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Scaling Limits of Integrable Quantum Field Theories and Non-Local Chiral Models

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Abstract

In this thesis, short distance scaling limits of integrable quantum field theoretic models are explored. We consider an integrable model as a representation of two abstract Zamolodchikov-Faddeev algebras on an S-symmetric Fock space which are related in a specific manner. The defining datum of such an algebra is an R-matrix, namely an involutive, unitary solution of the Yang-Baxter equation with spectral parameters. We show how such R-matrices S (on the tensor product of Hilbert spaces $\mathcal{H} \otimes \mathcal{H}$) and R (on $\mathcal{K} \otimes \mathcal{K}$) can be combined into a box-sum $S \boxplus R$ and how this operation is reflected on the level of the Fock spaces $\mathcal{F}_S(\mathcal{H}), \mathcal{F}_R(\mathcal{K}), \mathcal{F}_{S \boxplus R}(\mathcal{H} \oplus \mathcal{K})$. The construction of chiral models as the short-distance scaling limit of such integrable models is outlined and the implications of such equivalences are discussed in this one-dimensional setting. In particular, we investigate the local observable content of the resulting chiral models. It is shown how the R-matrix relates to an algebra of observables localised at infinity, and how this algebra encodes the local observable content. In a specific example, we show how a deformation procedure produces strongly non-local models without strictly local observables.

Chapter 1

Introduction

In recent years the landscape of quantum field theory (QFT) research has grown in breadth. Results in mathematics from seemingly unrelated fields are being realised as relevant to the study of high energy physics. The application of abstract group theory provides interesting insights into the underlying data of scattering theory [CL21, LPW19], while free probability theory yields applications to abstract algebraic formulations of QFT.

It is true that the predictions of QFT thus far seem to be accurate when compared to experimental results. However, it hasn't yet been settled the most convenient manner in which to describe and construct a framework for QFT, though currently many exist and are frequently exploited. The most common is reminiscent of a classical mechanical approach whereby a Lagrangian is studied. Though, this methodology is rife with problems that have plagued physicists for many years, a prominent one being the divergence of a perturbative series arising from the expansion of a Lagrangian.

The algebraic approach is one that is more recent [HK64, Haa96] which we will exploit in this current work. To condense the finer points for now, the system works by considering an inverse scattering problem (in a similar fashion to other constructive QFT processes such as the Form Factor program [KTTW77, BKW79, KW78]). We then characterise observables as self-adjoint elements of algebras given by a local net which assigns to each bounded region of the chosen spacetime an algebra of operators.

The model described via any framework is open to scrutiny in many directions. A relevant analysis is the ultraviolet scaling limit of the resulting theory, importantly profiting answers to asymptotic freedom of quantum chromodynamics (QCD). First proposed in the algebraic framework in [BV95] then advanced in [BDM09, BDM10] (with further applications found in [DM06, DMV04]) a scaling limit of a QFT model can be calculated via the function $\lambda \to A_\lambda$ of scaling algebras. The resulting theory is described by the GNS representation of the algebra generated by the scaling limit of the vacuum state on the starting algebra at a finite scale. As previously mentioned, the property of asymptotic freedom was shown for QCD via such scaling limits (though not in the algebraic framework) and it is then natural to wonder if the same may be shown for other models. The most prominent of which perhaps is the $O(N)$ σ-models which thus far have only hinted asymptotic freedom in generality [AFPT10], although not yet proved. This analysis is not purely restricted to the algebraic setting, however, with relations to the Lagrangian description being found in [BDM09].

The existence of locally measurable observables in a theory is another problem one may face in the constructive process for any program, though some headway has been made in this direction in recent years [Lec08, BC15, BDL90]. The question is equivalent to the size of an algebra $\mathcal{A}(\mathcal{O})$ of a bounded region of spacetime generated by the net $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ in the algebraic framework. Furthermore, this can be written as a problem in abstract algebra, in particular relative commutants of von Neumann algebras [Wie93, GLW97, BL04]. Examples have been constructed (Poincaré covariant models, and also chiral conformal models on the real line) where the algebras $\mathcal{A}(\mathcal{O})$ are isomorphic to the unique hyperfinite factor type III_1 [Lec05, BDF87]. As we will later describe, this is not the only possibility [BLM11], though it is the most desirable. The opposite extreme is that of a *singular inclusion*, meaning that the algebra $\mathcal{A}(\mathcal{O}) = \mathbb{C}1$ with 1 is the identity. This case is actually pathological from a quantum field theoretic perspective, implying that there are no measurable observables, but it is a fruitful area of investigation in the context of half-sided modular inclusions of von Neumann algebras. The existence of such a trivial object was only theorised until recently when in [LTU19] the authors constructed such a singular inclusion by exploiting the methods of free probability.

In the present work, we analyse the problems discussed above in the context of integrable models, and chiral models on the light ray. This thesis is organised as follows. In Chapter 2 we start from the definition of a two-particle scattering (S-)matrix S and construct an integrable QFT on two dimensional Minkowski space mirroring those described in more detail in [LS12]. In contrast to Lechner and Schützenhofer however, we suppress discussions of gauge symmetries which we briefly describe in Appendix (B). This description will serve as a brief overview of constructive quantum field theory, as well as describing the types of models under consideration for our short distance scaling procedure later on.

In Chapter 3 we define an abstract Zamolodchikov-Faddeev (ZF) algebra [ZZ79, SF78] and discuss the case of constant S-matrices. Applying results found in [LPW19, CL21] we show that S-matrices can exhibit a natural equivalence arising from representations of the symmetric group. This then induces an isomorphism between the representations governed by two equivalent S-matrices. The material here is found in the joint work [LS20] previously published with G. Lechner.

Chapter 4 is concerned with the scaling limits of the integrable models outlined in Chapter 2, a problem considered in the scalar case in [BLM11]. In particular, we first consider the limits of two-particle S-matrices where the limit values (constant matrices) can be thought of as similar objects to the constant S-matrices analysed in Chapter 3. Moving further, we construct a chiral theory on the light ray, expanding on the constructions in [BLM11] by considering multiple particle species. We then define multi-component fields that are localised in only the wedge-local sense. The fields we build are then used to derive potential obstructions to the existence of local observables. That is, we calculate operators that lead to a sufficient condition on the size of the local algebras $\mathcal{A}(\mathcal{O})$ showing that in the setting we find ourselves in, the spectrum of possibilities is greater than that in the scalar case.

In Chapter 5 we take the idea of obstructions to local operators as inspiration and describe this in the abstract setting via the *algebra at infinity.* In the context of von Neumann algebras and half-sided modular inclusions, we recall the notion of a Borchers triple in both one and two dimensions and how this relates to quantum field theory as a Hilbert space representation. Via a two dimensional Borchers triple, we illustrate a natural deformation procedure first introduced in [GL07] and extended in [BS08, BLS10] known as *warped convolution.* In the representation, the von Neumann algebras of a Borchers triple play the role of our algebra of observables and we analyse the continuity of such objects under deformation. We illustrate examples of discontinuity with respect to the algebra of observables and outline a framework for constructing a singular inclusion, providing further examples to the one existing already in the literature in an arguably simpler fashion. The results in this Chapter appear in the joint publication [LS22] with G. Lechner.

The results presented in this thesis and the cited joint articles with G. Lechner have already provided inspiration for further research in this area, for example in [dSL22].

Chapter 2

Operator Algebraic Quantum Field Theory

In this chapter, we will describe and construct a general two-particle S-matrix and use it to build a suitably symmetrised Fock space. Closely following the constructions in [LS12], we define data on such a Fock space and illustrate the relations they satisfy before showing how this relates to the operator algebraic formulation. The latter is described by the intersections of von Neumann algebras and may be split into three possible scenarios concerning the relative size of these algebras. We briefly outline these possibilities to give an overview of the subject matter under analysis in this work.

2.1 Two-Particle S-Matrices and S-Symmetric Fock Spaces

We restrict ourselves to two $(1+1)$ dimensional Minkowski Space, on which we identify the single particle space for a fixed species as $L^2(\mathbb{R}, d\mu(p))$, where $\mu(p) = (p^2+m^2)^{-1/2}dp$ is the usual Lorentz invariant measure (see Appendix A for a description of Minkowski space geometry). However, it is more convenient to describe wavefunctions of particles in terms of the rapidity θ which is a parameterisation of the one dimensional upper mass shall $H_+ := \{((p^2+m^2)^{-1/2}, p),\ p \in \mathbb{R}\}$ for $m > 0$. The rapidity is then related to the on-shell momentum by

$$p(\theta) := m \begin{pmatrix} \cosh(\theta) \\ \sinh(\theta) \end{pmatrix}. \tag{2.1}$$

Let $\tilde{\mathcal{H}}$ be a finite dimensional Hilbert space of dimension $d_{\tilde{\mathcal{H}}}$, then our single particle space for several particle species is given by

$$\mathcal{H} := L^2(\mathbb{R}, \mathrm{d}\theta) \otimes \tilde{\mathcal{H}} \cong L^2(\mathbb{R} \to \tilde{\mathcal{H}}, \mathrm{d}\theta).$$

On $L^2(\mathbb{R}, \mathrm{d}\theta)$, we have a spacetime symmetry given by an irreducible, unitary representation of the proper, orthochronus Poincaré group $\mathcal{P}_+^\uparrow$ [LS12]

$$\left(\tilde{U}_1(a,\lambda)\psi\right)(\theta) := e^{ip(\theta)\cdot a}\psi(\theta - \lambda), \qquad (a,\lambda) \in \mathcal{P}_+^\uparrow.$$

As one may observe, the action of $\tilde{U}_1(a,\lambda)$ is to act by a translation $a \in \mathbb{R}^2$ and boost $\lambda \in \mathbb{R}$. The extension to a representation on $\mathcal{H}$ is natural:

$$U_1(a,\lambda) := \tilde{U}_1(a,\lambda) \otimes 1_{\tilde{\mathcal{H}}}.$$

We choose an orthonormal basis e_α, $\alpha \in \{1, \ldots, d_{\tilde{\mathcal{H}}}\}$ of $\tilde{\mathcal{H}}$ then denote components of vectors $\Psi_1 \in \mathcal{H}$ by $\theta \to \Psi_1^\alpha(\theta)$. Moreover, our conventions for multi-index notation is

$$v^{\boldsymbol{\alpha}} = \langle e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}, v\rangle_{\tilde{\mathcal{H}}}, \quad \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$$

for vectors $v \in \tilde{\mathcal{H}}^{\otimes n}$. Throughout this work, we will denote by $\mathcal{B}(\mathcal{H})$ the set of bounded linear operators over a Hilbert space $\mathcal{H}$, then for a tensor $R \in \mathcal{B}\left(\tilde{\mathcal{H}}^{\otimes n}\right)$ ($n \in \mathbb{N}$), we write its matrix elements as

$$R^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} = \langle e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}, R\, e_{\beta_1} \otimes \cdots \otimes e_{\beta_n}\rangle_{\tilde{\mathcal{H}}}.$$

In addition, for linear operators $R \in \mathcal{B}(\tilde{\mathcal{H}}^{\otimes n})$ we will use the shorthand notation $R_{k,n} := 1_{\tilde{\mathcal{H}}}^{\otimes k-1} \otimes R \otimes 1_{\tilde{\mathcal{H}}}^{\otimes n-k-1}$.

The single particle structure is fully understood with the description of the TCP operator J_1 on $\mathcal{H}$. This is product of a space-time reflection acting by complex conjugation on $L^2(\mathbb{R}, \mathrm{d}\theta)$ and an index conjugation on $\tilde{\mathcal{H}}$, that is we have an involutive automorphism $\alpha \mapsto \overline{\alpha}$ for $\alpha \in \{1, \ldots, d_{\tilde{\mathcal{H}}}\}$. Therefore, the TCP operator reads

$$(J_1\Psi_1)^\alpha(\theta) := \overline{\Psi_1^{\overline{\alpha}}(\theta)}$$

and one can easily see this is an antiunitary involution which extends the representation U_1 to the proper Poincaré group with the space-time reflection $j(x_0, x_1) := (-x_0, -x_1)$ by setting $U_1(j) := J_1$ [LS12].

We now wish to describe the notion of an S-matrix. In massive integrable models, an S-matrix is a bounded linear operator which describes the scattering of two particles in interactions. The example of a factorising S-matrix describes interactions of two incoming and two outgoing particles where momenta are preserved. To aid the following definition, we use the notation $S(a,b) := \{z \in \mathbb{C} \ : \ a < \mathrm{Im}(z) < b\}$ for strips in the complex plane. Furthermore, we denote by $\overline{S(a,b)} := \{z \in \mathbb{C} \ : \ a \leq \mathrm{Im}(z) \leq b\}$ the closure of the set.

Definition 2.1. An S-matrix is a continuous bounded function $S : \overline{S(0,\pi)} \to \mathcal{B}(\tilde{\mathcal{H}} \otimes \tilde{\mathcal{H}})$ which is analytic in the interior of the strip and satisfies for arbitrary $\theta, \theta' \in \mathbb{R}$ and $\alpha, \beta, \delta, \gamma \in \{1, \dots, d_{\tilde{\mathcal{H}}}\}$,

i) Unitarity:

$$S(\theta)^* = S(\theta)^{-1},$$

ii) Hermitian Analyticity :

$$S(\theta)^{-1} = S(-\theta),$$

iii) Yang-Baxter Equation:

$$(S(\theta) \otimes 1_{\tilde{\mathcal{H}}})(1_{\tilde{\mathcal{H}}} \otimes S(\theta + \theta'))(S(\theta') \otimes 1_{\tilde{\mathcal{H}}}) = (1_{\tilde{\mathcal{H}}} \otimes S(\theta'))(S(\theta + \theta') \otimes 1_{\tilde{\mathcal{H}}})(1_{\tilde{\mathcal{H}}} \otimes S(\theta)),$$

iv) TCP Invariance:

$$S^{\alpha\beta}_{\delta\gamma}(\theta) = S^{\overline{\gamma}\overline{\delta}}_{\overline{\beta}\overline{\alpha}}(\theta),$$

v) Crossing Symmetry:

$$S^{\alpha\beta}_{\delta\gamma}(i\pi - \theta) = S^{\overline{\gamma}\alpha}_{\delta\overline{\beta}}(\theta),$$

vi) Translational Invariance:

$$[S(\theta), U_1(a,\lambda) \otimes U_1(a,\lambda)] = 0, \quad \text{for all } a \in \mathbb{R}^2, \lambda \in \mathbb{R}.$$

The family of all S-matrices on $\tilde{\mathcal{H}}$ will be denoted by $\mathcal{S}(\tilde{\mathcal{H}})$.

Physically, we interpret the S-matrix as an operator that describes the scattering behaviour of particles. Indeed, particles are described by their states which may change upon interaction with another particle. The incoming states before an interaction are connected to the outgoing states after an interaction by the scattering matrix of the theory. As we are exclusively working in the integrable setting with a factoring S-matrix, multi-particle interactions can be described by numerous two-particle interactions with the multi-particle S-matrix decomposing into a product of two-particle S-matrices (as is defined in Definition

(2.1)). With momentum and particle number conserved through an interaction of two particles, the transition between in(coming) and out(going) states is completely described via S.

It is worth noting that though we have described the properties above in a basis-dependent manner, it is possible to define the same operator on the same Hilbert space in a manifestly basis-independent manner [AL17]. For a full analysis and description of the physical reasoning behind the properties outlined above, we refer the interested reader to [Iag76, AAR01].

For $d_{\tilde{\mathcal{H}}} > 1$ the general solution to Definition (2.1) is unknown, however for the scalar case $d_{\tilde{\mathcal{H}}} = 1$ the solutions are all known [BLM11]. In the case of $d_{\tilde{\mathcal{H}}} > 1$ there do however exist model-specific examples. A simple one would be

$$S(\theta)^{\alpha\beta}_{\delta\gamma} = \omega(\theta)\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta}$$

where the function $\omega(\theta)$ is a solution to Definition (2.1) with $\dim(\tilde{\mathcal{H}}) = 1$ (a *scalar scattering function*). Again, this may take many forms such as that governing the Sinh-Gordon model with coupling constant $g \in \mathbb{R}$ [AFZ79]

$$\omega(\theta) = \frac{\sinh(\theta) - i\sin\left(\frac{\pi g^2}{4\pi + g^2}\right)}{\sinh(\theta) + i\sin\left(\frac{\pi g^2}{4\pi + g^2}\right)}.$$

We consider n-fold tensor products $\mathcal{H}^{\otimes n}$ of the single particle space $\mathcal{H}$ and introduce the permutation operators $\rho^S_{n,k} : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$

$$(\rho^S_{n,k}\Psi_n)(\boldsymbol{\theta}) := S(\theta_{k+1} - \theta_k)_{n,k}\Psi_n(\theta_1, \dots, \theta_{k+1}, \theta_k, \dots, \theta_n) \tag{2.2}$$

where $n \in \mathbb{N}$, $k \in \{1, \dots, n-1\}$, $\Psi_n \in \mathcal{H}^{\otimes n}$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$.

Nearest neighbour transpositions τ_k swapping the k and $(k+1)$-th elements are generating elements for the symmetric group $\mathfrak{S}_n$ of n letters, and for arbitrary $i_1 \dots, i_r \in \{1, \dots, n-1\}$ we define

$$\rho^S_n(\tau_{i_1} \cdots \tau_{i_r}) := \rho^S_{n,i_1} \cdots \rho^S_{n,i_r} \tag{2.3}$$

which is a unitary representation of $\mathfrak{S}_n$ on $\mathcal{H}^{\otimes n}$ [LM95].

The operator $P^S_n := \frac{1}{n!}\sum_{\pi \in \mathfrak{S}_n} \rho^S_n(\pi)$ is then the orthogonal projection onto the ρ_n invariant subspace and we define the S-symmetric Fock space $\mathcal{F}_S(\mathcal{H})$ over $\mathcal{H}$ as

$$\mathcal{F}_S(\mathcal{H}) := \bigoplus_{n=1}^{\infty} \mathcal{H}_n, \quad \mathcal{H}_n := P^S_n \mathcal{H}^{\otimes n},\ n \geq 1,\ \mathcal{H}_0 := \mathbb{C}.$$

Elements of the Fock space $\mathcal{F}_S(\mathcal{H})$ are then sequences $\Psi = (\Psi_0, \Psi_1, \ldots)$, where $\Psi_n \in \mathcal{H}^{\otimes n}$ which is subject to the symmetry

$$\Psi_n^{\boldsymbol{\alpha}}(\boldsymbol{\theta}) = S^{\alpha_k \alpha_{k+1}}_{\beta_k \beta_{k+1}}(\theta_{k+1} - \theta_k)\Psi_n^{\alpha_1 \cdots \alpha_{k-1}\beta_k \beta_{k+1}\alpha_{k+2}\cdots\alpha_n}(\theta_1, \ldots, \theta_{k+1}, \theta_k, \ldots, \theta_n), \tag{2.4}$$

with norm

$$\|\Psi\|^2 = \sum_{n=0}^{\infty} \int \mathrm{d}^n\boldsymbol{\theta}\ \overline{\Psi_n^{\boldsymbol{\alpha}}(\boldsymbol{\theta})}\Psi_n^{\boldsymbol{\alpha}}(\boldsymbol{\theta}) < \infty.$$

We will occasionally make reference to the orthogonal projection $P^S : \bigoplus_n \mathcal{H}^{\otimes n} \to \mathcal{F}_S(\mathcal{H})$, and the subspace $\mathcal{D}_S \subset \mathcal{F}_S(\mathcal{H})$ of finite particle number where for $\Psi \in \mathcal{D}_S$, the sequence $(\Psi_0, \Psi_1, \ldots)$ terminates for some finite n.

The space $\mathcal{F}_S(\mathcal{H})$ can be equipped with the second quantisation of the operators U_1, J_1 which leave it invariant [LS12] and take the natural definiton:

$$(U(a,\lambda)\Psi)_n^{\boldsymbol{\alpha}}(\boldsymbol{\theta}) := \exp\left(i\sum_{j=1}^{n} p(\theta_j)\cdot a\right)\Psi_n^{\boldsymbol{\alpha}}(\theta_1 - \lambda, \ldots, \theta_n - \lambda), \tag{2.5}$$

$$(J\Psi)_n^{\boldsymbol{\alpha}}(\boldsymbol{\theta}) := \overline{\Psi_n^{\overline{\alpha_n}\cdots\overline{\alpha_1}}(\theta_n, \ldots, \theta_1)} \tag{2.6}$$

for $\Psi \in \mathcal{D}_S$ and $(a,\lambda) \in \mathcal{P}_+^{\uparrow}$. It is straightforward to notice that $JU(a,\lambda)J = U(-a,\lambda)$ from these definitions.

On $\mathcal{F}_S(\mathcal{H})$ we have the unique (up to scalar multiplication) U-invariant vector $\Omega_S = (1, 0, 0, \ldots)$ which we call the vacuum vector (so called as it represents the physical, empty vacuum of particle number zero).

On the unsymmetrised (Boltzmann) Fock space $\widehat{\mathcal{H}} := \bigoplus_n \mathcal{H}^{\otimes n}$, we have creation/annihilation operators $a^\dagger(\varphi), a(\varphi)$ ($\varphi \in \mathcal{H}$) which raise and lower the particle number by one, respectively, and they are linearly extended from

$$a^\dagger(\varphi)\psi_1 \otimes \cdots \otimes \psi_n := \sqrt{n+1}\ \varphi \otimes \psi_1 \otimes \cdots \otimes \psi_n,$$

$$a(\varphi)\psi_1 \otimes \cdots \otimes \psi_n := \sqrt{n}\ \langle \varphi, \psi_1\rangle_{\mathcal{H}}\ \psi_2 \otimes \cdots \otimes \psi_n, \qquad a(\varphi)\Omega_S := 0.$$

for $\psi_1, \ldots, \psi_n \in \mathcal{H}$ to $\mathcal{H}_n$ and then to the subspace of finite particle number $\mathcal{D}_S$ where they satisfy $a(\varphi)^* \supset a^\dagger(\varphi)$.

The projections of the creation/annihilation operators onto $\mathcal{F}_S(\mathcal{H})$ are denoted by

$$z_S^\dagger(\varphi) := P^S a^\dagger(\varphi) P^S \quad \text{and} \quad z_S(\varphi) := P^S a(\varphi) P^S, \quad (\varphi \in \mathcal{H}).$$

Their explicit action on $\Psi \in \mathcal{D}_S$ is given by

$$\left[z_S^\dagger(\varphi)\Psi\right]_n^{\boldsymbol{\alpha}}(\boldsymbol{\theta}) = \frac{1}{\sqrt{n}}\sum_{k=1}^{n} \rho_n^S(\sigma_k)^{\boldsymbol{\alpha}}_{\beta\delta_1\cdots\delta_{n-1}}\varphi(\theta_k)^{\beta}\Psi_{n-1}^{\delta_1\cdots\delta_{n-1}}(\theta_1, \ldots, \hat{\theta}_k, \ldots, \theta_n) \tag{2.7a}$$

$$\left[z_S(\varphi)\Psi\right]_n^{\boldsymbol{\alpha}}(\boldsymbol{\theta}) = \sqrt{n+1}\int d\theta' \overline{\varphi_\beta(\theta')}\Psi_{n+1}^{\beta\boldsymbol{\alpha}}(\theta',\boldsymbol{\theta}), \tag{2.7b}$$

where $\hat{\theta}_k$ indicates that this variable is omitted, and the permutations $\sigma_k \in \mathfrak{S}_n$ is defined as $\tau_{k-1}\tau_{k-2}\cdots\tau_1$ for $k \geq 1$ and $\sigma_1 = \mathrm{id}$.

