INTERLEAVING MAYER-VIETORIS SPECTRAL SEQUENCES

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Abstract. We discuss the Mayer-Vietoris spectral sequence as an invariant in the context of persistent homology. In particular, we introduce the notion of ε-acyclic carriers and ε-acyclic equivalences between filtered regular CW-complexes and study stability conditions for the associated spectral sequences. We also look at the Mayer-Vietoris blowup complex and the geometric realization, finding stability properties under compatible noise; as a result we prove a version of an approximate nerve theorem. Adapting work by Serre, we find conditions under which ε-interleavings exist between the spectral sequences associated to two different covers.

1. Introduction

One of the benefits of homology as a topological invariant over, for example, the homotopy groups, is its computability via long exact sequences. The classical Mayer-Vietoris exact sequence has been used in countless examples to compute $\mathrm{H}_k(X)$ from a decomposition of a space X into two open subsets U and V. When we generalise this concept to open covers $(U_i)_{i\in I}$ consisting of more than just two subsets, the relations between the parts $\mathrm{H}_k(U_i)$ become more intricate and are encoded in the Mayer-Vietoris spectral sequence. These sequences first appeared in work of Leray and later Serre, and they proved to be one of the most powerful tools in pure algebraic topology. Applications of spectral sequences in applied algebraic topology, however, are still a young subject.

In [26] it was proven that the Persistence Mayer-Vietoris spectral sequence can be used to compute persistent homology. The starting point is a filtered simplicial complex X together with a cover by subcomplexes $\mathcal{U}$. Then, one computes $\mathrm{PH}_i(\mathcal{U}_\sigma)$ for all $i \geq 0$ and $\sigma \in N_{\mathcal{U}}$. Here, notice that $N_{\mathcal{U}}$ is the nerve of the cover $\mathcal{U}$ whose simplices $\sigma \in N_{\mathcal{U}}$ are subsets from $\mathcal{U}$; this leads to the notation $\mathcal{U}_\sigma = \bigcap_{U\in\sigma} U$. The Mayer-Vietoris spectral sequence starts from these groups and the morphisms induced by inclusions and converges to $\mathrm{PH}_i(X)$. As pointed out in [27], the additional insight gained from the cover $\mathcal{U}$ can be used for example for multiscale feature detection. Similar information was also explored much earlier in [28] in the form of *localized homology.*

Motivated by these results, we study the spectral sequence $E^*_{p,q}(X,\mathcal{U})$ and answer the following questions:

- Let a pair $(X,\mathcal{U})$ consisting of a space, X, and a cover for X, $\mathcal{U}$. The Mayer-Vietoris spectral sequence $E^*_{p,q}(X,\mathcal{U})$ converges to $\mathrm{PH}_*(\Delta^{\mathcal{U}}(X))$. Is $\mathrm{PH}_*(\Delta^{\mathcal{U}}(X))$ stable? Can this result be generalised?

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- Suppose that the data in each covering set $\mathcal{U}_\sigma$ for $\sigma \in N_\mathcal{U}$ is modified slightly. If the underlying cover $\mathcal{U}$ is ignored, then we would not expect $E^*_{p,q}(X,\mathcal{U})$ to be stable. Are there natural coherence conditions between changes in the sets $\mathcal{U}_\sigma$ that imply stability? If so, what do we mean by stability of spectral sequences?
- Let $\mathcal{U}$ and $\mathcal{V}$ be covers of the same space X. Can we compare $E^*_{p,q}(X,\mathcal{U})$ and $E^*_{p,q}(X,\mathcal{V})$ up to ε-interleavings?

To explain why the first question is important and how it is linked to spectral sequences, we note that $E^*_{p,q}(X,\mathcal{U})$ *converges* to the *target* persistent homology $\mathrm{PH}_*(\Delta^\mathcal{U}(X))$ (this is usually denoted by $E^*_{p,q}(X,\mathcal{U}) \Rightarrow \mathrm{PH}_*(\Delta^\mathcal{U}(X))$). The blowup complex $\Delta^\mathcal{U}(X)$ already appeared in the context of topological data analysis in [16] and [28]. It is homotopy equivalent to a *homotopy colimit*, and therefore enjoys good properties with respect to local homotopy equivalences. For example, if we assume that $\mathcal{U}_\sigma$ is contractible for all $\sigma \in N_\mathcal{U}$, then we can use [13, Proposition 4G.2] to recover Leray's Nerve Theorem. That is, there are homotopy equivalences

$$X \simeq \Delta^\mathcal{U}(X) \simeq \Delta^\mathcal{U}(*) = N(\mathcal{U}),$$

where $*$ denotes the constant *complex of spaces* on $\mathcal{U}$, see [13, App. 4.G]. The fundamental importance of this result in applied topology is underlined by the persistent Nerve lemma presented in [5]. It is worth mentioning the Approximate Nerve Theorem [12] and the Generalized Nerve Theorem [3], which are approximate versions of the Leray Theorem within the context of persistence. In particular, in [12] the spectral sequence $E^*_{p,q}(X,\mathcal{U}) \Rightarrow \mathrm{PH}_*(X)$ is examined, and it is studied how much it differs from another spectral sequence $E^*_{p,q}(*,\mathcal{U}) \Rightarrow \mathrm{PH}_*(N(\mathcal{U}))$, by careful inspection of all pages as well as the extension problem.

Throughout the paper we focus on the category **RCW-cpx** of *regularly filtered regular CW complexes* as well as the subcategory **FCW-cpx** of *filtered regular CW complexes*, see subsection 2.1. Instead of restricting our attention to a space X together with a cover $\mathcal{U}$, we look at regular diagrams $\mathcal{D}$ in **RCW-cpx** over a simplicial complex K. There is a natural replacement for the Mayer-Vietoris blowup complex in this setting, denoted by $\Delta_K(\mathcal{D})$, as explained in [13, App. 4.G.]. This object also appears in the context of semisimplicial spaces, where it is called the *geometric realization* [9]; in fact, it has an associated spectral sequence [9, Sub. 1.4.]. As we explain in Sec. 3, there are good reasons why it is worth taking this more general perspective. In particular, we consider the spectral sequence

$$E^2_{p,q}(\mathcal{D}) \Rightarrow \mathrm{PH}_{p+q}(\Delta_K\mathcal{D}) \ .$$

In order to address the first two questions, we introduce the notion of acyclic carriers to define *ε-acyclic equivalences*. Using the Acyclic Carrier Theorem we show the following: Let X and Y be two objects in **RCW-cpx**. If there exists an ε-acyclic equivalence $F^\varepsilon : X \rightrightarrows Y$, then $\mathrm{PH}_*(X)$ is ε-interleaved with $\mathrm{PH}_*(Y)$ (see Lemma 4.7 and Proposition 4.2 for a stronger statement in **FCW-cpx**). These equivalences provide a very flexible notion that works in different contexts as the examples 4.5, 4.6 and 4.8 show.

We address the first question in the following way. Let $\mathcal{D}$ and $\mathcal{L}$ be two diagrams over the same simplicial complex K and assume that for all $\sigma \in K$ there are ε-acyclic equivalences $F^\varepsilon_\sigma : \mathcal{D}(\sigma) \rightrightarrows \mathcal{L}(\sigma)$ which satisfy a compatibility condition with respect to composition in the poset category associated to K, see Proposition 5.2 for details. Then, there is an ε-acyclic equivalence $F^\varepsilon : \Delta_K(\mathcal{D}) \rightrightarrows \Delta_K(\mathcal{L})$. This result implies stability in the targets of convergence of the spectral sequences. We use this result to show a 'Strong Approximate Multinerve Theorem' in Theorem 5.3. Later, in section 6, we introduce (ε, n)-interleavings, which are given by spectral sequence morphisms that start at some page n together with a shift by a persistence parameter $\varepsilon > 0$. Assuming the same conditions as in the geometric realization case, we can obtain a $(\varepsilon, 1)$-interleaving between $E^*_{p,q}(\mathcal{D})$ and $E^*_{p,q}(\mathcal{L})$, see Proposition 5.2. This result appears in Theorem 6.5 and a specialised strong statement for covers of spaces in **FCW-cpx** is given in Proposition 6.4.

As for the third question about the comparison of the spectral sequences associated to two covers $\mathcal{U}$ and $\mathcal{V}$ of a space X, we rely on work of Serre from the fifties, in which he studied the relation between the Čech cohomology of two different covers [24]; here we adapt this work in the context of cosheaves and cosheaf homology. Take a cosheaf $\mathcal{F}$ of abelian groups on X and assume that there is a refinement $\mathcal{V} \prec \mathcal{U}$. Serre showed that the refinement morphism induced on Čech homology $\rho^{\mathcal{U}\mathcal{V}} : \check{\mathcal{H}}_*(\mathcal{V}, \mathcal{F}) \to \check{\mathcal{H}}_*(\mathcal{U}, \mathcal{F})$ is independent of the particular choice of morphism in the cochains. In [24] it was also shown that $\rho^{\mathcal{U}\mathcal{V}}$ can be factored through a construction that uses a double complex associated to both covers $C_{p,q}(\mathcal{U}, \mathcal{V}; \mathcal{F})$, see [24, Proposition 4, Sec. 29]. This construction introduces two double complex spectral sequences ${}^{\mathrm{I}}E^*_{p,q}(\mathcal{U}, \mathcal{V}; \mathcal{F})$ and ${}^{\mathrm{II}}E^*_{p,q}(\mathcal{U}, \mathcal{V}; \mathcal{F})$, both of which converge to $\check{\mathcal{H}}_*(\mathcal{U} \cap \mathcal{V}; \mathcal{F}) \simeq \check{\mathcal{H}}_*(\mathcal{V}; \mathcal{F})$. Here one might study conditions on ${}^{\mathrm{II}}E^*_{p,q}(\mathcal{U}, \mathcal{V}; \mathcal{F})$ to find when an inverse of $\rho^{\mathcal{U}\mathcal{V}}$ exists. As an application, Serre obtained an analogous result to the Leray Theorem in the context of sheaves [24, Theorem 1 in §29].

We start our analysis of the third question in Sec. 7. In case $\mathcal{V} \prec \mathcal{U}$ there is a unique morphism induced by the refinement map on the second page

$$\rho^{\mathcal{U}\mathcal{V}} : E^*_{p,q}(X, \mathcal{V}) \to E^*_{p,q}(X, \mathcal{U}) .$$

On the other hand, Theorem 7.10 tells us under what conditions there exists an ε-shifted morphism $\psi : E^*_{p,q}(X, \mathcal{U}) \to E^*_{p,q}(X, \mathcal{V})[\varepsilon]$ so that $\rho^{\mathcal{U}\mathcal{V}}$ and ψ form an $(\varepsilon, 2)$-interleaving between $E^*_{p,q}(X, \mathcal{U})$ and $E^*_{p,q}(X, \mathcal{V})$. Finally, in Proposition 7.12 we give a means of obtaining an $(\varepsilon, 2)$-interleaving between $E^*_{p,q}(X, \mathcal{U})$ and $E^*_{p,q}(X, \mathcal{V})$ through the computation of local Mayer-Vietoris spectral sequences $E^*_{p,q}(\mathcal{U}_\sigma, \mathcal{V}_{|\mathcal{U}_\sigma})$ for all $\sigma \in N_{\mathcal{U}}$. Since the open regions $\mathcal{U}_\sigma$ are assumed to be 'small' in comparison to X, this gives a means of using local calculations to deduce the interleaving. As Corollary 7.14 we present the case when $\mathcal{V}$ does not need to refine $\mathcal{U}$.

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2. Background

2.1. Regular CW-complexes with filtrations.

Recall the definition of CW-complex from [13, Chapter 0]. In contrast to the usual treatment of CW-complexes, but in line with the structure we are dealing with in TDA, we consider the cell decomposition as part of the data of our CW-complexes. For a CW-complex X, if c is an open cell in X we follow the notation from [7] and denote this by $c \in X$. We denote by X^n the set of n-dimensional cells from X and we denote by $X^{\leq n}$

the n-skeleton from X. Recall that X has a natural filtration given by its skeleta $X^0 \subseteq X^{\leq 1} \subseteq \cdots \subseteq X^{\leq N} \subseteq \cdots$, and a *cellular morphism* $f : X \to Y$ respects this filtration, in the sense that it restricts to morphisms $f^m : X^{\leq m} \to Y^{\leq m}$ for all $m \geq 0$. We work with regular CW-complexes, which are CW-complexes where the attaching maps are homeomorphisms. It is recommended to consult [7, 17] for properties and results related to regular CW-complexes. An intuitive way of understanding incidences of cells in regular complexes is through the barycentric subdivision, as explained in [11, §2.1]. Given a pair of cells $a \in X^n$ and $b \in X^{n-1}$, we denote by $[b : a]$ the degree of attaching map $\partial a \to \overline{b}/\partial b$.

Definition 2.1. A cellular morphism $f : X \to Y$ is a *regular morphism* whenever the closure $\overline{f(a)}$ is a subcomplex of Y for all cells $a \in X$. For such a morphism and a pair $a \in X^n$ and $b \in Y^n$, we denote by $[b : f(a)]$ the degree of the morphism f restricted to the open cell a and mapping into the open cell b.

We write **CW-cpx** to denote the category of finite regular CW-complexes and regular morphisms. Denote by $\mathbf{R}$ the ordered category $(\mathbb{R}, \leq)$ of real numbers. We focus on functors $X : \mathbf{R} \to \mathbf{CW\text{-}cpx}$ which we call *regularly filtered CW complexes*, and we denote their category by **RCW-cpx**. We say that an object $X \in \mathbf{RCW\text{-}cpx}$ is *tame*, whenever X is constant along a finite number of right open intervals decomposing the poset $\mathbf{R}$. For $X \in \mathbf{RCW\text{-}cpx}$, we write X_r for the regular CW-complex $X(r)$ for all $r \in \mathbf{R}$. On the other hand we write $X(r \leq s)$ to denote the morphisms $X_r \to X_s$ for all $r \leq s$ in $\mathbf{R}$; we call such morphisms *structure maps*. The reader might find an example of such a regularly filtered complex in Appendix A. If the morphisms $X(r \leq s) : X_r \to X_s$ are injections preserving the cellular structure for all $r \leq s$ in $\mathbf{R}$, then we call X a *filtered CW-complex*, denoting by **FCW-cpx** the corresponding subcategory of **RCW-cpx**. Notice that objects in **FCW-cpx** can be seen as a pair $(\mathbf{colim}\, X_*, f)$ where $\mathbf{colim}\, X_*$ is a regular CW-complex and $f : \mathbf{colim}\, X_* \to \mathbb{R}$ is a filtration function.

Throughout this text, we work with a fixed field $\mathbb{F}$. Given $X \in \mathbf{RCW\text{-}cpx}$, we define the persistent homology in degree n as the functor $\mathrm{PH}_n(X) : \mathbf{R} \to \mathbf{vect}$ given by computing cellular homology $\mathrm{PH}_n(X)_r = \mathrm{H}_n^{\mathrm{cell}}(X_r; \mathbb{F})$ for all $r \in \mathbf{R}$. As X_r is finite, the vector space $\mathrm{PH}_n(X)_r$ is finite dimensional for all $r \in \mathbf{R}$. If in addition X is tame, $\mathrm{PH}_n(X)$ only changes at a finite number of points $r \in \mathbf{R}$. We call the category of functors $\mathbf{R} \to \mathbf{vect}_{\mathbb{F}}$ *persistence modules* and denote it by **PMod**. Given $a \in (0, \infty)$ and $X \in \mathbf{RCW\text{-}cpx}$, we write $X[a]$ for the element of **RCW-cpx** such that $X[a]_r = X_{r+a}$ for all $r \in \mathbf{R}$. We use Σ^ε to denote the *ε-shift functor* $\Sigma^\varepsilon : \mathbf{RCW\text{-}cpx} \to \mathrm{Hom}(\mathbf{RCW\text{-}cpx})$ which sends $X \in \mathbf{RCW\text{-}cpx}$ to $\Sigma^\varepsilon X : X \to X[\varepsilon]$, where $\varepsilon \geq 0$. Also, for any morphism of filtered CW-complexes $f : A \to B$, one can check that $f[\varepsilon] \circ \Sigma^\varepsilon A = \Sigma^\varepsilon B \circ f$, where we use $f[\varepsilon] : A[\varepsilon] \to B[\varepsilon]$. Similarly, there are shift functors for persistence modules $\Sigma^\varepsilon : \mathbf{PMod} \to \mathrm{Hom}(\mathbf{PMod})$ for $\varepsilon \geq 0$.

Remark 2.2. Notice that the standard algorithm for the computation of persistent homology cannot be applied to objects in **RCW-cpx**. However, if X is tame and one successfully computes the coefficients for the morphisms $C_*^{\mathrm{cell}}(X_r) \to C_*^{\mathrm{cell}}(X_s)$ for all $r \leq s$ in $\mathbf{R}$, then one can use `Image_Kernel` from [26] to obtain a *barcode basis* for the filtered cellular complex $C_*^{\mathrm{cell}}(X)$. Then we compute homology of the persistence morphisms given by the differentials $d_n : C_n^{\mathrm{cell}}(X) \to C_{n-1}^{\mathrm{cell}}(X)$ by the use of `Image_Kernel`. See [26] for an explanation.

2.2. **Acyclic carriers.** Fix a field $\mathbb{F}$. We say that $X \in \mathbf{CW\text{-}cpx}$ is $\mathbb{F}$-*acyclic* if the reduced homology $\widetilde{\mathrm{H}}^*(X;\mathbb{F})$ with $\mathbb{F}$-coefficients vanishes in all dimensions; as the field is understood from the context, we just say that X is *acyclic*. Consider two objects Φ and Γ from **CW-cpx** with their respective pairs of chains and differentials $\left(C^{\mathrm{cell}}_*(\Phi), \delta^\Phi\right)$ and $\left(C^{\mathrm{cell}}_*(\Gamma), \delta^\Gamma\right)$. Let $\langle \cdot, \cdot \rangle_\Phi$ and $\langle \cdot, \cdot \rangle_\Gamma$ denote the inner products on $C^{\mathrm{cell}}_*(\Phi)$ and $C^{\mathrm{cell}}_*(\Gamma)$, where the cells form an orthonormal basis. We define a relation $\prec$ on Φ by setting $\tau \prec \sigma$ if $\langle \tau, \delta^\Phi(\sigma)\rangle_\Phi \neq 0$ and by taking the transitive closure. We denote by $\preceq$ the partial order generated by $\prec$. Thus, $\tau \prec \sigma$ does not necessarily imply $\dim(\tau) + 1 = \dim(\sigma)$. Also, notice that $\langle \tau, \delta^\Phi(\sigma)\rangle_\Phi = [\tau : \sigma]$, see the cellular boundary formula from [13, Sec. 2.2].

Definition 2.3. A *carrier* $F : \Phi \rightrightarrows \Gamma$ is a map from the set of cells of Φ to subcomplexes of Γ that is semicontinuous in the sense that for any pair $\tau \prec \sigma$ in Φ, $F(\tau) \subseteq F(\sigma)$. A carrier $F : \Phi \rightrightarrows \Gamma$ is called *acyclic*, if for every $\sigma \in \Phi$, $F(\sigma)$ is a nonempty acyclic subcomplex of Γ.

Given a chain map $w_p : C^{\mathrm{cell}}_p(\Phi) \to C^{\mathrm{cell}}_{p+r}(\Gamma)$ of degree $r = 0, 1$, we say that it is carried by F if for all cells $\sigma \in \Phi_p$

$$\{\gamma \in \Gamma_{p+r} \mid \langle w_p(\sigma), \gamma\rangle_\Gamma \neq 0\} \subseteq F(\sigma) \ ,$$

where we followed the notation from [21].

The next statement is an application of [20, Theorem 13.4]. In Proposition 4.2 we prove a version of this statement that applies to filtered CW-complexes. Notice that this theorem works for carriers which are $\mathbb{F}$-acyclic and which do not necessarily need to be $\mathbb{Z}$-acyclic; see the proof of Proposition 4.2.

Theorem 2.4. *Let $F : \Phi \rightrightarrows \Gamma$ be an acyclic carrier between CW-complexes Φ and Γ. Then we have that*

- **existence:** *there is a chain map carried by F,*
- **equivalence:** *if F carries two chain maps ϕ and φ, then F carries a chain homotopy between ϕ and φ.*

Given two acyclic carriers $F, G : \Phi \rightrightarrows \Gamma$, we write $F \subseteq G$ whenever $F(\sigma) \subseteq G(\sigma)$ for all $\sigma \in \Phi$. Given another pair of acyclic carriers $F' : \Phi \rightrightarrows \Gamma$ and $G' : \Gamma \rightrightarrows \Psi$, we also define the composition carrier $G' \circ F' : \Phi \rightrightarrows \Psi$, where each $\sigma \in \Phi$ is sent to

$$G' \circ F'(\sigma) := \bigcup_{\tau \in F(\sigma)} H(\tau) \ .$$

In particular, notice that if f is carried by F' and g is carried by G', then $g \circ f$ is 'carried' by $G' \circ F'$. However, this composition does not need to be acyclic.

Example 2.5. Consider a regular morphism $f : \Phi \to \Gamma$. We define the (not necessarily acyclic) carrier $F_f : \Phi \rightrightarrows \Gamma$ induced by f that sends $\sigma \in \Phi$ to $\overline{f(\sigma)}$. By continuity of f, for any pair $\tau \prec \sigma$ in Φ, we have that $\overline{f(\tau)} \subseteq \overline{f(\sigma)}$. Also, $\overline{f(\sigma)} \neq \emptyset$ since it must contain at least a point. Given an acyclic carrier $G : \Gamma \rightrightarrows \Psi$, we denote by $G(f(\sigma))$ the composition of carriers $G \circ F_f(\sigma)$ for all $\sigma \in \Phi$. This comes up very often in this text and whenever we are looking at the composition $G \circ F_f$ we assume that it is acyclic. Note that F_f is acyclic if f is an embedding of the regular CW-complex Φ as a subcomplex of Γ. The hypothesis that f is regular is key to define the carrier F_f. If we considered a more general continuous morphism $f : \Phi \to \Gamma$, a possible strategy would be to use outer approximations [14, 21]. However, for simplicity, we restrict to regular morphisms in this article.

2.3. **Regular diagrams of filtered complexes.** First, recall a few gluing constructions that one can perform in algebraic topology. For a brief introduction to these, see [13, App. 4.G]. They are also relevant in Kozlov's approach [15], where diagrams of spaces over trisps are studied.

Let K be a simplicial complex. We view K as a category whose objects are given by the simplices $\sigma \in K$. For any pair of simplices $\tau, \sigma \in K$ such that $\tau \preceq \sigma$, there is a unique arrow $\tau \to \sigma$ in K. We are particularly interested in K^{op}, the *opposite category* of K whose arrows are given by reversing the arrows of K. The example one should have in mind here is the case where K is the nerve of a cover of a cellular complex. Splitting the input data up by the cover then provides a diagram over the nerve where higher intersections of covering sets are included into smaller degree intersections. We formalise these constructions in the following definition.

Definition 2.6. Let K be a simplicial complex. A functor $\mathcal{D} : K^{\mathrm{op}} \to \mathbf{CW\text{-}cpx}$ is called a *regular diagram of CW-complexes* and its category is denoted by $\mathbf{RDiag}(K)$; notice here that, for any pair of simplices $\tau \preceq \sigma$ of K, the morphism $\mathcal{D}(\tau \preceq \sigma) : \mathcal{D}(\sigma) \to \mathcal{D}(\tau)$ is regular; we call such morphisms *face maps*. Given a pair of diagrams $\mathcal{D}, \mathcal{L} \in \mathbf{RDiag}(K)$, a morphism of diagrams $\varphi : \mathcal{D} \to \mathcal{L}$ is a natural transformation; i.e. the commutativity relation

$$\mathcal{D}(\tau \preceq \sigma) \circ \varphi(\sigma) = \varphi(\tau) \circ \mathcal{D}(\tau \preceq \sigma)$$

holds for any pair $\tau \preceq \sigma$ of simplices in K.

Example 2.7. Let L be a simplicial complex and suppose that it is covered by a pair of nontrivial subcomplexes L_0 and L_1. Consider a pair of vertices $v, w \in L_0 \cap L_1$ and suppose that both are connected by a pair of paths γ_0 and γ_1 within the respective 1-skeletons of L_0 and L_1. Further, we ask that these paths are simple, in the sense that they do not self intersect. Now, consider a diagram $\mathcal{D} \in \mathbf{RDiag}(\Delta^1)$ given by the closures of the paths on the vertices $\mathcal{D}(0) = \overline{\gamma_0}$ and $\mathcal{D}(1) = \overline{\gamma_1}$, while $\mathcal{D}([0,1]) = \Delta^1$, the standard one simplex. We define the face maps of $\mathcal{D}$, for $i = 0, 1$, as the regular morphism mapping $0 \mapsto v$ and $1 \mapsto w$, while $[0,1]$ is sent to γ_i. On the other hand, we consider a diagram $\mathcal{L} \in \mathbf{RDiag}(\Delta^1)$ which is given by the cover $\{L_0, L_1\}$; that is, we define $\mathcal{L}(0) = L_0$, $\mathcal{L}(1) = L_1$ while $\mathcal{L}([0,1]) = L_0 \cap L_1$; also, the face maps of $\mathcal{L}$ are given by inclusions. Then, we might consider a morphism of diagrams $\varphi : \mathcal{D} \to \mathcal{L}$ given by inclusions $\mathcal{D}(0) \hookrightarrow \mathcal{L}(0)$ and $\mathcal{D}(1) \hookrightarrow \mathcal{L}(1)$, while $\mathcal{D}([0,1]) = \Delta^1$ is sent to some path within $L_0 \cap L_1$ so that φ is well-defined. In fact, φ can only be well-defined whenever $\gamma_0 = \gamma_1$. Later, in definition 5.1, introduce (ε, K)-acyclic carriers; in this case, one might be able to consider such a carrier $F^{\varepsilon} : \mathcal{D} \rightrightarrows \mathcal{L}$ so that γ_0 and γ_1 are only required to lie within some acyclic complex.

