



Continuum limits for discrete Dirac operators on 2D square lattices

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Abstract

We discuss the continuum limit of discrete Dirac operators on the square lattice in \mathbb{R}^2 as the mesh size tends to zero. To this end, we propose the most natural and simplest embedding of $\ell^2(\mathbb{Z}_h^d)$ into $L^2(\mathbb{R}^d)$, which enables us to compare the discrete Dirac operators with the continuum Dirac operators in the same Hilbert space $L^2(\mathbb{R}^2)^2$. In particular, we prove that the discrete Dirac operators converge to the continuum Dirac operators in the strong resolvent sense. Potentials are assumed to be bounded and uniformly continuous functions on \mathbb{R}^2 and allowed to be complex matrix-valued. We also prove that the discrete Dirac operators do not converge to the continuum Dirac operators in the norm resolvent sense. This is closely related to the observation that the Liouville theorem does not hold in discrete complex analysis.

Keywords Discrete Dirac operators · Dirac operators on square lattices · Discrete Fourier transform · Continuum limits · Spectrum · Complex potentials

Mathematics Subject Classification Primary 47A10; Secondary 47B37 · 47B93

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1 Introduction

This paper is concerned with the discrete Dirac operator $\mathbb{D}_{m,h} + V_h$ defined by

$$\mathbb{D}_{m,h} + V_h = \begin{pmatrix} m & i\partial_{1,h}^* + \partial_{2,h}^* \\ -i\partial_{1,h} + \partial_{2,h} & -m \end{pmatrix} + V_h \quad \text{in } \ell^2(\mathbb{Z}_h^2)^2, \quad (1.1)$$

which is a discrete analogue of the two-dimensional Dirac operator defined by

$$\mathbb{D}_m + V = -i\sigma_1 \frac{\partial}{\partial x_1} - i\sigma_2 \frac{\partial}{\partial x_2} + m\sigma_3 + V(x) \quad \text{in } L^2(\mathbb{R}^2)^2, \quad (1.2)$$

where $m \geq 0$ and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1.3)$$

and V is a complex matrix valued function. For the definition of $\ell^2(\mathbb{Z}_h^2)^2$, see (2.1) in Sect. 2; the finite difference operators $\partial_{j,h}$ and $\partial_{j,h}^*$ ($j \in \{1, 2\}$) are defined in (2.3) and (2.4) in Sect. 2, respectively; for V_h , see (5.3) in Sect. 5.

We remark that both operators $\mathbb{D}_{m,h}$ in (1.1) and \mathbb{D}_m in (1.2) possess supersymmetry structure (see [28, Chapter 5], [29, Chapter 3]). The discrete Dirac operator (1.1) can be rewritten in a form analogous to (1.2),

$$\mathbb{D}_{m,h} = -i\sigma_1 \begin{pmatrix} \partial_{1,h} & 0 \\ 0 & -\partial_{1,h}^* \end{pmatrix} - i\sigma_2 \begin{pmatrix} \partial_{2,h} & 0 \\ 0 & -\partial_{2,h}^* \end{pmatrix} + m\sigma_3 + V_h.$$

It is widely recognized that 2D Dirac operators, especially in the massless case, have been the object of extensive research in the context of graphene since its discovery in 2004, see [5] or [21] for an exposition. In particular, we would like to mention the work [20], which reported that electron transport in graphene is essentially governed by a massless Dirac equation and that a variety of unusual phenomena are characteristic of two-dimensional Dirac fermions. These are the main reasons why we focus on the two-dimensional case, although it is apparent that the methods and ideas to be developed below in the present paper are directly applicable to the one-dimensional and the three-dimensional cases. The discussions in these two cases will appear elsewhere.

It is natural to make an attempt to show that the discrete operator (1.1) converges to the continuum operator (1.2) as the mesh size h of the lattice \mathbb{Z}_h^2 tends to 0. However, there is a difficulty in that these two operators work in completely different Hilbert spaces. For example, it is not immediately obvious how one can make sense of the expression $(\mathbb{D}_{m,h} + V_h) - (\mathbb{D}_m + V)$. For this reason, it is necessary to embed $\ell^2(\mathbb{Z}_h^2)^2$ onto an appropriate subspace of $L^2(\mathbb{R}^2)^2$.

In this paper, we propose a simple and natural embedding of $\ell^2(\mathbb{Z}_h^d)$ into $L^2(\mathbb{R}^d)$ by assigning to each element in $\ell^2(\mathbb{Z}_h^d)$ a step function in $L^2(\mathbb{R}^d)$:

$$[J_h f](x) := \sum_{n \in \mathbb{Z}^d} f(hn) \chi_{I_{n,h}}(x) \quad (f \in \ell^2(\mathbb{Z}_h^d)) \quad (1.4)$$

where $\chi_{I_{n,h}}$ is a characteristic function of the set

$$I_{n,h} := \{x \mid hn_j \leq x_j < h(n_j + 1), j \in \{1, \dots, d\}\}$$

(see Sect. 2.1). We find it is important that the discrete Fourier transform can be naturally defined for $J_h f$ (see Sect. 2.2). Also, the use of step functions is desirable from the point of view of numerical analysis. This idea of embedding $\ell^2(\mathbb{Z}_h^d)$ into $L^2(\mathbb{R}^d)$ induces a subspace $L^2(\mathbb{Z}_h^d)$ of $L^2(\mathbb{R}^d)$. With this embedding, one can naturally define the difference operators $\partial_{j,h}$ and $\partial_{j,h}^*$ in $L^2(\mathbb{Z}_h^d)$, the subspace of step functions of the form (1.4) (cf. Sect. 2.1), and hence the discrete Dirac operators $\mathbb{D}_{m,h}$ in $L^2(\mathbb{Z}_h^2)^2$. For the reasons mentioned here, the discrete Dirac operators $\mathbb{D}_{m,h}$ in $L^2(\mathbb{Z}_h^2)^2$ are the exact counterparts of the discrete Dirac operators $\mathbb{D}_{m,h}$ in $\ell^2(\mathbb{Z}_h^2)^2$, so we can identify these two operators. In other words, we are able to regard the discrete Dirac operators $\mathbb{D}_{m,h} + V_h$ as an operator acting in $L^2(\mathbb{R}^2)^2$ with domain $L^2(\mathbb{Z}_h^d)^2$, and able to compare the discrete Dirac operators $\mathbb{D}_{m,h} + V_h$ with the continuum Dirac operators $\mathbb{D}_m + V$ in the same Hilbert space $L^2(\mathbb{R}^2)^2$. The purpose of the present paper is to show, with the embedding operator defined by (1.4), that the resolvents of the discrete Dirac operators (1.1) converge to the continuum Dirac operators (1.2) in the strong resolvent sense [24] as the mesh size h tends to 0 (see Theorem 4.2 in Sect. 4 and Theorem 5.1 in Sect. 5). In addition, we show that the discrete operator $\mathbb{D}_{m,h}$ does not converge to the continuum operator \mathbb{D}_m in the norm resolvent sense (see Theorem 4.3 in Sect. 4). As a motivation for the proof of the latter theorem, we observe that the Liouville theorem does not hold in discrete complex analysis (see Remark 4.4 in Sect. 4).

In connection with the embedding operator (1.4), we would like to mention the works by [7] and [19], in which the embedding operators are defined by

$$\sum_{n \in \mathbb{Z}^d} \rho((x - hn)/h) f(hn) \quad (f \in \ell^2(\mathbb{Z}_h^d)), \quad (1.5)$$

with ρ a smooth and (possibly rapidly) decreasing function. However, in this paper, we do not adopt this type of embedding of $\ell^2(\mathbb{Z}_h^d)$ into $L^2(\mathbb{R}^d)$, because of the following three reasons. Firstly, the embedding operator (1.5) depends on the choice of the function ρ . Secondly, the embedded functions defined by (1.5) are smooth, and no longer discrete objects. In fact, the discrete Fourier transform is not applicable to smooth functions. Thirdly, the difference operators working on smooth functions can be regarded as a mixture of discreteness and continuum, and may not be regarded as the exact counterparts of difference operators in $\ell^2(\mathbb{Z}_h^2)$. On the other hand, as was pointed out in [7], it is inevitable to introduce the embedding operator (1.5) and a

modification of the discrete Dirac operators in $\ell^2(\mathbb{Z}_h^2)^2$ if one would like to show the norm resolvent convergence.

We should like to remark that if one replaces the function ρ with the characteristic function $\chi_{I_{0,1}}$ (i.e., $\chi_{I_{n,h}}$ with $n = 0, h = 1$) then the embedding operator (1.5) coincides with our embedding operator (1.4).

When we apply our idea to the discrete Dirac operator (1.1) to discuss the continuum limit as the mesh size h tends to 0, we require a convergence theorem for the orthogonal projection P_h onto the closed subspace $L^2(\mathbb{Z}_h^2)^2$ of $L^2(\mathbb{R}^2)^2$. Also, we require a discrete Fourier transform on $L^2(\mathbb{Z}_h^2)$ (which is essentially the same, but not identical to the Fourier series with coefficients in $\ell^2(\mathbb{Z}_h^2)$), and need to prove a convergence theorem for the discrete Fourier transform as $h \rightarrow 0$. Indeed, we will establish both convergence theorems in the strong topology of $L^2(\mathbb{R}^d)$. Precise descriptions are given in Sects. 2.2 and 3. With these strong convergence theorems, we can prove that the resolvents of the discrete Dirac operator (1.1) strongly converge to those of the Dirac operator (1.2) in $L^2(\mathbb{R}^2)^2$.

In the literature, there have been few papers studying spectral properties of discrete Dirac operators on 2D or 3D lattices, while there have been many working on 1D lattices, see for example the recent works [1, 2, 4, 6, 8, 11, 12, 15–17, 22, 25–27]. We mention in passing that the discrete Dirac operator on a 1D lattice, when written in matrix form, is a tri-diagonal matrix and indeed unitarily equivalent to a discrete Schrödinger-type operator; for example, with the unitary operator $U : \ell^2(\mathbb{Z}_h) \rightarrow \ell^2(\mathbb{Z}_h)^2$ given as $(Uf)(nh) = (-1)^n \begin{pmatrix} f(2nh) \\ f((2n+1)h) \end{pmatrix}$,

$$U^* \begin{pmatrix} m & \partial_h^* \\ \partial_h & -m \end{pmatrix} U = h(-\partial_h^* \partial_h + q),$$

where $q(nh) = (-1)^n m/h - 2/h^2$.

To our knowledge, the only papers working on discrete Dirac operators in dimensions 2 and 3 are [7, 23]. The lack of works on the continuum limit of discrete analogs of quantum Hamiltonians, as far as we know, is hardly surprising in view of the fact that research on this topic began rather recently; see [7, 14, 19].

Finally, we would like to mention yet another idea of natural embedding. Indeed, we find the embedding in [14] is natural in the sense that it assigns to each element in $\ell^2(\mathbb{Z}_h^d)$ a discrete object in $\mathcal{S}'(\mathbb{R}^d)$ and discrete Fourier transform is naturally associated. With this embedding operator, continuum limits of lattice Schrödinger operators for various models were investigated in [14]. In particular, lattice Laplacians satisfying suitable assumptions were shown to converge to the 2D Dirac operators (1.2) with $m = 0$ and $V = 0$. Specifically for the hexagonal (graphene) lattice, see also [9].

The present paper is organised as follows. Section 2 illustrates the idea of embedding $\ell^2(\mathbb{Z}_h^d)$ into $L^2(\mathbb{R}^d)$ and shows how the finite difference operators in the embedded space can naturally be defined. It also describes how the discrete Fourier transform can be extended as an operator in $L^2(\mathbb{R}^d)$. In Sect. 3, convergence of discrete Fourier transform in $L^2(\mathbb{R}^d)$ is discussed. Resolvent convergence of the discrete Dirac operator without potential is discussed in Sect. 4, based on the results obtained in the previous

sections. Strong resolvent convergence of the discrete Dirac operators with potentials is discussed in Sect. 5.

2 Embedding of $\ell^2(\mathbb{Z}_h^d)$ into $L^2(\mathbb{R}^d)$ and discrete Fourier transform in $L^2(\mathbb{R}^d)$

In applications to the Dirac operator (1.1), the underlying Hilbert space is $\ell^2(\mathbb{Z}_h^d)^2 := \ell^2(\mathbb{Z}_h^d) \otimes \mathbb{C}^2$, which consists of \mathbb{C}^2 -valued functions on the 2-dimensional lattice. In this section, we focus on the space of complex-valued functions on the d -dimensional lattice, $\ell^2(\mathbb{Z}_h^d)$, and related spaces of functions on \mathbb{R}^d ; the results naturally extend to the corresponding spaces of \mathbb{C}^2 -valued functions.

The d -dimensional square lattice with the mesh size $h > 0$ is denoted by

$$\mathbb{Z}_h^d := \{hn \mid n \in \mathbb{Z}^d\}.$$

The Hilbert space

$$\ell^2(\mathbb{Z}_h^d) := \left\{ f \mid f : \mathbb{Z}_h^d \rightarrow \mathbb{C}, \sum_{n \in \mathbb{Z}^d} |f(hn)|^2 < \infty \right\}, \quad (2.1)$$

has the standard inner product

$$(f, g)_{\ell^2(\mathbb{Z}_h^d)} = \sum_{n \in \mathbb{Z}^d} f(hn) \overline{g(hn)}. \quad (2.2)$$

For $f \in \ell^2(\mathbb{Z}_h^d)$, define

$$[\partial_{j,h} f](hn) := \frac{1}{h} \left\{ f(h(n + e_j)) - f(hn) \right\} \quad (j \in \{1, \dots, d\}), \quad (2.3)$$

where $e_1 = (1, 0, \dots, 0), \dots, e_d = (0, \dots, 0, 1)$. The adjoint of $\partial_{j,h}$ is given by

$$[\partial_{j,h}^* f](hn) := \frac{1}{h} \left\{ f(h(n - e_j)) - f(hn) \right\} \quad (j \in \{1, \dots, d\}), \quad (2.4)$$

so

$$(\partial_{j,h} f, g)_{\ell^2(\mathbb{Z}_h^d)} = (f, \partial_{j,h}^* g)_{\ell^2(\mathbb{Z}_h^d)}, \quad \forall f, g \in \ell^2(\mathbb{Z}_h^d). \quad (2.5)$$

