \nabla h_+^{(t)}(\tilde{l}) - \frac{\partial}{\partial t} \nabla h_+^{(y)}(\tilde{l}) = [\nabla h_+^{(t)}(\tilde{l}), \nabla h_+^{(y)}(\tilde{l})] \tag{5.9}$$

for all $y, t \in \mathbb{R}$. On the other hand, condition (5.9) is equivalent to the condition of compatibility of two linear equations

$$\left(\frac{\partial}{\partial t} + \nabla h_+^{(t)}(\tilde{l})\right)\psi = 0, \qquad \left(\frac{\partial}{\partial y} + \nabla h_+^{(y)}(\tilde{l})\right)\psi = 0 \tag{5.10}$$

for a function $\psi \in C^2(\mathbb{R}^2 \times \mathbb{C} \times \mathbb{T}^n; \mathbb{C})$ for all $y, t \in \mathbb{R}$ and any $\lambda \in \mathbb{C}$. The reasoning presented above can be formulated in the form of the following main technical statement:

Proposition 5.1. *Assume that a seed-element $\tilde{l} \in \tilde{\mathcal{G}}^*$ and $h^{(y)}, h^{(t)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$ are Casimir functions with respect to the metric $(\cdot|\cdot)$ on the loop Lie algebra $\tilde{\mathcal{G}}$ and the natural coadjoint action on the loop coalgebra $\tilde{\mathcal{G}}^*$. Then the dynamical systems*

$$\partial \tilde{l}/\partial y = -ad^*_{\nabla h_+^{(y)}(\tilde{l})} \tilde{l}, \qquad \partial \tilde{l}/\partial t = -ad^*_{\nabla h_+^{(t)}(\tilde{l})} \tilde{l}$$

are commuting vector Hamiltonian fields for all $\lambda \in \mathbb{C}$ and $y, t \in \mathbb{R}$. Moreover, the condition of compatibility of these flows is equivalent to relations (5.10), where $\psi \in C^2(\mathbb{R}^2 \times \mathbb{C} \times \mathbb{T}^n; \mathbb{C})$ and the vector fields $\nabla h_+^{(t)}(\tilde{l})$ and $\nabla h_+^{(y)}(\tilde{l}) \in \tilde{\mathcal{G}}$ are given by relations (5.8) and (5.5).

Remark 5.1. As indicated above, expansion (5.4) is efficient if the chosen generating seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$ is singular as $|\lambda| \to \infty$. In the case where it is singular as $|\lambda| \to 0$, expression (5.4) takes the form

$$\nabla\gamma^{(p)}(l) \sim \lambda^{-p} \sum_{j\in\mathbb{Z}_+} \nabla\gamma_j^{(p)}(l)\lambda^j$$

for properly chosen integers $p \in \mathbb{Z}_+$ and reduced gradients of the Casimir function given by the generators of the Hamiltonian vector fields

$$\nabla h_-^{(y)}(l) := \lambda(\lambda^{-p_y-1}\nabla\gamma^{(p_y)}(l))_-,$$

$$\nabla h_-^{(t)}(l) := \lambda(\lambda^{-p_t-1}\nabla\gamma^{(p_t)}(l))_-$$

for properly chosen positive integers $p_y, p_t \in \mathbb{Z}_+$, and the corresponding Hamiltonian flows can be represented as

$$\partial\tilde{l}/\partial t = ad^*_{\nabla h_-^{(t)}(\tilde{l})}\tilde{l} \qquad \text{and} \qquad \partial\tilde{l}/\partial y = ad^*_{\nabla h_-^{(y)}(\tilde{l})}\tilde{l}.$$

6. Integrable Multidimensional Heavenly Lax–Sato-Type Systems and the Associated Equations of Conformal Structures

6.1. Einstein–Weyl Metric Equations. We denote $\tilde{\mathcal{G}}^* = \widetilde{\mathrm{diff}}(\mathbb{T}^1)^*$ and choose a seed element as follows:

$$\tilde{l} = (u_x\lambda - 2u_xv_x - u_y)\,dx + \left(\lambda^2 - v_x\lambda + v_y + v_x^2\right)d\lambda.$$

In metric (4.1), this element generates the gradient of Casimir invariants $h^{(p_t)}, h^{(p_y)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$ in the form

$$\nabla h^{(p_t)}(l) \sim \lambda^2(0,1)^\top + (-u_x, v_x)^\top\lambda + (u_y, u - v_y)^\top + O(\lambda^{-1}),$$

$$\nabla h^{(p_y)}(l) \sim \lambda(0,1)^\top + (-u_x, v_x)^\top + (u_y, -v_y)^\top\lambda^{-1} + O(\lambda^{-2})$$

as $|\lambda| \to \infty$ for $p_t = 2$ and $p_y = 1$. For the gradients of the Casimir functions $h^{(t)}, h^{(y)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$ given by Eqs. (5.5), we can get the corresponding generators of the Hamiltonian vector fields (5.8) and (5.5) on the coalgebra $\tilde{\mathcal{G}}^*$ in the form

$$\begin{gathered} A_{\nabla h_+^{(t)}} = \left\langle \nabla h_+^{(t)}(l), \frac{\partial}{\partial \mathrm{x}} \right\rangle = (\lambda^2 + \lambda v_x + u - v_y)\frac{\partial}{\partial x} + (-\lambda u_x + u_y)\frac{\partial}{\partial\lambda}, \\ A_{\nabla h_+^{(y)}} = \left\langle \nabla h_+^{(y)}(l), \frac{\partial}{\partial \mathrm{x}} \right\rangle = (\lambda + v_x)\frac{\partial}{\partial x} - u_x\frac{\partial}{\partial\lambda} \end{gathered} \tag{6.1}$$

satisfying the compatibility condition (5.9) equivalent to the integrable Einstein–Weyl equations [36]

$$\begin{gathered} u_{xt} + u_{yy} + (uu_x)_x + v_xu_{xy} - v_yu_{xx} = 0, \\ v_{xt} + v_{yy} + uv_{xx} + v_xv_{xy} - v_yv_{xx} = 0. \end{gathered} \tag{6.2}$$

It is known [33] that, for $v=0$, the invariant reduction (6.2) yields the well-known dispersion–free Kadomtsev–Petviashvili equation

$$(u_t + uu_x)_x + u_{yy} = 0, \tag{6.3}$$

for which the reduced representation (5.10) follows from (6.1) and takes the form of vector fields:

$$\begin{aligned} A_{\nabla h_+^{(t)}} &= (\lambda^2 + u)\frac{\partial}{\partial x} + (-\lambda u_x + u_y)\frac{\partial}{\partial \lambda}, \\ A_{\nabla h_+^{(y)}} &= \lambda\frac{\partial}{\partial x} - u_x\frac{\partial}{\partial \lambda}, \end{aligned} \tag{6.4}$$

satisfying the compatibility condition (5.9), which is equivalent to Eq. (6.3). As a partial result of (5.10) and (6.4), we conclude that the compatibility condition for the vector fields

$$\frac{\partial \psi}{\partial t} + (\lambda^2 + u)\frac{\partial \psi}{\partial x} + (-\lambda u_x + u_y)\frac{\partial \psi}{\partial \lambda} = 0,$$

$$\frac{\partial \psi}{\partial y} + \lambda\frac{\partial \psi}{\partial x} - u_x\frac{\partial \psi}{\partial \lambda} = 0$$

is satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{C} \times \mathbb{T}^n; \mathbb{C})$ and any $y, t \in \mathbb{R}, (x, \lambda) \in \mathbb{T}^1_{\mathbb{C}}$.

6.2. Modified Einstein–Weyl Metric Equations. These equations were introduced in [20] and have the form

$$u_{xt} = u_{yy} + u_x u_y + u_x^2 w_x + u u_{xy} + u_{xy} w_x + u_{xx} a,$$

$$w_{xt} = u w_{xy} + u_y w_x + w_x w_{xy} + a w_{xx} - a_y,$$

where $a_x := u_x w_x - w_{xy}$. In this case, we take

$$\tilde{\mathcal{G}}^* = \widetilde{\operatorname{diff}}(\mathbb{C} \times \mathbb{T}^n)$$

and choose a seed-element $\tilde{l} \in \tilde{\mathcal{G}}$ in the form

$$\begin{aligned} \tilde{l} = [&\lambda^2 u_x + (2u_x w_x + u_y + 3uu_x)\,\lambda + 2u_x\partial_x^{-1} u_x w_x + 2u_x \partial_x^{-1} u_y \\ &+ 3u_x {w_x}^2 + 2u_y w_x + 6uu_x w_x + 2uu_y + 3u^2 u_x - 2au_x]dx \\ &+ [\lambda^2 + (w_x + 3u)\,\lambda + 2\partial_x^{-1} u_x w_x + 2\partial_x^{-1} u_y + {w_x}^2 + 3uw_x + 3u^2 - a]d\lambda. \end{aligned}$$

It generates two Casimir invariants with respect to metric (4.1) $\gamma^{(j)} \in \mathrm{I}(\tilde{\mathcal{G}}^*),\ j = \overline{1,2},$ with the following gradients:

$$\begin{aligned} \nabla\gamma^{(2)}(l) \sim \lambda^2[&(u_x, -1)^\top + (uu_x + u_y, -u + w_x)^\top \lambda^{-1} \\ &+ (0, uw_x - a)^\top \lambda^{-2}] \;+\; O(\lambda^{-1}), \end{aligned}$$

$$\nabla\gamma^{(1)}(l) \sim \lambda[(u_x,-1)^\top + (0,w_x)^\top\lambda^{-1}] + O(\lambda^{-1})$$

as $|\lambda| \to \infty$ for $p_y = 1$ and $p_t = 2$. The corresponding gradients of the Casimir functions $h^{(t)}, h^{(y)} \in \mathrm{I}(\mathcal{G}^*)$ given by (5.5) generate the Hamiltonian vector fields

$$\begin{gathered}\nabla h_+^{(y)} := \nabla\gamma^{(1)}(l)|_+ = (u_x\lambda, -\lambda + w_x)^\top, \\ \nabla h_+^{(t)} = \nabla\gamma^{(2)}(l)|_+ = (u_x\lambda^2 + (uu_x+u_y)\lambda, -\lambda^2 + (w_x - u)\lambda + uw_x - a)^\top.\end{gathered} \tag{6.5}$$