The main object of study in this work are diagrams of filtered CW-complexes. These arise naturally in topological data analysis, for example whenever point clouds come equipped with a cover. We therefore make the following definition:

Definition 2.8. A *regularly filtered regular diagram of CW-complexes* $\mathcal{D}$ over K is a functor $\mathcal{D} : K^{\mathrm{op}} \to \mathbf{RCW\text{-}cpx}$; we denote this category by $\mathbf{RRDiag}(K)$. As in $\mathbf{RDiag}(K)$, morphisms in $\mathbf{RRDiag}(K)$ are given by natural transformations. We might consider the subcategory of $\mathbf{RRDiag}(K)$ given by functors $\mathcal{D} : K^{\mathrm{op}} \to \mathbf{FCW\text{-}cpx}$, which we call *filtered regular diagrams of CW-complexes* denoting the corresponding category by $\mathbf{FRDiag}(K)$. If for a diagram $\mathcal{D} \in \mathbf{FRDiag}(K)$ the face maps $\mathcal{D}(\tau \prec \sigma)$ are inclusions respecting the cellular structures for all $\tau \prec \sigma$

from K, then we call $\mathcal{D}$ a *fully filtered diagram of CW-complexes*, whose category we denote by $\mathbf{FFDiag}(K)$.

Example 2.9. Consider a filtered CW-complex X covered by filtered subcomplexes $\mathcal{U}$. We define $X^{\mathcal{U}}$ over the nerve $N_{\mathcal{U}}$ as $X^{\mathcal{U}}(\sigma) = \mathcal{U}_\sigma$ for all $\sigma \in N_{\mathcal{U}}$. This diagram $X^{\mathcal{U}}$ is part of $\mathbf{FFDiag}(N_{\mathcal{U}})$ since all morphisms $X^{\mathcal{U}}(\tau \preceq \sigma)$ are actually embeddings of subcomplexes. On the other hand, we can define the constant diagram $*^{\mathcal{U}}$ as $*^{\mathcal{U}}(\sigma)_r = *$ if $X^{\mathcal{U}}(\sigma)_r \neq \emptyset$ or $*^{\mathcal{U}}(\sigma)_r = \emptyset$ otherwise; for all $\sigma \in N_{\mathcal{U}}$ and all $r \in \mathbf{R}$. We also have that $*^{\mathcal{U}}$ is in $\mathbf{FFDiag}(N_{\mathcal{U}})$. Then, there is an obvious epimorphism of diagrams $X^{\mathcal{U}} \to *^{\mathcal{U}}$. Continuing with the same example, we can also define the complex of spaces $\pi_0^{\mathcal{U}}$ given by $\pi_0^{\mathcal{U}}(\sigma) = \pi_0(U_\sigma)$ for all $\sigma \in N_{\mathcal{U}}$; where for each $r \in \mathbf{R}$, $\pi_0(U_\sigma(r))$ denotes the discrete topological space given by the connected components of $U_\sigma(r)$. Thus, each $\pi_0(U_\sigma)$ is a disjoint union of points that are identified with each other as the filtration value increases and so it cannot be an element in **FCW-cpx**, but rather an element from **RCW-cpx**. Thus, in this case $\pi_0^{\mathcal{U}} \in \mathbf{RRDiag}(K)$. For all $r \in \mathbf{R}$, there is an epimorphism $X^{\mathcal{U}}(r) \to {\pi_0}^{\mathcal{U}}(r)$ sending each cell from $X^{\mathcal{U}}(r)$ to its respective connected component from ${\pi_0}^{\mathcal{U}}(r)$; these morphisms are consistent along $\mathbf{R}$. Altogether we have a sequence of epimorphisms $X^{\mathcal{U}} \to {\pi_0}^{\mathcal{U}} \to *^{\mathcal{U}}$.

2.4. **Geometric Realization.** For an abstract simplicial complex K, we denote by $|K|$ its underlying topological space. Given a simplex $\sigma \in K$, we write $|\sigma|$ to denote the number of vertices of σ. We use $\dim(\sigma)$ for the dimension of a simplex σ, that is $\dim(\sigma) = |\sigma| - 1$. We write by Δ^n the topological space associated to the standard n-simplex. Given a simplex $\sigma \in K$, we use the notation $\Delta^\sigma \coloneqq \Delta^{\dim(\sigma)}$ for simplicity. Given a pair $\tau \prec \sigma$ in K, we have a corresponding inclusion $\Delta^\tau \hookrightarrow \Delta^\sigma$. As a special case of a CW-complex, we denote by K^n and $K^{\leq n}$ the set of n-cells and the n-skeleton respectively.

Definition 2.10. Let $\mathcal{D} \in \mathbf{RDiag}(K)$. The *geometric realization* $\Delta_K \mathcal{D}$ of $\mathcal{D}$ is the object in **CW-cpx** defined as

$$\Delta_K \mathcal{D} = \bigsqcup_{\sigma \in K} \Delta^\sigma \times \mathcal{D}(\sigma) / \sim \ ,$$

where, for any pair $\tau \preceq \sigma$ in K the relation identifies a pair of points

$$(\Delta^\tau \hookrightarrow \Delta^\sigma)(x) \times y \sim x \times \mathcal{D}(\tau \preceq \sigma)(y)$$

for each pair of points $x \in \Delta^\tau$ and $y \in \mathcal{D}(\sigma)$. This $\Delta_K \mathcal{D}$ has a natural filtration given by $F^p \Delta_K \mathcal{D} = \bigcup_{\sigma \in K^{\leq p}} \Delta^\sigma \times \mathcal{D}(\sigma)$ for all $p \geq 0$. A cell $\tau \times c$ is a face of another cell $\sigma \times a$ if and only if $\tau \preceq \sigma$ and also $c \in \overline{\mathcal{D}(\tau \preceq \sigma)(a)}$. If the underlying simplicial complex K is clear from the context, we write $\Delta \mathcal{D}$ instead of $\Delta_K \mathcal{D}$.

Notice that Definition 2.10 also applies to diagrams $\mathcal{D} \in \mathbf{RRDiag}(K)$. We define $\Delta_K \mathcal{D}$ by setting $(\Delta_K \mathcal{D})_r \coloneqq \Delta_K(\mathcal{D}_r)$ for all $r \in \mathbf{R}$. Notice that our gluing conditions are consistent in this case as

$$\mathcal{D}(\tau \preceq \sigma) \circ \Sigma^t \mathcal{D}(\sigma)(y) = \Sigma^t \mathcal{D}(\tau) \circ \mathcal{D}(\tau \preceq \sigma)(y)$$

for any pair $\tau \preceq \sigma$ from K and all $t > 0$ and all points $y \in \mathcal{D}(\sigma)$. Altogether we obtain $\Delta_K(\mathcal{D}) \in$ **RCW-cpx**. Given a regular morphism $\mathcal{F} : \mathcal{D} \to \mathcal{L}$ of diagrams in $\mathbf{RRDiag}(K)$, there is an induced morphism on the geometric realization which we denote $\Delta\mathcal{F}$. Denote by $*^{\mathcal{D}}$ the diagram given by

$$*^{\mathcal{D}}(\sigma)_r = \begin{cases} * & \text{if } \mathcal{D}(\sigma)_r \neq \emptyset \\ \emptyset & \text{else} \end{cases}$$

and note that there is a homotopy equivalence $\Delta(*^{\mathcal{D}})_r \simeq |K^{\mathcal{D}}_r|$, where $K^{\mathcal{D}}$ is the filtered simplicial complex with the same underlying vertex set as K and $\sigma \in K^{\mathcal{D}}_r$ if and only if $\mathcal{D}(\sigma)_r \neq \emptyset$. The projection onto the simplex coordinates gives a *base projection* $p_b : \Delta\mathcal{D} \to \Delta(*^{\mathcal{D}}) \simeq |K^{\mathcal{D}}|$.

Example 2.11. Let $\mathcal{D} \in \mathbf{FRDiag}(K)$. We define the *multinerve* of $\mathcal{D}$ as

$$\mathrm{MNerv}(\mathcal{D}) = \Delta(\pi_0(\mathcal{D})) \ .$$

This object was first introduced in [6] in the case of $\pi_0^{\mathcal{U}}$ for a space X covered by $\mathcal{U}$. In [6] it was defined as a simplicial poset, a notion that is equivalent to that of a Δ-complex. There are epimorphisms $\Delta\mathcal{D} \to \mathrm{MNerv}(\mathcal{D}) \to \Delta(*^{\mathcal{D}}) \simeq |K|$.

Remark 2.12. Let $\mathcal{D}$ be a diagram of CW-complexes over the simplicial complex K. We can extend $\mathcal{D}$ to a diagram $\mathcal{D}'$ on the barycentric subdivision $\mathbf{Bd}\,(K)$ by defining $\mathcal{D}'(\tau_0 \prec \cdots \prec \tau_n) = \mathcal{D}(\tau_n)$ on an n-simplex $\tau_0 \prec \tau_1 \prec \cdots \prec \tau_n$ in $\mathbf{Bd}\,(K)$. A non-identity morphism in $\mathbf{Bd}\,(K)$ that has $\tau_0 \prec \tau_1 \prec \cdots \prec \tau_n$ as its codomain must have the same flag with one of the τ_k's left out as its domain. The diagram $\mathcal{D}'$ maps such a morphism to the identity in case $k \neq n$ or the morphism $\mathcal{D}(\tau_{n-1} \prec \tau_n)$ in case $k = n$. It is clear from the definition of the homotopy colimit via the simplicial replacement that the geometric realization $\Delta(\mathcal{D}')$ coincides with the definition of $\mathbf{hocolim}\,\mathcal{D}$; see [8, § 4] and also [15, Def. 15.8]; notice that in the category K, each flag is to be interpreted as a sequence of arrows $\tau_0 \leftarrow \tau_1 \leftarrow \cdots \leftarrow \tau_n$. A modified version of the homotopy equivalence $|K| \simeq |\mathbf{Bd}\,(K)|$ shows that $\Delta(\mathcal{D}) \simeq \Delta(\mathcal{D}')$. Hence, we could have worked with homotopy colimits all throughout, but we chose to work with the geometric realization since it is technically easier to handle and because in some instances it is the Mayer-Vietoris blowup complex, which has already appeared before in TDA [28]. An instance of a homotopy colimit in TDA can be found in Appendix B in [4].

Proposition 2.13. *Let $\mathcal{F} : \mathcal{D} \to \mathcal{L}$ be a morphism of diagrams in* $\mathbf{RDiag}(K)$*. If $\mathcal{F}(\sigma)$ is a homotopy equivalence for all $\sigma \in K$, then $\Delta\mathcal{F} : \Delta\mathcal{D} \to \Delta\mathcal{L}$ is a homotopy equivalence.*

One way to see this is to view $\Delta\mathcal{D}$ as a homotopy colimit (see Remark 2.12), which is a homotopy invariant functor on diagrams. Also, a proof of this result in the more general context of diagrams of spaces can be found in [13, Proposition 4G.1].

Example 2.14. Let $X \in \mathbf{CW\text{-}cpx}$ covered by $\mathcal{U}$ and recall the diagram $X^{\mathcal{U}}$ from Example 2.9. In this case $\Delta(X^{\mathcal{U}})$ is the Mayer-Vietoris blowup complex associated to the pair $(X,\mathcal{U})$ and it can be described as a subspace of the product $X \times |N_{\mathcal{U}}|$. This leads to the *fiber projection* $p_f : \Delta(X^{\mathcal{U}}) \to X$ and to the *base projection* $p_b : \Delta(X^{\mathcal{U}}) \to |N_{\mathcal{U}}|$. As shown in [13, Proposition 4G.2], p_f is a homotopy equivalence $\Delta(X^{\mathcal{U}}) \simeq X$. If each $X^{\mathcal{U}}(\sigma)$ is contractible for all $\sigma \in N_{\mathcal{U}}$, then p_b is also a homotopy equivalence by Proposition 2.13.

An interesting direction of research would be to use Proposition 2.13 to define compatible *collapses*, such as in Discrete Morse Theory (see [1, 21, 25]) and end up with a diagram of regular CW-complexes. This motivates the study of spectral sequences associated to such diagrams. We see further reasons in Section 3. On the other hand, given the importance of Proposition 2.13, we would like to adapt it to an approximate version in the context of diagrams in $\mathbf{RRDiag}(K)$. Instead of studying homotopy equivalences, we consider equivalences induced by acyclic carriers. This is done in Section 5.

2.5. **Spectral Sequences of Bounded Filtrations.** Let A_* be a graded module with differentials $d_n : A_n \to A_{n-1}$ for all $n \geq 1$, and such that $A_m = 0$ for all $m < 0$. Assume that there is a filtration $0 = F^{-1}A_* \subseteq F^0 A_* \subseteq F^1 A_* \subseteq \cdots \subseteq F^N A_* = A_*$ of A_* that is preserved by the differentials d_* in the sense that $d_n(F^p A) \subseteq F^p A$ for all $p \geq 0$. We say that A_* is a *filtered differential graded module* and denote this by the triple (A, d, F). Then there is a spectral sequence

$$E^1_{p,q} = \mathrm{H}_q\big(F^p A_* / F^{p-1} A_*\big) \Rightarrow \mathrm{H}_{p+q}\big(A_*\big)$$

for all $p, q \geq 0$, see [19, Theorem 2.6]. A morphism of spectral sequences is a sequence of bigraded morphisms $f^r : E^r_{p,q} \to \overline{E}^r_{p,q}$ that commute with the spectral sequence differentials, i.e. $d_r \circ f^r = d_r \circ f^r$ for all $r \geq 0$. Apart from that, these morphisms satisfy $f^{r+1} = H(f^r)$ for all $r \geq 0$.

Suppose that $(\overline{A}_*, \overline{d}, \overline{F})$ is another filtered differential graded module together with its corresponding spectral sequence $\overline{E}^r_{p,q}$. Consider a morphism $f : A_* \to B_*$ that commutes with the differential $f \circ d = \overline{d} \circ f$ and also preserves filtrations $f(F^p A_*) \subseteq \overline{F}^p(\overline{A}_*)$ for all $p \geq 0$. This induces a morphism of spectral sequences

$$E^r_{p,q} \to \overline{E}^r_{p,q}$$

by [19, Theorem 3.5]. We denote by **SpSq** the category of spectral sequences, while we denote by **PSpSq** the category of functors $F : \mathbf{R} \to \mathbf{SpSq}$.

3. Spectral Sequences for Geometric Realizations

Recall the persistent Mayer-Vietoris spectral sequence [26] associated to a pair $(X, \mathcal{U})$ of a space with a cover:

$$E^1_{p,q}(X, \mathcal{U}) = \bigoplus_{\sigma \in N^p_{\mathcal{U}}} \mathrm{PH}_q(X^{\mathcal{U}}(\sigma)) \Rightarrow \mathrm{PH}_{p+q}(\Delta X^{\mathcal{U}}) \simeq \mathrm{PH}_{p+q}(X)\ . \quad (1)$$

For the details about this spectral sequence in the persistent case we refer the reader to [26]. There are some limitations to the applicability of this spectral sequence to Vietoris-Rips complexes that were already pointed out in [27]: If we choose a cover of a point cloud $\mathbb{X}$ and then deduce a cover $\mathcal{U}$ of the associated Vietoris-Rips complex $\mathrm{VR}_*(\mathbb{X})$ by subcomplexes, then we can only recover $\mathrm{PH}_k(\mathrm{VR}(\mathbb{X}))$ from $\mathrm{PH}_k(\Delta \mathrm{VR}_*(\mathbb{X})^{\mathcal{U}})$ for filtration parameters below an upper bound R determined by the overlaps of the covering sets. In this section we present a regular diagram of CW-complexes that avoids this upper limit problem completely, see Example 3.6.

Before we solve our problem, we need to introduce some chain complexes. We come back to the case of filtrations later, but for now we focus on regular diagrams instead. Given a diagram $\mathcal{D}$ in $\mathbf{RDiag}(K)$, we denote by $\mathcal{D}(\tau \preceq \sigma)_*$ the induced morphism of cellular chain complexes $C^{\mathrm{cell}}_*(\mathcal{D}(\sigma)) \to C^{\mathrm{cell}}_*(\mathcal{D}(\tau))$. The cellular chain complex $C^{\mathrm{cell}}_*(\Delta\mathcal{D}, \delta^{\Delta})$ associated to $\Delta\mathcal{D}$ is defined as follows: For all $m \geq 0$ we have that $C^{\mathrm{cell}}_m(\Delta\mathcal{D})$ is a vector space generated by cells $\sigma \times c$ with $\dim(\sigma) = p$ and

$c \in \mathcal{D}(\sigma)_q$ so that $p+q=m$. On such a cell $\sigma \times c$ the differential δ^Δ is given by

$$\delta^\Delta(\sigma\times c) = \sum_{\sigma_i \prec \sigma}(-1)^i \left(\sum_{a \in \overline{\mathcal{D}(\sigma_i \preceq \sigma)(c)}} [a : \mathcal{D}(\sigma_i \preceq \sigma)(c)]\sigma_i \times a \right) + (-1)^{\dim(\sigma)} \sum_{b \in \overline{c} \setminus c} [b : c]\sigma \times b$$

where the first sum runs over the faces σ_i of σ. As shown in the proof of Lemma 3.1, the map δ^Δ is indeed a differential. In addition, notice that the filtration of $\Delta_K(\mathcal{D})$ carries over to $C^{\text{cell}}_*(\Delta_K\mathcal{D})$ by taking $F^pC_*(\Delta_K\mathcal{D}) \coloneqq C_*(F^p\Delta_K\mathcal{D})$ for all $p \geq 0$.

Now, consider the double complex $(C_{p,q}(\mathcal{D}), d^V, d^H)$ given by

$$C_{p,q}(\mathcal{D}) = \bigoplus_{\sigma \in K^p} C^{\text{cell}}_q\big(\mathcal{D}(\sigma)\big)$$

for all $p, q \geq 0$. The vertical differential is defined by the direct sum of chain differentials $d^V_{p,q} = (-1)^p \bigoplus_{\sigma \in K^p} d^\sigma_q$ where d^σ_* denotes the differential from $C^{\text{cell}}_*(\mathcal{D}(\sigma))$ for all $\sigma \in K^p$; of course $d^V \circ d^V = 0$. The horizontal differential is given by the Čech differential $d^H_{p,q}$ which is defined for a cell $a \in \mathcal{D}(\sigma)$ as $\sum_{\sigma_i \prec \sigma}(-1)^i \mathcal{D}(\sigma_i \prec \sigma)_*(a)$, where $\mathcal{D}(\sigma_i \prec \sigma)_*$ denotes the induced chain morphism $C^{\text{cell}}_*(\mathcal{D}(\sigma)) \to C^{\text{cell}}_*(\mathcal{D}(\sigma_i))$ for all faces σ_i from σ. Also, $d^H \circ d^H = 0$ by functoriality of $C^{\text{cell}}_*(\cdot)$ and the fact that $\mathcal{D}(\rho \prec \tau)_*\mathcal{D}(\tau \prec \sigma)_* = \mathcal{D}(\rho \prec \sigma)_*$ for any three simplices $\rho \prec \tau \prec \sigma$. Note that for each pair of indices $i < j$, the face map $\mathcal{D}(\sigma_{ij} \preceq \sigma)_*$ appears twice with respective coefficients $(-1)^i(-1)^j$ and $(-1)^i(-1)^{j-1}$; which have opposite sign and cancel out. On the other hand, anticommutativity $d^V \circ d^H = -d^H \circ d^V$ follows since $\mathcal{D}(\tau \prec \sigma)_*$ is a chain morphism for all $\tau \prec \sigma$ from K.

Now, we consider the double complex spectral sequence from [19, Section 2.4]. Given $\mathcal{D}$ in $\mathbf{RDiag}(K)$ there is a spectral sequence

$$E^1_{p,q}(\mathcal{D}) = \bigoplus_{\sigma \in K^p} \mathrm{H}_q(\mathcal{D}(\sigma)) \Rightarrow \mathrm{H}_{p+q}(S^{\text{Tot}}_*(\mathcal{D}))$$

where $S^{\text{Tot}}(\mathcal{D})$ is the *total complex* defined as $S^{\text{Tot}}_n(\mathcal{D}) = \bigoplus_{p+q=n} C_{p,q}(\mathcal{D})$ together with a differential $d^{\text{Tot}} = d^V + d^H$. Also, recall that the total complex has a filtration induced by the vertical filtration on $C_{p,q}(\mathcal{D})$ given by

$$F^m S^{\text{Tot}}_*(\mathcal{D}) = \bigoplus_{\substack{p+q=n \\ p \leq m}} C_{p,q}(\mathcal{D})$$

for all integers $m \geq 0$, see [26] for an explanation. Next, we relate this total complex to the geometric realization from Definition 2.10.

Lemma 3.1. *There is an isomorphism* $C^{\text{cell}}_*(\Delta\mathcal{D}, \delta^\Delta) \simeq S^{\text{Tot}}_*(\mathcal{D})$ *which preserves filtration. That is,* $F^pC^{\text{cell}}_*(\Delta\mathcal{D}, \delta^\Delta) \simeq F^pS^{\text{Tot}}_*(\mathcal{D})$ *for all* $p \geq 0$.

Proof. First we define a chain morphism $\psi : C^{\text{cell}}_m(\Delta\mathcal{D}) \to S^{\text{Tot}}_m(\mathcal{D})$ generated by the assignment: a cell $\sigma \times c \in (\Delta\mathcal{D})_m$ with $\sigma \in K^p$ and $c \in \mathcal{D}(\sigma)^q$ for integers $p+q=m$, is sent to $\psi(\sigma \times c) = (c)_\sigma \in S^{\text{Tot}}_m(\mathcal{D})$; where by $(c)_\sigma$ we refer to the vector from $S^{\text{Tot}}_m(\mathcal{D})$ which is zero in all components except at the component indexed by σ, where it is equal to c. On the other hand, ψ is a chain morphism since we have the equality

$$\psi\big(\delta^{\Delta}(\sigma \times c)\big) = \sum_{\sigma_i \prec \sigma} (-1)^i \Big(\sum_{a \in \overline{\mathcal{D}(\sigma_i \preceq \sigma)(c)}} ([a : \mathcal{D}(\sigma_i \preceq \sigma)(c)]a)_{\sigma_i} \Big)$$
$$+ (-1)^{\dim(\sigma)} \sum_{b \in \overline{c} \setminus c} ([b : c]b)_\sigma = \sum_{\sigma_i \prec \sigma} (-1)^i (\mathcal{D}(\sigma_i \preceq \sigma)_*(c))_{\sigma_i} + (-1)^{\dim(\sigma)} (d_q^\sigma(c))_\sigma$$
$$= (d^H + d^V)((c)_\sigma) = d^{\mathrm{Tot}}((c)_\sigma) \; .$$

One can see that ψ is injective, and admits an inverse $\psi^{-1} : S_m^{\mathrm{Tot}}(\mathcal{D}) \to C_m^{\mathrm{cell}}(\Delta\mathcal{D})$ that sends $(\sigma)_c$ to $\sigma \times c$. Notice that by definition ψ sends a chain in $F^p C_n^{\mathrm{cell}}(\Delta\mathcal{D})$ to a chain in $F^p S_n^{\mathrm{Tot}}(\mathcal{D})$ for all $p \geq 0$ and in particular it preserves filtration. □

Remark 3.2. Continuing with Remark 2.12, as both $\Delta_{\mathbf{Bd}\,(K)}\mathcal{D}'$ and $\mathbf{hocolim}\,(\mathcal{D})$ refer to the same space, we could have considered the homotopy colimit spectral sequence

$$E^1_{p,q}(\mathbf{Bd}\,(K), \mathcal{D}') = \bigoplus_{\sigma \in \mathbf{Bd}\,(K)^p} \mathrm{H}_q(\mathcal{D}'(\sigma)) \Rightarrow \mathrm{H}_{p+q}(\mathbf{hocolim}\,\mathcal{D}) \; .$$

Let us construct a diagram of spaces whose geometric realization is homeomorphic to $|K|$ for any finite simplicial complex K. We start by taking a finite partition $\mathcal{P}$ of the vertex set $V(K)$ and denote by $K(U)$ the maximal subcomplex of K with vertices in $U \in \mathcal{P}$. We denote by $\Delta^{\mathcal{P}}$ the standard simplex with vertices in $\mathcal{P}$. For a simplex $\tau \in K$, we define $\mathcal{P}(\tau) \in \Delta^{\mathcal{P}}$ to be the simplex consisting of all partitioning sets $U \in \mathcal{P}$ such that $\tau \cap U \neq \emptyset$. In particular if $U \in \mathcal{P}(\tau)$, then it determines a standard simplex $\tau(U) \in K(U)$ of dimension $|\tau \cap U| - 1 \geq 0$ whose vertices are precisely those from $\tau \cap U$, so that there is an inclusion $\Delta^{\tau(U)} \hookrightarrow |K(U)|$. For a vertex $v \in K$, we denote by $\mathcal{P}(v)$ the partitioning set from $\mathcal{P}$ which contains v.