From the above, we can also read off the explicit action of the respective distributional kernels

$$z_S^\dagger(\varphi) = \int d\theta z_{S,\alpha}^\dagger(\theta)\varphi_\alpha(\theta) \quad \text{and} \quad z_S(\varphi) = \int d\theta z_{S,\alpha}(\theta)\overline{\varphi_\alpha(\theta)} \tag{2.8}$$

as

$$\left[z_{S,\beta}^\dagger(\theta')\Psi\right]_n^{\boldsymbol{\alpha}}(\boldsymbol{\theta}) = \frac{1}{\sqrt{n}}\sum_{k=1}^n \rho_n^S(\sigma_k)^{\boldsymbol{\alpha}}_{\mu\delta_1\cdots\delta_{n-1}}\delta_\mu^\beta\delta(\theta'-\theta_k)\Psi_{n-1}^{\delta_1\cdots\delta_{n-1}}(\theta_1,\ldots,\hat{\theta}_k,\ldots,\theta_n) \tag{2.9a}$$

$$\left[z_{S,\beta}(\theta')\Psi\right]_n^{\boldsymbol{\alpha}}(\boldsymbol{\theta}) = \sqrt{n+1}\Psi_{n+1}^{\beta\boldsymbol{\alpha}}(\theta',\boldsymbol{\theta}). \tag{2.9b}$$

The projected creation/annihilation operators $z_S^\dagger, z_S$ together with the identity $1_{\mathcal{H}}$ and their polynomials form an algebra denoted by $\mathcal{P}_S$.

Lemma 2.2. [LS12] *The distributional kernels $z_\alpha^\#(\theta)$ satisfy*

$$z_{S,\alpha}(\theta)z_{S,\beta}(\theta') = S_{\delta\gamma}^{\beta\alpha}(\theta-\theta')z_{S,\gamma}(\theta')z_{S,\delta}(\theta) \tag{2.10a}$$

$$z_{S,\alpha}(\theta)z_{S,\beta}^\dagger(\theta') = S_{\beta\delta}^{\alpha\gamma}(\theta'-\theta)z_{S,\gamma}^\dagger(\theta')z_{S,\delta}(\theta) + \delta_\beta^\alpha\delta(\theta'-\theta)\cdot 1_{\mathcal{H}} \tag{2.10b}$$

for all $\alpha, \beta \in \{1,\ldots,d_{\tilde{\mathcal{H}}}\}$.

Remark 1. These relations are similar to those that define the Zamolodchikov-Faddeev algebra [ZZ79] however here a spectral parameter is included. We will describe this algebra in more detail in the scalar case in the next chapter.

As an aside and for use later on, we also define the TCP-reflected creation/annihilation operators on $\mathcal{F}_S(\mathcal{H})$

$$z_S^\dagger(\varphi)' := Jz_S^\dagger(J_1\varphi)J, \quad z_S(\varphi)' := Jz_S(J_1\varphi)J. \tag{2.11}$$

We refer the reader to [LS12] for a deeper analysis into these operators and the proof of the following results.

Lemma 2.3. [LS12] *Let $\varphi, \psi \in \mathcal{H}$, $\Psi \in \mathcal{D}_S$ and $n \in \mathbb{N}_0$. Then*

a) For $(a,\lambda) \in \mathcal{P}_+^\uparrow$ we have

$$U(a,\lambda)z_S(\varphi)U(a,\lambda)^* = z_S(U_1(a,\lambda)\varphi)$$

and similarly for $z_S^\dagger$ and their TCP reflected versions.

b)

$$\left[z_S(\varphi)', z_S(\psi)\right]\Psi_n = 0, \tag{2.12a}$$

$$\left[z_S^\dagger(\varphi)', z_S^\dagger(\psi)\right]\Psi_n = 0, \tag{2.12b}$$

$$\left[z_S(\varphi)', z_S^\dagger(\psi)\right]\Psi_n = K_n^{\varphi,\psi}\Psi_n, \tag{2.12c}$$

$$\left[z_S^\dagger(\varphi)', z_S(\psi)\right]\Psi_n = L_n^{\varphi,\psi}\Psi_n, \tag{2.12d}$$

where the multiplication operators $K_n^{\varphi,\psi}, L_n^{\varphi,\psi}$ on $\mathcal{D}_n$ have the action

$$K_n^{\varphi,\psi}(\boldsymbol{\theta})^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} = +\int \mathrm{d}\theta' \overline{\varphi_\gamma(\theta')}\rho^S_{n+1}(\sigma_{n+1})^{\boldsymbol{\alpha}\gamma}_{\delta\boldsymbol{\beta}}\psi_\delta(\theta'), \tag{2.12e}$$

$$L_n^{\varphi,\psi}(\boldsymbol{\theta})^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} = -\int \mathrm{d}\theta' \varphi_\gamma(\theta')\overline{\rho^S_{n+1}(\sigma_{n+1})^{\boldsymbol{\beta}\gamma}_{\delta\boldsymbol{\alpha}}\psi_\delta(\theta')}, \tag{2.12f}$$

Proposition 2.4. *The vacuum vector Ω_S is cyclic for the algebra $\mathcal{P}_S$. That is, $\mathcal{P}_S\Omega_S$ is dense in $\mathcal{F}_S(\mathcal{H})$.*

Proof. Assume $\Psi \in \mathcal{F}_S(\mathcal{H})$ is orthogonal to $\mathcal{P}_S\Omega_S$. For any $n \in \mathbb{N}_0$ and $f_1, \ldots, f_n \in \mathcal{H}$, we have

$$\begin{aligned} 0 &= \langle \Psi, z_S^\dagger(f_1)\cdots z_S^\dagger(f_n)\Omega_S\rangle \\ &= \sqrt{n!}\langle \Psi, P_n^S\left(f_1 \otimes \cdots \otimes f_n\right)\rangle \\ &= \sqrt{n!}\langle \Psi, f_1 \otimes \cdots \otimes f_n\rangle \end{aligned}$$

where we have used the fact that the projections P_n^S are self-adjoint, and the vector Ψ is symmetrized as in (2.4). Now, since $f_1 \otimes \cdots \otimes f_n$ forms a total set in $\mathcal{H}^{\otimes n}$ we must conclude that $\Psi = 0$. □

2.2 Multi-Component Fields and Algebraic QFT

Now that we have an understanding of the data available in an S-symmetric Fock space, we advance further to the definition of two quantum fields ϕ and ϕ' on two dimensional Minkowski space and discuss the algebraic aspects of this construction.

We take test functions $f \in \mathscr{S}(\mathbb{R}^2) \otimes \tilde{\mathcal{H}}$ having several components $f_\alpha(x) := (e_\alpha, f(x))$. Their positive and negative frequency wave functions in rapidity space are given in terms of a Fourier transform:

$$f^\pm_\alpha(\theta) := \tilde{f}_\alpha(\pm p(\theta)) = \frac{1}{2\pi}\int d^2x\ f_\alpha(x)e^{\pm ip(\theta)\cdot x} \tag{2.13}$$

Clearly, the above expression is a linear and well-defined transformation, moreover, the component functions $f_\alpha^\pm$ are in $L^2(\mathbb{R}, \mathrm{d}\theta)$ for $f_\alpha \in \mathscr{S}(\mathbb{R}^2)$.

We then define our quantum fields for $f \in \mathscr{S}(\mathbb{R}^2) \otimes \tilde{\mathcal{H}}$ by

$$\phi_S(f) := z_S^\dagger(f^+) + z_S(J_1 f^-), \tag{2.14}$$

$$\phi_S'(f) := z_S^\dagger(f^+)' + z_S(J_1 f^-)'. \tag{2.15}$$

Proposition 2.5. [LS12] *Let $f \in \mathscr{S}(\mathbb{R}^2) \otimes \tilde{\mathcal{H}}$ and $\Psi \in \mathcal{D}_S$. Then*

a) The map $f \mapsto \phi_S(f)\Psi$ is linear and continuous.

b) All vectors in $\mathcal{D}_S$ are entire analytic for $\phi_S(f)$. For $f = f^*$ (where $(f^*)_\alpha(x) := \overline{f_{\overline{\alpha}}(x)}$), the operator $\phi_S(f)$ is essentially self-adjoint.

c) ϕ_S transforms covariantly with $\mathcal{P}_+^\uparrow$, that is

$$\phi_S(U_1(a,\lambda)f)\Psi = U(a,\lambda)\phi_S(f)U(a,\lambda)^*\Psi, \quad (a,\lambda) \in \mathcal{P}_+^\uparrow.$$

d) $\phi_S(J_1 f) = J\phi_S'(f)J$.

e) The vacuum vector Ω_S is cyclic for ϕ_S.

f) ϕ_S is local if and only if $S = F$.

All statements are analogous for the reflected field ϕ_S'.

As stated above, the fields ϕ_S, ϕ_S' are local in the usual sense for the free case, however it can be shown that they are relatively wedge-local as in the scalar case [Lec03, LS12]. We recall the definition of the right wedge region in Minkowski space

$$W_R := \{x \in \mathbb{R}^2 \;:\; x_1 > |x_0|\} \tag{2.16}$$

and the set of all wedges are denoted by $\mathcal{W}$ which consist of all wedges produced by the orbit of W_R under the natural action of $\mathcal{P}_+$ on $\mathbb{R}^2$. It is clear by the definition of W_R that is invariant under boosts and hence $\mathcal{W}$ consists of all translations of W_R and the left wedge $W_L := jW_R = -W_R$.

Since a point localisation is not possible, we instead show that the fields ϕ_S, ϕ_S' are localised in these wedge regions by assigning $\phi_S'(f)$ the localisation region $(W_R + \mathrm{supp}(f))''$ where the dash indicates the causal complement - this region amounts to the smallest wedge region containing the support of f. Similarly, we assign $\phi_S(g)$ the localisation region $(W_L + \mathrm{supp}(g))''$. The relevant locality property is described below.

Theorem 2.6. [LS12] *The fields ϕ_S and ϕ'_S are relatively wedge-local. That is, for any $a \in \mathbb{R}^2$, $f \in \mathscr{S}(W_R + a) \otimes \tilde{\mathcal{H}}$, $g \in \mathscr{S}(W_L + a) \otimes \tilde{\mathcal{H}}$ and $\Psi \in \mathcal{D}_S$ we have*

$$[\phi'_S(f), \phi_S(g)] = 0. \tag{2.17}$$

With these notions in hand we may begin to connect this construction with the ideas of the algebraic framework of quantum field theory as described in Appendix A. The wedge regions in $\mathcal{W}$ will act as our regions $\mathcal{O}$ and we can construct von Neumann algebras from them. For any $x \in \mathbb{R}^2$:

$$\mathcal{A}_S(W_L + x) := \left\{ e^{i\phi_S(f)} \;:\; f = f^* \in \mathscr{S}(W_L + x) \otimes \tilde{\mathcal{H}} \right\}'', \tag{2.18}$$

$$\mathcal{A}_S(W_R + x) := \left\{ e^{i\phi'_S(f)} \;:\; f = f^* \in \mathscr{S}(W_R + x) \otimes \tilde{\mathcal{H}} \right\}''. \tag{2.19}$$

In the context of algebras the dash indicates the commutant with respect to $\mathcal{B}(\mathcal{H})$, and by the Double Commutant Theorem [Arv76], the double commutant $\mathcal{A}''$ of an algebra $\mathcal{A}$ is a von Neumann algebra. So, to any wedge region $W \in \mathcal{W}$, we associate a von Neumann algebra $\mathcal{A}_S(W)$ which has a number of basic properties.

Proposition 2.7. [LS12] *Let $S \in \mathcal{S}(\mathcal{H})$ and $W_1, W_2 \in \mathcal{W}$. Then*

a) Isotony: $\mathcal{A}_S(W) \subset \mathcal{A}_S(W_2)$ for $W_1 \subset W_2$,

b) Covariance: $U(a,\lambda)\mathcal{A}_S(W_1)U(a,\lambda)^* = \mathcal{A}_S(\Lambda_\lambda W_1 + a), \quad (a,\lambda) \in \mathcal{P}_+^\uparrow$,

c) Locality: $\mathcal{A}_S(W_1) \subset \mathcal{A}(W_2)'$ for $W_1 \subset W_2'$,

d) Cyclicity: The vacuum vector Ω_S is cyclic and separating for $\mathcal{A}_S(W)$.

Given this understanding of defining algebras of observables localised in wedge regions, it is natural to question how to construct an algebra of observables that are localised in smaller regions $\mathcal{O}$ of spacetime, and to do so we consider a double cone $\mathcal{O}_{x,y} := (W_L + x) \cap (W_R + y)$ for $x - y \in W_R$ (to ensure a non-empty intersection). We then take

$$\mathcal{A}_S(\mathcal{O}_{x,y}) := \mathcal{A}_S(W_L + x) \cap \mathcal{A}_S(W_R + y) \tag{2.20}$$

and define algebras associated with arbitrary regions $\mathcal{O} \subset \mathbb{R}^2$ by additivity of those above. This construction then defines a local net $\mathcal{O} \mapsto \mathcal{A}_S(\mathcal{O})$ on $\mathbb{R}^2$.

The relative size of the algebras $\mathcal{A}_S(\mathcal{O})$ for general S is an open question, however progress has been made in this direction [BL04, Lec08]. There are three cases surrounding this question:

1) “The cyclic case”: The vacuum vector Ω_S is cyclic for $\mathcal{A}_S(\mathcal{O})$, and therefore the algebra is non-trivial.

2) “The intermediate case”: $\mathcal{A}_S(\mathcal{O})$ is non-trivial, but it does not have Ω_S as a cyclic vector.

3) “The singular case”: $\mathcal{A}_S(\mathcal{O})$ is trivial, that is $\mathcal{A}_S(\mathcal{O}) = \mathbb{C} \cdot 1$.

The first case is what one would find in a local field theory, examples include [Jos65, Ara]. The nuclear modularity condition [BL04, Lec08] is a sufficient condition for a field theory to belong to this class, however, the calculations are difficult to carry out for theories that are less simplistic. The second case has been observed in local theories with gauge charges [BF82]. The third case is pathological and not something one would observe in a local field theory. Such an example has been constructed previously by applying free probabilities [LTU19]. Though this scenario is not desirable, it has applications and intrigue in its own right, particularly in the analysis of half-sided modular inclusions. In Chapter 5 we construct further examples of these singular cases.

Chapter 3

Constant S-matrices and Isomorphic ZF Algebras

In the previous chapter, we dealt with the general construction that appears in the Fock representation of a ZF algebra (on a Hilbert space). In particular, we had the presence of a spectral parameter θ as an independent variable of an S-matrix S understood as the rapidity of a particle. Though a number of interesting and physically relevant examples are indeed described by a θ-dependent S-matrix, one may also consider a constant S. The constant cases provide more simple examples of quantum field theories. On the other hand, the matrices themselves can instead be thought of as specific constant values of a θ-dependent version, in particular, the value $S(0)$ and the limits $\lim_{\theta\to\pm\infty} S(\theta)$ which play roles in the scaling limits of integrable models. In their own right, it has been shown that rapidity independent S-matrices have been derived as results of the Yang-Baxter equation with defects [CAFG02].

The results in this chapter are the central discussion in the joint work [LS20] with G. Lechner.

3.1 An Abstract ZF Algebra

We begin by abstractly defining a version of the well-known Zamolodchikov-Faddeev (ZF) algebra (cf. [ZZ79, Fad95]). Let $\mathcal{L}$ be a separable Hilbert space (of arbitrary dimension) and S a set of d^4 ($d \in \mathbb{N}$) complex numbers whose elements are labelled by the symbols $S^{\alpha\beta}_{\delta\gamma}$ where $\alpha, \beta, \delta, \gamma \in \{1, \ldots, d\}$. The symbols $1_{\mathcal{Z}(S,\mathcal{L})}, Z_1(f), Z_2(f), \ldots, Z_d(f)$ ($f \in \mathcal{L}$) generate the unital $\star$-algebra $\mathcal{Z}(S,\mathcal{L})$ and obey the following exchange relations:

$$Z_\alpha(f)Z_\beta(g) = S^{\beta\alpha}_{\delta\gamma} Z_\gamma(g)Z_\delta(f), \tag{3.1}$$

$$Z_\alpha(f)Z^*_\beta(g) = S^{\alpha\gamma}_{\beta\delta} Z^*_\gamma(g)Z_\delta(f) + \delta^\alpha_\beta \cdot \langle f,g\rangle_{\mathcal{L}} \cdot 1_{\mathcal{Z}(S,\mathcal{L})}, \tag{3.2}$$

where we adopt Einstein summation convention over the repeated indices.

Remark 2. To connect with the discussions of the previous chapter, we could instead consider the single particle Hilbert space to be a finite dimensional Hilbert space $\tilde{\mathcal{H}}$ (i.e., the scalar case). Then S can be viewed as a linear map over $\mathcal{H}$ with dimension d. The numbers $S^{\alpha\beta}_{\delta\gamma}$ are the matrix elements $\langle e_\alpha \otimes e_\beta, S(e_\delta \otimes e_\gamma)\rangle$ if $(e_\alpha)_{\alpha=1}^{d_{\tilde{\mathcal{H}}}}$ is an orthonormal basis of $\tilde{\mathcal{H}}$. ZF algebras are relevant to many areas of mathematics, in particular integrable models of quantum field theory, see for example [LS12]. A variation of this algebra can be defined by omitting (3.2), which is known as a Wick algebra. Their representations have also been studied in [KOP+22, JSW94, DVL18].

There is no implication from this abstract definition that there exists a Hilbert space representation of $\mathcal{Z}(S,\mathcal{L})$. For example, in [JSW94, p. 18] it is shown that S can be chosen in such a way that the generated Wick algebra admits no Hilbert space representation. It will become clear that properties of S indicate whether or not a GNS representation of $\mathcal{Z}(S,\mathcal{L})$ can be constructed.

Wick ordering (also referred to as "normal" ordering in some works) is useful in the analysis of algebras of this type. For an arbitrary element $X \in \mathcal{Z}(S,\mathcal{L})$, the relations (3.1) and (3.2) allow one to shift the ordering of the individual generating elements of $\mathcal{Z}(S,\mathcal{L})$ in X. Repeatedly doing so transforms X into Wick ordered form with an additional additive constant:

$$\sum_{\boldsymbol{\eta},\boldsymbol{\xi}} \zeta_{\boldsymbol{\eta},\boldsymbol{\xi}} Z^*_{\boldsymbol{\eta}}(f_{\boldsymbol{\eta}}) Z_{\boldsymbol{\xi}}(g_{\boldsymbol{\xi}}) + \sigma \cdot 1_{\mathcal{Z}(S,\mathcal{L})} \tag{3.3}$$

where $\zeta_{\boldsymbol{\eta},\boldsymbol{\xi}} \in \mathbb{C}$ and $\sigma \in \mathbb{C}$. The multi-index notation we have adopted here can be read as, for example,

$$Z^*_{\boldsymbol{\eta}}(f_{\boldsymbol{\eta}}) = Z^*_{\eta_1}(f_{\eta_1}) Z^*_{\eta_2}(f_{\eta_2}) \cdots Z^*_{\eta_N}(f_{\eta_N}),$$

where all $f_{\eta_n} \in \mathcal{L}$ and $|\boldsymbol{\eta}| = N \in \mathbb{N}$. Note that every element of $\mathcal{Z}(S,\mathcal{L})$ can be written in the form (3.3). The Wick ordered form is typically not unique; if there exist multiple Z or Z^* elements in a product, the application of (3.2) gives a distinct, but equally Wick ordered element. However, the term with $|\boldsymbol{\eta}| = |\boldsymbol{\xi}| = 0$ *is* unique.

To complete the description of the algebra $\mathcal{Z}(S,\mathcal{L})$, we describe a linear functional over it.

Definition 3.1. We define a normalised linear functional $\omega : \mathcal{Z}(S,\mathcal{L}) \to \mathbb{C}$ by the properties

i)
$$\omega(1_{\mathcal{Z}(S,\mathcal{L})}) = 1, \tag{3.4}$$

ii)
$$\omega(Z^*_\alpha(f) \cdot X) = 0, \tag{3.5}$$

iii)
$$\omega(X \cdot Z_\alpha(f)) = 0, \tag{3.6}$$

for all $\alpha \in \{1,\dots,d\}$, $f \in \mathcal{L}$ any $X \in \mathcal{Z}(S,\mathcal{L})$.

Defining a second functional as $\lambda(X) := \overline{\omega(X^*)}$ and applying uniqueness, we see that ω is Hermitian, but it is not positive.

Examples 3.1. We consider here some simple examples of $\mathcal{Z}(S,\mathcal{L})$.

Choosing $S^{\beta\alpha}_{\delta\gamma} = \pm\delta^\alpha_\delta\delta^\beta_\gamma$ (the tensor flip in finite dimensions), where δ is the Kronecker delta, the relations (3.1) and (3.2) now read (for $f,g \in \mathcal{L}$)

$$Z_\alpha(f)Z_\beta(g) = \pm Z_\beta(g)Z_\alpha(f), \tag{3.7}$$

$$Z_\alpha(f)Z^*_\beta(g) = \pm Z^*_\beta(g)Z_\alpha(f) + \delta^\alpha_\beta \cdot \langle f,g\rangle_\mathcal{L} \tag{3.8}$$

Let $(e_\alpha)^d_{\alpha=1}$ be an orthonormal basis of $\mathbb{C}^d$, then we may realise that

$$Z_\alpha(f) =: a(e_\alpha \otimes f)$$

satisfy the governing relations of the $\mathrm{CCR}(\mathbb{C}^d \otimes \mathcal{L})$ (+) and $\mathrm{CAR}(\mathbb{C}^d \otimes \mathcal{L})$ (−) algebras [BR81, EK98], respectively. In the Fock representation, these correspond to the Bose (+) and Fermi (−) Fock spaces.

If instead we choose $S^{\alpha\beta}_{\delta\gamma} = -\delta^\alpha_\delta\delta^\beta_\gamma$, we have

$$Z_\alpha(f)Z_\beta(g) = -Z_\alpha(g)Z_\beta(f) \tag{3.9}$$

$$Z_\alpha(f)Z^*_\beta(g) = \delta^\alpha_\beta \left(-\sum_\delta Z^*_\delta(g)Z_\delta(f) + 1_{\mathcal{Z}(1,\mathcal{L})} \right). \tag{3.10}$$

This example is explored in detail for the case of $\mathcal{L} = \mathbb{C}$ in [JSW94, p. 48] in a Wick algebraic setting where it is known as a “degenerate case”.

3.2 Constant S-matrices and Binary Operations

Definition (2.1) outlines the properties of a θ-dependent S-matrix, which indicates how we may define a constant analogue. Throughout this work we will use a tilde to denote finite dimensional Hilbert spaces such as $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{K}}$ (with no tilde indicating a more general space), and also operators over these spaces such as $\tilde{S}$ and $\tilde{R}$.

Definition 3.2. A constant S-matrix $\tilde{S} \in \mathcal{B}(\tilde{\mathcal{H}} \otimes \tilde{\mathcal{H}})$ is a constant $d_{\tilde{\mathcal{H}}} \times d_{\tilde{\mathcal{H}}}$ matrix which for all $\alpha, \beta, \delta, \gamma \in \{1, \ldots, d_{\tilde{\mathcal{H}}}\}$ satisfies:

i) Unitarity:
$$\tilde{S}^* = \tilde{S}^{-1},$$

ii) Involutivity:
$$\tilde{S}^{-1} = \tilde{S},$$

iii) Yang-Baxter equation:
$$\left(\tilde{S} \otimes 1_{\tilde{\mathcal{H}}}\right)(1_{\tilde{\mathcal{H}}} \otimes \tilde{S})(\tilde{S} \otimes 1_{\tilde{\mathcal{H}}}) = (1_{\tilde{\mathcal{H}}} \otimes \tilde{S})(\tilde{S} \otimes 1_{\tilde{\mathcal{H}}})(1_{\tilde{\mathcal{H}}} \otimes \tilde{S}),$$

iv) TCP Invariance :
$$\tilde{S}^{\alpha\beta}_{\delta\gamma} = \tilde{S}^{\bar{\gamma}\bar{\delta}}_{\bar{\beta}\bar{\alpha}},$$

v) Crossing Symmetry:
$$\tilde{S}^{\alpha\beta}_{\delta\gamma} = \tilde{S}^{\bar{\gamma}\alpha}_{\delta\bar{\beta}}.$$

The family of all constant S-matrices on $\tilde{\mathcal{H}}$ will be denoted by $\mathcal{S}_c(\tilde{\mathcal{H}})$.