By using (6.5), we arrive at a consistent system of linear Lax equations

$$\frac{\partial\psi}{\partial y} + (-\lambda + w_x)\frac{\partial\psi}{\partial x} + u_x\lambda\frac{\partial\psi}{\partial\lambda} = 0,$$

$$\frac{\partial\psi}{\partial t} + (-\lambda^2 + (\ w_x - u)\lambda + uw_x - a)\frac{\partial\psi}{\partial x} + (u_x\lambda^2 + (uu_x + u_y)\lambda)\frac{\partial\psi}{\partial\lambda} = 0,$$

which is true for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{C} \times \mathbb{T}^n; \mathbb{C})$ and any $y, t \in \mathbb{R},\ (\lambda, x) \in \mathbb{C} \times \mathbb{T}^n$.

6.3. System of Dunajski Heavenly Equations. These equations were proposed in [35] and generalize the corresponding antiself-dual Einstein vacuum equation related to the Plebański metric and the well-known second heavenly Plebański equation [8, 32]. To study the integrability of the Dunajski equations

$$\begin{gathered}u_{x_1t} + u_{yx_2} + u_{x_1x_1}u_{x_2x_2} - u_{x_1x_2}^2 - v = 0, \\ v_{x_1t} + v_{x_2y} + u_{x_1x_1}v_{x_2x_2} - 2u_{x_1x_2}v_{x_1x_2} = 0,\end{gathered} \tag{6.6}$$

where $(u, v) \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{R}^2),\ (y, t; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2$, we define

$$\tilde{\mathcal{G}}^* = \widetilde{\mathrm{diff}}(\mathbb{C} \times \mathbb{T}^2)^*.$$

As a seed-element $\bar{l} \in \tilde{\mathcal{G}}^*$, we take

$$\tilde{l} = (\lambda + v_{x_1} - u_{x_1x_1} + u_{x_1x_2})dx_1 + (\lambda + v_{x_2} + u_{x_2x_2} - u_{x_1x_2})dx_2 + (\lambda - x_1 - x_2)d\lambda.$$

In metric (4.1), the gradients of two functionally independent Casimir invariants $h^{(p_y)}$, $h^{(p_y)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$ as $|\lambda| \to \infty$ can be found in the asymptotic form as follows:

$$\begin{gathered}\nabla h^{(p_y)}(l) \sim \lambda(1,0,0)^\top + (-u_{x_1x_2}, u_{x_1x_1}, -v_{x_1})^\top\ + O(\lambda^{-1}), \\ \nabla h^{(p_t)}(l) \sim \lambda(0,-1,0)^\top + (u_{x_2x_2}, -u_{x_1x_2}, v_{x_2})^\top\ + O(\lambda^{-1})\end{gathered} \tag{6.7}$$

for $p_t = 1 = p_y$. Computing the generators of the Hamiltonian vector fields

$$\nabla h_+^{(y)} := \nabla h^{(p_y)}(l)|_+ = (\lambda - u_{x_1x_2}, u_{x_1x_1}, -v_{x_1})^\top,$$

$$\nabla h_+^{(t)} := \nabla h^{(p_t)}(l)|_+ = (u_{x_2x_2}, -\lambda - u_{x_1x_2}, v_{x_2})^\top,$$

which follow from the gradients of the Casimir functions (6.7), we get the vector fields

$$A_{\nabla h_+^{(t)}} = \left\langle \nabla h_+^{(t)}, \frac{\partial}{\partial \mathrm{x}} \right\rangle = u_{x_2x_2}\frac{\partial}{\partial x_1} - (\lambda + u_{x_1x_2})\frac{\partial}{\partial x_2} + v_{x_2}\frac{\partial}{\partial \lambda},$$

$$A_{\nabla h_+^{(y)}} = \left\langle \nabla h_+^{(y)}, \frac{\partial}{\partial \mathrm{x}} \right\rangle = (\lambda - u_{x_1x_2})\frac{\partial}{\partial x_1} + u_{x_1x_1}\frac{\partial}{\partial x_2} - v_{x_1}\frac{\partial}{\partial \lambda}.$$

The vector fields (6.8) satisfy the Lax compatibility condition (5.9), which is equivalent to the following consistent relations for vector fields:

$$\begin{aligned}&\frac{\partial\psi}{\partial t} + u_{x_2x_2}\frac{\partial\psi}{\partial x_1} - (\lambda + u_{x_1x_2})\frac{\partial\psi}{\partial x_2} + v_{x_2}\frac{\partial\psi}{\partial\lambda} = 0,\\ &\frac{\partial\psi}{\partial y} + (\lambda - u_{x_1x_2})\frac{\partial\psi}{\partial x_1} + u_{x_1x_1}\frac{\partial\psi}{\partial x_2} - v_{x_1}\frac{\partial\psi}{\partial\lambda} = 0,\end{aligned} \tag{6.8}$$

which are satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{C} \times \mathbb{T}^2; \mathbb{C})$ for any $(y, t) \in \mathbb{R}^2$ and all $(\lambda; x_1, x_2) \in \mathbb{C} \times \mathbb{T}^2$. As indicated in [34], the Dunajski equations (6.6) generalize both dispersion-free Kadomtsev–Petviashvili equations, and the second Plebański equation is also a Lax integrable Hamiltonian system.

6.4. First Generating Equation of the Conformal Structure: $\boldsymbol{u_{yt} + u_{xt}u_y - u_tu_{xy} = 0}$**.** A seed-element $\tilde{l} \in \tilde{\mathcal{G}}^* = \widetilde{\mathrm{diff}}(\mathbb{T}^1_{\mathbb{C}})^*$ of the form

$$\tilde{l} = \big[u_t^{-2}(1-\lambda)\lambda^{-1} + u_y^{-2}\lambda(\lambda-1)^{-1}\big]dx,$$

where $u \in C^2(\mathbb{R}^2 \times \mathbb{T}^1; \mathbb{R})$, $x \in \mathbb{T}^1$, $\lambda \in \mathbb{C}\backslash\{0, 1\}$, and d denotes the total differential, generates two independent Casimir functionals $\gamma^{(1)}$ and $\gamma^{(2)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$ whose gradients have the following asymptotic expansions:

$$\nabla\gamma^{(1)}(l) \sim u_y + O(\mu^2)$$

as $|\mu| \to 0,\ \mu := \lambda - 1$, and

$$\nabla\gamma^{(2)}(l) \sim u_t + O(\lambda^2)$$

as $|\lambda| \to 0$. The commutativity condition

$$[X^{(y)}, X^{(t)}] = 0 \tag{6.9}$$

for the vector fields

$$X^{(y)} := \partial/\partial y + \nabla h^{(y)}(\tilde{l}), \qquad X^{(t)} := \partial/\partial t + \nabla h^{(t)}(\tilde{l}), \tag{6.10}$$

where

$$\nabla h^{(y)}(\tilde{l}) := -\big(\mu^{-1}\nabla\gamma^{(1)}(\tilde{l})\big)\big|_- = -\frac{u_y}{\lambda - 1}\frac{\partial}{\partial x},$$

$$\nabla h^{(t)}(\tilde{l}) := -\big(\lambda^{-1}\nabla\gamma^{(2)}(\tilde{l})\big)\big|_- = -\frac{u_t}{\lambda}\frac{\partial}{\partial x},$$

leads to the heavenly equations

$$u_{yt} + u_{xt}u_y - u_{xy}u_t = 0.$$

Their Lax–Sato representation is the compatibility condition for the first-order partial differential equations

$$\frac{\partial\psi}{\partial y} - \frac{u_y}{\lambda - 1}\frac{\partial\psi}{\partial x} = 0,$$

$$\frac{\partial\psi}{\partial t} - \frac{u_t}{\lambda}\frac{\partial\psi}{\partial x} = 0,$$

where $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^1_{\mathbb{C}}; \mathbb{C})$.