We define the $(K, \mathcal{P})$-*join* diagram $\mathcal{J}^K_{\mathcal{P}} : (\Delta^{\mathcal{P}})^{\mathrm{op}} \to \mathbf{FCW\text{-}cpx}$ for all $\sigma \subseteq \mathcal{P}$ by assigning the subspace formed by the union of products of images

$$\mathcal{J}^K_{\mathcal{P}}(\sigma) = \bigcup_{\substack{\rho \in K \\ \mathcal{P}(\rho) = \sigma}} \prod_{U \in \sigma} \mathrm{Im}\Big(\Delta^{\rho(U)} \hookrightarrow |K(U)|\Big) \; ,$$

for all $\sigma \in \Delta^{\mathcal{P}}$; by definition, notice that $\mathcal{J}^K_{\mathcal{P}}(\sigma) \subseteq \prod_{U \in \sigma} |K(U)|$. Additionally, notice that $\mathcal{J}^K_{\mathcal{P}}(U) = |K(U)|$ for all $U \in \mathcal{P}$. However, $\mathcal{J}^K_{\mathcal{P}}(\sigma)$ could even be empty for $\sigma \in \Delta^{\mathcal{P}}$ with $\dim(\sigma) > 0$. For any pair $\tau \preceq \sigma$ in $\Delta^{\mathcal{P}}$, we consider the projection $\pi_{\tau \preceq \sigma} : \prod_{U \in \sigma} |K(U)| \to \prod_{U \in \tau} |K(U)|$, that forgets all product components which are indexed by vertices of σ that are not vertices of τ. We claim that $\pi_{\tau \preceq \sigma}$ restricts to a well-defined face map $\mathcal{J}^K_{\mathcal{P}}(\tau \preceq \sigma) : \mathcal{J}^K_{\mathcal{P}}(\sigma) \to \mathcal{J}^K_{\mathcal{P}}(\tau)$. In order to show this, we consider an arbitrary simplex $\rho \in K$ such that $\mathcal{P}(\rho) = \sigma$. Next, we consider the face $\lambda(\tau) \preceq \rho$ which is spanned by the vertices from $\rho \cap U$ for all $U \in \tau$, so that $\mathcal{P}(\lambda(\tau)) = \tau$ and also $\lambda(\tau)(U) = \rho(U)$ for all $U \in \mathcal{P}$. Then, we obtain the following equality

$$\pi_{\tau \preceq \sigma}\bigg(\prod_{U \in \sigma} \mathrm{Im}\Big(\Delta^{\rho(U)} \hookrightarrow |K(U)|\Big) \bigg) = \prod_{U \in \tau} \mathrm{Im}\Big(\Delta^{\lambda(\tau)(U)} \hookrightarrow |K(U)|\Big) \; ,$$

so that the face maps are well-defined, as claimed.

Lemma 3.3. *Let K be a simplicial complex together with a partition $\mathcal{P}$ of its vertex set $V(K)$. There is a CW-complex homeomorphism $\Delta(\mathcal{J}^K_{\mathcal{P}}) \simeq |K|$.*

Proof. Consider the continuous map $f : \Delta(\mathcal{J}_{\mathcal{P}}^K) \to |K|$ defined by mapping a point

$$\left(\sum_{U \in \mathcal{P}(\tau)} y_U U, \left(\sum_{v \in U} x_v v \right)_{U \in \mathcal{P}(\tau)} \right) \in \Delta^{\mathcal{P}(\tau)} \times \prod_{U \in \mathcal{P}(\tau)} \Delta^{\tau(U)} \Big/ \sim$$

to $\sum_{v \in \tau} y_{\mathcal{P}(v)} x_v v$ in Δ^τ for all $\tau \in K$; where we have values $0 \leq y_U \leq 1$ and $0 \leq x_v \leq 1$ for all $U \in \mathcal{P}(\tau)$ and all $v \in U$, and such that $\sum_{U \in \mathcal{P}(\tau)} y_U = 1$ and $\sum_{v \in U} x_v = 1$ for all $U \in \mathcal{P}$. On the other hand, let $\sum_{v \in \tau} x_v v \in \Delta^\tau$ be a point such that $0 \leq x_v \leq 1$ for all $v \in \Delta^\tau$ and such that $\sum_{v \in \tau} x_v = 1$. Then we can define the inverse continuous map

$$f^{-1}\left(\sum_{v \in \tau} x_v v \right) = \left(\sum_{U \in \mathcal{P}(\tau)} \left(\sum_{v \in U} x_v \right) U, \left(\psi_U \left(\sum_{v \in \tau} x_v v \right) \right)_{U \in \mathcal{P}(\tau)} \right)$$

where we consider a map $\psi_U : \Delta^\tau \to \Delta^{\tau(U)}$ given by

$$\psi_U\left(\sum_{v \in \tau} x_v v \right) = \begin{cases} \sum_{v \in \tau(U)} \left(\dfrac{x_v}{\sum_{v \in \tau(U)} x_v} \right) v \text{ if } \sum_{v \in \tau(U)} x_v \neq 0 \\ * \in \Delta^{\tau(U)} \text{ otherwise, where } * \text{ denotes any point (see below).} \end{cases} .$$

By the equivalence relation used to define $\Delta(\mathcal{J}_{\mathcal{P}}^K)$, the product factor $\Delta^{\tau(U)}$ is collapsed to a single point for the subset of points whose U-coordinate in $\Delta^{\mathcal{P}(\tau)}$ vanishes. If $\sum_{v \in \tau(U)} x_v = 0$, then $x_v = 0$ for all $v \in \tau(U)$ and the U-coordinate of the point $\sum_{v \in \tau} x_v v$ in $\Delta^{\mathcal{P}(\tau)}$ is 0. It is straightforward to check that f and f^{-1} are well-defined and consistent along K. □

Example 3.4. Consider the simplicial complex K depicted in the top left part of Figure 1, formed by gluing a 2-simplex to a 4 simplex along an edge. We consider a partition of the vertex set of K into the two subsets $\mathcal{P} = \{U, V\}$, where points in U are indicated by black circles and points in V are indicated by red squares. In the top right of Figure 1, we depict the standard 1-simplex $\Delta^{\mathcal{P}}$ together with the diagram $J_{\mathcal{P}}^K$ over it. In particular, notice that $J_{\mathcal{P}}^K([U, V])$ is a subset of the product $|K(U)| \times |K(V)|$ and that the morphisms $J_{\mathcal{P}}^K([U, V]) \to J_{\mathcal{P}}^K(V) = |K(V)|$ and $J_{\mathcal{P}}^K([U, V]) \to J_{\mathcal{P}}^K(U) = |K(U)|$ are both projections. In addition, notice that $\mathcal{J}_{\mathcal{P}}^K([U, V])$ has five vertices corresponding to the five different edges connecting vertices from U to V, five edges corresponding to five 2-simplices containing vertices in both U and V and a single 2-cell corresponding to the unique 4-simplex in K. Finally, the bottom left of Figure 1 shows the geometric realization $\Delta J_{\mathcal{P}}^K$.

Observe that $\mathcal{J}_{\mathcal{P}}^K$ is a diagram of *prodsimplicial* complexes as in [15, Def. 2.43], which are in particular regular CW-complexes. By the observations above we can therefore consider the associated double complex spectral sequence

$$E^1_{p,q}(\mathcal{J}_{\mathcal{P}}^K) = \bigoplus_{\sigma \in \Delta^{\mathcal{P}}} \mathrm{H}_q(\mathcal{J}_{\mathcal{P}}^K(\sigma)) \Rightarrow \mathrm{H}_{p+q}(\Delta \mathcal{J}_{\mathcal{P}}^K) \simeq \mathrm{H}_{p+q}(K) .$$

Next, we show that the "size" of K is the same as the "size" of the diagram $\mathcal{J}_{\mathcal{P}}^K$. For this, recall that each simplex $\sigma \in K$ corresponds to a unique simplex $\mathcal{P}(\sigma) \in \Delta^{\mathcal{P}}$. This is different to the case of a cover, $\mathcal{U}$, for K, where a simplex in K might correspond to several simplices from the nerve $N_{\mathcal{U}}$. Here, we write $\#L$ for the number of cells in a complex L.

Proposition 3.5. *$\#K = \sum_{\sigma \in \Delta^{\mathcal{P}}} \#\mathcal{J}_{\mathcal{P}}^K(\sigma)$.*

Proof. Consider an assignment $\psi : K \to \bigsqcup_{\sigma \in \Delta^{\mathcal{P}}} \mathcal{J}_{\mathcal{P}}^K(\sigma)$ given by sending $\rho \in K$ to $(\rho(U))_{U \in \mathcal{P}(\rho)} \in \mathcal{J}_{\mathcal{P}}^K(\mathcal{P}(\rho))$; where $(\rho(U))_{U \in \mathcal{P}(\rho)} \in \prod_{U \in \mathcal{P}(\rho)} |K(U)|$. By the definition of $\mathcal{J}_{\mathcal{P}}^K$, ψ is well-defined and surjective. Also, ψ is injective as the vertex set from $\rho \in K$ is uniquely determined by the simplices $\rho(U)$ for all $U \in \mathcal{P}(\rho)$. □

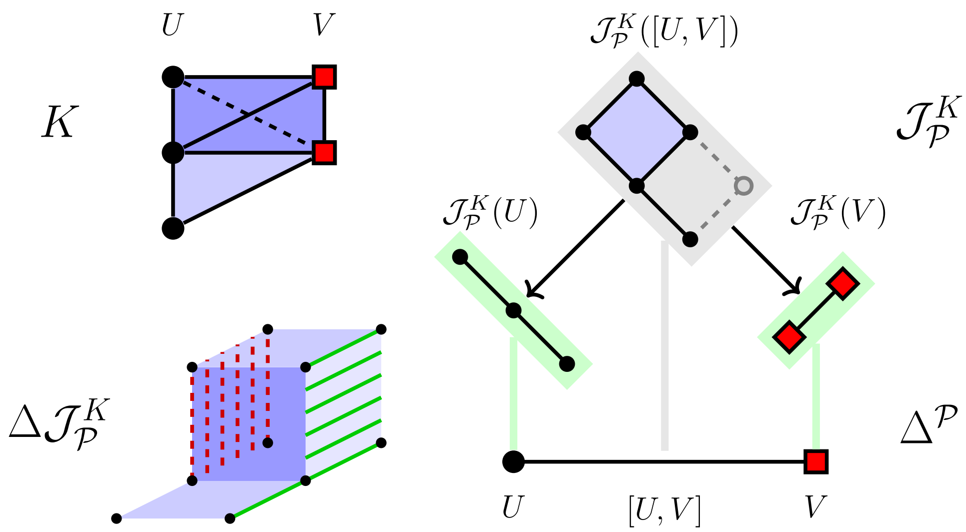


FIGURE 1. Depiction of K, $\mathcal{J}_{\mathcal{P}}^K$ and $\Delta\mathcal{J}_{\mathcal{P}}^K$ from Example 3.4. Over the edge $[U, V]$, we consider $\mathcal{J}_{\mathcal{P}}^K([U, V]) \subset |K(U)| \times |K(V)|$, where we add dashed lines to illustrate the embedding into the product. On the bottom left we depict $\Delta\mathcal{J}_{\mathcal{P}}^K$, where each red dashed line and each green line is collapsed to a single point.

Now, let us consider a filtered simplicial complex $K_* \in$ **FCW-cpx** such that its vertex set $V(K_*)$ is fixed throughout all values of $\mathbf{R}$. Let $\mathcal{P}$ be a partition of $V(K_*)$. We define the filtered regular diagram $\mathcal{J}_{\mathcal{P}}^K \in \mathbf{FRDiag}(\mathcal{P})$ by sending $r \in \mathbf{R}$ to $\mathcal{J}_{\mathcal{P}}^{K_r}$. These diagrams inherit the shift morphisms ΣK_* from K_* in the following way: Let $\sigma \in \Delta^{\mathcal{P}}$ and notice that we have restrictions $\Sigma^{s-r} K_{|U} : |K_r(U)| \to |K_s(U)|$ for all $U \in \sigma$, so that we have induced morphisms

$$\prod_{U \in \sigma} \Sigma^{s-r} K_{|U} : J_{\mathcal{P}}^{K_r}(\sigma) \to J_{\mathcal{P}}^{K_s}(\sigma)$$

for all $\sigma \in \Delta^{\mathcal{P}}$. In turn, these induce a shift morphism on $\Delta J_{\mathcal{P}}^K$ which respect filtrations, so that we have a commutative diagram

$$\begin{array}{ccccc} E^*_{p,q}(J_{\mathcal{P}}^{K_r}) & \Longrightarrow & \Delta J_{\mathcal{P}}^{K_r} & \xrightarrow{\simeq} & K_r \\ \downarrow & & \downarrow & & \downarrow \\ E^*_{p,q}(J_{\mathcal{P}}^{K_s}) & \Longrightarrow & \Delta J_{\mathcal{P}}^{K_s} & \xrightarrow{\simeq} & K_s \end{array}$$

and thus $\mathrm{PH}_*(\Delta\mathcal{J}_{\mathcal{P}}^K) \simeq \mathrm{PH}_*(K_*)$. For each simplex $\sigma \in \Delta^{\mathcal{P}}$ one can see $\mathcal{J}_{\mathcal{P}}^K(\sigma)$ as a filtered simplicial complex, so that

$$E^1_{p,q}(\mathcal{J}_{\mathcal{P}}^K) = \bigoplus_{\sigma \in (\Delta^{\mathcal{P}})^p} \mathrm{PH}_q(\mathcal{J}_{\mathcal{P}}^K(\sigma)) \Rightarrow \mathrm{PH}_{p+q}(K) .$$

Example 3.6. Consider a point cloud $\mathbb{X}$, a partition $\mathcal{P}$ and consider its Vietoris Rips complex $\mathrm{VR}_*(\mathbb{X}) \in$ **FCW-cpx**. In this case we have a fixed partition of the vertex set of $\mathrm{VR}_*(\mathbb{X})$, which allows us to consider the spectral sequence:

$$E^1_{p,q}\big(\mathcal{J}_{\mathcal{P}}^{\mathrm{VR}_*(\mathbb{X})}\big) = \bigoplus_{\sigma \in \Delta^{\mathcal{P}}} \mathrm{PH}_q(\mathcal{J}_{\mathcal{P}}^{\mathrm{VR}_*(\mathbb{X})}(\sigma)) \Rightarrow \mathrm{PH}_{p+q}(\mathrm{VR}_*(\mathbb{X})) \ .$$

This is very convenient as it avoids the main difficulties with the Mayer-Vietoris blowup complex associated to a cover. Namely, one recovers $\mathrm{PH}_*(K)$ completely without any bounds depending on the cover overlaps. In addition, notice that $\Delta\mathcal{J}_{\mathcal{P}}^{\mathrm{VR}_*(\mathbb{X})}$ has the same number of cells than $\mathrm{VR}_*(\mathbb{X})$, contrary to the Mayer-Vietoris blowup complex, whose number of cells is much larger, as shown in Proposition 3.5.

The $(K, \mathcal{P})$-join diagram is related to [23, Example 4]. There the motivation behind the filtrations is given by a consistency radius and a filtration based on the differences between local measurements. The same example appears (without a filtration) as one of the opening examples in [13, Appendix 4.G].

4. ε-acyclic carriers

The following definition encodes our notion of 'noise'.

Definition 4.1. Let $X, Y \in$ **RCW-cpx**. An ε-acyclic carrier $F^\varepsilon_* : X_* \rightrightarrows Y[\varepsilon]_*$ is a family of acyclic carriers $F^\varepsilon_a : X_a \rightrightarrows Y_{a+\varepsilon}$ for all $a \in \mathbf{R}$ such that

$$Y(a+\varepsilon \le b+\varepsilon)F^\varepsilon_a(c) \subseteq F^\varepsilon_b(X(a \le b)(c))$$

for all cells c of X_a and $a, b \in \mathbf{R}$ with $a \le b$.

The proposition below is an adaptation of [20, Theorem 13.4] or [7, Lemma 2.4] to the context of tame filtered CW-complexes.

Proposition 4.2. *Let $X_*, Y_* \in$ **FCW-cpx** be tame, and assume that there exists an ε-acyclic carrier*

$$F^\varepsilon_* : X_* \rightrightarrows Y[\varepsilon]_* \ .$$

Then there exist chain morphisms $f^\varepsilon_a : C_*(X_a) \to C_*(Y_{a+\varepsilon})$ carried by F^ε_a for all $a \in \mathbf{R}$, so that $Y(a+\varepsilon \le b+\varepsilon) \circ f^\varepsilon_a = f^\varepsilon_b \circ X(a \le b)$. Furthermore, given another such sequence of morphisms $g^\varepsilon_a : C_*(X_a) \to C_*(Y_{a+\varepsilon})$, there exist chain homotopy equivalences $H^\varepsilon_a : g^\varepsilon_a \simeq f^\varepsilon_a$ which are carried by F^ε_a for all $a \in \mathbf{R}$.

Proof. Let $b \in \mathbf{R}$ and assume that f^ε_a has already been defined for all values $a < b$, where we allow for $b = -\infty$. We first define f^ε_b on all cells which are in the image of $X(a < b)$ for any $a < b$ using the definition

$$f^\varepsilon_b \circ X(a < b) = Y(a+\varepsilon < b+\varepsilon) \circ f^\varepsilon_a \ .$$

Notice that the assumption that $X_a \subseteq X_b$ is crucial for this to work. By hypotheses, given a cell $c \in \mathrm{Im}(X(a < b))$, its image $f^\varepsilon_b(c)$ is then contained in

$$Y(a+\varepsilon < b+\varepsilon)F^\varepsilon_a(\widetilde{c}) \subseteq F^\varepsilon_b(X(a < b)(\widetilde{c})) \ ,$$

where $\widetilde{c} \in X_a$ is such that $c = X(a < b)(\widetilde{c})$. Hence, f^ε_b satisfies the carrier condition. Next we define f^ε_b on the remaining cells in

$$\widetilde{X_b} = X_b \setminus \big(\bigcup_{a<b} X(a < b)\big) \ .$$

We proceed to prove this by induction. First, choose a 0-cell $f^\varepsilon_b(v) \in F^\varepsilon_b(v)$ for each remaining 0-cell $v \in \widetilde{X_b}$, and notice that $d_* f^\varepsilon_b(v) = 0 = f^\varepsilon_b(d_* v)$, where we use d_* for the chain complex differentials. Next, by induction, assume that for a fixed $p \geq 0$, the p-cells $s \in X_b$ have image $f^\varepsilon_b(s)$ carried by $F^\varepsilon_b(s)$ and such that $d_* \circ f^\varepsilon_b(s) = f^\varepsilon_b \circ d_*(s)$. We would like to extend f^ε_b to the $(p+1)$-cells. By semicontinuity, given such a cell $c \in X_b$, its boundary $d_* c$ is contained in $F^\varepsilon_b(c)$. On the other hand, notice that by linearity and the induction hypotheses $d_* f^\varepsilon_b(d_* c) = f^\varepsilon_b(d_* d_* c) = 0$, thus $f^\varepsilon_b(d_* c)$ is a cycle in $F^\varepsilon_b(c)$. By acyclicity, there exists $h \in F^\varepsilon_b(c)$ such that $d_* h = f^\varepsilon_b(d_* c)$ and thus we can define $f^\varepsilon_b(c) = h$. Altogether, we have defined a chain morphism f^ε_b which is carried by F^ε_b.

Since X_* is tame, there exist a finite sequence of values $a_1 < a_2 < \cdots < a_N$ such that $X_s = X_{a_i}$ for all $s \in (a_{i-1}, a_i)$ where we define $a_0 = -\infty$ and $a_{N+1} = \infty$. We apply the construction of f^ε_b for all values b ranging over a_i from $i = 1$ up to $i = N$. This determines the chain morphism $f^\varepsilon_* : C_*(X_*) \to C_*(Y[\varepsilon]_*)$, where we set $f^\varepsilon_s = f^\varepsilon_{a_i}$ for all $s \in (a_{i-1}, a_i]$ where $i = 1, 2, \ldots, N$ and also $f^\varepsilon_t = f^\varepsilon_{a_N}$ for all $t > a_N$.

Now, assume that g^ε_b is also carried by F^ε_b for all $b \in \mathbb{R}$. Following [18, Sec. 12.3], we define the chain complex $\mathcal{I}$ given by $\mathcal{I}_0 = \langle [0], [1] \rangle$ and $\mathcal{I}_1 = \langle [0,1] \rangle$ and $\mathcal{I}_k = 0$ for $k > 0$. This is the cellular chain complex of the unit interval I decomposed into two 0-cells and one 1-cell. A chain homotopy $h^\varepsilon_b : f^\varepsilon_b \simeq g^\varepsilon_b$ corresponds to a chain map $h^\varepsilon_b \colon C^{\mathrm{cell}}_*(X_b) \otimes \mathcal{I} \to C^{\mathrm{cell}}_*(Y_b)$ such that $h^\varepsilon_b(x, [0]) = f^\varepsilon_b(x)$ and $h^\varepsilon_b(x, [1]) = g^\varepsilon_b(x)$ for all $x \in X_b$. Let $H^\varepsilon_b(c, i) = F^\varepsilon_b(c)$ for a cell $(c, i) \in X \times I$. By assumption, $H^\varepsilon \colon X \times I \rightrightarrows Y$ is an ε-acyclic carrier. Note that $C^{\mathrm{cell}}_*(X_b) \otimes \mathcal{I} \cong C^{\mathrm{cell}}_*(X_b \times I)$. Replicating the first part of the proof we can now extend any map $h^\varepsilon_b \colon C^{\mathrm{cell}}_*(X_b) \otimes \mathcal{I}_0 \to C^{\mathrm{cell}}_*(Y_b)$ with the above properties to all cells of $X \times I$. □

Definition 4.3. Let $X_*, Y_* \in$ **RCW-cpx**. A *shift* carrier is an ε-acyclic carrier $I^\varepsilon_X : X_* \rightrightarrows X_{*+\varepsilon}$ carrying the standard shift $\Sigma^\varepsilon X_*$. Let two ε-acyclic carriers

$$F^\varepsilon : X_* \rightrightarrows Y_{*+\varepsilon} \ ,$$
$$G^\varepsilon : Y_* \rightrightarrows X_{*+\varepsilon} \ ,$$

together with shift carriers $I^{2\varepsilon}_X$ and $I^{2\varepsilon}_Y$. We say that X_* and Y_* are *ε-acyclic equivalent* whenever we have inclusions $G^\varepsilon \circ F^\varepsilon \subseteq I^{2\varepsilon}_X$ and $F^\varepsilon \circ G^\varepsilon \subseteq I^{2\varepsilon}_Y$.

The motivation for the definition of ε-acyclic equivalences is the following lemma:

Proposition 4.4. *Let X_* and Y_* be two tame elements from* **FCW-cpx** *which are* ε-acyclic equivalent. *Then* $\mathrm{PH}(X_*)$ *and* $\mathrm{PH}(Y_*)$ *are ε-interleaved.*

Proof. By Proposition 4.2 we know that there exist two chain maps $f^\varepsilon_* \colon C_*(X_*) \to C_*(Y_{*+\varepsilon})$ and $g^\varepsilon \colon C_*(Y_*) \to C_*(X_{*+\varepsilon})$ carried by F^ε and G^ε respectively. By hypothesis the compositions $g^\varepsilon \circ f^\varepsilon$ and $f^\varepsilon \circ g^\varepsilon$ are carried by corresponding shift carriers $I^{2\varepsilon}_X$ and $I^{2\varepsilon}_Y$. Thus, using the second part of Proposition 4.2 we obtain chain homotopies $g^\varepsilon \circ f^\varepsilon \simeq \Sigma^{2\varepsilon} C_*(X)$ and $f^\varepsilon \circ g^\varepsilon \simeq \Sigma^{2\varepsilon} C_*(Y)$. Altogether, in homology these compositions are equal to the corresponding shifts, and $\mathrm{PH}_*(X_*)$ and $\mathrm{PH}_*(Y_*)$ are ε-interleaved. □

Example 4.5. Consider two finite metric spaces $\mathbb{X}$ and $\mathbb{Y}$. Let $d_H(\mathbb{X}, \mathbb{Y})$ be their Hausdorff distance and set $\varepsilon = 2 d_H(\mathbb{X}, \mathbb{Y})$. Given a subcomplex $K \subseteq \mathrm{VR}(\mathbb{X})$, we denote its vertex set by $\mathbb{X}(K) \subseteq \mathbb{X}$. Likewise for a simplex $\sigma \in \mathrm{VR}(\mathbb{X})$, we write $\mathbb{X}(\sigma) \subseteq \mathbb{X}$ for the vertices spanning σ. Define a carrier $F^\varepsilon : \mathrm{VR}(\mathbb{X}) \rightrightarrows \mathrm{VR}(\mathbb{Y})$ by mapping a simplex $\sigma \in \mathrm{VR}(\mathbb{X})_a$ to

$$F^\varepsilon(\sigma) = |\sup\{K \subseteq \mathrm{VR}(\mathbb{Y})_{a+\varepsilon} \mid d_{\mathrm{H}}(\mathbb{X}(\sigma), \mathbb{Y}(K)) \leq \varepsilon/2\}|$$

This is clearly semicontinuous. If $v_0, \ldots, v_n$ are vertices in $F^\varepsilon(\sigma)$, then by definition $\{v_0, \ldots, v_n\}$ is an n-simplex in $F^\varepsilon(\sigma)$. Therefore we have $F^\varepsilon(\sigma) \simeq \Delta^N$ for some $N \in \mathbb{N}$, which is acyclic. In particular, F^ε is an ε-acyclic carrier. Interchanging the roles of $\mathbb{X}$ and $\mathbb{Y}$ we also obtain an ε-acyclic carrier $G^\varepsilon : \mathrm{VR}(\mathbb{Y}) \rightrightarrows \mathrm{VR}(\mathbb{X})$. Similarly, we define for a simplex $\sigma \in \mathrm{VR}(\mathbb{X})_a$ the shift carrier

$$I^{2\varepsilon}_{\mathbb{X}}(\sigma) = |\sup\{K \subseteq \mathrm{VR}(\mathbb{X})_{a+2\varepsilon} \mid d_{\mathrm{H}}(\mathbb{X}(\sigma), \mathbb{X}(K)) \leq \varepsilon\}|$$

Analogously one defines $I^{2\varepsilon}_{\mathbb{Y}}$. Since $G^\varepsilon \circ F^\varepsilon \subseteq I^{2\varepsilon}_{\mathbb{X}}$ and $F^\varepsilon \circ G^\varepsilon \subseteq I^{2\varepsilon}_{\mathbb{Y}}$, Proposition 4.4 implies that $\mathrm{PH}_*(\mathrm{VR}(\mathbb{X}))$ and $\mathrm{PH}_*(\mathrm{VR}(\mathbb{Y}))$ are ε-interleaved. This is similar to the proof using *correspondences*, see [22, Proposition 7.8, Sec. 7.3].