It is worth remarking that if a matrix is crossing symmetric, it is also then TCP invariant by the above definition. The latter is the result of applying the crossing symmetry relation twice, however, due to their physical interpretations, we make specific mention of each.

We recall that an R-matrix is a unitary solution to the (quantum) Yang-Baxter equation and here we denote the family of R-matrices on $\tilde{\mathcal{H}}$ as $\mathcal{R}(\tilde{\mathcal{H}})$ and the set of involutive R-matrices as $\mathcal{R}_0(\tilde{\mathcal{H}}) \subset \mathcal{R}(\tilde{\mathcal{H}})$. A constant S-matrix is therefore an involutive R-matrix satisfying additional symmetries implying that $\mathcal{S}_c(\tilde{\mathcal{H}}) \subset \mathcal{R}_0(\tilde{\mathcal{H}})$.

On these sets we can define two binary operations [LPW19] which we recall below.

Definition 3.3. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and let $S \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, $R \in \mathcal{B}(\mathcal{K} \otimes \mathcal{K})$. Then we define

i) $S \boxplus R \in \mathcal{B}((\mathcal{H} \oplus \mathcal{K}) \otimes (\mathcal{H} \oplus \mathcal{K}))$ as

$$S \boxplus R = S \oplus R \oplus F \text{ on } (\mathcal{H} \oplus \mathcal{K}) \otimes (\mathcal{H} \oplus \mathcal{K}) = (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{K} \otimes \mathcal{K}) \oplus (\mathcal{H} \otimes \mathcal{K}) \oplus (\mathcal{K} \otimes \mathcal{H}),$$

where the operator $F = F_{(\mathcal{H} \otimes \mathcal{K}) \oplus (\mathcal{K} \otimes \mathcal{H})}$ is the tensor flip, and in the case above acts on the space $(\mathcal{H} \otimes \mathcal{K}) \oplus (\mathcal{K} \otimes \mathcal{H})$. More explicitly we may describe this operator as $F_{(\mathcal{H} \otimes \mathcal{K}) \oplus (\mathcal{K} \otimes \mathcal{H})} = F_{(\mathcal{H} \otimes \mathcal{K})} \oplus F_{(\mathcal{K} \otimes \mathcal{H})}$.

ii) $S \boxtimes R \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K} \otimes \mathcal{H} \otimes \mathcal{K})$ as

$$S \boxtimes R = F_2 (S \otimes R) F_2$$

where F_2 is a tensor flip acting on the second and third tensor slots.

We remark here that the S-matrix on $\mathcal{H} = L^2(\mathbb{R}) \otimes \tilde{\mathcal{H}}$ considered in Chapter 2 can actually be written more explicitly as $S = F \boxtimes \tilde{S}$ where F is the tensor flip on $L^2(\mathbb{R})$ and $\tilde{S} \in \mathcal{S}_c(\tilde{\mathcal{H}})$.

These two operations give a notion of addition and multiplication for involutive R-matrices and in analogy to that of scalar operations, we have the distributivity property of multiplication across a sum.

Lemma 3.4. *Let $\mathcal{H}, \mathcal{K}, \mathcal{L}$ be Hilbert spaces, $S \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}), R \in \mathcal{B}(\mathcal{K} \otimes \mathcal{K})$, and $F_{\mathcal{L}}$ be the tensor flip on $\mathcal{L} \otimes \mathcal{L}$. Then*

$$F_{\mathcal{L}} \boxtimes (S \boxplus R) = (F_{\mathcal{L}} \boxtimes S) \boxplus (F_{\mathcal{L}} \boxtimes R)\,. \tag{3.11}$$

Proof. We first note that the domains of the left and right of (3.11) are isomorphic:

$$\begin{aligned}\mathcal{D}\left(F_{\mathcal{L}} \boxtimes (S \boxplus R)\right) &= (\mathcal{L} \otimes (\mathcal{H} \oplus \mathcal{K}))^{\otimes 2} \\ &\cong ((\mathcal{L} \otimes \mathcal{H}) \oplus (\mathcal{L} \otimes \mathcal{K}))^{\otimes 2} = \mathcal{D}((F_{\mathcal{L}} \boxtimes S) \boxplus (F_{\mathcal{L}} \boxtimes R)),\end{aligned}$$

whereby $\mathcal{D}(S)$ we mean the domain of S.

In order to check that each side of (3.11) does indeed map to the same vector from a vector in their respective domain, we analyse each orthogonal component of their domains and check how each operator acts. These separate components are $(\mathcal{L} \otimes \mathcal{H})^{\otimes 2}, (\mathcal{L} \otimes \mathcal{K})^{\otimes 2}, \mathcal{L} \otimes \mathcal{H} \otimes \mathcal{L} \otimes \mathcal{K}$ and $\mathcal{L} \otimes \mathcal{K} \otimes \mathcal{L} \otimes \mathcal{H}$.

Let $h_1, h_2 \in \mathcal{H}, k_1, k_2 \in \mathcal{K}, l_1, l_2 \in \mathcal{L}$ and we firstly consider the case of $(\mathcal{L} \otimes \mathcal{H})^{\otimes 2}$. The left hand side of (3.11) acts as

$$\begin{aligned}\left(F_{\mathcal{L}} \boxtimes (S \boxplus R)\right)\left((l_1 \otimes h_1) \otimes (l_2 \otimes h_2)\right) &= \left(F_{\mathcal{L}} \otimes (S \boxplus R)\right)\left((l_1 \otimes l_2) \otimes (h_1 \otimes h_2)\right) \\ &= (l_2 \otimes l_1) \otimes S(h_1 \otimes h_2).\end{aligned}$$

Similarly for the right hand side of (3.11) we note that by definition of the box-sum, the $F_{\mathcal{L}} \boxtimes R)$ term drops out and we calculate

$$
\begin{aligned}
(F_{\mathcal{L}} \boxtimes S)\,(l_1 \otimes h_1 \otimes l_2 \otimes h_2) &= (F_{\mathcal{L}} \otimes S)\,(l_1 \otimes l_2 \otimes h_1 \otimes h_2) \\
&= (l_2 \otimes l_1) \otimes S(h_1 \otimes h_2).
\end{aligned}
$$

In this case, we can see we have equality between the two operators.

For the space $(\mathcal{L} \otimes \mathcal{K})^{\otimes 2}$ the arguments are identical by taking $S \to R, h_1 \to k_1, h_2 \to k_2$.

The "mixed" cases are those remaining: $\mathcal{L} \otimes \mathcal{H} \otimes \mathcal{L} \otimes \mathcal{K}$ and $\mathcal{L} \otimes \mathcal{K} \otimes \mathcal{L} \otimes \mathcal{H}$. Due to the presence of only one $\mathcal{H}$ and $\mathcal{K}$ factor in both spaces, by definition of the box-sum, the operators reduce to a series of tensor flips and it is easy to realise that we have

$$
F_2(F_{\mathcal{L}} \otimes F_{\mathcal{H} \oplus \mathcal{K}})F_2 = F_{(\mathcal{L} \otimes \mathcal{H}) \otimes (\mathcal{L} \otimes \mathcal{K})} \text{ and } F_2(F_{\mathcal{L}} \otimes F_{\mathcal{K} \oplus \mathcal{H}})F_2 = F_{(\mathcal{L} \otimes \mathcal{K}) \otimes (\mathcal{L} \otimes \mathcal{H})}.
$$

Both sides of (3.11) then act in the same way on each orthogonal part of their isomorphic domains, therefore they are equal. □

In the specific case at hand of finite-dimensional R-matrices, these operations also preserve a number of desirable properties.

Proposition 3.5. *Let $\tilde{\mathcal{H}}, \tilde{\mathcal{K}}$ be finite dimensional Hilbert spaces and let $\tilde{S} \in \mathcal{S}_c(\tilde{\mathcal{H}})$, $\tilde{R} \in \mathcal{S}_c(\tilde{\mathcal{K}})$. Then*

i) $\tilde{S} \boxplus \tilde{R} \in \mathcal{S}_c(\tilde{\mathcal{H}} \oplus \tilde{\mathcal{K}})$,

ii) $\tilde{S} \boxtimes \tilde{R} \in \mathcal{R}_0(\tilde{\mathcal{H}} \otimes \tilde{\mathcal{K}})$.

Proof. i) Unitarity and involutivity (and therefore also self-adjointness and invertibility) all follow once one realises that F possesses all of these properties and the flip on $(\tilde{\mathcal{H}} \otimes \tilde{\mathcal{K}}) \oplus (\tilde{\mathcal{K}} \otimes \tilde{\mathcal{H}})$ leaves its domain and the spaces $\tilde{\mathcal{H}} \otimes \tilde{\mathcal{H}}$ and $\tilde{\mathcal{K}} \otimes \tilde{\mathcal{K}}$ invariant. The inheritance of crossing symmetry (and therefore TCP symmetry) is also clear given that $S \boxplus R$ is crossing symmetric in each part of its domain. Finally, [LPW19, Prop 4.3] shows that $S \boxplus R$ as defined here solves the Yang-Baxter equation.

ii) Similarly to the previous part, unitarity, self-adjointness, involutivity and invertibility are inherited from the box product, moreover the Yang-Baxter equation is stable under tensor products and conjugation by the self-adjoint tensor flip F_2 [LPW19].

□

It is worth noting that it is not true in general that $S \boxtimes R \in \mathcal{S}_c(\tilde{\mathcal{H}} \otimes \tilde{\mathcal{K}})$ – as a counter-example, one may take $S, R = F$, then

$$1 = S_{21}^{12} R_{11}^{11} \neq S_{12}^{11} R_{21}^{11} = 0$$

violating crossing symmetry.

3.3 Equivalences of S-matrices and Induced Isomorphisms

With an understanding of constant S-matrices, R-matrices and the operations on them, we now analyse equivalences that arise between such objects. From motivations originating in [Tho64], it was discussed in [AL17] and later more concretely examined in [LPW19,CL21] that involutive R-matrices exhibit a natural equivalence. The derivation of such an equivalence is outside the subject matter of this thesis, but we refer the interested reader to [LPW19,CL21] for a detailed analysis of this.

In the case of R-matrices over $\tilde{\mathcal{H}}$ for $\dim(\tilde{\mathcal{H}}) = 2$, all equivalence relations are known and understood [LPW19] and can be described by three types (which we define later). The results in this direction can be compared and verified with that of [Hie92] where all solutions to the Yang-Baxter equation in this two-dimensional setting were calculated and one may easily further restrict to the involutive setting from this.

For arbitrary dimension, these three types of equivalence relations still remain valid, and one may use them to generate equivalence classes of involutive R-matrices, however, it is not yet understood if there exists other forms of equivalence relations beyond these three types.

Before describing the nature of this equivalence, we first make note of a particular form of R-matrix which will form a basis of such for the set of equivalence classes we will generate.

For the following, we will write 1_{a^2} for the identity on a vector space of dimension a.

Definition 3.6. Let $\tilde{\mathcal{H}}$ be a finite dimensional Hilbert space of dimension $d_{\tilde{\mathcal{H}}}$, $n \in \mathbb{N}_0$ and $\varepsilon_i = \{-1, +1\}$ for $i \in \{1, \ldots, n\}$. Then a *normal form* R-matrix N is of the form

$$N := \boxplus_{i=1}^{n} \varepsilon_i 1_{d_i^2} \tag{3.12}$$

where $\sum_{i=1}^{n} d_i^s = d_{\tilde{\mathcal{H}}}^2$.

We will mainly concern ourselves with three forms of equivalences which we may work with explicitly by considering the representations of the symmetric group generated by the R-matrix under consideration. Similarly to the parameter dependent case in (2.2), (2.3) a representation of the symmetric group of n letters $\rho_n^{\tilde{S}}$ is defined on the generating transposition elements τ_i for $0 < i < n$ by

$$\rho_n^{\tilde{S}}(\tau_i) := 1^{\otimes i-1} \otimes \tilde{S} \otimes 1^{n-i-1}.$$

Definition 3.7. Let $\tilde{\mathcal{H}}, \tilde{\mathcal{K}}$ be finite dimensional Hilbert spaces and $\tilde{S} \in \mathcal{R}_0(\tilde{\mathcal{H}}), \tilde{R} \in \mathcal{R}_0(\tilde{\mathcal{K}})$, then $\tilde{S}$ and $\tilde{R}$ are said to be equivalent, denoted as $\tilde{S} \sim \tilde{R}$, if and only if for each $n \in \mathbb{N}$ the representations $\rho_n^{\tilde{S}}$ and $\rho_n^{\tilde{R}}$ are unitarily equivalent.

In a more specific description, for $\tilde{S} \in \mathcal{R}_0(\tilde{\mathcal{H}})$ and $\tilde{R} \in \mathcal{R}_0(\tilde{\mathcal{K}})$, if $\tilde{S} \sim \tilde{R}$, the definition says that for all $n \in \mathbb{N}$ there exists a unitary intertwining operator $Y_n^{\tilde{S},\tilde{R}} : \tilde{\mathcal{H}}_n \to \tilde{\mathcal{K}}_n$ such that

$$Y_n^{\tilde{S},\tilde{R}} \rho_n^{\tilde{S}}(\pi) = \rho_n^{\tilde{R}}(\pi) Y_n^{\tilde{S},\tilde{R}}, \quad (\pi \in \mathfrak{S}_n). \tag{3.13}$$

The second quantisation of the above we denote as $Y^{\tilde{S},\tilde{R}} = \bigoplus_{n \geq 0} Y_n^{\tilde{S},\tilde{R}}$.

In the general case, the form of $Y_n^{\tilde{S},\tilde{R}}$ is unknown, however as mentioned previously, there are some known examples which we will describe - the following three types of equivalences also generate the full set of equivalence relations for the case of $\dim(\tilde{\mathcal{H}}) = \dim(\tilde{\mathcal{K}}) = 2$.

- **Type 1:** There exists a unitary operator U on $\tilde{\mathcal{H}}$ such that $(U \otimes U)\tilde{S}(U^* \otimes U^*) = \tilde{R}$. Then $\tilde{S} \sim \tilde{R}$ and we may choose
$$Y_n^{\tilde{S},\tilde{R}} = U^{\otimes n}.$$
- **Type 2:** There exists a unitary U on $\tilde{\mathcal{H}}$ whose tensor square commutes with $\tilde{S}$, i.e. $[\tilde{S}, U \otimes U] = 0$, and $(1 \otimes U)\tilde{S}(1 \otimes U^*) = \tilde{R}$. Then $\tilde{S} \sim \tilde{R}$ and we may choose
$$Y_n^{\tilde{S},\tilde{R}} = 1 \otimes U \otimes \cdots \otimes U^{\otimes n-1}.$$
- **Type 3:** Let $F_{\tilde{\mathcal{H}}}$ be the tensor flip on $\tilde{\mathcal{H}}$ such that $F_{\tilde{\mathcal{H}}} \tilde{S} F_{\tilde{\mathcal{H}}} = \tilde{R}$. Then $\tilde{S} \sim \tilde{R}$ and we may choose
$$Y_n^{\tilde{S},\tilde{R}} = \rho_n^{F_{\tilde{\mathcal{H}}} \tilde{S} F_{\tilde{\mathcal{H}}}}(\iota_n^{-1}) \rho_n^{\tilde{S}}(\iota_n),$$
where ι_n is the total inversion permutation of n letters.

In two dimensions, any involutive R-matrix is equivalent to a normal form R-matrix by either types $1,2$ or 3 [LPW19] and as mentioned, further examples of equivalences in higher dimensions and the description of the full set $\mathcal{R}_0(\tilde{\mathcal{H}})$ is not known.

In Chapter 1 we saw how a parameter dependent S-matrix governed the exchange relations for a ZF algebra and was deeply imprinted in the data produced by a Fock representation. We consider now finite dimensional Hilbert spaces $\tilde{\mathcal{H}}, \tilde{\mathcal{K}}$ with dimensions $d_{\tilde{\mathcal{H}}}, d_{\tilde{\mathcal{K}}}$, respectively, and a more general (separable) Hilbert space $\mathcal{L}$ (as in Chapter 2, one may identify this specifically with $L^2(\mathbb{R})$, however for our needs here it is not necessary to specify this space). We take $\tilde{S} \in \mathcal{S}_c(\tilde{\mathcal{H}})$ and $\tilde{R} \in \mathcal{S}_c(\tilde{\mathcal{K}})$ such that $\tilde{S} \sim \tilde{R}$ and define $\mathcal{S}_c(\mathcal{H}) \ni S := F \boxtimes \tilde{S}$, $\mathcal{S}_c(\mathcal{K}) \ni R := F \boxtimes \tilde{R}$ on $\mathcal{H} := \mathcal{L} \otimes \tilde{\mathcal{H}}$ and $\mathcal{K} := \mathcal{L} \otimes \tilde{\mathcal{K}}$, respectively, where we now write F to be the tensor flip on $\mathcal{L}$ dropping the subscript for simplicity.

To show the stability of our notion of equivalence over a box-multiplication, we introduce the following operator $\mathfrak{F}_n : \mathcal{H}^{\otimes n} \to \mathcal{L}^{\otimes n} \otimes \tilde{\mathcal{H}}^{\otimes n}$ acting explicitly by

$$\mathfrak{F}_n \left(\bigotimes_{i=1}^{n} (f_i \otimes h_i) \right) = \left(\bigotimes_{i=1}^{n} f_i \right) \otimes \left(\bigotimes_{i=1}^{n} h_i \right), \quad f_i \in \mathcal{L}, h_i \in \tilde{\mathcal{H}} . \tag{3.14}$$

It should be clear from the above expression that $\mathfrak{F}_n$ is a product of unitary, involutive operators $F_{i,j}$ and is therefore unitary itself for any n. Moreover, we define the second quantisation of $\mathfrak{F}_n$:

$$\mathfrak{F} := \bigoplus_{n \geq 0} \mathfrak{F}_n . \tag{3.15}$$

Lemma 3.8. *Let $\mathcal{H}, \mathcal{K}$ be separable Hilbert spaces, and $S \in \mathcal{R}_0(\mathcal{H}), R \in \mathcal{R}_0(\mathcal{K})$. Then the representation of the symmetric group, $\rho_n^{S \boxtimes R}$, generated by $S \boxtimes R$ is unitarily equivalent to $\rho_n^S \otimes \rho_n^R$ for any $n \in \mathbb{N}$.*

Proof. It is enough to show the result for the generating elements τ_k of $\mathfrak{S}_n$ and the rest follows by the properties of a representation. Let $(h_\alpha)_{\alpha \in \mathbb{N}}$ and $(k_\beta)_{\beta \in \mathbb{N}}$ be orthonormal bases of $\mathcal{H}$ and $\mathcal{K}$ respectively then by employing the permutation operator $\mathfrak{F}_n$ an element in the domain of $\rho_n^{S \boxtimes R}$ is mapped to an element in the domain of $\rho_n^S \otimes \rho_n^R$. The action of the latter operator is given by

$$\begin{aligned}
&\rho_n^S(\tau_k) \otimes \rho_n^R(\tau_k) \left(\bigotimes_{i=1}^{n} h_{\alpha_i} \right) \otimes \left(\bigotimes_{i=1}^{n} k_{\beta_i} \right) \\
&= S_{\delta\gamma}^{\alpha_k \alpha_{k+1}} R_{\eta\xi}^{\beta_k \beta_{k+1}} \left(\left(\bigotimes_{i=1}^{k-1} h_{\alpha_i} \right) \otimes h_\delta \otimes h_\gamma \otimes \left(\bigotimes_{i=k+2}^{n} h_{\alpha_i} \right) \right) \\
&\qquad \otimes \left(\left(\bigotimes_{i=1}^{k-1} k_{\beta_i} \right) \otimes k_\eta \otimes k_\xi \otimes \left(\bigotimes_{i=k+2}^{n} k_{\beta_i} \right) \right),
\end{aligned}$$

where the implicit sums converge in the norm topology. Applying $\mathfrak{F}_n^*$ and its linearity gives the action of $\rho_n^{S\boxtimes R}$ as stated. □

Proposition 3.9. *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_1, \mathcal{K}_2$ be Hilbert spaces and $S_1 \in \mathcal{R}_0(\mathcal{H}_1), S_2 \in \mathcal{R}_0(\mathcal{H}_2), R_1 \in \mathcal{R}_0(\mathcal{K}_1), R_2 \in \mathcal{R}_0(\mathcal{K}_2)$ such that $S_1 \sim S_2$ and $R_1 \sim R_2$. Then $S_1 \boxtimes R_1 \sim S_2 \boxtimes R_2$ with an intertwiner $Y^{S_1\boxtimes R_1, S_2\boxtimes R_2}$ which is given by*

$$Y^{S_1\boxtimes R_1, S_2\boxtimes R_2} = \mathfrak{F}^* \left(Y^{S_1,S_2} \otimes Y^{R_1,R_2} \right) \mathfrak{F}.$$

Proof. This is clear from the definition of $\mathfrak{F}_n$ and Lemma (3.8). □

Returning for the moment to the discussion of representations as in Chapter 2, one may show how these equivalences interact in the representation.

Example 3.1. Let $\tilde{\mathcal{H}}, \tilde{\mathcal{K}}$ be finite dimensional Hilbert spaces and $\tilde{S} \in \mathcal{R}_0(\tilde{\mathcal{H}}), \tilde{R} \in \mathcal{R}_0(\tilde{\mathcal{K}})$ such that $\tilde{S} \sim \tilde{R}$ in the type 1 sense. We may then choose their intertwiner to be of the form

$$Y_n^{\tilde{S},\tilde{R}} = \left(Y_1^{\tilde{S},\tilde{R}} \right)^{\otimes n}. \tag{3.16}$$

For a separable Hilbert space $\mathcal{L}$ we write $\mathcal{H} = \mathcal{L} \otimes \tilde{\mathcal{H}}, \mathcal{K} = \mathcal{L} \otimes \tilde{R}$ and $S = F \boxtimes \tilde{S}, R = F \boxtimes \tilde{R}$ and we then have an expression for the intertwiner $Y_n^{S,R}$ from Proposition (3.9). Then

$$Y^{S,R} \pi_S(Z_\alpha(f)) \left(Y^{S,R} \right)^* = \pi_R \left(Y_1^{S,R} Z_\alpha(f) \right). \tag{3.17}$$

We can verify this by calculation - let $f \in \mathcal{L}$, then

$$\begin{aligned}
&\mathfrak{F}^* \left(Y^{\tilde{S},\tilde{R}} \otimes 1 \right) \mathfrak{F} z_{S,\alpha}(f) \mathfrak{F}^* \left(\left(Y^{\tilde{S},\tilde{R}} \right)^* \otimes 1 \right) \mathfrak{F} \\
&\qquad = \mathfrak{F}^* \left(Y^{\tilde{S},\tilde{R}} \otimes 1 \right) z_{\tilde{S}}(e_\alpha) \otimes z_F(f) \left(\left(Y^{\tilde{S},\tilde{R}} \right)^* \otimes 1 \right) \mathfrak{F} \\
&\qquad = \mathfrak{F}^* \left(Y^{\tilde{S},\tilde{R}} z_{\tilde{S}}(e_\alpha) \left(Y^{\tilde{S},\tilde{R}} \right)^* \otimes z_F(f) \right) \mathfrak{F}
\end{aligned}$$

It remains to calculate the action of $Y^{\tilde{S},\tilde{R}} z_{\tilde{S}}(e_\alpha) \left(Y^{\tilde{S},\tilde{R}} \right)^*$. Let $\psi \in \mathcal{F}_S(\mathcal{H})$ then

$$\begin{aligned}
Y_{n+1}^{\tilde{S},\tilde{R}} z_{\tilde{S}}^*(e_\alpha) \left(Y_n^{\tilde{S},\tilde{R}} \right)^* \psi_n &= Y_{n+1}^{\tilde{S},\tilde{R}} P_{n+1}^S e_\alpha \otimes \left(\left[Y_1^{\tilde{S},\tilde{R}} \right]^* \right)^{\otimes n} \psi_n \\
&= P_{n+1}^R \left(Y_1^{\tilde{S},\tilde{R}} \right)^{\otimes(n+1)} \left(e_\alpha \otimes \left(\left[Y_1^{\tilde{S},\tilde{R}} \right]^* \right)^{\otimes n} \psi_n \right) \\
&= P_{n+1}^R \left(Y_1^{\tilde{S},\tilde{R}} e_\alpha \right) \otimes \psi_n \\
&= z_{\tilde{R}}^* \left(Y_1^{\tilde{S},\tilde{R}} e_\alpha \right) \psi_n.
\end{aligned}$$

Thus the intertwiner $Y^{\tilde{S},\tilde{R}}$ also acts as an isomorphism between elements of the polynomial algebras $\mathcal{P}_S, \mathcal{P}_R$.