2.1 Embedding of $\ell^2(\mathbb{Z}_h^d)$ into $L^2(\mathbb{R}^d)$

We introduce an embedding of $\ell^2(\mathbb{Z}_h^d)$ into $L^2(\mathbb{R}^d)$ by assigning to $f \in \ell^2(\mathbb{Z}_h^d)$ the step function

$$[J_h f](x) := \sum_{n \in \mathbb{Z}^d} f(hn) \chi_{I_{n,h}}(x) \in L^2(\mathbb{R}^d), \tag{2.6}$$

where $\chi_{I_{n,h}}$ is a characteristic function of the set

$$I_{n,h} := \{x \mid hn_j \leq x_j < h(n_j + 1), j \in \{1, \dots, d\}\}; \tag{2.7}$$

clearly, $\sum_{n \in \mathbb{Z}^d} \chi_{I_{n,h}} \equiv 1$. Note that

$$\|J_h f\|_{L^2(\mathbb{R}^d)} = h^{d/2} \|f\|_{\ell^2(\mathbb{Z}_h^d)}, \tag{2.8}$$

so $h^{-d/2} J_h$ is an isometry from $\ell^2(\mathbb{Z}_h^d)$ into $L^2(\mathbb{R}^d)$. Since $\ell^2(\mathbb{Z}_h^d)$ is a Hilbert space, the image $J_h(\ell^2(\mathbb{Z}_h^d))$ is a closed subspace of the Hilbert space $L^2(\mathbb{R}^d)$. We thus have an orthogonal decomposition

$$L^2(\mathbb{R}^d) = L^2(\mathbb{Z}_h^d) \oplus L^2(\mathbb{Z}_h^d)^\perp, \quad L^2(\mathbb{Z}_h^d) := J_h(\ell^2(\mathbb{Z}_h^d)). \tag{2.9}$$

(The notation $L^2(\mathbb{Z}_h^d)$ already appeared in [14], but there it essentially denotes $\ell^2(\mathbb{Z}_h^d)$; for details, see [14, Subsection 2.1].) The orthogonal projection P_h of $L^2(\mathbb{R}^d)$ onto $L^2(\mathbb{Z}_h^d)$ can be described, for general $\varphi \in L^2(\mathbb{R}^d)$, as $P_h \varphi = J_h \tilde{\varphi}$, where

$$\tilde{\varphi}(hn) = \frac{1}{h^d} \int_{I_{n,h}} \varphi(x) dx \quad (n \in \mathbb{Z}^d). \tag{2.10}$$

Remark 2.1 The following example illustrates the action of the projection P_h . Let $\varphi \in L^2(\mathbb{R})$ be defined by $\varphi(x) = 0$ ($|x| \geq 2h$), $\varphi(x) = x + 2h$ ($-2h < x < -h$), $\varphi(x) = h$ ($|x| \leq h$), and $\varphi(x) = -x + 2h$ ($h < x < 2h$). This function can be decomposed as $\varphi = f_h + g_h$ with

$$f_h(x) = \begin{cases} h & \text{if } -h \leq x < h, \\ \frac{h}{2} & \text{if } -2h \leq x < -h \text{ or } h \leq x < 2h, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g_h(x) = \begin{cases} x + \frac{3h}{2} & \text{if } -2h \leq x < -h, \\ -x + \frac{3h}{2} & \text{if } h \leq x < 2h, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $f_h \in L^2(\mathbb{Z}_h)$ and $g_h \in L^2(\mathbb{Z}_h)^\perp$, so $P_h \varphi = f_h$.

As the above embedding gives a one-to-one relationship between the elements of $\ell^2(\mathbb{Z}_h^d)$ and of $L^2(\mathbb{Z}_h^d)$, we can define the finite difference operators $\partial_{j,h}$ and $\partial_{j,h}^*$ on $L^2(\mathbb{Z}_h^d)$ by applying them, as defined in (2.3) and (2.4), to the corresponding element of $\ell^2(\mathbb{Z}_h^d)$, such that

$$\partial_{j,h} J_h[f] := J_h[\partial_{j,h} f] \quad \partial_{j,h}^* J_h[f] := J_h[\partial_{j,h}^* f] \quad (f \in \ell^2(\mathbb{Z}_h^d)). \quad (2.11)$$

Then we again have

$$(\partial_{j,h} f, g)_{L^2(\mathbb{Z}_h^d)} = (f, \partial_{j,h}^* g)_{L^2(\mathbb{Z}_h^d)} \quad (f, g \in L^2(\mathbb{Z}_h^d)). \quad (2.12)$$

in analogy to (2.5).

2.2 Discrete Fourier transform

Let

$$\mathbb{T}_{1/h}^d = [-\pi/h, \pi/h]^d = \{\xi \in \mathbb{R}^d \mid |\xi|_\infty \leq \pi/h\}, \quad (2.13)$$

where $|\xi|_\infty = \max\{|\xi_1|, \dots, |\xi_d|\}$.

(Although we use a notation alluding to the interpretation, natural in the following, of this set as a flat d -dimensional torus of side length $2\pi/h$, we emphasize that it is a bounded interval in \mathbb{R}^d .) As the functions

$$e_n(\xi) = \left(\frac{h}{2\pi}\right)^{d/2} e^{-ihn \cdot \xi} \quad (\xi \in \mathbb{T}_{1/h}^d; n \in \mathbb{Z}^d)$$

form an orthonormal basis of $L^2(\mathbb{T}_{1/h}^d)$, any function $f \in \ell^2(\mathbb{Z}_h^d)$ serves as a collection of Fourier coefficients for a d -dimensional Fourier series in $L^2(\mathbb{T}_{1/h}^d)$,

$$\sum_{n \in \mathbb{Z}^d} f(hn) e_n(\xi) \quad (\xi \in \mathbb{T}_{1/h}^d).$$

In view of the bijection between $\ell^2(\mathbb{Z}_h^d)$ and $L^2(\mathbb{Z}_h^d)$, this motivates the following definition of a *discrete Fourier transform* $\mathcal{F}_h : L^2(\mathbb{Z}_h^d) \rightarrow L^2(\mathbb{T}_{1/h}^d)$,

$$\begin{aligned} [\mathcal{F}_h J_h f](\xi) &:= h^{d/2} \sum_{n \in \mathbb{Z}^d} f(hn) e_n(\xi) \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}^d} e^{-ihn \cdot \xi} f(hn) \chi_{I_{n,h}}(x) dx \quad (\xi \in \mathbb{T}_{1/h}^d) \end{aligned} \quad (2.14)$$

for $f \in \ell^2(\mathbb{Z}_h^d)$. By Parseval's identity for the orthonormal basis $\{e_n \mid n \in \mathbb{Z}^d\}$,

$$\|\mathcal{F}_h J_h f\|_{L^2(\mathbb{T}_{1/h}^d)}^2 = h^d \sum_{n \in \mathbb{Z}^d} |f(hn)|^2 = h^d \|f\|_{\ell^2(\mathbb{Z}_h^d)}^2 = \|J_h f\|_{L^2(\mathbb{Z}_h^d)}^2 \quad (2.15)$$

for any $f \in \ell^2(\mathbb{Z}_h^d)$, so \mathcal{F}_h is a unitary operator.

Its inverse is the operator $\overline{\mathcal{F}}_h : L^2(\mathbb{T}_{1/h}^d) \rightarrow L^2(\mathbb{Z}_h^d)$,

$$[\overline{\mathcal{F}}_h u](x) = \sum_{n \in \mathbb{Z}^d} \left\{ (2\pi)^{-d/2} \int_{\mathbb{T}_{1/h}^d} e^{ihn \cdot \xi} u(\xi) d\xi \right\} \chi_{I_{n,h}}(x) \quad (x \in \mathbb{R}^d). \quad (2.16)$$

for $u \in L^2(\mathbb{T}_{1/h}^d)$.

By direct computations, we have

$$[\mathcal{F}_h(\partial_{j,h} f)](\xi) = \frac{1}{h} (e^{ih\xi_j} - 1) [\mathcal{F}_h f](\xi) \quad (f \in L^2(\mathbb{Z}_h^d)) \quad (2.17)$$

and

$$[\mathcal{F}_h(\partial_{j,h}^* f)](\xi) = \frac{1}{h} (e^{-ih\xi_j} - 1) [\mathcal{F}_h f](\xi) \quad (f \in L^2(\mathbb{Z}_h^d)). \quad (2.18)$$

Extending functions by 0 outside $\mathbb{T}_{1/h}^d$, the space $L^2(\mathbb{T}_{1/h}^d)$ naturally forms a closed subspace of $L^2(\mathbb{R}^d)$; it is the range of the orthogonal projection $Q_{1/h}$ defined as the operator of multiplication with the characteristic function of $\mathbb{T}_{1/h}^d$.

Using the projections $P_h \in \mathbf{B}(L^2(\mathbb{R}^d))$ and $Q_{1/h} \in \mathbf{B}(L^2(\mathbb{R}^d))$, we can extend \mathcal{F}_h and its inverse $\overline{\mathcal{F}}_h$ to become elements of $\mathbf{B}(L^2(\mathbb{R}^d))$ by setting

$$\mathcal{F}_h := Q_{1/h} \mathcal{F}_h P_h, \quad \overline{\mathcal{F}}_h := P_h \overline{\mathcal{F}}_h Q_{1/h}. \quad (2.19)$$

Here $\mathbf{B}(L^2(\mathbb{R}^d))$ denotes the Banach space of all bounded linear operators in $L^2(\mathbb{R}^d)$, equipped with the uniform operator topology. Note that \mathcal{F}_h is a partial isometry from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ with the initial set $L^2(\mathbb{Z}_h^d)$ and the final set $L^2(\mathbb{T}_{1/h}^d)$, and that $\overline{\mathcal{F}}_h$ is a partial isometry from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ with the initial set $L^2(\mathbb{T}_{1/h}^d)$ and the final set $L^2(\mathbb{Z}_h^d)$. Clearly,

$$\overline{\mathcal{F}}_h \mathcal{F}_h = P_h, \quad \mathcal{F}_h \overline{\mathcal{F}}_h = Q_{1/h}, \quad (2.20)$$

and

$$P_h^\perp \overline{\mathcal{F}}_h = \mathbf{0}, \quad Q_{1/h}^\perp \mathcal{F}_h = \mathbf{0}. \quad (2.21)$$

3 Convergence of discrete Fourier transforms

As in the previous section, we will work in \mathbb{C} -valued functions on d dimensional Euclidean space \mathbb{R}^d .

For $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz space of rapidly decreasing functions, we define

$$\varphi_h(x) = \sum_{n \in \mathbb{Z}^d} \varphi(hn) \chi_{I_{n,h}}(x). \quad (3.1)$$

It is clear that $\varphi_h \in L^2(\mathbb{Z}_h^d)$ and $P_h \varphi_h = \varphi_h$. However, we emphasize that $P_h \varphi \neq \varphi_h$ in general. In fact, for each $h > 0$ one can easily choose a function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $(\varphi_h, \varphi - \varphi_h)_{L^2(\mathbb{R}^d)} \neq 0$.

Remark 3.1 As was noted in (2.8), we have $\|\varphi_h\|_{L^2(\mathbb{R}^d)} = h^{d/2} \|\varphi_h\|_{\ell^2(\mathbb{Z}_h^d)}$.

Lemma 3.1 Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $k \in \mathbb{N}$.

(i) There exists a constant $C_{\varphi k}$, depending only on φ and k , such that

$$|\varphi_h(x) - \varphi(x)| \leq C_{\varphi k} \langle x \rangle^{-k} h \quad \left(x \in \mathbb{R}^d, 0 < h < \frac{1}{\sqrt{d}} \right), \quad (3.2)$$

where $\langle x \rangle = \sqrt{1 + |x|^2}$. In particular, φ_h is rapidly decreasing, i.e., $|\varphi_h(x)| \leq C_{\varphi k} \langle x \rangle^{-k}$ for any $k \in \mathbb{N}$.

(ii) There exists a constant C_φ , depending only on φ , such that

$$\|\varphi_h - \varphi\|_{L^2} \leq C_\varphi h \quad \left(0 < h < \frac{1}{\sqrt{d}} \right). \quad (3.3)$$

Proof We prove statement (i); then statement (ii) follows as a straightforward consequence, taking $k > d/2$. Let $n \in \mathbb{Z}^d$. Then, for $x \in I_{n,h}$,

$$\varphi(hn) - \varphi(x) = (hn - x) \cdot \int_0^1 (\nabla \varphi)(t(hn - x) + x) dt.$$

Consequently,

$$\begin{aligned} & |\varphi(hn) - \varphi(x)| \langle x \rangle^k \\ & \leq |hn - x| \int_0^1 \frac{\langle x \rangle^k}{\langle t(hn - x) + x \rangle^k} |(\nabla \varphi)(t(hn - x) + x)| \langle t(hn - x) + x \rangle^k dt \\ & \leq \left(\sup_{y \in \mathbb{R}^d} |(\nabla \varphi)(y)| \langle y \rangle^k \right) |hn - x| \int_0^1 \frac{\langle x \rangle^k}{\langle t(hn - x) + x \rangle^k} dt. \end{aligned}$$

Now if $|x| \geq 2\sqrt{d}h$, then

$$|t(hn - x) + x| \geq |x| - |hn - x| \geq |x| - \sqrt{d}h \geq \frac{|x|}{2},$$

so

$$\langle t(hn - x) + x \rangle \geq \sqrt{1 + \frac{|x|^2}{4}} = \sqrt{1 + \frac{\langle x \rangle^2 - 1}{4}} \geq \frac{\langle x \rangle}{2};$$

if $|x| < 2\sqrt{dh}$, then (trivially) $\langle t(hn - x) + x \rangle \geq 1$ and $\langle x \rangle < \sqrt{1 + 4dh^2} < \sqrt{5}$. In either case,

$$\int_0^1 \frac{\langle x \rangle^k}{\langle t(hn - x) + x \rangle^k} dt \leq \sqrt{5}^k,$$

and, noting that $|hn - x| \leq \sqrt{dh}$, the inequality (3.2) follows.

The statement of Lemma 3.1 (i) has the following immediate consequence. □

Corollary 3.1 *Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $k \in \mathbb{N}$. There exists a constant $C_{\varphi k}$, depending only on φ and k , such that*

$$\sum_{n \in \mathbb{Z}^d} |\varphi(hn)| \chi_{I_{n,h}}(x) \leq |\varphi(x)| + C_{\varphi k} h \langle x \rangle^{-k} \quad (x \in \mathbb{R}^d) \tag{3.4}$$

for $0 < h < 1/\sqrt{d}$.

In what follows, we shall use the notation

$$\mathcal{S}_h^{\text{step}}(\mathbb{R}^d) = \left\{ \varphi_h = \sum_{n \in \mathbb{Z}^d} \varphi(hn) \chi_{I_{n,h}} \mid \varphi \in \mathcal{S}(\mathbb{R}^d) \right\}, \tag{3.5}$$

$$\mathcal{S}_{0+}^{\text{step}}(\mathbb{R}^d) = \bigcup_{h>0} \mathcal{S}_h^{\text{step}}(\mathbb{R}^d). \tag{3.6}$$

Note that, unlike $\mathcal{S}_h^{\text{step}}(\mathbb{R}^d)$, the set of functions $\mathcal{S}_{0+}^{\text{step}}(\mathbb{R}^d)$ is not a vector space. As we shall see in Lemma 3.4 in Sect. 3.1, each $\varphi_h \in \mathcal{S}_{0+}^{\text{step}}(\mathbb{R}^d)$ allows an explicit expression of its Fourier transform in a certain sense.

The following lemma is a direct consequence of Lemma 3.1(ii).