Example 4.6. Consider $\mathbb{R}^N$ together with a 1-Lipschitz function $f : \mathbb{R}^N \to \mathbb{R}$ with constant $\varepsilon > 0$. On the other hand, consider the lattices $\mathbb{Z}^N$ and $r\mathbb{Z}^N + l$ for a pair of vectors $r, l \in \mathbb{R}^N$ such that the coordinates of r satisfy $0 < r_i \leq 1$ for all $1 \leq i \leq N$. Then we take their corresponding cubical complexes $\mathcal{C}(\mathbb{Z}^N)$ and $\mathcal{C}(r\mathbb{Z}^N + l)$ thought as embedded in $\mathbb{R}^N$. The function f induces a natural filtration for these cubical complexes: a vertex $v \in \mathcal{C}(\mathbb{Z}^N)$ is contained in $\mathcal{C}(\mathbb{Z}^N)_{f(v)}$, while a cell $a \in \mathcal{C}(\mathbb{Z}^N)$ appears at the maximum filtration value on its vertices. There is an ε-acyclic carrier $F^\varepsilon : \mathcal{C}(\mathbb{Z}^N) \rightrightarrows \mathcal{C}(r\mathbb{Z}^N + l)$ sending each cell $a \in \mathcal{C}(\mathbb{Z}^N)$ to the smallest subcomplex $F^\varepsilon(a)$ containing all $b \in \mathcal{C}(r\mathbb{Z}^N + l)$ such that $\overline{b} \cap a \neq \emptyset$. In an analogous way the inverse acyclic carrier can be defined, and the compositions $F^\varepsilon \circ G^\varepsilon$ and $G^\varepsilon \circ F^\varepsilon$ define the shift carriers. Thus, using Proposition 4.4, one shows that $\mathrm{PH}_*(\mathcal{C}(\mathbb{Z}^N))$ and $\mathrm{PH}_*(\mathcal{C}(r\mathbb{Z}^N + l))$ are ε-interleaved.

An important assumption of Proposition 4.2 is that we are dealing with tame filtered CW-complexes. However, what if we considered a pair of elements $X_*, Y_* \in$ **RCW-cpx** instead? In this context, we notice that given an ε-acyclic carrier $F^\varepsilon : X_* \to Y_*[\varepsilon]$, it is not necessarily true that the compositions

$$Y(a+\varepsilon \leq b+\varepsilon)F^\varepsilon_a(c) \text{ and } F^\varepsilon_b(X(a \leq b)(c))$$

are still acyclic for all pairs $a \leq b$ from $\mathbf{R}$. Thus, whenever we talk about ε-acyclic carriers $F^\varepsilon : X_* \to Y_*[\varepsilon]$ in this context we assume that $F^\varepsilon_b(X(a \leq b)(c))$ is acyclic for all pairs $a, b \in \mathbf{R}$ with $a \leq b$ and all cells $c \in X(a)$.

Corollary 4.7. *Let $X_*, Y_* \in$ **RCW-cpx** be a pair of elements such that both are ε-acyclic equivalent in the above sense. Then $d_I(\mathrm{PH}_*(X_*), \mathrm{PH}_*(Y_*)) \leq \varepsilon$.*

Proof. For each persistence value $a \in \mathbf{R}$, we use Theorem 2.4 twice to obtain a pair of chain morphisms $f_a : C^{\mathrm{cell}}_a(X) \to C^{\mathrm{cell}}_{a+\varepsilon}(Y)$ and $g_{a+\varepsilon} : C^{\mathrm{cell}}_{a+\varepsilon}(Y) \to C^{\mathrm{cell}}_{a+2\varepsilon}(X)$. In a similar way we obtain a pair of chain homotopies $g_{a+\varepsilon} \circ f_a \simeq (\Sigma^{2\varepsilon} C^{\mathrm{cell}}_*(X))_a$ and $f_{a+\varepsilon} \circ g_a \simeq (\Sigma^{2\varepsilon} C^{\mathrm{cell}}_*(Y))_a$ so that we have equalities between the induced homology morphisms $[g_{a+\varepsilon}] \circ [f_a] = [(\Sigma^{2\varepsilon} C^{\mathrm{cell}}_*(X))_a]$ and $[f_{a+\varepsilon}] \circ [g_a] = [(\Sigma^{2\varepsilon} C^{\mathrm{cell}}_*(Y))_a]$ for all $a \in \mathbf{R}$. Now, for a pair of values $a \leq b$ from $\mathbf{R}$, it is not necessarily true that $Y(a+\varepsilon \leq b+\varepsilon) \circ f_a = f_b \circ X(a \leq b)$. However, since $Y(a+\varepsilon \leq b+\varepsilon) \circ f_a$ and $f_b \circ X(a \leq b)$ are both included in $F^\varepsilon_b(X(a \leq b)(c))$ by hypotheses, then by applying Theorem 2.4 again there is a chain homotopy equivalence $Y(a+\varepsilon \leq b+\varepsilon) \circ f_a \simeq f_b \circ X(a \leq b)$, which implies

$$[Y(a+\varepsilon \leq b+\varepsilon)] \circ [f_a] = [f_b] \circ [X(a \leq b)] \ ,$$

and we have defined a persistence morphism $[f_*] : \mathrm{PH}_*(X_*) \to \mathrm{PH}_*(Y_*[\varepsilon])$. Similarly, we can also put together the g_a for all $a \in \mathbf{R}$ so that we obtain a morphism $[g_*] : \mathrm{PH}_*(Y_*) \to \mathrm{PH}_*(X_*[\varepsilon])$. This leads to the claimed ε-interleaving. □

Example 4.8. In Appendix A, we describe a filtered CW-complex X, a regularly filtered CW-complex Y, together with a pair of 0-acyclic carriers (i.e. $\varepsilon = 0$) $F : Y \rightrightarrows X$ and $G : X \rightrightarrows Y$ which, together with the compositions $G \circ F$ and $G \circ F$ as shift carriers, define a 0-acyclic equivalence between Y and X. Therefore, by Corollary 4.7 we obtain isomorphisms $\mathrm{PH}_n(X) \cong \mathrm{PH}_n(Y)$ for all $n \geq 0$. In this case, notice that Y is much smaller than X; thus it is worth considering the Regularly Filtered complex Y in place of X. Next, we briefly describe how one could use ε-equivalences. In this case, one could have considered a filtered complex $\widetilde{X}$ which is equal to X_* outside the intervals $(i - \varepsilon, i + \varepsilon)$ for values $i = 1, 2, 3, 4$ and for some $\varepsilon < 1/2$. Notice that in this case one should be able to obtain an ε-acyclic equivalence between $\widetilde{X}$ and Y, so that by Corollary 4.7 $\mathrm{PH}_n(\widetilde{X})$ and $\mathrm{PH}_n(Y)$ are ε-interleaved for all $n \geq 0$.

Remark 4.9. Notice that our notion of acyclicity is different from that in [3] and [12]. In [12] a filtered complex K_* is called ε-acyclic whenever the induced homology maps $\mathrm{H}_*(K_r) \to \mathrm{H}_*(K_{r+\varepsilon})$ vanish for all $r \in \mathbb{R}$. In this case, one can still (trivially) define acyclic carriers between $*$ and K_*. The problem arises when defining the shift carrier $I_K^{A\varepsilon}$ for some constant $A > 0$, which does not exist in general. One can however, adapt the proof of Proposition 4.2 so that there is a chain morphism $\psi^{\varepsilon(\dim(K_r)+1)} : C_*(K_r) \to C_*(K_{r+\varepsilon(\dim(K_r)+1)})$; and that this coincides up to chain homotopy with the composition through $C_*(*)$. One does this by following the same proof as in Proposition 4.2, but increasing the filtration value by ε each time we assume that some cycle lies in an acyclic carrier. Thus, if we have $\dim(K) = \sup_{r\in\mathbb{R}}(\dim(K_r)) < \infty$, then one could say that there is an $\varepsilon(\dim(K)+1)$-*approximate* chain homotopy equivalence between $C(*)$ and $C(K_*)$.

5. Interleaving Geometric Realizations

Next, we focus on acyclic carrier equivalences between a pair of diagrams $\mathcal{D}, \mathcal{L} \in \mathbf{RRDiag}(K)$. We start by taking ε-acyclic carriers $F^\varepsilon_\sigma : \mathcal{D}(\sigma) \rightrightarrows \mathcal{L}(\sigma)$ for all $\sigma \in K$ which have to be compatible in the following sense: For any pair $\tau \preceq \sigma$ and any cell $c \in \mathcal{D}(\sigma)$, there is an inclusion

$$\mathcal{L}(\tau \preceq \sigma)(F^\varepsilon_\sigma(c)) \subseteq F^\varepsilon_\tau(\mathcal{D}(\tau \preceq \sigma)(c)) \tag{2}$$

and we assume in addition that $F^\varepsilon_\tau(\mathcal{D}(\tau \preceq \sigma)\Sigma^r\mathcal{D}(\sigma)(c))$ is acyclic for all $r \geq 0$. This compatibility leads to 'local' diagrams of spaces. That is, given a pair of values $a \in \mathbf{R}$ and $r \geq 0$ and a cell $c \in \mathcal{D}(\sigma)_a$, we consider an object $F^{r,\varepsilon}_{\sigma\times c} \in \mathbf{RDiag}(\Delta^\sigma)$. It is given by the space $F^{r,\varepsilon}_{\sigma\times c}(\tau) = F^\varepsilon_\tau\big(\mathcal{D}(\tau \preceq \sigma)\Sigma^r\mathcal{D}(\sigma)(c)\big)$ for all faces $\tau \preceq \sigma$. For any sequence $\rho \preceq \tau \preceq \sigma$ in K, there are morphisms in $F^{r,\varepsilon}_{\sigma\times c}$ given by restricting morphisms from $\mathcal{L}$

$$\begin{array}{ccccc}
\tau & \longrightarrow & F^{r,\varepsilon}_{\sigma\times c}(\tau) & = & F^\varepsilon_\tau\big(\mathcal{D}(\tau \preceq \sigma)\Sigma^r\mathcal{D}(\sigma)(c)\big) \\
\preceq\big\uparrow & & \big\downarrow & & \big\downarrow \mathcal{L}(\rho\preceq\tau) \\
\rho & \longrightarrow & F^{r,\varepsilon}_{\sigma\times c}(\rho) & = & F^\varepsilon_\rho\big(\mathcal{D}(\rho \preceq \sigma)\Sigma^r\mathcal{D}(\sigma)(c)\big)
\end{array}.$$

Using condition (2) repeatedly on the cells from $L = \mathcal{D}(\tau \preceq \sigma)\Sigma^r\mathcal{D}(\sigma)(c)$, we see that we have an inclusion

$$\mathcal{L}(\rho \preceq \tau)\big(F^{\varepsilon}_{\tau}(L)\big) \subseteq F^{\varepsilon}_{\sigma}\big(\mathcal{D}(\rho \preceq \tau)(L)\big)\ .$$

Thus the diagram $F^{r,\varepsilon}_{\sigma\times c}$ is indeed well-defined, and we may consider the geometric realization $\Delta F^{r,\varepsilon}_{\sigma\times c}$. By hypothesis each $F^{r,\varepsilon}_{\sigma\times c}(\tau)$ is acyclic for all $\tau \preceq \sigma$, so that the first page of the spectral sequence $E^{*}_{p,q}(F^{r,\varepsilon}_{\sigma\times c}) \Rightarrow \mathrm{H}_{p+q}(\Delta F^{r,\varepsilon}_{\sigma\times c})$ is equal to

$$E^{1}_{p,q}(F^{r,\varepsilon}_{\sigma\times c}) = \bigoplus_{\tau\in(\Delta^{\sigma})^{p}} \mathrm{H}_q(F^{r,\varepsilon}_{\sigma\times c}(\tau)) = \begin{cases} \bigoplus_{\tau\in(\Delta^{\sigma})^{p}} \mathbb{F} \text{ if } q = 0, \\ 0 \text{ otherwise.} \end{cases}$$

In fact, computing the homology with respect to the horizontal differentials on the first page corresponds to computing the homology of Δ^{σ}. Thus, $E^{2}_{p,q}(F^{r,\varepsilon}_{\sigma\times c})$ is zero everywhere except at $p = q = 0$ where it is equal to $\mathbb{F}$. Thus, the spectral sequence collapses on the second page, and $\Delta F^{r,\varepsilon}_{\sigma\times c}$ is acyclic. We use the notation $F^{\varepsilon}_{\sigma\times c} = F^{0,\varepsilon}_{\sigma\times c}$.

Definition 5.1. Let $\mathcal{D}$ and $\mathcal{L}$ be two diagrams in $\mathbf{RRDiag}(K)$. Suppose that there are ε-acyclic carriers $F^{\varepsilon}_{\sigma} : \mathcal{D}(\sigma) \rightrightarrows \mathcal{L}(\sigma)$ for all $\sigma \in K$, and such that

$$\mathcal{L}(\tau \preceq \sigma)\big(F^{\varepsilon}_{\sigma}(c)\big) \subseteq F^{\varepsilon}_{\sigma}\big(\mathcal{D}(\tau \preceq \sigma)(c)\big)$$

for all $c \in \mathcal{D}(\sigma)$ and in addition $F^{\varepsilon}_{\tau}(\mathcal{D}(\tau \preceq \sigma)\Sigma^r\mathcal{D}(\sigma)(c))$ is acyclic for all $r \geq 0$. Then we say that the set $\{F^{\varepsilon}_{\sigma}\}_{\sigma\in K}$ is a (ε, K)-acyclic carrier between $\mathcal{D}$ and $\mathcal{L}$. We denote this by $F^{\varepsilon} : \mathcal{D} \rightrightarrows \mathcal{L}$.

Theorem 5.2. *Let $\mathcal{D}$ and $\mathcal{L}$ be two diagrams in* $\mathbf{RRDiag}(K)$*. Suppose that there are (ε, K)-acyclic carriers $F^{\varepsilon} : \mathcal{D} \rightrightarrows \mathcal{L}$ and $G^{\varepsilon} : \mathcal{L} \rightrightarrows \mathcal{D}$, together with a pair of shift (ε, K)-acyclic carriers $I^{2\varepsilon}_{\mathcal{D}} : \mathcal{D} \rightrightarrows \mathcal{D}$ and $I^{2\varepsilon}_{\mathcal{L}} : \mathcal{L} \rightrightarrows \mathcal{L}$, and such that these restrict to acyclic equivalences*

$$G^{\varepsilon}_{\tau} \circ F^{\varepsilon}_{\tau} \subseteq (I^{2\varepsilon}_{\mathcal{D}})_{\tau} \text{ and } F^{\varepsilon}_{\tau} \circ G^{\varepsilon}_{\tau} \subseteq (I^{2\varepsilon}_{\mathcal{L}})_{\tau}$$

for each simplex $\tau \in K$. Then there is an ε-acyclic equivalence $F^{\varepsilon} : \Delta\mathcal{D} \rightrightarrows \Delta\mathcal{L}$ which preserves filtration. That is, there are ε-acyclic equivalences $F^pF^{\varepsilon} : F^p\Delta\mathcal{D} \rightrightarrows F^p\Delta\mathcal{L}$ for all $p \geq 0$.

Proof. Let $\sigma\times c \in \Delta\mathcal{D}$ be a cell, where c is an m-cell in $\mathcal{D}(\sigma)$. Define an acyclic carrier $F^{\varepsilon} : \Delta\mathcal{D} \rightrightarrows \Delta\mathcal{L}$ by sending $\sigma\times c$ to the acyclic carrier $\Delta F^{\varepsilon}_{\sigma\times c}$, which is a subcomplex of $\Delta\mathcal{L}$. Let us first check semicontinuity. For any pair of cells $\tau \times a \preceq \sigma \times c$ in $\Delta\mathcal{D}$, the cell a is contained in the subcomplex $\overline{\mathcal{D}(\tau \preceq \sigma)(c)}$, and by continuity of $\mathcal{D}(\rho \preceq \tau)$ we have that $\mathcal{D}(\rho \preceq \tau)(a) \subseteq \overline{\mathcal{D}(\rho \preceq \sigma)(c)}$. Thus there are inclusions

$$F^{\varepsilon}_{\rho}(\mathcal{D}(\rho \preceq \tau)(a)) \subseteq F^{\varepsilon}_{\rho}(\overline{\mathcal{D}(\rho \preceq \sigma)(c)}) = F^{\varepsilon}_{\rho}(\mathcal{D}(\rho \preceq \sigma)(c))$$

for all $\rho \preceq \tau$. More concisely, $F^{\varepsilon}_{\tau\times a}(\rho) \subseteq F^{\varepsilon}_{\sigma\times c}(\rho)$ for all $\rho \preceq \tau$. As a consequence $\Delta F^{\varepsilon}_{\tau\times a} \subseteq \Delta F^{\varepsilon}_{\sigma\times c}$ and semicontinuity holds.

Next, notice that $F^{\varepsilon}\big(\Sigma^r\Delta\mathcal{D}(\sigma \times c)\big) = F^{\varepsilon}\big(\sigma \times \Sigma^r\mathcal{D}(\sigma)(c)\big) = \Delta F^{r,\varepsilon}_{\sigma\times c}$ which is an acyclic carrier. In order for F^{ε} to be an ε-acyclic carrier, it remains to show the inclusion $\Sigma^r\Delta\mathcal{L} \circ F^{\varepsilon} \subseteq F^{\varepsilon} \circ \Sigma^r\Delta\mathcal{D}$ for all $r \geq 0$. For this, take $\sigma \times c \in \Delta\mathcal{D}$ and see that

$$\Sigma^r \Delta\mathcal{L} \circ F^\varepsilon(\sigma \times c) = \Sigma^r \Delta\mathcal{L}\Big(\bigcup_{\tau\preceq\sigma} \tau \times F^\varepsilon_\tau\big(\mathcal{D}(\tau \preceq \sigma)(c)\big)\Big)$$
$$= \bigcup_{\tau\preceq\sigma} \tau \times \Sigma^r\mathcal{L}(\tau)\big(F^\varepsilon_\tau\big(\mathcal{D}(\tau \preceq \sigma)(c)\big)\big) \subseteq \bigcup_{\tau\preceq\sigma} \tau \times F^\varepsilon_\tau\big(\Sigma^r\mathcal{D}(\tau)\mathcal{D}(\tau \preceq \sigma)(c)\big)$$
$$= \bigcup_{\tau\preceq\sigma} \tau \times F^\varepsilon_\tau\big(\mathcal{D}(\tau \preceq \sigma)\Sigma^r\mathcal{D}(\sigma)(c)\big) = F^\varepsilon\big(\sigma \times \Sigma^r\mathcal{D}(\sigma)(c)\big) = F^\varepsilon \circ \Sigma^r\Delta\mathcal{D}(\sigma \times c)\ .$$

Similarly, one can define an ε-acyclic carrier $G^\varepsilon : \Delta\mathcal{L} \rightrightarrows \Delta\mathcal{D}$ sending $\sigma \times c \in \Delta\mathcal{L}$ to $\Delta G^\varepsilon_{\sigma\times c}$. In addition, we define respective shift ε-acyclic carriers $I^{2\varepsilon}_\mathcal{D} : \Delta\mathcal{D} \rightrightarrows \Delta\mathcal{D}$ and $I^{2\varepsilon}_\mathcal{L} : \Delta\mathcal{L} \rightrightarrows \Delta\mathcal{L}$, sending respectively $\sigma \times c \in \Delta\mathcal{D}$ to $\Delta(I^{2\varepsilon}_\mathcal{D})_{\sigma\times c}$ and $\tau \times a \in \Delta\mathcal{L}$ to $\Delta(I^{2\varepsilon}_\mathcal{L})_{\tau\times a}$. Then we have

$$G^\varepsilon \circ F^\varepsilon(\sigma \times c) = G^\varepsilon(\Delta F^\varepsilon_{\sigma\times c}) = G^\varepsilon\Big(\bigcup_{\tau\preceq\sigma} \tau \times F^\varepsilon_\tau\big(\mathcal{D}(\tau \preceq \sigma)(c)\big)\Big)$$
$$= \bigcup_{\rho\preceq\tau\preceq\sigma} \rho \times G^\varepsilon_\rho\Big(\mathcal{L}(\rho \preceq \tau)F^\varepsilon_\tau\big(\mathcal{D}(\tau \preceq \sigma)(c)\big)\Big)$$
$$\subseteq \bigcup_{\rho\preceq\sigma} \rho \times G^\varepsilon_\rho F^\varepsilon_\rho\big(\mathcal{D}(\rho \preceq \sigma)(c)\big) \subseteq \Delta(I^{2\varepsilon}_\mathcal{D})_{\sigma\times c} = I^{2\varepsilon}_\mathcal{D}(\sigma \times c),$$

where we have used the commutativity condition and equivalence of F^ε_ρ and G^ε_ρ. Consequently $G^\varepsilon \circ F^\varepsilon \subseteq I^{2\varepsilon}_\mathcal{D}$; the other inclusion $F^\varepsilon \circ G^\varepsilon \subseteq I^{2\varepsilon}_\mathcal{L}$ follows by symmetry. Altogether, we have obtained an ε-equivalence $F^\varepsilon : \Delta\mathcal{D} \rightrightarrows \Delta\mathcal{L}$. Finally, notice that for all $p \geq 0$ and for each cell $\sigma\times c \in F^p\Delta\mathcal{D}$, its carrier $\Delta F^\varepsilon_{\sigma\times c}$ is contained in $F^p\Delta\mathcal{D}$ and so it preserves filtration. The same follows for the other acyclic carriers. □

Let $X \in$ **FCW-cpx** together with a cover $\mathcal{U}$. Recall the definitions of the diagrams $X^\mathcal{U}$ and $\pi_0^\mathcal{U}$ over $N_\mathcal{U}$ from Example 2.9. Let $d_I\big(\mathrm{PH}_*(X^\mathcal{U}(\sigma)), \mathrm{PH}_*(\pi_0^\mathcal{U}(\sigma))\big) \leq \varepsilon$ for all $\sigma \in N_\mathcal{U}$. This example has been of interest before, see for Example [12] or [3]. As mentioned in Remark 4.9, our notion of ε-acyclicity is much stronger than that from [12]. This is why we obtain a result closer to the *Persistence Nerve Theorem* from [5] than to the *Approximate Nerve Theorem* from [12].

Given a diagram $\mathcal{D} \in \mathbf{FRDiag}(K)$, recall the diagram $\pi_0\mathcal{D}$ from Example 2.11. We may define an (ε, K)-acyclic carrier $\pi_0^\varepsilon\mathcal{D} : \mathcal{D} \rightrightarrows \pi_0\mathcal{D}$ where we send cells to their corresponding connected component classes. The compatibility condition $\pi_0(\mathcal{D}(\tau \preceq \sigma))(\pi_0^\varepsilon\mathcal{D}(\mathcal{D}(\sigma))) \subseteq \pi_0^\varepsilon\mathcal{D}(\mathcal{D}(\tau))$ also follows for any pair of simplices $\tau \preceq \sigma$ from K.

Corollary 5.3 (Strong Approximate Multinerve Theorem)**.** *Consider a diagram $\mathcal{D}$ in $\mathbf{FRDiag}(K)$. Assume that there is a (ε, K)-acyclic carrier $F^\varepsilon : \pi_0\mathcal{D} \rightrightarrows \mathcal{D}$ such that the composition $F^\varepsilon_\sigma \circ \pi_0^\varepsilon\mathcal{D}_\sigma$ carries the shift morphism $\Sigma^{2\varepsilon}\mathcal{D}_\sigma$ for all $\sigma \in K$. Then, there is an ε-acyclic equivalence $F^\varepsilon : \mathrm{MNerv}(\mathcal{D}) \rightrightarrows \Delta\mathcal{D}$. Consequently,*

$$d_I(\mathrm{PH}_*(\mathrm{MNerv}(\mathcal{D})), \mathrm{PH}_*(\Delta\mathcal{D})) \leq \varepsilon\ .$$

Proof. The shift $(2\varepsilon, K)$-carrier $I^{2\varepsilon}_{\pi_0\mathcal{D}}$ sends points to points, while the other $I^{2\varepsilon}_\mathcal{D}$ is defined as the composition $F^\varepsilon_\sigma \circ \pi_0^\varepsilon\mathcal{D}_\sigma$, which is a $(2\varepsilon, K)$-acyclic carrier by hypotheses. Thus, by Proposition 5.2 there exists an ε-acyclic equivalence $F^\varepsilon : \mathrm{MNerv}(\mathcal{D}) \rightrightarrows \Delta\mathcal{D}$. □

Example 5.4. Consider a filtered simplicial complex L_* together with a partition of its vertex set $\mathcal{P}$. Assume that the $(L_*, \mathcal{P})$-join diagram $\mathcal{J}^{L_*}_\mathcal{P}$ is such that there exists a (ε, K)-acyclic carrier $F^\varepsilon\colon \pi_0\mathcal{J}_{\mathcal{P}}^{L_*} \rightrightarrows \mathcal{J}_{\mathcal{P}}^{L_*}$ such that $F^\varepsilon_\sigma \circ \pi_0^\varepsilon\mathcal{J}_{\mathcal{P}}^{L_*}(\sigma)$ is a carrier for $\Sigma^{2\varepsilon}\mathcal{J}_{\mathcal{P}}^{L_*}(\sigma)$ for all $\sigma \in \Delta^P$. Then, by Corollary 5.3, there is an ε-acyclic equivalence $\Delta\pi_0\big(\mathcal{J}_{\mathcal{P}}^{L_*}\big) \rightrightarrows \Delta\mathcal{J}_{\mathcal{P}}^{L_*}$ so that

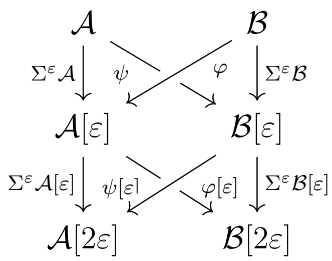
$$d_I\big(\mathrm{PH}_*(\mathrm{MNerv}(\mathcal{J}_{\mathcal{P}}^{L_*})), \mathrm{PH}_*(L_*)\big) \leq \varepsilon \ .$$

Acyclic carriers have been used in [14] and in [21] for approximating continuous morphisms by means of simplicial maps. Here we have used the same tools to obtain an approximate homotopy colimit theorem. The acyclic carrier theorem is an instance of the more general acyclic Model theorem, see [10, Sec. 2]. An interesting future research direction would be to see how that general result can bring new insights into applied topology.