Though the above illustrates how a type 1 equivalence gives rise to an isomorphism of representations, it is not true that any equivalence type gives rise to such an isomorphism between representations. As a counter-example, consider $S = -F_{\mathcal{H}} \sim -1 \boxplus -1 = R$ in the two dimensional setting which are type 2 equivalent. In the Fock representation of $\mathcal{Z}(S, \mathcal{H})$, we have anti-commutation between all annihilation operators as in the Fermi case [Lec03], i.e.

$$\left\{ z_{\tilde{S},\alpha}(f), z_{\tilde{S},\beta}(g) \right\} = 0$$

for $\alpha, \beta \in \{1, 2\}$ and all $f, g \in \mathcal{L}$. Such an anti-commutation should be preserved if there were to exist an isomorphism between the representations π_S, π_R, however, the definition of R implies this is not the case. For some choices of α and β, we actually have commutation between $\pi_R(Z_\alpha(f)), \pi_R(Z_\beta(g))$, for example:

$$\left[z_{\tilde{R},1}(f), z_{\tilde{R},2}(g) \right] = 0$$

for all $f, g \in \mathcal{L}$. Clearly the product $z_{\tilde{R},1}(f) z_{\tilde{R},2}(g)$ is not zero and thus the representations $\pi_{\tilde{S}}, \pi_{\tilde{R}}$ are not isomorphic in this case.

3.4 Exponential Relations and Generalisations

In addition to a natural isomorphism arising via equivalent R-matrices, the box-sum operation indicates a generalisation to the well-known relation [BPS97]

$$\mathcal{F}_{F_{\tilde{\mathcal{H}} \oplus \tilde{\mathcal{K}}}}(\tilde{\mathcal{H}} \oplus \tilde{\mathcal{K}}) \cong \mathcal{F}_{F_{\tilde{\mathcal{H}}}}(\tilde{\mathcal{H}}) \otimes \mathcal{F}_{F_{\tilde{\mathcal{K}}}}(\tilde{\mathcal{K}}) \tag{3.18}$$

for finite dimensional $\tilde{\mathcal{H}}, \tilde{\mathcal{K}}$ which extends to an isomorphism of the representing data. In particular, we find that for Hilbert spaces $\mathcal{H}, \mathcal{K}$ and $S \in \mathcal{R}_0(\mathcal{H}), R \in \mathcal{R}_0(\mathcal{K})$ that

$$\mathcal{F}_{S \boxplus R}(\mathcal{H} \oplus \mathcal{K}) \cong \mathcal{F}_S(\mathcal{H}) \otimes \mathcal{F}_R(\mathcal{K}) \tag{3.19}$$

where as before the isomorphism extends to one between the data on each space.

The representation of the Zamolodchikov algebra $\mathcal{Z}(S \boxplus R, \mathcal{H} \oplus \mathcal{K})$ on the left hand side of (3.19) is already described in Chapter 2 via the GNS construction (more transparently by applying the distributivity property of the box-times over the box-sum as in Lemma (3.4)). This comes with a pre-equipped vacuum vector $\Omega_{S \boxplus R}$ upon which we can generate a polynomial algebra $\mathcal{P}_{S \boxplus R}$ from the identity element $1_{\mathcal{H} \oplus \mathcal{K}}$ and creation/annihilation operators $z^\dagger_{S \boxplus R}, z_{S \boxplus R}$. The latter obey exchange relations according to the following

$$z_{S \boxplus R,\alpha}(f \oplus 0) z_{S \boxplus R,\beta}(g \oplus 0) = S^{\beta\alpha}_{\delta\gamma} z_{S \boxplus R,\gamma}(g \oplus 0) z_{S \boxplus R,\delta}(f \oplus 0), \tag{3.20}$$

$$z_{S\boxplus R,\alpha}(f\oplus 0)z^\dagger_{S\boxplus R,\beta}(g\oplus 0) = S^{\alpha\gamma}_{\beta\delta} z^\dagger_{S\boxplus R,\gamma}(g\oplus 0)z_{S\boxplus R,\delta}(f\oplus 0) + \delta^\alpha_\beta\langle f,g\rangle\cdot 1_{\mathcal{H}\oplus\mathcal{K}}, \tag{3.21}$$

$$z_{S\boxplus R,\xi}(0\oplus f)z_{S\boxplus R,\eta}(0\oplus g) = R^{\eta\xi}_{\varepsilon\pi} z_{S\boxplus R,\pi}(0\oplus g_\pi)z_{S\boxplus R,\varepsilon}(0\oplus f), \tag{3.22}$$

$$z_{S\boxplus R,\xi}(0\oplus f)z^\dagger_{S\boxplus R,\eta}(0\oplus g) = R^{\xi\pi}_{\eta\varepsilon} z^\dagger_{S\boxplus R,\pi}(0\oplus g)z_{S\boxplus R,\varepsilon}(0\oplus f) + \delta^\xi_\eta\langle f,g\rangle\cdot 1_{\mathcal{H}\oplus\mathcal{K}}, \tag{3.23}$$

$$z_{S\boxplus R,\alpha}(f\oplus 0)z_{S\boxplus R,\eta}(0\oplus g) = z_{S\boxplus R,\eta}(0\oplus g)z_{S\boxplus R,\alpha}(f\oplus 0), \tag{3.24}$$

$$z_{S\boxplus R,\alpha}(f\oplus 0)z^\dagger_{S\boxplus R,\eta}(0\oplus g) = z^\dagger_{S\boxplus R,\eta}(0\oplus g)z_{S\boxplus R,\alpha}(f\oplus 0). \tag{3.25}$$

This data provides us with the natural representation $\pi_{S\boxplus R} : \mathcal{Z}(S\boxplus R, \mathcal{H}\oplus\mathcal{K}) \to \mathcal{P}_{S\boxplus R}$.

Consider now the tensor product

$$\mathcal{F}_S(\mathcal{H})\otimes\mathcal{F}_R(\mathcal{K})$$

where on the individual Fock spaces $\mathcal{F}_S(\mathcal{H}), \mathcal{F}_R(\mathcal{K})$ there act creation/annihilation operators $z^\dagger_S, z_S, z^\dagger_R, z_R$ (endomorphically on the spaces of finite particle number $\mathcal{D}_S, \mathcal{D}_R$) and there exist unique vacuum vectors Ω_S, Ω_R.

Using this prerequisite data, we may define analogous operators to $z_{S\boxplus R}$ and $z^\dagger_{S\boxplus R}$ on the space $\mathcal{F}_S(\mathcal{H})\otimes\mathcal{F}_R(\mathcal{K})$:

$$z_{S,R}\left(f_\alpha\oplus g_\xi\right) := z_{S,\alpha}(f)\otimes 1_\mathcal{K} + 1_\mathcal{H}\otimes z_{R,\xi}(g), \quad f,g\in\mathcal{L}, \tag{3.26}$$

$$\Omega_{S,R} := \Omega_S\otimes\Omega_R. \tag{3.27}$$

As before, the polynomial algebra $\mathcal{P}_{S,R}$ is then defined as the algebra generated by the identity element $1_\mathcal{H}\otimes 1_\mathcal{K}$ and the creation/annihilation operators $z^\dagger_{S,R}, z_{S,R}$.

Lemma 3.10. *The vacuum vector $\Omega_{S,R}$ is cyclic for the polynomial algebra $\mathcal{P}_{S,R}$. That is, $\mathcal{P}_{S,R}\Omega_{S,R}$ is dense in $\mathcal{F}_S(\mathcal{H})\otimes\mathcal{F}_R(\mathcal{K})$.*

Proof. Let $\psi\in\mathcal{F}_S(\mathcal{H})\otimes\mathcal{F}_R(\mathcal{K})$ be orthogonal to $\mathcal{P}_{S,R}\Omega_{S,R}$. Then for any $i,j\in\mathbb{N}_0$ and vectors $f_1,\ldots,f_i\in\mathcal{H}, g_1,\ldots,g_j\in\mathcal{K}$

$$\begin{aligned}&\langle\psi_{i,j}\ ,\ z^\dagger_{S,R}(f_1\oplus 0)\cdots z^\dagger_{S,R}(f_i\oplus 0)z^\dagger_{S,R}(0\oplus g_1)\cdots z^\dagger_{S,R}(0\oplus g_j)\Omega_S\otimes\Omega_R\rangle\\ &=\sqrt{i!j!}\langle\psi_{i,j}\ ,\ P^S_i\otimes P^R_j\left(f_1\otimes\cdots\otimes f_i\otimes g_1\otimes\cdots\otimes g_j\right)\rangle\\ &=\sqrt{i!j!}\langle\psi_{i,j}\ ,\ f_1\otimes\cdots\otimes f_i\otimes g_1\otimes\cdots\otimes g_j\rangle\end{aligned}$$

where $\psi_{i,j}$ is the i,j-th component of ψ and each index corresponds to the particle number in each tensor slot in $\mathcal{F}_S(\mathcal{H})\otimes\mathcal{F}_R(\mathcal{K})$ and we have used the self-adjoint property of $P^S\otimes P^R$ and that the vector ψ is invariant under this projection. Vectors of the form $f_1\otimes\cdots\otimes f_i\otimes g_1\otimes\cdots\otimes g_j$ form a total set in $\mathcal{H}^{\otimes i}\otimes\mathcal{K}^{\otimes j}$ and hence we conclude that $\psi = 0$. □

Finally, we are now able to construct a representation of the algebra $\mathcal{Z}(S \boxplus R, \mathcal{H} \oplus \mathcal{K})$ on $\mathcal{F}_S(\mathcal{H}) \otimes \mathcal{F}_R(\mathcal{K})$ and prove the relation (3.19).

Theorem 3.11. *Let $\tilde{\mathcal{H}}, \tilde{\mathcal{K}}$ be Hilbert spaces of finite dimensions $d_{\tilde{\mathcal{H}}}, d_{\tilde{\mathcal{K}}}$, respectively, and $S \in \mathcal{R}_0(\mathcal{H}), R \in \mathcal{R}_0(\mathcal{K})$, then:*

a) The map $\pi_{S,R} : \mathcal{Z}(S \boxplus R, \mathcal{H} \oplus \mathcal{K}) \to \mathcal{P}_{S,R}$

$$\pi_{S,R}\left(1_{\mathcal{Z}(S\boxplus R,\mathcal{L})}\right) := 1_{\mathcal{H}\otimes\mathcal{K}},$$

$$\pi_{S,R}(Z_\alpha(f)) := \begin{cases} z_{S,R}(f_\alpha \oplus 0), & \alpha \in \{1, \dots, d_{\tilde{\mathcal{H}}}\} \\ z_{S,R}(0 \oplus f_{\alpha - d_{\tilde{\mathcal{H}}}}), & \alpha \in \{d_{\tilde{\mathcal{H}}} + 1, \dots, d_{\tilde{\mathcal{H}}} + d_{\tilde{\mathcal{K}}}\} \end{cases}$$

extends to a unital $\star$-representation of $\mathcal{Z}(S \boxplus R, \mathcal{L})$ on $\mathcal{F}_S^0(\mathcal{H}) \otimes \mathcal{F}_R^0(\mathcal{K})$ with cyclic vector $\Omega_{S,R}$ and

$$\omega_{S,R}(X) = \langle \Omega_{S,R} \ , \ \pi_{S,R}(X)\Omega_{S,R} \rangle, \quad X \in \mathcal{Z}(S \boxplus R, \mathcal{H} \oplus \mathcal{K}). \tag{3.28}$$

b) There exists a unitary operator $V : \mathcal{F}_{S\boxplus R}(\mathcal{H} \oplus \mathcal{K}) \to \mathcal{F}_S(\mathcal{H}) \otimes \mathcal{F}_R(\mathcal{K})$ such that

$$V\Omega_{S\boxplus R} = \Omega_{S,R}, \qquad V\pi_{S\boxplus R}(X)V^* = \pi_{S,R}(X), \quad X \in \mathcal{Z}(S \boxplus R, \mathcal{H} \oplus \mathcal{K}). \tag{3.29}$$

Proof. a) We show first that the operators $z^\dagger_{S,R}, z_{S,R}$ satisfy the same relations as $z^\dagger_{S\boxplus R}, z_{S\boxplus R}$ as outlined in (3.20)-(3.25), firstly noting that

$$z_{S,R}(f_\alpha \oplus 0) = z_{S,\alpha}(f) \otimes 1_{\mathcal{K}}, \quad z_{S,R}(0 \oplus f_\xi) = 1_{\mathcal{K}} \otimes z_{R,\xi}(f), \quad (f \in \mathcal{L}),$$

and similarly for $z^\dagger_{S,R}$.

A priori, the operators $z^\dagger_S, z_S$ and $z^\dagger_R, z_R$ and their distributional kernels satisfy exchange relations (2.10a), (2.10b) governed by S and R, respectively. Let $f, g \in \mathcal{L}$ then

$$\begin{aligned} z_{S,R}(f_\alpha \oplus 0) z_{S,R}(g_\beta \oplus 0) &= z_{S,\alpha}(f) z_{S,\beta}(g) \otimes 1_{\mathcal{K}} \\ &= S^{\beta\alpha}_{\delta\gamma} z_{S,\gamma}(g) z_{S,\delta}(f) \otimes 1_{\mathcal{K}} \\ &= S^{\beta\alpha}_{\delta\gamma} z_{S,R}(g_\gamma \oplus 0) z_{S,R}(f_\delta \oplus 0), \end{aligned}$$

which gives (3.20). The relation (3.22) follows in an analogous way. Similarly

$$\begin{aligned} z_{S,R}(f_\alpha \oplus 0) z^\dagger_{S,R}(g_\beta \oplus 0) &= z_{S,\alpha}(f) z^\dagger_{S,\beta}(g) \otimes 1_{\mathcal{K}} \\ &= S^{\alpha\gamma}_{\beta\delta} z^\dagger_{S,\gamma}(g) z_{S,\delta}(f) \otimes 1_{\mathcal{K}} + \delta^\alpha_\beta \langle f, g \rangle 1_{\mathcal{H}} \otimes 1_{\mathcal{K}} \\ &= S^{\alpha\gamma}_{\beta\delta} z^\dagger_{S,R}(g_\gamma \oplus 0) z_{S,R}(f_\delta \oplus 0) + \delta^\alpha_\beta \langle f, g \rangle 1_{\mathcal{H}} \otimes 1_{\mathcal{K}}. \end{aligned}$$

As for (3.24), (3.25) we see that since $z_{S,R}(f_\alpha \oplus 0)$ and $z_{S,R}(0 \oplus g_\eta)$ operate on different tensor slots, they commute. Lemma (3.10) and the action of the annihilation operator $z_{S,R}$ on $\Omega_{S,R}$ implies that the functional $\omega_{S,R}$ as defined in (3.28) satisfies i)-iii) in Definition (3.1) and this coincides with ω.

b) Let $X \in \mathcal{Z}(S \boxplus R, \mathcal{H} \oplus \mathcal{K})$. Then

$$\begin{aligned}\|\pi_{S,R}(X)\Omega_{S,R}\|^2 &= \langle \pi_{S,R}(X)\Omega_{S,R}, \pi_{S,R}(X)\Omega_{S,R}\rangle \\ &= \langle \Omega_{S,R}, \pi_{S,R}(X^*X)\Omega_{S,R}\rangle \\ &= \omega(X^*X) \\ &= \|\pi_{S\boxplus R}(X)\Omega_{S\boxplus R}\|^2.\end{aligned}$$

This shows that the map $V : \mathcal{P}_{S\boxplus R}\Omega_{S\boxplus R} \to \mathcal{P}_{S,R}\Omega_{S,R}$, $V\pi_{S\boxplus R}(X)\Omega_{S\boxplus R} := \pi_{S,R}(X)\Omega_{S,R}$ ($X \in \mathcal{Z}(S \boxplus R, \mathcal{H} \oplus \mathcal{K})$) is well-defined and isometric. Moreover, for $X, Y \in \mathcal{Z}(S \boxplus R, \mathcal{H} \oplus \mathcal{K})$

$$\begin{aligned}V\pi_{S\boxplus R}(X)V^*\pi_{S,R}(Y)\Omega_{S,R} &= V\pi_{S\boxplus R}(X)\pi_{S\boxplus R}(Y)\Omega_{S\boxplus R} \\ &= V\pi_{S\boxplus R}(XY)\Omega_{S\boxplus R} \\ &= \pi_{S,R}(XY)\Omega_{S,R} \\ &= \pi_{S,R}(X)\pi_{S,R}(Y)\Omega_{S,R}.\end{aligned}$$

Cyclicity of $\Omega_{S\boxplus R}, \Omega_{S,R}$ for the representations $\pi_{S\boxplus R}$ and $\pi_{\tilde{S},\tilde{R}}$, respectively, indicates that V extends to a unitary $\mathcal{F}_{S\boxplus R}(\mathcal{H} \oplus \mathcal{K}) \to \mathcal{F}_S(\mathcal{H}) \otimes \mathcal{F}_R(\mathcal{K})$ satisfying (3.29).

□

For simple R-matrices, this result decomposes Fock spaces over higher dimensional Hilbert spaces into a tensor product of many Fock spaces over smaller dimensional Hilbert spaces. This is, in particular, applicable to matrices in normal form (3.12).

Corollary 3.12. *Let* $\tilde{\mathcal{H}} = \bigoplus_{i=1}^n \tilde{\mathcal{H}}_i$, *dim*$(\tilde{\mathcal{H}}_i) = d_i$ *and let* $\boxplus_{i=1}^n \varepsilon_i 1_{d_i^2} = N \in R_0(\mathcal{H})$ *be an involutive R-matrix in normal form. Then there exists a unitary*

$$V : \mathcal{F}_S(\mathcal{H}) \to \bigotimes_{i=1}^n \mathcal{F}_{\varepsilon_i}(\mathcal{H}_i).$$

As in Chapter 2 where we have the specific choice of $\mathcal{L} = L^2(\mathbb{R})$, the Definitions (2.13), (4.41) provides us with a definition of fields $\phi_{S\boxplus R}$. The isomorphism is now of relevance in this context and we have the necessary data to construct a field operator unitarily equivalent to $\phi_{S\boxplus R}$

$$\phi_{S,R}(f \oplus g) = \phi_S(f) \otimes 1_{\mathcal{K}} + 1_{\mathcal{H}} \otimes \phi_R(g),$$

for $f \in \mathscr{S}(\mathbb{R}^2) \otimes \tilde{\mathcal{H}}$ and $g \in \mathscr{S}(\mathbb{R}^2) \otimes \tilde{\mathcal{K}}$. By definition, it should be clear that this field inherits many properties from $\phi_{S \boxplus R}$ as in Proposition (2.5). Moreover, defining the reflected field as

$$\phi'_{S,R}(f \oplus g) = \phi'_S(f) \otimes 1_{\mathcal{K}} + 1_{\mathcal{H}} \otimes \phi'_R(g),$$

it is straightforward to see that the fields $\phi_{S,R}, \phi'_{S,R}$ are relatively wedge local given the correct localisation properties of their arguments as in Theorem (2.6). Moving to the algebraic aspect, we recall these form part of a definition of von Neumann algebras, in particular for any $x \in \mathbb{R}^2$

$$\mathcal{A}_{S \boxplus R}(W_L + x) := \left\{ e^{i\phi_{S \boxplus R}(f \oplus g)} \ : \ f = f^* \in \mathscr{S}(W_L + x) \otimes \tilde{\mathcal{H}}, \ g = g^* \in \mathscr{S}(W_L + x) \otimes \tilde{\mathcal{K}} \right\}'', \tag{3.30}$$

$$\mathcal{A}_{S \boxplus R}(W_R + x) := \left\{ e^{i\phi'_{S \boxplus R}(f \oplus g)} \ : \ f = f^* \in \mathscr{S}(W_R + x) \otimes \tilde{\mathcal{H}}, \ g = g^* \in \mathscr{S}(W_R + x) \otimes \tilde{\mathcal{K}} \right\}'', \tag{3.31}$$

with these being subject to the same properties as in Proposition (2.7). Moreover we define

$$\mathcal{A}_{S,R}(W_L + x) := \left\{ e^{i\phi_{S,R}(f \oplus g)} \ : \ f = f^* \in \mathscr{S}(W_L + x) \otimes \tilde{\mathcal{H}}, \ g = g^* \in \mathscr{S}(W_L + x) \otimes \tilde{\mathcal{K}} \right\}'' \tag{3.32}$$

$$\mathcal{A}_{S,R}(W_R + x) := \left\{ e^{i\phi'_{S,R}(f \oplus g)} \ : \ f = f^* \in \mathscr{S}(W_R + x) \otimes \tilde{\mathcal{H}}, \ g = g^* \in \mathscr{S}(W_R + x) \otimes \tilde{\mathcal{K}} \right\}''. \tag{3.33}$$

The latter algebras are generated by exponentials of fields which have the form of a sum of operators which clearly commute, in particular

$$e^{i\phi_{S,R}(f \oplus g)} = e^{i(\phi_S(f) \otimes 1_{\mathcal{K}} + 1_{\mathcal{H}} \otimes \phi_R(g))} = e^{i(\phi_S(f) \otimes 1_{\mathcal{K}})} e^{i(1_{\mathcal{H}} \otimes \phi_R(g))}$$

by standard properties of the exponential.

Lemma 3.13. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $A \in \mathcal{B}(\mathcal{H}), B \in \mathcal{B}(\mathcal{K})$. Then*

i) $e^{A \otimes 1_{\mathcal{K}}} = e^A \otimes 1_{\mathcal{K}}$,

ii) $e^{1_{\mathcal{H}} \otimes B} = 1_{\mathcal{H}} \otimes e^B$.

Proof. We proceed by calculation

$$e^{A \otimes 1_{\mathcal{K}}} = \sum_{n=0}^{\infty} \frac{(A \otimes 1_{\mathcal{K}})^n}{n!} = \sum_{n=0}^{\infty} \frac{(A^n \otimes 1_{\mathcal{K}})}{n!} = \left(\sum_{n=0}^{\infty} \frac{A^n}{n!} \right) \otimes 1_{\mathcal{K}} = e^A \otimes 1_{\mathcal{K}}$$

The proof for ii) follows in an identical way. □

This idea can also be extended to the operators $e^{i(\phi_S(f) \otimes 1_{\mathcal{K}})}$, $e^{i(1_{\mathcal{H}} \otimes \phi_R(g))}$ by the properties listed in Proposition (2.5). In particular, we note that on the subspaces of finite particle number $\mathcal{D}_S \subset \mathcal{F}_S(\mathcal{H}), \mathcal{D}_R \subset \mathcal{F}_R(\mathcal{K})$ the field operators $\phi_S(f), \phi_R(g)$ are essentially self-adjoint for the choice of input vectors $f = f^*, g = g^*$ as in (3.32). Moreover, the elements of $\mathcal{D}_S, \mathcal{D}_R$ contain entire analytic vectors for $\phi_S(f), \phi_R(g)$, respectively, on which the expressions in Lemma (3.13) converge.