Lemma 3.2 $\mathcal{S}_{0+}^{\text{step}}(\mathbb{R}^d)$ is a dense subset of $L^2(\mathbb{R}^d)$.

We can now prove the strong convergence of the orthogonal projectors P_h and $Q_{1/h}$ to the identity.

Lemma 3.3 *For any $u \in L^2(\mathbb{R}^d)$, $\|P_h u - u\|_{L^2(\mathbb{R}^d)} \rightarrow 0$ and $\|Q_{1/h} u - u\|_{L^2(\mathbb{R}^d)} \rightarrow 0$ as $h \rightarrow 0$.*

Proof Since $\|P_h\|_{\mathbf{B}(L^2(\mathbb{R}^d))} = 1$ for all $h > 0$ and $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, it is sufficient to prove that for $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\|P_h\varphi - \varphi\|_{L^2(\mathbb{R}^d)} \rightarrow 0$ as $h \rightarrow 0$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and, for each $h > 0$, let $\varphi_h \in L^2(\mathbb{Z}_h^d)$ be the step function defined in (3.1). Then $P_h\varphi_h = \varphi_h$, so

$$\begin{aligned} \|P_h\varphi - \varphi\|_{L^2(\mathbb{R}^d)} &\leq \|P_h\varphi - P_h\varphi_h\|_{L^2(\mathbb{R}^d)} + \|\varphi_h - \varphi\|_{L^2(\mathbb{R}^d)} \\ &\leq (\|P_h\|_{\mathbf{B}(L^2(\mathbb{R}^d))} + 1) \|\varphi_h - \varphi\|_{L^2(\mathbb{R}^d)} \rightarrow 0 \quad (h \rightarrow 0) \end{aligned}$$

by Lemma 3.1(ii).

Furthermore, for any $u \in L^2(\mathbb{R}^d)$

$$\|Q_{1/h}u - u\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d \setminus \mathbb{T}_{1/h}^d} |u|^2 \rightarrow 0 \quad (h \rightarrow 0).$$

□

Remark 3.2 It is clear from the proof of Lemma 3.3 that the projection $Q_{1/h}$ does not converge to the identity operator in the operator norm as $h \rightarrow 0$. The same is true for the projection P_h . Indeed, for any $h > 0$ there is a function $\varphi \in L^2(\mathbb{R}^d)$ such that $\|\varphi\|_{L^2(\mathbb{R}^d)} = 1$ and $\int_{I_{n,h}} \varphi = 0$ for all $n \in \mathbb{Z}^d$, e.g.

$$\varphi(x) = \sum_{n \in \mathbb{Z}^d} c_n \chi_{I_{n,h}}(x) \prod_{j=1}^d (x_j - h(n_j + \frac{1}{2})) \quad (x \in \mathbb{R}^d)$$

with suitable $(c_n)_{n \in \mathbb{Z}^d}$. Then, using (2.10), we find

$$\|P_h\varphi - \varphi\|_{L^2(\mathbb{R}^d)} = \|0 - \varphi\|_{L^2(\mathbb{R}^d)} = 1,$$

so $\|P_h - I\|_{\mathbf{B}(L^2(\mathbb{R}^d))} \geq 1$ for any $h > 0$.

3.1 Convergence of \mathcal{F}_h and $\overline{\mathcal{F}}_h$

As usual, the Fourier transform \mathcal{F} on $L^2(\mathbb{R}^d)$ and its inverse $\overline{\mathcal{F}}$ arise by extension of the integral operators

$$[\mathcal{F}\varphi](\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} \varphi(x) dx \quad (\xi \in \mathbb{R}^d; \varphi \in \mathcal{S}(\mathbb{R}^d)) \quad (3.7)$$

and

$$[\overline{\mathcal{F}}\varphi](x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \varphi(\xi) d\xi \quad (x \in \mathbb{R}^d; \varphi \in \mathcal{S}(\mathbb{R}^d)), \quad (3.8)$$

respectively. We emphasize that we can compare \mathcal{F}_h with \mathcal{F} and $\overline{\mathcal{F}}_h$ with $\overline{\mathcal{F}}$ on $L^2(\mathbb{R}^d)$, as we have extended the discrete Fourier transform and its inverse to all of $L^2(\mathbb{R}^d)$ in (2.19).

Lemma 3.4 *Let $\varphi_h \in \mathcal{S}_{0+}^{\text{step}}(\mathbb{R}^d)$. Then*

$$[\mathcal{F}\varphi_h](\xi) = \left\{ (2\pi)^{-d/2} \sum_{n \in \mathbb{Z}^d} \varphi(hn) h^d e^{-ihn \cdot \xi} \right\} \prod_{j=1}^d a(h\xi_j) \quad (\xi \in \mathbb{R}^d),$$

where

$$a(\theta) = \frac{1 - e^{-i\theta}}{i\theta} \quad (\theta \in \mathbb{R}).$$

Proof Let $n \in \mathbb{Z}^d$ and $h > 0$. A direct computation shows that

$$\int_{h,h} e^{-ix \cdot \xi} dx = \prod_{j=1}^d e^{-ihn_j \xi_j} \frac{e^{-ih\xi_j} - 1}{-i\xi_j} = h^d e^{-ihn \cdot \xi} \prod_{j=1}^d a(h\xi_j). \quad (3.9)$$

We then have

$$\begin{aligned} (\xi) &= (2\pi)^{-d/2} \sum_{n \in \mathbb{Z}^d} \varphi(hn) \int_{h,h} e^{-ix \cdot \xi} dx \\ &= (2\pi)^{-d/2} \sum_{n \in \mathbb{Z}^d} \varphi(hn) h^d e^{-ihn \cdot \xi} \prod_{j=1}^d a(h\xi_j). \end{aligned} \quad (3.10)$$

This completes the proof. □

An immediate consequence of (2.14) and Lemma 3.4 is the following corollary, which we expect will be useful from the view point of numerical analysis of discrete approximations of Fourier transform.

Corollary 3.2 *Let $\varphi_h \in \mathcal{S}_{0+}^{\text{step}}(\mathbb{R}^d)$. Then*

$$[\mathcal{F}_h\varphi_h - \mathcal{F}\varphi_h](\xi) = \left(1 - \prod_{j=1}^d a(h\xi_j) \right) [\mathcal{F}_h\varphi_h](\xi) \quad (\xi \in \mathbb{T}_{1/h}^d). \quad (3.11)$$

Remark 3.3 Note that $a(\theta) \rightarrow 1$ as $\theta \rightarrow 0$, and that $|a(\theta)| \leq 1$ for all θ .

Remark 3.4 One can deduce that $\mathcal{F}_h\varphi$ converges locally in L^2 to $\mathcal{F}\varphi$ for any $\varphi \in L^2(\mathbb{R}^d)$ in the following manner.

Since $\|\mathcal{F}_h\|_{\mathbf{B}(L^2(\mathbb{R}^d))} = 1$ for all $h > 0$, and since $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, it is sufficient to prove the local convergence in L^2 for $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and let φ_h be given by (3.1). Then, by (2.14) and (3.4),

$$\begin{aligned} |[\mathcal{F}_h\varphi_h](\xi)| &= (2\pi)^{-d/2} \left| \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}^d} e^{-ihn \cdot \xi} \varphi(hn) \chi_{I_{n,h}}(x) dx \right| \\ &\leq (2\pi)^{-d/2} \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}^d} |\varphi(hn)| \chi_{I_{n,h}}(x) dx \\ &\leq (2\pi)^{-d/2} \int_{\mathbb{R}^d} (|\varphi(x)| + \text{const}_\varphi \langle x \rangle^{-d-1}) dx =: C_\varphi < \infty. \end{aligned} \tag{3.12}$$

Taken together with Corollary 3.2, this estimate implies that

$$\|[\mathcal{F}_h - \mathcal{F}]\varphi_h\|_{L^2(K)} \leq C_\varphi \sup_{\xi \in K} \left| 1 - \prod_{j=1}^d a(h\xi_j) \right| \rightarrow 0 \quad \text{as } h \rightarrow 0$$

for any compact subset K of \mathbb{R}^d . The decomposition

$$[\mathcal{F}_h - \mathcal{F}]\varphi = \mathcal{F}_h(\varphi - \varphi_h) + [\mathcal{F}_h - \mathcal{F}]\varphi_h + \mathcal{F}(\varphi_h - \varphi), \tag{3.13}$$

together with Lemma 3.1 (ii), gives the local convergence in L^2 . However, Lemma 3.6 below shows a stronger convergence of \mathcal{F}_h .

We close Sect. 3 with the (fairly straightforward) proofs of convergences of $\overline{\mathcal{F}}_h$ and \mathcal{F}_h respectively.

Lemma 3.5 For any $u \in L^2(\mathbb{R}^d)$, $\|\overline{\mathcal{F}}_h u - \overline{\mathcal{F}}u\|_{L^2(\mathbb{R}^d)} \rightarrow 0$ as $h \rightarrow 0$.

Proof Since $\|\overline{\mathcal{F}}_h\|_{\mathbf{B}(L^2(\mathbb{R}^d))} = 1$ for all $h > 0$, and since $C_0^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, it is sufficient to prove the assertion for $u \in C_0^\infty(\mathbb{R}^d)$.

Let $u \in C_0^\infty(\mathbb{R}^d)$ and choose $h_* > 0$ so that $\mathbb{T}_{1/h}^d \supset \text{supp}[u]$ for all $h \in (0, h_*)$. Then we have, by (2.16) and by the fact that $\mathbb{T}_{1/h}^d \supset \text{supp}[u]$,

$$\begin{aligned} \overline{\mathcal{F}}_h u &= \sum_{n \in \mathbb{Z}^d} \left\{ (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ihn \cdot \xi} u(\xi) d\xi \right\} \chi_{I_{n,h}} \\ &= \sum_{n \in \mathbb{Z}^d} \varphi(hn) \chi_{I_{n,h}} \end{aligned} \tag{3.14}$$

for all $h \in (0, h_*)$, where we set $\varphi := \overline{\mathcal{F}}u$. This equality implies that

$$|\overline{\mathcal{F}}_h u(x) - \overline{\mathcal{F}}u(x)|^2 = \sum_{n \in \mathbb{Z}^d} |\varphi(hn) - \varphi(x)|^2 \chi_{I_{n,h}}(x) \quad (x \in \mathbb{R}^d). \quad (3.15)$$

In view of the fact that $\varphi \in \mathcal{S}(\mathbb{R}^d)$, it follows from (3.15) and Lemma 3.1(i) that $\|\overline{\mathcal{F}}_h u - \overline{\mathcal{F}}u\|_{L^2(\mathbb{R}^d)} \rightarrow 0$ as $h \rightarrow 0$. \square

Lemma 3.6 For any $\varphi \in L^2(\mathbb{R}^d)$,

$$\|\mathcal{F}_h \varphi - \mathcal{F}\varphi\|_{L^2(\mathbb{R}^d)} \rightarrow 0 \text{ as } h \rightarrow 0. \quad (3.16)$$

Proof We first prove that for any $u \in L^2(\mathbb{R}^d)$

$$\|[\mathcal{F}_h - \mathcal{F}] \overline{\mathcal{F}}Q_{1/h}u\|_{L^2(\mathbb{R}^d)} \rightarrow 0 \text{ as } h \rightarrow 0. \quad (3.17)$$

In fact, we see that the left hand side of (3.17) is equal to

$$\|\mathcal{F}_h \overline{\mathcal{F}}Q_{1/h}u - Q_{1/h}u\|_{L^2(\mathbb{R}^d)}, \quad (3.18)$$

which is bounded by

$$\|\overline{\mathcal{F}}Q_{1/h}u - \overline{\mathcal{F}}_h u\|_{L^2(\mathbb{R}^d)}. \quad (3.19)$$

Here we have used the fact that $Q_{1/h} = \mathcal{F}_h \overline{\mathcal{F}}_h$ (recall (2.20)), and the fact that $\|\mathcal{F}_h\|_{\mathbf{B}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))} = 1$. Since

$$Q_{1/h}u = (Q_{1/h}u - u) + u,$$

(3.19) is estimated by $\|Q_{1/h}u - u\|_{L^2(\mathbb{R}^d)} + \|\overline{\mathcal{F}}u - \overline{\mathcal{F}}_h u\|_{L^2(\mathbb{R}^d)}$. The fact (3.17) now follows from Lemmas 3.5 and 3.3.

We next prove (3.16). For $\varphi \in L^2(\mathbb{R}^d)$, we put $u = \mathcal{F}\varphi$. We decompose

$$\varphi = \overline{\mathcal{F}}Q_{1/h}u + \overline{\mathcal{F}}(u - Q_{1/h}u).$$

We then see that

$$\begin{aligned} \|\mathcal{F}_h \varphi - \mathcal{F}\varphi\|_{L^2(\mathbb{R}^d)} &\leq \|(\mathcal{F}_h - \mathcal{F})\overline{\mathcal{F}}Q_{1/h}u\|_{L^2(\mathbb{R}^d)} \\ &\quad + \|(\mathcal{F}_h - \mathcal{F})\overline{\mathcal{F}}(u - Q_{1/h}u)\|_{L^2(\mathbb{R}^d)}. \end{aligned} \quad (3.20)$$

Then it is clear that (3.16) follows from (3.17), the fact that

$$\|(\mathcal{F}_h - \mathcal{F})\overline{\mathcal{F}}\|_{\mathbf{B}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))} \leq 2, \quad (3.21)$$

and Lemma 3.3. \square

4 Resolvent convergences of $\mathbb{D}_{m,h}$

The continuum Dirac operator we shall consider in this section is

$$\mathbb{D}_m = -i\sigma_1 \frac{\partial}{\partial x_1} - i\sigma_2 \frac{\partial}{\partial x_2} + m\sigma_3 \text{ in } L^2(\mathbb{R}^2)^2, \quad (4.1)$$

where $m \geq 0$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.2)$$

It is well-known that \mathbb{D}_m is a self-adjoint operator in $L^2(\mathbb{R}^2)^2$ with domain $H^1(\mathbb{R}^2)^2$, the Sobolev space of order 1 of \mathbb{C}^2 -valued functions.

The discrete Dirac operator $\mathbb{D}_{m,h}$ we shall consider is

$$\mathbb{D}_{m,h} = \begin{pmatrix} m & i\partial_{1,h}^* + \partial_{2,h}^* \\ -i\partial_{1,h} + \partial_{2,h} & -m \end{pmatrix} \text{ in } L^2(\mathbb{Z}_h^2)^2, \quad (4.3)$$

where difference operators $\partial_{j,h}$ and $\partial_{j,h}^*$ ($j \in \{1, 2\}$) are as defined in (2.11). It is evident that $\mathbb{D}_{m,h}$ is a bounded self-adjoint operator in $L^2(\mathbb{Z}_h^2)^2$. We mention in passing that (4.3) is a Dirac operator with supersymmetry in the abstract sense (see [28, Chapter 5]). It can be rewritten in the form

$$\mathbb{D}_{m,h} = -i\sigma_1 \begin{pmatrix} \partial_{1,h} & 0 \\ 0 & -\partial_{1,h}^* \end{pmatrix} - i\sigma_2 \begin{pmatrix} \partial_{2,h} & 0 \\ 0 & -\partial_{2,h}^* \end{pmatrix} + m\sigma_3,$$

which is comparable with (1.2).