6. Interleaving Spectral Sequences

Definition 6.1. Let $\mathcal{A}$ and $\mathcal{B}$ from **SpSq**. A n-spectral sequence morphism $f : \mathcal{A} \to \mathcal{B}$ is a spectral sequence morphism $f : \mathcal{A} \to \mathcal{B}$ which is defined from page n.

Definition 6.2. Given two objects $\mathcal{A}$ and $\mathcal{B}$ in **PSpSq**. We say that $\mathcal{A}$ and $\mathcal{B}$ are (ε, n)-interleaved whenever there exist two n-morphisms $\psi : \mathcal{A} \to \mathcal{B}[\varepsilon]$ and $\varphi : \mathcal{B} \to \mathcal{A}[\varepsilon]$ such that the following diagram commutes



(3)

for all pages $r \geq n$. This interleaving defines a pseudometric in **PSpSq**

$$d_I^n\big(\mathcal{A}, \mathcal{B}\big) \coloneqq \inf\big\{\varepsilon \mid \mathcal{A} \text{ and } \mathcal{B} \text{ are } (\varepsilon, n)\text{-interleaved}\big\} \ .$$

Proposition 6.3. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are (ε, n)-interleaved. Then these are (ε, m)-interleaved for all $m \geq n$. In particular, we have that*

$$d_I^m\big(\mathcal{A}, \mathcal{B}\big) \leq d_I^n\big(\mathcal{A}, \mathcal{B}\big)$$

for any pair of integers $m \geq n$.

Proof. Follows directly from the definitions. □

We start now by considering Mayer-Vietoris spectral sequences. Under some conditions which are a special case of Theorem 5.2, one can obtain one-page stability. In fact, this stability is due to morphisms directly defined on the underlying double complexes, which is a very strong property.

Proposition 6.4. *Let X and Y be two tame elements in* **FCW-cpx** *together with a pair of respective finite covers $\mathcal{U}$ and $\mathcal{V}$ by subcomplexes so that $K = N_{\mathcal{U}} = N_{\mathcal{V}}$. Suppose that there are (ε, K)-acyclic carriers $F^\varepsilon : X^{\mathcal{U}} \rightrightarrows Y^{\mathcal{V}}$ and $G^\varepsilon : Y^{\mathcal{V}} \rightrightarrows X^{\mathcal{U}}$, together with a pair of shift (ε, K)-acyclic carriers $I^{2\varepsilon}_{X^{\mathcal{U}}} : X^{\mathcal{U}} \rightrightarrows X^{\mathcal{U}}$ and $I^{2\varepsilon}_{Y^{\mathcal{V}}} : Y^{\mathcal{V}} \rightrightarrows Y^{\mathcal{V}}$, and such that these restrict to acyclic equivalences*

$$G^\varepsilon_\tau \circ F^\varepsilon_\tau \subseteq (I^{2\varepsilon}_{X^{\mathcal{U}}})_\tau \text{ and } F^\varepsilon_\tau \circ G^\varepsilon_\tau \subseteq (I^{2\varepsilon}_{Y^{\mathcal{V}}})_\tau$$

for each simplex $\tau \in K$. Then there are a pair of double complex morphisms $\phi^\varepsilon : C_{*,*}(X,\mathcal{U}) \to C_{*,*}(Y,\mathcal{V})[\varepsilon]$ and $\psi^\varepsilon : C_{*,*}(Y,\mathcal{V}) \to C_{*,*}(X,\mathcal{U})[\varepsilon]$ inducing a first page interleaving between $E^*_{*,*}(X,\mathcal{U})$ and $E^*_{*,*}(Y,\mathcal{V})$.

Proof. Unpacking the definitions this means we have to give chain homomorphisms

$$(\phi^\varepsilon_\sigma)_r \colon C_*(X^{\mathcal{U}}(\sigma)_r) \to C_*(Y^{\mathcal{V}}(\sigma)_{r+\varepsilon}) \ ,$$
$$(\psi^\varepsilon_\sigma)_r \colon C_*(Y^{\mathcal{V}}(\sigma)_r) \to C_*(X^{\mathcal{U}}(\sigma)_{r+\varepsilon})$$

that are natural in $\sigma \in K$ and in $r \in \mathbf{R}$. Since K is a post category, these can be constructed inductively as follows: As in Proposition 4.2 we may define ϕ^ε_σ on all simplices $\sigma \in K$ of dimension $\dim(\sigma) = \dim(K)$. Note that $(\phi^\varepsilon_\sigma)_r$ is carried by $(F^\varepsilon_\sigma)_r$ for all $r \in \mathbf{R}$. Assume by (reverse) induction that ϕ^ε_τ are defined and carried by F^ε_τ for all $\tau \in K$ with $n \leq \dim(\tau) \leq \dim(K)$ in such a way that for all cofaces $\tau \preceq \sigma$ the naturality condition $\phi^\varepsilon_\tau \circ X^{\mathcal{U}}(\tau \prec \sigma) = Y^{\mathcal{V}}(\tau \prec \sigma)[\varepsilon] \circ \phi^\varepsilon_\sigma$ holds. Now let $\tau \in K$ have dimension $\dim(\tau) = n - 1 \geq 0$. The naturality condition on the simplices fixes ϕ^ε_τ on the filtered subcomplex $X^\tau = \bigcup_{\tau \prec \sigma} \mathrm{Im}\left(X^{\mathcal{U}}(\tau \prec \sigma)\right)$, where the union is taken over all cofaces σ of τ. Here notice that we can assume that ϕ^ε_τ is well-defined since the previous choices of ϕ^ε_σ for all cofaces $\tau \prec \sigma$ are consistent due to the fact that for each cell $c \in X^\tau$ there exists a unique maximal simplex $\sigma \in N_{\mathcal{U}}$ such that $c \in X^{\mathcal{U}}(\sigma)$. In addition, notice that by hypotheses $Y^{\mathcal{V}}(\tau \prec \sigma)((F^\varepsilon_\sigma)(c)) \subseteq F^\varepsilon_\tau(X^{\mathcal{U}}(\tau \prec \sigma)(c))$ for all $a \in \mathbf{R}$ and $c \in X^{\mathcal{U}}(\sigma)$, so that our definition of ϕ^ε_τ in X^τ is indeed carried by F^ε_τ. We then proceed as in Proposition 4.2 to define $(\phi^\varepsilon_\tau)_a$ on all simplices in the subset $X^{\mathcal{U}}(\tau)_a \setminus X^\tau_a$ for all $a \in \mathbf{R}$. The resulting chain map $(\phi^\varepsilon_\tau)_a$ is carried by $(F^\varepsilon_\tau)_a$ for all $a \in \mathbf{R}$. Since $X^{\mathcal{U}}$ is tame, we only need finitely many steps to obtain a morphism $\phi^\varepsilon_\tau : C_*(X^{\mathcal{U}}(\tau)) \to C_*(Y^{\mathcal{V}}(\tau)[\varepsilon])$ that satisfies the induction hypotheses.

Thus, we obtain double complex morphisms $\phi^\varepsilon_{p,q} : C_{p,q}(X,\mathcal{U}) \to C_{p,q}(Y,\mathcal{V})[\varepsilon]$ for all $p, q \geq 0$ by adding up our defined local morphisms

$$\phi^\varepsilon_{p,q} : \bigoplus_{\sigma \in K^p} \phi^\varepsilon_\sigma : \bigoplus_{\sigma \in K^p} C_q(X^{\mathcal{U}}(\sigma)) \longrightarrow \bigoplus_{\sigma \in K^p} C_q(Y^{\mathcal{V}}(\sigma))[\varepsilon] \ .$$

Notice that $\phi^\varepsilon_{p,q}$ commute both with horizontal and vertical differentials since we assumed that each ϕ^ε_σ is a chain morphism and these satisfy a naturality condition with respect to K. Thus, this double complex morphism induces a spectral sequence morphism $\phi^\varepsilon_{p,q} : E^*_{p,q}(X^{\mathcal{U}}) \to E^*_{p,q}(Y^{\mathcal{V}})[\varepsilon]$. By doing the same construction, we can obtain local chain morphisms $\psi^\varepsilon_\sigma : C_*(Y^{\mathcal{V}}(\sigma)) \to C_*(X^{\mathcal{U}}(\sigma))[\varepsilon]$ so that by Proposition 4.2 we have equalities $[\psi^\varepsilon_\sigma] \circ [\phi^\varepsilon_\sigma] = [\Sigma^{2\varepsilon} C_*(X^{\mathcal{U}}(\sigma))]$ and also $[\phi^\varepsilon_\sigma] \circ [\psi^\varepsilon_\sigma] = [\Sigma^{2\varepsilon} C_*(Y^{\mathcal{V}}(\sigma))]$ for all $\sigma \in K$. Then we can construct a double complex morphism $\psi^\varepsilon_{p,q} : C_{p,q}(Y,\mathcal{V}) \to C_{p,q}(X,\mathcal{U})[\varepsilon]$ inducing an "inverse" spectral sequence morphism $\psi^\varepsilon_{p,q} : E^*_{p,q}(Y,\mathcal{V}) \to E^*_{p,q}(X,\mathcal{U})[\varepsilon]$. These are such that from the first page, $\phi^\varepsilon_{*,*}$ and $\psi^\varepsilon_{*,*}$ form a $(\varepsilon, 1)$-interleaving of spectral sequences. ☐

Notice that the proof of Proposition 6.4 relies heavily on the fact that the diagrams we are considering come from a cover. This allows us to define a pair of double complex morphisms that are compatible along the common indexing nerve. However, in Theorem 5.2 we observed that, under some conditions, the geometric realizations of regularly filtered regular diagrams are stable. Does this stability carry over to the associated spectral sequences? The next theorem shows that this is indeed the case.

Theorem 6.5. *Let $\mathcal{D}$ and $\mathcal{L}$ be two diagrams in $\mathbf{RRDiag}(K)$. Suppose that there are (ε, K)-acyclic carriers $F^\varepsilon : \mathcal{D} \rightrightarrows \mathcal{L}$ and $G^\varepsilon : \mathcal{L} \rightrightarrows \mathcal{D}$, together with a pair of shift*

(ε, K)-acyclic carriers $I^{2\varepsilon}_{\mathcal{D}} : \mathcal{D} \rightrightarrows \mathcal{D}$ and $I^{2\varepsilon}_{\mathcal{L}} : \mathcal{L} \rightrightarrows \mathcal{L}$, and such that these restrict to acyclic equivalences

$$G^{\varepsilon}_{\tau} \circ F^{\varepsilon}_{\tau} \subseteq (I^{2\varepsilon}_{\mathcal{D}})_{\tau} \text{ and } F^{\varepsilon}_{\tau} \circ G^{\varepsilon}_{\tau} \subseteq (I^{2\varepsilon}_{\mathcal{L}})_{\tau}$$

for each simplex $\tau \in K$. Then

$$d^1_I(E(\mathcal{D}, K), E(\mathcal{L}, K)) \leq \varepsilon\,.$$

Proof. Recall from Theorem 5.2 that there is a filtration-preserving ε-acyclic carrier $F^{\varepsilon} : \Delta_K\mathcal{D} \rightrightarrows \Delta_K\mathcal{L}$. Given $r \in \mathbf{R}$, this implies that there is a chain complex morphism $f^{\varepsilon}_r : C_*(\Delta\mathcal{D})_r \to C_*(\Delta\mathcal{L})_{r+\varepsilon}$ carried by F^{ε}_r and which respects filtrations in the sense that $f^{\varepsilon}_r(F^pC_*(\Delta\mathcal{D})_r) \subseteq F^pC_*(\Delta\mathcal{L})_{r+\varepsilon}$ for all $p \geq 0$. By Lemma 3.1 this defines a morphism $f^{\varepsilon}_r : S^{\mathrm{Tot}}_*(\mathcal{D})_r \to S^{\mathrm{Tot}}_*(\mathcal{L})_{r+\varepsilon}$ which respects filtrations. Altogether we deduce that f^{ε}_r determines a morphism of spectral sequences $f^{\varepsilon}_r : E^*_{p,q}(\mathcal{D})_r \to E^*_{p,q}(\mathcal{L})_{r+\varepsilon}$. Similarly as in Lemma 4.7 the commutativity

$$\Sigma^s E^*_{p,q}(\mathcal{L})_{r+\varepsilon} \circ f^{\varepsilon}_r = f^{\varepsilon}_{r+s} \circ \Sigma^s E^*_{p,q}(\mathcal{D})_r \tag{4}$$

does not need to hold for all $r \in \mathbf{R}$ and all $s \geq 0$. However, by definition of ε-acyclic carrier, there is an inclusion $\Sigma^s\Delta\mathcal{L} \circ F^{\varepsilon} \subseteq F^{\varepsilon} \circ \Sigma^s\Delta\mathcal{D}$ where the superset is acyclic, so that $\Sigma^sC_*(\Delta\mathcal{L})_{r+\varepsilon} \circ f^{\varepsilon}_r$ and $f^{\varepsilon}_{r+s} \circ \Sigma^sC_*(\Delta\mathcal{D})_r$ are both carried by the filtration preserving acyclic carrier $F^{\varepsilon} \circ \Sigma^s\Delta\mathcal{D}_r$. This implies that there exist chain homotopies $h^{\varepsilon}_r : C_n(\Delta\mathcal{D})_r \to C_{n+1}(\Delta\mathcal{L})_{r+s+\varepsilon}$ which respect filtrations and such that

$$f^{\varepsilon}_{r+s} \circ \Sigma^sC_*(\Delta\mathcal{D})_r - \Sigma^sC_*(\Delta\mathcal{L})_{r+\varepsilon} \circ f^{\varepsilon}_r = \delta^{\Delta} \circ h^{\varepsilon}_r + h^{\varepsilon}_r \circ \delta^{\Delta}\ .$$

for all $r \in \mathbf{R}$ and all $s \geq 0$. Recall that the zero page terms are given as quotients on successive filtration terms $E^0_{p,q}(\mathcal{D})_r = F^pS^{\mathrm{Tot}}_{p+q}(\mathcal{D})_r/F^{p-1}S^{\mathrm{Tot}}_{p+q}(\mathcal{D})_r$, for all $r \in \mathbf{R}$ and all integers $p, q \geq 0$. Thus, by Lemma 3.1, these chain homotopies carry over to $S^{\mathrm{Tot}}_*(\mathcal{D})_r$ and the commutativity relation from equation (4) holds from the first page onwards.

Similarly, we can define spectral sequence morphisms $g^{\varepsilon}_r : E^*_{p,q}(\mathcal{L})_r \to E^*_{p,q}(\mathcal{D})_{r+\varepsilon}$ for all $r \in \mathbf{R}$ which commute with the shift morphisms from the first page. Also, by inspecting the shift carriers, we can obtain equalities of 1-spectral sequence morphisms $g^{\varepsilon}_{r+\varepsilon} \circ f^{\varepsilon}_r = \Sigma^{2\varepsilon}E^*_{p,q}(\mathcal{D})_r$ and also $f^{\varepsilon}_{r+\varepsilon} \circ g^{\varepsilon}_r = \Sigma^{2\varepsilon}E^*_{p,q}(\mathcal{L})_r$ for all $r \in \mathbf{R}$, and the result follows. □

Example 6.6. Consider a pair of point clouds $\mathbb{X}, \mathbb{Y} \in \mathbb{R}^N$, together with partitions $\mathcal{P}$ and $\mathcal{Q}$ for $\mathbb{X}$ and $\mathbb{Y}$ respectively. Also, assume that there is an isomorphism $\phi : \Delta^{\mathcal{P}} \to \Delta^{\mathcal{Q}}$ such that $d_H(\mathbb{X} \cap V, \mathbb{Y} \cap \phi(V)) < \varepsilon$ for all $V \in \mathcal{P}$. As defined in Example 4.5, there are ε-acyclic carrier equivalences $F^{\varepsilon}_V : \mathrm{VR}_*(\mathbb{X}\cap V) \rightrightarrows \mathrm{VR}_*(\mathbb{Y}\cap V)$ for all $V \in \mathcal{U}$. Now suppose that, for some $\eta > 0$, if $\mathcal{J}^{\mathrm{VR}_*(\mathbb{X})}_{\mathcal{P}}(\sigma)_r \neq \emptyset$ then $\mathcal{J}^{\mathrm{VR}_*(\mathbb{Y})}_{\mathcal{P}}(\phi(\sigma))_{r+\eta} \neq \emptyset$ for all $\sigma \in \Delta^{\mathcal{P}}$ and all $r \in \mathbf{R}$. For any $\sigma \in \Delta^{\mathcal{P}}$, one can define $(\varepsilon + \eta)$-acyclic carriers $\widetilde{F}^{(\varepsilon+\eta)}_{\sigma} : \mathcal{J}^{\mathrm{VR}_*(\mathbb{X})}_{\mathcal{P}}(\sigma) \rightrightarrows \mathcal{J}^{\mathrm{VR}_*(\mathbb{Y})}_{\mathcal{Q}}(\sigma)$ by sending a cell $\prod_{V\in\sigma} \tau_V \in \mathcal{J}^{\mathrm{VR}_*(\mathbb{X})}_{\mathcal{P}}(\sigma)_r$ to $\prod_{V\in\sigma} \Sigma^{\eta}\mathrm{VR}_*(\mathbb{Y} \cap V)\big(F^{\varepsilon}_V(\tau_V)\big) \in \mathcal{J}^{\mathrm{VR}_*(\mathbb{Y})}_{\mathcal{Q}}(\sigma)_{r+(\varepsilon+\eta)}$ for all $r \in \mathbf{R}$. Similarly, we assume the converse that $\mathcal{J}^{\mathrm{VR}_*(\mathbb{Y})}_{\mathcal{P}}(\widetilde{\sigma})_r \neq \emptyset$ implies $\mathcal{J}^{\mathrm{VR}_*(\mathbb{X})}_{\mathcal{P}}(\phi^{-1}(\widetilde{\sigma}))_{r+\eta} \neq \emptyset$ for all $\widetilde{\sigma} \in \Delta^{\mathcal{Q}}$ and all $r \in \mathbf{R}$. With an analogous definition to that of $\widetilde{F}^{(\varepsilon+\eta)}_{\sigma}$, we obtain 'inverses' for the carriers $\widetilde{F}^{(\varepsilon+\eta)}_{\sigma}$, so that these become $(\varepsilon + \eta)$-acyclic equivalences. One can check that these are compatible along $\Delta^{\mathcal{P}}$ and $\Delta^{\mathcal{Q}}$, so that by Theorem 6.5 $d^1_I\big(E^*_{*,*}(\mathcal{J}^{\mathrm{VR}_*(\mathbb{X})}_{\mathcal{P}}, \Delta^{\mathcal{P}}), E^*_{*,*}(\mathcal{J}^{\mathrm{VR}_*(\mathbb{Y})}_{\mathcal{Q}}, \Delta^{\mathcal{Q}})\big) \leq \varepsilon + \eta$.

7. Interleavings with respect to different covers

7.1. **Refinement Induced Interleavings.** In the previous sections we considered general diagrams in $\mathbf{FRDiag}(K)$ for some simplicial complex K. We now focus on the situation where we have a filtered complex X together with a cover $\mathcal{U}$, which provides a diagram $X^{\mathcal{U}} : N_{\mathcal{U}} \to \mathbf{FCW\text{-}cpx}$. The associated spectral sequence is denoted by $E^*_{*,*}(X,\mathcal{U})$, as done at the start of section 3. We want to measure how $E^*_{*,*}(X,\mathcal{U})$ changes depending on $\mathcal{U}$ and follow ideas from [24] to achieve this. First we consider a refinement $\mathcal{V} \prec \mathcal{U}$, which means that for all $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subseteq U$. In particular, one can choose a morphism $\rho^{\mathcal{U},\mathcal{V}} : N_{\mathcal{V}} \to N_{\mathcal{U}}$ such that $\mathcal{V}_\sigma \subseteq \mathcal{U}_{\rho\sigma}$ for all $\sigma \in N_{\mathcal{V}}$. This choice is of course not necessarily unique. We would like to compare the Mayer-Vietoris spectral sequences of both covers. For this, we recall the definition of the Čech chain complex outlined in the introduction of [26], which leads to the following isomorphism on the terms from the 0-page

$$E^0_{p,q}(X,\mathcal{U}) = \check{\mathcal{C}}_p(\mathcal{U}; C^{\mathrm{cell}}_q) := \bigoplus_{\sigma \in N^p_{\mathcal{U}}} C^{\mathrm{cell}}_q(\mathcal{U}_\sigma) \simeq \bigoplus_{s \in X^q} f^{\sigma(s,\mathcal{U})}_* \Big(C^{\mathrm{cell}}_p\big(\Delta^{\sigma(s,\mathcal{U})}\big)\Big) . \quad (5)$$

Here, $\sigma(s,\mathcal{U})$ is the simplex of maximal dimension in $N_{\mathcal{U}}$ such that $s \in X^{\mathcal{U}}(\sigma(s,\mathcal{U}))$, and $f^{\sigma(s,\mathcal{U})} \colon \Delta^{\sigma(s,\mathcal{U})} \hookrightarrow N_{\mathcal{U}}$ denotes the inclusion. The isomorphism in (5) is given by sending a generator $(a)_\sigma \in \bigoplus_{\sigma \in N^p_{\mathcal{U}}} C^{\mathrm{cell}}_q(\mathcal{U}_\sigma)$ to its transpose $(\sigma)_a$, for all cells $a \in X$ and all $\sigma \in N_{\mathcal{U}}$.

Returning to a refinement $\mathcal{V} \prec \mathcal{U}$ and a morphism $\rho^{\mathcal{U},\mathcal{V}} : N_{\mathcal{V}} \to N_{\mathcal{U}}$, there is an induced double complex morphism $\rho^{\mathcal{U},\mathcal{V}}_{p,q} : C_{p,q}(X,\mathcal{V}) \to C_{p,q}(X,\mathcal{U})$ given by

$$\rho^{\mathcal{U},\mathcal{V}}_{p,q}((\sigma)_a) = \begin{cases} (\rho^{\mathcal{U},\mathcal{V}}\sigma)_a & \text{if } \dim(\rho^{\mathcal{U},\mathcal{V}}\sigma) = p, \\ 0 & \text{otherwise,} \end{cases}$$

for all generators $(\sigma)_a \in C_{p,q}(X,\mathcal{V})$ with $\sigma \in N^p_{\mathcal{V}}$ and $a \in X^q$.

Lemma 7.1. *$\rho^{\mathcal{U},\mathcal{V}}_{*,*}$ is a morphism of double complexes. Thus, it induces a morphism of spectral sequences*

$$\rho^{\mathcal{U},\mathcal{V}}_{p,q} : E^*_{p,q}(X,\mathcal{V}) \to E^*_{p,q}(X,\mathcal{U})$$

dependent on the choice of $\rho^{\mathcal{U},\mathcal{V}}$.

Proof. Let $\delta^{\mathcal{V}}$ and $\delta^{\mathcal{U}}$ denote the respective Čech differentials from $\check{\mathcal{C}}_p(\mathcal{V}; C^{\mathrm{cell}}_q)$ and $\check{\mathcal{C}}_p(\mathcal{U}; C^{\mathrm{cell}}_q)$. The refinement $\rho^{\mathcal{U},\mathcal{V}} : N_{\mathcal{V}} \to N_{\mathcal{U}}$ induces a chain morphism $\rho^{\mathcal{U},\mathcal{V}}_* : C^{\mathrm{cell}}_*(N_{\mathcal{V}}) \to C^{\mathrm{cell}}_*(N_{\mathcal{U}})$, so that we have commutativity $\rho^{\mathcal{U},\mathcal{V}}_{*,*} \circ \delta^{\mathcal{V}} = \delta^{\mathcal{U}} \circ \rho^{\mathcal{U},\mathcal{V}}_{*,*}$. This implies that $\rho^{\mathcal{U},\mathcal{V}}_{*,*}$ commutes with the horizontal differential d^H. For commutativity with d^V, we consider a generating chain $(\sigma)_a \in E^0_{p,q}(X,\mathcal{V})$ with $\sigma \in N^p_{\mathcal{V}}$ and $a \in X^q$. Then, if $\dim(\rho^{\mathcal{U},\mathcal{V}}\sigma) = p$, we have

$$\begin{aligned} \rho^{\mathcal{U},\mathcal{V}}_{p,q-1} \circ d^V\big((\sigma)_a\big) = \rho^{\mathcal{U},\mathcal{V}}_{p,q-1}\Big((-1)^p \sum_{b \preceq \overline{a}} ([b:a]\sigma)_b\Big) &= (-1)^p \sum_{b \preceq \overline{a}} \big([b:a]\rho^{\mathcal{U},\mathcal{V}}\sigma\big)_b \\ &= (-1)^p d^{\mathrm{cell}}_q\big((\rho^{\mathcal{U},\mathcal{V}}\sigma)_a\big) = d^V \circ \rho^{\mathcal{U},\mathcal{V}}_{p,q}\big((\sigma)_a\big) \end{aligned}$$

and for $\dim(\rho^{\mathcal{U},\mathcal{V}}\sigma) < p$ commutativity follows since both terms vanish.