6.5. Second Generating Equation of Conformal Structure: $u_{xt} + u_x u_{yy} - u_y u_{xy} = 0$. For a seed-element

$$\tilde{l} \in \tilde{\mathcal{G}}^* = \widetilde{\operatorname{diff}}(\mathbb{C} \times \mathbb{T}^1)^*$$

of the form

$$\tilde{l} = \big[u_x^2 + 2u_x^2(u_y + \alpha)\lambda^{-1} + u_x^2(3u_y^2 + 4\alpha u_y + \beta)\lambda^{-2}\big]dx,$$

where $u \in C^2(\mathbb{T}^1 \times \mathbb{R}^2; \mathbb{R}),\ x \in \mathbb{T}^1,\ \lambda \in \mathbb{C}\backslash\{0\}$, and $\alpha, \beta \in \mathbb{R}$, there exists a unique independent Casimir functional $\gamma^{(1)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$ with the following asymptotic expansion of its functional gradient as $|\lambda| \to 0$:

$$\nabla\gamma^{(1)}(l) \sim c_0 u_x^{-1} + (-c_0 u_y + c_1)u_x^{-1}\lambda + (-c_1 u_y + c_2)u_x^{-1}\lambda^2 + O(\lambda^3),$$

where $c_r \in \mathbb{R},\ r = \overline{1,2}$. If we assume that $c_0 = 1,\ c_1 = 0$, and $c_2 = 0$, then we get two functionally independent gradient elements

$$\nabla h^{(y)}(\tilde{l}) := -(\lambda^{-1}\nabla\gamma^{(1)}(\tilde{l}))|_- = -\frac{1}{\lambda u_x}\frac{\partial}{\partial x},$$

$$\nabla h^{(t)}(\tilde{l}) := (\lambda^{-2}\nabla\gamma^{(1)}(\tilde{l}))|_- = \left(\frac{1}{\lambda^2 u_x} - \frac{u_y}{\lambda u_x}\right)\frac{\partial}{\partial x}.$$

The corresponding commutativity condition (6.9) for the vector fields (6.10) leads to the heavenly equations

$$u_{xt} + u_x u_{yy} - u_y u_{xy} = 0.$$

The linearized Lax–Sato representation of these equations is given by the following system of first-order equations:

$$\frac{\partial\psi}{\partial y} - \frac{1}{\lambda u_x}\frac{\partial\psi}{\partial x} = 0,$$

$$\frac{\partial \psi}{\partial t} + \left(\frac{1}{\lambda^2 u_x} - \frac{u_y}{\lambda u_x} \right) \frac{\partial \psi}{\partial x} = 0$$

for the linear vector fields with a function $\psi \in C^2(\mathbb{R}^2 \times \mathbb{C} \times \mathbb{T}^1; \mathbb{C})$.

6.6. Inverse First Reduced Heavenly Shabat Equation. A seed-element $\tilde{l} \in \tilde{\mathcal{G}}^* = \widetilde{\mathrm{diff}}(\mathbb{T}^1_{\mathbb{C}})^*$ of the form

$$\tilde{l} = (a_0 u_y^{-2} u_x^2 (\lambda + 1)^{-1} + a_1 u_x^2 + a_1 u_x^2 \lambda) dx,$$

where $u \in C^2(\mathbb{T}^1 \times \mathbb{R}^2; \mathbb{R})$, $x \in \mathbb{T}^1$, $\lambda \in \mathbb{C} \setminus \{-1\}$, and $a_0, a_1 \in \mathbb{R}$, generates two independent Casimir functionals $\gamma^{(1)}$ and $\gamma^{(2)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$ whose gradients have the following asymptotic expansions:

$$\nabla \gamma^{(1)}(l) \sim u_y u_x^{-1} - u_y u_x^{-1} \mu + O(\mu^2)$$

as $|\mu| \to 0,\ \mu := \lambda + 1$, and

$$\nabla \gamma^{(2)}(l) \sim u_x^{-1} + O(\lambda^{-2})$$

as $|\lambda| \to \infty$. Setting

$$\nabla h^{(y)}(\tilde{l}) := \left(\mu^{-1} \nabla \gamma^{(1)}(\tilde{l}) \right)\big|_- = -\frac{\lambda}{\lambda + 1} \frac{u_y}{u_x} \frac{\partial}{\partial x},$$

$$\nabla h^{(t)}(\tilde{l}) := \left(\lambda \nabla \gamma^{(2)}(\tilde{l}) \right)\big|_+ = \frac{\lambda}{u_x} \frac{\partial}{\partial x},$$

we conclude that the commutativity condition (6.9) for the vector fields (6.10) yields the following heavenly equation:

$$u_{xy} + u_y u_{tx} - u_{ty} u_x = 0,$$

which can be obtained as a result of the following simultaneous change of independent variables: $\mathbb{R} \ni x \to t \in \mathbb{R}$, $\mathbb{R} \ni y \to \in \mathbb{R}$, and $\mathbb{R} \ni t \to y \in \mathbb{R}$ in the first reduced heavenly Shabat equation. The corresponding Lax–Sato representation is specified by the compatibility condition for the first-order equations for the vector fields

$$\frac{\partial \psi}{\partial y} - \frac{\lambda}{\lambda + 1} \frac{u_y}{u_x} \frac{\partial \psi}{\partial x} = 0,$$

$$\frac{\partial \psi}{\partial t} + \frac{\lambda}{u_x} \frac{\partial \psi}{\partial x} = 0,$$

where $\psi \in C^2(\mathbb{R}^2 \times \mathbb{C} \times \mathbb{T}^n; \mathbb{C})$.

6.7. First Plebański Equation and Its Generalizations. A seed-element

$$\tilde{l} \in \tilde{\mathcal{G}}^* = \widetilde{\mathrm{diff}}(\mathbb{C} \times \mathbb{T}^2)^*$$

of the form

$$\tilde{l} = \lambda^{-1}(u_{yx_1} dx_1 + u_{yx_2} dx_2) + (u_{tx_1} dx_1 + u_{tx_2} dx_2) = \lambda^{-1} du_y + du_t, \tag{6.11}$$

where $u \in C^2(\mathbb{T}^2 \times \mathbb{R}^2;\mathbb{R}),\ (x_1,x_2)\in\mathbb{T}^2,\ \lambda\in\mathbb{C}\backslash\{0\}$, and d denotes the total differential, generates two independent Casimir functionals $\gamma^{(1)}$ and $\gamma^{(2)} \in\ \mathrm{I}(\tilde{\mathcal{G}}^*)$ whose gradients have the following asymptotic expansions:

$$\begin{aligned}\nabla\gamma^{(1)}(l) &\sim (-u_{yx_2}, u_{yx_1},)^\top + O(\lambda),\\ \nabla\gamma^{(2)}(l) &\sim (-u_{tx_2}, u_{tx_1})^\top + O(\lambda)\end{aligned} \tag{6.12}$$

as $|\lambda|\to 0$. The commutativity condition (6.9) for the vector fields (6.10), where

$$\nabla h^{(y)}(\tilde{l}) := (\lambda^{-1}\nabla\gamma^{(1)}(\tilde{l}))\,\big|_- = -\frac{u_{yx_2}}{\lambda}\frac{\partial}{\partial x_1} + \frac{u_{yx_1}}{\lambda}\frac{\partial}{\partial x_2},$$

$$\nabla h^{(t)}(\tilde{l}) := (\lambda^{-1}\nabla\gamma^{(2)}(\tilde{l}))\,\big|_- = -\frac{u_{tx_2}}{\lambda}\frac{\partial}{\partial x_1} + \frac{u_{tx_1}}{\lambda}\frac{\partial}{\partial x_2},$$

yields the first Plebański equation [5]:

$$u_{yx_1}u_{tx_2} - u_{yx_2}u_{tx_1} = 1.$$

Its Lax–Sato representation implies the compatibility condition for the first-order partial differential equations

$$\frac{\partial\psi}{\partial y} - \frac{u_{yx_2}}{\lambda}\frac{\partial\psi}{\partial x_1} + \frac{u_{yx_1}}{\lambda}\frac{\partial\psi}{\partial x_2} = 0,$$

$$\frac{\partial\psi}{\partial t} - \frac{u_{tx_2}}{\lambda}\frac{\partial\psi}{\partial x_1} + \frac{u_{tx_1}}{\lambda}\frac{\partial\psi}{\partial x_2} = 0,$$

where $\psi\in C^\infty(\mathbb{R}^2\times\mathbb{C}\times\mathbb{T}^2;\mathbb{C})$.

In view of the fact that the determining condition for the Casimir invariants is symmetric and equivalent to a system of first-order inhomogeneous linear differential equations for the covector function $l = (l_1,l_2)^\top$, the corresponding seed-element can be also chosen in a different form. Moreover, form (6.11) is independent of the space dimension of the torus $\mathbb{T}^n$, which enables us to describe the corresponding generalized conformal metric equations of any dimension.

In particular, we note that the asymptotic expansions (6.12) are also true for the invariant seed-elements

$$\tilde{l} = \lambda^{-1}du_y + du_t.$$

The Lie-algebraic scheme described above can be generalized to an arbitrary dimension $n = 2k$, where $k\in\mathbb{N}$ and $n>2$. In this case, we get $2k$ independent Casimir functionals $\gamma^{(j)}\in\mathrm{I}(\tilde{\mathcal{G}}^*)$, where

$$\tilde{\mathcal{G}}^* = \widetilde{\mathrm{diff}}(\mathbb{T}^{2k})^*, \quad j = \overline{1,2k},$$

with the following asymptotic expansions of their gradients:

$$\nabla\gamma^{(1)}(l) \sim \Big(-u_{yx_2}, u_{yx_1}, \underbrace{0,\ldots,0}_{2k-2}\Big)^\top + O(\lambda),$$

$$\nabla\gamma^{(2)}(l) \sim \Big(-u_{tx_2}, u_{tx_1}, \underbrace{0,\ldots,0}_{2k-2}\Big)^\top + O(\lambda),$$

$$\nabla\gamma^{(3)}(l) \sim \Big(0, 0, -u_{yx_4}, u_{yx_3}, \underbrace{0,\ldots,0}_{2k-4}\Big)^\top + O(\lambda),$$

$$\nabla\gamma^{(4)}(l) \sim \Big(0, 0, -u_{tx_4}, u_{tx_3}, \underbrace{0,\ldots,0}_{2k-4}\Big)^\top + O(\lambda),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\nabla\gamma^{(2k-1)}(l) \sim \Big(\underbrace{0,\ldots,0}_{2k-2}, -u_{yx_{2k}}, u_{yx_{2k-1}}\Big)^\top + O(\lambda),$$

$$\nabla\gamma^{(2k)}(l) \sim \Big(\underbrace{0,\ldots,0}_{2k-2}, -u_{tx_{2k}}, u_{tx_{2k-1}}\Big)^\top + O(\lambda).$$