In accordance with the decomposition (2.9) of $L^2(\mathbb{R}^d)$, we can compare \mathbb{D}_m and $\mathbb{D}_{m,h} \oplus \mathbf{0}_h$ in the same Hilbert space $L^2(\mathbb{R}^2)^2$, where $\mathbf{0}_h$ is the null operator on $(L^2(\mathbb{Z}_h^2)^2)^\perp$. In particular, we investigate the difference

$$(\mathbb{D}_{m,h} - z)^{-1} \oplus \mathbf{0}_h - (\mathbb{D}_m - z)^{-1} \text{ as } h \rightarrow 0. \quad (4.4)$$

We define $\widehat{\mathbb{D}}_m := \mathcal{F}\mathbb{D}_m\overline{\mathcal{F}}$, which is the operator of multiplication with the matrix-valued function

$$\widehat{\mathbb{D}}_m(\xi) = \begin{pmatrix} m & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & -m \end{pmatrix} \quad (\xi \in \mathbb{R}^2) \quad (4.5)$$

in $L^2(\mathbb{R}^2)^2$.

With help of (2.17), (2.18) and (4.3), we also define $\widehat{\mathbb{D}}_{m,h} := \mathcal{F}_h\mathbb{D}_{m,h}\overline{\mathcal{F}}_h$, where we abbreviate $\mathcal{F}_h := \mathcal{F}_h \otimes \mathbf{1}_{\mathbb{C}^2} \in \mathbf{B}(L^2(\mathbb{Z}_h^2)^2, L^2(\mathbb{T}_{1/h}^2)^2)$ and $\overline{\mathcal{F}}_h := \overline{\mathcal{F}}_h \otimes \mathbf{1}_{\mathbb{C}^2} \in \mathbf{B}(L^2(\mathbb{T}_{1/h}^2)^2, L^2(\mathbb{Z}_h^2)^2)$. The operator $\widehat{\mathbb{D}}_{m,h} \in \mathbf{B}(L^2(\mathbb{T}_{1/h}^2)^2)$ is the operator of

multiplication with the matrix-valued function

$$\widehat{\mathbb{D}}_{m,h}(\xi) = \begin{pmatrix} m & \frac{i(e^{-ih\xi_1} - 1) + (e^{-ih\xi_2} - 1)}{h} \\ \frac{-i(e^{ih\xi_1} - 1) + (e^{ih\xi_2} - 1)}{h} & -m \end{pmatrix} \quad (4.6)$$

($\xi \in \mathbb{T}_{1/h}^2$). With the notation of (2.19), we see that

$$\mathbb{D}_{m,h} \oplus \mathbf{0}_h = \overline{\mathcal{F}}_h(\widehat{\mathbb{D}}_{m,h} \oplus \mathbf{0}_h)\mathcal{F}_h \text{ in } L^2(\mathbb{R}^2)^2. \quad (4.7)$$

Therefore, the difference in (4.4) can be written as

$$\begin{aligned} & (\mathbb{D}_{m,h} - z)^{-1} \oplus \mathbf{0}_h - (\mathbb{D}_m - z)^{-1} \\ &= \overline{\mathcal{F}}_h((\widehat{\mathbb{D}}_{m,h} - z)^{-1} \oplus \mathbf{0}_h)\mathcal{F}_h - \overline{\mathcal{F}}(\widehat{\mathbb{D}}_m - z)^{-1}\mathcal{F}. \end{aligned} \quad (4.8)$$

To investigate the matrices $\widehat{\mathbb{D}}_m(\xi)$ and $\widehat{\mathbb{D}}_{m,h}(\xi)$, we start by noting the following.

Lemma 4.1 *Let $m \geq 0$ and $\zeta \in \mathbb{C}$, and assume that either $m > 0$ or $\zeta \neq 0$. Then, with the unitary matrix*

$$U_m(\zeta) = \frac{1}{\sqrt{2}\sqrt{\mu_m(\zeta)^2 + m\mu_m(\zeta)}} \begin{pmatrix} \mu_m(\zeta) + m & -\bar{\zeta} \\ \zeta & \mu_m(\zeta) + m \end{pmatrix}, \quad (4.9)$$

where $\mu_m(\zeta) = \sqrt{|\zeta|^2 + m^2}$, we have

$$U_m(\zeta)^* \begin{pmatrix} m & \bar{\zeta} \\ \zeta & -m \end{pmatrix} U_m(\zeta) = \begin{pmatrix} \mu_m(\zeta) & 0 \\ 0 & -\mu_m(\zeta) \end{pmatrix}. \quad (4.10)$$

Note that the columns of the matrix on the right hand side of (4.9) are the eigenvectors of the matrix

$$\begin{pmatrix} m & \bar{\zeta} \\ \zeta & -m \end{pmatrix}. \quad (4.11)$$

Applying Lemma 4.1 to the matrix $\widehat{\mathbb{D}}_m(\xi)$ in (4.5) with $\zeta = \xi_1 + i\xi_2$, we see that the eigenvalues of the matrix $\widehat{\mathbb{D}}_m(\xi)$ are given by

$$\pm \lambda_m(\xi) := \pm \sqrt{|\xi|^2 + m^2}, \quad (4.12)$$

and that the unitary transformation defined by $U_m(\xi_1 + i\xi_2)^*[\mathcal{F}\varphi](\xi)$ ($\varphi \in L^2(\mathbb{R}^2)^2$) brings the Dirac operator \mathbb{D}_m in (1.2) into the form of the operator of multiplication

with the diagonal matrix-valued function

$$\begin{pmatrix} \lambda_m(\xi) & 0 \\ 0 & -\lambda_m(\xi) \end{pmatrix} \tag{4.13}$$

in $L^2(\mathbb{R}^2)^2$. This unitary transformation is a two-dimensional version of the Foldy-Wouthuysen-Tani transformation; see [10], [28, Section 1.4], [3, Section 2.1]. As is well-known, one can infer from (4.13) that the Dirac operator \mathbb{D}_m is absolutely continuous and that its spectrum is given by $\sigma(\mathbb{D}_m) = (-\infty, -m] \cup [m, \infty)$.

In a similar manner, applying Lemma 4.1 to the matrix $\widehat{\mathbb{D}}_{m,h}(\xi)$ in (4.6) with

$$\zeta = \frac{-i(e^{ih\xi_1} - 1) + (e^{ih\xi_2} - 1)}{h}, \tag{4.14}$$

we see that the eigenvalues of the matrix $\widehat{\mathbb{D}}_{m,h}(\xi)$ are given by

$$\pm \lambda_{m,h}(\xi) := \pm \sqrt{h^{-2}\omega(h\xi) + m^2}, \tag{4.15}$$

where

$$\begin{aligned} \omega(\xi) &= 4 + 2 \sin(\xi_1 - \xi_2) - 2(\sin \xi_1 + \cos \xi_1) + 2(\sin \xi_2 - \cos \xi_2) \\ &= 2(1 - \cos \xi_1)(1 + \sin \xi_2) + 2(1 - \cos \xi_2)(1 - \sin \xi_1). \end{aligned} \tag{4.16}$$

Then the unitary transformation defined by $U_m(\zeta)*[\mathcal{F}_h f](\xi)$ ($f \in L^2(\mathbb{Z}_h^2)^2$), with ζ as specified in (4.14), brings the discrete Dirac operator $\mathbb{D}_{m,h}$ in (4.3) into the form of the operator of multiplication with the diagonal matrix-valued function

$$\begin{pmatrix} \lambda_{m,h}(\xi) & 0 \\ 0 & -\lambda_{m,h}(\xi) \end{pmatrix} \tag{4.17}$$

in $L^2(\mathbb{T}_1^2/h)^2$.

Remark 4.1 It is interesting that the Laplacian on the hexagonal lattice considered in [14] has a similar form to the discrete Dirac operator (4.3) in the massless case $m = 0$. As a result, a function similar to $\omega(\xi)$ appears in [14, subsection 8.1].

Simple calculations show that for $\xi \in \mathbb{T}_1^2$

$$\begin{aligned} \nabla \omega(\xi) = 0 &\iff \\ \xi \in &\left\{ (0, 0), \left(\frac{\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{3\pi}{4}, \frac{3\pi}{4}\right), \left(\frac{\pi}{4}, -\frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{3\pi}{4}\right), \left(-\frac{3\pi}{4}, -\frac{\pi}{4}\right) \right\}, \end{aligned}$$

and that the function $\omega : \mathbb{T}_1^2 \rightarrow \mathbb{R}$ has the following properties:

- (1) ω attains its minimum value 0 at $(0, 0)$ and at $\left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$;

- (2) ω attains its unique maximum $6 + 4\sqrt{2}$ at $\left(-\frac{3\pi}{4}, \frac{3\pi}{4}\right)$;
- (3) The saddle points of ω are $\left(\frac{\pi}{4}, -\frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{3\pi}{4}\right), \left(-\frac{3\pi}{4}, -\frac{\pi}{4}\right)$.

Summing up, we have shown the following.

Theorem 4.1 *The discrete Dirac operator $\mathbb{D}_{m,h}$ is a bounded self-adjoint operator in $L^2(\mathbb{Z}_h^2)^2$ with purely absolutely continuous spectrum*

$$\sigma(\mathbb{D}_{m,h}) = \left[-\sqrt{h^{-2}(6 + 4\sqrt{2}) + m^2}, -m\right] \cup \left[m, \sqrt{h^{-2}(6 + 4\sqrt{2}) + m^2}\right].$$

Remark 4.2 As was mentioned in the introduction, some spectral properties of 2D and 3D discrete Dirac operators with a fixed mesh size were already discussed in [23, Theorem 2.1], where the statement of Theorem 4.1 is shown in the special case $h = 1$.

To prove the convergence theorems on $\mathbb{D}_{m,h}$, we need to examine the function ω in more detail. It follows from (4.16) that

$$0 \leq \omega(\xi) \leq 2|\xi|^2 \quad (\xi \in \mathbb{R}^2) \tag{4.18}$$

and that

$$\omega(\xi) \geq \frac{2 - \sqrt{2}}{8} |\xi|^2 \quad \left(|\xi|_\infty \leq \frac{\pi}{4}\right) \tag{4.19}$$

(Obviously, the number $(2 - \sqrt{2})/8$ can be replaced by any smaller positive constant, but we fix this value for the sake of definiteness.) Also, it follows from (4.16) that, setting $\alpha = \left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$,

$$\begin{aligned} \omega(\alpha + \xi) &= 4 - 2 \sin(\xi_1 - \xi_2) + 2(\sin \xi_1 - \cos \xi_1) - 2(\sin \xi_2 + \cos \xi_2) \\ &= 2(1 - \cos \xi_2)(1 + \sin \xi_1) + 2(1 - \cos \xi_1)(1 - \sin \xi_2), \end{aligned} \tag{4.20}$$

which, together with (4.18), implies that

$$0 \leq \omega(\alpha + \xi) \leq 2|\xi|^2 \quad (\xi \in \mathbb{R}^2) \tag{4.21}$$

and that

$$\omega(\alpha + \xi) \geq \frac{2 - \sqrt{2}}{8} |\xi|^2 \quad \left(|\xi|_\infty \leq \frac{\pi}{4}\right). \tag{4.22}$$

Now, for any $\varepsilon \in (0, \pi^2/128)$ we divide \mathbb{T}_1^2 into two disjoint subsets,

$$\begin{aligned} \mathbb{T}_1^2 &= E(\varepsilon) \cup F(\varepsilon), \\ E(\varepsilon) &:= \{ \xi \in \mathbb{T}_1^2 \mid \omega(\xi) \geq \varepsilon \}, \quad F(\varepsilon) := \{ \xi \in \mathbb{T}_1^2 \mid \omega(\xi) < \varepsilon \}. \end{aligned} \quad (4.23)$$

In view of (4.18), (4.19), (4.21) and (4.22), one can see that $F(\varepsilon)$ consists of two disjoint components $F_0(\varepsilon)$ and $F_1(\varepsilon)$ satisfying

$$B\left(0, \sqrt{\frac{\varepsilon}{2}}\right) \subset F_0(\varepsilon) \subset B(0, 4\sqrt{\varepsilon}), \quad (4.24)$$

and

$$B\left(\alpha, \sqrt{\frac{\varepsilon}{2}}\right) \subset F_1(\varepsilon) \subset B(\alpha, 4\sqrt{\varepsilon}), \quad (4.25)$$

where $B(a, r) = \{ \xi \in \mathbb{R}^2 \mid |a - \xi| < r \}$ is the ball with center a and radius $r > 0$. Hence we have a disjoint decomposition of \mathbb{T}_1^2 :

$$\mathbb{T}_1^2 = E(\varepsilon) \cup F_0(\varepsilon) \cup F_1(\varepsilon). \quad (4.26)$$

Accordingly, we have a disjoint decomposition of $\mathbb{T}_{1/h}^2$:

$$\mathbb{T}_{1/h}^2 = E(\varepsilon, 1/h) \cup F_0(\varepsilon, 1/h) \cup F_1(\varepsilon, 1/h), \quad (4.27)$$

where

$$\begin{aligned} E(\varepsilon, 1/h) &:= \{ \xi \in \mathbb{T}_{1/h}^2 \mid h\xi \in E(\varepsilon) \}, \\ F_j(\varepsilon, 1/h) &:= \{ \xi \in \mathbb{T}_{1/h}^2 \mid h\xi \in F_j(\varepsilon) \} \quad (j = 0, 1). \end{aligned} \quad (4.28)$$

In the following proposition, we refer to the decomposition $L^2(\mathbb{R}^2)^2 = L^2(\mathbb{T}_{1/h}^2)^2 \oplus L^2(\mathbb{R}^2 \setminus \mathbb{T}_{1/h}^2)^2$; remember that $Q_{1/h}$, the operator of multiplication with $\chi_{\mathbb{T}_{1/h}^2}$, is the orthogonal projection onto the first direct summand.

Proposition 4.1 *Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then, for any $u \in L^2(\mathbb{R}^2)^2$,*

$$\lim_{h \rightarrow 0} \left\| \left[(\widehat{\mathbb{D}}_{m,h} - z)^{-1} \oplus \mathbf{0}_h - (\widehat{\mathbb{D}}_m - z)^{-1} \right] u \right\|_{L^2(\mathbb{R}^2)^2} = 0. \quad (4.29)$$

The proof of Proposition 4.1 can be found after the proof of Lemma 4.5.

Lemma 4.2 *Let $h > 0$. Then we have*

$$\left\| \widehat{\mathbb{D}}_{m,h}(\xi) - \widehat{\mathbb{D}}_m(\xi) \right\|_{\mathbf{B}(\mathbb{C}^2)} \leq \frac{h}{2} |\xi|^2 \quad (\xi \in \mathbb{T}_{1/h}^2), \quad (4.30)$$

where $\|M\|_{\mathbf{B}(\mathbb{C}^2)}$ denotes the operator norm of a 2×2 matrix M as a linear operator in \mathbb{C}^2 .