A morphism of double complexes gives rise to a morphism of the vertical filtration. By [19, Theorem 3.5] this induces a morphism of spectral sequences $\rho^{\mathcal{U},\mathcal{V}}_{*,*}$. □

Since $\rho^{\mathcal{U},\mathcal{V}} : N_{\mathcal{V}} \to N_{\mathcal{U}}$ is not unique, the induced morphism $\rho^{\mathcal{U},\mathcal{V}}_{*,*}$ on the 0-page does not need to be unique either. We have, however, the following:

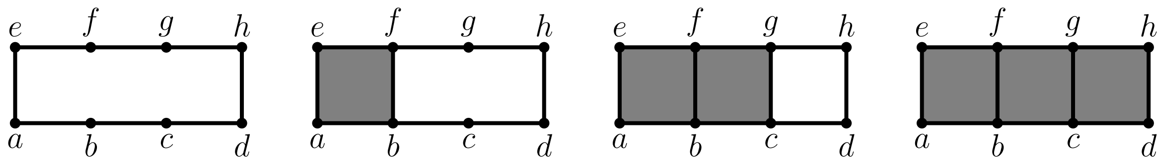


FIGURE 2. Cubical complex $\mathcal{C}_*$ at values 0,1,2 and 3.

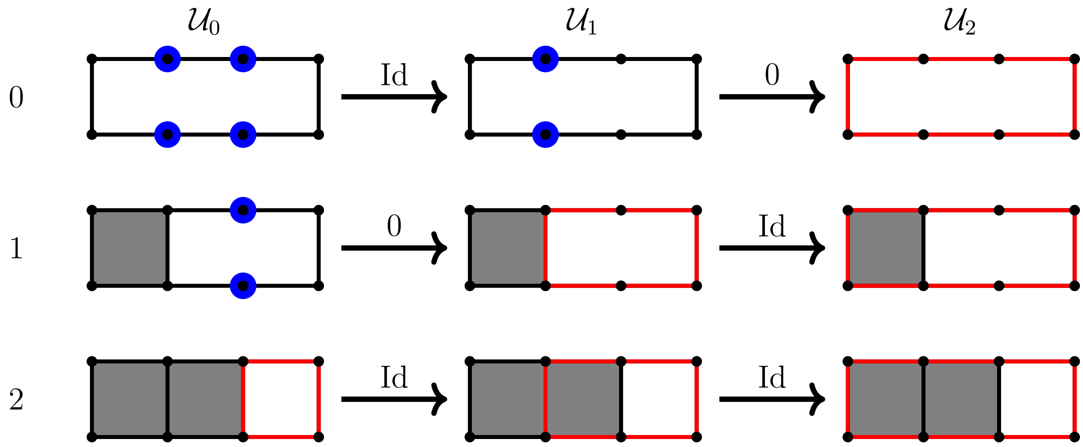


FIGURE 3. Cubical complex $\mathcal{C}_*$ with covers $\mathcal{U}_0$, $\mathcal{U}_1$ and $\mathcal{U}_2$, and with filtration values 0,1 and 2. Blue dots represent classes in $E^2_{1,0}(\mathcal{C},\mathcal{U}_i)$ and red loops represent classes on $E^2_{0,1}(\mathcal{C},\mathcal{U}_i)$, for $i = 0, 1, 2$.

Proposition 7.2. *The 2-morphism obtained by restricting $\rho^{\mathcal{U},\mathcal{V}}_{*,*}$ is independent of the particular choice of refinement map $\rho^{\mathcal{U},\mathcal{V}} : N_{\mathcal{V}} \to N_{\mathcal{U}}$.*

Proof. We have to show that $\rho^{\mathcal{U},\mathcal{V}}_{*,*}$ is independent of the particular choice of the refinement morphism. First, define a carrier $R : N_{\mathcal{V}} \rightrightarrows N_{\mathcal{U}}$ by the assignment

$$\sigma \mapsto R(\sigma) = \left\{\nu \in N_{\mathcal{U}} \mid V_\sigma \subseteq U_\nu\right\} .$$

The geometric realization of the subcomplex $R(\sigma)$ is homeomorphic to a standard simplex, in particular contractible, so R is acyclic. Note that $\rho^{\mathcal{U},\mathcal{V}}_{*,*}$ is carried by R. Hence, by Theorem 2.4 for any pair of refinement maps $\rho^{\mathcal{U},\mathcal{V}}, \tau^{\mathcal{U},\mathcal{V}} : N_{\mathcal{V}} \to N_{\mathcal{U}}$, there exists a chain homotopy $k_* : C_n(N_{\mathcal{V}}) \to C_{n+1}(N_{\mathcal{U}})$ carried by R, so that

$$k_*\delta^{\mathcal{V}} + \delta^{\mathcal{U}}k_* = \tau^{\mathcal{U},\mathcal{V}}_* - \rho^{\mathcal{U},\mathcal{V}}_*$$

for all $n \geq 0$ and where $\tau^{\mathcal{U},\mathcal{V}}_*$ and $\rho^{\mathcal{U},\mathcal{V}}_*$ are induced morphisms of chain complexes $C_*(N_{\mathcal{V}}) \to C_*(N_{\mathcal{U}})$. In particular, using the same notation, this translates into chain homotopies $k_* : E^0_{p,q}(X,\mathcal{V}) \to E^0_{p+1,q}(X,\mathcal{U})$ on the 0-page such that

$$k_*\delta^{\mathcal{V}} + \delta^{\mathcal{U}}k_* = \tau^{\mathcal{U},\mathcal{V}}_{*,*} - \rho^{\mathcal{U},\mathcal{V}}_{*,*}$$

Thus, $\tau^{\mathcal{U},\mathcal{V}}_{*,*} = \rho^{\mathcal{U},\mathcal{V}}_{*,*}$ from the second page onward. □

Example 7.3. Consider a filtered cubical complex $\mathcal{C}_*$. At value 0, $\mathcal{C}_*$ is given by the vertices on $\mathcal{R}^2$ at the coordinates $a = (0,0), b = (1,0), c = (2,0), d = (3,0), e = (0,1), f = (1,1), g = (2,1), h = (3,1)$, together with all edges contained in the boundary of the rectangle $adhe$. Then, at value 1 there appears the edge bf with the face $abfe$. At value 2 the edge gc with the face $fgcb$, and finally at value 3 the face *ghdc* appears. This is depicted on Figure 2. Then, consider the cover $\mathcal{U}_0$ by three subcomplexes on the squares $A = (a, b, f, e)$, $B = (b, c, g, f)$ and $C = (c, d, h, g)$. Also, we consider the cover $\mathcal{U}_1$ given by A and $C \cup B$, and $\mathcal{U}_2$ given by all $\mathcal{C}_*$. The induced morphisms on second-page terms at different filtration values are either null or the identity, as illustrated on Figure 3.

A consequence of Proposition 7.2 is that if we have a space X together with covers $\mathcal{U} \prec \mathcal{V} \prec \mathcal{U}$, then by uniqueness the morphism on the second page induced by the consecutive inclusions coincides with the identity. This gives rise to the next result.

Proposition 7.4. *Suppose a pair of covers $\mathcal{U}$ and $\mathcal{V}$ of X are a refinement of one another. Then there is a 2-spectral sequence isomorphism $E^2_{*,*}(X,\mathcal{U}) \simeq E^2(X,\mathcal{V})$.*

This corollary implies that for any cover $\mathcal{U}$ of X, the cover $\mathcal{U} \cup X$ obtained by adding the extra covering element X is such that the second page $E^2_{p,q}(X, \mathcal{U} \cup X)$ has only the first column nonzero.

Lemma 7.5. *Consider a cover $\mathcal{U}$ of a space X, and suppose that $X \in \mathcal{U}$. Then $E^2_{p,q}(X,\mathcal{U}) = 0$ for all $p > 0$.*

Proof. This follows from the observation that the cover $\{X\}$ consisting of a single element satisfies $\{X\} \prec \mathcal{U} \prec \{X\}$. Using Proposition 7.4 we therefore obtain isomorphisms $E^2_{p,q}(X,\mathcal{U}) \simeq E^2_{p,q}(X, \{X\})$, and the result follows. □

Suppose that none of the two covers $\mathcal{V}$ and $\mathcal{U}$ refines the other. One can still compare them using the common refinement $\mathcal{V} \cap \mathcal{U} = \big\{V \cap U\big\}_{V \in \mathcal{V}, U \in \mathcal{U}}$ which is a cover of X. Thus, there are two refinement morphisms

$$E^2_{p,q}(X,\mathcal{U}) \xleftarrow{\rho^{\mathcal{U},\mathcal{V}\cap\mathcal{U}}_{p,q}} E^2_{p,q}(X, \mathcal{V} \cap \mathcal{U}) \xrightarrow{\rho^{\mathcal{V},\mathcal{V}\cap\mathcal{U}}_{p,q}} E^2_{p,q}(X,\mathcal{V}). \tag{6}$$

Following [24, Sec. 28] we can now build the double complex $C_{p,q}(\mathcal{V}, \mathcal{U}, \mathrm{PH}_k)$ which, for each $k \geq 0$, is given by

$$\begin{array}{ccc}
\displaystyle\bigoplus_{\substack{\sigma \in N^{p+1}_{\mathcal{V}} \\ \tau \in N^{q}_{\mathcal{U}}}} \mathrm{PH}_k(\mathcal{V}_\sigma \cap \mathcal{U}_\tau) & \xleftarrow{(-1)^{p+1}\delta^{\mathcal{U}}} & \displaystyle\bigoplus_{\substack{\sigma \in N^{p+1}_{\mathcal{V}} \\ \tau \in N^{q+1}_{\mathcal{U}}}} \mathrm{PH}_k(\mathcal{V}_\sigma \cap \mathcal{U}_\tau) \\
\Big\downarrow \delta^{\mathcal{V}} & & \Big\downarrow \delta^{\mathcal{V}} \\
\displaystyle\bigoplus_{\substack{\sigma \in N^{p}_{\mathcal{V}} \\ \tau \in N^{q}_{\mathcal{U}}}} \mathrm{PH}_k(\mathcal{V}_\sigma \cap \mathcal{U}_\tau) & \xleftarrow{(-1)^{p}\delta^{\mathcal{U}}} & \displaystyle\bigoplus_{\substack{\sigma \in N^{p}_{\mathcal{V}} \\ \tau \in N^{q+1}_{\mathcal{U}}}} \mathrm{PH}_k(\mathcal{V}_\sigma \cap \mathcal{U}_\tau)
\end{array}$$

for any pair of integers $p, q \geq 0$. From this double complex we can study the two associated spectral sequences

$$^{\mathrm{I}}E^1_{p,q}(\mathcal{V}, \mathcal{U}; \mathrm{PH}_k) = \bigoplus_{\sigma \in N^p_{\mathcal{V}}} \check{\mathcal{H}}_q\big(\mathcal{V}_\sigma \cap \mathcal{U}; \mathrm{PH}_k\big) \ ,$$

$$^{\mathrm{II}}E^1_{p,q}(\mathcal{V}, \mathcal{U}; \mathrm{PH}_k) = \bigoplus_{\tau \in N^q_{\mathcal{U}}} \check{\mathcal{H}}_p\big(\mathcal{V} \cap \mathcal{U}_\tau; \mathrm{PH}_k\big) \ ,$$

whose common target of convergence is $\check{\mathcal{H}}_n(\mathcal{V} \cap \mathcal{U}; \mathrm{PH}_k)$ with $p + q = n$. For details about the spectral sequence associated to a double complex, the reader is recommended to look at [19, Theorem 2.15].

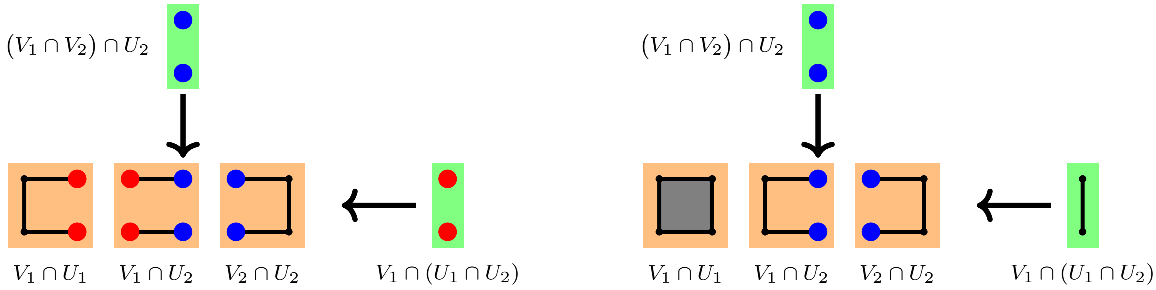


FIGURE 4. $C_{p,q}(\mathcal{V}, \mathcal{U}, \mathrm{PH}_k)$ at filtration values 0 and 1.

Example 7.6. Consider the cubical complex $\mathcal{C}_*$ from Example 7.3. Set $\mathcal{U} = \mathcal{U}_1$, that is, $\mathcal{U}$ is the cover by the sets $U_1 = A$ and $U_2 = B \cup C$. On the other hand, consider $\mathcal{V}$ to be formed of $V_1 = A \cup B$ and $V_2 = C$. The double complex $C_{p,q}(\mathcal{V}, \mathcal{U}, \mathrm{PH}_k)$ is illustrated on Figure 4 for filtration values 0 and 1, and for $k = 0$. We encourage the reader to work out the refinement morphisms from (6) and see that these are actually projections.

Consider the nerve $N_{\mathcal{V}\cap\mathcal{U}}$ as a subset of the product of nerves $N_{\mathcal{V}} \times N_{\mathcal{U}}$. We have then two projections $\pi^{\mathcal{V}} : N_{\mathcal{V}\cap\mathcal{U}} \to N_{\mathcal{V}}$ and $\pi^{\mathcal{U}} : N_{\mathcal{V}\cap\mathcal{U}} \to N_{\mathcal{U}}$, both of which induce chain morphisms $\pi^{\mathcal{V}}_* : C_*(N_{\mathcal{V}\cap\mathcal{U}}) \to C_*(N_{\mathcal{V}})$ and $\pi^{\mathcal{U}}_* : C_*(N_{\mathcal{V}\cap\mathcal{U}}) \to C_*(N_{\mathcal{U}})$. For example, $\pi^{\mathcal{V}}_*$ is given by $\pi^{\mathcal{V}}_*(\sigma \times \tau) = \sigma$ if $\dim(\tau) = 0$ or $\pi^{\mathcal{V}}_*(\sigma \times \tau) = 0$ otherwise, for all $\sigma \in N_{\mathcal{V}}, \tau \in N_{\mathcal{U}}$. These induce a pair of morphisms

$$\bigoplus_{\sigma \in N^p_{\mathcal{V}}} C^{\mathrm{cell}}_k(\mathcal{V}_\sigma) \xleftarrow{\pi^{\mathcal{V}}_{p,k}} \bigoplus_{\substack{\sigma \in N^p_{\mathcal{V}} \\ \tau \in N^q_{\mathcal{U}}}} C^{\mathrm{cell}}_k(\mathcal{V}_\sigma \cap \mathcal{U}_\tau) \xrightarrow{\pi^{\mathcal{U}}_{q,k}} \bigoplus_{\tau \in N^q_{\mathcal{U}}} C^{\mathrm{cell}}_k(\mathcal{U}_\tau)\,,$$

for any pair of integers $p, q \geq 0$. The induced map $\pi^{\mathcal{V}}_{p,k}$ on $C_k(\mathcal{V}_\sigma \cap \mathcal{U}_\tau)$ satisfies $\pi^{\mathcal{V}}_{p,k}((\sigma \times \tau)_a) = (\pi^{\mathcal{V}}_*(\sigma \times \tau))_a$ for all $\sigma \in N^p_{\mathcal{V}}, \tau \in N_{\mathcal{U}}$ and all $a \in (\mathcal{V}_\sigma \cap \mathcal{U}_\tau)^k$. The map $\pi^{\mathcal{U}}_{*,*}$ acts similarly. By definition $\pi^{\mathcal{U}}_{*,*}$ and $\pi^{\mathcal{V}}_{*,*}$ both commute with the Čech differentials $\delta^{\mathcal{U}}$ and $\delta^{\mathcal{V}}$ respectively. Let $\sigma \in N^p_{\mathcal{V}}$ and $\tau \in N^0_{\mathcal{U}}$. Then we have

$$\begin{array}{ccc} (\sigma \times \tau)_a & \xrightarrow{\quad\pi^{\mathcal{V}}_{*,*}\quad} & (\sigma)_a \\ \big\downarrow d_n & & \big\downarrow d_n \\ \sum_{b \in \overline{a}} \left([b : a]\sigma \times \tau\right)_b & \xrightarrow{\pi^{\mathcal{V}}_{*,*}} & \sum_{b \in \overline{a}} \left([b : a]\sigma\right)_b \end{array}$$

for all cells $a \in (\mathcal{V}_\sigma \cap \mathcal{U}_\tau)^k$. This implies that $\pi^{\mathcal{V}}_{*,*}$ commutes with d_n and the same holds for $\pi^{\mathcal{U}}_{*,*}$. We obtain a morphism $\pi^{\mathcal{V}}_{p,k} \colon \check{C}_p(\mathcal{V}\cap\mathcal{U}; C^{\mathrm{cell}}_k) \to \check{C}_p(\mathcal{V}; C^{\mathrm{cell}}_k)$ commuting with d_* and $\delta^{\mathcal{V}\cap\mathcal{U}}$ and $\delta^{\mathcal{V}}$. This induces $\kappa^{\mathcal{V}}_{p,k} \colon \check{C}_p(\mathcal{V} \cap \mathcal{U}; \mathrm{PH}_k) \to \check{C}_p(\mathcal{V}; \mathrm{PH}_k)$ and, in turn, this induces $\theta^{\mathcal{V},\mathcal{V}\cap\mathcal{U}}_{p,k} \colon \check{\mathcal{H}}_p(\mathcal{V} \cap \mathcal{U}; \mathrm{PH}_k) \to \check{\mathcal{H}}_p(\mathcal{V}; \mathrm{PH}_k)$.

There is a very natural way of understanding how much $\theta^{\mathcal{V},\mathcal{V}\cap\mathcal{U}}_{p,k}$ fails to be an isomorphism. To start, notice that $\kappa^{\mathcal{V}}_{p,k}$ is equal to the composition

$$\check{C}_p(\mathcal{V} \cap \mathcal{U}; \mathrm{PH}_k) \longrightarrow\!\!\!\!\rightarrow {}^{\mathrm{I}}E^0_{p,0}(\mathcal{V}, \mathcal{U}; \mathrm{PH}_k) \xrightarrow{{}^{\mathrm{I}}\pi^{\mathcal{V}}_{p,k}} \check{C}_p(\mathcal{V}; \mathrm{PH}_k)\,,$$

where the first morphism forgets the summands with $\tau \notin N^0_{\mathcal{U}}$; the second morphism is the restriction of $\kappa^{\mathcal{V}}_{p,k}$ to the remaining terms. Next, we take for each simplex

$\sigma \in N^p_{\mathcal{V}}$, the Mayer-Vietoris spectral sequence for $\mathcal{V}_\sigma$ covered by $\mathcal{V}_\sigma \cap \mathcal{U}$

$$M^2_{q,k}(\mathcal{V}_\sigma \cap \mathcal{U}) \Rightarrow \mathrm{PH}_{q+k}(\mathcal{V}_\sigma),$$

where we changed the notation from $E^2_{q,k}(\mathcal{V}_\sigma, \mathcal{V}_\sigma \cap \mathcal{U})$ to $M^2_{q,k}(\mathcal{V}_\sigma \cap \mathcal{U})$ as it helps distinguishing this spectral sequence from ${}^{\mathrm{I}}E^*_{p,q}$. Then, we write more compactly

$${}^{\mathrm{I}}E^1_{p,q}(\mathcal{V}, \mathcal{U}; \mathrm{PH}_k) = \bigoplus_{\sigma \in N^p_{\mathcal{V}}} M^2_{q,k}(\mathcal{V}_\sigma \cap \mathcal{U}) \ .$$

Taking ${}^{\mathrm{I}}E^1_{p,0}(\mathcal{V}, \mathcal{U}; \mathrm{PH}_k)$ as a chain complex, ${}^{\mathrm{I}}\pi^{\mathcal{V}}_{p,k}$ induces a chain morphism

$${}^{\mathrm{I}}\pi^{\mathcal{V}}_{p,k} : {}^{\mathrm{I}}E^1_{p,0}(\mathcal{V}, \mathcal{U}; \mathrm{PH}_k) \to \check{\mathcal{C}}_p(\mathcal{V}; \mathrm{PH}_k)$$

for all $p \geq 0$. In particular, the restriction of ${}^{\mathrm{I}}\pi^{\mathcal{V}}_{p,k}$ to the summand $M^2_{0,k}(\mathcal{V}_\sigma \cap \mathcal{U})$ equals the composition

$$M^2_{0,k}(\mathcal{V}_\sigma \cap \mathcal{U}) \twoheadrightarrow M^\infty_{0,k}(\mathcal{V}_\sigma \cap \mathcal{U}) \hookrightarrow \mathrm{PH}_k(\mathcal{V}_\sigma) \,.$$

Notice that PH_0 is a cosheaf, and in this case $M^2_{0,0}(\mathcal{V}_\sigma \cap \mathcal{U}) = \mathrm{PH}_0(\mathcal{V}_\sigma)$ for all $\sigma \in N^p_{\mathcal{V}}$. This implies that ${}^{\mathrm{I}}\pi^{\mathcal{V}}_{p,0}$ is an isomorphism for all $p \geq 0$. By the same argument, there is another chain morphism for all $q \geq 0$

$${}^{\mathrm{II}}\pi^{\mathcal{U}}_{q,k} : {}^{\mathrm{II}}E^1_{0,q}(\mathcal{V}, \mathcal{U}; \mathrm{PH}_k) \to \check{\mathcal{C}}_q(\mathcal{U}; \mathrm{PH}_k) \ .$$

Going back to the morphism $\theta^{\mathcal{V}, \mathcal{V}\cap\mathcal{U}}_{p,k}$, it is given by the composition

$$\check{\mathcal{H}}_p(\mathcal{V} \cap \mathcal{U}; \mathrm{PH}_k) \twoheadrightarrow {}^{\mathrm{I}}E^\infty_{p,0}(\mathcal{V}, \mathcal{U}; \mathrm{PH}_k) \hookrightarrow {}^{\mathrm{I}}E^2_{p,0}(\mathcal{V}, \mathcal{U}, \mathrm{PH}_k) \xrightarrow{{}^{\mathrm{I}}\pi^{\mathcal{V}}_{p,k}} \check{\mathcal{H}}_p(\mathcal{V}; \mathrm{PH}_k) \,.$$

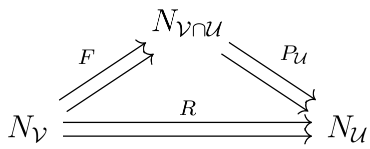
Using Lemma 7.5, if $\mathcal{V} \prec \mathcal{U}$ then $M^2_{q,k}(\mathcal{V}_\sigma \cap \mathcal{U}) = 0$ for all $q > 0$ and ${}^{\mathrm{I}}\pi^{\mathcal{V}}_{p,k}$ becomes an isomorphism. In addition, ${}^{\mathrm{I}}E^1_{p,q} = 0$ for all $q > 0$ and the first two arrows in the above factorisation of $\theta^{\mathcal{V}, \mathcal{V}\cap\mathcal{U}}_{p,k}$ are isomorphisms. Altogether, the inverse $(\theta^{\mathcal{V}, \mathcal{V}\cap\mathcal{U}}_{p,k})^{-1}$ is well-defined, and by composition we define morphisms $\theta^{\mathcal{U}, \mathcal{V}}_{p,k} = \theta^{\mathcal{U}, \mathcal{V}\cap\mathcal{U}}_{p,k} \circ (\theta^{\mathcal{V}, \mathcal{V}\cap\mathcal{U}}_{p,k})^{-1}$. Here notice that $\theta^{\mathcal{U}, \mathcal{V}\cap\mathcal{U}}_{p,k}$ is defined in an analogous way to $\theta^{\mathcal{V}, \mathcal{V}\cap\mathcal{U}}_{p,k}$, but it factors through ${}^{\mathrm{II}}\pi^{\mathcal{U}}_{q,k}$ instead of ${}^{\mathrm{I}}\pi^{\mathcal{V}}_{p,k}$. The following proposition should also follow from applying an appropriate version of the universal coefficient theorem to [24, Proposition 4.4]. Instead, we prove the dual statement of this proposition by means of acyclic carriers.

