DOI 10.1007/s11253-023-02140-7
Ukrainian Mathematical Journal, Vol. 74, No. 9, February, 2023 (Ukrainian Original Vol. 74, No. 9, September, 2022)

GEOMETRIC STRUCTURES ON THE ORBITS OF LOOP DIFFEOMORPHISM GROUPS AND RELATED HEAVENLY-TYPE HAMILTONIAN SYSTEMS. II

O. E. Hentosh,[1] **Ya. A. Prykarpatskyy,**[2,3] **A. A. Balinsky,**[4] **and A. K. Prykarpatski**[5] UDC 517.9

We present a review of differential-geometric and Lie-algebraic approaches to the study of a broad class of nonlinear integrable differential systems of "heavenly" type associated with Hamiltonian flows on the spaces conjugated to the loop Lie algebras of vector fields on the tori. These flows are generated by the corresponding orbits of the coadjoint action of the diffeomorphism loop group and satisfy the Lax–Sato-type vector-field compatibility conditions. The corresponding hierarchies of conservation laws and their relationships with Casimir invariants are analyzed. We consider typical examples of these systems and establish their complete integrability by using the developed Lie-algebraic construction. We also describe new generalizations of the integrable dispersion-free systems of heavenly type for which the corresponding generating elements of the orbits have factorized structures, which allows their extension to the multidimensional case.

1. Multidimensional Systems of the Heavenly Type: Modified Lie-Algebraic Scheme

Let $\widetilde{\mathrm{Diff}}_{\pm}(\mathbb{T}^n),\ n \in \mathbb{Z}_+$, be subgroups of the diffeomorphism loop group

$$\widetilde{\mathrm{Diff}}(\mathbb{T}^n) := \{\mathbb{C} \supset \mathbb{S}^1 \to \mathrm{Diff}(\mathbb{T}^n)\}$$

holomorphically extended to the interior $\mathbb{D}^1_+ \subset \mathbb{C}$ and exterior $\mathbb{D}^1_- \subset \mathbb{C}$ of the central unit disk $\mathbb{D}^1 \subset \mathbb{C}^1$, respectively, so that

$$\tilde{g}(\infty) = 1 \in \mathrm{Diff}(\mathbb{T}^n)$$

for any $\tilde{g}(\lambda) \in \widetilde{\mathrm{Diff}}_-(\mathbb{T}^n),\ \lambda \in \mathbb{D}^1_-$. The corresponding Lie subalgebras

$$\widetilde{\mathrm{diff}}_{\pm}(\mathbb{T}^n) \simeq \widetilde{\mathrm{Vect}}_{\pm}(\mathbb{T}^n)$$

of the diffeomorphism loop subgroups $\widetilde{\mathrm{Diff}}_{\pm}(\mathbb{T}^n)$ form vector fields on $\mathbb{T}^n$ that are holomorphic on the domains $\mathbb{D}^1_{\pm} \subset \mathbb{C}^1$, respectively, where, for any $\tilde{a}(\lambda) \in \widetilde{\mathrm{diff}}_-(\mathbb{T}^n)$, we have $\tilde{a}(\infty) = 0$.

The Lie algebra $\widetilde{\mathrm{diff}}(\mathbb{T}^n)$ can be split into the direct sum of two Lie subalgebras:

$$\widetilde{\mathrm{diff}}(\mathbb{T}^n) = \widetilde{\mathrm{diff}}_+(\mathbb{T}^n) \oplus \widetilde{\mathrm{diff}}_-(\mathbb{T}^n).$$

[1] Institute for Applied Problems in Mechanics and Mathematics, National Academy of Sciences of Ukraine, Lviv, Ukraine; e-mail: ohen@ukr.net.
[2] Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv, Ukraine; University of Agriculture in Kraków, Poland; e-mail: yarpry@imath.kiev.ua.
[3] Corresponding author.
[4] Mathematics Institute at the Cardiff University, Cardiff, United Kingdom; e-mail: BalinskyA@cardiff.ac.uk.
[5] Institute of Mathematics at the Kraków University of Technology, Kraków, Poland; e-mail: pryk.anat@cybergal.com.

Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 74, No. 9, pp. 1182–1200, September, 2022. Ukrainian DOI: 10.37863/umzh.v74i9.7234. Original article submitted June 7, 2022.

 0041-5995/23/7409–1348

Its regular adjoint space $\widetilde{\mathrm{diff}}(\mathbb{T}^n)^*$ with respect to the convolution:

$$(\tilde{l}|\tilde{a}) := \operatorname*{res}_{\lambda\in\mathbb{C}}(l(x;\lambda)|a(x;\lambda))_{H^0}, \tag{1.1}$$

where

$$(l(x;\lambda)|a(x;\lambda))_{H^0} := \int_{\mathbb{T}^n} dx\langle l(x;\lambda), a(x;\lambda)\rangle$$

is the ordinary scalar product in the Hilbert space $H^0 := L_2(\mathbb{T}^n;\mathbb{R}^n)$ for any elements $\tilde{l} \in \widetilde{\mathrm{diff}}(\mathbb{T}^n)^*$ and $\tilde{a} \in \widetilde{\mathrm{diff}}(\mathbb{T}^n)$ of the form

$$\tilde{a} = \sum_{j=1}^{n} a^{(j)}(x;\lambda)\frac{\partial}{\partial x_j} := \left\langle a(x;\lambda), \frac{\partial}{\partial x}\right\rangle,$$

$$\tilde{l} = \sum_{j=1}^{n} l_j(x;\lambda)dx_j := \langle l(x;\lambda), dx\rangle,$$

can be identified with the Lie algebra $\widetilde{\mathrm{diff}}(\mathbb{T}^n)^*$. Here,

$$\frac{\partial}{\partial x} := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)^{\intercal}$$

denotes the operator of gradient in the Euclidean space $(\mathbb{E}^n;\langle\cdot,\cdot\rangle)$. The Lie commutator of any vector fields $\tilde{a},\tilde{b} \in \widetilde{\mathrm{diff}}(\mathbb{T}^n)$ can be found according to the rule

$$\begin{aligned}[\tilde{a},\tilde{b}] &= \tilde{a}\tilde{b} - \tilde{b}\tilde{a} \\ &= \left\langle\left\langle a(x;\lambda), \frac{\partial}{\partial x}\right\rangle b(x;\lambda), \frac{\partial}{\partial x}\right\rangle - \left\langle\left\langle b(x;\lambda), \frac{\partial}{\partial x}\right\rangle a(x;\lambda), \frac{\partial}{\partial x}\right\rangle.\end{aligned}$$

In addition, we have the following identification of regular adjoint subspaces:

$$\widetilde{\mathrm{diff}}_+(\mathbb{T}^n)^* \simeq \widetilde{\mathrm{diff}}_-(\mathbb{T}^n), \qquad \widetilde{\mathrm{diff}}_-(\mathbb{T}^n)^* \simeq \widetilde{\mathrm{diff}}_+(\mathbb{T}^n),$$

where any $\tilde{l}(\lambda) \in \widetilde{\mathrm{diff}}_-(\mathbb{T}^n)^*$ satisfies the condition $\tilde{l}(0) = 0$.

We now construct the loop Lie algebra

$$\tilde{\mathcal{G}} := \widetilde{\mathrm{diff}}(\mathbb{T}^n) \ltimes \widetilde{\mathrm{diff}}(\mathbb{T}^n)^*$$

as the semidirect sum of the Lie algebra $\widetilde{\mathrm{diff}}(\mathbb{T}^n)$ and its regular adjoint space $\widetilde{\mathrm{diff}}(\mathbb{T}^n)^*$ on which the Lie commutator, for any pair of elements $(\tilde{a}_1 \ltimes \tilde{l}_1), (\tilde{a}_2 \ltimes \tilde{l}_2) \in \tilde{\mathcal{G}}$, is given by the rule

$$[\tilde{a}_1 \ltimes \tilde{l}_1, \tilde{a}_2 \ltimes \tilde{l}_2\,] := [\tilde{a}_1, \tilde{a}_2] \ltimes (ad^*_{\tilde{a}_2}\tilde{l}_1 - ad^*_{\tilde{a}_1}\tilde{2}_1), \tag{1.2}$$

where

$$ad^*_{\widetilde{\mathrm{diff}}(\mathbb{T}^n)} \colon \widetilde{\mathrm{diff}}(\mathbb{T}^n)^* \to \widetilde{\mathrm{diff}}(\mathbb{T}^n)^*,$$

$$(ad^*_{\tilde{a}}\tilde{l}|\tilde{b}) := (\tilde{l}|[\tilde{a},\tilde{b}]) \quad \text{for} \quad \tilde{l} \in \widetilde{\mathrm{diff}}(\mathbb{T}^n)^* \quad \text{and any} \quad \tilde{a},\tilde{b} \in \widetilde{\mathrm{diff}}(\mathbb{T}^n),$$

is a standard coadjoint mapping of the Lie algebra $\widetilde{\mathrm{diff}}(\mathbb{T}^n)$ on its regular adjoint space $\widetilde{\mathrm{diff}}(\mathbb{T}^n)^*$ with respect to convolution (1.1). On the Lie algebra $\tilde{\mathcal{G}}$, we can introduce an ad-invariant nondegenerate scalar product in the form

$$(\tilde{a}_1 \ltimes \tilde{l}_1|\tilde{a}_2 \ltimes \tilde{l}_2) := (\tilde{l}_2|\tilde{a}_1) + (\tilde{l}_1|\tilde{a}_2), \tag{1.3}$$

where $\tilde{a}_1 \ltimes \tilde{l}_1$ and $\tilde{a}_2 \ltimes \tilde{l}_2 \in \tilde{\mathcal{G}}$, which enables one to identify the regular adjoint space $\tilde{\mathcal{G}}^*$ with respect to (1.3) for the algebra $\tilde{\mathcal{G}}$ with this Lie algebra, i.e., $\tilde{\mathcal{G}}^* \simeq \tilde{\mathcal{G}}$.

We can split the Lie algebra $\tilde{\mathcal{G}}$ into the direct sum of two subalgebras [1–3]: $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$, where

$$\tilde{\mathcal{G}}_+ := \widetilde{\mathrm{diff}}(\mathbb{T}^n)_+ \ltimes \widetilde{\mathrm{diff}}(\mathbb{T}^n)^*_-, \qquad \tilde{\mathcal{G}}_- := \widetilde{\mathrm{diff}}(\mathbb{T}^n)_- \ltimes \widetilde{\mathrm{diff}}(\mathbb{T}^n)^*_+.$$

This enables us to introduce a new Lie commutator on $\tilde{\mathcal{G}}$ in the form

$$[\tilde{w}_1,\tilde{w}_2]_{\mathcal{R}} := [\mathcal{R}\tilde{w}_1,\tilde{w}_2] + [\tilde{w}_1,\mathcal{R}\tilde{w}_2],$$

where $\tilde{w}_1,\tilde{w}_2 \in \tilde{\mathcal{G}}$, $\mathcal{R} := (P_+ - P_-)/2$ is the standard $\mathcal{R}$-operator homomorphism [10, 11, 15] on $\tilde{\mathcal{G}}$ and, by definition, $P_\pm \colon \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}_\pm \subset \tilde{\mathcal{G}}$. Thus, we can apply the classical AKS (Adler–Konstant–Symes) theory to the Lie algebra $\tilde{\mathcal{G}}$ in order to construct Hamiltonian systems on the regular adjoint space $\tilde{\mathcal{G}}^* \simeq \tilde{\mathcal{G}}$ with the use of hierarchies of Casimir invariants for the base Lie commutator (1.2).