Proof By using the inequality

$$|e^{i\theta} - 1 - i\theta| \leq \frac{\theta^2}{2} \quad (\theta \in \mathbb{R}), \tag{4.31}$$

we see that

$$\begin{aligned} & \left| \frac{-i(e^{ih\xi_1} - 1) + (e^{ih\xi_2} - 1)}{h} - (\xi_1 + i\xi_2) \right| \\ & \leq \frac{1}{h} \left\{ \frac{(h\xi_1)^2}{2} + \frac{(h\xi_2)^2}{2} \right\} = \frac{h}{2} |\xi|^2 \quad (\xi \in \mathbb{R}^2), \end{aligned} \tag{4.32}$$

which, together with (4.5) and (4.6), implies the lemma. □

Since $h|\xi| \leq \sqrt{2}\pi$ for $\xi \in \mathbb{T}_{1/h}^2$, the above lemma immediately gives the following estimate.

Lemma 4.3 *Let $h > 0$. Then we have*

$$\|\widehat{\mathbb{D}}_{m,h}(\xi) - \widehat{\mathbb{D}}_m(\xi)\|_{\mathbf{B}(\mathbb{C}^2)} \leq \frac{\sqrt{2}\pi}{2} |\xi| \quad (\xi \in \mathbb{T}_{1/h}^2). \tag{4.33}$$

Lemma 4.4 *For any $z \in \mathbb{C} \setminus \mathbb{R}$, there exists a constant C_z such that*

$$\|(\widehat{\mathbb{D}}_m(\xi) - z)^{-1}\|_{\mathbf{B}(\mathbb{C}^2)} \leq C_z (|\xi|^2 + m^2 + |\Im z|^2)^{-1/2} \quad (\xi \in \mathbb{R}^2). \tag{4.34}$$

Proof We first note that the matrix $\widehat{\mathbb{D}}_m(\xi) - z$ is unitarily equivalent to the matrix

$$\begin{pmatrix} \lambda_m(\xi) - z & 0 \\ 0 & -\lambda_m(\xi) - z \end{pmatrix}, \tag{4.35}$$

as was discussed after Lemma 4.1. Hence it is clear that

$$\|(\widehat{\mathbb{D}}_m(\xi) - z)^{-1}\|_{\mathbf{B}(\mathbb{C}^2)} = \max \left\{ \frac{1}{|\lambda_m(\xi) - z|}, \frac{1}{|-\lambda_m(\xi) - z|} \right\}. \tag{4.36}$$

Now let $z_0 = i \Im z$. Then

$$\frac{|z_0 - t|^2}{|z - t|^2} = \frac{(\Im z)^2 + t^2}{(\Im z)^2 + (t - \Re z)^2}$$

is a continuous function of $t \in \mathbb{R}$ that tends to 1 as $t \rightarrow \pm\infty$. It is therefore bounded, so there exists a constant C_z such that $|z_0 - t| \leq C_z |z - t|$ for all $t \in \mathbb{R}$, which together with (4.36) gives (4.34). □

Lemma 4.5 *Let $z \in \mathbb{C} \setminus \mathbb{R}$, and let $\varepsilon > 0$. Then for any $h \in \left(0, \frac{\sqrt{\varepsilon}}{2|\Re z|}\right)$ we have*

$$\|(\widehat{\mathbb{D}}_{m,h}(\xi) - z)^{-1}\|_{\mathbf{B}(\mathbb{C}^2)} \leq \min\left(\frac{1}{|\Im z|}, \frac{2h}{\sqrt{\varepsilon}}\right) \quad (\xi \in E(\varepsilon, 1/h)). \quad (4.37)$$

Proof As a consequence of the spectral theorem for self-adjoint operators,

$$\|(\widehat{\mathbb{D}}_{m,h}(\xi) - z)^{-1}\|_{\mathbf{B}(\mathbb{C}^2)} \leq \frac{1}{|\Im z|} \quad (\xi \in \mathbb{T}_{1/h}^2) \quad (4.38)$$

for any $z \in \mathbb{C} \setminus \mathbb{R}$. Therefore, it is sufficient to prove the inequality

$$\|(\widehat{\mathbb{D}}_{m,h}(\xi) - z)^{-1}\|_{\mathbf{B}(\mathbb{C}^2)} \leq \frac{2h}{\sqrt{\varepsilon}} \quad (\xi \in E(\varepsilon, 1/h)). \quad (4.39)$$

To this end, we note the fact that the matrix $\widehat{\mathbb{D}}_{m,h}(\xi) - z$ is unitarily equivalent to the matrix

$$\begin{pmatrix} \lambda_{m,h}(\xi) - z & 0 \\ 0 & -\lambda_{m,h}(\xi) - z \end{pmatrix}, \quad (4.40)$$

as was discussed after Lemma 4.1. By the reverse triangle inequality,

$$\begin{aligned} |\lambda_{m,h}(\xi) \pm z| &> \sqrt{h^{-2}\omega(h\xi) + m^2} - |\Re z| \\ &> h^{-1}\sqrt{\varepsilon} - |\Re z| \quad (\text{for } \xi \in E(\varepsilon, 1/h)). \end{aligned} \quad (4.41)$$

If h satisfies the inequality $0 < h < \sqrt{\varepsilon}/(2|\Re z|)$, then the inequality (4.39) follows from (4.41). \square

Proof of Proposition 4.1 Let $u \in L^2(\mathbb{R}^2)^2$. Then

$$\begin{aligned} &\|((\widehat{\mathbb{D}}_{m,h} - z)^{-1} \oplus \mathbf{0}_h - (\widehat{\mathbb{D}}_m - z)^{-1})u\|_{L^2(\mathbb{R}^2)^2}^2 \\ &= \int_{\mathbb{T}_{1/h}^2} \left| \left((\widehat{\mathbb{D}}_{m,h}(\xi) - z)^{-1} - (\widehat{\mathbb{D}}_m(\xi) - z)^{-1} \right) u(\xi) \right|_{\mathbb{C}^2}^2 d\xi \\ &\quad + \int_{\mathbb{R}^2 \setminus \mathbb{T}_{1/h}^2} \left| (\widehat{\mathbb{D}}_m(\xi) - z)^{-1} u(\xi) \right|_{\mathbb{C}^2}^2 d\xi. \end{aligned}$$

As the operator norm of $(\widehat{\mathbb{D}}_m(\xi) - z)^{-1}$ can be estimated by $|\Im z|^{-1}$, the second integral tends to 0 as $h \rightarrow 0$. Let $\varepsilon \in (0, \pi^2/128)$. In view of (4.27), we divide the first integral into three terms,

$$\left(\int_{E(\varepsilon, 1/h)} + \int_{F_0(\varepsilon, 1/h)} + \int_{F_1(\varepsilon, 1/h)} \right) \left| (\widehat{\mathbb{D}}_{m,h}(\xi) - z)^{-1} - (\widehat{\mathbb{D}}_m(\xi) - z)^{-1} \right| u(\xi) \Big|_{\mathbb{C}^2}^2 d\xi$$

$$=: I(\varepsilon, h) + II_0(\varepsilon, h) + II_1(\varepsilon, h). \tag{4.42}$$

Since

$$\begin{aligned} & (\widehat{\mathbb{D}}_{m,h}(\xi) - z)^{-1} - (\widehat{\mathbb{D}}_m(\xi) - z)^{-1} \\ &= (\widehat{\mathbb{D}}_{m,h}(\xi) - z)^{-1} (\widehat{\mathbb{D}}_m(\xi) - \widehat{\mathbb{D}}_{m,h}(\xi)) (\widehat{\mathbb{D}}_m(\xi) - z)^{-1}, \end{aligned} \tag{4.43}$$

we get

$$\begin{aligned} I(\varepsilon, h) &\leq \int_{E(\varepsilon, 1/h)} \|(\widehat{\mathbb{D}}_{m,h}(\xi) - z)^{-1}\|_{\mathbf{B}(\mathbb{C}^2)}^2 \\ &\quad \times \|\widehat{\mathbb{D}}_m(\xi) - \widehat{\mathbb{D}}_{m,h}(\xi)\|_{\mathbf{B}(\mathbb{C}^2)}^2 \|(\widehat{\mathbb{D}}_m(\xi) - z)^{-1}\|_{\mathbf{B}(\mathbb{C}^2)}^2 |u(\xi)|_{\mathbb{C}^2}^2 d\xi \\ &\leq \int_{E(\varepsilon, 1/h)} \left\{ \min\left(\frac{1}{|\Im z|}, \frac{2h}{\sqrt{\varepsilon}}\right) \right\}^2 \\ &\quad \times \left(\frac{\sqrt{2}\pi}{2} |\xi|\right)^2 C_z^2 (|\xi|^2 + m^2 + |\Im z|^2)^{-1} |u(\xi)|_{\mathbb{C}^2}^2 d\xi \\ &\leq \left\{ \min\left(\frac{1}{|\Im z|}, \frac{2h}{\sqrt{\varepsilon}}\right) \right\}^2 \left(\frac{\sqrt{2}\pi}{2}\right)^2 C_z^2 \int_{\mathbb{R}^2} |u(\xi)|_{\mathbb{C}^2}^2 d\xi, \end{aligned} \tag{4.44}$$

where we have used Lemmas 4.3, 4.4 (with the same constant C_z) and 4.5. It follows from (4.44) that

$$\lim_{h \rightarrow 0} I(\varepsilon, h) = 0. \tag{4.45}$$

The second term in (4.42) can be estimated in a similar manner:

$$\begin{aligned} II_0(\varepsilon, h) &\leq \int_{F_0(\varepsilon, 1/h)} \|(\widehat{\mathbb{D}}_{m,h}(\xi) - z)^{-1}\|_{\mathbf{B}(\mathbb{C}^2)}^2 \\ &\quad \times \|\widehat{\mathbb{D}}_m(\xi) - \widehat{\mathbb{D}}_{m,h}(\xi)\|_{\mathbf{B}(\mathbb{C}^2)}^2 \|(\widehat{\mathbb{D}}_m(\xi) - z)^{-1}\|_{\mathbf{B}(\mathbb{C}^2)}^2 |u(\xi)|_{\mathbb{C}^2}^2 d\xi \\ &\leq \int_{F_0(\varepsilon, 1/h)} \left(\frac{1}{|\Im z|}\right)^2 \\ &\quad \times \left(\frac{h}{2} |\xi|^2\right)^2 C_z^2 (|\xi|^2 + m^2 + |\Im z|^2)^{-1} |u(\xi)|_{\mathbb{C}^2}^2 d\xi, \end{aligned} \tag{4.46}$$

where we have used (4.38), Lemmas 4.2 and 4.4. If $\xi \in F_0(\varepsilon, 1/h)$, then by (4.24) we see that $h|\xi| < 4\sqrt{\varepsilon}$, hence that

$$\left(\frac{h}{2} |\xi|^2\right)^2 C_z^2 (|\xi|^2 + m^2 + |\Im z|^2)^{-1} < 4\varepsilon C_z^2. \tag{4.47}$$

It follows from (4.45) and (4.47) that

$$\limsup_{h \rightarrow 0} II_0(\varepsilon, h) \leq \left(\frac{2C_z}{|\Im z|} \|u\|_{L^2_\xi} \right)^2 \varepsilon. \tag{4.48}$$

Also, the third term in (4.42) can be estimated in a similar manner:

$$\begin{aligned} II_1(\varepsilon, h) &\leq \int_{F_1(\varepsilon, 1/h)} \|(\widehat{\mathbb{D}}_{m,h}(\xi) - z)^{-1}\|_{\mathbf{B}(\mathbb{C}^2)}^2 \\ &\quad \times \|\widehat{\mathbb{D}}_m(\xi) - \widehat{\mathbb{D}}_{m,h}(\xi)\|_{\mathbf{B}(\mathbb{C}^2)}^2 \|\widehat{\mathbb{D}}_m(\xi) - z\|_{\mathbf{B}(\mathbb{C}^2)}^{-1} |u(\xi)|_{\mathbb{C}^2}^2 d\xi \\ &\leq \int_{F_1(\varepsilon, 1/h)} \left(\frac{1}{|\Im z|} \right)^2 \\ &\quad \times \left(\frac{\sqrt{2}\pi}{2} |\xi| \right)^2 C_z^2 (|\xi|^2 + m^2 + |\Im z|^2)^{-1} |u(\xi)|_{\mathbb{C}^2}^2 d\xi \\ &\leq \left(\frac{\sqrt{2}\pi C_z}{2|\Im z|} \right)^2 \int_{F_1(\varepsilon, 1/h)} |u(\xi)|_{\mathbb{C}^2}^2 d\xi \end{aligned} \tag{4.49}$$

where we have used (4.38), Lemmas 4.3 and 4.4. Since, by (4.25),

$$F_1(\varepsilon, 1/h) \subset \{ \xi \in \mathbb{R}^2 \mid |\xi| \geq h^{-1}(|\alpha| - 4\sqrt{\varepsilon}) \}, \tag{4.50}$$

we get

$$\lim_{h \rightarrow 0} II_1(\varepsilon, h) = 0. \tag{4.51}$$

We can deduce from (4.42), together with (4.45), (4.48) and (4.51), that

$$\begin{aligned} \limsup_{h \rightarrow 0} \int_{\mathbb{T}_{1/h}^2} |(\widehat{\mathbb{D}}_{m,h}(\xi) - z)^{-1} - (\widehat{\mathbb{D}}_m(\xi) - z)^{-1}| u(\xi)|_{\mathbb{C}^2}^2 d\xi \\ \leq \left(\frac{2C_z}{|\Im z|} \|u\|_{L^2_\xi} \right)^2 \varepsilon. \end{aligned} \tag{4.52}$$

This completes the proof of conclusion of Proposition 4.1, since $\varepsilon > 0$ was arbitrarily small. □

Theorem 4.2 *Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$s\text{-}\lim_{h \rightarrow 0} \{(\mathbb{D}_{m,h} - z)^{-1} \oplus \mathbf{0}_h\} = (\mathbb{D}_m - z)^{-1} \text{ in } L^2(\mathbb{R}^2)^2. \tag{4.53}$$

Proof Let $\varphi \in L^2(\mathbb{R}^2)^2$. It follows from (4.8) that

$$\{(\mathbb{D}_{m,h} - z)^{-1} \oplus \mathbf{0}_h - (\mathbb{D}_m - z)^{-1}\} \varphi$$

$$\begin{aligned}
&= \overline{\mathcal{F}}_h \{ (\widehat{\mathbb{D}}_{m,h} - z)^{-1} \oplus \mathbf{0}_h \} \mathcal{F}_h \varphi - \overline{\mathcal{F}} (\widehat{\mathbb{D}}_m - z)^{-1} \mathcal{F} \varphi \\
&= \overline{\mathcal{F}}_h \{ (\widehat{\mathbb{D}}_{m,h} - z)^{-1} \oplus \mathbf{0}_h \} (\mathcal{F}_h - \mathcal{F}) \varphi
\end{aligned} \tag{4.54}$$

$$+ (\overline{\mathcal{F}}_h - \overline{\mathcal{F}}) \{ (\widehat{\mathbb{D}}_{m,h} - z)^{-1} \oplus \mathbf{0}_h \} \mathcal{F} \varphi \tag{4.55}$$

$$+ \overline{\mathcal{F}} \{ (\widehat{\mathbb{D}}_{m,h} - z)^{-1} \oplus \mathbf{0}_h - (\widehat{\mathbb{D}}_m - z)^{-1} \} \mathcal{F} \varphi. \tag{4.56}$$

The L^2 norm of the term in (4.54) can be estimated by

$$\| (\widehat{\mathbb{D}}_{m,h} - z)^{-1} \oplus \mathbf{0}_h \|_{\mathbf{B}(L^2(\mathbb{R}^2)^2)} \| (\mathcal{F}_h - \mathcal{F}) \varphi \|_{L^2(\mathbb{R}^2)^2} \tag{4.57}$$

$$\leq \left(\frac{1}{|\Im z|} \right) \| (\mathcal{F}_h - \mathcal{F}) \varphi \|_{L^2(\mathbb{R}^2)^2} \tag{4.58}$$

which, by Lemma 3.6, tends to 0 as $h \rightarrow 0$.