Theorem 3.14. *Let $S \in \mathcal{S}_c(\tilde{\mathcal{H}}), R \in \mathcal{S}_c(\tilde{\mathcal{K}})$, $x \in \mathbb{R}^2$ and the von Neumann algebra $\mathcal{A}_{S\boxplus R}(W_L + x)$ be defined as in* (3.30)*. Then*

$$\mathcal{A}_{S\boxplus R}(W_L + x) \cong \mathcal{A}_S(W_L + x) \otimes \mathcal{A}_R(W_L + x).$$

Proof. Clearly the algebras $\mathcal{A}_{S\boxplus R}(W_L + x), \mathcal{A}_{S,R}(W_L + x)$ are isomorphic given that the generating fields $\phi_{S\boxplus R}, \phi_{S,R}$ are unitarily equivalent. Lemma (3.13) then provides the isomorphism between $\mathcal{A}_{S,R}(W_L + x)$ and the tensor product $\mathcal{A}_S(W_L + x) \otimes \mathcal{A}_R(W_L + x)$. □

This theorem with corollary (3.12) implies that given a model described by a tensor product of Bose/Fermi spaces the resulting local algebra is nothing more than an algebraic tensor product of better understood algebras [Lec07] simplifying matters greatly.

In conclusion for this Chapter, we have outlined a possible simplification for potentially complicated quantum field theories - beginning with a constant S-matrix that is type one equivalent to its normal form, by isomorphisms we may describe the theory on a tensor product of Bose/Fermi Fock spaces. Though this hints towards the method of [Luk95], our method is slightly more natural arising only from isomorphisms of R-matrices.

Chapter 4

Scaling Limits of Integrable Models

Having discussed simplistic constant S-matrices in the previous chapter, we advance to the more general situation touched upon briefly in Chapter 2 of rapidity dependent S-matrices as in Definition (2.1). Though many specific examples of integrable models in (1+1)-dimensions are known [BR18, BFK05, BK03], in some cases the details of the resulting theory are ill understood and open questions surround them.

A fruitful area of discussion in this direction is in analysing the ultraviolet scaling limit of a model. A well studied application of such investigations is the proof of asymptotic freedom of the theory of Quantum Chromodynamics (QCD) [GW73] which is now accepted as the theory of strong interactions. Though there exist many methods to calculate a scaling limit depending on the description of the theory presented, we will study the scaling algebras, and in particular the scaling of the generating wedge-local field operators. That is, the algebra generated by functions $\lambda \mapsto A_\lambda$ of the scaling parameter λ whose values take on that of observables of the base theory. The scaling limit of the theory is then identified as the GNS representation of the scaling algebra generated from the scaling limit of the vacuum state on the unscaled algebra.

4.1 Limits of S-matrices

In the general analysis of quantum field theories the limits of the governing S-matrix is not necessary data, nor particularly informative. However, in the context of the short distance scaling limit it becomes an important quantity describing the twisting between the tensor product of chiral components and their structure [BLM11].

Scaling distances in Minkowski space as $x \mapsto \lambda x$ momenta are then scaled as $p \mapsto \lambda^{-1}p$ and hence the rapidity variable (the dependent parameter of an S-matrix in our description) scales according to $\theta = \sinh^{-1}(\frac{p}{m}) \mapsto \sinh^{-1}(\frac{p}{\lambda m})$ which implies we are concerned with the limits $S_\pm := \lim_{\theta\to\infty} S(\theta)$ in our analysis.

Throughout this chapter, we will again consider a Hilbert space $\mathcal{H} := L^2(\mathbb{R}) \otimes \tilde{\mathcal{H}}$ where we identify the previously more general and separable Hilbert space $\mathcal{L}$ with $L^2(\mathbb{R})$ for simplicity and $\tilde{\mathcal{H}}$ is of finite dimension.

Definition 4.1.

a) An S-matrix $S \in \mathcal{S}(\tilde{\mathcal{H}})$ is called *regular* if there exists a $\kappa > 0$ such that S (and its constituent elements) continues to a bounded analytic function in the strip $S(-\kappa, \pi + \kappa)$. The subset of all regular S-matrices is denoted by $\mathcal{S}_{\text{reg}}(\tilde{\mathcal{H}}) \subset \mathcal{S}(\tilde{\mathcal{H}})$.

b) A regular S-matrix $S \in \mathcal{S}_{\text{reg}}(\tilde{\mathcal{H}})$ is called an *S-matrix with a limit* if the two limits $S_\pm := \lim_{\theta\to\pm\infty} S(\theta)$ exist. The set of all S-matrices with a limit is denoted by $\mathcal{S}_{\text{lim}}(\tilde{\mathcal{H}}) \subset \mathcal{S}_{\text{reg}}(\tilde{\mathcal{H}})$.

The question of the size of $\mathcal{S}_{\text{lim}}(\tilde{\mathcal{H}})$ is not a trivial one in the higher dimensional (in terms of the S-matrix) case we consider here. In the scalar case, this is already settled - in fact, one may write down the complete set $\mathcal{S}_{\text{lim}}(\mathbb{C})$ and show a number of further results, such as the limits of any such function having coinciding limits [BLM11]. For the higher dimensional case, we have no such clear cut result and the situation is a much richer one.

For reference, we note that the definition of a scalar scattering function can be retrieved from Definition (2.1) by setting $\tilde{\mathcal{H}} = \mathbb{C}$. The properties of a scalar scattering function can be written in one line as:

$$\overline{S(\theta)} = S(\theta)^{-1} = S(\theta + i\pi) = S(-\theta), \quad \theta \in \mathbb{R}. \tag{4.1}$$

We state the following result already alluded to.

Proposition 4.2. [BLM11]

a) The set $\mathcal{S}_{\lim}(\mathbb{C})$ *consists precisely of the functions*

$$S(z) = \varepsilon \cdot \prod_{k=1}^{N} \frac{\sinh(z) - \sinh(b_k)}{\sinh(z) + \sinh(b_k)}, \quad z \in \overline{S(0,\pi)}, \tag{4.2}$$

where $\varepsilon = \pm 1, N \in \mathbb{N}_0$ *and* $\{b_1, \ldots, b_N\}$ *is a set of complex numbers in the strip* $0 < Im(b_1), \ldots, Im(b_N) < \frac{\pi}{2}$ *such that for each* b_k*, also* $-\overline{b_k}$ *is also contained in* $\{b_1, \ldots, b_N\}$*.*

b) For each $S \in \mathcal{S}_{\lim}(\mathbb{C})$*, the two limits* S_-, S_+ *coincide and are equal to* ± 1*.*

An analogous result for any finite dimensional $\tilde{\mathcal{H}}$ and arbitrary $S \in \mathcal{S}_{\lim}(\mathcal{H})$ is unknown and would be a difficult feat given that the Yang-Baxter equation results in a set of $(d_{\tilde{\mathcal{H}}})^6$ cubic equations for the elements of S. Instead, we consider a simple, but non-trivial form of S-matrices, the so-called diagonal ones.

Definition 4.3. A *diagonal* S-matrix S_D is of the form

$$S_D(\theta)^{\alpha\beta}_{\gamma\delta} = \omega_{\alpha\beta}(\theta)\delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma}, \tag{4.3}$$

(with no summation over α and β). The continuous, bounded functions $\omega_{\alpha\beta} : \overline{S(0,\pi)} \to \mathbb{C}$ are analytic in the interior of the strip and satisfy

$$\overline{\omega_{\alpha\beta}(\theta)} = \omega_{\alpha\beta}(\theta)^{-1} = \omega_{\beta\alpha}(-\theta) = \omega_{\beta\alpha}(\theta + i\pi). \tag{4.4}$$

It has been shown that matrices of this form solve the Yang-Baxter equation with no further restriction on the functions $\omega_{\alpha\beta}$ [AL17], hence these are really examples of S-matrices as defined by Definition (2.1).

For $\alpha = \beta$ we see that (4.4) read exactly (4.1), however for the remaining cases we have a slightly different set of symmetry conditions which we now discuss.

Definition 4.4. The set of functions $\mathcal{G}_{\lim}$ is the set of regular, bounded and continuous functions $G : \overline{S(0,\pi)} \to \mathbb{C}$ that are analytic in the interior of the strip and satisfy for all $\theta \in \mathbb{R}$:

$$|G(\theta)| = 1, \; G(\theta) = \overline{G(i\pi + \theta)}, \; \lim_{\theta \to \pm\infty} G(\theta) \text{ exist.} \tag{4.5}$$

As it turns out, these functions can also be completely classified in a similar fashion to the scalar scattering functions.

Proposition 4.5. *The set $\mathcal{G}_{\lim}$ consists precisely of the functions*

$$G(z) = \varepsilon \prod_{k=1}^{N} \frac{e^z - e^{b_k}}{e^z - e^{\overline{b_k}}} \cdot \frac{e^z - e^{\overline{b_k}+i\pi}}{e^z - e^{b_k - i\pi}}, \tag{4.6}$$

where $\varepsilon = \pm 1$, $N \in \mathbb{N}_0$ and $\{b_1, \ldots, b_N\}$ is a set of complex numbers in the strip $0 < \mathrm{Im}(b_1), \ldots, \mathrm{Im}(b_N) \leq \pi/2$.

Moreover, for each $G \in \mathcal{G}_{\lim}$ we have that $\lim_{\theta\to\infty} G(\theta) = \lim_{\theta\to-\infty} G(\theta) = \pm 1$, $\theta \in \mathbb{R}$.

Proof. For $z \in \mathbb{R}$ each factor

$$g_{b_k} : \zeta \mapsto \pm \frac{e^z - e^{b_k}}{e^z - e^{\overline{b_k}}} \cdot \frac{e^z - e^{\overline{b_k}+i\pi}}{e^z - e^{b_n - i\pi}}$$

satisfies $|g_{b_k}(z)| = 1$ (which is trivial to see) and also $g_{b_k}(z) = \overline{g_{b_k}(i\pi + z)}$. Indeed,

$$\begin{aligned} g_{b_k}(i\pi + z) &= \pm \frac{-e^z - e^{b_k}}{-e^z - e^{\overline{b_k}}} \cdot \frac{-e^z - e^{\overline{b_k}+i\pi}}{-e^z - e^{b_n - i\pi}} \\ &= \pm \frac{e^z + e^{b_k}}{e^z + e^{\overline{b_k}}} \cdot \frac{e^z + e^{\overline{b_k}+i\pi}}{e^z + e^{b_n - i\pi}} \\ &= \pm \frac{e^z - e^{b_k - i\pi}}{e^z - e^{\overline{b_k}+i\pi}} \cdot \frac{e^z - e^{\overline{b_k}}}{e^z - e^{b_n}} = \overline{g_{b_k}(z)}. \end{aligned}$$

Given the location of the poles in the above expression, we can surmise that for a sufficiently small $\delta > 0$, the factor g_{z_k} is analytic and bounded in the strip $S(-\mathrm{Im}(z_k) + \delta, \pi + \mathrm{Im}(z_k) - \delta) \supset S(0, \pi)$. The product (4.6) is finite, so there exists a $\kappa > 0$ such that G is analytic and bounded in the strip $S(-\kappa, \pi + \kappa)$. Clearly the limits $\lim_{\theta\to\infty} G(\theta), \lim_{\theta\to-\infty} G(\theta)$ coincide from the expression which shows that $G \in \mathcal{G}_{\lim}$.

Now we pick an arbitrary $G \in \mathcal{G}_{\lim}$ and show that it is of the form (4.6). Let $\varepsilon_1 := \lim_{\theta\to\infty} G(\theta), \varepsilon_2 := \lim_{\theta\to-\infty} G(\theta)$, then from the regularity properties of G, we have that $G(\theta + i\lambda) \to \varepsilon_1$ as $\theta \to \infty$ uniformly in $\lambda \in [0, \pi]$ [Tit39] and similar for the opposite limit given that the strip $S(0, \pi)$ is biholomorphic to the unit disc. Moreover, since $|G(\theta)| = 1$ for all real θ, then $|\varepsilon_1| = 1$, but since $G(\theta) = \overline{G(i\pi + \theta)}$ also, it must mean that $\varepsilon_1 = \pm 1$ since $\lim_{\theta\to\infty} \overline{G(i\pi + \theta)} = \lim_{\theta\to\infty} G(i\pi + \theta)$.

The function G is continuous on the closed strip $\overline{S(0, \pi)}$ and of unit modulus on the boundary a priori. These properties together with the uniformity of the limit throughout the strip allows us to conclude that G has only finitely many zeroes in $S(0, \pi)$ [Fat23]. Let $b_1, \ldots, b_N$ be the zeroes of G whose imaginary parts μ satisfy $0 < \mu \leq \frac{\pi}{2}$ then for every zero b_i, there is also a corresponding zero $\overline{b_i} + i\pi$. We construct the finite Blaschke product

$$B(z) = \varepsilon_1 \prod_{k=1}^{N} \frac{e^z - e^{b_k}}{e^z - e^{\overline{b_k}}} \cdot \frac{e^z - e^{\overline{b_k}+i\pi}}{e^z - e^{b_n - i\pi}}$$

and note that it has precisely the same number of zeroes as G and also $B(\theta + i\lambda) \to \varepsilon_1$ for $\theta \to \pm\infty$ uniformly in $\lambda \in [0, \pi]$.

Now define a new function F by

$$F = G \cdot B^{-1}.$$

Clearly F has no zeroes in $S(0, \pi)$ and also $F(\theta + i\lambda) \to 1$ for $\theta \to \infty$ uniformly for $\lambda \in [0, \pi]$ as regularity is clearly inherited by this product. As F is continuous on $\overline{S(0, \pi)}$ and of unit modulus on the boundary of the strip (these two properties hold for G and B^{-1}, hence also holds for their product) so it is bounded above and below, and there exists a $K > 0$ such that $K < |F(\theta)| < 1$ for $\theta \in \overline{S(0, \pi)}$. We may meromorphically extend F, however since F has no zeroes in $\overline{S(0, \pi)}$ it is actually an analytic continuation. On this continuation, the boundedness of F on $\overline{S(0, \pi)}$ it implies that it also holds that $|F(\zeta)| < K^{-1} < \infty$ for all $\zeta \in S(-\pi, \pi)$. Taking $\theta \mapsto \theta - i\pi$ in (4.5) we have that

$$G(\theta - i\pi) = G(\theta + i\pi)$$

for $\theta \in \mathbb{R}$ and the same clearly holds for B^{-1}. Hence F extends to a $2\pi i$-periodic, entire function which is bounded and so constant by Liouville's Theorem[1]. Thus we have that $F(\theta) = \lim_{\theta \to \infty} F(\theta) = 1$, so $G = FB = B$. □

As one can see, the sets $\mathcal{S}_{\lim}(\mathbb{C})$, $\mathcal{G}_{\lim}$ are somewhat comparable in that they both consist entirely of functions in the form of finite Blaschke products. Furthermore, given the symmetries satisfied by the functions belonging to each set, we can observe that $\mathcal{S}_{\lim}(\mathbb{C}) \subset \mathcal{G}_{\lim}$.

Theorem 4.6. *The limits* $\lim_{\theta \to \infty} S_D(\theta)$, $\lim_{\theta \to -\infty} S_D(\theta)$ *exist if and only if* $\omega_{\alpha\alpha} \in \mathcal{S}_{\lim}(\mathbb{C})$ *and* $\omega_{\alpha\beta} \in \mathcal{G}_{\lim}$ *(*$\alpha \neq \beta, \omega_{\alpha\beta}(\theta) = \overline{\omega_{\beta\alpha}(-\theta)}$*) for all* α, β.

Proof. The conditions in (4.4) for $\alpha = \beta$ read

$$\overline{\omega_{\alpha\alpha}(\theta)} = \omega_{\alpha\alpha}(\theta)^{-1} = \omega_{\alpha\alpha}(-\theta) = \omega_{\alpha\alpha}(\theta + i\pi), \quad (\theta \in \mathbb{R})$$

which with the existence of the limits $\lim_{\theta \to \infty} \omega_{\alpha\alpha}(\theta), \lim_{\theta \to -\infty} \omega_{\alpha\alpha}(\theta)$ implies that $\omega_{\alpha\alpha}$ must belong to $\mathcal{S}_{\lim}(\mathbb{C})$.

For $\alpha \neq \beta$ we have in particular

$$\overline{\omega_{\alpha\beta}(\theta)} = \overline{\omega_{\beta\alpha}(i\pi - \theta)} = \omega_{\beta\alpha}(-\theta),$$

[1]Liouville's Theorem states that any bounded, entire function must be constant.

hence we have functions that satisfy

$$|\omega_{\alpha\beta}(\theta)| = 1, \quad \omega_{\alpha\beta}(\theta) = \omega_{\alpha\beta}(i\pi + \theta)$$

for $\theta \in \mathbb{R}$. Since the limits $\lim_{\theta\to\infty} \omega_{\alpha\beta}(\theta), \lim_{\theta\to-\infty} \omega_{\alpha\beta}(\theta)$ exist, they have the form of (4.6).

The opposite direction is trivial by the definition of functions in $\mathcal{S}_{\lim}(\mathbb{C})$ and $\mathcal{G}_{\lim}$. $\square$

This analysis of diagonal S-matrices can be viewed as a generalisation to the previous results in the scalar case as they are simple, but non-trivial examples of an S-matrix. An example of a more complex form would be the S-matrix $S_{\sigma,N}$ governing the $O(N)$ σ-models which has the explicit form

$$S_{\sigma,N}(\theta)^{\alpha_1\alpha_2}_{\beta_1\beta_2} := \sigma_1(\theta)\delta^{\alpha_1}_{\alpha_2}\delta^{\beta_1}_{\beta_2} + \sigma_2(\theta)\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1} + \sigma_3(\theta)\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}, \tag{4.7}$$

with the functions

$$\sigma_2(\theta) := Q(\theta)Q(i\pi - \theta), \quad Q(\theta) := \frac{\Gamma\left(\frac{1}{N-2} - i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} - i\frac{\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{N-2} - i\frac{\theta}{2\pi}\right)\Gamma\left(-i\frac{\theta}{2\pi}\right)},$$

$$\sigma_1(\theta) := -\frac{2\pi i}{(N-2)}\frac{\sigma_2(\theta)}{i\pi - \theta},$$

$$\sigma_3(\theta) := \sigma_1(i\pi - \theta) = -\frac{2\pi i}{(N-2)}\frac{\sigma_2(\theta)}{\theta}$$

where Γ denotes the complex Gamma function.

It is well known that this S-matrix satisfies our definition, and in particular the Yang-Baxter equation and one may also calculate its limiting behaviour since it is regular in our sense [LS12].

Before proceeding, we recall that for $z \in \mathbb{C}\backslash\mathbb{Z}_-$ (where $\mathbb{Z}_-$ denotes all non-positive integers)

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z).$$

Moreover, for large enough $|z|$ and $|\arg(z)| < \pi - \varepsilon$ $(\varepsilon > 0)$ then we have Stirling's formula:

$$\Gamma(z) \sim \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z,$$

where $\sim$ indicates that the ratio of both sides converges to one or asymptotically converges.

Proposition 4.7. *Let $S_{\sigma,N}$ be defined as in* (4.7) *for any $N > 2$, then*

$$\lim_{\theta\to\pm\infty} S_{\sigma,N}(\theta) = F_{\mathcal{H}}.$$

Proof. Let $z := \frac{1}{N-2} + i\frac{\theta}{2\pi}$ and $w = i\frac{\theta}{2\pi} = i\mathrm{Im}(z)$, then

$$\begin{aligned}
\sigma_2(\theta) &= \frac{\Gamma(\overline{z})\Gamma\left(\frac{1}{2}+\overline{w}\right)\Gamma\left(\frac{1}{2}+z\right)\Gamma(1+w)}{\Gamma\left(\frac{1}{2}+\overline{z}\right)\Gamma(\overline{w})\Gamma(1+z)\Gamma\left(\frac{1}{2}+w\right)} \\
&= \frac{\Gamma(\overline{z})\Gamma\left(\frac{1}{2}+\overline{w}\right)\Gamma\left(\frac{1}{2}+z\right)w\Gamma(w)}{\Gamma\left(\frac{1}{2}+\overline{z}\right)\Gamma(\overline{w})z\Gamma(z)\Gamma\left(\frac{1}{2}+w\right)} \\
&= \frac{\Gamma(\overline{z})}{z\Gamma(z)}\frac{w\Gamma(w)}{\Gamma(\overline{w})}\frac{\Gamma\left(\frac{1}{2}+z\right)}{\Gamma\left(\frac{1}{2}+\overline{z}\right)}\frac{\Gamma\left(\frac{1}{2}+\overline{w}\right)}{\Gamma\left(\frac{1}{2}+w\right)} \\
&= \frac{(\Gamma(\overline{z}))^2}{z(\Gamma(z))^2}\frac{w(\Gamma(w))^2}{(\Gamma(\overline{w}))^2}\frac{2^{1-2z}\Gamma(2z)}{2^{1-2\overline{z}}\Gamma(2\overline{z})}\frac{2^{1-2\overline{w}}\Gamma(2\overline{w})}{2^{1-2w}\Gamma(2w)} \\
&= \frac{(\Gamma(\overline{z}))^2}{z(\Gamma(z))^2}\frac{w(\Gamma(w))^2}{(\Gamma(\overline{w}))^2}\frac{2^{4\overline{w}}\Gamma(2z)}{\Gamma(2\overline{z})}\frac{2^{4w}\Gamma(2\overline{w})}{\Gamma(2w)} \\
&= \frac{(\Gamma(\overline{z}))^2}{z(\Gamma(z))^2}\frac{w(\Gamma(w))^2}{(\Gamma(\overline{w}))^2}\frac{\Gamma(2z)}{\Gamma(2\overline{z})}\frac{\Gamma(2\overline{w})}{\Gamma(2w)} \\
&\sim \frac{\left(\frac{\overline{z}}{e}\right)^{2\overline{z}}}{\overline{z}\left(\frac{z}{e}\right)^{2z}}\frac{\overline{w}\left(\frac{w}{e}\right)^{2w}}{\left(\frac{\overline{w}}{e}\right)^{2w}}\frac{\sqrt{\overline{z}}\left(\frac{2z}{e}\right)^{2z}}{\sqrt{z}\left(\frac{2\overline{z}}{e}\right)^{2\overline{z}}}\frac{\sqrt{w}\left(\frac{2\overline{w}}{e}\right)^{2\overline{w}}}{\sqrt{\overline{w}}\left(\frac{2w}{e}\right)^{2w}} \\
&= \frac{|w|2^{-2\overline{z}+2z}2^{-2w+2\overline{w}}}{|z|} \\
&= \frac{|w|}{|z|} = \frac{|\theta|}{\sqrt{\frac{4\pi^2}{(N-2)^2}+\theta^2}} \to 1 \quad (\theta\to\pm\infty).
\end{aligned}$$

From the definitions of σ_1, σ_3 given in terms of σ_2, it is then straightforward to see that

$$\sigma_1(\theta), \sigma_3(\theta) \to 0 \quad (\theta\to\pm\infty).$$

□

The complex seeming nature of the model defined by $S_{\sigma,N}$ therefore has a very simple limiting behaviour implying that their ultraviolet scaling limit is simpler in some respects. In particular, we can deduce that the resulting model of a short distance scaling limit results in an untwisted tensor product of chiral models. Despite this example having coinciding limits, it is not immediately clear that this is always the case, however generally speaking the limit values S_+, S_- enjoy a number of symmetries.