To describe the corresponding Lie-algebraic scheme in detail, we determine the Casimir invariants $h \in I(\tilde{\mathcal{G}}^*)$. By definition, these invariants satisfy the relation

$$ad^*_{\nabla h(\tilde{l};\tilde{a})}(\tilde{l};\tilde{a}) = 0,$$

which can be rewritten in the commutator form as follows:

$$\left[\nabla h(\tilde{l};\tilde{a}), \tilde{a} \ltimes \tilde{l}\right] = 0, \tag{1.4}$$

where

$$\nabla h(\tilde{l};\tilde{a}) := \nabla h_{\tilde{l}} \ltimes \nabla h_{\tilde{a}} \in \widetilde{\mathrm{diff}}(\mathbb{T}^n) \ltimes \widetilde{\mathrm{diff}}(\mathbb{T}^n)^* = \tilde{\mathcal{G}}$$

is the gradient of the Casimir invariant $h \in I(\tilde{\mathcal{G}}^*)$ at the point $(\tilde{l};\tilde{a}) \in \tilde{\mathcal{G}}^* \simeq \tilde{\mathcal{G}}$. Relation (1.4) is equivalent to the system of differential-algebraic equations

$$[\nabla h_{\tilde{l}};\tilde{a}] = 0,$$

$$ad^*_{\nabla h_{\tilde{l}}}\tilde{l} - ad^*_{\tilde{a}}\nabla h_{\tilde{a}} = 0.$$

In the explicit form, these equations can be rewritten as follows:

$$\begin{gathered}\langle \nabla h_l, \partial/\partial x\rangle\, a - \langle a, \partial/\partial x\rangle\, \nabla h_l = 0,\\ \langle \partial/\partial x, \nabla h_l\rangle\, l + \langle l, (\partial/\partial x)\nabla h_l)\rangle - \langle \partial/\partial x, a\rangle\, \nabla h_a - \langle \nabla h_a, (\partial/\partial x) a\rangle = 0,\end{gathered} \tag{1.5}$$

where

$$\begin{gathered}\nabla h_{\tilde l} := \langle \nabla h_l, \partial/\partial x\rangle\,, \quad \tilde l := \langle l, dx\rangle\,,\\ \nabla h_{\tilde a} := \langle \nabla h_a, dx\rangle\,, \quad \tilde a := \langle a, \partial/\partial x\rangle\,.\end{gathered}$$

The system of linear equations (1.5) for a given element $\tilde a \ltimes \tilde l \in \tilde{\mathcal G}$, which is singular as $|\lambda| \to \infty$, can be solved by using the asymptotic expansions

$$\nabla h_l \sim \sum_{j\in\mathbb Z_+} \nabla h_l^{(j)} \lambda^{-j}, \qquad \nabla h_a \sim \sum_{j\in\mathbb Z_+} \nabla h_a^{(j)} \lambda^{-j}\,, \tag{1.6}$$

which enable us to get the infinite hierarchy of gradients

$$\nabla h^{(p)}(\tilde l; \tilde a) = \lambda^p \nabla h(\tilde l; \tilde a) \in \tilde{\mathcal G}, \qquad p \in \mathbb Z_+,$$

for the corresponding Casimir invariants $h^{(p)} \in I(\tilde{\mathcal G}^*),\ p \in \mathbb Z_+$. If the given element $\tilde a \ltimes \tilde l \in \tilde{\mathcal G}$ is singular as $|\lambda| \to 0,$ then the system of linear equations (1.5) can be solved by using the asymptotic expansions

$$\nabla h_l \sim \sum_{j\in\mathbb Z_+} \nabla h_l^{(j)} \lambda^{j}, \qquad \nabla h_a \sim \sum_{j\in\mathbb Z_+} \nabla h_a^{(j)} \lambda^{j}, \tag{1.7}$$

which enable us to construct an infinite hierarchy of gradients

$$\nabla h^{(p)}(\tilde l; \tilde a) = \lambda^{-p} \nabla h(\tilde a, \tilde l) \in \tilde{\mathcal G}, \qquad p \in \mathbb Z_+,$$

for the corresponding Casimir invariants $h^{(p)} \in I(\tilde{\mathcal G}^*),\ p \in \mathbb Z_+$.

Further, we assume that the gradients

$$\nabla h^{(y)}(\tilde a; \tilde l) := \lambda^{p_y} \nabla h^{(1)}(\tilde a, \tilde l) \qquad \text{and} \qquad \nabla h^{(t)}(\tilde a; \tilde l) := \lambda^{p_y} \nabla h^{(2)}(\tilde a; \tilde l) \in \tilde{\mathcal G}$$

are found for two Casimir invariants $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal G}^*)$ (not necessarily different) with some integer $p_y, p_t \in \mathbb Z$ satisfying Eq. (1.5). By using the classical AKS theory, we construct two commuting flows for the evolutionary parameters $y, t \in \mathbb R$ in the regular adjoint space $\tilde{\mathcal G}^* \simeq \tilde{\mathcal G}$:

$$\frac{\partial}{\partial y}\tilde a = -\left[\nabla h^{(y)}_{\tilde l,+}, \tilde a\right], \qquad \frac{\partial}{\partial t}\tilde a = -\left[\nabla h^{(t)}_{\tilde l,+}, \tilde a\right] \tag{1.8}$$

and

$$\frac{\partial}{\partial y}\tilde l = -ad^*_{\nabla h^{(y)}_{\tilde l,+}}\tilde l + ad^*_{\tilde a}\big(\nabla h^{(y)}_{\tilde a,+}\big), \qquad \frac{\partial}{\partial t}\tilde l = -ad^*_{\nabla h^{(t)}_{\tilde l,+}}\tilde l + ad^*_{\tilde a}\big(\nabla h^{(t)}_{\tilde a,+}\big), \tag{1.9}$$

where

$$\left(\nabla h^{(y)}_{\tilde{l},+} \ltimes \nabla h^{(y)}_{\tilde{a},+}\right) := P_+ \nabla h^{(y)}(\tilde{a};\tilde{l}) \in \tilde{\mathcal{G}}_+ \qquad \text{and} \qquad \left(\nabla h^{(t)}_{\tilde{l},+} \ltimes \nabla h^{(t)}_{\tilde{a},+}\right) := P_+ \nabla h^{(t)}(\tilde{a};\tilde{l}) \in \tilde{\mathcal{G}}_+$$

are the projections of the corresponding asymptotic expansions (1.6) and (1.7). For the chosen element $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}^* \simeq \tilde{\mathcal{G}}$, flows (1.8) and (1.9) are caused by the Hamiltonian flows

$$\frac{\partial}{\partial y}(\tilde{a} \ltimes \tilde{l}) = \{\tilde{a} \ltimes \tilde{l}, h^{(y)}\}_{\mathcal{R}}, \qquad \frac{\partial}{\partial t}(\tilde{a} \ltimes \tilde{l}) = \{\tilde{a} \ltimes \tilde{l}, h^{(t)}\}_{\mathcal{R}} \tag{1.10}$$

generated by the $\mathcal{R}$-deformed Lie–Poisson bracket [10–12, 15]:

$$\{h, f\}_{\mathcal{R}} := \Big(\tilde{a} \ltimes \tilde{l}, [\nabla h(\tilde{l};\tilde{a}), \nabla f(\tilde{l},\tilde{a})]_{\mathcal{R}}\Big) \tag{1.11}$$

in the regular adjoint space $\tilde{\mathcal{G}}^* \simeq \tilde{\mathcal{G}}$. Here, $h, f \in D(\tilde{\mathcal{G}}^*)$ are Fréchet-smooth functionals. The condition of commutativity of these flows is equivalent to the following system of two equations:

$$\left[\nabla h^{(y)}_{\tilde{l},+}, \nabla h^{(t)}_{\tilde{l},+}\right] - \frac{\partial}{\partial t}\nabla h^{(y)}_{\tilde{l},+} + \frac{\partial}{\partial y}\nabla h^{(t)}_{\tilde{l},+} = 0 \tag{1.12}$$

and

$$ad^*_{\tilde{a}}\tilde{P} = 0,$$

$$\tilde{P} = ad^*_{\nabla h^{(y)}_{\tilde{l},+}}(\nabla h^{(t)}_{\tilde{a},+}) - ad^*_{\nabla h^{(t)}_{\tilde{l},+}}(\nabla h^{(y)}_{\tilde{a},+}) - \frac{\partial}{\partial t}\nabla h^{(y)}_{\tilde{a},+} + \frac{\partial}{\partial y}\nabla h^{(t)}_{\tilde{a},+}$$

for any element $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}$.

Thus, the following statement is true:

Proposition 1.1. *Hamiltonian flows (1.10) in a regular adjoint space $\tilde{\mathcal{G}}^* \simeq \tilde{\mathcal{G}}$ generate the systems of commuting evolutionary equations (1.8) and (1.9). The commutativity condition for the evolutionary equations (1.8) is equivalent to the Lax–Sato compatibility condition (1.12) for a certain system of nonlinear partial differential equations of the heavenly type.*

We generalize the described scheme of construction of Hamiltonian flows in the regular adjoint space $\tilde{\mathcal{G}}^*$ as follows:

We parametrize the Lie algebra $\tilde{\mathcal{G}}$ by using the point product $\tilde{\mathcal{G}}^{\mathbb{S}^1} := \prod_{z \in \mathbb{S}^1} \tilde{\mathcal{G}}$ and consider its central extension by the Maurer–Cartan 2-cocycle $\tilde{\omega}_2 : \tilde{\mathcal{G}} \times \tilde{\mathcal{G}} \to \mathbb{C}$:

$$\tilde{\omega}_2(\tilde{a}_1 \ltimes \tilde{l}_1, \tilde{a}_2 \ltimes \tilde{l}_2) := \int_{\mathbb{S}^1} [(l_1, \partial\tilde{a}_2/\partial z) - (l_2, \partial\tilde{a}_1/\partial z)],$$

where $\tilde{a}_1 \ltimes \tilde{l}_1,\ \tilde{a}_2 \ltimes \tilde{l}_2 \in \tilde{\mathcal{G}}$. On the central extension $\tilde{\mathfrak{G}} := \tilde{\mathcal{G}} \oplus \mathbb{C}$, the commutator is given by the rule

$$\big[(\tilde{a}_1 \ltimes \tilde{l}_1; \alpha_1), (\tilde{a}_2 \ltimes \tilde{l}_2; \alpha_1)\big] := \big([\tilde{a}_1, a_2] \ltimes (ad^*_{\tilde{a}_1}\tilde{l}_2 - ad^*_{\tilde{a}_2}\tilde{l}_1); \tilde{\omega}_2(\tilde{a}_1 \ltimes \tilde{l}_1, \tilde{a}_2 \ltimes \tilde{l}_2)\big)$$

for any pair of elements $(\tilde{a}_1 \ltimes \tilde{l}_1; \alpha_1), (\tilde{a}_2 \ltimes \tilde{l}_2; \alpha_1) \in \tilde{\mathfrak{G}}$.