The term in (4.55) can be written as

$$(\overline{\mathcal{F}}_h - \overline{\mathcal{F}}) \{ (\widehat{\mathbb{D}}_{m,h} - z)^{-1} \oplus \mathbf{0}_h \} \mathcal{F} \varphi \tag{4.59}$$

$$= (\overline{\mathcal{F}}_h - \overline{\mathcal{F}}) \{ (\widehat{\mathbb{D}}_m - z)^{-1} \oplus \mathbf{0}_h \} \mathcal{F} \varphi \tag{4.60}$$

$$+ (\overline{\mathcal{F}}_h - \overline{\mathcal{F}}) \{ (\widehat{\mathbb{D}}_{m,h} - z)^{-1} \oplus \mathbf{0}_h - (\widehat{\mathbb{D}}_m - z)^{-1} \} \mathcal{F} \varphi, \tag{4.61}$$

where, by Lemma 3.5, the L^2 norm of the term (4.60) tends to 0 as $h \rightarrow 0$, and the L^2 norm of the term (4.61) is bounded by

$$2 \| \{ (\widehat{\mathbb{D}}_{m,h} - z)^{-1} \oplus \mathbf{0}_h - (\widehat{\mathbb{D}}_m - z)^{-1} \} \mathcal{F} \varphi \|_{L^2(\mathbb{R}^2)^2}, \tag{4.62}$$

because $\| \overline{\mathcal{F}}_h \|_{\mathbf{B}(L^2(\mathbb{R}^2)^2, L^2(\mathbb{R}^2)^2)} = 1$. Proposition 4.1 implies that L^2 norm of the term in (4.62) tends to 0 as $h \rightarrow 0$. Therefore, the L^2 norm of the term in (4.55) tends to 0 as $h \rightarrow 0$.

Finally, Proposition 4.1 immediately implies that L^2 norm of the term in (4.56) tends to 0 as $h \rightarrow 0$. \square

Remark 4.3 In view of the strong convergence of the orthogonal projectors P_h (see Lemma 3.3), the strong limits of $(\mathbb{D}_{m,h} \oplus \mathbf{0}_h - z)^{-1}$ and of $(\mathbb{D}_{m,h} - z)^{-1} \oplus \mathbf{0}_h$ are the same.

We conclude this section by proving that $(\mathbb{D}_{m,h} \oplus \mathbf{0}_h - z)^{-1}$ does not converge in the operator norm sense to $(\mathbb{D}_m - z)^{-1}$ as $h \rightarrow 0$. We would like to mention that the proof of Theorem 4.3 below is based on the idea demonstrated in [31] and [30].

Theorem 4.3 *Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\begin{aligned}
&\liminf_{h \rightarrow 0} \| (\mathbb{D}_{m,h} \oplus \mathbf{0}_h - z)^{-1} - (\mathbb{D}_m - z)^{-1} \|_{\mathbf{B}(L^2(\mathbb{R}^2)^2)} \\
&\geq \max \left(\frac{1}{|m - z|}, \frac{1}{|m + z|} \right).
\end{aligned} \tag{4.63}$$

Proof For $h > 0$, consider the function $u_h = \begin{pmatrix} y_h \\ 0 \end{pmatrix} \in L^2(\mathbb{R}^2)^2$, where

$$y_h(x) = \sqrt{h} e^{i\frac{\pi}{2h}(x_1-x_2)} e^{-h(x_1^2+x_2^2)} \quad (x \in \mathbb{R}^2). \quad (4.64)$$

Then $\|u_h\|_{L^2(\mathbb{R}^2)^2} = \sqrt{\frac{\pi}{2}}$ and

$$(\mathcal{F}y_h)(\xi) = \frac{1}{2\sqrt{h}} e^{-\frac{1}{4h}[(\xi_1-\frac{\pi}{2h})^2+(\xi_2+\frac{\pi}{2h})^2]} \quad (\xi \in \mathbb{R}^2). \quad (4.65)$$

Further, by (4.5) we find that

$$(\mathbb{D}_m - z)^{-1}u_h = \overline{\mathcal{F}} \frac{1}{m^2 - z^2 + \xi_1^2 + \xi_2^2} \begin{pmatrix} m + z & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & -m + z \end{pmatrix} \begin{pmatrix} \mathcal{F}y_h \\ 0 \end{pmatrix}, \quad (4.66)$$

and, as the Fourier transform is an isometry on $L^2(\mathbb{R}^2)$, we conclude that

$$\begin{aligned} & \|(\mathbb{D}_m - z)^{-1}u_h\|_{L^2(\mathbb{R}^2)^2}^2 \\ &= \int_{\mathbb{R}^2} \frac{|m + z|^2 + \xi_1^2 + \xi_2^2}{|m^2 - z^2 + \xi_1^2 + \xi_2^2|^2} \frac{1}{4h} e^{-[(\xi_1-\frac{\pi}{2h})^2+(\xi_2+\frac{\pi}{2h})^2]/2h} d\xi \\ &\leq \int_{\mathbb{R}^2} \frac{C_z}{|(m^2 - z^2) + (\eta_1 + \frac{\pi}{2h})^2 + (\eta_2 - \frac{\pi}{2h})^2|} \frac{1}{4h} e^{-(\eta_1^2+\eta_2^2)/2h} d\eta. \end{aligned} \quad (4.67)$$

Here we have used the fact that

$$\frac{|m + z|^2 + \xi_1^2 + \xi_2^2}{|m^2 - z^2 + \xi_1^2 + \xi_2^2|} \rightarrow 1 \quad \text{as } \xi = (\xi_1, \xi_2) \rightarrow \infty,$$

and the fact that $|m^2 - z^2 + \xi_1^2 + \xi_2^2| \geq c_z > 0$ for $\forall \xi \in \mathbb{R}^2$. In fact, we have

$$|m^2 - z^2 + \xi_1^2 + \xi_2^2| \geq \begin{cases} |\Im z|^2 & \text{if } z \text{ is pure imaginary,} \\ 2|(\Re z)(\Im z)| & \text{if } z \text{ is not pure imaginary.} \end{cases}$$

The second integral in (4.67) can be written in the form

$$\int_{\mathbb{R}^2} \frac{C_z}{|(m^2 - z^2) + (\sqrt{h}\eta_1 + \frac{\pi}{2h})^2 + (\sqrt{h}\eta_2 - \frac{\pi}{2h})^2|} \frac{1}{4} e^{-(\eta_1^2+\eta_2^2)/2} d\eta. \quad (4.68)$$

The integrand in (4.68) tends to 0 pointwise as $h \rightarrow 0$ and is bounded above by the function $\frac{C_z}{|m^2 - z^2|} \frac{1}{4} e^{-(\eta_1^2+\eta_2^2)/2}$, so by the dominated convergence theorem

$$\lim_{h \rightarrow 0} \|(\mathbb{D}_m - z)^{-1}u_h\|_{L^2(\mathbb{R}^2)^2} = 0. \quad (4.69)$$

Now, in order to apply the discrete Dirac operator, we project u_h into $L^2(\mathbb{Z}_h^2)^2$. We see that $P_h y_h = \sum_{n \in \mathbb{Z}^2} \tilde{y}_h(hn) \chi_{I_{n,h}}$, where

$$\tilde{y}_h(hn) = \sqrt{h} \left(\int_{n_1}^{n_1+1} e^{i\frac{\pi}{2}t} e^{-h^3 t^2} dt \right) \left(\int_{n_2}^{n_2+1} e^{-i\frac{\pi}{2}t} e^{-h^3 t^2} dt \right) \quad (n = (n_1, n_2) \in \mathbb{Z}^2)$$

by (2.10). Integration by parts and the formula

$$\int_n^{n+1} 2at e^{-at^2} dt = e^{-an^2} - e^{-a(n+1)^2} \quad (n \in \mathbb{Z})$$

show that

$$\begin{aligned} & \int_n^{n+1} e^{\pm i\frac{\pi}{2}t} e^{-h^3 t^2} dt \quad (n \in \mathbb{Z}) \\ &= \pm \frac{2}{i\pi} \left(e^{\pm i\frac{\pi}{2}(n+1)} e^{-h^3(n+1)^2} - e^{\pm i\frac{\pi}{2}n} e^{-h^3 n^2} \right) \pm \frac{2}{i\pi} \int_n^{n+1} e^{\pm i\frac{\pi}{2}t} 2h^3 t e^{-h^3 t^2} dt \\ &= \frac{2}{\pi} (1 \pm i) e^{\pm i\frac{\pi}{2}n} e^{-h^3 n^2} \pm \frac{4h^3}{i\pi} \int_n^{n+1} \left(e^{\pm i\frac{\pi}{2}t} - e^{\pm i\frac{\pi}{2}(n+1)} \right) t e^{-h^3 t^2} dt \\ &=: \frac{2}{\pi} (1 \pm i) e^{\pm i\frac{\pi}{2}n} e^{-h^3 n^2} \pm \frac{4h^3}{i\pi} K^\pm(n, h). \end{aligned}$$

Hence

$$P_h u_h = \begin{pmatrix} \frac{8}{\pi^2} (y_h)_h + R_h \\ 0 \end{pmatrix} = \frac{8}{\pi^2} (u_h)_h + \begin{pmatrix} R_h \\ 0 \end{pmatrix}, \tag{4.70}$$

where we use the notation of (3.1) for $(y_h)_h$, $(u_h)_h$ and set $R_h := R_{h,a} + R_{h,b} + R_{h,c}$ with

$$\begin{aligned} R_{h,a}(n, h) &= \sqrt{h} \frac{8h^3}{\pi^2} (i-1) e^{i\frac{\pi}{2}n_1} e^{-h^3 n_1^2} K^-(n_2, h) \\ R_{h,b}(n, h) &= -\sqrt{h} \frac{8h^3}{\pi^2} (i+1) e^{-i\frac{\pi}{2}n_2} e^{-h^3 n_2^2} K^+(n_1, h) \quad (n \in \mathbb{Z}^2) \\ R_{h,c}(n, h) &= \sqrt{h} \frac{16h^6}{\pi^2} K^+(n_1, h) K^-(n_2, h). \end{aligned}$$

Using the asymptotics

$$\sum_{k \in \mathbb{Z}} e^{-\beta k^2} = \sqrt{\frac{\pi}{\beta}} + O(1), \quad \sum_{k=1}^{\infty} k^2 e^{-\beta k^2} = \frac{\sqrt{\pi}}{4\beta^{3/2}} + O(\beta^{-1}) \quad (\beta \rightarrow 0), \tag{4.71}$$

we find

$$\sum_{n \in \mathbb{Z}} e^{-h^3 n^2} = \sqrt{\frac{\pi}{h^3}} + O(1) \quad (4.72)$$

and

$$\begin{aligned} \sum_{n \in \mathbb{Z}} |h^{\frac{3}{2}} K^\pm(n, h)|^2 &\leq \sum_{n \in \mathbb{Z}} \left(2 \int_n^{n+1} h^{\frac{3}{2}} |t| e^{-(h^3/2)t^2/2} e^{-h^3 t^2/2} dt \right)^2 \\ &\leq \frac{4}{e} \sum_{n \in \mathbb{Z}} \left(\int_n^{n+1} e^{-h^3 t^2/2} dt \right)^2 \\ &\leq \frac{8}{e} \sum_{n=0}^{\infty} e^{-h^3 n^2} = \frac{4\sqrt{\pi}}{e} h^{-\frac{3}{2}} + O(1) \quad (h \rightarrow 0), \end{aligned}$$

where we have used the inequality $h^{\frac{3}{2}} |t| e^{-(h^3/2)t^2/2} \leq 1/\sqrt{e}$ in the second inequality. Hence, bearing in mind (2.8),

$$\|R_{h,a}\|_{L^2(\mathbb{Z}_h^2)}^2, \|R_{h,b}\|_{L^2(\mathbb{R}^2)}^2 \leq h^3 \frac{256}{\sqrt{2}\pi^3 e} + O(h^{9/2}) \quad (4.73)$$

and $\|R_{h,c}\|_{L^2(\mathbb{Z}_h^2)}^2 = O(h^6)$, which gives $\lim_{h \rightarrow 0} \|R_h\|_{L^2(\mathbb{Z}_h^2)} = 0$, while

$$\left\| \frac{8}{\pi^2} (y_h)_h \right\|_{L^2(\mathbb{Z}_h^2)} = \frac{8}{\sqrt{2}\pi^{3/2}} + o(1) \quad (h \rightarrow 0). \quad (4.74)$$

Now, applying the discrete Dirac operator to $(u_h)_h = \begin{pmatrix} (y_h)_h \\ 0 \end{pmatrix}$, we find

$$(\mathbb{D}_{m,h} - z)(u_h)_h = (m - z)(u_h)_h + \begin{pmatrix} 0 \\ (-i\partial_{1,h} + \partial_{2,h})(y_h)_h \end{pmatrix}, \quad (4.75)$$

where

$$\begin{aligned} (-i\partial_{1,h} + \partial_{2,h})(y_h)_h(hn) &= (-i\partial_{1,h} + \partial_{2,h})\sqrt{h} \frac{8}{\pi^2} e^{i\frac{\pi}{2}(n_1 - n_2)} e^{-h^3(n_1^2 + n_2^2)} \\ &= \frac{1}{\sqrt{h}} \frac{8}{\pi^2} e^{i\frac{\pi}{2}(n_1 - n_2)} e^{-h^3(n_1^2 + n_2^2)} \left((e^{-h^3(2n_1+1)} - 1) - i(e^{-h^3(2n_2+1)} - 1) \right) \\ &= \frac{1}{\sqrt{h}} \frac{8}{\pi^2} e^{i\frac{\pi}{2}(n_1 - n_2)} \\ &\quad \times \left((e^{-h^3(n_1+1)^2} - e^{-h^3 n_1^2}) e^{-h^3 n_2^2} - i e^{-h^3 n_1^2} (e^{-h^3(n_2+1)^2} - e^{-h^3 n_2^2}) \right) \\ &=: \tilde{R}_h(n, h) \quad (n = (n_1, n_2) \in \mathbb{Z}^2). \end{aligned}$$