Proposition 4.8. *Let $S \in \mathcal{S}_{lim}(\mathcal{H})$ and $S_\pm := \lim_{\theta\to\pm\infty} S(\theta)$ then*

i) $S_\pm^* = S_\mp$,

ii) $S_\pm^c = S_\mp$, *where* $(S^c)^{\alpha\beta}_{\delta\gamma} = S^{\bar{\delta}\alpha}_{\gamma\bar{\beta}}$,

iii) $(S_\pm)^{\alpha\beta}_{\delta\gamma} = (S^l_\pm)^{\alpha\beta}_{\delta\gamma} := \overline{(S_\pm)^{\gamma\beta}_{\delta\alpha}}$, $(S_\pm)^{\alpha\beta}_{\delta\gamma} = (S^r_\pm)^{\alpha\beta}_{\delta\gamma} := \overline{(S_\pm)^{\alpha\delta}_{\beta\gamma}}$,

in addition to being unitary solutions to the Yang-Baxter equation.

Proof. By definition, S is unitary on the real line. That is, $S(\theta)^* = S(\theta)^{-1}$ for $\theta \in \mathbb{R}$ and in the limit $\theta \to \pm\infty$ this property remains for both $S_\pm$, clearly. Similarly, S satisfies the Yang-Baxter equation:

$$(1 \otimes S(\theta))(S(\theta+\theta') \otimes 1)(1 \otimes S(\theta')) = (S(\theta') \otimes 1)(1 \otimes S(\theta+\theta'))(S(\theta) \otimes 1)$$

for $\theta, \theta' \in \mathbb{R}$. We're free to take $\theta = \theta'$ for example, and in this case, the above now reads

$$(1 \otimes S(\theta))(S(2\theta) \otimes 1)(1 \otimes S(\theta)) = (S(\theta) \otimes 1)(1 \otimes S(2\theta))(S(\theta) \otimes 1)$$

which in the limit $\theta \to \pm\infty$ becomes

$$(1 \otimes S(\pm\infty))(S(\pm\infty) \otimes 1)(1 \otimes S(\pm\infty)) = (S(\pm\infty) \otimes 1)(1 \otimes S(\pm\infty))(S(\pm\infty) \otimes 1)$$

showing that $S_\pm$ also solve the Yang-Baxter equation.

Unitarity and Hermitian analyticity of S culminate to $S(\theta)^* = S(-\theta)$ which gives (i) in the limit $\theta \to \pm\infty$.

The crossing symmetry of S reads as $S(\theta) = S^c(i\pi - \theta)$ and now by regularity in the limit we arrive at

$$S_\pm = \lim_{\theta\to\pm\infty} S(\theta) = \lim_{\theta\to\pm\infty} S^c(i\pi - \theta) = \lim_{\theta\to\pm\infty} S^c(-\theta)$$

showing (ii).

Part (iii) is a direct consequence of parts (i) and (ii) and can be easily retrieved by calculating $(S^c_\pm)^*$. □

Corollary 4.9. *Let $S \in \mathcal{S}_{lim}$ and $S_\pm := \lim_{\theta\to\pm\infty} S(\theta)$ then the following statements are equivalent.*

i) $S_\pm$ *is involutive,*

ii) $S_\pm$ *is crossing symmetric, i.e.* $S^c_\pm = S_\pm$, *iii)* $S_+ = S_-$.

Proof. i) $\implies$ ii): If S_+ is involutive, together with unitarity implies that $S_\pm = S_\pm^*$, which combining parts i), ii) of Proposition (4.8) gives $S_\pm = S_\pm^c$.

ii) $\implies$ iii): This follows immediately from Proposition (4.8) ii).

iii) $\implies$ i): Since $S_\pm$ is self-adjoint and unitary, this immediately follows. □

We denote the set of all constant R-matrices R such that $R = \lim_{\theta\to\infty} S(\theta)$ for some $S \in \mathcal{S}_{\mathrm{lim}}(\mathcal{H})$ as $\mathcal{R}_{\mathrm{lim}}(\tilde{\mathcal{H}})$. Of course, this also includes the opposite limit of S as $\theta \to -\infty$ if we note that also $S_- \in \mathcal{S}_{\mathrm{lim}}(\mathcal{H})$.

In preparation for calculating the short distance scaling behaviour of quantum fields, we must note the behaviour of the space-time scaling $x \mapsto \lambda x$ on an element $S \in \mathcal{S}_{\mathrm{lim}}(\mathcal{H})$. For such data, it is easier to work with the momenta $p = m\sinh(\theta), q = m\sinh(\theta')$ as arguments for S rather than the rapidities which more clearly maintains a mass dependency. Indeed,

$$\theta - \theta' = \sinh^{-1}\left(\frac{p}{m}\right) - \sinh^{-1}\left(\frac{q}{m}\right) = \sinh^{-1}\left(\frac{p\omega_p^m - q\omega_p^m}{m^2}\right)$$

where we define the energies $\omega_p^m := \sqrt{p^2 + m^2}$, $\omega_q^m := \sqrt{q^2 + m^2}$. Now, for any $m > 0$ and $S \in \mathcal{S}_{\mathrm{lim}}(\mathcal{H})$ we have a modified matrix-valued function $S_m : \mathbb{R}^2 \to \mathcal{B}(\tilde{\mathcal{H}} \otimes \tilde{\mathcal{H}})$

$$S_m(p,q) := S\left(\sinh^{-1}\left(\frac{p\omega_p^m - q\omega_p^m}{m^2}\right)\right).$$

The properties of S_m are irrelevant to our needs, however, we do note that clearly, we have the scaling relation

$$S_m(\lambda^{-1}p, \lambda^{-1}q) = S_{\lambda m}(p,q), \quad \lambda > 0. \tag{4.8}$$

The mass zero limit can be calculated in precisely the same manner as in the scalar case [BLM11] when each element of $S_{\lambda m}$ is considered individually, however, care must be taken with the potential for distinct limits at each infinity unlike the scalar case.

Proposition 4.10. *Let $S \in \mathcal{S}_{lim}(\mathcal{H})$ and $m > 0$. Then for $p, q \in \mathbb{R}$:*

$$S_0(p,q) := \lim_{\lambda\to 0} S_{\lambda m}(p,q) = \begin{cases} S(\log(p) - \log(q)) \ , & p > 0, q > 0; \\ S(\log(-q) - \log(-p)) \ , & p < 0, q < 0; \\ S(0) \ , & p = q = 0; \\ S_+ \ , & p > 0, q = 0, \ \ p = 0, q < 0, \ \ or\ p < 0, q > 0; \\ S_- \ , & p > 0, q = 0, \ \ p = 0, q > 0 \ \ or\ p > 0, q < 0. \end{cases} \tag{4.9}$$

Proof. For any $p, q \in \mathbb{R}$, we have the limit

$$\lim_{\lambda\to 0}(p\omega_q^{\lambda m} - q\omega_p^{\lambda m}) = p|q| - q|p| = \begin{cases} 2pq \ , & p < 0, q > 0; \\ -2pq \ , & p > 0, q < 0; \\ 0 \ , & p \cdot q \geq 0. \end{cases}$$

Hence

$$(\lambda^2 m^2)^{-1}(p\omega_q^{\lambda m} - q\omega_p^{\lambda m}) \to \begin{cases} -\infty \ , & p < 0, q > 0; \\ \infty \ , & p > 0, q < 0 \end{cases}$$

for $\lambda \to 0$ and the limit for $S_{\lambda m}$ for this configuration of momenta is justified. For the remaining cases, we apply l'Hospital's rule to calculate the limit

$$\lim_{\lambda\to 0}\frac{p\omega_q^{\lambda m} - q\omega_p^{\lambda m}}{\lambda^2 m^2} = \lim_{\lambda\to 0}\frac{\frac{p\lambda m^2}{\omega_q^{\lambda m}} - \frac{q\lambda m^2}{\omega_p^{\lambda m}}}{2\lambda m^2} = \frac{1}{2}\lim_{\lambda\to 0}\left(\frac{p}{\omega_q^{\lambda m}} - \frac{q}{\omega_p^{\lambda m}}\right)$$

$$= \begin{cases} 0 \ , & p = q = 0; \\ \operatorname{sign}(p)\cdot\infty \ , & p \neq 0, q = 0; \\ -\operatorname{sign}(q)\cdot\infty \ , & p = 0, q \neq 0; \\ \frac{1}{2}\left(\frac{p}{|q|} - \frac{q}{|p|}\right) \ , & p \cdot q > 0. \end{cases}$$

Inputting these expressions into the composite function $S \circ \sinh^{-1}$ gives the result. □

The limit S_0 is clearly not independent of S and from this, we deduce that the short distance limit behaviour of the model will depend on the original scattering matrix.

Following a similar structure as was described in Chapter 2, we again may rewrite all data in terms of momenta to track the mass m, in particular for $\varphi \in \mathcal{H}, \Psi \in \mathcal{D}_S$

$$[y_{m,S}(\varphi)\Psi]_n^{\boldsymbol{\alpha}}(\boldsymbol{p}) := \sqrt{n+1}\int \frac{\mathrm{d}q}{\omega_q^m}\varphi_\delta(q)\Psi_{n+1}^{\delta\boldsymbol{\alpha}}(q, \boldsymbol{p})$$

and similarly for $y_{m,S}^\dagger(\varphi) = y_{m,S}(J_1\varphi)^*$.

Remark 3. We note that the operators defined above are related to the operators $z_S, z_S^\dagger$ by

$$y_{m,S}(\varphi) = z_S(\varphi).$$

As is the case in rapidity space, the distributional kernels are related to these operators by integrating against a function

$$y_{m,S}(\varphi) = \int \frac{\mathrm{d}p}{\omega_p^m} y_{m,S,\alpha}(p)\varphi_\alpha(p), \quad y_{m,S}^\dagger(\varphi) = \int \frac{\mathrm{d}p}{\omega_p^m} y_{m,S,\alpha}^\dagger(p)\varphi_\alpha(p)$$

For completeness, we write down the exchange relations obeyed by these operators in momentum space as an analogy to those in (2.10a), (2.10b):

$$y_{m,S,\alpha}(p)z_{m,S,\beta}(q) - S_m(p,q)^{\beta\alpha}_{\delta\gamma} y_{m,S,\gamma}(q)y_{m,S,\delta}(p) = 0 \tag{4.10a}$$

$$y_{m,S,\alpha}(p)y^{\dagger}_{m,S,\beta}(q) - S_m(p,q)^{\alpha\gamma}_{\beta\delta} y^{\dagger}_{m,S,\gamma}(q)y_{m,S,\delta}(p) = \omega^m_p \delta^{\alpha}_{\beta}\delta(p-q)\cdot 1_{\mathcal{H}} \tag{4.10b}$$

We also rewrite our transformations of test functions (2.13) for $f \in \mathscr{S}(\mathbb{R}^2) \otimes \tilde{\mathcal{H}}$ as

$$f^{m\pm}_{\alpha}(p) := \frac{1}{2\pi}\int d^2x f_{\alpha}(x)e^{\pm i(\omega^m_p,p)\cdot x}$$

and our explicitly mass-dependent field operator is given by

$$\phi_{m,S}(f) = y^{\dagger}_{m,S}(f^{m+}) + y_{m,S}(f^{m-}).$$

In a similar approach to [BLM11] we wish to analyse the effect of a short distance scaling on quantum fields and consider the rescaled, unsmeared field operators

$$\phi_{m,S}(\lambda x)$$

and evaluate on a scaled test function f_{λ},

$$f^{\lambda}_{\alpha}(x) := \lambda^{-2}f_{\alpha}(\lambda^{-1}x), \quad \lambda > 0, x \in \mathbb{R}^2.$$

In momentum space the scaling has the effect of scaling the mass and the momentum:

$$f^{m\pm}_{\lambda,\alpha} = f^{\lambda m\pm}_{\alpha}(\lambda p).$$

It is relevant now to consider the n-point functions of these rescaled fields

$$\mathscr{W}^{n,\lambda}_{m}(f_1,\dots,f_n) := \langle \Omega_S, \phi_{m,S}((f_1)_{\lambda})\cdots\phi_{m,S}((f_n)_{\lambda})\Omega_S\rangle \tag{4.11}$$

for $f_1,\dots,f_n \in \mathcal{H}$. By the reconstruction theorem [SW00] one may completely recover a field theory from the n-point functions defined above, hence the short distance scaling limit of these then provides an indication of the state of the scaling limit of the model, prompting the following result.

Theorem 4.11. *Let $m > 0$, $S \in \mathcal{S}_{lim}(\tilde{\mathcal{H}})$ and $f_1,\dots,f_n \in \mathscr{S}(\mathbb{R}^2 \to \tilde{\mathcal{H}})$ with $(f_j)^{0\pm}_{\alpha}(0) = 0$ for all $j \in \{1,\dots,n\}$ and $\alpha \in \{1,\dots,d_{\tilde{\mathcal{H}}}\}$. Then*

$$\lim_{\lambda\to 0}\mathscr{W}^{n,\lambda}_{m}(f_1,\dots,f_n) = \mathscr{W}^{n,1}_{0}(f_1,\dots,f_n).$$

The same result holds for the n-point functions generated by the reflected fields $\phi'_{m,S}, \phi'_{0,S}$.

Proof. The field operators $\phi_{m,S}, \phi_{0,S}$ are both defined as sums of creation/annihilation operators which raise and lower the particle numbers by one, respectively. As such, a product of an odd number of field operators gives a vacuum expectation value of zero, hence we may assume that $n = 2k$ for some $k \in \mathbb{N}_0$. Moreover, we first consider a product of $2k$ creation/annihilation operators ordered in reverse Wick ordered form [Wic50] - that is, all annihilation operators on the left, and all creation on the right. Recalling that the projection P^S is self-adjoint, we compute

$$\begin{aligned}
&\langle \Omega_S,\ y_{m,S}((f_1)^{m-}_\lambda)\cdots y_{m,S}((f_k)^{m-}_\lambda) y^\dagger_{m,S}((f_{k+1})^{m+}_\lambda)\cdots y^\dagger_{m,S}((f_{2k})^{m+}_\lambda)\Omega_S\rangle_{\mathcal{F}_S(\mathcal{H})} \\
&= \langle y^\dagger_{m,S}(J_1(f_k)^{m-}_\lambda)\cdots y^\dagger_{m,S}(J_1(f_1)^{m-}_\lambda)\Omega_S, y^\dagger_{m,S}((f_{k+1})^{m+}_\lambda)\cdots y^\dagger_{m,S}((f_{2k})^{m+}_\lambda)\Omega_S\rangle_{\mathcal{F}_S(\mathcal{H})} \\
&= k!\langle \overline{(f_k)^{m-}_\lambda \otimes \cdots \otimes (f_1)^{m-}_\lambda}, P^{S_m}_k((f_{k+1})^{m+}_\lambda \otimes \cdots \otimes (f_{2k})^{m+}_\lambda)\rangle_{\mathcal{H}} \\
&= k! \int \frac{\mathrm{d}p_1}{\omega^m_{p_1}}\cdots\frac{\mathrm{d}p_k}{\omega^m_{p_k}} \left((f_k)^{m-}_\lambda \otimes \cdots \otimes (f_1)^{m-}_\lambda\right)(\lambda p_k, \dots, \lambda p_1)(P^{S_m}_k((f_{k+1})^{m+}_\lambda \otimes \cdots \otimes (f_{2k})^{m+}_\lambda))(\lambda p_1, \dots, \lambda p_k)
\end{aligned}$$

We have the relation (4.8) for the scaling of S_m and also that $\frac{\mathrm{d}\lambda p}{\omega^{\lambda m}_{\lambda p}} = \frac{\mathrm{d}p}{\omega^{\lambda m}_p}$ we proceed by a change of variables $\lambda p_j \to p_j$:

$$\begin{aligned}
&\langle \Omega_S,\ y_{m,S}((f_1)^{m-}_\lambda)\cdots y_{m,S}((f_k)^{m-}_\lambda) y^\dagger_{m,S}((f_{k+1})^{m+}_\lambda)\cdots y^\dagger_{m,S}((f_{2k})^{m+}_\lambda)\Omega_S\rangle_{\mathcal{H}} \\
&= k! \int \frac{\mathrm{d}p_1}{\omega^{\lambda m}_{p_1}}\cdots\frac{\mathrm{d}p_k}{\omega^{\lambda m}_{p_k}} \left((f_k)^{\lambda m-} \otimes \cdots \otimes (f_1)^{\lambda m-}\right)(p_k, \dots, p_1)(P^{S_m}_k((f_{k+1})^{\lambda m+} \otimes \cdots \otimes (f_{2k})^{\lambda m+}))(p_1, \dots, p_k)
\end{aligned}$$

We look to apply Lebesgue dominated convergence to compute this limit, and as such we first apply the triangle inequality and consider the boundedness of one of the integrands in the above sums. We further note that the integrand tends to the same expression for $m = 0$ in the limit - the divergence from the contributions of the reciprocals of the energy quantities $\omega^{\lambda m}_{p_j}$ in this limit are corrected by our assumption on the test functions. In particular, for $m = 0$ the energy expression $\omega^m_{p_i}$ simplifies to $|p_i|$ which vanishes at zero. However, we assume that each component of the test functions also vanishes at zero and so we avoid any divergence.

The Schwartz class of functions come with an a priori λ-independent bound, the S-matrix S is unitary, and hence all components are bounded by at most 1. Finally, noting that $|\omega^{\lambda m}_j| \geq |p_j|$ we can produce a λ-independent bound on the integrand and apply dominated convergence to conclude

$$\begin{aligned}
&\lim_{\lambda\to 0}\langle \Omega_S,\ y_{m,S}((f_1)^{m-}_\lambda)\cdots y_{m,S}((f_k)^{m-}_\lambda) y^\dagger_{m,S}((f_{k+1})^{m+}_\lambda)\cdots y^\dagger_{m,S}((f_{2k})^{m+}_\lambda)\Omega_S\rangle_{\mathcal{H}} \\
&= \int \frac{\mathrm{d}p_1}{|p_1|}\cdots\frac{\mathrm{d}p_k}{|p_k|} \left((f_k)^{0-} \otimes \cdots \otimes (f_1)^{0-}\right)(p_k, \dots, p_1)(P^{S_m}_k((f_{k+1})^{0+} \otimes \cdots \otimes (f_{2k})^{0+}))(p_1, \dots, p_k) \\
&= \langle \Omega_S, y_{0,S}((f_1)^{0-})\cdots y_{0,S}((f_k)^{0-}) y^\dagger_{0,S}((f_{k+1})^{0+})\cdots y^\dagger_{0,S}((f_{2k})^{0+})\Omega_S\rangle_{\mathcal{H}}.
\end{aligned} \tag{4.12}$$

In general, an element of the $2k$ point function will not have this particular ordering, however, the creation/annihilation operators obey specific exchange relations that we may invoke to reshuffle these operators into a form similar to the above. Starting from the general $2k$ point function

$$\mathscr{W}_m^{2k,\lambda}(f_1,\ldots,f_{2k}) = \langle \Omega_S, (y_{m,S}((f_1)_\lambda^{m-}) + y_{m,S}^\dagger((f_1)_\lambda^{m+}))\cdots(y_{m,S}((f_{2k})_\lambda^{m-}) + y_{m,S}^\dagger((f_{2k})_\lambda^{m+}))\Omega_S\rangle.$$

Expanding out and applying linearity of the scalar product, this amounts to a sum of 2^{2k} expectation values, each involving a product of $2k$ creation/annihilation operators. Clearly, those which do not contain an identical number of both creation and annihilation operators give zero in the vacuum expectation (this should be obvious given that the annihilation operator acting on the vacuum gives zero), hence we need only consider the cases where we have k many creation operators and k many annihilation operators in no particular order.

Focusing on one of these terms, we recall the exchange relations these operators obey (4.10a), (4.10b) allowing us to shuffle the order until we arrive at a product in reverse Wick ordered form. Each exchange as in (4.10a) introduces a multiplicative S term (which is bounded by at most one by the unitarity of the S-matrix) and a reshuffling of indices and momenta. As already discussed, the former can be bounded independently of λ and is of no concern to the dominated convergence argument, and the latter simply introduces a sum over many terms, all of which can be bounded in the same way as those previously and these indices and momenta orderings remain unchanged in the expression for both $m > 0$ and $m = 0$. Therefore, neither of these disrupts the conditions for dominated convergence as in the previous argument.

Applying the relation (4.10b) introduces a similar multiplicative S factor and further reshuffling of indices and momenta, however, we further gain an additive contraction term of the form $\omega_p^m \delta_\beta^\alpha \delta(p-q) \cdot 1_{\mathcal{H}}$. These extra integrations are subject to the same bounding conditions as those considered previously and hence we may apply dominated convergence to these inner integrals in the same fashion. Thus, the limit (4.12) applies analogously to the arbitrarily ordered products of $y_{m,S}$ and $y_{m,S}^\dagger$ and the assertion holds. $\square$

This result indicates that the short distance scaling limit of the n-point functions of the finite scale fields $\phi_{m,S}$ is given by the n-point functions of the fields $\phi_{0,S}$ and hence by the reconstruction theorem, the scaling limit of the (m,S) model is really the $(0,S)$ model.

The proceeding discussion here is identical to the scalar case in [BLM11] so we keep things

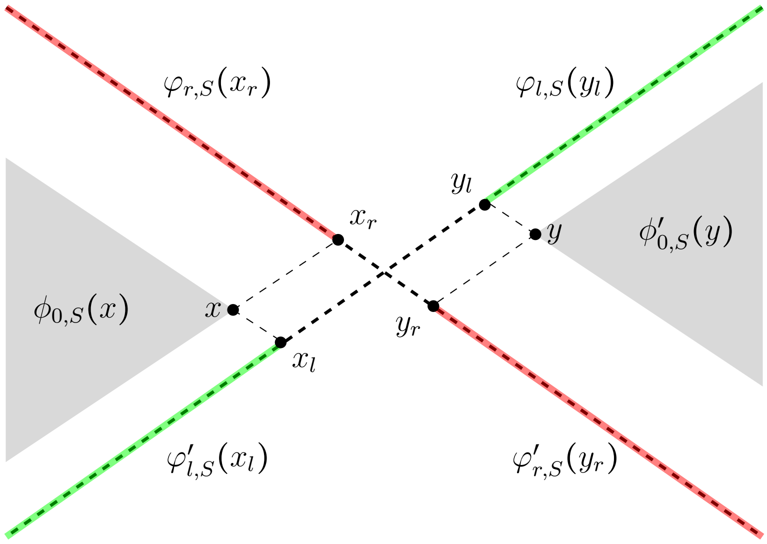

Figure 4.1: The fields $\phi_{0,S}, \phi'_{0,S}$ and their localisation regions which then decompose into half-line localised chiral fields.

brief, outlining our motivation for the following section.

As in the scalar case the wedge-localised fields $\phi_{0,S}$ obtained via the scaling limit are chiral and split into sums of half-line localised fields on the light ray which we illustrate in Figure (4.1).

In Figure (4.1), the subscripts of r, l indicate right/left moving field components. As in [BLM11] the field $\phi_{0,S}$ written in terms of its operator valued distributional form can be written as a sum of the fields illustrated above, in particular

$$\phi_{0,S}(x) = \frac{1}{\sqrt{2\pi}} \left(\varphi_r(x_r) + \varphi'_l(x_l)\right) \tag{4.13}$$

where $x_l = x_0 + x_1, x_r = x_0 - x_1$ are the left/right components of the coordinate $x = (x_0, x_1)$ and the chiral fields

$$\varphi_{r,S}(x_r) := \int_0^\infty \frac{\mathrm{d}p}{\sqrt{2\pi p}} \left(e^{ipx_r} y^\dagger_{0,S}(p) + e^{-ipx_r} y_{0,S}(p)\right), \tag{4.14a}$$

$$\varphi'_{l,S}(x_l) := \int_0^\infty \frac{\mathrm{d}p}{\sqrt{2\pi p}} \left(e^{ipx_l} y^\dagger_{0,S}(-p) + e^{-ipx_l} y y_{0,S}(-p)\right) \tag{4.14b}$$

are one-dimensional and are localised on the right/left light ray components on two-dimensional Minkowski space. In calculations later needed for these fields, one should consider the derivative of these field distributions to avoid the infrared divergence arising from the $1/p$ terms in the integrands. In the proceeding section, we will do just that but for now, we content ourselves with the above definitions.