For any smooth functionals $h, f \in D(\mathfrak{G}^*)$, the $\mathcal{R}$-deformed Lie–Poisson bracket (1.11) in the regular adjoint space $\tilde{\mathfrak{G}}^*$ has the form

$$\{h,f\}_{\mathcal{R}} := \big(\tilde{a} \ltimes \tilde{l}, [\nabla h(\tilde{l};\tilde{a}), \nabla f(\tilde{l};\tilde{a})]_{\mathcal{R}}\big) + \tilde{\omega}_2\big(\mathcal{R}\nabla h(\tilde{l};\tilde{a}), \nabla f(\tilde{l},\tilde{a})\big) + \tilde{\omega}_2\big(\nabla h(\tilde{l};\tilde{a}), \mathcal{R}\nabla f(\tilde{l};\tilde{a})\big). \tag{1.13}$$

The corresponding Casimir invariants $h^{(p)} \in I(\tilde{\mathfrak{G}}^*),\ p \in \mathbb{Z}_+,$ are determined by the standard Lie–Poisson bracket as follows:

$$\begin{gathered}\{h^{(p)}, f\} = 0,\\ \big(\tilde{a} \ltimes \tilde{l}, [\nabla h^{(p)}(\tilde{l},\tilde{a}), \nabla f(\tilde{a},\tilde{l})]\big) + \tilde{\omega}_2(\nabla h^{(p)}(\tilde{a},\tilde{l}), \nabla f(\tilde{a},\tilde{l})),\end{gathered} \tag{1.14}$$

for all smooth functionals $f \in D(\tilde{\mathfrak{G}}^*)$. By using equality (1.14), we conclude that the gradients $\nabla h^{(p)} \in \tilde{\mathfrak{G}}$ of the Casimir invariants $h^{(p)} \in I(\tilde{\mathfrak{G}}^*),\ p \in \mathbb{Z}_+,$ satisfy the equations

$$[\nabla h_{\tilde{l}}\tilde{a}] - \frac{\partial}{\partial z}\nabla h_{\tilde{l}} = 0, \qquad ad^*_{\nabla h_{\tilde{l}}}\tilde{l} - ad^*_{\tilde{a}}\nabla h_{\tilde{a}} - \frac{\partial}{\partial z}\nabla h_{\tilde{a}} = 0$$

for any chosen element $\tilde{a} \ltimes \tilde{l} \in \bar{\mathcal{G}}^*$. By using some of the obtained Casimir invariants $h^{(y)}, h^{(t)} \in I(\tilde{\mathfrak{G}}^*)$ and relation (1.13), we construct the following commuting Hamiltonian flows in the regular adjoint space $\tilde{\mathfrak{G}}^*$:

$$\frac{\partial}{\partial y}(\tilde{a} \ltimes \tilde{l}) = \big\{\tilde{a} \ltimes \tilde{l}, h^{(y)}\big\}_{\mathcal{R}}, \qquad \frac{\partial}{\partial t}(\tilde{a} \ltimes \tilde{l}) = \big\{\tilde{a} \ltimes \tilde{l}, h^{(t)}\big\}_{\mathcal{R}}, \tag{1.15}$$

which are equivalent to the system of evolutionary equations

$$\frac{\partial}{\partial y}\tilde{a} = -\big[\nabla h^{(y)}_{\tilde{l},+}, \tilde{a}\big] + \frac{\partial}{\partial z}\nabla h^{(y)}_{\tilde{l},+}, \qquad \frac{\partial}{\partial t}\tilde{a} = -\big[\nabla h^{(t)}_{\tilde{l},+}, \tilde{a}\big] + \frac{\partial}{\partial z}\nabla h^{(t)}_{\tilde{l},+}, \tag{1.16}$$

and

$$\begin{gathered}\frac{\partial}{\partial y}\tilde{l} = -ad^*_{\nabla h^{(y)}_{\tilde{l},+}}\tilde{l} + ad^*_{\tilde{a}}(\nabla h^{(y)}_{\tilde{a},+}) + \frac{\partial}{\partial z}\nabla h^{(y)}_{\tilde{a},+},\\ \frac{\partial}{\partial t}\tilde{l} = -ad^*_{\nabla h^{(t)}_{\tilde{l},+}}\tilde{l} + ad^*_{\tilde{a}}(\nabla h^{(t)}_{\tilde{a},+}) + \frac{\partial}{\partial z}\nabla h^{(t)}_{\tilde{a},+}.\end{gathered} \tag{1.17}$$

The condition of commutativity of these flows is given by the following system of two equations:

$$\Big[\nabla h^{(y)}_{\tilde{l},+}, \nabla h^{(t)}_{\tilde{l},+}\Big] - \frac{\partial}{\partial t}\nabla h^{(y)}_{\tilde{l},+} + \frac{\partial}{\partial y}\nabla h^{(t)}_{\tilde{l},+} = 0$$

and

$$\frac{\partial \tilde{P}}{\partial z} + ad^*_{\tilde{a}}\tilde{P} = 0,$$

$$\tilde{P} = ad^*_{\nabla h^{(y)}_{\tilde{l},+}}(\nabla h^{(t)}_{\tilde{a},+}) - ad^*_{\nabla h^{(t)}_{\tilde{l},+}}(\nabla h^{(y)}_{\tilde{a},+}) - \frac{\partial}{\partial t}\nabla h^{(y)}_{\tilde{a},+} + \frac{\partial}{\partial y}\nabla h^{(t)}_{\tilde{a},+}$$

for any $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}$. The first of these equations can be regarded as the Lax-type compatibility condition for the evolutionary equations (1.16). As a consequence, we can formulate the following statement:

Proposition 1.2. *The Hamiltonian flows (1.15) in the regular adjoint space $\tilde{\mathfrak{G}}^* \simeq \tilde{\mathfrak{G}}$ generate systems of commuting evolutionary equations (1.16) and (1.17). The commutativity condition for the evolutionary equations (1.16) is, in fact, the compatibility condition for the set of linear vector-field equations*

$$\partial\psi/\partial y + \nabla h^{(y)}_{\tilde{l},+}\psi = 0, \qquad \partial\psi/\partial z + \tilde{a}\psi = 0, \qquad \partial\psi/\partial t + \nabla h^{(t)}_{\tilde{l},+}\psi = 0 \tag{1.18}$$

for all $(y,t;\lambda,z,x) \in \mathbb{R}^2 \times (\mathbb{C} \times \mathbb{S}^1 \times \mathbb{T}^n)$ and the function $\psi \in C^2(\mathbb{R}^2 \times (\mathbb{C} \times \mathbb{S}^1 \times \mathbb{T}^n);\mathbb{C})$; moreover, on the orbits of coadjoint action of the Lie algebra $\tilde{\mathfrak{G}}$ it is reduced to a system of nonlinear partial differential equations of heavenly type.

According to the Lie-algebraic approach described above, every Casimir invariant $h^{(j)} \in I(\mathfrak{G}^*)$, $j \in \mathbb{Z}_+$, for the element $\tilde{a} \ltimes \tilde{l} \in \mathfrak{G}^*$ generates, in the regular adjoint space $\mathfrak{G}^*$, the following hierarchy of commuting Hamiltonian flows:

$$\begin{gathered} \frac{d}{dt_j}\tilde{a} = -[\nabla h^{(j)}_{\tilde{l},+}, \tilde{a}] + \frac{\partial}{\partial z}\nabla h^{(j)}_{\tilde{l},+}, \\ \frac{d}{dt_j}\tilde{l} = -ad^*_{\nabla h^{(j)}_{\tilde{l},+}}\tilde{l} + ad^*_{\tilde{a}}(\nabla h^{(j)}_{\tilde{a},+}) + \frac{\partial}{\partial z}\nabla h^{(j)}_{\tilde{a},+}. \end{gathered} \tag{1.19}$$

The hierarchy of flows (1.19) can be represented with the help of a generating vector field

$$\frac{\partial}{\partial t} := \sum_{j\in\mathbb{Z}_+} \mu^{-j}\frac{\partial}{\partial t_j}$$

as follows:

$$\frac{d}{dt}\tilde{a}(\lambda) = -\frac{\mu}{\mu-\lambda}[\nabla h_{\tilde{l}}(\mu), \tilde{a}(\lambda)| + \frac{\mu}{\mu-\lambda}\frac{\partial}{\partial z}\nabla h_{\tilde{l}}(\mu) \tag{1.20}$$

and

$$\begin{aligned} \frac{d}{dt}(\tilde{l}(\lambda)|\tilde{Y}(\lambda)) = {} & \frac{\mu}{\mu-\lambda}(\tilde{l}(\lambda)|[\nabla h_{\tilde{l}}(\mu)), \tilde{Y}(\lambda)]) \\ & + \frac{\mu}{\mu-\lambda}\big(\nabla h_{\tilde{a}}(\lambda)|[\tilde{a}(\mu), \tilde{Y}(\lambda)]\big) - \frac{\mu}{\mu-\lambda}\left(\frac{\partial}{\partial z}\nabla h_{\tilde{l}}(\mu)|\tilde{Y}(\lambda)\right) \\ = {} & \left(i_{\frac{\mu}{\mu-\lambda}\nabla h_{\tilde{l}}(\mu)} d\tilde{l}(\lambda)|\tilde{Y}(\lambda)\right) + \left(d\, i_{\frac{\mu}{\mu-\lambda}\nabla h_{\tilde{l}}(\mu)}\tilde{l}(\lambda)|\tilde{Y}(\lambda)\right) \\ & - \frac{\mu}{\mu-\lambda}(\langle d/dx, \nabla h_{\tilde{l}}(\mu)\rangle\tilde{l}(\lambda)|\tilde{Y}(\lambda)) - \frac{\mu}{\mu-\lambda}\left(\frac{\partial}{\partial z}\nabla h_{\tilde{a}}(\mu)|\tilde{Y}(\lambda)\right) \\ & - \left(i_{\frac{\mu}{\mu-\lambda}\tilde{a}(\mu)} d\nabla h_{\tilde{a}}(\lambda)|\tilde{Y}(\lambda)\right) + \left(d\, i_{\frac{\mu}{\mu-\lambda}\tilde{a}(\mu)}\nabla h_{\tilde{a}}(\lambda)|\tilde{Y}(\lambda)\right), \end{aligned} \tag{1.21}$$

where $\tilde{Y}(\lambda) \in \mathfrak{G}$ and $\mu \in \mathbb{C}$ is such that $|\lambda/\mu| < 1$ as $|\lambda|,\ |\mu| \to \infty$. Here,

$$\widetilde{\operatorname{diff}}(\mathbb{T}^n) \simeq \tilde{\Gamma}(\mathbb{T}^n) \quad \text{and} \quad \widetilde{\operatorname{diff}}(\mathbb{T}^n)^* \simeq \tilde{\Lambda}^1(\mathbb{T}^n).$$

For the generating element $\tilde{a}(\lambda) \ltimes \tilde{l}(\lambda) \in \tilde{\Gamma}(\mathbb{T}^n) \ltimes \tilde{\Lambda}^1(\mathbb{T}^n)$, relations (1.20) and (1.21) can be rewritten in the form

$$\frac{d}{dt}\tilde{a}(\lambda) = \frac{\partial}{\partial z}\tilde{K}(\mu, \lambda) \tag{1.22}$$

and

$$\begin{aligned}\frac{d}{dt}\tilde{l}(\lambda) &= \frac{\mu}{\mu-\lambda}\langle d/dx, \nabla h_l(\mu)\rangle\tilde{l}(\lambda) + L_{\tilde{A}(\mu,\lambda)}\nabla h_{\tilde{a}}(\mu) + \frac{\partial}{\partial z}\nabla h_{\tilde{a}}(\mu) \\ &:= \operatorname{div}\tilde{K}(\mu,\lambda)\,\tilde{l}(\lambda) + \frac{d}{dz}\nabla h_{\tilde{a}}(\mu),\end{aligned} \tag{1.23}$$

where

$$d/dt := \partial/\partial t + L_{\tilde{K}(\mu,\lambda)}, \qquad d/dz := \frac{\mu}{\mu-\lambda}\partial/\partial z + L_{\tilde{A}(\mu,\lambda)},$$

and also

$$L_{\tilde{K}(\mu,\lambda)} = i_{\tilde{K}(\mu,\lambda)}d + di_{\tilde{K}(\mu,\lambda)} \qquad \text{and} \qquad L_{\tilde{A}(\mu,\lambda)} = i_{\tilde{A}(\mu,\lambda))}d + di_{\tilde{A}(\mu,\lambda)}$$

are the Cartan expressions [11, 13, 14] for the derivatives along the vector fields

$$\tilde{K}(\mu,\lambda) := \frac{\mu}{\mu-\lambda}\nabla h_{\tilde{l}}(\mu) = \frac{\mu}{\mu-\lambda}\left\langle \nabla h_l(\mu), \frac{d}{dx}\right\rangle$$

and

$$\tilde{A}(\mu,\lambda) := \frac{\mu}{\mu-\lambda}\tilde{a}(\mu) = \frac{\mu}{\mu-\lambda}\left\langle a(\mu), \frac{d}{dx}\right\rangle,$$

respectively, where $|\lambda/\mu| < 1$ as $|\mu|,\ |\lambda| \to \infty$, which are equivalent to the hierarchies of the Lax–Sato equations [5, 6, 8, 9, 16–18] for the generating flows (1.23) and (1.22). These flows can be interpreted by using the classical Lagrange–d'Alembert principle [13] in exactly the same way as in [4].