Then, using the formula

$$\sum_{n_1 \in \mathbb{Z}} |e^{-h^3(n_1+1)^2} - e^{-h^3n_1^2}|^2 = 2 \sum_{n_1=0}^{\infty} |e^{-h^3(n_1+1)^2} - e^{-h^3n_1^2}|^2$$

and the estimate

$$|e^{-h^3(2n_1+1)} - 1| = \left| \int_0^{2n_1+1} (-h^3) e^{-h^3s} ds \right| \leq h^3 (2n_1 + 1) \quad (n_1 \geq 0), \tag{4.76}$$

we obtain for $\tilde{R}_{h,a}(n, h) := \frac{1}{\sqrt{h}} \frac{8}{\pi^2} e^{i\frac{\pi}{2}n_1} (e^{-h^3(n_1+1)^2} - e^{-h^3n_1^2}) e^{-h^3n_2^2}$

$$\begin{aligned} \|\tilde{R}_{h,a}\|_{L^2(\mathbb{Z}_h^2)}^2 &\leq h^7 \frac{64}{\pi^4} \left(2 \sum_{n_1=0}^{\infty} (2n_1 + 1)^2 e^{-2h^3n_1^2} \right) \left(\sum_{n_2 \in \mathbb{Z}} e^{-2h^3n_2^2} \right) \\ &\leq h^7 \frac{64}{\pi^4} \left(2 \sum_{n_1=0}^{\infty} e^{-2h^3n_1^2} + 16 \sum_{n_1=0}^{\infty} n_1^2 e^{-2h^3n_1^2} \right) \left(\sum_{n_2 \in \mathbb{Z}} e^{-2h^3n_2^2} \right) \\ &= h \frac{64}{\pi^3} + o(h) \quad (h \rightarrow 0) \end{aligned}$$

and similarly for $\tilde{R}_{h,b}(n, h) := \frac{1}{\sqrt{h}} \frac{8}{\pi^2} e^{-i\frac{\pi}{2}n_2} e^{-h^3n_1^2} (e^{-h^3(n_2+1)^2} - e^{-h^3n_2^2})$, so

$$\lim_{h \rightarrow 0} \|\tilde{R}_h\|_{L^2(\mathbb{Z}_h^2)} = 0. \tag{4.77}$$

We are now ready to complete the proof. We first note that

$$(\mathbb{D}_{m,h} \oplus \mathbf{0}_h - z)^{-1} = (\mathbb{D}_{m,h} - z)^{-1} \oplus \left(-\frac{1}{z} \right). \tag{4.78}$$

Now we observe that

$$(\mathbb{D}_{m,h} - z) P_h u_h = \frac{8}{\pi^2} (m - z) (u_h)_h + \frac{8}{\pi^2} \begin{pmatrix} 0 \\ \tilde{R}_h \end{pmatrix} + (\mathbb{D}_{m,h} - z) \begin{pmatrix} R_h \\ 0 \end{pmatrix}, \tag{4.79}$$

so, applying (4.70) to the $(u_h)_h$ on the right hand side of (4.79) and multiplying the both sides of (4.79) by $\frac{1}{m-z} (\mathbb{D}_{m,h} - z)^{-1}$, we infer that

$$(\mathbb{D}_{m,h} - z)^{-1} P_h u_h = \frac{1}{m - z} P_h u_h - \frac{1}{m - z} (\mathbb{D}_{m,h} - z)^{-1} \frac{8}{\pi^2} \begin{pmatrix} 0 \\ \tilde{R}_h \end{pmatrix} - \frac{1}{m - z} \begin{pmatrix} R_h \\ 0 \end{pmatrix}$$

$$+ (\mathbb{D}_{m,h} - z)^{-1} \begin{pmatrix} R_h \\ 0 \end{pmatrix}.$$

As $\|(\mathbb{D}_{m,h} - z)^{-1}\|_{\mathbf{B}(L^2(\mathbb{Z}_h^2)^2)} \leq 1/|\Im z|$, it follows that

$$\begin{aligned} & \|(\mathbb{D}_{m,h} - z)^{-1} P_h u_h - \frac{1}{m-z} P_h u_h\|_{L^2(\mathbb{R}^2)^2} \\ & \leq \frac{8}{|m-z| |\Im z| \pi^2} \|\tilde{R}_h\|_{L^2(\mathbb{Z}_h^2)} + \left(\frac{1}{|m-z|} + \frac{1}{|\Im z|} \right) \|R_h\|_{L^2(\mathbb{Z}_h^2)} \rightarrow 0 \quad (h \rightarrow 0). \end{aligned}$$

Consequently,

$$\begin{aligned} & \|u_h\|_{L^2(\mathbb{R}^2)^2} \|(\mathbb{D}_{m,h} \oplus \mathbf{0}_h - z)^{-1} - (\mathbb{D}_m - z)^{-1}\|_{\mathbf{B}(L^2(\mathbb{R}^2)^2)} \\ & \geq \|(\mathbb{D}_{m,h} \oplus \mathbf{0}_h - z)^{-1} u_h - (\mathbb{D}_m - z)^{-1} u_h\|_{L^2(\mathbb{R}^2)^2} \\ & \geq \|(\mathbb{D}_{m,h} - z)^{-1} P_h u_h - z^{-1} (1 - P_h) u_h\|_{L^2(\mathbb{R}^2)^2} - \|(\mathbb{D}_m - z)^{-1} u_h\|_{L^2(\mathbb{R}^2)^2} \\ & = \sqrt{\|(\mathbb{D}_{m,h} - z)^{-1} P_h u_h\|_{L^2(\mathbb{R}^2)^2}^2 + |z^{-1}| \|(1 - P_h) u_h\|_{L^2(\mathbb{R}^2)^2}^2} + o(1) \\ & \geq \frac{1}{|m-z|} \|u_h\|_{L^2(\mathbb{R}^2)^2} + o(1) \quad (h \rightarrow 0). \end{aligned}$$

This inequality implies that

$$\liminf_{h \rightarrow 0} \|(\mathbb{D}_{m,h} \oplus \mathbf{0}_h - z)^{-1} - (\mathbb{D}_m - z)^{-1}\|_{\mathbf{B}(L^2(\mathbb{R}^2)^2)} \geq \frac{1}{|m-z|}. \quad (4.80)$$

If we replace $u_h = \begin{pmatrix} y_h \\ 0 \end{pmatrix}$ with $v_h = \begin{pmatrix} 0 \\ y_h \end{pmatrix}$ and make the similar arguments as above, we get

$$\liminf_{h \rightarrow 0} \|(\mathbb{D}_{m,h} \oplus \mathbf{0}_h - z)^{-1} - (\mathbb{D}_m - z)^{-1}\|_{\mathbf{B}(L^2(\mathbb{R}^2)^2)} \geq \frac{1}{|m+z|}. \quad (4.81)$$

We now arrived at the inequality (4.63). \square

Remark 4.4 The lack of norm resolvent convergence shown in Theorem 4.3 is closely related to the fact that, unlike the continuous Dirac operator, the discrete Dirac operator does not control the gradient: while a simple calculation shows that $\|\nabla u\|_{L^2(\mathbb{R}^2)^2} \leq \|\mathbb{D}_m u\|_{L^2(\mathbb{R}^2)^2}$ for all $u \in H^1(\mathbb{R}^2)^2$, there is no constant $C > 0$ such that $\sqrt{\|\partial_{1,h} u\|_{\ell^2(\mathbb{Z}_h^2)}^2 + \|\partial_{2,h} u\|_{\ell^2(\mathbb{Z}_h^2)}^2} \leq C \|\mathbb{D}_{m,h} u\|_{\ell^2(\mathbb{Z}_h^2)}$. This, in turn, is connected to the fact that the Liouville theorem does not hold in discrete complex analysis. Indeed, the function $y : \mathbb{Z} + i\mathbb{Z} \rightarrow \mathbb{Z} + i\mathbb{Z}$, $y(n_1 + in_2) = i^{n_1 - n_2}$ satisfies the discrete Cauchy-Riemann equation $(\partial_{1,1} + i\partial_{2,1})y = 0$ in the whole lattice of Gaussian integers (and thus is ‘monodiffric’, see [13, 18]) and is bounded, but not constant.

The functions y_h in the proof of Theorem 4.3 arise from this function y by a natural extension to all of $\mathbb{R}^2 \cong \mathbb{C}$, scaling to the lattice with spacing h and multiplication with a suitable Gaussian to place the functions into Schwartz space.

5 Strong resolvent convergence of $\mathbb{D}_{m,h} + V_h$

In this section, we shall discuss the continuum Dirac operators

$$\mathbb{D}_m + V = -i\sigma_1 \frac{\partial}{\partial x_1} - i\sigma_2 \frac{\partial}{\partial x_2} + m\sigma_3 + V(x) \text{ in } L^2(\mathbb{R}^2)^2, \tag{5.1}$$

where V is a complex 2×2 matrix-valued potential. More precisely, we make the following

Assumption (V). $V : \mathbb{R}^2 \rightarrow \mathbb{C}^{2 \times 2}$ is a matrix-valued function each element of which is a bounded and uniformly continuous function.

Remark 5.1 It is apparent that electro-magnetic Dirac operators

$$\sigma \cdot (-i\nabla - a(x)) + m\beta + q(x)$$

can be written in the form (5.1). Indeed, one can take V to be $-\sigma \cdot a(x) + q(x)$.

We note that V can be decomposed into its Hermitian and skew-Hermitian parts, $V = V_{\mathfrak{R}} + iV_{\mathfrak{I}}$, where

$$V_{\mathfrak{R}}(x) = \frac{V(x) + V(x)^*}{2}, \quad iV_{\mathfrak{I}}(x) = \frac{V(x) - V(x)^*}{2} \quad (x \in \mathbb{R}^2). \tag{5.2}$$

It is evident that under the assumption (V), the operator V of multiplication with the matrix-valued function V is a bounded operator in $L^2(\mathbb{R}^2)^2$ and that the operator $\mathbb{D}_m + V$ is well-defined. In particular, if $V_{\mathfrak{I}} = 0$, then $\mathbb{D}_m + V$ is a self-adjoint operator in $L^2(\mathbb{R}^2)^2$ with domain $H^1(\mathbb{R}^2)^2$.

In analogy to (5.1), we consider the discrete Dirac operator $\mathbb{D}_{m,h} + V_h$ in $L^2(\mathbb{Z}_h^2)^2$, where $\mathbb{D}_{m,h}$ is the operator introduced in (4.3) and V_h is the operator of multiplication by

$$V_h(x) = \sum_{n \in \mathbb{Z}^2} V(hn) \chi_{I_{n,h}}(x) \quad (x \in \mathbb{R}^2). \tag{5.3}$$

It is clear that $\mathbb{D}_{m,h} + V_h$ is a bounded operator in $L^2(\mathbb{Z}_h^2)^2$. In the same manner as in (5.2), we split $V_h = V_{\mathfrak{R},h} + iV_{\mathfrak{I},h}$.

Theorem 5.1 Under the assumption (V),

(i) both the spectra $\sigma(\mathbb{D}_{m,h} + V_h)$ and $\sigma(\mathbb{D}_m + V)$ are subsets of the strip

$$\left\{ z \in \mathbb{C} \mid |\Im z| \leq \sup_{x \in \mathbb{R}^2} \|V_{\mathfrak{I}}(x)\|_{\mathbf{B}(\mathbb{C}^2)} \right\};$$

(ii) for z with $|\Im z| > \sup_{x \in \mathbb{R}^2} \|V_{\mathfrak{I}}(x)\|_{\mathbf{B}(\mathbb{C}^2)}$

$$\text{s-lim}_{h \rightarrow 0} \{ (\mathbb{D}_{m,h} + V_h - z)^{-1} \oplus \mathbf{0}_h \} = (\mathbb{D}_m + V - z)^{-1} \text{ in } L^2(\mathbb{R}^2)^2. \tag{5.4}$$

Note that in the self-adjoint case $V_{\mathfrak{J}} = 0$, (5.4) holds for $z \in \mathbb{C} \setminus \mathbb{R}$.

We prepare the proof of Theorem 5.1 by providing the following auxiliary statements.

Lemma 5.1 *Suppose that (V) holds. Then*

$$\sigma(\mathbb{D}_m + V) \subset \left\{ z \in \mathbb{C} \mid |\Im m z| \leq \sup_{x \in \mathbb{R}^2} \|V_{\mathfrak{J}}(x)\|_{\mathbf{B}(\mathbb{C}^2)} \right\}.$$

Moreover,

$$(\mathbb{D}_m + V - z)^{-1} = (\mathbb{D}_m + V_{\mathfrak{R}} - z)^{-1} \{ I_{L^2(\mathbb{R}^2)^2} + i V_{\mathfrak{J}} (\mathbb{D}_m + V_{\mathfrak{R}} - z)^{-1} \}^{-1} \quad (5.5)$$

for all z with $|\Im m z| > \sup_{x \in \mathbb{R}^2} \|V_{\mathfrak{J}}(x)\|_{\mathbf{B}(\mathbb{C}^2)}$.

Proof Let $|\Im m z| > \sup_{x \in \mathbb{R}^2} \|V_{\mathfrak{J}}(x)\|_{\mathbf{B}(\mathbb{C}^2)}$. Since $\mathbb{D}_m + V_{\mathfrak{R}}$ is self-adjoint, it follows that $z \in \rho(\mathbb{D}_m + V_{\mathfrak{R}})$, the resolvent set of $\mathbb{D}_m + V_{\mathfrak{R}}$. This enables us to write

$$\mathbb{D}_m + V - z = \{ I_{L^2(\mathbb{R}^2)} + i V_{\mathfrak{J}} (\mathbb{D}_m + V_{\mathfrak{R}} - z)^{-1} \} (\mathbb{D}_m + V_{\mathfrak{R}} - z). \quad (5.6)$$

Since $\|V_{\mathfrak{J}}\|_{\mathbf{B}(L^2(\mathbb{R}^2)^2)} \leq \sup_{x \in \mathbb{R}^2} \|V_{\mathfrak{J}}(x)\|_{\mathbf{B}(\mathbb{C}^2)}$, we see that

$$\|V_{\mathfrak{J}} (\mathbb{D}_m + V_{\mathfrak{R}} - z)^{-1}\|_{\mathbf{B}(L^2(\mathbb{R}^2)^2)} < 1,$$

so the operator on the right hand side of (5.6) is invertible in $L^2(\mathbb{R})^2$, and therefore $z \in \rho(\mathbb{D}_m + V)$.