Previously having calculated the limit of S in momentum space (4.9) and recalling the exchange relations outlined in (4.10a) and (4.10b), it is easy to see that

$$y_{0,S,\alpha}(-p)y_{0,S,\beta}(q) - (S_+)^{\beta\alpha}_{\delta\gamma}\, y_{0,S,\gamma}(q)y_{0,S,\delta}(-p), \tag{4.15a}$$

$$y_{0,S,\alpha}(-p)y^\dagger_{0,S,\beta}(q) - (S_-)^{\alpha\gamma}_{\beta\delta}\, y^\dagger_{0,S,\gamma}(q)y_{0,S,\delta}(-p). \tag{4.15b}$$

A significant difference we can notice to the scalar case here is that the relations are no longer governed by the same quantity - instead the first involves the positive limit S_+ and the second its adjoint S_-, the opposite limit. In the case of an involutive limit, we know that these are one and the same (Corollary (4.9)) and we are in a similar situation. Moreover, for the cases of $S_\pm = F_{\mathcal{H}}$, $S_\pm = -F_{\mathcal{H}}$ we see that the above relations are nothing more than the CCR/CAR relations, respectively implying that the fields $\varphi'_{l,S}, \varphi_{r,S}$ commute/anticommute.

For the latter, simple cases we arrive at the same conclusion as for the scalar case - the scaled model can be written as the (twisted for $S_\pm = -F_{\mathcal{H}}$) tensor product of two chiral models defined as a representation of a ZF algebra. For other involutive limits, we can apply the results of Chapter 2 to see that such an argument may still be possible if the limit S_+ is Type 1 equivalent to its normal form - in which case, the tensor product decomposition constructed previously holds and we may consider again these chiral models as a further step on each tensor slot.

For more complicated limit values that are not involutive, very little is known and will therefore not be discussed.

4.2 Constructing a Chiral Theory for a General S-matrix

Independently of previous discussions, we construct a one dimensional model on the real line which largely mirrors that of the two-dimensional case constructed in Chapter 2. There are, however, changes to the structure - for example, we will see that we gain a translation-dilation covariant model, and the localisation regions of the fields and algebras will instead be half-lines and intervals as opposed to wedges and double cones.

Our starting data is a finite dimensional Hilbert space $\tilde{\mathcal{H}}$ of dimension $d_{\tilde{\mathcal{H}}}$, a single particle Hilbert space

$$\mathcal{H} := L^2(\mathbb{R}, \mathrm{d}\beta) \otimes \tilde{\mathcal{H}}$$

where the variable β is related to the momentum by

$$p = e^{\beta}$$

and an S-matrix $S \in \mathcal{S}_{\lim}(\mathcal{H})$. On $L^2(\mathbb{R}, \mathrm{d}\beta)$ we have a representation $\tilde{V}_1$ of the affine group $\mathcal{G}_0$ which consists of translations and dilations on the real line, i.e. for a group element $g_{a,b} \in \mathcal{G}_0, a, b \in \mathbb{R}$ a point is transformed by $x \mapsto e^a x + b$. The action of the representation is

$$(\tilde{V}_1(g_{a,b})f)(\beta) := e^{ibe^{\beta}} \cdot f(\beta + a), \tag{4.16}$$

where $g_{a,b} \in \mathcal{G}_0$.

To see this is indeed a representation, we check that the group law is preserved. To this end, let $g_{c,d} \in \mathcal{G}_0$ with $c, d \in \mathbb{R}$, then

$$\begin{aligned} (\tilde{V}_1(g_{a,b})\tilde{V}_1(g_{c,d})f)(\beta) &= e^{ibe^{\theta}}(\tilde{V}_1(g_{c,d})f)(\beta + a) \\ &= e^{ibe^{\beta}} e^{ie^a de^{\beta}} f(\beta + a + c) \\ &= e^{i(b+e^a d)e^{\beta}} f(\beta + a + c) \\ &= (\tilde{V}_1(g_{a+c,e^a d+b})f)(\beta). \end{aligned} \tag{4.17}$$

Lemma 4.12. *The representation of $\mathcal{G}_0$ defined in (4.16) on $L^2(\mathbb{R}, \mathrm{d}\beta)$ is strongly continuous, irreducible, unitary and the restriction to only translations has a generator with spectrum within $\mathbb{R}_+$ (positive energy).*

Proof. First we prove the unitarity: Let $f, g \in \mathscr{S}(\mathbb{R})$ and $g_{a,b} \in \mathcal{G}_0$, then we compute

$$\begin{aligned} \langle f, \tilde{V}_1(g_{a,b})g\rangle &= \int_{\mathbb{R}} \mathrm{d}\beta \overline{f(\beta)}(\tilde{V}_1(g_{a,b})g)(\beta) \\ &= \int_{\mathbb{R}} \mathrm{d}\beta \overline{f(\beta)} e^{ibe^{\beta}} g(\beta + a) \\ &= \int_{\mathbb{R}} \mathrm{d}\beta \overline{e^{-ibe^{\beta}} f(\beta)} g(\beta + a) \\ &= \int_{\mathbb{R}} \mathrm{d}\beta' \overline{e^{-ibe^{\beta' - a}} f(\beta' - a)} g(\beta') \;\; =: \langle \tilde{V}_1^*(g_{a,b})f, g\rangle. \end{aligned}$$

Thus we can see that the adjoint acts as

$$\tilde{V}_1^*(g_{a,b}) = \tilde{V}_1\left(g_{-a,-be^{-a}}\right) = \tilde{V}_1(g_{a,b}^{-1}).$$

By the group law, we observed in equation (4.17) we conclude we have unitarity.

Next, we prove the strong continuity, but note that a unitary representation is strongly continuous if and only if it is weakly continuous [Cas13]. Let $\mathcal{G}_0 \ni g_{a_n,b_n} \to g_{a,b} \in \mathcal{G}_0$ $(n \to \infty)$, then for any $f, g \in \mathscr{S}(\mathbb{R})$

$$\begin{aligned}
\lim_{n\to\infty} \langle f, \tilde{V}_1(g_{a_n,b_n})g\rangle &= \lim_{n\to\infty} \int_{\mathbb{R}} \mathrm{d}\beta \overline{f(\beta)} e^{ib_n e^{\beta}} g(\beta + a_n) \\
&= \int_{\mathbb{R}} \mathrm{d}\beta \lim_{n\to\infty} \overline{f(\beta)} e^{ib_n e^{\beta}} g(\beta + a_n) \\
&= \int_{\mathbb{R}} \mathrm{d}\beta \overline{f(\beta)} e^{ibe^{\beta}} g(\beta + a) \\
&= \langle f, \tilde{V}_1(g_{a,b})g\rangle,
\end{aligned}$$

where we have used Lebesgue's dominated convergence theorem with $|e^{ib_n e^{\beta}}| = 1$ and an a priori bound on the functions f, g as they are both of Schwartz class. In particular, we may pick

$$|g(\beta)| \leq \frac{C_1}{1+\beta^2}, \quad |f(\beta + a_n)| \leq C_2,$$

for $C_1, C_2 \in \mathbb{R}$, and the resulting product of bounds is integrable and we may exchange the limit and integral. This now holds on a dense set of functions in $L^2(\mathbb{R}, \mathrm{d}\beta)$ and hence the representation is weakly continuous and thus strongly continuous.

Irreducibility for this particular representation on this Hilbert space is shown in [Vil68] but we outline the proof here. A unitary representation on $L^2(\mathbb{R}, \mathrm{d}\beta)$ is irreducible if and only if the only bounded linear operators on $L^2(\mathbb{R}, \mathrm{d}\beta)$ that commute with all $\tilde{V}_1(g_{a,b})$ are multiples of the identity [BdlH19]. Now, let $A \in \mathcal{B}(L^2(\mathbb{R}, \mathrm{d}\beta))$ and we note that to commute with all $\tilde{V}_1(g_{a,b})$ for all $g_{a,b} \in \mathcal{G}_0$, it means that it must commute with both $\tilde{V}_1(g_{0,b})$, $b \in \mathbb{R}$ and $\tilde{V}_1(g_{a,0})$, $a \in \mathbb{R}$. Focusing on the former initially, we have that for any $f \in L^2(\mathbb{R}, \mathrm{d}\beta)$

$$\left(A\tilde{V}_1(g_{0,b})f\right)(\beta) = \left(\tilde{V}_1(g_{0,b})Af\right)(\beta).$$

By definition, $\tilde{V}_1(g_{0,b})$ acts by multiplying by the function $e^{ibe^{\beta}}$, and for an arbitrary operator A to commute with all multiplication operators of this form, A itself must be a multiplication operator [Nas82], acting as $(Af)(\beta) = z(\beta)f(\beta)$ for some $z \in L^\infty(\mathbb{R})$. Commuting with the translations then implies

$$z(\beta)f(\beta + a) = z(\beta + a)f(\beta + a) \quad \forall f \in L^2(\mathbb{R}, \mathrm{d}\beta).$$

This indicates that $z(\beta) = z(\beta + a)$ for all $a \in \mathbb{R}$, implying that z must be constant and moreover, A acts by multiplying by a constant and so it is simply a multiple of the identity. So we conclude that the only operators commuting with all $\tilde{V}_1(g_{a,b})$ are multiples of the identity and hence the representation is irreducible.

To see that we have positivity of the generator of translations, we again note that translations (denoting this restriction of $\tilde{V}_1(g_{a,b})$ to translations as $\tilde{V}_1(g_{0,x}) = \tilde{V}_1^{\tau}(x)$) act as a multiplication operator

$$(\tilde{V}_1^{\tau}(x)f)(\beta) = e^{ixe^{\beta}} f(\beta),$$

from which we can identify the generator as being the multiplication operator that operates by multiplying by e^{β} which is positive. □

This symmetry representation can be extended to the full single particle Hilbert space $\mathcal{H}$ by way of a tensor product

$$V_1(g_{a,b}) := \tilde{V}_1(g_{a,b}) \otimes 1_{\tilde{\mathcal{H}}}, \quad g_{a,b} \in \mathcal{G}_0 \tag{4.18}$$

which clearly retains all features of $\tilde{V}_1$ as stated in Lemma (4.12).

We retain the same conventions as in Chapter 2 index notation - choosing an orthonormal basis e^{α} for $\alpha \in \{1, \dots, d_{\tilde{\mathcal{H}}}\}$ of $\tilde{\mathcal{H}}$ and denote components of vectors $\Psi_1 \in \mathcal{H}$ by $\beta \mapsto \Psi_1^{\alpha}(\beta)$ and similarly for multi-index notation of vectors and tensors.

We can implement a reflection in the affine symmetry with $j(x) := -x$ which acts antiunitarily on $\mathcal{H}$

$$(V_1(j)\Psi)_1^{\alpha}(\beta) := (J_1\Psi)_1^{\alpha}(\beta) = \overline{\Psi_1^{\overline{\alpha}}(\beta)}, \quad \Psi_1 \in \mathcal{H}_1 .$$

This can be seen to act reflectively by its commutation relations with the translations, i.e. for $f \in \mathcal{H}, a \in \mathbb{R}$

$$\begin{aligned}
(V_1(j)V_1^{\tau}(a)V_1(j)f)_1^{\alpha}(\beta) &= \overline{(V_1^{\tau}(a)V_1(j)f)_1^{\overline{\alpha}}(\beta)} \\
&= \overline{e^{iae^{\beta}}(V_1(j)f)_1^{\overline{\alpha}}(\beta)} \\
&= e^{-iae^{\beta}} f_1^{\alpha}(\beta) \\
&= (V_1^{\tau}(-a)f)_1^{\alpha}(\beta).
\end{aligned}$$

On the n-fold tensor space $\mathcal{H}_n := \mathcal{H}^{\otimes n} = L^2(\mathbb{R}^n, \mathrm{d}\beta) \otimes \tilde{\mathcal{H}}^{\otimes n}$ for $n > 1$ we can naturally extend this representation by defining

$$[V(g_{a,b})\Psi]_n^{\boldsymbol{\alpha}}(\boldsymbol{\beta}) := e^{ib\sum_{j=1}^{n} e^{\beta_j}} \Psi_n^{\boldsymbol{\alpha}}(\beta_1 + a, \dots, \beta_n + a)$$

and the reflection

$$[J\Psi]_n^{\boldsymbol{\alpha}}(\boldsymbol{\beta}) := \overline{\Phi_n^{\overline{\alpha_n}\cdots\overline{\alpha_1}}(\beta_n, \dots, \beta_1)}.$$

Our chosen S-matrix defines a representation of the symmetric group on n letters $\mathfrak{S}_n$ defined first on the generating transposition elements τ_k for $0 < k < n$ which swap the k and $(k+1)$-th elements

$$(\rho_n^S(\tau_k)\Psi)_n^{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = S(\beta_{k+1} - \beta_k)_{\xi_k \xi_{k+1}}^{\alpha_k \alpha_{k+1}} \Psi_n^{\alpha_1 \cdots \xi_k \xi_{k+1} \cdots \alpha_n}(\beta_1, \ldots, \beta_{k+1}, \beta_k, \ldots, \beta_n).$$

As before, this produces a unitary representation of $\mathfrak{S}_n$ on $\mathcal{H}_n$. Since any β-dependence is suppressed in the notation of the left hand side, in future calculations we will explicitly write $(\rho_n^S(\pi)\Psi)_n(\boldsymbol{\beta}) = S_n^\pi(\boldsymbol{\beta})$ for transparency, but for constant S we retain the former notation. We also note the following useful result.

Lemma 4.13. [LS12]

Let $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$. Then

$$S_n^{\sigma_k}(\boldsymbol{\beta})_{\boldsymbol{\nu}}^{\boldsymbol{\alpha}} = \sum_{\eta_1, \ldots, \eta_k} \delta_{\eta_k}^{\alpha_k} \delta_{\eta_1}^{\nu_1} \prod_{l=1}^{k-1} S_{\eta_l \nu_{l+1}}^{\alpha_l \eta_{l+1}}(\beta_k - \beta_l) \cdot \delta_{\nu_{k+1}}^{\alpha_{k+1}} \cdots \delta_{\nu_n}^{\alpha_n}. \tag{4.19}$$

Proof. We proceed by induction in k and begin with the base step of $k = 1$. Here we have that $\sigma_1 = \mathrm{id}$ implying that $\rho_n^S(\sigma_1) = 1$ and so $S_n^{\sigma_1}(\boldsymbol{\beta}) = 1_n$. Checking this against (4.19) we see that

$$S_n^{\sigma_1}(\boldsymbol{\beta})_{\boldsymbol{\nu}}^{\boldsymbol{\alpha}} = \sum_{\eta_1} \delta_{\eta_1}^{\alpha_1} \delta_{\eta_1}^{\nu_1} \delta_{\nu_2}^{\alpha_2} \cdots \delta_{\nu_n}^{\alpha_n} = \delta_{\nu_1}^{\alpha_1} \cdots \delta_{\nu_n}^{\alpha_n} = (1_{\mathcal{H}})_{\boldsymbol{\nu}}^{\boldsymbol{\alpha}}.$$

We move to the induction step and assume the formula (4.19) holds for some k, taking into account that $\sigma_{k+1} = \tau_k \sigma_k$ we calculate on some $\Psi_n \in \mathcal{H}_n$.

$$\begin{aligned}
&\left[\rho_n^S(\sigma_{k+1})\Psi_n\right]^{\boldsymbol{\alpha}}(\boldsymbol{\beta}) \\
&= S_{\xi \nu_{k+1}}^{\alpha_k \alpha_{k+1}}(\beta_{k+1} - \beta_k) \left[\rho_n^S(\sigma_k)\Psi_n\right]^{\alpha_1 \cdots \alpha_{k-1} \xi \nu_{k+1} \cdots \alpha_n}(\beta_1, \ldots, \beta_{k+1}, \beta_k, \ldots, \beta_n) \\
&= S_{\xi \nu_{k+1}}^{\alpha_k \alpha_{k+1}}(\beta_{k+1} - \beta_k) \sum_{\eta_1, \ldots, \eta_k} \delta_{\eta_k}^{\xi} \delta_{\eta_1}^{\nu_1} \prod_{l=1}^{k-1} S_{\eta_l \nu_{l+1}}^{\alpha_l \eta_{l+1}}(\beta_{k+1} - \beta_l) \Psi_n^{\nu_1 \ldots \nu_{k+1} \alpha_{k+2} \ldots \alpha_n}(\beta_{k+1}, \beta_1, \ldots, \hat{\beta}_{k+1}, \ldots, \beta_n) \\
&= \sum_{\eta_1, \ldots, \eta_k} \delta_{\eta_{k+1}}^{\alpha_{k+1}} \delta_{\eta_1}^{\nu_1} S_{\eta_k \nu_{k+1}}^{\alpha_k \eta_{k+1}}(\beta_{k+1} - \beta_k) \prod_{l=1}^{k-1} S_{\eta_l \nu_{l+1}}^{\alpha_l \eta_{l+1}}(\beta_{k+1} - \beta_l) \Psi_n^{\nu_1 \ldots \nu_{k+1} \alpha_{k+2} \ldots \alpha_n}(\beta_{k+1}, \beta_1, \ldots, \hat{\beta}_{k+1}, \ldots, \beta_n) \\
&= \sum_{\eta_1, \ldots, \eta_k} \delta_{\eta_{k+1}}^{\alpha_{k+1}} \delta_{\eta_1}^{\nu_1} \prod_{l=1}^{k} S_{\eta_l \nu_{l+1}}^{\alpha_l \eta_{l+1}}(\beta_{k+1} - \beta_l) \Psi_n^{\nu_1 \ldots \nu_{k+1} \alpha_{k+2} \ldots \alpha_n}(\beta_{k+1}, \beta_1, \ldots, \hat{\beta}_{k+1}, \ldots, \beta_n),
\end{aligned}$$

which coincides with formula (4.19). □

Defining the orthogonal projection $P^S := \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \rho_n^S(\pi)$ onto the S-symmetric subspace of $\mathcal{H}_n$ we construct the S-symmetric Fock space $\mathcal{F}_S(\mathcal{H})$ over $\mathcal{H}$ as

$$\mathcal{F}_S(\mathcal{H}) := \bigoplus_{n=1}^{\infty} P_n^S \mathcal{H}_n, \quad \mathcal{H}_0 := \mathbb{C}$$

on which we have a V_1 invariant vacuum vector $\Omega_S = 1 \oplus 0 \oplus 0 \oplus ...$ and the space of finite particle number $\mathcal{D}_n^S$ (the space of terminating sequences $(\Psi_0, \Psi_1, \ldots, \Psi_n, 0, 0, \ldots)$ for some $n > 0$).

Elements of this Fock space are subject to the symmetry condition

$$\Psi_n^{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = S_{\xi_k \xi_{k+1}}^{\alpha_k \alpha_{k+1}}(\beta_{k+1} - \beta_k) \Psi_n^{\alpha_1 \cdots \xi_k \xi_{k+1} \cdots \alpha_n}(\beta_1, \ldots, \beta_{k+1}, \beta_k, \ldots, \beta_n),$$

and having finite norm

$$\|\Psi\|^2 = \sum_{n=0}^{\infty} \sum_{\boldsymbol{\alpha}} \int d^n \boldsymbol{\beta} \, |\Psi_n^{\boldsymbol{\alpha}}(\boldsymbol{\beta})|^2 < \infty.$$

In this mode the S-symmetrised creation and annihilation operators $y_S, y_S^\dagger$ take a similar form to those discussed previously, but we define them fully here. For $\varphi \in \mathcal{H}$ and $\Psi \in \mathcal{D}_n^S$ they act as

$$(y_S^\dagger(\varphi)\Psi)_n := \sqrt{n} P_n^S(\varphi \otimes \Psi_{n-1}). \quad y_S(\varphi) := y_S^\dagger(\varphi)^*, \tag{4.20}$$

with explicit action given by [LS12]:

$$[y_S(\varphi)\Psi]_n^{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = \sqrt{n+1} \int \mathrm{d}\beta' \overline{\varphi_\delta(\beta')} \Psi_{n+1}^{\delta \boldsymbol{\alpha}}(\beta', \boldsymbol{\beta}), \tag{4.21}$$

$$\left[y_S^\dagger(\varphi)\Psi\right]_n^{\boldsymbol{\alpha}} = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} S_n^{\sigma_k}(\boldsymbol{\beta})_{\delta \xi_1 \cdots \xi_{n-1}}^{\boldsymbol{\alpha}} \varphi^\delta(\beta_k) \Psi_{n-1}^{\xi_1 \cdots \xi_{n-1}}(\beta_1, \ldots, \hat{\beta}_k, \ldots, \beta_n) \tag{4.22}$$

for $n \geq 1$, where $\hat{\beta}_k$ indicates this variable is omitted and the permutations $\sigma_k \in \mathfrak{S}_n$ are defined as $\sigma_k := \tau_{k-1}\tau_{k-2}\cdots\tau_1$ and $\sigma_1 = 1$. For $n = 0$,

$$\left[y_S^\dagger(\varphi)\Psi\right]_0 = 0.$$

The distributional kernels $z_\alpha^\#(\beta)$ related to these operators by

$$y_S^\dagger(\varphi) = \int \mathrm{d}\beta \ \varphi_\alpha(\beta) y_{S,\alpha}^\dagger(\beta) \ , \quad y_S(\varphi) = \int \mathrm{d}\beta \ \overline{\varphi_\alpha(\beta)} y_{S,\alpha}(\beta) \tag{4.23}$$

and by comparing these to expressions (4.21), (4.22) we can see the explicit action of the distributional kernels:

$$\left[y_{S,\alpha}(\tilde{\beta})\Psi\right]_n^{\boldsymbol{\xi}}(\boldsymbol{\beta}) = \sqrt{n+1} \Psi_n^{\alpha \boldsymbol{\xi}}(\tilde{\beta}, \boldsymbol{\beta}), \tag{4.24}$$

$$\left[y_{S,\alpha}^\dagger(\tilde{\beta})\Psi\right]_n^{\boldsymbol{\xi}}(\boldsymbol{\beta}) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} S_n^{\sigma_k}(\boldsymbol{\beta})_{\delta \gamma_1 \cdots \gamma_{n-1}}^{\boldsymbol{\xi}} \delta_\delta^\alpha \delta(\tilde{\beta} - \beta_k) \Psi_{n-1}^{\gamma_1 \cdots \gamma_{n-1}}(\beta_1, \ldots, \widehat{\beta_k}, \ldots, \beta_n) \tag{4.25}$$

From these expressions, it is easy to see that these operators transform covariantly under the symmetry V. Indeed, let $g_{a,b} \in \mathcal{G}_0$ and $\Psi_n \in \mathcal{D}_n^S$, then

$$\begin{aligned}
[V(g_{a,b})y_S(\varphi)V(g_{a,b})^*\Psi]_n^{\boldsymbol{\alpha}}(\boldsymbol{\beta}) &= e^{ib\sum_{j=1}^n e^{\beta_j}}\left[y_S(\varphi)V(g_{a,b})^*\Psi\right]_n^{\boldsymbol{\alpha}}(\beta_1+a,\ldots,\beta_n+a) \\
&= \sqrt{n+1}e^{ib\sum_{j=1}^n e^{\beta_j}}\int \mathrm{d}\beta'\overline{\varphi_\delta(\beta')}\left[V(g_{a,b})^*\Psi\right]_{n+1}^{\delta\boldsymbol{\alpha}}(\beta',\beta_1+a,\ldots,\beta_n+a) \\
&= \sqrt{n+1}\int \mathrm{d}\beta' e^{-ibe^{\beta'}}\overline{\varphi_\delta(\beta')}\Psi_{n+1}^{\delta\boldsymbol{\alpha}}(\beta'-a,\boldsymbol{\beta}) \\
&= \sqrt{n+1}\int \mathrm{d}\beta'\overline{e^{ibe^{\beta'}}\varphi_\delta(\beta'+a)}\Psi_{n+1}^{\delta\boldsymbol{\alpha}}(\beta',\boldsymbol{\beta}) \\
&= \left[y_S(V_1(g_{a,b}))\Psi\right]_n^{\boldsymbol{\alpha}}(\boldsymbol{\beta}).
\end{aligned}$$

Covariance of $y_S^\dagger$ follows by taking the adjoint.