The proposed Lie-algebraic scheme can be used to construct a broad class of integrable multidimensional systems of heavenly type on function spaces.

1.1. Example: New Modified Mikhalev–Pavlov Equation in the Four-Dimensional Space. If the generating element $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}^*$ of Hamiltonian flows in the regular adjoint space $\tilde{\mathfrak{G}}^*$ is chosen in the form

$$\tilde{a} \ltimes \tilde{l} = ((u_x + v_x\lambda - \lambda^2)\partial/\partial x \ltimes (w_x + \zeta_x\lambda)dx), \tag{1.24}$$

where $u, v, w, \zeta \in C^2(\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{T}^1; \mathbb{R})$, then the asymptotic expansions of coordinates of the gradient of the corresponding unique functionally independent Casimir invariant $h \in I(\tilde{\mathfrak{G}}^*)$ as $|\lambda| \to \infty$ can be represented in the form

$$\begin{aligned}\nabla h_{\tilde{l}} &\simeq 1 - v_x\lambda^{-1} - u_x\lambda^{-2} - v_z\lambda^{-3} - (u_z + v_xv_z - 2(\partial_x^{-1}v_{xx}v_z))\lambda^{-4} \\ &+ v_y\lambda^{-5} - (-u_y - v_xv_y + 2(\partial_x^{-1}v_{xx}v_y))\lambda^{-6} + \ldots,\end{aligned}$$

$$\nabla h_{\tilde{a}} \simeq -\zeta_x\lambda^{-1} - w_x\lambda^{-2} - \zeta_z\lambda^{-3} - (w_z - \zeta_x v_z + 2v_x\zeta_z + (\partial_x^{-1}v_x\zeta_x)_z)\lambda^{-4}$$

$$+ \zeta_y\lambda^{-5} - (-w_y + \zeta_x v_y - 2v_x\zeta_y + (\partial_x^{-1}v_x\zeta_x)_y)\lambda^{-6} + \ldots .$$

In the case where

$$\nabla h_{\tilde{l},+}^{(y)} := \lambda^4 - v_x\lambda^3 - u_x\lambda^2 - v_z\lambda - (u_z + v_xv_z - 2(\partial_x^{-1}v_{xx}v_z)),$$

$$\nabla h_{\tilde{a},+}^{(y)} := -\zeta_x\lambda^3 - w_x\lambda^2 - \zeta_z\lambda - (w_z - \zeta_x v_z + 2v_x\zeta_z - (\partial_x^{-1}v_x\zeta_x)_z)$$

and

$$\nabla h_{\tilde{l},+}^{(t)} := \lambda^6 - v_x\lambda^5 - u_x\lambda^4 - v_z\lambda^3 - (u_z + v_xv_z - 2(\partial_x^{-1}v_{xx}v_z))\lambda^2$$

$$+ v_y\lambda - (-u_y - v_xv_y + 2(\partial_x^{-1}v_{xx}v_y)),$$

$$\nabla h_{\tilde{a},+}^{(t)} := -\zeta_x\lambda^5 - w_x\lambda^4 - \zeta_z\lambda^3 - (w_z - \zeta_x v_z + 2v_x\zeta_z - (\partial_x^{-1}v_x\zeta_x)_z)\lambda^2$$

$$+ \zeta_y\lambda - (-w_y + \zeta_x v_y - 2v_x\zeta_y + (\partial_x^{-1}v_x\zeta_x)_y),$$

the commutativity condition for the Hamiltonian flows (1.15) is reduced to the following system of equations:

$$\begin{aligned}
u_{zt} + u_{yy} &= -u_yu_{xz} + u_zu_{xy} - v_yv_{xy} + v_zv_{xt} - u_zv_yv_{xx} + u_yv_zv_{xx} \\
&\quad - v_x^2v_zv_{xy} + v_x^2v_yv_{xz} - 2eu_{xy} - 2su_{xz} + 2e_t - 2s_y + 2ev_yv_{xx} + 2sv_zv_{xx}, \\
v_{zt} + v_{yy} &= -u_yv_{xz} + u_zv_{xy} - v_yu_{xz} + v_zu_{xy} \\
&\quad - 2ev_{xy} - 2sv_{xz} - 2v_xv_yv_{xz} + 2v_xv_zv_{xy}, \\
-u_{xy} - u_{zz} &= u_xu_{xz} - u_zu_{xx} - u_{xx}v_xv_z \\
&\quad + u_xv_{xz}v_x - u_xv_{xx}v_z + (v_xv_z)_z + 2u_{xx}e - 2e_z, \\
-v_{xy} - v_{zz} &= u_{xz}v_x - u_zv_{xx} - u_{xx}v_z + u_xv_{xz} - 2v_{xx}v_xv_z + v_x^2v_{xz} + 2v_{xx}e, \\
-u_{xt} + u_{yz} &= -u_xu_{xy} + u_yu_{xx} + u_{xx}v_xv_y \\
&\quad - u_xv_{xy}v_x + u_xv_{xx}v_y - (v_xv_y)_z + 2u_{xx}s - 2s_z, \\
-v_{xt} + v_{yz} &= -u_{xy}v_x + u_yv_{xx} + u_{xx}v_y \\
&\quad - u_xv_{xy} + 2v_{xx}v_xv_y - v_x^2v_{xy} + 2v_{xx}s, \\
e_x &= v_{xx}v_z, \qquad s_x = -v_{xx}v_y.
\end{aligned} \tag{1.25}$$

As a result of the reduction $v = 0,$ we get a collection of independent differential relations obtained in [18–20], namely, two four-dimensional (in the space variables) equations

$$u_{zt} + u_{yy} = -u_y u_{xz} + u_z u_{xy} \tag{1.26}$$

and

$$-u_{xt} + u_{yz} = -u_x u_{xy} + u_y u_{xx} \tag{1.27}$$

and one three-dimensional (in the space variables) equation

$$-u_{xy} - u_{zz} = u_x u_{xz} - u_z u_{xx}. \tag{1.28}$$

In particular, in the case of reduction of independent space variables $x \to y \in \mathbb{R},\ t \to z \in \mathbb{R},$ Eq. (1.27) becomes trivial, while Eqs. (1.26) and (1.28) are reduced to the Mikhalev–Pavlov-type equation

$$u_{zz} + u_{yy} = -u_y u_{yz} + u_z u_{yy}. \tag{1.29}$$

Proposition 1.3. *The modified system of Mikhalev–Pavlov equations (1.25) admits a vector-field Lax–Sato representation with a "spectral" parameter $\lambda \in \mathbb{C}$ and the generating element $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}^*$ of the form (1.24).*

The generating element (1.24) can be rewritten in the form

$$\tilde{a} \ltimes \tilde{l} := \frac{\partial \tilde{\eta}}{\partial x}\frac{\partial}{\partial x} \ltimes d\tilde{\rho}, \qquad \tilde{\eta} = u + v\lambda - \lambda^2 x, \qquad \tilde{\rho} = w + \zeta\lambda, \tag{1.30}$$

which is explicitly connected with the space of moduli of gauge connectedness for the coadjoint action of the corresponding Casimir invariants. This enables one to construct multidimensional generalizations of system (1.30) by choosing the generating element in the form

$$\tilde{a} \ltimes \tilde{l} := \langle \nabla\tilde{\eta}, \nabla \rangle \ltimes d\tilde{\rho}, \tag{1.31}$$

where $\tilde{\eta}, \tilde{\rho} \in \Omega^0(\mathbb{T}^n) \otimes \mathbb{C},\ n \in \mathbb{N}.$ Case (1.31) will be analyzed elsewhere.

1.2. Example: New Modified Martínez-Alonso–Shabat Equation in the Five-Dimensional Space. For a generating element $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}^*$ of the form

$$\tilde{a} \ltimes \tilde{l} = (((u_{x_1} + cu_{x_2}) + c\lambda)\partial/\partial x_1 + ((v_{x_1} + cv_{x_2}) + c\lambda)\partial/\partial x_2) \ltimes ((w_{x_1} + cw_{x_2})dx_1 + (\zeta_{x_1} + c\zeta_{x_2})dx_2), \tag{1.32}$$

where $u, v, w, \zeta \in C^2(\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{T}^2; \mathbb{R}),\ c \in \mathbb{R}\backslash\{0\},$ we obtain the following asymptotic expansions of coordinates for the gradients of the corresponding two independent Casimir invariants $h^{(1)}, h^{(2)} \in I(\tilde{\mathfrak{G}}^*)$ as $|\lambda| \to \infty$:

$$\nabla h^{(1)}_{\tilde{l}} \simeq \begin{pmatrix} 1 + (u_{x_1} + cu_{x_2})\lambda^{-1} - u_z\lambda^{-2} + \dots \\ c + (v_{x_1} + cv_{x_2})\lambda^{-1} - v_z\lambda^{-2} + \dots \end{pmatrix},$$

$$\nabla h^{(1)}_{\tilde a} \simeq \begin{pmatrix} (w_{x_1}+cw_{x_2})\lambda^{-1} - w_z\lambda^{-2}+\ldots \\ (\zeta_{x_1}+c\zeta_{x_2})\lambda^{-1}-\zeta_z\lambda^{-2}+\ldots \end{pmatrix},$$

and

$$\nabla h^{(2)}_{\tilde l} \simeq \begin{pmatrix} 1+(u_{x_1}-cu_{x_2})\lambda^{-1}+\kappa\lambda^{-2}+\ldots \\ -c+(v_{x_1}-cv_{x_2})\lambda^{-1}+\omega\lambda^{-2}+\ldots \end{pmatrix},$$

$$\nabla h^{(2)}_{\tilde a} \simeq \begin{pmatrix} (w_{x_1}-cw_{x_2})\lambda^{-1}+\varrho\lambda^{-2}+\ldots \\ (\zeta_{x_1}-c\zeta_{x_2})\lambda^{-1}+\chi\lambda^{-2}+\ldots \end{pmatrix},$$