In the same manner as in the proof of Lemma 5.1, one can prove a similar statement for $\mathbb{D}_{m,h} + V_h$. Recall that $\mathbb{D}_{m,h} + V_h$ is a bounded operator acting in $L^2(\mathbb{Z}_h^2)^2$. \square

Lemma 5.2 *Under assumption (V),*

$$\sigma(\mathbb{D}_{m,h} + V_h) \subset \left\{ z \in \mathbb{C} \mid |\Im m z| \leq \sup_{x \in \mathbb{R}^2} \|V_{\mathfrak{J}}(x)\|_{\mathbf{B}(\mathbb{C}^2)} \right\}.$$

Moreover,

$$\begin{aligned} & (\mathbb{D}_{m,h} + V_h - z)^{-1} \\ &= (\mathbb{D}_{m,h} + V_{\mathfrak{R},h} - z)^{-1} \{ I_{L^2(\mathbb{Z}_h^2)^2} + i V_{\mathfrak{J},h} (\mathbb{D}_{m,h} + V_{\mathfrak{R},h} - z)^{-1} \}^{-1} \end{aligned} \quad (5.7)$$

for all z with $|\Im m z| > \sup_{x \in \mathbb{R}^2} \|V_{\mathfrak{J}}(x)\|_{\mathbf{B}(\mathbb{C}^2)}$.

Lemma 5.3 *Under the assumption (V), $V_h \oplus \mathbf{0}_h (= V_h P_h) \rightarrow V$ strongly in $L^2(\mathbb{R}^2)^2$.*

Proof In this proof, we distinguish the multiplication operator V_h in $L^2(\mathbb{R}^2)^2$ from the embedded version of the multiplication operator V_h in $L^2(\mathbb{Z}_h^2)^2$, denoted by $V_h \oplus \mathbf{0}_h$.

Let $\varphi \in L^2(\mathbb{R}^2)^2$. In view of the fact that $\{V_h \oplus \mathbf{0}_h\}\varphi = V_h P_h \varphi$, we infer that

$$\begin{aligned} \|[V_h \oplus \mathbf{0}_h] - V\]\varphi\|_{L^2(\mathbb{R}^2)^2} &\leq \|V_h(P_h \varphi - \varphi)\|_{L^2(\mathbb{R}^2)^2} + \|(V_h - V)\varphi\|_{L^2(\mathbb{R}^2)^2} \\ &\leq \left\{ \sup_{x \in \mathbb{R}^2} \|V(x)\|_{\mathbf{B}(\mathbb{C}^2)} \right\} \|P_h \varphi - \varphi\|_{L^2(\mathbb{R}^2)^2} \\ &\quad + \sup_{n \in \mathbb{Z}^2} \left\{ \sup_{x \in I_{n,h}} \|V(x) - V(hn)\|_{\mathbf{B}(\mathbb{C}^2)} \right\} \|\varphi\|_{L^2(\mathbb{R}^2)^2}. \end{aligned}$$

By virtue of Lemma 3.3 and the assumption that the function V is bounded and uniformly continuous, it follows that $V_h \oplus \mathbf{0}_h \rightarrow V$ strongly in $L^2(\mathbb{R}^2)^2$ as $h \rightarrow 0$. \square

Remark 5.2 As shown in Lemma 5.3, $V_h \oplus \mathbf{0}_h$ converges to V strongly, but not in the operator norm unless $V \equiv 0$. Indeed, if $V \not\equiv 0$, then there is some open subset $\Omega \subset \mathbb{R}^2$, some $v \in \mathbb{C}^2$ with $\|v\|_{\mathbb{C}^2} = 1$ and a constant $C > 0$ such that $\|V(x)v\|_{\mathbb{C}^2} \geq C$ for all $x \in \Omega$. Let $h > 0$ be so small that for some $n \in \mathbb{Z}^2$, $I_{n,h} \subset \Omega$, and set

$$\varphi(x) = \chi_{I_{n,h}}(x) \prod_{j=1}^2 \left(x_j - h \left(n + \frac{1}{2} \right) \right) v \quad (x \in \mathbb{R}^2).$$

Then

$$\begin{aligned} \|(V_h \oplus 0 - V)\varphi\|_{L^2(\mathbb{R}^2)} &= \|V_h P_h \varphi - V\varphi\|_{L^2(\mathbb{R}^2)} = \|V\varphi\|_{L^2(\mathbb{R}^2)} \\ &= \left\{ \int_{I_{n,h}} \|V(x)v\|_{\mathbb{C}^2}^2 \prod_{j=1}^2 |x_j - h \left(n_j + \frac{1}{2} \right)|^2 dx \right\}^{1/2} \geq C \|\varphi\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Therefore $\|V_h \oplus 0 - V\|_{\mathbf{B}(L^2(\mathbb{R}^2))} \geq C > 0$ for all sufficiently small $h > 0$.

This remark shows that even at the level of the potential operator V , we cannot expect norm convergence, which is slightly counterintuitive, as V_h , as a function, does converge in $\|\cdot\|_\infty$ norm to V .

Applying the above lemma to V and to V^* and using (5.2), we obtain the following convergence results for the Hermitian and skew-Hermitian parts separately.

Corollary 5.1 *Suppose that (V) is verified. Then $V_{\mathfrak{H},h} \oplus \mathbf{0}_h \rightarrow V_{\mathfrak{H}}$ strongly in $L^2(\mathbb{R}^2)^2$ and $V_{\mathfrak{J},h} \oplus \mathbf{0}_h \rightarrow V_{\mathfrak{J}}$ strongly in $L^2(\mathbb{R}^2)^2$.*

Furthermore, we shall use the following two abstract lemmas. We omit the quite straightforward proof of the first of them.

Lemma 5.4 *Let \mathcal{H} be a Hilbert space. Let S_h and T_h belong to $\mathbf{B}(\mathcal{H})$ for each $h > 0$, and suppose that S_h and T_h strongly converge to S and $T \in \mathbf{B}(\mathcal{H})$ respectively as $h \rightarrow 0$. If $\sup_{h>0} \|S_h\|_{\mathbf{B}(\mathcal{H})} < \infty$, then $S_h T_h$ strongly converges to ST as $h \rightarrow 0$.*

Lemma 5.5 *Let \mathcal{H} be a Hilbert space. Suppose that \mathcal{H} has an orthogonal decomposition $\mathcal{H} = \mathcal{X}_h \oplus \mathcal{X}_h^\perp$ for each $h > 0$ and that the orthogonal projection P_h onto \mathcal{X}_h strongly converges to $I_{\mathcal{H}}$ as $h \rightarrow 0$. Let A_h , for each $h > 0$, and A be invertible operators in \mathcal{X}_h and in \mathcal{H} respectively such that $A_h \oplus 0$ strongly converges to A as $h \rightarrow 0$. If $\sup_{h>0} \|A_h^{-1}\|_{\mathbf{B}(\mathcal{X}_h)} < \infty$, then $A_h^{-1} \oplus 0$ strongly converges to A^{-1} as $h \rightarrow 0$.*

Proof Let $\varphi \in \mathcal{H}$. By hypothesis, we see that

$$(A_h \oplus \mathbf{0}_h)A^{-1}\varphi \rightarrow AA^{-1}\varphi = \varphi \text{ in } \mathcal{H}$$

as $h \rightarrow 0$. Hence we obtain

$$\begin{aligned} & \|A^{-1}\varphi - (A_h^{-1} \oplus \mathbf{0}_h)\varphi\|_{\mathcal{H}} \\ & \leq \|(I_{\mathcal{H}} - P_h)A^{-1}\varphi\|_{\mathcal{H}} + \|(A_h^{-1} \oplus \mathbf{0}_h)((A_h \oplus \mathbf{0}_h)A^{-1}\varphi - \varphi)\|_{\mathcal{H}} \rightarrow 0 \quad (h \rightarrow 0) \end{aligned}$$

by the uniform boundedness of A_h^{-1} . \square

Proof of Theorem 5.1 Statement (i) was shown in Lemma 5.1 and Lemma 5.2. For (ii), let $z \in \mathbb{C}$, $|\Im z| > \sup_{x \in \mathbb{R}^2} \|V_{\mathcal{J}}\|_{\mathbf{B}(\mathbb{C}^2)}$. Then Lemmas 5.3, 5.4, and Theorem 4.2, together with the fact that

$$\|V_h \oplus \mathbf{0}_h\|_{\mathbf{B}(L^2(\mathbb{R}^2)^2)} \leq \sup_{x \in \mathbb{R}^2} \|V(x)\|_{\mathbf{B}(\mathbb{C}^2)} < \infty \quad (5.8)$$

imply that

$$\{V_h(\mathbb{D}_{m,h} - z)^{-1}\} \oplus \mathbf{0}_h = \{V_h \oplus \mathbf{0}_h\} \{(\mathbb{D}_{m,h} - z)^{-1} \oplus \mathbf{0}_h\} \rightarrow V(\mathbb{D}_m - z)^{-1} \quad (5.9)$$

strongly in $L^2(\mathbb{R}^2)^2$ as $h \rightarrow 0$, so also

$$\{I_{L^2(\mathbb{Z}_h^2)} + V_h(\mathbb{D}_{m,h} - z)^{-1}\} \oplus \mathbf{0}_h \rightarrow I_{L^2(\mathbb{R}^2)} + V(\mathbb{D}_m - z)^{-1} \quad (5.10)$$

strongly in $L^2(\mathbb{R}^2)^2$ as $h \rightarrow 0$.

Since z lies in the resolvent set of both $\mathbb{D}_{m,h}$ and $\mathbb{D}_{m,h} + V_h$ (see Lemma 5.2), we see that the right hand (and therefore the left hand) side of

$$I_{L^2(\mathbb{Z}_h^2)^2} + V_h(\mathbb{D}_{m,h} - z)^{-1} = (\mathbb{D}_{m,h} + V_h - z)(\mathbb{D}_{m,h} - z)^{-1} \quad (5.11)$$

is invertible in $L^2(\mathbb{Z}_h^2)^2$, so

$$(\mathbb{D}_{m,h} + V_h - z)^{-1} = (\mathbb{D}_{m,h} - z)^{-1} \{I_{L^2(\mathbb{Z}_h^2)^2} + V_h(\mathbb{D}_{m,h} - z)^{-1}\}^{-1}.$$

Now, in order to apply Lemma 5.5 with

$$A_h = I_{L^2(\mathbb{Z}_h^2)^2} + V_h(\mathbb{D}_{m,h} - z)^{-1} \text{ in } L^2(\mathbb{Z}_h^2)^2$$

and

$$A = I_{L^2(\mathbb{R}^2)^2} + V(\mathbb{D}_m - z)^{-1} \text{ in } L^2(\mathbb{R}^2)^2,$$

we require uniform boundedness of A_h^{-1} . To this end, we note that one can write

$$A_h^{-1} = I_{L^2(\mathbb{Z}_h^2)^2} - V_h(\mathbb{D}_{m,h} + V_h - z)^{-1}, \tag{5.12}$$

and analogously

$$\left(I_{L^2(\mathbb{Z}_h^2)^2} + iV_{\mathfrak{J},h}(\mathbb{D}_{m,h} + V_{\mathfrak{R},h} - z)^{-1} \right)^{-1} = I_{L^2(\mathbb{Z}_h^2)^2} - iV_{\mathfrak{J},h}(\mathbb{D}_{m,h} + V_h - z)^{-1};$$

so we find, using (5.7),

$$\begin{aligned} & \|(\mathbb{D}_{m,h} + V_h - z)^{-1}\|_{\mathbf{B}(L^2(\mathbb{Z}_h^2)^2)} \\ &= \left\| (\mathbb{D}_{m,h} + V_{\mathfrak{R},h} - z)^{-1} \left(I_{L^2(\mathbb{Z}_h^2)^2} + iV_{\mathfrak{J},h}(\mathbb{D}_{m,h} + V_{\mathfrak{R},h} - z)^{-1} \right)^{-1} \right\|_{\mathbf{B}(L^2(\mathbb{Z}_h^2)^2)} \\ &\leq \|(\mathbb{D}_{m,h} + V_{\mathfrak{R},h} - z)^{-1}\|_{\mathbf{B}(L^2(\mathbb{Z}_h^2)^2)} \|I_{L^2(\mathbb{Z}_h^2)^2} - iV_{\mathfrak{J},h}(\mathbb{D}_{m,h} + V_h - z)^{-1}\|_{\mathbf{B}(L^2(\mathbb{Z}_h^2)^2)} \\ &\leq \frac{1}{|\mathfrak{Im} z|} \left(1 + \sup_{x \in \mathbb{R}^2} \|V_{\mathfrak{J}}(x)\|_{\mathbf{B}(\mathbb{C}^2)} \|(\mathbb{D}_{m,h} + V_h - z)^{-1}\|_{\mathbf{B}(L^2(\mathbb{Z}_h^2)^2)} \right), \end{aligned}$$

giving

$$\|(\mathbb{D}_{m,h} + V_h - z)^{-1}\|_{\mathbf{B}(L^2(\mathbb{Z}_h^2)^2)} \leq \frac{1}{|\mathfrak{Im} z| - \sup_{x \in \mathbb{R}^2} \|V_{\mathfrak{J}}(x)\|_{\mathbf{B}(\mathbb{C}^2)}}$$

and further by (5.12)

$$\begin{aligned} \|A_h^{-1}\|_{\mathbf{B}(L^2(\mathbb{Z}_h^2)^2)} &\leq 1 + \sup_{x \in \mathbb{R}^2} \|V_{\mathfrak{J}}(x)\|_{\mathbf{B}(\mathbb{C}^2)} \|(\mathbb{D}_{m,h} + V_h - z)^{-1}\|_{\mathbf{B}(L^2(\mathbb{Z}_h^2)^2)} \\ &\leq \frac{|\mathfrak{Im} z|}{|\mathfrak{Im} z| - \sup_{x \in \mathbb{R}^2} \|V_{\mathfrak{J}}(x)\|_{\mathbf{B}(\mathbb{C}^2)}} \end{aligned}$$

for all $h > 0$. Thus we can conclude, with the help of (5.10) and Lemma 5.5, that

$$\{I_{L^2(\mathbb{Z}_h^2)^2} + V_h(\mathbb{D}_{m,h} - z)^{-1}\}^{-1} \oplus \mathbf{0}_h \rightarrow \{I_{L^2(\mathbb{R}^2)^2} + V(\mathbb{D}_m - z)^{-1}\}^{-1} \tag{5.13}$$

strongly in $L^2(\mathbb{R}^2)^2$.

Noting that

$$\|(\mathbb{D}_{m,h} - z)^{-1} \oplus \mathbf{0}_h\|_{L^2(\mathbb{R}^2)^2} \leq \frac{1}{|\Im m z|}$$

for all $h > 0$, we see from Lemma 5.4, Theorem 4.2 and (5.13) that

$$(\mathbb{D}_{m,h} + V_h - z)^{-1} \oplus \mathbf{0}_h \rightarrow (\mathbb{D}_m - z)^{-1} \{I_{L^2(\mathbb{R}^2)} + V(\mathbb{D}_m - z)^{-1}\}^{-1}$$

strongly in $L^2(\mathbb{R}^2)^2$. This completes the proof. \square

Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

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