Assume that

$$\nabla h^{(y)}(\tilde{l}) := \Big(\lambda^{-1}\Big(\nabla\gamma^{(1)}(\tilde{l}) + \ldots + \nabla\gamma^{(2k-1)}(\tilde{l})\Big)\Big)\Big|_- = -\sum_{m=1}^{k}\left(\frac{u_{yx_{2m}}}{\lambda}\frac{\partial}{\partial x_{2m-1}} - \frac{u_{yx_{2m-1}}}{\lambda}\frac{\partial}{\partial x_{2m}}\right),$$

$$\nabla h^{(t)}(\tilde{l}) := \Big(\lambda^{-1}\Big(\nabla\gamma^{(2)}(\tilde{l}) + \ldots + \nabla\gamma^{(2k)}(\tilde{l})\Big)\Big)\Big|_- = -\sum_{m=1}^{k}\left(\frac{u_{tx_{2m}}}{\lambda}\frac{\partial}{\partial x_{2m-1}} - \frac{u_{tx_{2m-1}}}{\lambda}\frac{\partial}{\partial x_{2m}}\right).$$

Thus, the commutativity condition (6.9) for the vector fields (6.10) yields the following multidimensional analogs of the first heavenly Plebański equation:

$$\sum_{m=1}^{k}(u_{yx_{2m-1}}u_{tx_{2m}} - u_{yx_{2m}}u_{tx_{2m-1}}) = 1.$$

6.8. Modified Heavenly Plebański Equation and Its Generalizations. For a seed-element $\tilde{l} \in \tilde{\mathcal{G}}^* = \widetilde{\mathrm{diff}}(\mathbb{T}^2)^*$ of the form

$$\tilde{l} = (\lambda^{-1}u_{x_1y} + u_{x_1x_1} - u_{x_1x_2} + \lambda)dx_1 + (\lambda^{-1}u_{x_2y} + u_{x_1x_2} - u_{x_2x_2} + \lambda)dx_2 = d(\lambda^{-1}u_y + u_{x_1} - u_{x_2} + \lambda x_1 + \lambda x_2), \tag{6.13}$$

where $d\lambda = 0,\ u \in C^2(\mathbb{T}^2 \times \mathbb{R}^2; \mathbb{R}),\ (x_1, x_2) \in \mathbb{T}^2$, and $\lambda \in \mathbb{C}\backslash\{0\}$, there exist two independent Casimir functionals $\gamma^{(1)}$ and $\gamma^{(2)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$ with the following asymptotic expansions of the gradients:

$$\nabla\gamma^{(1)}(l) \sim (u_{yx_2}, -u_{yx_1})^\top + O(\lambda)$$

as $|\lambda| \to 0$ and

$$\nabla\gamma^{(2)}(l) \sim (0, -1)^\top + (-u_{x_2x_2}, u_{x_1x_2})^\top \lambda^{-1} + O(\lambda^{-2})$$

as $|\lambda| \to \infty$. In the case where

$$\nabla h^{(y)}(\tilde{l}) := (\lambda^{-1}\nabla\gamma^{(1)}(\tilde{l}))|_- = \frac{u_{yx_2}}{\lambda}\frac{\partial}{\partial x_1} - \frac{u_{yx_1}}{\lambda}\frac{\partial}{\partial x_2},$$

$$\nabla h^{(t)}(\tilde{l}) := (\lambda\nabla\gamma^{(2)}(\tilde{l}))|_+ = -u_{x_2x_2}\frac{\partial}{\partial x_1} + (u_{x_1x_2} - \lambda)\frac{\partial}{\partial x_2},$$

the commutativity condition (6.9) for the vector fields (6.10) yields the modified heavenly Plebański equation [5]:

$$u_{yt} - u_{yx_1}u_{x_2x_2} + u_{yx_2}u_{x_1x_2} = 0 \tag{6.14}$$

with the Lax–Sato representation given by the first-order partial differential equations

$$\frac{\partial\psi}{\partial y} - \frac{u_{yx_2}}{\lambda}\frac{\partial\psi}{\partial x_1} + \frac{u_{yx_1}}{\lambda}\frac{\partial\psi}{\partial x_2} = 0,$$

$$\frac{\partial\psi}{\partial t} - u_{x_2x_2}\frac{\partial\psi}{\partial x_1} + (u_{x_1x_2} - \lambda)\frac{\partial\psi}{\partial x_2} = 0$$

for the functions $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^2_{\mathbb{C}}; \mathbb{C})$.

The differential-geometric form of seed-element (6.13) is also independent of the dimension of additional space variables on the torus $\mathbb{T}^n,\ n > 2$. This leads to the natural problem of determination of the corresponding multidimensional generalizations of the modified heavenly Plebański equation (6.14).

Selecting a seed-element $\tilde{l} \in \tilde{\mathcal{G}}^* = \widetilde{\mathrm{diff}}(\mathbb{T}^{2k})^*$ in the form (6.13), where $u \in C^2(\mathbb{T}^{2k} \times \mathbb{R}^2; \mathbb{R})$, we arrive at the following asymptotic expansions for the gradients of $2k \in \mathbb{N}$ independent Casimir functionals $\gamma^{(j)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$, where $\tilde{\mathcal{G}}^* = \widetilde{\mathrm{diff}}(\mathbb{T}^{2k})^*,\ j = \overline{1, 2k}$:

$$\nabla\gamma^{(1)}(l) \sim \Big(-u_{yx_2}, u_{yx_1}, \underbrace{0, \ldots, 0}_{2k-2}\Big)^\top + O(\lambda),$$

$$\nabla\gamma^{(3)}(l) \sim \Big(0, 0, -u_{yx_4}, u_{yx_3}, \underbrace{0, \ldots, 0}_{2k-4}\Big)^\top + O(\lambda),$$

. . . .

$$\nabla\gamma^{(2k-1)}(l) \sim \Big(\underbrace{0, \ldots, 0}_{2k-2}, -u_{yx_{2k}}, u_{yx_{2k-1}}\Big)^\top + O(\lambda)$$

as $|\lambda| \to 0$ and

$$\nabla\gamma^{(2)}(l) \sim \Big(0,-1,\underbrace{0,\ldots,0}_{2k-2}\Big)^{\top} + \Big(-u_{x_2x_2},u_{x_1x_2},\underbrace{0,\ldots,0}_{2k-2}\Big)^{\top}\lambda^{-1}+O(\lambda^{-2}),$$

$$\nabla\gamma^{(4)}(l) \sim \Big(0,0,-u_{x_4x_2},u_{x_3x_2},\underbrace{0,\ldots,0}_{2k-4}\Big)^{\top}\lambda^{-1}+O(\lambda^{-2}),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\nabla\gamma^{(2k)}(l) \sim \Big(\underbrace{0,\ldots,0}_{2k-2},-u_{x_{2k}x_2},u_{x_{2k-1}x_2}\Big)^{\top}\lambda^{-1}+O(\lambda^{-2})$$

as $|\lambda| \to \infty$. In the case where

$$\nabla h^{(y)}(\tilde l) := -(\lambda^{-1}\Big(\nabla\gamma^{(1)}(\tilde l)+\ldots+\nabla\gamma^{(2k-1)}(\tilde l))\Big)\Big|_{-}$$

$$=\sum_{m=1}^{k}\left(\frac{u_{yx_{2m}}}{\lambda}\frac{\partial}{\partial x_{2m-1}}-\frac{u_{yx_{2m-1}}}{\lambda}\frac{\partial}{\partial x_{2m}}\right),$$

$$\nabla h^{(t)}(\tilde l) := \Big(\lambda\Big(\nabla\gamma^{(2)}(\tilde l)+\ldots+\nabla\gamma^{(2k)}(\tilde l)\Big)\Big)\Big|_{+}$$

$$=-u_{x_2x_2}\frac{\partial}{\partial x_1}+(u_{x_1x_2}-\lambda)\frac{\partial}{\partial x_2}-\sum_{m=2}^{k}\left(u_{x_{2m}x_2}\frac{\partial}{\partial x_{2m-1}}-u_{x_{2m-1}x_2}\frac{\partial}{\partial x_{2m}}\right),$$

the commutativity condition (6.9) for the vector fields (6.10) yields the following multidimensional analogs of the modified heavenly Plebański equation:

$$u_{yt}-\sum_{m=1}^{k}(u_{yx_{2m}}u_{x_2x_{2m-1}}-u_{yx_{2m-1}}u_{x_2x_{2m}})=0.$$