where

$$\begin{aligned} \kappa_{x_1}+c\kappa_{x_2} &= -(u_{zx_1}-cu_{zx_2}) \\ &\quad + 2c(u_{x_1}u_{x_1x_2}-u_{x_2}u_{x_1x_1}+v_{x_1}u_{x_2x_2}-v_{x_2}u_{x_1x_2}), \\ \omega_{x_1}+c\omega_{x_2} &= -(v_{zx_1}-cv_{zx_2}) \\ &\quad + 2c(u_{x_1}v_{x_1x_2}-u_{x_2}v_{x_1x_1}+v_{x_1}v_{x_2x_2}-v_{x_2}v_{x_1x_2}), \end{aligned} \tag{1.33}$$

and

$$\begin{aligned} \varrho_{x_1}+c\varrho_{x_2} &= -(w_{zx_1}-cw_{zx_2}) \\ &\quad + 2c(u_{x_1}w_{x_1x_2}-u_{x_2}w_{x_1x_1}+2w_{x_2}u_{x_1x_1}-2w_{x_1}u_{x_1x_2} \\ &\quad + v_{x_1}w_{x_2x_2}-v_{x_2}w_{x_1x_2}+w_{x_2}v_{x_1x_2}-w_{x_2}v_{x_2x_2}+\zeta_{x_2}v_{x_1x_1}-\zeta_{x_1}v_{x_1x_2}), \\ \chi_{x_1}+c\chi_{x_2} &= -(\zeta_{zx_1}-c\zeta_{zx_2}) \\ &\quad + 2c(v_{x_1}\zeta_{x_2x_2}-v_{x_2}\zeta_{x_1x_2}+2\zeta_{x_2}v_{x_1x_2}-2\zeta_{x_1}v_{x_2x_2} \\ &\quad + u_{x_1}\zeta_{x_1x_2}-u_{x_2}\zeta_{x_1x_1}+\zeta_{x_2}u_{x_1x_1}-\zeta_{x_1}u_{x_1x_2}+w_{x_2}u_{x_1x_2}-w_{x_1}u_{x_2x_2}). \end{aligned}$$

If

$$\nabla h^{(y)}_{\tilde l,+} := \begin{pmatrix} \lambda^2+(u_{x_1}+cu_{x_2})\lambda-u_z \\ c\lambda^2+(v_{x_1}+cv_{x_2})\lambda-v_z \end{pmatrix}, \qquad \nabla h^{(y)}_{\tilde a,+} := \begin{pmatrix} (w_{x_1}+cw_{x_2})\lambda-w_z \\ (\zeta_{x_1}+c\zeta_{x_2})\lambda-\zeta_z \end{pmatrix}$$

and

$$\nabla h^{(t)}_{\tilde l,+} := \begin{pmatrix} \lambda^2+(u_{x_1}-cu_{x_2})\lambda+\kappa \\ -c\lambda^2+(v_{x_1}-cv_{x_2})\lambda+\omega \end{pmatrix}, \qquad \nabla h^{(t)}_{\tilde a,+} := \begin{pmatrix} (w_{x_1}-cw_{x_2})\lambda+\varrho \\ (\zeta_{x_1}-c\zeta_{x_2})\lambda+\chi \end{pmatrix},$$

then the commutativity condition for the Hamiltonian flows (1.15) is reduced to the following system of equations of heavenly type:

$$
\begin{aligned}
u_{zt} + \kappa_y &= -u_{zx_1}\kappa - u_{zx_2}\omega + u_z\kappa_{x_1} + v_z\kappa_{x_2},\\
v_{zt} + \omega_y &= -v_{zx_1}\kappa - v_{zx_2}\omega + u_z\omega_{x_1} + v_z\omega_{x_2},\\
u_{yx_1} + cu_{yx_2} &= -(u_{x_1} + cu_{x_2})u_{zx_1} - (v_{x_1} + cv_{x_2})u_{zx_2}\\
&\quad + (u_{x_1x_1} + cu_{x_1x_2})u_z + (u_{x_1x_2} + cu_{x_2x_2})v_z - u_{zz},\\
v_{yx_1} + cv_{yx_2} &= -(u_{x_1} + cu_{x_2})v_{zx_1} - (v_{x_1} + cv_{x_2})v_{zx_2}\\
&\quad + (v_{x_1x_1} + cv_{x_1x_2})u_z + (v_{x_1x_2} + cv_{x_2x_2})v_z - v_{zz},\\
u_{tx_1} + cu_{tx_2} &= (u_{x_1} + cu_{x_2})\kappa_{x_1} + (v_{x_1} + cv_{x_2})\kappa_{x_2}\\
&\quad - (u_{x_1x_1} + cu_{x_1x_2})\kappa - (u_{x_1x_2} + cu_{x_2x_2})\omega + \kappa_z,\\
v_{tx_1} + cv_{tx_2} &= (u_{x_1} + cu_{x_2})\omega_{x_1} + (v_{x_1} + cv_{x_2})\omega_{x_2}\\
&\quad - (v_{x_1x_1} + cv_{x_1x_2})\kappa - (v_{x_1x_2} + cv_{x_2x_2})\omega + \omega_z.
\end{aligned}
\tag{1.34}
$$

Thus, the following assertion is true:

Proposition 1.4. *The constructed system of equations of heavenly type (1.34), (1.33) admits a vector-field Lax–Sato representation with "spectral" parameter $\lambda \in \mathbb{C}$, which is connected with the element $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}^*$ in the form (1.32).*

For $v = u,\ \omega = \kappa$, and $c = 1$, system (1.34), (1.33) can be reduced to the following system:

$$
\begin{aligned}
u_{zt} + \kappa_y &= -(u_{zx_1} + u_{zx_2})\kappa + u_z(\kappa_{x_1} + \kappa_{x_2}),\\
\kappa_{x_1} + \kappa_{x_2} &= -(u_{zx_1} - u_{zx_2}) - 2((u_{x_1}u_{x_2})_{x_1} - (u_{x_1}u_{x_2})_{x_2}).
\end{aligned}
\tag{1.35}
$$

The presence of additional constraints $u_z = u_{x_1} + u_{x_2}$ for Eqs. (1.35) leads to the following system:

$$
\begin{aligned}
(u_{\tilde{t}x_1} + u_{\tilde{t}x_2}) - (u_{\tilde{y}x_1} - u_{\tilde{y}x_2}) &= u_{x_1x_2}(u_{x_1} - u_{x_2}) - u_{x_1x_1}u_{x_2} + u_{x_2x_2}u_{x_1} - u_{x_1x_2}(u_{x_1}^2 - u_{x_2}^2)\\
&\quad - u_{x_1x_1}u_{x_2}(u_{x_1} + u_{x_2}) + u_{x_2x_2}u_{x_1}(u_{x_1} + u_{x_2})\\
&\quad - 2\rho_{\tilde{y}} + (u_{x_1x_1} + 2u_{x_1x_2} + u_{x_2x_2})\rho,\\
\rho_{x_1} + \rho_{x_2} &= (u_{x_1}u_{x_2})_{x_1} - (u_{x_1}u_{x_2})_{x_2},
\end{aligned}
$$

where $\tilde{t} = 2t$ and $\tilde{y} = 2y$, which can be regarded as a modification of the system of Martínez-Alonso–Shabat-type equations of heavenly type [21].

2. Multidimensional Systems of "Heavenly" Type: Generalized Lie-Algebraic Structures

For subsequent generalizations of the Lie-algebraic scheme connected with the loop group $\widetilde{\mathrm{Diff}}(\mathbb{T}^n)$ on the torus $\mathbb{T}^n,\ n \in \mathbb{Z}_+$, we can use the approach proposed in [5].

Since the Lie algebra $\widetilde{\mathrm{diff}}(\mathbb{T}^n)$ is formed by elements of a loop group analytically extended from the circle $\mathbb{S}^1 := \partial\mathbb{D}^1$, which is the boundary of the central unit disk $\mathbb{D}^1 \subset \mathbb{C}$, with the help of a complex "spectral" variable $\lambda \in \mathbb{C}$, both to the interior $\mathbb{D}^1_+ \subset \mathbb{C}$ and to the exterior $\mathbb{D}^1_- \subset \mathbb{C}$ of the disk $\mathbb{D}^1 \subset \mathbb{C}$, it is analytically invariant under the diffeomorphism group of the circle $\mathrm{Diff}(\mathbb{S}^1)$. This property enables one to consider the extended Lie algebra

$$\mathrm{diff}(\mathbb{T}^n \times \mathbb{C}) = \widetilde{\mathrm{diff}}(\mathbb{T}^n \times \mathbb{D}^1_+) \oplus \widetilde{\mathrm{diff}}(\mathbb{T}^n \times \mathbb{D}^1_-)$$

of holomorphic vector fields on the Cartesian product $\mathbb{T}^n_{\mathbb{C}} := \mathbb{D}^1_{\pm} \times \mathbb{T}^n$ whose elements are vector fields of the form

$$\bar a(x;\lambda) := a_0(x;\lambda)\frac{\partial}{\partial\lambda} + \left\langle a(x;\lambda), \frac{\partial}{\partial \mathrm{x}}\right\rangle = \sum_{j=1}^{n} a_j(x;\lambda)\frac{\partial}{\partial x_j},$$

where $x \in \mathbb{T}^n,\ a(\lambda;x) \in \mathbb{E}\times\mathbb{E}^n$ are vectors on $\mathbb{E}\times\mathbb{E}^n$ holomorphic in $\lambda \in \mathbb{D}^1_{\pm}$, and in addition,

$$\frac{\partial}{\partial \mathrm{x}} := \left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)^{\intercal}$$

denotes the gradient operator in the Euclidean space $\mathbb{E}\times(\mathbb{E}^n;\langle\cdot,\cdot\rangle)$ with respect to the vector variable $\mathrm{x} := (\lambda, x) \in \mathbb{T}^n_{\mathbb{C}}$.

Consider a semidirect sum

$$\bar{\mathcal{G}} := \mathrm{diff}(\mathbb{T}^n\times\mathbb{C}) \ltimes \mathrm{diff}(\mathbb{T}^n\times\mathbb{C})^*$$

of the loop Lie algebra $\mathrm{diff}(\mathbb{T}^n\times\mathbb{C})$ and its regular adjoint space $\mathrm{diff}(\mathbb{T}^n\times\mathbb{C})^*$ with respect to the convolution:

$$(\bar l|\bar a) := \operatorname*{res}_{\lambda\in\mathbb{C}}(l(\mathrm{x})|a(\mathrm{x}))_{H^0}$$

for any $\bar l \in \mathrm{diff}(\mathbb{T}^n\times\mathbb{C})^*$ and $\bar a \in \mathrm{diff}(\mathbb{T}^n\times\mathbb{C})$. Here, each element $\bar l \in \mathrm{diff}(\mathbb{T}^n\times\mathbb{C})^*$ has the form

$$\bar l := \langle l(x;\lambda), d\mathrm{x}\rangle = l_0(x;\lambda)d\lambda + \sum_{j=1}^{n} l_j(x;\lambda)dx_j.$$

The commutator on the loop Lie algebra $\bar{\mathcal{G}}$ for any its elements $\bar a_1 \ltimes \bar l_1,\ \bar a_2 \ltimes \bar l_2 \in \bar{\mathcal{G}}$ is given by the rule

$$[\bar a_1 \ltimes \bar l_1, \bar a_2 \ltimes \bar l_2] := [\bar a_1, a_2] \ltimes ad^*_{a_1}\bar l_2 - ad^*_{a_2}\bar l_1.$$

By using the decomposition of the Lie algebra $\bar{\mathcal{G}}$ into the direct sum of two Lie subalgebras: $\bar{\mathcal{G}} = \bar{\mathcal{G}}_+ \oplus \bar{\mathcal{G}}_-$, we can construct the R-deformed commutator as follows:

$$\big[\bar a_1 \ltimes \bar l_1, \bar a_2 \ltimes \bar l_2\big]_R := \big[R(\bar a_1 \ltimes \bar l_1),\ \bar a_2 \ltimes \bar l_2\big] + \big[\,\bar a_1 \ltimes \bar l_1,\ R(\bar a_2 \ltimes \bar l_2)\big],$$

where $\bar a_1 \ltimes \bar l_1,\ \bar a_2 \ltimes \bar l_2 \in \bar{\mathcal{G}},\ R := (P_+ - P_-)/2$, and $P_\pm\bar{\mathcal{G}} := \bar{\mathcal{G}}_\pm \subset \bar{\mathcal{G}}$.