Proposition 7.7. *Suppose that $\mathcal{V} \prec \mathcal{U}$, and let $\rho^{\mathcal{U}, \mathcal{V}}$ denote a refinement map. The morphism $\theta^{\mathcal{U}, \mathcal{V}}_{p,k} : E^2_{p,k}(X, \mathcal{V}) \to E^2_{p,k}(X, \mathcal{U})$ coincides with the standard morphism induced by $\rho^{\mathcal{U}, \mathcal{V}}$.*

Proof. Since $\mathcal{V} \prec \mathcal{U}$, the morphism $\theta^{\mathcal{V}, \mathcal{V}\cap\mathcal{U}}_{p,k} : \check{\mathcal{H}}_p(\mathcal{V} \cap \mathcal{U}, \mathrm{PH}_k) \to \check{\mathcal{H}}_p(\mathcal{V}, \mathrm{PH}_k)$ is an isomorphism. Now consider the diagram

$$\begin{array}{ccccc}
\check{\mathcal{H}}_p(\mathcal{V}; \mathrm{PH}_k) & & \xrightarrow{\rho^{\mathcal{U}, \mathcal{V}}_{p,k}} & & \check{\mathcal{H}}_p(\mathcal{U}; \mathrm{PH}_k) \\
\uparrow \simeq & & & & \uparrow {}^{\mathrm{II}}\pi^{\mathcal{U}}_{p,k} \\
\check{\mathcal{H}}_p(\mathcal{V} \cap \mathcal{U}; \mathrm{PH}_k) & \twoheadrightarrow & {}^{\mathrm{II}}E^\infty_{0,p}(\mathcal{V}, \mathcal{U}; \mathrm{PH}_k) & \hookrightarrow & {}^{\mathrm{II}}E^2_{0,p}(\mathcal{V}, \mathcal{U}; \mathrm{PH}_k) \,.
\end{array}$$

To check that it commutes we study the following triangles of acyclic carriers



where R is defined in Proposition 7.2. The carrier F is given for every $\sigma \in N_{\mathcal{V}}$ by $F(\sigma) = \Delta^\sigma \times |R(\sigma)|$. In fact, F defines an acyclic equivalence by considering the inverse carrier $P_{\mathcal{V}} : N_{\mathcal{V}\cap\mathcal{U}} \rightrightarrows N_{\mathcal{V}}$ sending $\sigma \times \tau$ to Δ^σ. In this case the shift carrier $I_{\mathcal{V}} : N_{\mathcal{V}} \rightrightarrows N_{\mathcal{V}}$ is given by the assignment $\sigma \mapsto \Delta^\sigma$, and $I_{\mathcal{V}\cap\mathcal{U}} : N_{\mathcal{V}\cap\mathcal{U}} \rightrightarrows N_{\mathcal{V}\cap\mathcal{U}}$ is given by $\sigma \times \tau \mapsto \Delta^\sigma \times \Delta^{\tau\cup\tau'}$; where $\tau' \in N_{\mathcal{U}}$ is such that $|R(\sigma)| = \Delta^{\tau'} \subseteq N_{\mathcal{U}}$. Here, we need to show that $\Delta^\sigma \times \Delta^{\tau\cup\tau'}$ is a subcomplex of $N_{\mathcal{V}\cap\mathcal{U}}$. First notice that, by hypotheses, $\mathcal{V}_\sigma \cap \mathcal{U}_\tau \neq \emptyset$ and, by definition of $R(\sigma)$, we have $\mathcal{V}_\sigma \subseteq \mathcal{U}_{\tau'}$. Consequently $\mathcal{V}_\sigma \cap \big(\mathcal{U}_\tau \cap \mathcal{U}_{\tau'}\big) \neq \emptyset$, which accounts to $\Delta^\sigma \times \Delta^{\tau\cup\tau'}$ being a subcomplex of $N_{\mathcal{V}\cap\mathcal{U}}$.

Since F is acyclic, there exists $\nu_* : C_*(N_{\mathcal{V}}) \to C_*(N_{\mathcal{V}\cap\mathcal{U}})$ carried by F and inducing a chain morphism $f_* : \check{\mathcal{C}}_p(\mathcal{V}, C_k^{\text{cell}}) \to \check{\mathcal{C}}_p(\mathcal{V}\cap\mathcal{U}, C_k^{\text{cell}})$ by the assignment $(\sigma)_s \mapsto (\nu_*(\sigma))_s$ for all cells $s \in X$ and all $\sigma \in N_{\mathcal{V}}$. On the other hand, recall that $\theta_{p,k}^{\mathcal{V},\mathcal{V}\cap\mathcal{U}}$ is induced by $\pi_{p,k}^{\mathcal{V}}$, which is given as an assignment $(\sigma \times \tau)_s \to (\pi_*^{\mathcal{V}}(\sigma \times \tau))_s$. As $\pi_*^{\mathcal{V}}$ is carried by $P_{\mathcal{V}}$ and, as noted earlier, F defines an acyclic equivalence, it follows that $\pi_*^{\mathcal{V}} \circ \nu_*$ is the identity in $C_*(N_{\mathcal{V}})$ up to boundary. Thus, $\pi_{p,k}^{\mathcal{V}} \circ f_*$ is the identity in $\check{\mathcal{C}}_p(\mathcal{V}, C_k^{\text{cell}})$ up to the Čech boundary $\check{\delta}_{\mathcal{V}}$. This implies that $f_* = \big(\theta_{p,k}^{\mathcal{V},\mathcal{V}\cap\mathcal{U}}\big)^{-1}$ as morphisms $\check{\mathcal{H}}_p(\mathcal{V}, \mathrm{PH}_k) \to \check{\mathcal{H}}_p(\mathcal{V}\cap\mathcal{U}, \mathrm{PH}_k)$. Consequently, $\theta_{p,k}^{\mathcal{U},\mathcal{V}}$ is induced by the assignment $(\sigma)_s \mapsto (\pi_*^{\mathcal{U}} \circ \nu_*(\sigma))_s$ for all $\sigma \in N_{\mathcal{V}}$ and all $s \in X$; where $\pi_*^{\mathcal{U}} \circ \nu_*$ is carried by $P_{\mathcal{U}}F = R$. Altogether, as $\rho^{\mathcal{U},\mathcal{V}}$ is carried by R, we obtain the equality $\theta_{p,k}^{\mathcal{U},\mathcal{V}} = \rho_{p,k}^{\mathcal{U},\mathcal{V}}$ as morphisms $\check{\mathcal{H}}_p(\mathcal{V}, \mathrm{PH}_k) \to \check{\mathcal{H}}_p(\mathcal{U}, \mathrm{PH}_k)$. □

Still assuming that $\mathcal{V} \prec \mathcal{U}$, we now look for conditions for the existence of an inverse $\varphi_{p,k}^{\mathcal{V},\mathcal{U}} : E_{p,k}^2(X,\mathcal{U}) \to E_{p,k}^2(X,\mathcal{V})$ of $\theta_{p,k}^{\mathcal{U},\mathcal{V}}$.

Proposition 7.8. *Suppose that $\mathcal{V} \prec \mathcal{U}$. If $M_{p,k}^2(\mathcal{V}\cap\mathcal{U}_\tau) = 0$ for all $p > 0$, $k \geq 0$ and all $\tau \in N_{\mathcal{U}}^q$, then the maps $\theta_{*,*}^{\mathcal{U},\mathcal{V}}$ induce a 2-isomorphism of spectral sequences*

$$E_{*,*}^{\geq 2}(X,\mathcal{U}) \simeq E_{*,*}^{\geq 2}(X,\mathcal{V}).$$

Proof. By Proposition 7.2 and Proposition 7.7 we can choose a refinement map $\rho^{\mathcal{U},\mathcal{V}} : N_{\mathcal{V}} \to N_{\mathcal{U}}$ giving a morphism of spectral sequences

$$\rho_{*,*}^{\mathcal{U},\mathcal{V}} : E_{*,*}^{\geq 2}(X,\mathcal{V}) \to E_{*,*}^{\geq 2}(X,\mathcal{U})$$

that coincides with $\theta_{*,*}^{\mathcal{U},\mathcal{V}}$. Our assumption about $M_{p,k}^2$ implies ${}^{\mathrm{II}}E_{p,q}^2(\mathcal{V},\mathcal{U};\mathrm{PH}_k) = 0$ for all $p > 0$, which in turn, gives

$$\mathrm{Ker}\Big(\check{\mathcal{H}}_q(\mathcal{V}\cap\mathcal{U};\mathrm{PH}_k) \twoheadrightarrow {}^{\mathrm{II}}E_{0,q}^\infty(\mathcal{V},\mathcal{U};\mathrm{PH}_k)\Big) = 0 \tag{7}$$

and

$$\mathrm{Coker}\Big({}^{\mathrm{II}}E_{0,q}^\infty(\mathcal{V},\mathcal{U};\mathrm{PH}_k) \hookrightarrow {}^{\mathrm{II}}E_{0,q}^2(\mathcal{V},\mathcal{U},\mathrm{PH}_k)\Big) = 0. \tag{8}$$

Now note that ${}^{\mathrm{II}}\pi_{q,k}^{\mathcal{U}}$ yields an isomorphism ${}^{\mathrm{II}}E_{0,q}^2(\mathcal{V},\mathcal{U},\mathrm{PH}_k) \simeq \check{\mathcal{H}}_q(\mathcal{U},\mathrm{PH}_k)$. This shows that $\theta_{q,k}^{\mathcal{U},\mathcal{V}}$ is a composition of isomorphisms; thus the statement follows. □

We now relax the conditions in Proposition 7.8 and use the relations of *left-interleaving* and *right-interleaving* of persistence modules (denoted by $\sim_L^\varepsilon$ and $\sim_R^\varepsilon$, respectively) to achieve this (see [12, Sec. 4]). We have to adapt [12, Proposition 4.14].

Lemma 7.9. *Suppose that we have persistence modules A, B and C, and a parameter $\varepsilon \geq 0$ such that $A \sim_R^\varepsilon B$ and $B \sim_L^\varepsilon C$. Denote by Φ the morphism $\Phi : A \to C$ given by the composition $A \twoheadrightarrow B \hookrightarrow C$. Then there exists $\Psi : C \to A[2\varepsilon]$ such that Φ and Ψ define a 2ε-interleaving $A \sim^{2\varepsilon} C$.*

Proof. By hypothesis, we have a sequence

$$\mathcal{E}_1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \mathcal{E}_2$$

which is exact in A and C and where $\mathcal{E}_1 \sim^\varepsilon 0$ and $\mathcal{E}_2 \sim^\varepsilon 0$. Then, let $v \in C$ and notice that $\Sigma^\varepsilon C(v) \in \mathrm{Im}(g)$. Thus, there exists a unique vector $w \in B$ such that $g(w) = \Sigma^\varepsilon C(v)$. On the other hand, there exists $z \in A$, not necessarily unique, such that $f(z) = w$. This defines a unique element $\Sigma^\varepsilon A(z) \in A$.To see this, suppose that another $z' \in A$ is such that $f(z') = w$.Then $f(z - z') = 0$ and $z - z' \in \mathrm{Ker}(f)$, which implies $0 = \Sigma^\varepsilon A(z - z') = \Sigma^\varepsilon A(z) - \Sigma^\varepsilon A(z')$, and then $\Sigma^\varepsilon A(z) = \Sigma^\varepsilon A(z')$. Altogether, we set $\Psi = \Sigma^\varepsilon A \circ \Phi^{-1} \circ \Sigma^\varepsilon C$, which is well-defined. □

Recall that for $\mathcal{V} \prec \mathcal{U}$ we have that $\check{\mathcal{H}}_q(\mathcal{V}; \mathrm{PH}_k) \simeq \check{\mathcal{H}}_q(\mathcal{V} \cap \mathcal{U}; \mathrm{PH}_k)$ for all $k \geq 0$ and $q \geq 0$. There is a natural way to relax (7) and (8) to the persistent case. We assume that for $\varepsilon \geq 0$, there are right and left interleavings

$$\check{\mathcal{H}}_q(\mathcal{V} \cap \mathcal{U}; \mathrm{PH}_k) \sim_R^\varepsilon {}^{\mathrm{II}}E^\infty_{0,q}(\mathcal{V}, \mathcal{U}; \mathrm{PH}_k) \sim_L^\varepsilon {}^{\mathrm{II}}E^2_{0,q}(\mathcal{V}, \mathcal{U}, \mathrm{PH}_k). \tag{9}$$

If we define $\Phi_{q,k} : \check{\mathcal{H}}_q(\mathcal{V} \cap \mathcal{U}; \mathrm{PH}_k) \to {}^{\mathrm{II}}E^2_{0,q}(\mathcal{V}, \mathcal{U}, \mathrm{PH}_k)$ to be the composition of the associated persistence morphisms as in Lemma 7.9, then there exists

$$\Psi_{q,k} : {}^{\mathrm{II}}E^2_{0,q}(\mathcal{V}, \mathcal{U}, \mathrm{PH}_k) \to \check{\mathcal{H}}_q(\mathcal{V} \cap \mathcal{U}; \mathrm{PH}_k)[2\varepsilon],$$

such that $\Phi_{q,k}$ and $\Psi_{q,k}$ define a 2ε-interleaving. We repeat this argument for the local Mayer-Vietoris spectral sequences. Assume that for some $\nu \geq 0$ there are interleavings

$${}^{\mathrm{II}}E^1_{0,q}(\mathcal{V}, \mathcal{U}, \mathrm{PH}_k) \sim_R^\nu \bigoplus_{\tau \in N^q_{\mathcal{U}}} M^\infty_{k,0}(\mathcal{V} \cap \mathcal{U}_\tau) \sim_L^\nu \bigoplus_{\tau \in N^q_{\mathcal{U}}} \mathrm{PH}_k(\mathcal{U}_\tau). \tag{10}$$

Let $\Pi_{q,k} : {}^{\mathrm{II}}E^1_{0,q}(\mathcal{V}, \mathcal{U}, \mathrm{PH}_k) \to \bigoplus_{\tau \in N^q_{\mathcal{U}}} \mathrm{PH}_k(\mathcal{U}_\tau)$ be the composition of the associated morphisms. By Lemma 7.9 there exists $\Xi_{q,k}$ such that $\Pi_{q,k}$ and $\Xi_{q,k}$ define a 2ν-interleaving. By slight abuse of notation we continue to denote the induced 2ν-interleaving between ${}^{\mathrm{II}}E^2_{0,q}(\mathcal{V}, \mathcal{U}, \mathrm{PH}_k)$ and $\check{\mathcal{H}}_q(\mathcal{U}; \mathrm{PH}_*)$ by $\Pi_{q,k}$ and $\Xi_{q,k}$. Altogether we have that $\theta^{\mathcal{U},\mathcal{V}}_{q,k} = \Pi_{q,k} \circ \Phi_{q,k} \circ (\theta^{\mathcal{V},\mathcal{V}\cap\mathcal{U}}_{q,k})^{-1}$ and in this situation there is an 'inverse' $\psi^{\mathcal{V},\mathcal{U}}_{q,k} = \theta^{\mathcal{V},\mathcal{V}\cap\mathcal{U}}_{q,k} \circ \Psi_{q,k} \circ \Xi_{q,k}$, which increases the persistence values by $2(\varepsilon + \nu)$.

Theorem 7.10. *Suppose that $\mathcal{V} \prec \mathcal{U}$ and for $\varepsilon \geq 0$ and $\nu \geq 0$ the interleavings in (9) and (10) hold. Then*

$$\psi^{\mathcal{V},\mathcal{U}}_{p,q} : E^*_{p,q}(X, \mathcal{U}) \to E^*_{p,q}(X, \mathcal{V})[2(\varepsilon + \nu)]$$

is a 2-morphism of spectral sequences such that $\theta^{\mathcal{U},\mathcal{V}}_{p,q}$ and $\psi^{\mathcal{V},\mathcal{U}}_{p,q}$ define a second page $2(\varepsilon + \nu)$-interleaving between $E^*_{p,q}(X, \mathcal{U})$ and $E^*_{p,q}(X, \mathcal{V})$.

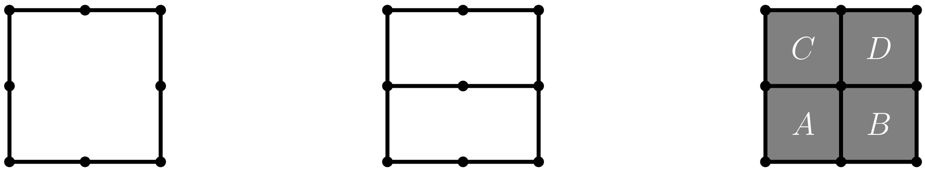


FIGURE 5. Cubical complex $\mathcal{C}_*$ at values 0,1 and $1+\varepsilon$.

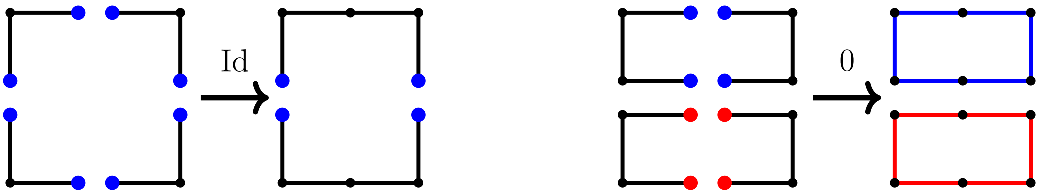
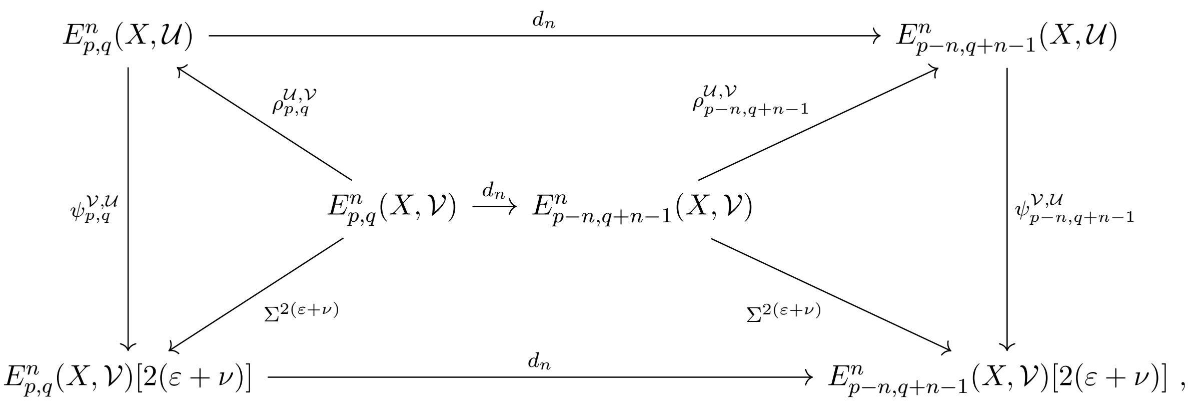


FIGURE 6. Morphisms $\theta^{\mathcal{U},\mathcal{V}}_{1,0}$ along $[0,1)$ and along $[1,1+\varepsilon)$.

Proof. The only thing that remains to be proved is that $\psi^{\mathcal{V},\mathcal{U}}_{p,q}$ commutes with the spectral sequence differentials d_n for all $n \geq 2$. Since these differentials commute with the shift morphisms $\Sigma^{2(\varepsilon+\nu)}$, this follows from considering the diagram



in which the two trapeziums and the two triangles commute. □

Example 7.11. Consider a cubical complex $\mathcal{C}_*$ as shown in Fig. 5, together with the covers $\mathcal{V} = \{\overline{A}, \overline{B}, \overline{C}, \overline{D}\}$ and $\mathcal{U} = \{\overline{A \cup B}, \overline{C \cup D}\}$, see Fig. 5 for the cells A,B,C and D. In this case, we have

$$\check{\mathcal{H}}_1(\mathcal{V};\mathrm{PH}_0) \simeq \check{\mathcal{H}}_1(\mathcal{V}\cap\mathcal{U};\mathrm{PH}_0) \simeq \mathrm{I}(0,1+\varepsilon)\oplus\mathrm{I}(1,1+\varepsilon) \sim^{\varepsilon} \mathrm{I}(0,1) \simeq {}^{\mathrm{II}}E^2_{0,1}(\mathcal{V},\mathcal{U},\mathrm{PH}_0)$$

and also

$${}^{\mathrm{II}}E^1_{0,0}(\mathcal{V},\mathcal{U},\mathrm{PH}_1) \simeq 0 \sim^{\varepsilon} \mathrm{I}(1,1+\varepsilon)\oplus\mathrm{I}(1,1+\varepsilon) \simeq \bigoplus_{\dim(\tau)=0} \mathrm{PH}_1(\mathcal{U}_\tau).$$

These interleavings are shown in Fig. 6. Theorem 7.10 implies that there is a 4ε-interleaving between $E^*_{p,q}(X,\mathcal{U})$ and $E^*_{p,q}(X,\mathcal{V})$. Notice that in this example, the nontrivial interleaved terms are in different positions of the spectral sequences. Therefore we can improve the upper bound to 2ε. We use this observation later in Proposition 7.12.

7.2. **Interpolating covers and spectral sequence interleavings.** Consider $X \in$ **FCW-cpx**, together with a pair of covers $\mathcal{W}$ and $\mathcal{U}$ so that $\mathcal{W} \prec \mathcal{U}$. Motivated by the interleaving constructed in Theorem 7.10 we take a closer look at the following finite sequence of covers interpolating between $\mathcal{W}$ and a cover that both refines and is refined by $\mathcal{U}$. Let the strict r-th intersections of $\mathcal{U}$ be the family of sets $\mathcal{U}^r = \{\mathcal{U}_\tau\}_{\tau \in N^r_{\mathcal{U}}}$ for all $r \geq 0$. We define the $(r, \mathcal{W}, \mathcal{U})$-*interpolation* as the covering set $\mathcal{W}^r = \mathcal{W} \cup \mathcal{U}^r$. In particular, note that the $(0, \mathcal{W}, \mathcal{U})$-interpolation has the property that $\mathcal{W}^0 \prec \mathcal{U} \prec \mathcal{W}^0$, and consequently $E^2_{p,q}(X, \mathcal{U}) \simeq E^2_{p,q}(X, \mathcal{W}^0)$. In addition if $\mathcal{U}$ is a finite cover, then we have $\mathcal{U}^N = \emptyset$ for $N \geq 0$ sufficiently large and consequently $\mathcal{W}^N = \mathcal{W}$.

Proposition 7.12 (Local Checks)**.** *Let $\mathcal{W} \prec \mathcal{U}$ be a pair of covers for X, where $\mathcal{U}$ is finite. Let $N \geq 0$ be such that $\mathcal{U}^N = \emptyset$. For every $0 \leq r \leq N$, we assume that there exist $\varepsilon_r \geq 0$ and $\nu_r \geq 0$ such that for all $\tau \in N^r_{\mathcal{U}}$*

$$E^2_{0,q}\big(\mathcal{U}_\tau, \mathcal{W}^{r+1}_{|\mathcal{U}_\tau}\big) \sim^{\nu_r}_R E^\infty_{0,q}\big(\mathcal{U}_\tau, \mathcal{W}^{r+1}_{|\mathcal{U}_\tau}\big) \sim^{\nu_r}_L \mathrm{PH}_q(\mathcal{U}_\tau)$$

and also

$$d_I(E^2_{p,q}(\mathcal{U}_\tau, \mathcal{W}^{r+1}_{|\mathcal{U}_\tau}), 0) \leq \varepsilon_r \ .$$

for all $p > 0$, and $q \geq 0$. Then we have that

$$d^2_I\big(E^*_{p,q}(X, \mathcal{W}^k), E^*_{p,q}(X, \mathcal{W}^{k+1})\big) \leq 2\max(\varepsilon_r, \nu_r).$$

Therefore, by using the triangle inequality, we obtain

$$d^2_I\big(E^*_{p,q}(X, \mathcal{U}), E^*_{p,q}(X, \mathcal{W})\big) \leq \sum_{k=0}^{N} 2\max(\varepsilon_r, \nu_r) \ .$$

Proof. We need to consider the spectral sequence ${}^{\mathrm{II}}E^2_{p,q}(\mathcal{W}^{r+1}, \mathcal{W}^r; \mathrm{PH}_k)$. Note that, by the construction of $\mathcal{W}^r$, for each $\tau \in N^r_{\mathcal{U}}$ with $\dim(\tau) > 0$ the set $\mathcal{W}^r_\tau$ is contained in one of the open sets from $\mathcal{W}^{r+1}$. By Lemma 7.5 this implies that ${}^{\mathrm{II}}E^1_{p,q}(\mathcal{W}^{r+1}, \mathcal{W}^r; \mathrm{PH}_k) = 0$ for all $p > 0$ and $q > 0$ and $k \geq 0$. Moreover, we have that ${}^{\mathrm{II}}E^1_{0,q}(\mathcal{W}^{r+1}, \mathcal{W}^r; \mathrm{PH}_k) = \bigoplus_{\tau \in N^q_{\mathcal{W}^r}} \mathrm{PH}_k(\mathcal{W}^r_\tau)$ for all $q > 0$ and $k \geq 0$. The resulting spectral sequence is shown in Fig. 7.