6.9. Heavenly Husain Equation and Its Generalizations. A seed-element $\tilde l \in \tilde{\mathcal G}^* = \widetilde{\operatorname{diff}}(\mathbb T^2)^*$ of the form

$$\tilde l = \frac{d(u_y+iu_t)}{\lambda-i}+\frac{d(u_y-iu_t)}{\lambda+i}=\frac{2(\lambda du_y-du_t)}{\lambda^2+1}, \tag{6.15}$$

where $i^2=-1,\ d\lambda=0,\ u\in C^2(\mathbb T^2\times\mathbb R^2;\mathbb R),\ (x_1,x_2)\in\mathbb T^2$, and $\lambda\in\mathbb C\backslash\{-i;i\}$, generates two independent Casimir functionals $\gamma^{(1)}$ and $\gamma^{(2)}\in\ \mathrm I(\tilde{\mathcal G}^*)$ with the following asymptotic expansions for the gradients:

$$\nabla\gamma^{(1)}(l) \sim \frac12(-u_{yx_2}-iu_{tx_2},u_{yx_1}+iu_{tx_1})^{\top}+O(\mu),\quad \mu:=\lambda-i,$$

as $|\mu| \to 0$ and

$$\nabla \gamma^{(2)}(l) \sim \frac{1}{2}(-u_{yx_2} + iu_{tx_2}, u_{yx_1} - iu_{tx_1})^\top + O(\xi), \quad \xi := \lambda + i,$$

as $|\xi| \to 0$. In the case where

$$\begin{aligned}
\nabla h^{(y)}(\tilde{l}) &:= (\mu^{-1}\nabla\gamma^{(1)}(\tilde{l}) + \xi^{-1}\nabla\gamma^{(2)}(\tilde{l}))|_- \\
&= \frac{1}{2\mu}\left((-u_{yx_2} - iu_{tx_2})\frac{\partial}{\partial x_1} + (u_{yx_1} + iu_{tx_1})\frac{\partial}{\partial x_2}\right) \\
&\quad + \frac{1}{2\xi}\left((-u_{yx_2} + iu_{tx_2})\frac{\partial}{\partial x_1} + (u_{yx_1} - iu_{tx_1})\frac{\partial}{\partial x_2}\right) \\
&= \frac{u_{tx_2} - \lambda u_{yx_2}}{\lambda^2+1}\frac{\partial}{\partial x_1} + \frac{\lambda u_{yx_1} - u_{tx_1}}{\lambda^2+1}\frac{\partial}{\partial x_2},
\end{aligned}$$

$$\begin{aligned}
\nabla h^{(t)}(\tilde{l}) &:= (-\mu^{-1}i\nabla\gamma^{(1)}(\tilde{l}) + \xi^{-1}i\nabla\gamma^{(2)}(\tilde{l}))|_- \\
&= \frac{1}{2\mu}\left((-u_{tx_2} + iu_{yx_2})\frac{\partial}{\partial x_1} + (u_{tx_1} - iu_{yx_1})\frac{\partial}{\partial x_2}\right) \\
&\quad + \frac{1}{2\xi}\left(-(u_{tx_2} + iu_{yx_2})\frac{\partial}{\partial x_1} + (u_{tx_1} + iu_{yx_1})\frac{\partial}{\partial x_2}\right) \\
&= -\frac{u_{yx_2} + \lambda u_{tx_2}}{\lambda^2+1}\frac{\partial}{\partial x_1} + \frac{u_{yx_1} + \lambda u_{tx_1}}{\lambda^2+1}\frac{\partial}{\partial x_2},
\end{aligned}$$

the commutativity condition (6.9) for the vector fields (6.10) gives the heavenly Husain equation [5]

$$u_{yy} + u_{tt} + u_{yx_1}u_{tx_2} - u_{yx_2}u_{tx_1} = 0 \tag{6.16}$$

with the Lax–Sato representation given by the following first-order partial differential equations:

$$\begin{aligned}
&\frac{\partial\psi}{\partial y} + \frac{u_{tx_2} - \lambda u_{yx_2}}{\lambda^2+1}\frac{\partial\psi}{\partial x_1} + \frac{\lambda u_{yx_1} - u_{tx_1}}{\lambda^2+1}\frac{\partial\psi}{\partial x_2} = 0, \\
&\frac{\partial\psi}{\partial t} - \frac{u_{yx_2} + \lambda u_{tx_2}}{\lambda^2+1}\frac{\partial\psi}{\partial x_1} + \frac{u_{yx_1} + \lambda u_{tx_1}}{\lambda^2+1}\frac{\partial\psi}{\partial x_2} = 0,
\end{aligned}$$

where $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^2_{\mathbb{C}}; \mathbb{C})$.

The differential-geometric form of the seed-element (6.15) is also independent of the dimension of additional space variables on the torus $\mathbb{T}^n,\ n > 2$, which opens the problem of determination of the corresponding multidimensional generalizations of the heavenly Husain equation (6.16).

For a seed-element $\tilde{l} \in \tilde{\mathcal{G}}^* = \widetilde{\mathrm{diff}}(\mathbb{T}^{2k})^*$ chosen in the form (6.15), where $u \in C^2(\mathbb{T}^{2k} \times \mathbb{R}^2; \mathbb{R})$, we obtain the following asymptotic expansions for the gradients of $2k \in \mathbb{N}$ independent Casimir functionals $\gamma^{(j)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$, where $\tilde{\mathcal{G}}^* = \widetilde{\mathrm{diff}}(\mathbb{T}^{2k})^*,\ j = \overline{1, 2k}$:

$$\nabla\gamma^{(1)}(l) \sim \frac{1}{2}\Big(-u_{yx_2} - iu_{tx_2}, u_{yx_1} + iu_{tx_1}, \underbrace{0, \ldots, 0}_{2k-2}\Big)^\top + O(\mu),$$

$$\nabla\gamma^{(3)}(l) \sim \frac{1}{2}\Big(0, 0, -u_{yx_4} - iu_{tx_4}, u_{yx_3} + iu_{tx_3}, \underbrace{0, \ldots, 0}_{2k-4}\Big)^\top + O(\mu),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\nabla\gamma^{(2k-1)}(l) \sim \frac{1}{2}\Big(\underbrace{0, \ldots, 0}_{2k-2}, -u_{yx_{2k}} - iu_{tx_{2k}}, u_{yx_{2k-1}} + iu_{tx_{2k-1}}\Big)^\top + O(\mu)$$

as $|\mu| \to 0$ and

$$\nabla\gamma^{(2)}(l) \sim \frac{1}{2}\Big(-u_{yx_2} + iu_{tx_2}, u_{yx_1} - iu_{tx_1}, \underbrace{0, \ldots, 0}_{2k-2}\Big)^\top + O(\xi),$$

$$\nabla\gamma^{(4)}(l) \sim \frac{1}{2}\Big(0, 0, -u_{yx_4} + iu_{tx_4}, u_{yx_3} - iu_{tx_3}, \underbrace{0, \ldots, 0}_{2k-4}\Big)^\top + O(\xi),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\nabla\gamma^{(2k)}(l) \sim \frac{1}{2}\Big(\underbrace{0, \ldots, 0}_{2k-2}, -u_{yx_{2k}} + iu_{tx_{2k}}, u_{yx_{2k-1}} - iu_{tx_{2k-1}}\Big)^\top + O(\xi)$$

as $|\xi| \to 0$. In the case where

$$\begin{aligned}\nabla h^{(y)}(\tilde{l}) &:= \sum_{m=1}^{k} (\mu^{-1}\nabla\gamma^{(2m-1)}(\tilde{l}) + \xi^{-1}\nabla\gamma^{(2m)}(\tilde{l}))\big|_- \\ &= \sum_{m=1}^{k}\left(\frac{u_{tx_{2m}} - \lambda u_{yx_{2m}}}{\lambda^2+1}\frac{\partial}{\partial x_{2m-1}} + \frac{\lambda u_{yx_{2m-1}} - u_{tx_{2m-1}}}{\lambda^2+1}\frac{\partial}{\partial x_{2m}}\right),\end{aligned}$$

$$\begin{aligned}\nabla h^{(t)}(\tilde{l}) &:= \sum_{m=1}^{k} i(-\mu^{-1}\nabla\gamma^{(2m-1)}(\tilde{l}) + \xi^{-1}\nabla\gamma^{(2m)}(\tilde{l}))\big|_- \\ &= \sum_{m=1}^{k}\left(-\frac{u_{yx_{2m}} + \lambda u_{tx_{2m}}}{\lambda^2+1}\frac{\partial}{\partial x_{2m-1}} + \frac{u_{yx_{2m-1}} + \lambda u_{tx_{2m-1}}}{\lambda^2+1}\frac{\partial}{\partial x_{2m}}\right),\end{aligned}$$

the commutativity condition (6.9) for the vector fields (6.10) yields the multidimensional analogs of the heavenly

Husain equation:

$$u_{yy} + u_{tt} + \sum_{m=1}^{k} (u_{yx_{2m-1}} u_{tx_{2m}} - u_{yx_{2m}} u_{x_2 x_{2m-1}}) = 0.$$

6.10. General Heavenly Monge Equation and Its Generalizations. A seed-element

$$\tilde{l} \in \tilde{\mathcal{G}}^* = \widetilde{\mathrm{diff}}(\mathbb{C} \times \mathbb{T}^4)^*$$

of the form

$$\tilde{l} = du_y + \lambda^{-1}(dx_1 + dx_2),$$

where $u \in C^2(\mathbb{T}^4 \times \mathbb{R}^2; \mathbb{R}),\ (x_1, x_2, x_3, x_4) \in \mathbb{T}^4$, and $\lambda \in \mathbb{C}\backslash\{0\}$, generates four independent Casimir functionals $\gamma^{(1)},\ \gamma^{(2)},\ \gamma^{(3)}$, and $\gamma^{(4)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$ whose gradients have the following asymptotic expansions:

$$\nabla\gamma^{(1)}(l) \sim (0, 1, 0, 0)^\top$$
$$+ \big(-u_{yx_2} - (\partial_{x_2} - \partial_{x_1})^{-1} u_{yx_2x_1}, (\partial_{x_2} - \partial_{x_1})^{-1} u_{yx_2x_1}, 0, 0\big)^\top \lambda + O(\lambda^2),$$

$$\nabla\gamma^{(2)}(l) \sim (1, 0, 0, 0)^\top$$
$$+ (\partial_{x_1} - \partial_{x_2})^{-1} u_{yx_1x_2}, -u_{yx_1} - (\partial_{x_1} - \partial_{x_2})^{-1} u_{yx_1x_2}, 0, 0)^\top \lambda + O(\lambda^2),$$