On the Lie algebra $\bar{\mathcal{G}},$ we can introduce an ad-invariant nondegenerate scalar product:

$$(\bar{a} \ltimes \bar{l}|\bar{r} \ltimes \bar{m}) := \operatorname*{res}_{\lambda\in\mathbb{C}}(\bar{a} \ltimes \bar{l}|\bar{r} \ltimes \bar{m})_{H^0},$$

where, by definition,

$$(\bar{a} \ltimes \bar{l}|\bar{r} \ltimes \bar{m})_{H^0} = (\bar{m}|\bar{a})_{H^0} + (\bar{l}|\bar{r})_{H^0} \tag{2.1}$$

for any pair of elements $\bar{a} \ltimes \bar{l},\ \bar{r} \ltimes \bar{m} \in \bar{\mathcal{G}}$. By using this product, we can identify its regular adjoint space $\bar{\mathcal{G}}^*$ with the Lie algebra: $\bar{\mathcal{G}}^* \simeq \bar{\mathcal{G}}$.

For any smooth functionals $f, g \in \mathrm{D}(\bar{\mathcal{G}}^*),$ we can construct the following two Lie–Poisson brackets:

$$\{f, g\} := \Big(\bar{a} \ltimes \bar{l}|[\nabla f(\bar{l};\bar{a}), \nabla g(\bar{l};\bar{a})]\Big)$$

and

$$\{f, g\}_R := \Big(\bar{a} \ltimes \bar{l}|[\nabla f(\bar{l};\bar{a}), \nabla g(\bar{l};\bar{a})]_R\Big), \tag{2.2}$$

where

$$\nabla f(\bar{l};\bar{a}) := \nabla f_{\bar{l}} \ltimes \nabla f_{\bar{a}} \simeq \langle \nabla f(l;a), (\partial/\partial \mathrm{x}, d\mathrm{x})^{\intercal}\rangle \in \bar{\mathcal{G}}$$

and

$$\nabla g(\bar{l};\bar{a}) := \nabla g_{\bar{l}} \ltimes \nabla g_{\bar{a}} \simeq \langle \nabla g(l;a), (\partial/\partial \mathrm{x}, d\mathrm{x})^{\intercal}\rangle \in \bar{\mathcal{G}}^*$$

are the gradients of the functionals $f, g \in \mathrm{D}(\bar{\mathcal{G}}^*)$ with respect to the scalar product (2.1) at the point $\bar{a}\ltimes\bar{l} \in \bar{\mathcal{G}}^* \simeq \bar{\mathcal{G}}$. Here,

$$\nabla f_{\bar{l}} = \langle \nabla f_l, \partial/\partial \mathrm{x}\rangle, \quad \nabla f_{\bar{a}} = \langle \nabla f_a, d\mathrm{x}\rangle \quad \text{and} \quad \nabla g_{\bar{l}} = \langle \nabla g_l, \partial/\partial \mathrm{x}\rangle, \quad \nabla g_{\bar{a}} = \langle \nabla g_a, d\mathrm{x}\rangle.$$

Assume that a smooth functional $h \in I(\bar{\mathcal{G}}^*)$ is the Casimir invariant, i.e.,

$$ad^*_{\nabla h(\bar{l},\bar{a})}(\bar{a} \ltimes \bar{l}) = 0 \tag{2.3}$$

for a chosen element $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}^* \simeq \bar{\mathcal{G}}$. Since, for any element $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}^* \simeq \bar{\mathcal{G}}$ and any smooth functional $f \in \mathrm{D}(\bar{\mathcal{G}}^*),$ the coadjoint map has the form

$$ad^*_{\nabla f(\bar{l},\bar{a})}(\bar{a} \ltimes \bar{l}) = ([\nabla h_{\tilde{l}}, \tilde{a}] \ltimes (ad^*_{\nabla h_{\tilde{l}}}\tilde{l} + ad^*_{\tilde{a}}\nabla h_{\tilde{a}}),$$

we can rewrite condition (2.3) as follows:

$$[\nabla h_{\tilde{l}}, \tilde{a}] = 0, \qquad ad^*_{\nabla h_{\tilde{l}}}\tilde{l} - ad^*_{\tilde{a}}\nabla h_{\tilde{a}} = 0.$$

Moreover, the Casimir invariant $h \in I(\bar{\mathcal{G}}^*)$ satisfies the system of equations

$$\begin{gathered}\langle \nabla h_l, \partial/\partial \mathrm{x}\rangle\, a - \langle a, \partial/\partial \mathrm{x}\rangle\, \nabla h_l = 0,\\ \langle \partial/\partial \mathrm{x}, \circ\nabla h_l\rangle\, l + \langle l, (\partial/\partial \mathrm{x}\nabla h_l)\rangle + \langle \partial/\partial \mathrm{x}, \circ a\rangle\, \nabla h_a + \langle a, (\partial/\partial \mathrm{x}\nabla h_a)\rangle = 0.\end{gathered} \tag{2.4}$$

For any Casimir invariant $h \in \mathrm{D}(\bar{\mathcal{G}}^*)$, the system of equations (2.4) can be solved analytically. If the element $\bar{l} \ltimes \bar{a} \in \bar{\mathcal{G}}^*$ has a singularity as $|\lambda| \to \infty$, then, for every $p \in \mathbb{Z}_+$, we determine this solution by using the asymptotic expansion

$$\nabla h^{(p)}(l;a) \sim \lambda^p \sum_{j \in \mathbb{Z}_+} \big(\nabla h^{(p)}_{l;j}; \nabla h^{(p)}_{a;j}\big)\lambda^{-j} \tag{2.5}$$

and substitute it in the system of equations (2.4). Hence, the asymptotic solutions of system (2.4) can be found with the help of recurrence relations.

Further, we assume that, for some Casimir invariants $h^{(y)}, h^{(t)} \in I(\bar{\mathcal{G}}^*)$ involutive with respect to the Lie–Poisson bracket (2.2), the generators of Hamiltonian vector fields can be chosen in the form

$$\nabla h^{(y)}(\bar{l};\bar{a})_+ := \big(\nabla h^{(p_y)}(\bar{l};\bar{a})\big)_+, \qquad \nabla h^{(t)}(\bar{l};\bar{a})_+ := \big(\nabla h^{(p_t)}(\bar{l};\bar{a})\big)_+, \tag{2.6}$$

where

$$\nabla h^{(y)}(\bar{l};\bar{a})_+ := \big(\nabla h^{(y)}_{\bar{l},+} \ltimes \nabla h^{(y)}_{\bar{a},+}\big) \in \bar{\mathcal{G}}_+, \qquad \nabla h^{(t)}(\bar{l};\bar{a})_+ := \big(\nabla h^{(t)}_{\bar{l},+} \ltimes \nabla h^{(t)}_{\bar{a},+}\big) \in \bar{\mathcal{G}}_+,$$

$$p^{(y)},\ p^{(t)} \in \mathbb{Z}_+.$$

In view of the Lie–Poisson bracket (2.2), these invariants generate the commuting Hamiltonian flows

$$\begin{aligned} \frac{\partial}{\partial y}(\bar{a} \ltimes \bar{l}) &= -ad^*_{\nabla h^{(y)}(\bar{l},\bar{a})_+}(\bar{a} \ltimes \bar{l}), \\ \frac{\partial}{\partial t}(\bar{a} \ltimes \bar{l}) &= -ad^*_{\nabla h^{(t)}(\bar{l},\bar{a})_+}(\bar{a} \ltimes \bar{l}) \end{aligned} \tag{2.7}$$

for any element $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}^* \simeq \bar{\mathcal{G}}$ with respect to the evolutionary parameters $y, t \in \mathbb{R}$. The constructed flows (2.6) can be represented in the form

$$\begin{aligned} \partial a/\partial y &= -\left\langle \nabla h^{(y)}_l, \frac{\partial}{\partial \mathrm{x}} \right\rangle a + \left\langle a, \frac{\partial}{\partial \mathrm{x}} \right\rangle \nabla h^{(y)}_l, \\ \partial a/\partial t &= -\left\langle \nabla h^{(t)}_l, \frac{\partial}{\partial \mathrm{x}} \right\rangle a + \left\langle a, \frac{\partial}{\partial \mathrm{x}} \right\rangle \nabla h^{(t)}_l \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \partial l/\partial y = &-\left\langle \frac{\partial}{\partial \mathrm{x}}, \nabla h^{(y)}_l \right\rangle l - \left\langle l, \left(\frac{\partial}{\partial \mathrm{x}} \nabla h^{(y)}_l\right) \right\rangle \\ &+ \left\langle \frac{\partial}{\partial \mathrm{x}}, a \right\rangle \nabla h^{(y)}_a + \left\langle a, \left(\frac{\partial}{\partial \mathrm{x}} \nabla h^{(y)}_a\right) \right\rangle, \\ \partial l/\partial t = &-\left\langle \frac{\partial}{\partial \mathrm{x}}, \nabla h^{(t)}_l \right\rangle l - \left\langle l, \left(\frac{\partial}{\partial \mathrm{x}} \nabla h^{(t)}_l\right) \right\rangle \\ &+ \left\langle \frac{\partial}{\partial \mathrm{x}}, a \right\rangle \nabla h^{(t)}_a + \left\langle a, \left(\frac{\partial}{\partial x} \nabla h^{(t)}_a\right) \right\rangle. \end{aligned} \tag{2.9}$$

The commutativity condition for the flows (2.6) is equivalent to the system of equalities

$$\left[\nabla h^{(y)}_{\bar{l},+}, \nabla h^{(t)}_{\bar{l},+}\right] - \frac{\partial}{\partial t}\nabla h^{(y)}_{\bar{l},+} + \frac{\partial}{\partial y}\nabla h^{(t)}_{\bar{l},+} = 0 \tag{2.10}$$

and

$$ad^*_{\bar{a}}\bar{P} = 0,$$

$$\bar{P} = ad^*_{\nabla h^{(y)}_{\bar{l},+}}(\nabla h^{(t)}_{\bar{a},+}) - ad^*_{\nabla h^{(t)}_{\bar{l},+}}(\nabla h^{(y)}_{\bar{a},+}) - \frac{\partial}{\partial t}\nabla h^{(y)}_{\bar{a},+} + \frac{\partial}{\partial y}\nabla h^{(t)}_{\bar{a},+},$$

where $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}$. In addition, equality (2.10) is the compatibility condition for the following three linear vector-field equations:

$$\frac{\partial \psi}{\partial y} + \nabla h^{(y)}_{\bar{l},+}\psi = 0, \qquad \langle a, \partial/\partial \mathrm{x}\rangle\psi = 0, \qquad \frac{\partial \psi}{\partial t} + \nabla h^{(t)}_{\bar{l},+}\psi = 0, \tag{2.11}$$

for some function $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^n_{\mathbb{C}}; \mathbb{C})$, all $y, t \in \mathbb{R}$, and any $\mathrm{x} \in \mathbb{T}^n_{\mathbb{C}}$. The obtained results can be formulated as the following statement:

Proposition 2.1. *Let $h^{(y)}, h^{(t)} \in I(\bar{\mathcal{G}}^*)$ be Casimir invariants for the loop Lie algebra $\bar{\mathcal{G}}$ with respect to the scalar product $(\cdot,\cdot)$ at a point $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}^*$ of the regular adjoint space $\bar{\mathcal{G}}^* \simeq \bar{\mathcal{G}}$ of this Lie algebra. Then evolutions (2.7) on $\bar{\mathcal{G}}^*$ are commuting Hamiltonian flows equivalent to the system of evolutionary equations (2.8), (2.9). The commutativity condition for the evolutionary equations (2.8) is the condition of compatibility of three linear vector-field equations (2.11).*

Note that, in the case where the generating element $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}^*$ is singular as $|\lambda| \to 0$, the asymptotic expansion (2.5) should be replaced by the formula

$$\nabla h^{(p)}(\bar{l}, \bar{a}) \sim \lambda^{-p} \sum_{j \in \mathbb{Z}_+} \nabla h^{(p)}_j(\bar{l}, \bar{a})\lambda^j$$

for every $p \in \mathbb{Z}_+$. The corresponding commuting Hamiltonian flows for the chosen integers $p_y, p_t \in \mathbb{Z}_+$ have the form

$$\frac{\partial}{\partial t}(\bar{a} \ltimes \bar{l}) = ad^*_{\nabla h^{(t)}(\bar{l},\bar{a})_-}(\bar{a} \ltimes \bar{l}),$$

$$\frac{\partial}{\partial y}(\bar{a} \ltimes \bar{l}) = ad^*_{\nabla h^{(y)}(\bar{l},\bar{a})_-}(\bar{a} \ltimes \bar{l}),$$

where

$$\nabla h^{(y)}(\bar{l}, \bar{a})_- := \lambda(\lambda^{-p_y-1}\nabla h^{(p_y)}(\bar{l}, \bar{a}))_-,$$

$$\nabla h^{(t)}(\bar{l}, \bar{a})_- := \lambda(\lambda^{-p_t-1}\nabla h^{(p_t)}(\bar{l}, \bar{a}))_-,$$

and $y, t \in \mathbb{R}$.

As in Sec. 3, we consider the central extension of the point product $\bar{\mathcal{G}}^{\mathbb{S}^1} := \prod_{z\in\mathbb{S}^1} \bar{\mathcal{G}}$ of the holomorphic loop Lie algebra $\bar{\mathcal{G}}$ by the Maurer–Cartan 2-cocycle $\bar{\omega}_2 : \bar{\mathcal{G}} \times \bar{\mathcal{G}} \to \mathbb{C}$ of the form

$$\bar{\omega}_2(\bar{a}_1 \ltimes \bar{l}_1, \bar{a}_2 \ltimes \bar{l}_2) := \int\limits_{\mathbb{S}^1} \left[(\bar{l}_1, \partial\bar{a}_2/\partial z)_1 - (\bar{l}_2, \partial\bar{a}_1/\partial z)_1 \right],$$

where $\bar{a}_1 \ltimes \bar{l}_1, \bar{a}_2 \ltimes \bar{l}_2 \in \bar{\mathcal{G}}$.

For any smooth functionals $h, f \in D(\bar{\mathfrak{G}}^*)$ on the regular space $\bar{\mathfrak{G}}^*$ adjoint to the centrally extended holomorphic loop algebra $\bar{\mathfrak{G}} := \bar{\mathcal{G}} \oplus \mathbb{C}$, the $\mathcal{R}$-deformed Lie–Poisson bracket has the form

$$\{h, f\}_{\mathcal{R}} := \big(\bar{a} \ltimes \bar{l}, [\nabla h(\bar{l}, \bar{a}), \nabla f(\bar{l}, \bar{a})]_{\mathcal{R}}\big) + \bar{\omega}_2\big(\mathcal{R}\nabla h(\bar{l}, \bar{a}), \nabla f(\bar{l}, \bar{a})\big) + \bar{\omega}_2\big(\nabla h(\bar{l}, \bar{a}), \mathcal{R}\nabla f(\bar{l}, \bar{a})\big). \tag{2.12}$$

The corresponding Casimir invariants $h^{(p)} \in I(\bar{\mathfrak{G}}^*)$, $p \in \mathbb{Z}_+$, are determined by the standard Lie–Poisson bracket as follows:

$$\{h^{(p)}, f\} := \big(\bar{a} \ltimes \bar{l}, [\nabla h^{(p)}(\bar{l}, \bar{a}), \nabla f(\bar{l}, \bar{a})]\big) + \bar{\omega}_2\big(\nabla h^{(p)}(\bar{l}, \bar{a}), \nabla f(\bar{l}, \bar{a})\big) = 0$$

for any smooth functional $f \in D(\bar{\mathfrak{G}}^*)$.

It follows from equality (1.18) that the gradients $\nabla h^{(p)} \in \bar{\mathcal{G}}$ of the Casimir invariants $h^{(p)} \in I(\bar{\mathfrak{G}}^*)$, $p \in \mathbb{Z}_+$, satisfy the equations

$$[\nabla h_{\bar{l}}, \bar{a}] - \frac{\partial}{\partial z}\nabla h_{\bar{l}} = 0, \qquad ad^*_{\nabla h_{\bar{l}}}\bar{l} - ad^*_{\bar{a}}\nabla h_{\bar{a}} - \frac{\partial}{\partial z}\nabla h_{\bar{a}} = 0$$

at the point $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}^*$.

For some Casimir invariants $h^{(y)}, h^{(t)} \in I(\bar{\mathfrak{G}}^*)$, we use the Lie–Poisson bracket (2.12) to construct the commuting Hamiltonian flows on $\bar{\mathfrak{G}}^*$:

$$\frac{\partial}{\partial y}(\bar{a} \ltimes \bar{l}) = \{\bar{a} \ltimes \bar{l}, h^{(y)}\}_{\mathcal{R}}, \qquad \frac{\partial}{\partial t}(\bar{a} \ltimes \bar{l}) = \{\bar{a} \ltimes \bar{l}, h^{(t)}\}_{\mathcal{R}}. \tag{2.13}$$

These flow are equivalent to the evolutionary equations

$$\begin{gathered} \frac{\partial}{\partial y}\bar{a} = -\big[\nabla h^{(y)}_{\bar{l},+}, \bar{a}\big] + \frac{\partial}{\partial z}\nabla h^{(y)}_{\bar{l},+}, \\ \frac{\partial}{\partial t}\bar{a} = -\big[\nabla h^{(t)}_{\bar{l},+}, \bar{a}\big] + \frac{\partial}{\partial z}\nabla h^{(t)}_{\bar{l},+} \end{gathered} \tag{2.14}$$

and

$$\begin{gathered} \frac{\partial}{\partial y}\bar{l} = -ad^*_{\nabla h^{(y)}_{\bar{l},+}}\bar{l} + ad^*_{\bar{a}}(\nabla h^{(y)}_{\bar{a},+}) + \frac{\partial}{\partial z}\nabla h^{(y)}_{\bar{a},+}, \\ \frac{\partial}{\partial t}\bar{l} = -ad^*_{\nabla h^{(t)}_{\bar{l},+}}\bar{l} + ad^*_{\bar{a}}(\nabla h^{(t)}_{\bar{a},+}) + \frac{\partial}{\partial z}\nabla h^{(t)}_{\bar{a},+}. \end{gathered} \tag{2.15}$$

The commutativity condition for these flows is equivalent to the system of equations

$$\left[\nabla h^{(y)}_{\bar{l},+}, \nabla h^{(t)}_{\bar{l},+}\right] - \frac{\partial}{\partial t}\nabla h^{(y)}_{\bar{l},+} + \frac{\partial}{\partial y}\nabla h^{(t)}_{\bar{l},+} = 0 \tag{2.16}$$

and

$$\frac{\partial \bar{P}}{\partial z} + ad^*_{\bar{a}}\bar{P} = 0,$$

$$\bar{P} = ad^*_{\nabla h^{(y)}_{\bar{l},+}}(\nabla h^{(t)}_{\bar{a},+}) - ad^*_{\nabla h^{(t)}_{\bar{l},+}}(\nabla h^{(y)}_{\bar{a},+}) - \frac{\partial}{\partial t}\nabla h^{(y)}_{\bar{a},+} + \frac{\partial}{\partial y}\nabla h^{(t)}_{\bar{a},+}$$

for any $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}$.

Hence, the following statement is true :

Proposition 2.2. *The Hamiltonian flows (2.13) on the regular adjoint space* $\bar{\mathfrak{G}}^*$ *generate systems of commuting evolutionary equations (2.14), (2.15). The commutativity condition for the evolutionary equations (2.14) is equivalent to the Lax–Sato representation (2.16) for some system of nonlinear partial differential equations of the heavenly type and coincides with the condition of compatibility of three linear vector-field equations*

$$\frac{\partial \psi}{\partial y} + \nabla h^{(y)}_{\bar{l},+}\psi = 0, \qquad \frac{\partial \psi}{\partial z} + \langle a, \partial/\partial \mathrm{x}\rangle\psi = 0, \qquad \frac{\partial \psi}{\partial t} + \nabla h^{(t)}_{\bar{l},+}\psi = 0$$

for all $(y,t,z;\mathrm{x}) \in (\mathbb{R}^2 \times \mathbb{S}^1) \times \mathbb{T}^n_{\mathbb{C}}$ *and some function* $\psi \in C^2((\mathbb{R}^2 \times \mathbb{S}^1) \times \mathbb{T}^n_{\mathbb{C}}; \mathbb{C})$.