As a consequence of these observations condition (10) holds for these indices with $\nu = 0$. In addition, ${}^{\mathrm{II}}E^2_{0,q}(\mathcal{W}^{r+1}, \mathcal{W}^r; \mathrm{PH}_k) = E^2_{q,k}(X, \mathcal{W}^r)$ holds for all $q \geq 2$ and $k \geq 0$ (see Fig. 7 and 8). In particular, there is only one possible non-trivial differential for each entry in the bottom row as indicated in Fig. 8. Note that our hypothesis $d_I(E^2_{p,q}(\mathcal{U}_\tau, \mathcal{W}^{r+1}_{|\mathcal{U}_\tau}), 0) \leq \varepsilon_r$ applies to the entries in the first column with $p > 0$ and gives left and right interleavings of the form

$$\check{\mathcal{H}}_q(\mathcal{W}^{r+1} \cap \mathcal{W}^r; \mathrm{PH}_k) \sim^{\varepsilon_r}_R {}^{\mathrm{II}}E^\infty_{0,q}(\mathcal{W}^{r+1}, \mathcal{W}^r; \mathrm{PH}_k) \sim^{\varepsilon_r}_L {}^{\mathrm{II}}E^2_{0,q}(\mathcal{W}^{r+1}, \mathcal{W}^r; \mathrm{PH}_k)$$

for all $q > 0$ and $k \geq 0$. Hence, condition (9) holds with value ε_r.

Let us look now at the case $q = 0$. Here we have $\check{\mathcal{H}}_0(\mathcal{W}^{r+1} \cap \mathcal{W}^r; \mathrm{PH}_k) = {}^{\mathrm{II}}E^2_{0,0}(\mathcal{W}^{r+1}, \mathcal{W}^r; \mathrm{PH}_k)$ and consequently (9) holds with value $\varepsilon = 0$. Next, by hypothesis, for all $k \geq 0$ we have right and left interleavings

$$M^2_{0,k}\big(\mathcal{U}_\tau \cap \mathcal{W}^{r+1}\big) \sim^{\nu_r}_R M^\infty_{0,k}\big(\mathcal{U}_\tau \cap \mathcal{W}^{r+1}\big) \sim^{\nu_r}_L \mathrm{PH}_k(\mathcal{U}_\tau) \ ,$$

for all $\tau \in N^r_{\mathcal{U}}$. Thus by taking the direct sum of these interleavings we obtain

$${}^{\mathrm{II}}E^1_{0,0}(\mathcal{W}^{r+1}, \mathcal{W}^r; \mathrm{PH}_k) \sim^{\nu_r}_R \bigoplus_{\tau \in N^{\mathcal{W}^r}_0} M^\infty_{0,k}(\mathcal{W}^r_\tau \cap \mathcal{W}^{r+1}) \sim^{\nu_r}_L E^1_{0,k}(X, \mathcal{W}^r) \ .$$

and condition (10) also holds for $q = 0$. The result now follows from Theorem 7.10.

$$
\begin{array}{llll}
{}^{\mathrm{II}}E^1_{2,0}(\mathcal{W}^{r+1},\mathcal{W}^r;\mathrm{PH}_k) & 0 & 0 & \ddots \\
{}^{\mathrm{II}}E^1_{1,0}(\mathcal{W}^{r+1},\mathcal{W}^r;\mathrm{PH}_k) & 0 & 0 & 0 \\
{}^{\mathrm{II}}E^1_{0,0}(\mathcal{W}^{r+1},\mathcal{W}^r;\mathrm{PH}_k) \xleftarrow{d_1} & \bigoplus_{\tau\in N^1_{\mathcal{W}^r}} \mathrm{PH}_k(\mathcal{W}^r_\tau) \longleftarrow & \bigoplus_{\tau\in N^2_{\mathcal{W}^r}} \mathrm{PH}_k(\mathcal{W}^r_\tau) \longleftarrow & \bigoplus_{\tau\in N^3_{\mathcal{W}^r}} \mathrm{PH}_k(\mathcal{W}^r_\tau)
\end{array}
$$

FIGURE 7. First page of ${}^{\mathrm{II}}E^*_{p,q}(\mathcal{W}^{r+1},\mathcal{W}^r;\mathrm{PH}_k)$.

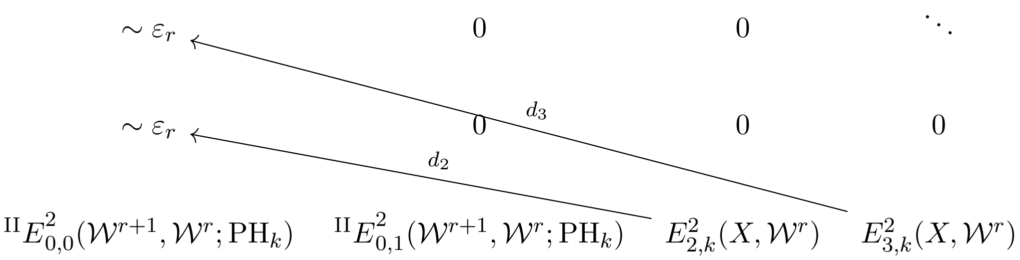


FIGURE 8. Second page of ${}^{\mathrm{II}}E^*_{p,q}(\mathcal{W}^{r+1},\mathcal{W}^r;\mathrm{PH}_k)$ together with higher differentials.

Notice that we can slightly improve the statement of Theorem 7.10 here: For each term in the bottom row of the spectral sequence in this particular example only one of the two conditions (9) and (10) is nontrivial, and the proof of Theorem 7.10 carries over with $2\max(\varepsilon_r,\nu_r)$ replacing $2(\varepsilon_r+\nu_r)$. □

Remark 7.13. Notice that for reasonable cases the parameters ν_r are bounded above by $K\varepsilon_r$ for some constant $K>0$ by a result from [12]. Nevertheless, we would like to keep ν_r and ε_r separated here, since we hope to compute it from $M^*_{p,k}\big(\mathcal{U}_\tau,\mathcal{W}^{r+1}_{|\mathcal{U}_\tau}\big)$ for $\tau\in N^r_{\mathcal{U}}$ hereby get more accurate estimates. Intuitively, asking for ε_r and ν_r to be small is equivalent to asking for cycle representatives in covers from $\mathcal{W}^r$ to be approximately contained in covering sets from $\mathcal{W}^{r+1}$.

Finally, we would like to compare two separate covers $\mathcal{U}$ and $\mathcal{V}$ and have an estimate for the interleaving distance between the associated spectral sequences. The main idea of Proposition 7.12 is to translate this comparison problem into a few local checks that can be run in parallel. We formalize this in the following Corollary.

Corollary 7.14 (Stability of Covers)**.** *Consider two pairs $(X,\mathcal{U})$ and $(X,\mathcal{V})$, where X is a space and $\mathcal{U}$ and $\mathcal{V}$ are covers. Let $\mathcal{W}=\mathcal{U}\cap\mathcal{V}$ and denote by $\mathcal{W}^r_{\mathcal{U}}$ and $\mathcal{W}^r_{\mathcal{V}}$ the respective $(r,\mathcal{W},\mathcal{U})$ and $(r,\mathcal{W},\mathcal{V})$ interpolations. For every $0\le r\le N$, we assume that there exist $\varepsilon_r,\varepsilon'_r\ge 0$ and $\nu_r,\nu'_r\ge 0$ such that for all $\tau\in N^r_{\mathcal{U}}$ and $\sigma\in N^r_{\mathcal{V}}$*

$$E^2_{0,q}\big(\mathcal{U}_\tau,\mathcal{W}^{r+1}_{\mathcal{U}}\big)\sim^{\nu_r}_R E^\infty_{0,q}\big(\mathcal{U}_\tau,\mathcal{W}^{r+1}_{\mathcal{U}}\big)\sim^{\nu_r}_L \mathrm{PH}_q(\mathcal{U}_\tau),$$
$$E^2_{0,q}\big(\mathcal{V}_\sigma,\mathcal{W}^{r+1}_{\mathcal{V}}\big)\sim^{\nu'_r}_R E^\infty_{0,q}\big(\mathcal{V}_\sigma,\mathcal{W}^{r+1}_{\mathcal{V}}\big)\sim^{\nu'_r}_L \mathrm{PH}_q(\mathcal{V}_\sigma),$$

for all $r \geq 0$*, and also*

$$d_I(E^2_{p,q}(\mathcal{U}_\tau, \mathcal{W}^{r+1}_{\mathcal{U}}), 0) \leq \varepsilon_r \qquad , \qquad d_I(E^2_{p,q}(\mathcal{V}_\sigma, \mathcal{W}^{r+1}_{\mathcal{V}}), 0) \leq \varepsilon'_r$$

for all $p > 0$*, and* $q \geq 0$*. Then we have that*

$$d^2_I\big(E^*_{p,q}(X, \mathcal{U}), E^*_{p,q}(X, \mathcal{V})\big) \leq R(\mathcal{U}, \mathcal{V})$$

where $R(\mathcal{U}, \mathcal{V}) = \max\Big(\sum_{r=0}^{N} 2\max(\varepsilon_r, \nu_r), \sum_{r=0}^{N} 2\max(\varepsilon'_r, \nu'_r)\Big)$.

Proof. By Lemma 7.1 there are double complex morphisms given by the refinement maps

$$\check{\mathcal{C}}_p(\mathcal{U}, C^{\text{cell}}_q) \xleftarrow{\rho^{\mathcal{U},\mathcal{W}}_{p,q}} \check{\mathcal{C}}_p(\mathcal{W}, C^{\text{cell}}_q) \xrightarrow{\rho^{\mathcal{V},\mathcal{W}}_{p,q}} \check{\mathcal{C}}_p(\mathcal{V}, C^{\text{cell}}_q) \ .$$

In turn, these induce 2-morphisms of spectral sequences

$$E^2_{p,q}(X, \mathcal{U}) \xleftarrow{\rho^{\mathcal{U},\mathcal{W}}_{p,q}} E^2_{p,q}(X, \mathcal{W}) \xrightarrow{\rho^{\mathcal{V},\mathcal{W}}_{p,q}} E^2_{p,q}(X, \mathcal{V}) \ .$$

Let $\psi^{\mathcal{U},\mathcal{W}}_{p,q}$ and $\psi^{\mathcal{V},\mathcal{W}}_{p,q}$ be the 'inverses' of $\rho^{\mathcal{U},\mathcal{W}}_{p,q}$ and $\rho^{\mathcal{V},\mathcal{W}}_{p,q}$, respectively, witnessing the interleavings of the two spectral sequences (see Theorem 7.10 and Proposition 7.12). The result follows from considering the commutative diagram

$$\begin{array}{ccccc}
E^2_{p,q}(X, \mathcal{U}) & \xleftarrow{\rho^{\mathcal{U},\mathcal{W}}_{p,q}} & E^2_{p,q}(X, \mathcal{W}) & \xrightarrow{\rho^{\mathcal{V},\mathcal{W}}_{p,q}} & E^2_{p,q}(X, \mathcal{V}) \\
\downarrow{\scriptstyle \Sigma^{R(\mathcal{V},\mathcal{U})}} & \searrow{\scriptstyle \psi^{\mathcal{W},\mathcal{U}}_{p,q}} & \downarrow{\scriptstyle \Sigma^{R(\mathcal{V},\mathcal{U})}} & \swarrow{\scriptstyle \psi^{\mathcal{W},\mathcal{V}}_{p,q}} & \downarrow{\scriptstyle \Sigma^{R(\mathcal{V},\mathcal{U})}} \\
E^2_{p,q}(X, \mathcal{U})[R(\mathcal{V}, \mathcal{U})] & \xleftarrow{\rho^{\mathcal{U},\mathcal{W}}_{p,q}} & E^2_{p,q}(X, \mathcal{W})[R(\mathcal{V}, \mathcal{U})] & \xrightarrow{\rho^{\mathcal{V},\mathcal{W}}_{p,q}} & E^2_{p,q}(X, \mathcal{V})[R(\mathcal{V}, \mathcal{U})]
\end{array}$$

where all arrows are 2-morphisms of spectral sequences. □

8. Outlook

We expect spectral sequences associated to the geometric realizations of diagrams of CW-complexes to have a natural use in the distributed computation of persistent homology. The first future research direction is to develop further examples and use cases that benefit from the theory developed in this article.

The ε-acyclic carriers and equivalences which we introduced here in the context of persistent homology are of course based on acyclic carriers, which are similar to the ones used for example in [2, Theorem 6] to prove a generalisation of the Nerve Theorem. A possible future research direction might be to ask for conditions on the acyclic carriers with the goal of obtaining similar results as those from [2] within the category of regularly filtered diagrams.

The bounds obtained in section 7 for the interleavings between the second pages of two spectral sequences can certainly be improved; one possible direction is to explore similar examples as those in [12, § 9] where the authors found sharp bounds.

In general, we think that spectral sequences deserve a more prominent role in applied algebraic topology and hope that the tools we developed here will motivate further study.

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Appendix A. Example of Acyclic Equivalence in **RCW-cpx**

Consider a filtered regular CW complex X which is constant along $\mathbf{R}$, except at values $1, 2, 3$ and 4, where it changes; see Figure 9. In order to describe X, we use the notation $(CD)_1$ for the edge between C and D, $(FGIJ)_2$ for a two cell whose vertices are F, G, I, J and so on. By regularity of X, and since we do not define multiple edges between the same pair of vertices, X is determined by:

$$\begin{aligned}
X_1 &= \{A, B, C, D, E, F, H\} \cup \{(AH)_1, (BC)_1, (CD)_1, (EF)_1\} \\
X_2 &= X_1 \cup \{G\} \cup \{(AB)_1, (DE)_1, (FG)_1, (GH)_1\} \\
X_3 &= X_2 \cup \{I, J\} \cup \{(BI)_1, (CJ)_1, (FJ)_1, (GI)_1, (IJ)_1\} \cup \{(FGIJ)_2\} \\
X_4 &= X_3 \cup \{K\} \cup \{(AK)_1, (CK)_1, (EK)_1, (GK)_1\} \\
&\quad \cup \{(ABCK)_2, (CDEK)_2, (EFGK)_2, (AKGH)_2\} \ .
\end{aligned}$$

where $X_0 = \emptyset$; this is shown in Figure 9, which illustrates X. Of course, as X is a filtered complex, the structure maps of X are given by inclusions $X_s \hookrightarrow X_t$ for all $s < t$ from $\mathbf{R}$. Next, we describe the regularly filtered CW-complex Y, which is constant along $\mathbf{R}$, except at values $1, 2, 3$ and 4, where it changes; this is also depicted in Figure 9. We define Y_* by:

$$\begin{aligned}
Y_1 =& \{\alpha, \beta, \gamma\} \\
Y_2 =& Y_1 \cup \{(\alpha\beta)_1, (\alpha\gamma)_1, (\beta\gamma)_1\} \\
Y_3 =& (Y_2 \setminus \{(\alpha\gamma)_1\}) \cup \{\delta, \tau\} \cup \{(\gamma\tau)_1, (\tau\delta)_1, (\alpha\delta)_1, (\beta\delta)_1, (\beta\tau)_1\} \\
Y_4 =& Y_3 \setminus \{\alpha, (\alpha\beta)_1, (\alpha\delta)_1\}
\end{aligned}$$

and $Y_0 = \emptyset$.

The structure maps of Y are defined as follows, where we use the overline notation $\overline{*}$ to denote the closure of some cell:

- $Y(1 \leq 2)$ is an inclusion,
- $Y(2 \leq 3)$ restricts to an inclusion in the subcomplex $\overline{(\alpha\beta)_1} \cup \overline{(\beta\gamma)_1}$, while $\overline{(\alpha\gamma)_1}$ is sent to $\overline{(\alpha\delta)_1} \cup \overline{(\delta\tau)_1} \cup \overline{(\tau\gamma)_1}$.
- $Y(3 \leq 4)$ restricts to the identity in $Y_3 \setminus \{(\alpha\beta)_1, \alpha, (\alpha\delta)_1\}$ while it maps the vertex α to γ, the edge $(\alpha\beta)_1$ to $(\beta\gamma)_1$ and the edge $(\alpha\delta)_1$ to $\{(\gamma\tau)_1, \tau, (\tau\delta)_1\}$.

One might check that Y is well-defined according to section 2.1. Next, we proceed to define an acyclic carrier $F : Y \rightrightarrows X$, which we depict in Figure 10, as follows:

- $F_1(\alpha) = \overline{(AH)_1}$, $\quad F_1(\beta) = \overline{(BC)_1} \cup \overline{(CD)_1}$, $\quad F_1(\gamma) = \overline{(EF)_1}$,

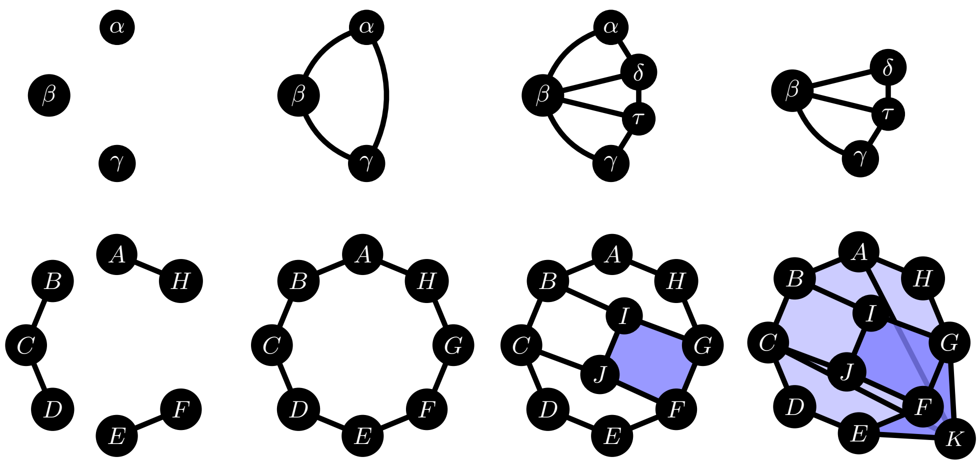

FIGURE 9. The spaces Y_i are shown at the top and X_i are at the bottom for values $i = 1, 2, 3, 4$. In filtration value 4, a cone with vertex in K is attached along the octahedron at the boundary of X_3; notice that we used 2-cells which are not 2-simplices.

- $F_2((\alpha\beta)_1) = F_1(\alpha) \cup F_1(\beta) \cup \{(AB)_1\}$,
  $F_2((\alpha\gamma)_1) = F_1(\alpha) \cup F_1(\gamma) \cup \{(HG)_1, G, (FG)_1\}$,
  $F_2((\beta\gamma)_1) = F_1(\beta) \cup F_1(\gamma) \cup \{(DE)_1\}$,
- $F_3(\delta) = G$, $F_3(\tau) = F$, $F_3((\alpha\delta)_1) = \overline{(AH)_1} \cup \overline{(HG)_1}$,
  $F_3((\delta\tau)_1) = \overline{(IJFG)_2}$, $F_3((\gamma\tau)_1) = \overline{(EF)_1}$,
  $F_3((\beta\delta)_1) = \overline{(BC)_1} \cup \overline{(CD)_1} \cup \overline{(BI)_1} \cup \overline{(IG)_1}$
  $F_3((\beta\tau)_1) = \overline{(BC)_1} \cup \overline{(CD)_1} \cup \overline{(CJ)_1} \cup \overline{(JF)_1}$,
- $F_4(\gamma) = F_4((\beta\gamma)_1) = F_4((\gamma\tau)_1) = \overline{\mathrm{St}(K)}$.

If we did not define a carrier, this is because we assume it is continued from an earlier definition. On the other hand, we define the carrier $G : X \rightrightarrows Y$ as follows:

- $G_1(A) = G_1(H) = G_1((AH)_1) = \alpha$, $G_1(E) = G_1(F) = G_1((EF)_1) = \gamma$,
  $G_1(B) = G_1(C) = G_1(D) = G_1((BC)_1) = G_1((CD)_1) = \beta$,
- $G_2((AB)_1) = \overline{(\alpha\beta)_1}$, $G_2((DE)_1) = \overline{(\beta\gamma)_1}$,
  $G_2((HG)_1) = G_2(G) = G_2((GF)_1) = \overline{(\alpha\gamma)_1}$,
- Define $A_3 = \{I, J, G, (IJ)_1, (GI)_1, (FJ)_1, (HG)_1, (GF)_1, (FGIJ)_2\}$,
  then $\forall \sigma \in A_3$, we have $G_3(\sigma) = \overline{(\alpha\delta)_1} \cup \overline{(\delta\tau)_1} \cup \overline{(\tau\gamma)_1}$,
  $G_3((BI)_1) = \overline{(\beta\delta)_1}$, $G_3((CJ)_1) = \overline{(\beta\tau)_1}$
- $\forall \sigma \in X_4 \setminus \{(BI)_1, (CJ)_1\}$, $G_4(\sigma) = \overline{(\beta\gamma)_1} \cup \overline{(\gamma\tau)_1} \cup \overline{(\tau\delta)_1}$.

We define the shift carriers on X and Y by composition, that is, $I_X^0 = G \circ F$ and $I_Y^0 = F \circ G$, which in this particular case lead to well-defined acyclic carriers as one can check; to illustrate this, we write a couple of compositions:

$$G_3 \circ F_3((\beta\tau)_1) = \overline{(\alpha\delta)_1} \cup \overline{(\delta\tau)_1} \cup \overline{(\tau\gamma)_1} \cup \overline{(\beta\tau)_1} \ ,$$

$$F_3 \circ G_3((IJ)_1) = \overline{(AH)_1} \cup \overline{(HG)_1} \cup \overline{(IJFG)_2} \cup \overline{(EF)_1} \ .$$

One can check that the conditions from Definition 4.3 are satisfied and so by Corollary 4.7 we obtain isomorphisms $\mathrm{PH}_*(X) \cong \mathrm{PH}_*(Y)$.

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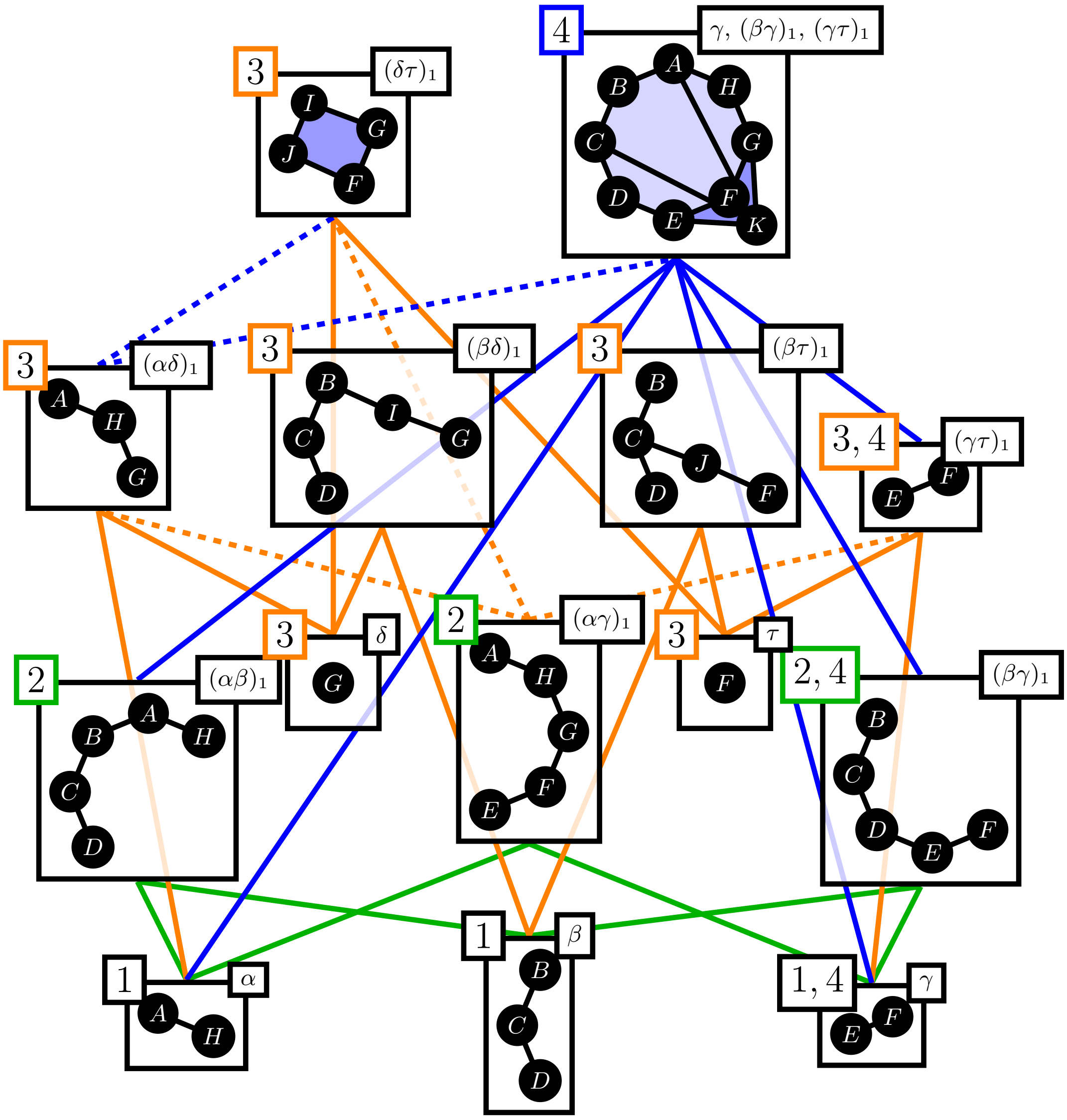

FIGURE 10. We depict the acyclic carriers from F. For each acyclic carrier we include its initial filtration value within a square on the top left while we write the cell(s) it corresponds to within a square on the top right; sometimes we write a pair of numbers a, b to indicate that the carrier applies for the filtration values in $[a, b)$ and that a new carrier is defined at b. Solid lines connecting the middle top of a box to the middle bottom of another box indicate that the containment relation must hold, where the carrier in the lower box needs to be embedded into the carrier on the upper box. We use dashed lines for containment relations involving a union of carriers, e.g. $F_3((\alpha\delta)_1) \subseteq F_4((\gamma\tau)_1) \cup F_4((\delta\tau)_1)$.