$$\nabla\gamma^{(3)}(l) \sim (0, 0, -u_{yx_4}, u_{yx_3})^\top + O(\lambda^2),$$

$$\nabla\gamma^{(4)}(l) \sim (0, 0, -u_{tx_4}, u_{tx_3})^\top$$
$$+ \big(u_{yx_3} u_{tx_4} - u_{yx_4} u_{tx_3}, 0, u_{yx_4} u_{tx_1} - u_{yx_1} u_{tx_4}, u_{yx_1} u_{tx_3} - u_{yx_3} u_{tx_1}\big)^\top \lambda + O(\lambda^2)$$

as $|\lambda| \to 0$. In the case where

$$\nabla h^{(y)}(\tilde{l}) := (\lambda^{-1}(\nabla\gamma^{(1)}(\tilde{l}) + \nabla\gamma^{(3)}(\tilde{l})))|_-$$
$$= 0\frac{\partial}{\partial x_1} + \frac{1}{\lambda}\frac{\partial}{\partial x_2} - \frac{u_{yx_4}}{\lambda}\frac{\partial}{\partial x_3} + \frac{u_{yx_3}}{\lambda}\frac{\partial}{\partial x_4},$$

$$\nabla h^{(t)}(\tilde{l}) := \big(\lambda^{-1}(-\nabla\gamma^{(2)}(\tilde{l}) + \nabla\gamma^{(4)}(\tilde{l}))\big)\ |_-$$
$$= -\frac{1}{\lambda}\frac{\partial}{\partial x_1} + 0\frac{\partial}{\partial x_2} - \frac{u_{tx_4}}{\lambda}\frac{\partial}{\partial x_3} + \frac{u_{tx_3}}{\lambda}\frac{\partial}{\partial x_4},$$

the commutativity condition (6.9) for the vector fields (6.10) yields the following general heavenly Monge equation [6]:

$$u_{yx_1} + u_{tx_2} + u_{yx_3} u_{tx_4} - u_{yx_4} u_{tx_3} = 0$$

with the Lax–Sato representation given by the following first-order partial differential equations:

$$\frac{\partial\psi}{\partial y}+\frac{1}{\lambda}\frac{\partial\psi}{\partial x_2}-\frac{u_{yx_4}}{\lambda}\frac{\partial\psi}{\partial x_3}+\frac{u_{yx_3}}{\lambda}\frac{\partial\psi}{\partial x_4}=0,$$

$$\frac{\partial\psi}{\partial t}-\frac{1}{\lambda}\frac{\partial\psi}{\partial x_1}-\frac{u_{tx_4}}{\lambda}\frac{\partial\psi}{\partial x_3}+\frac{u_{tx_3}}{\lambda}\frac{\partial\psi}{\partial x_4}=0,$$

where $\psi\in C^2(\mathbb{R}^2\times\mathbb{C}\times\mathbb{T}^n;\mathbb{C})$ and $\lambda\in\mathbb{C}\backslash\{0\}$.

In view of the fact that the condition imposed on the Casimir invariants is equivalent to a system of homogeneous linear differential equations for the covector function $l=(l_1,l_2,l_3,l_4)^\intercal$, the corresponding seed-element can be chosen in a different form. Thus, if the expression

$$\tilde{l}=du_t+\lambda^{-1}(dx_1+dx_2)$$

is regarded as a seed-element, then it generates four independent Casimir functionals $\gamma^{(1)}$, $\gamma^{(2)}$, $\gamma^{(3)}$, and $\gamma^{(4)}\in\mathrm{I}(\tilde{\mathcal{G}}^*)$ whose gradients have the following asymptotic expansions:

$$\begin{aligned}\nabla\gamma^{(1)}(l)&\sim(0,1,0,0)^\top\\&\quad+\big(-u_{tx_2}-(\partial_{x_2}-\partial_{x_1})^{-1}u_{tx_2x_1},(\partial_{x_2}-\partial_{x_1})^{-1}u_{tx_2x_1},0,0\big)^\top\lambda+O(\lambda^2),\\\nabla\gamma^{(2)}(l)&\sim(1,0,0,0)^\top\\&\quad+\big((\partial_{x_1}-\partial_{x_2})^{-1}u_{tx_1x_2},-u_{tx_1}-(\partial_{x_1}-\partial_{x_2})^{-1}u_{tx_1x_2},0,0\big)^\top\lambda+O(\lambda^2),\\\nabla\gamma^{(3)}(l)&\sim(0,0,-u_{tx_4},u_{tx_3})^\top\\&\quad+\big(0,u_{tx_3}u_{yx_4}-u_{tx_4}u_{yx_3},u_{tx_4}u_{yx_2}-u_{tx_2}u_{yx_4},u_{tx_2}u_{yx_3}-u_{tx_3}u_{yx_2}\big)^\top\lambda+O(\lambda^2),\\\nabla\gamma^{(4)}(l)&\sim(0,0,-u_{yx_4},u_{yx_3})^\top+O(\lambda^2)\end{aligned}$$

as $|\lambda|\to 0$. For a seed-element of the form

$$\tilde{l}=du_y+du_t+\lambda^{-1}(dx_1+dx_2), \tag{6.17}$$

the asymptotic expansions for the gradients of four independent Casimir functionals $\gamma^{(1)}$, $\gamma^{(2)}$, $\gamma^{(3)}$, and $\gamma^{(4)}\in\mathrm{I}(\tilde{\mathcal{G}}^*)$ can be represented in the form:

$$\begin{aligned}\nabla\gamma^{(1)}(l)&\sim(0,1,0,0)^\top\\&\quad+\big(-(u_{yx_2}+u_{tx_2})-(\partial_{x_2}-\partial_{x_1})^{-1}(u_{yx_2x_1}+u_{tx_2x_1}),\\&\qquad(\partial_{x_2}-\partial_{x_1})^{-1}(u_{yx_2x_1}+u_{tx_2x_1}),0,0\big)^\top\lambda\;+\;O(\lambda^2),\end{aligned}$$

$$\nabla\gamma^{(2)}(l) \sim (1,0,0,0)^\top + \big((\partial_{x_1}-\partial_{x_2})^{-1}(u_{yx_1x_2}+u_{tx_1x_2}), -(u_{yx_1}+u_{tx_1})-(\partial_{x_1}-\partial_{x_2})^{-1}(u_{yx_1x_2}+u_{tx_1x_2}),0,0\big)^\top\lambda + O(\lambda^2),$$

$$\nabla\gamma^{(3)}(l) \sim (0,0,-u_{yx_4},u_{yx_3})^\top + \big(0, u_{tx_3}u_{yx_4}-u_{tx_4}u_{yx_3}, u_{tx_4}u_{yx_2}-u_{tx_2}u_{yx_4}, u_{tx_2}u_{yx_3}-u_{tx_3}u_{yx_2}\big)^\top\lambda + O(\lambda^2),$$

$$\nabla\gamma^{(4)}(l) \sim (0,0,-u_{tx_4},u_{tx_3})^\top + \big(u_{yx_3}u_{tx_4}-u_{yx_4}u_{tx_3}, 0, u_{yx_4}u_{tx_1}-u_{yx_1}u_{tx_4}, u_{yx_1}u_{tx_3}-u_{yx_3}u_{tx_1}\big)^\top\lambda + O(\lambda^2)$$

as $|\lambda|\to 0$.

The scheme outlined above can be generalized for all $n=2k$, where $k\in\mathbb{N}$ and $n>2$. In this case, we have $2k$ independent Casimir functionals $\gamma^{(j)}\in \mathrm{I}(\tilde{\mathcal{G}}^*)$, where

$$\tilde{\mathcal{G}}^* = \widetilde{\operatorname{diff}}(\mathbb{C}\times\mathbb{T}^{2k})^*, \quad j=\overline{1,2k},$$

and the asymptotic expansions for their gradients are given by the formulas:

$$\nabla\gamma^{(1)}(l) \sim (0,1,\underbrace{0,\ldots,0}_{2k-2})^\top + \big(-(u_{yx_2}+u_{tx_2})-(\partial_{x_2}-\partial_{x_1})^{-1}(u_{yx_2x_1}+u_{tx_2x_1}), (\partial_{x_2}-\partial_{x_1})^{-1}(u_{yx_2x_1}+u_{tx_2x_1}),\underbrace{0,\ldots,0}_{2k-2}\big)^\top\lambda \;+\; O(\lambda^2),$$

$$\nabla\gamma^{(2)}(l) \sim (1,0,\underbrace{0,\ldots,0}_{2k-2})^\top + \big((\partial_{x_1}-\partial_{x_2})^{-1}(u_{yx_1x_2}+u_{tx_1x_2}), -(u_{yx_1}+u_{tx_1})-(\partial_{x_1}-\partial_{x_2})^{-1}(u_{yx_1x_2}+u_{tx_1x_2}),0,0\big)^\top\lambda + O(\lambda^2),$$

$$\nabla\gamma^{(3)}(l) \sim (0,0,-u_{yx_4},u_{yx_3},\underbrace{0,\ldots,0}_{2k-4})^\top$$

$$+ \big(0, u_{tx_3}u_{yx_4} - u_{tx_4}u_{yx_3},$$

$$u_{tx_4}u_{yx_2} - u_{tx_2}u_{yx_4}, u_{tx_2}u_{yx_3} - u_{tx_3}u_{yx_2}, \underbrace{0,\dots,0}_{2k-4}\big)^\top \lambda + O(\lambda^2),$$

$$\nabla\gamma^{(4)}(l) \sim (0,0,-u_{tx_4}, u_{tx_3}, \underbrace{0,\dots,0}_{2k-4})^\top$$