2.1. Example: New Generalized Mikhalev–Pavlov Equation in the Four-Dimensional Space. If the generating element $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}^*$ has the form

$$\bar{a} \ltimes \bar{l} = \big((u_x - \lambda)\partial/\partial x + v_x \partial/\partial\lambda\big) \ltimes (w_x dx + \xi_x d\lambda), \tag{2.17}$$

where

$$u, v, w, \xi \in C^2(\mathbb{R}^2 \times (\mathbb{S}^1 \times \mathbb{T}^1); \mathbb{R}),$$

then the asymptotic expansions for the coordinates of gradients of the Casimir invariants $h^{(p)} \in I(\bar{\mathfrak{G}}^*)$, $p \in \mathbb{Z}_+$, as $|\lambda| \to \infty$, take the form

$$\nabla h_{\tilde{l}} \simeq \lambda^p \begin{pmatrix} 1 - u_x\lambda^{-1} + (-u_z + (p-1)v)\lambda^{-2} + (u_y + (p-2)(-u_x v + \kappa))\lambda^{-3} + \ldots \\ -v_x\lambda^{-1} - v_z\lambda^{-2} + (v_y - (p-2)v_x v)\lambda^{-3} + \ldots \end{pmatrix},$$

$$\nabla h_{\tilde{a}} \simeq \lambda^p \begin{pmatrix} -w_x\lambda^{-1} - w_z\lambda^{-2} + (w_y - (p-2)(wv)_x)\lambda^{-3} + \ldots \\ -\xi_x\lambda^{-1} - (\xi_z + (p-1)w)\lambda^{-2} + (\xi_y - (p-2)(-u_x w + v\xi_x + \omega))\lambda^{-3} + \ldots \end{pmatrix},$$

where $p \in \mathbb{Z}_+$ and, in addition,

$$\kappa_x = v_z + u_x v_x, \qquad \omega_x = w_z - u_x w_x - v_x \xi_x. \tag{2.18}$$

In the case where

$$\nabla h^{(y)}_{\tilde{l},+} := \begin{pmatrix} \lambda^2 - u_x\lambda + (-u_z + v) \\ -v_x\lambda - v_z \end{pmatrix},$$

$$\nabla h^{(y)}_{\tilde{a},+} := \begin{pmatrix} -w_x\lambda - w_z \\ -\xi_x\lambda - (\xi_z + w) \end{pmatrix}$$

and

$$\nabla h^{(t)}_{\tilde{l},+} := \begin{pmatrix} \lambda^3 - u_x\lambda^2 + (-u_z + 2v)\lambda + (u_y - u_x v + \kappa) \\ -v_x\lambda^2 - v_z\lambda + (v_y - v_x v) \end{pmatrix},$$

$$\nabla h^{(t)}_{\tilde{a},+} := \begin{pmatrix} -w_x\lambda^2 - w_z\lambda + (w_y - (wv)_x) \\ -\xi_x\lambda^2 - (\xi_z + 2w)\lambda + (\xi_y + u_x w - v\xi_x - \omega) \end{pmatrix},$$

the commutativity condition for the Hamiltonian flows (2.13) is reduced to the following system of evolutionary equations:

$$\begin{gathered} u_{zt} + u_{yy} = -u_y u_{zx} + u_z u_{xy} - u_{xy} v - u_{zz} v - \kappa u_{xz}, \\ v_{zt} + v_{yy} = v v_x^2 - v_z^2 - v v_{xy} - v v_{zz} - u_y v_{xz} + u_z v_{xy} - u_z v_x^2 - \kappa v_{xz}, \\ -u_{xy} - u_{zz} = u_x u_{xz} - u_z u_{xx} + u_{xx} v, \\ -v_{xy} - v_{zz} = v_x^2 + v_{xx} v + u_x v_{xz} - u_z v_{xx}, \\ -u_{xt} + u_{yz} = -u_x u_{xy} + u_y u_{xx} + u_{xz} v + u_{xx}\kappa, \\ -v_{xt} + v_{yz} = -u_x v_{xy} + u_y v_{xx} + u_x v_x^2 + v_{xz} v + \kappa v_{xx} + 2 v_x v_z. \end{gathered} \tag{2.19}$$

For $v = 0,$ we again arrive at system (1.29).

Thus, the following statement is true:

Proposition 2.3. *The generalized Mikhalev–Pavlov system (2.19), (2.18) admits a vector-field Lax–Sato representation (2.16) with "spectral" parameter $\lambda \in \mathbb{C}$ generated by an element $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}^*$ in the form (2.17).*

Element (2.17) can be rewritten in the form

$$\tilde{a} \ltimes \tilde{l} := \left(\frac{\partial \tilde{\xi}}{\partial x}\frac{\partial}{\partial x} + \frac{\partial \tilde{\xi}_0}{\partial \lambda}\frac{\partial}{\partial \lambda} \right) \ltimes \left(d_x\tilde{\rho} + \frac{\partial \tilde{\rho}_0}{\partial \lambda} \right),$$

$$\tilde{\eta} = u - \lambda x, \qquad \tilde{\eta}_0 = \lambda v_x, \qquad \tilde{\rho} = w, \qquad \tilde{\rho}_0 = \lambda \xi_x,$$

where $d := d_x + d_\lambda,$ connected with the geometry of the space of moduli of gauge connectedness corresponding to the coadjoint actions of the Casimir invariants.

By using an element $\tilde{a} \ltimes \tilde{l} \in \bar{\mathcal{G}}^*$ of the form

$$\tilde{a} \ltimes \tilde{l} := \left(\langle \nabla_x \tilde{\eta}, \nabla_x \rangle + \frac{\partial \tilde{\eta}_0}{\partial \lambda} \frac{\partial}{\partial \lambda} \right) \ltimes \left(d_x \tilde{\rho} + \frac{\partial \tilde{\rho}_0}{\partial \lambda} \right),$$

where $\tilde{\eta}, \tilde{\eta}_0, \tilde{\rho}, \tilde{\rho}_0 \in \Omega^0(\mathbb{T}^n) \otimes \mathbb{C},\ n \in \mathbb{N},\ N > 2,$ we can get a Lax–Sato integrable multidimensional analog of the generalized Mikhalev–Pavlov system (2.19), (2.18) in the $(n+3)$-dimensional space, where $n > 2.$ This case will be considered elsewhere.

3. Conclusions

We present a review of differential-geometric and Lie-algebraic approaches to the construction of Lax–Sato integrable differential systems of equations of heavenly type based on the development of the Adler–Konstant–Symes structure and the R-operator structures associated with this structure. We propose the generalization of this structure to the case of central extensions of the loop Lie algebras $\tilde{\mathcal{G}} := \widetilde{\mathrm{diff}}(\mathbb{T}^n) \ltimes \widetilde{\mathrm{diff}}(\mathbb{T}^n)^*$ and their holomorphic extensions

$$\bar{\mathcal{G}} := \mathrm{diff}(\mathbb{T}^n \times \mathbb{C}) \ltimes \mathrm{diff}(\mathbb{T}^n \times \mathbb{C})^*.$$

Within the framework of the developed approach, the corresponding systems of equations of heavenly type follow from the condition of commutativity of two Hamiltonian flows on regular spaces adjoint to these Lie algebras as one-parameter orbits of the coadjoint action of holomorphic group of diffeomorphisms. In particular, in numerous cases, the solutions of these systems can be found by using the corresponding version of the method of inverse scattering problem [7]. The proposed approach enables us to obtain the Lax–Sato integrable modified and generalized heavenly Mikhalev–Pavlov systems in the four-dimensional space, a modified Martínez-Alonso–Shabat heavenly system in the five-dimensional space, and the corresponding hierarchies of conservation laws. It becomes also possible to consider the problem of construction of their integrable analogs in the spaces of higher dimensions. In addition, on the basis of the proposed approach, it is possible to find the Hamiltonian representations for the constructed heavenly systems as a result of reduction of the R-deformed Lie–Poisson bracket to the phase space of the generating element. This will be done elsewhere.

The authors are deeply grateful to M. Błaszak, E. Tseslins'kyi, and B. Szablikowski for useful discussions of the results obtained in this paper during the symposium on Integrable Systems (June 29–30, 2018, Poznań (Poland)) and to J. Goldin and A. Odzijewich for constructive comments and remarks during the XXXVIIth International Seminar on Geometric Methods in Physics (June 30–July 7, 2019, Białowieża (Poland)).

REFERENCES

1. V. Ovsienko and C. Roger, "Looped cotangent Virasoro algebra and non-linear integrable systems in dimension $2+1$," *Comm. Math. Phys.*, **273**, No. 2, 357–378 (2007).
2. V. Ovsienko, "Bi-Hamiltonian nature of the equation $u_{tx} = u_{xy}u_y - u_{yy}u_x$," *Adv. Pure Appl. Math.*, **1**, No. 1, 7–10 (2008); arXiv:0802.1818v1 (2008).
3. A. Sergyeyev and B. M. Szablikowski, "Central extensions of cotangent universal hierarchy: $(2+1)$-dimensional bi-Hamiltonian systems," *Phys. Lett. A*, **372**, No. 47, 7016–7023 (2008).
4. A. K. Prykarpatski, O. Ye. Hentosh, and Ya. A. Prykarpatsky, "The differential-geometric and algebraic aspects of the Lax–Sato theory," *Mathematics*, **5**, No. 4, MDPI, Basel, Switzerland (2017).
5. O. Ye. Hentosh, Ya. A. Prykarpatsky, D. Blackmore, and A. K. Prykarpatski, "Dispersionless completely integrable heavenly type Hamiltonian flows and their differential-geometric structure," *Symmetry, Integrability, and Geometry: Methods and Appl.*, **15**, Article 079 (2019); https: doi.org/10.3842/SIGMA.2019.079.

6. O. Ye. Hentosh, Ya. A. Prykarpatsky, D. Blackmore, and A. K. Prykarpatski, "Lie-algebraic structure of Lax–Sato integrable heavenly equations and the Lagrange–d'Alembert principle," *J. Geom. Phys.*, **120**, 208–227 (2017); https://doi.org/10.1016/j.geomphys.2017.06.003.
7. S. V. Manakov and P. M. Santini, "Inverse scattering problem for vector fields and the Cauchy problem for the heavenly equation," *Phys. Lett. A*, **359**, No. 6, 613–619 (2006).
8. K. Takasaki and T. Takebe, "$SDiff(2)$ Toda equation – hierarchy, tau function, and symmetries," *Lett. Math. Phys.*, **23**, No. 3, 205–214 (1991).
9. K. Takasaki and T. Takebe, "Integrable hierarchies and dispersionless limit," *Rev. Math. Phys.*, **7**, No. 5, 743–808 (1995).
10. L. A. Takhtadzhyan and L. D. Faddeev, *Hamiltonian Approach to the Theory of Solitons* [in Russian], Nauka, Moscow (1986).
11. D. Blackmore, A. K. Prykarpatsky, and V. Hr. Samoylenko, *Nonlinear Dynamical Systems of Mathematical Physics: Spectral and Symplectic Integrability Analysis*, World Scientific, Hackensack (2011).
12. M. A. Semenov-Tyan-Shanskii, "What is a classical r-matrix?," *Funkts. Anal. Prilozhen.*, **17**, No. 4, 17–33 (1983).
13. R. Abraham and J. Marsden, *Foundations of Mechanics*, Addison-Wesley, Redwood City, CA (1978).
14. C. Godbillon, *Geometrie Differentielle et Mecanique Analytique*, Hermann, Paris (1969).
15. M. Blaszak, *Multi-Hamiltonian Theory of Dynamical Systems*, Springer, Berlin (1998).
16. L. V. Bogdanov, "Interpolating differential reductions of multidimensional integrable hierarchies," *Teor. Mat. Fiz.*, **167**, No. 3, 705–713 (2011).
17. L. V. Bogdanov and B. G. Konopelchenko, "On the heavenly equation and its reductions," *J. Phys. A*, **39**, 11793–11802 (2006).
18. L. V. Bogdanov and M. V. Pavlov, "Linearly degenerate hierarchies of quasiclassical SDYM type," *J. Math. Phys.*, **58**, No. 9, Article 093505 (2017).
19. B. Doubrov and E. V. Ferapontov, "On the integrability of symplectic Monge–Ampère equations," *J. Geom. Phys.*, **60**, 1604–1616 (2010); arXiv:0910.3407v2 [math.DG] 13 Apr 2010.
20. E. V. Ferapontov and J. Moss, *Linearly Degenerate PDEs and Quadratic Line Complexes*; arXiv:1204.2777v1 [math.DG] 12 Apr 2012.
21. L. Martínez Alonso and A. B. Shabat, "Hydrodynamic reductions and solutions of a universal hierarchy," *Teor. Mat. Fiz.*, **140**, No. 2, 216–229 (2004).