$$+ \big(u_{yx_3}u_{tx_4} - u_{yx_4}u_{tx_3}, 0,$$

$$u_{yx_4}u_{tx_1} - u_{yx_1}u_{tx_4}, u_{yx_1}u_{tx_3} - u_{yx_3}u_{tx_1}, \underbrace{0,\dots,0}_{2k-4}\big)^\top \lambda + O(\lambda^2),$$

$$\nabla\gamma^{(2k-1)}(l) \sim (\underbrace{0,\dots,0}_{2k-4}, 0, 0, -u_{yx_{2k}}, u_{yx_{2k-1}})^\top$$

$$+ \big(\underbrace{0,\dots,0}_{2k-4}, 0, u_{tx_{2k-1}}u_{yx_{2k}} - u_{tx_{2k}}u_{yx_{2k-1}},$$

$$u_{tx_{2k}}u_{yx_2} - u_{tx_2}u_{yx_{2k}}, u_{tx_2}u_{yx_{2k-1}} - u_{tx_{2k-1}}u_{yx_2}\big)^\top \lambda + O(\lambda^2),$$

$$\nabla\gamma^{(2k)}(l) \sim \big(\underbrace{0,\dots,0}_{2k-4}, 0, 0, -u_{tx_{2k}}, u_{tx_{2k-1}}\big)^\top$$

$$+ \big(\underbrace{0,\dots,0}_{2k-4}, u_{yx_{2k-1}}u_{tx_{2k}} - u_{yx_{2k}}u_{tx_{2k-1}}, 0,$$

$$u_{yx_{2k}}u_{tx_1} - u_{yx_1}u_{tx_{2k}}, u_{yx_1}u_{tx_{2k-1}} - u_{yx_{2k-1}}u_{tx_1}\big)^\top \lambda + O(\lambda^2)$$

if a seed-element $\tilde{l} \in \tilde{\mathcal{G}}^*$ is chosen in the form (6.17). If

$$\nabla h^{(y)}(\tilde{l}) := \big(\lambda^{-1}(\nabla\gamma^{(1)}(\tilde{l}) + \nabla\gamma^{(3)}(\tilde{l}) + \ldots + \nabla\gamma^{(2k-1)}(\tilde{l}))\big)\,\big|_-$$

$$= 0\frac{\partial}{\partial x_1} + \frac{1}{\lambda}\frac{\partial}{\partial x_2} - \frac{u_{yx_4}}{\lambda}\frac{\partial}{\partial x_3}$$

$$+ \frac{u_{yx_3}}{\lambda}\frac{\partial}{\partial x_4} + \ldots - \frac{u_{yx_{2k}}}{\lambda}\frac{\partial}{\partial x_{2k-1}} + \frac{u_{yx_{2k-1}}}{\lambda}\frac{\partial}{\partial x_{2k}}$$

$$= 0\frac{\partial}{\partial x_1} + \frac{1}{\lambda}\frac{\partial}{\partial x_2} - \sum_{j=2}^{k}\left(\frac{u_{yx_{2j}}}{\lambda}\frac{\partial}{\partial x_{2j-1}} - \frac{u_{yx_{2j-1}}}{\lambda}\frac{\partial}{\partial x_{2j}}\right),$$

$$\nabla h^{(t)}(\tilde{l}) := \big(\lambda^{-1}(-\nabla\gamma^{(2)}(\tilde{l}) + \nabla\gamma^{(4)}(\tilde{l}) + \ldots + \nabla\gamma^{(2k)}(\tilde{l}))\big)\,\big|_-$$

$$= -\frac{1}{\lambda}\frac{\partial}{\partial x_1} + 0\frac{\partial}{\partial x_2} - \frac{u_{tx_4}}{\lambda}\frac{\partial}{\partial x_3}$$

$$+ \frac{u_{tx_3}}{\lambda}\frac{\partial}{\partial x_4} + \ldots - \frac{u_{tx_{2k}}}{\lambda}\frac{\partial}{\partial x_{2k-1}} + \frac{u_{tx_{2k-1}}}{\lambda}\frac{\partial}{\partial x_{2k}}$$

$$= -\frac{1}{\lambda}\frac{\partial}{\partial x_1} + 0\frac{\partial}{\partial x_2} - \sum_{j=2}^{k}\left(\frac{u_{tx_{2k}}}{\lambda}\frac{\partial}{\partial x_{2k-1}} - \frac{u_{tx_{2k-1}}}{\lambda}\frac{\partial}{\partial x_{2k}}\right),$$

then the compatibility condition (6.9) for the vector fields (6.10) yields the following multidimensional analogs of the general Monge equation:

$$u_{yx_1} + u_{tx_2} + \sum_{j=2}^{k}(u_{yx_{2j-1}}u_{tx_{2j}} - u_{yx_{2j}}u_{tx_{2j-1}}) = 0.$$

7. Superanalogs of the Heavenly Whitham Equation

Assume that the element $\tilde{l} \in \tilde{\mathcal{G}}^*$, where

$$\tilde{\mathcal{G}} := \tilde{\mathrm{diff}}(\mathbb{T}^{1|N}) = \tilde{\mathrm{diff}}_+(\mathbb{T}^{1|N}) \oplus \tilde{\mathrm{diff}}_-(\mathbb{T}^{1|N}),$$

is a loop Lie algebra of superconformal diffeomorphisms of the group $\tilde{\mathrm{Diff}}(\mathbb{T}^{1|N})$ of vector fields on a $(1,1)|N$-dimensional supertorus

$$\mathbb{T}^{(1,1)|N} := \mathbb{C} \times \mathbb{T}^1 \times \Lambda_1^N$$

(see [10]) embedded in the finite-dimensional Grassmann algebra $\Lambda := \Lambda_0 \oplus \Lambda_1$ over $\mathbb{C}$, $\Lambda_0 \supset \mathbb{R}$, which yields the following asymptotic expansions for the gradients of Casimir invariants: $h^{(1)}, h^{(2)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$:

$$\nabla h^{(1)}(l) \sim w_y + O(\lambda) \tag{7.1}$$

as $|\lambda| \to 0$ and

$$\nabla h^{(2)}(l) \sim 1 - w_x\lambda^{-1} + O(\lambda^{-2}) \tag{7.2}$$

as $|\lambda| \to \infty$. Then the compatibility condition for the Hamiltonian flows

$$\begin{aligned} d\tilde{l}/dy &= ad^*_{\nabla h_-^{(y)}(\tilde{l})}\tilde{l}, \qquad \nabla h_-^{(y)}(l) = -(\lambda^{-1}\nabla h^1(l))_- = -w_y\lambda^{-1}, \\ d\tilde{l}/dt &= -ad^*_{\nabla h_+^{(t)}(\tilde{l})}\tilde{l}, \qquad \nabla h_+^{(t)}(l) = -(\lambda\nabla h^{(2)}(l))_+ = -\lambda + w_x, \end{aligned} \tag{7.3}$$

yields the following heavenly-type equations:

$$w_{yt} = w_x w_{yx} - w_y w_{xx} - \frac{1}{2}\sum_{i=1}^{N}(D_{\vartheta_i}w_x)(D_{\vartheta_i}w_y), \tag{7.4}$$

where $w \in C^2(\mathbb{R}^2 \times \mathbb{T}^{(1,1)|N}; \Lambda_0)$ and

$$D_{\vartheta_i} := \partial/\partial\vartheta_i + \vartheta_i\partial/\partial x, \quad i = \overline{1, N},$$

are the superderivatives with respect to the anticommuting variables $\vartheta_i \in \Lambda_1,\ i = \overline{1, N}$.

This equation can be regarded as a supergeneralization of the heavenly Whitham equation [11, 12, 30] for any $N \in \mathbb{N}$. The compatibility condition for the first-order partial differential equations

$$\psi_y + \frac{1}{\lambda}\left(w_y\psi_x + \frac{1}{2}\sum_{i=1}^{N}(D_{\vartheta_i}w_y)(D_{\vartheta_i}\psi)\right) = 0,$$

$$\psi_t + (-\lambda + w_x)\psi_x + \frac{1}{2}\sum_{i=1}^{N}(D_{\vartheta_i}w_x)(D_{\vartheta_i}\psi) = 0,$$

where $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^{(1,1)|N}; \Lambda_0)$ and $\lambda \in \mathbb{C}\backslash\{0\}$, yields the corresponding Lax–Sato representation for the heavenly-type equation (7.4).

Moreover, as a result of simple calculations, we obtain the corresponding seed-element $\tilde{l} := ldx \in \tilde{\mathcal{G}}^*$ from the equation for the Casimir invariant. For any $N \in \mathbb{N}$, this element can be represented in the following form:

$$l = Ca^{-\frac{4-N}{2}}, \quad a := \nabla h(l).$$

Here, the scalar function $C = C(x; \vartheta)$ satisfies the linear homogeneous differential equation

$$C_x = \langle DC, Q\rangle,$$

where

$$Q = (Q_1, \ldots, Q_N) \quad \text{and} \quad Q_i = \frac{(-1)^N}{2}(D_{\vartheta_i} \ln\, a),$$

in the superspace

$$\mathbb{R}^{2^{N-1}|2^{N-1}} \simeq \Lambda_0^{2^{N-1}} \times \Lambda_1^{2^{N-1}}.$$

Moreover, $C \in C^\infty(\mathbb{T}^{(1,1)|N}; \Lambda_1)$ if N is an odd natural number and $C \in C^\infty(\mathbb{T}^{(1,1)|N}; \Lambda_0)$ if N is an even integer. For $N = 1$, we have

$$l = C_1(\partial_x^{-1}D_{\theta_1}a^{-\frac{1}{2}})a^{-\frac{3}{2}},$$

where $C_1 \in \mathbb{R}$ is a real constant.

If $N = 1$ and $C_1 = 1$, then the corresponding seed-element $\tilde{l} \in \tilde{\mathcal{G}}^*$ connected with the asymptotic expansions (7.1) and (7.2) can be reduced to the form

$$\tilde{l} = [\lambda^{-1}(\partial_x^{-1}D_{\theta_1}w_y^{-\frac{1}{2}})w_y^{-\frac{3}{2}} + \xi_x/2 + \theta_1(2u_x + \lambda)]dx,$$

where $w := u + \theta_1\xi,\ u \in C^\infty(\mathbb{R}^2 \times \mathbb{S}^1; \Lambda_0)$ and $\xi \in C^\infty(\mathbb{R}^2 \times \mathbb{S}^1; \Lambda_1)$.

8. Hamiltonian Flows Associated with Hydrodynamic Chaplygin Systems

Consider a hydrodynamic Chaplygin system [1, 17, 23]

$$u_t = -uu_x - kv_x v^{-3},$$
$$v_t = -(uv)_x, \tag{8.1}$$

where $k \in \mathbb{R}$ is a constant parameter and $(u, v) \in M \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^2)$ are 2π-periodic dynamic variables for the evolutionary parameter $t \in \mathbb{R}$ on the functional manifold M. To describe the geometric structure of system (8.1), we define a loop Lie algebra $\tilde{\mathcal{G}} := \widetilde{\operatorname{diff}}(\mathbb{T}^1)$ on the manifold $\mathbb{C} \times \mathbb{T}^1$ and choose a seed-element $\tilde{l} \in \tilde{\mathcal{G}}^*$ in the form

$$\tilde{l} = \left[\left(\frac{1}{8}\alpha_x + uu_x\right)\lambda + \frac{1}{2}u_x\lambda^3\right]dx$$
$$+ \left[\frac{3}{8}(\alpha + 4u^2) + \frac{5}{2}u\lambda^2 + \lambda^4\right]d\lambda,$$

where

$$\alpha := kv^{-2} + u^2.$$

Then we determine the asymptotic expansions for some Casimir functionals $h^{(y)}, h^{(t)}$, and $h^{(s)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$:

$$\nabla h^{(t)}(l) := \nabla h^{(2)}(l), \qquad \nabla h^{(y)}(l) := \nabla h^{(4)}(l), \qquad \nabla h^{(s)}(l) := \nabla h^{(6)}(l),$$

where

$$\nabla h^{(2)}(l) = \begin{pmatrix} -2 \\ 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 \\ u_x \end{pmatrix}\lambda^1 + \begin{pmatrix} u \\ 0 \end{pmatrix}\lambda^9 + O(\lambda^{-1}),$$

$$\nabla h^{(4)}(l) = \begin{pmatrix} -8 \\ 0 \end{pmatrix}\lambda^4 + \begin{pmatrix} 0 \\ 4u_x \end{pmatrix}\lambda^3 + \begin{pmatrix} -4u \\ 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 \\ \alpha_x \end{pmatrix}\lambda^1 + \begin{pmatrix} \alpha \\ 0 \end{pmatrix}\lambda^0 + O(\lambda^{-1})$$

and

$$\nabla h^{(6)}(l) = \begin{pmatrix} -2 \\ 0 \end{pmatrix}\lambda^6 + \begin{pmatrix} 0 \\ u_x \end{pmatrix}\lambda^5 + \begin{pmatrix} -3u \\ 0 \end{pmatrix}\lambda^4$$
$$+ \begin{pmatrix} 0 \\ \alpha_x/4 + uu_x \end{pmatrix}\lambda^3 + \begin{pmatrix} -\alpha/4 - 1/2u^2 \\ 0 \end{pmatrix}\lambda^2$$
$$+ \begin{pmatrix} 0 \\ -(u\alpha)_x/8 \end{pmatrix}\lambda^1 + \begin{pmatrix} u\alpha/8 \\ 0 \end{pmatrix}\lambda^0 + O(\lambda^{-1})$$

as $\lambda \to \infty$.

The corresponding Lax–Sato generators of the vector fields are given by the formulas

$$\begin{aligned}\nabla h_+^{(t)}(l) &:= (\nabla h^{(2)}(l))|_+ = \begin{pmatrix}-2\\0\end{pmatrix}\lambda^2 + \begin{pmatrix}0\\u_x\end{pmatrix}\lambda^1 + \begin{pmatrix}u\\0\end{pmatrix}\lambda^0,\\ \nabla h_+^{(y)}(l) &:= \left(\nabla h^{(4)}(l)\right)\big|_+\\ &= \begin{pmatrix}-8\\0\end{pmatrix}\lambda^4 + \begin{pmatrix}0\\4u_x\end{pmatrix}\lambda^3 + \begin{pmatrix}-4u\\0\end{pmatrix}\lambda^2 + \begin{pmatrix}0\\\alpha_x\end{pmatrix}\lambda^1 + \begin{pmatrix}\alpha\\0\end{pmatrix}\lambda^0\end{aligned} \tag{8.2}$$

and

$$\begin{aligned}\nabla h_+^{(s)}(l) &:= (\nabla h^{(6)}(l))|_+\\ &= \begin{pmatrix}-2\\0\end{pmatrix}\lambda^6 + \begin{pmatrix}0\\u_x\end{pmatrix}\lambda^5 + \begin{pmatrix}-3u\\0\end{pmatrix}\lambda^4\\ &\quad + \begin{pmatrix}0\\\alpha_x/4 + uu_x\end{pmatrix}\lambda^3 + \begin{pmatrix}-\alpha/4 - 1/2u^2\\0\end{pmatrix}\lambda^2\\ &\quad + \begin{pmatrix}0\\-(u\alpha)_x/8\end{pmatrix}\lambda^1 + \begin{pmatrix}u\alpha/8\\0\end{pmatrix}\lambda^0\end{aligned} \tag{8.3}$$

as $\lambda \to \infty$.

By using relations (8.2) and (8.3), we obtain the following evolutionary flows:

$$\partial\tilde{l}/\partial t = -ad^*_{\nabla h_+^{(t)}(\tilde{l})}\tilde{l} \sim \left.\begin{aligned} u_t &= -(u^2 - kv^{-2})_x\\ v_t &= -(uv)_x\end{aligned}\right\} \tag{8.4}$$

for the evolutionary parameter $t \in \mathbb{R}$, which are equivalent to the hydrodynamic system (8.1),

$$\partial\tilde{l}/\partial y = -ad^*_{\nabla h_+^{(y)}(\tilde{l})}\tilde{l} \sim \left.\begin{aligned} u_y &= -[uv(u^2 + kv^{-2})]_x\\ v_y &= -[(u^2 + kv^{-2})v]_x\end{aligned}\right\} \tag{8.5}$$

for the evolutionary parameter $y \in \mathbb{R}$, and

$$\partial\tilde{l}/\partial s = -ad^*_{\nabla h_+^{(s)}(\tilde{l})}\tilde{l} \sim \left.\begin{aligned} u_s &= -\left(-3\alpha^2 + 4u^4\right)_x/12\\ v_s &= -[(u^2 + kv^{-2})uv]_x/3\end{aligned}\right\} \tag{8.6}$$

for the evolutionary parameter $s \in \mathbb{R}$. All these flows are mutually commuting,

$$[\partial/\partial t + \nabla h_+^{(t)}(l), \partial/\partial y + \nabla h_+^{(y)}(l)] = 0,$$

$$[\partial/\partial t + \nabla h_+^{(t)}(l), \partial/\partial s + \nabla h_+^{(s)}(l)] = 0,$$

$$[\partial/\partial s + \nabla h_+^{(s)}(l), \partial/\partial y + \nabla h_+^{(y)}(l)] = 0,$$

Lax–Sato-type vector fields on the manifold $\mathbb{C} \times \mathbb{T}^1$ for all parameters $t,\ y,$ and $s \in \mathbb{R}$ and yield three new compatible systems of heavenly integrable dispersion-free differential equations. The obtained result can be formulated as the following theorem:

Theorem 8.1. *The hydrodynamic Chaplygin system (8.4) is equivalent to the completely integrable Hamiltonian system (8.6) on the space* $\tilde{\mathcal{G}}^*$ *conjugate to the loop Lie algebra* $\tilde{\mathcal{G}} \simeq \widetilde{\mathrm{diff}}(\mathbb{T}^1)$ *of vector fields on the manifold* $\mathbb{C} \times \mathbb{T}^1$. *The associated Casimir functionals on* $\tilde{\mathcal{G}}^*$ *generate an infinite hierarchy of additional commuting Hamiltonian systems of the form (8.5) and (8.6) and the Lax–Sato-type vector fields on* $\mathbb{C} \times \mathbb{T}^1$, *which gives new dispersion heavenly equations.*

As shown in [3], the hydrodynamic Chaplygin system (8.4) is closely related to the class of completely integrable Monge-type equations whose geometric structure was analyzed in [6] by using another approach based on the properties of embedding of the Grassmann manifold of general differential equations defined on jet-submanifolds. This observation reduces the problem of determination of the relationship between different geometric approaches to the description of completely integrable dispersion-free differential systems.

Finally, we note that the Lie-algebraic scheme proposed by Ovsienko [26, 27] can be generalized by analyzing a broader class of integrable heavenly equations represented in the form of consistent Hamiltonian flows on the semisimple product of the Lie algebra $\tilde{\mathcal{G}}$ of holomorphic vector fields on the torus $\mathbb{C} \times \mathbb{T}^n$ by its regular conjugate space supplemented with the Maurer–Cartan cocycle $\tilde{\mathcal{G}}^*$. This structure will be considered in the second part of the present survey.

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