Proposition 4.14. *The distributional kernels $y_\alpha^{\#}(\beta)$ satisfy*

$$y_{S,\alpha}(\beta)y_{S,\xi}(\beta') - S_{\delta\gamma}^{\xi\alpha}(\beta-\beta')y_{S,\gamma}(\beta')y_{S,\delta}(\beta) = 0, \tag{4.26}$$

$$y_{S,\alpha}^\dagger(\beta)y_{S,\xi}^\dagger(\beta') - S_{\alpha\xi}^{\gamma\delta}(\beta-\beta')y_{S,\gamma}^\dagger(\beta)y_{S,\delta}^\dagger(\beta') = 0 \tag{4.27}$$

$$y_{S,\alpha}(\beta)y_{S,\xi}^\dagger(\beta') - S_{\xi\delta}^{\alpha\gamma}(\beta'-\beta)y_{S,\gamma}^\dagger(\beta)y_{S,\delta}(\beta') = \delta_\xi^\alpha\delta(\beta-\beta')\cdot 1_{\mathcal{H}}. \tag{4.28}$$

Proof. Let $\Psi \in \mathcal{D}_S$, then

$$\begin{aligned}
\left[y_{S,\alpha}(\beta)y_{S,\xi}(\beta')\Psi\right]_n^{\boldsymbol{\nu}}(\tilde{\boldsymbol{\beta}}) &= \sqrt{n+2}\sqrt{n+1}\Psi_{n+2}^{\xi\alpha\boldsymbol{\nu}}(\beta',\beta,\tilde{\boldsymbol{\beta}}) \\
&= \sqrt{n+2}\sqrt{n+1}S_{\delta\gamma}^{\xi\alpha}(\beta-\beta')\Psi_{n+2}^{\delta\gamma\boldsymbol{\nu}}(\beta,\beta',\boldsymbol{\beta}) \\
&= S_{\delta\gamma}^{\xi\alpha}(\beta-\beta')\left[y_{S,\gamma}(\beta')y_{S,\delta}(\beta)\Psi\right]_n^{\boldsymbol{\nu}}(\tilde{\boldsymbol{\beta}})
\end{aligned}$$

which shows (4.26). By taking adjoints of (4.26) and applying the Hermitian analyticity and unitarity of S one can easily show (4.27).

Similarly, we proceed by calculation for (4.28)

$$\begin{aligned}
&\left[y_{S,\alpha}(\beta)y_{S,\xi}^\dagger(\beta')\Psi\right]_n^{\boldsymbol{\nu}}(\tilde{\boldsymbol{\beta}}) - \left[S_{\xi\delta}^{\alpha\gamma}(\beta'-\beta)y_{S,\gamma}^\dagger(\beta')y_{S,\delta}(\beta)\Psi\right]_n^{\boldsymbol{\nu}}(\tilde{\boldsymbol{\beta}}) \\
&= \sqrt{n+1}\left[y_{S,\xi}^\dagger(\beta')\Psi\right]_{n+1}^{\alpha\boldsymbol{\nu}}(\beta,\tilde{\boldsymbol{\beta}}) \\
&\quad - \frac{1}{\sqrt{n}}S_{\xi\delta}^{\alpha\gamma}(\beta'-\beta)\sum_{k=1}^n S_n^{\sigma_k}(\tilde{\boldsymbol{\beta}})_{\mu\zeta_1\cdots\zeta_{n-1}}^{\boldsymbol{\nu}}\delta_\gamma^\mu\delta(\beta'-\tilde{\beta}_k)\left[y_{S,\delta}(\beta)\Psi\right]_{n-1}^{\zeta_1\cdots\zeta_{n-1}}(\tilde{\beta}_1,\ldots,\widehat{\tilde{\beta}_k},\ldots,\tilde{\beta}_n) \\
&= \delta_\xi^\alpha\delta(\beta'-\beta)\Psi_n^{\boldsymbol{\nu}}(\tilde{\boldsymbol{\beta}}) + \sum_{k=2}^{n+1} S_{n+1}^{\sigma_k}(\beta,\tilde{\boldsymbol{\beta}})_{\mu\zeta_1\cdots\zeta_n}^{\alpha\boldsymbol{\nu}}\delta_\xi^\mu\delta(\beta'-\tilde{\beta}_{k-1})\Psi_n^{\zeta_1\cdots\zeta_n}(\beta,\tilde{\beta}_1,\ldots,\widehat{\tilde{\beta}_{k-1}},\ldots,\tilde{\beta}_n) \\
&\quad - S_{\xi\delta}^{\alpha\gamma}(\beta'-\beta)\sum_{k=1}^n S_n^{\sigma_k}(\tilde{\boldsymbol{\beta}})_{\mu\zeta_1\cdots\zeta_{n-1}}^{\boldsymbol{\nu}}\delta_\gamma^\mu\delta(\beta'-\tilde{\beta}_k)\Psi_n^{\delta\zeta_1\cdots\zeta_{n-1}}(\beta,\tilde{\beta}_1,\ldots,\widehat{\tilde{\beta}_k},\ldots,\tilde{\beta}_n).
\end{aligned} \tag{4.29}$$

We may shift the index in the first sum by extracting the first permutation term. This means that

$$\sum_{k=2}^{n+1} \rho_{n+1}^S(\sigma_k) = \left(\sum_{k=1}^{n} 1 \otimes \rho_n^S(\sigma_k)\right) \rho_{n+1}^S(\tau_1).$$

Applying this to the first sum in the final line of the above, we have

$$\begin{aligned}
&\sum_{k=2}^{n+1} S_{n+1}^{\sigma_k}(\beta, \tilde{\boldsymbol{\beta}})^{\alpha\boldsymbol{\nu}}_{\mu\zeta_1\cdots\zeta_n} \delta_\xi^\mu \delta(\beta' - \tilde{\beta}_{k-1}) \Psi_n^{\zeta_1\cdots\zeta_n}(\beta, \tilde{\beta}_1, \ldots, \widehat{\tilde{\beta}}_{k-1}, \ldots, \tilde{\beta}_n) \\
&\quad = S_{\xi\delta}^{\alpha\zeta_1}(\beta' - \beta) \sum_{k=1}^{n} S_n^{\sigma_k}(\tilde{\boldsymbol{\beta}})^{\boldsymbol{\nu}}_{\gamma\zeta_2\cdots\zeta_n} \delta_{\zeta_1}^\gamma \delta(\beta' - \tilde{\beta}_k) \Psi_n^{\delta\zeta_2\cdots\zeta_n}(\beta, \tilde{\beta}_1, \ldots, \widehat{\tilde{\beta}}_{k-1}, \ldots, \tilde{\beta}_n).
\end{aligned}$$

One can see the position of the indices in the above match with that of the final term in (4.29) and they cancel leaving just the contraction term $\delta_\xi^\alpha \delta(\beta' - \beta) \cdot 1_{\mathcal{H}}$ as required. $\square$

We also define TCP-transformed creation and annihilation operators by employing the operator J:

$$y_S^\dagger(\varphi)' = J y_S^\dagger(J\varphi) J, \quad y_S(\varphi)' = J y_S(J\varphi) J. \tag{4.30}$$

We observe the explicit action of these operators before proceeding with their relative commutation relations. Let $\Psi_n \in \mathcal{H}_n$ and $f \in \mathcal{H}$ then

$$\begin{aligned}
\left[y_S^\dagger(\varphi)'\Psi\right]_n^{\boldsymbol{\alpha}}(\boldsymbol{\beta}) &= \left[J y_S^\dagger(J\varphi) J\Psi\right]_n^{\boldsymbol{\alpha}}(\boldsymbol{\beta}) \\
&= \overline{\left[y_S^\dagger(J\varphi) J\Psi\right]_n^{\overline{\alpha_n}\ldots\overline{\alpha_1}}(\beta_n, \ldots, \beta_1)} \\
&= \frac{1}{\sqrt{n}} \overline{\left(\sum_{k=1}^{n} S_n^{\sigma_k}(\boldsymbol{\beta})^{\overline{\alpha_n}\ldots\overline{\alpha_1}}_{\overline{\gamma}\overline{\eta_{n-1}}\ldots\overline{\eta_1}} \overline{(J\varphi)_{\overline{\gamma}}(\beta_k)} \left[J\Psi\right]_{n-1}^{\overline{\eta_{n-1}}\ldots\overline{\eta_1}}(\beta_n, \ldots, \hat{\beta}_k, \ldots, \beta_1)\right)} \\
&= \frac{1}{\sqrt{n}} \left(\sum_{k=1}^{n} \overline{S_n^{\sigma_k}(\boldsymbol{\beta})^{\overline{\alpha_n}\ldots\overline{\alpha_1}}_{\overline{\gamma}\overline{\eta_{n-1}}\ldots\overline{\eta_1}}} \varphi_\gamma(\beta_k) \Psi_{n-1}^{\eta_1\ldots\eta_{n-1}}(\beta_1, \ldots, \hat{\beta}_k, \ldots, \beta_n)\right)
\end{aligned}$$

and

$$\begin{aligned}
\left[y_S(\varphi)'\Psi\right]_n^{\boldsymbol{\alpha}}(\boldsymbol{\beta}) &= \left[J y_S(J\varphi) J\Psi\right]_n^{\boldsymbol{\alpha}}(\boldsymbol{\beta}) \\
&= \overline{\left[y_S(J\varphi) J\Psi\right]_n^{\overline{\alpha_n}\ldots\overline{\alpha_1}}(\beta_n, \ldots, \beta_1)} \\
&= \sqrt{n+1} \overline{\int \mathrm{d}\beta' \overline{(J\varphi)_{\overline{\xi}}(\beta')} \left[J\Psi\right]_{n+1}^{\overline{\xi}\overline{\alpha_n}\ldots\overline{\alpha_1}}(\beta', \beta_n, \ldots, \beta_1)} \\
&= \sqrt{n+1} \int \mathrm{d}\beta' \overline{\varphi_\xi(\beta')} \Psi_{n+1}^{\boldsymbol{\alpha}\xi}(\boldsymbol{\beta}, \beta').
\end{aligned}$$

In analogy to the massive model previously described, we can see that these operators create/annihilate "from the right" as opposed to the left. The description of these operators working in opposite directions motivates us to calculate their commutation relations.

Proposition 4.15. *Let $f, g \in \mathcal{H}_1$ and $\Psi_n \in \mathcal{H}_n$. Then*

$$[y_S(f)', y_S(g)]\,\Psi_n = 0, \tag{4.31}$$

$$\left[y_S^\dagger(f)', y_S^\dagger(g)\right]\Psi_n = 0, \tag{4.32}$$

$$\left[y_S(f)', y_S^\dagger(g)\right]\Psi_n = K_n^{f,g}\Psi_n, \tag{4.33}$$

$$\left[y_S^\dagger(f)', y_S(g)\right]\Psi_n = L_n^{f,g}\Psi_n. \tag{4.34}$$

where $K_n^{f,g}$ and $L_n^{f,g}$ are multiplication operators which multiply by the tensors

$$\left(K_n^{f,g}\right)^{\boldsymbol{\alpha}}_{\boldsymbol{\gamma}} = \int \mathrm{d}\beta' \overline{f_\xi(\beta')} S_{n+1}^{\sigma_{n+1}}(\boldsymbol{\beta}, \beta')^{\boldsymbol{\alpha}\xi}_{\delta\boldsymbol{\gamma}} g_\delta(\beta'), \tag{4.35}$$

$$\left(L_n^{f,g}\right)^{\boldsymbol{\alpha}}_{\boldsymbol{\gamma}} = -\int \mathrm{d}\beta' f_\xi(\beta') \overline{S_{n+1}^{\sigma_{n+1}}(\boldsymbol{\beta})^{\boldsymbol{\gamma}\xi}_{\delta\boldsymbol{\alpha}}}\,\overline{g_\delta(\beta')} \tag{4.36}$$

Proof. From the explicit actions of $y_S(f), y_S(f)'$ it is clear they commute, but we compute for clarity - let $\Psi_n \in \mathcal{H}_n$ and $f, g \in \mathcal{H}$, then

$$\begin{aligned}
&[y_S(f)' y_S(g)\Psi]^{\boldsymbol{\alpha}}_n(\boldsymbol{\beta}) - [y_S(g) y_S(f)'\Psi]^{\boldsymbol{\alpha}}_n(\boldsymbol{\beta}) \\
&= \sqrt{n+1}\int \mathrm{d}\beta' \overline{f_\xi(\beta')}\,[y_S(g)\Psi]^{\boldsymbol{\alpha}\xi}_{n+1}(\boldsymbol{\beta}, \beta') - \sqrt{n+1}\int d\tilde{\beta}\,\overline{g_\gamma(\tilde{\beta})}\,[y_S(f)'\Psi]^{\boldsymbol{\alpha}\gamma}_{n+1}(\tilde{\beta}, \boldsymbol{\beta}) \\
&= \sqrt{n+1}\sqrt{n}\int \mathrm{d}\beta' d\tilde{\beta}\,\overline{f_\xi(\beta')}\,\overline{g_\gamma(\tilde{\beta})}\,\Psi^{\gamma\boldsymbol{\alpha}\xi}_{n+2}(\tilde{\beta}, \boldsymbol{\beta}, \beta') \\
&\qquad - \sqrt{n+1}\sqrt{n}\int d\tilde{\beta}\mathrm{d}\beta'\,\overline{g_\gamma(\tilde{\beta}) f_\xi(\tilde{\beta})}\,\Psi^{\gamma\boldsymbol{\alpha}\xi}_{n+2}(\tilde{\beta}, \boldsymbol{\beta}, \beta') = 0
\end{aligned}$$

Clearly, all functions in the integrand are measurable and hence Fubini's theorem allows us to exchange the order of integration giving (4.31) and taking adjoints gives (4.32).

For (4.33) we calculate

$$\begin{aligned}
&\left[y_S(f)' y_S^\dagger(g)\Psi\right]^{\boldsymbol{\alpha}}_n(\boldsymbol{\beta}) - \left[y_S^\dagger(g) y_S(f)'\Psi\right]^{\boldsymbol{\alpha}}_n(\boldsymbol{\beta}) \\
&= \sqrt{n+1}\int \mathrm{d}\beta_{n+1}\overline{f_\xi(\beta_{n+1})}\left[y_S^\dagger(g)\Psi\right]^{\boldsymbol{\alpha}\xi}_{n+1}(\boldsymbol{\beta}, \beta_{n+1}) \\
&\qquad - \frac{1}{\sqrt{n}}\sum_{k=1}^{n} S_n^{\sigma_k}(\boldsymbol{\beta})^{\boldsymbol{\alpha}}_{\delta\gamma_1\cdots\gamma_{n-1}} g_\delta(\beta_k)\,[y_S(f)'\Psi]^{\gamma_1\cdots\gamma_{n-1}}_{n-1}(\beta_1, \ldots, \hat{\beta}_k, \ldots, \beta_n) \\
&= \int \mathrm{d}\beta_{n+1}\overline{f_\xi(\beta_{n+1})}\sum_{k=1}^{n+1} S_{n+1}^{\sigma_k}(\boldsymbol{\beta}, \beta')^{\boldsymbol{\alpha}\xi}_{\delta\boldsymbol{\gamma}} g_\delta(\beta_k)\Psi_n^{\boldsymbol{\gamma}}(\beta_1, \ldots, \hat{\beta}_k, \ldots, \beta_n, \beta_{n+1}) \\
&\qquad - \sum_{k=1}^{n} S_n^{\sigma_k}(\boldsymbol{\beta})^{\boldsymbol{\alpha}}_{\delta\gamma_1\cdots\gamma_{n-1}} g_\delta(\beta_k)\int \mathrm{d}\beta_{n+1}\overline{f_{\gamma_n}(\beta_{n+1})}\Psi_n^{\boldsymbol{\gamma}}(\beta_1, \ldots, \hat{\beta}_k, \ldots, \beta_n, \beta_{n+1}).
\end{aligned}$$

The permutation σ_k for $k < n+1$ is a permutation of at most n letters, so we can view it as leaving the $(n+1)$-th letter fixed while permuting the others with $S_{n+1}^{\sigma_k}(\boldsymbol{\beta}, \beta_{n+1})^{\boldsymbol{\alpha}\xi}_{\delta\boldsymbol{\gamma}} =$

$\delta^{\xi}_{\gamma_n} S^{\sigma_k}_n(\boldsymbol{\beta})^{\boldsymbol{\alpha}}_{\delta\gamma_1\cdots\gamma_{n-1}}$ hence all terms in the first sum with $k \le n$ get cancelled by the second sum. This leaves

$$\begin{aligned}\left([y_S(f)', y_S^\dagger(g)]\Psi\right)^{\alpha}_n(\boldsymbol{\beta}) &= \int \mathrm{d}\beta_{n+1}\overline{f_\xi(\beta_{n+1})}S^{\sigma_{n+1}}_{n+1}(\boldsymbol{\beta},\beta_{n+1})^{\boldsymbol{\alpha}\xi}_{\delta\boldsymbol{\gamma}} g_\delta(\beta_{n+1})\Psi^{\boldsymbol{\gamma}}_n(\beta_1,\dots,\beta_n)\\ &=: \left(K^{f,g}_n\Psi\right)^{\boldsymbol{\alpha}}_n\end{aligned}$$

For the final commutation relation, we observe that $y(f)^* \supset y^\dagger(f)$ and $y(f)'^* \supset y^\dagger(f)'$ [LS12], hence, by taking the adjoint of (4.33):

$$\left(\left[y^\dagger(f)', y(g)\right]\Psi\right)^{\boldsymbol{\alpha}}_n(\boldsymbol{\beta}) = \left(\left[y(f)', y^\dagger(g)\right]^*\Psi\right)^{\boldsymbol{\alpha}}_n(\boldsymbol{\beta}) = -\left(K^{f,g}_n(\boldsymbol{\beta})^*\right)^{\boldsymbol{\alpha}}_{\boldsymbol{\gamma}}\Psi^{\boldsymbol{\gamma}}_n(\boldsymbol{\beta}) = -\overline{K^{f,g}_n(\boldsymbol{\beta})^{\boldsymbol{\gamma}}_{\boldsymbol{\alpha}}}\Psi^{\boldsymbol{\gamma}}_n(\boldsymbol{\beta}),$$

so we have $L^{f,g}_n(\boldsymbol{\beta})^{\boldsymbol{\alpha}}_{\boldsymbol{\gamma}} = -\overline{K^{f,g}_n(\boldsymbol{\beta})^{\boldsymbol{\gamma}}_{\boldsymbol{\alpha}}}$ which we can read off the multiplication operator $L^{f,g}_n$. □

Remark. Note that in the above we have found an explicit formula for the tensors appearing in (4.35) and (4.36):

$$S^{\sigma_{n+1}}_{n+1}(\boldsymbol{\beta},\beta')^{\boldsymbol{\alpha}\xi}_{\delta\boldsymbol{\gamma}} = \sum_{\eta_1,\dots,\eta_{n+1}} \delta^{\xi}_{\eta_{n+1}}\delta^{\delta}_{\eta_1}\prod_{l=1}^{n} S^{\alpha_l\eta_{l+1}}_{\eta_l\gamma_l}(\beta'-\beta_l). \tag{4.37}$$

4.3 Multi-Component Fields

Before constructing a chiral field, we define two maps and discuss their properties. Let $f \in \mathscr{S}(\mathbb{R}) \otimes \tilde{\mathcal{H}}$ and $\psi \in C_0^\infty(\mathbb{R}) \otimes \tilde{\mathcal{H}}$ (where $C_0^\infty(\mathbb{R})$ is the space of complex-valued, smooth functions with compact support), then

$$(F^\pm f)_\alpha(\beta) := \pm i e^\beta \tilde{f}_\alpha(\pm e^\beta) = \pm\frac{ie^\beta}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}x\ f_\alpha(x)e^{\pm ixe^\beta}, \tag{4.38}$$

$$(G^\pm\psi)_\alpha(\xi) := \mp\frac{i}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathrm{d}x\ \psi_\alpha(x)e^{\mp ie^x\xi} \tag{4.39}$$

where $\tilde{f}$ is the usual Fourier transform of f.

Lemma 4.16.

a) $F^\pm : \mathscr{S}(\mathbb{R})\otimes\tilde{\mathcal{H}} \to \mathcal{H}$ *is linear and well-defined,*

b) $F^\pm$ *is injective,*

c) $F^\pm\big|_{C_0^\infty(I)}$ *has dense range for any interval* $I \subset \mathbb{R}$.

Proof. All proofs will be for F^+ since the statements for F^- are analogous.

a) Linearity is obvious by linearity of the integral. To show the transform is well-defined, we must show that F does indeed map into $\mathcal{H}_1$. Let $f \in \mathscr{S}(\mathbb{R}) \otimes \tilde{\mathcal{H}}$, then by applying the substitution $p = e^{\beta}$

$$2\pi \|(F^{+}f)_{\alpha}\|^2 = \int_{\mathbb{R}_+} dp/p \, \left|ip\tilde{f}_{\alpha}(p)\right|^2 = \int_{\mathbb{R}_+} dp \, p \left|\tilde{f}_{\alpha}(p)\right|^2 < \infty.$$

The last integral is finite since the Fourier transform is bijective on $\mathscr{S}(\mathbb{R})$. That is, $\tilde{f}_{\alpha}$ is more rapidly decreasing than polynomials of any order, hence there exists a $C > 0$ such that for all $p \in \mathbb{R}_+$ we have

$$|\tilde{f}_{\alpha}(p)| < \frac{C}{1+p^2},$$

and so we can see that $\|(F^{+}f)_{\alpha}\| < \infty$. Thus $F^{+}f \in \mathcal{H}$ for any $f \in \mathscr{S}(\mathbb{R}) \otimes \tilde{\mathcal{H}}$.

b) To show injectivity, assume that $F^{+}f = 0$. This implies that $\tilde{f}_{\alpha}(p) = 0$ for $p > 0$ and all α. However for $f_{\alpha} \in C_0^{\infty}(\mathbb{R})$ the transformed function $\tilde{f}_{\alpha}$ is entire, and so we can conclude that $\tilde{f}_{\alpha}$ is constant for each α [McM13] and since it vanishes on the positive real half-line, it must vanish everywhere. It follows for all $f_{\alpha} \in \mathscr{S}(\mathbb{R})$ since $C_0^{\infty}(\mathbb{R})$ is dense in $\mathscr{S}(\mathbb{R})$. Finally, injectivity of the Fourier Transform implies $f = 0$.

c) We proceed by contradiction and let $I \subset \mathbb{R}$ and $\psi \in \mathcal{H}$ be non-zero such that $\langle \psi, F^{+}f \rangle_{\mathcal{H}} = 0$ for all $f \in C_0^{\infty}(I) \otimes \tilde{\mathcal{H}}$. Then set $\varphi(\beta) = H(\beta)\psi(\beta)$ (H the Heaviside function) we can apply the Plancherel theorem for distributions:

$$0 = 2\pi\langle \psi, F^{+}f \rangle_{\mathcal{H}} = \sum_{\alpha} \int_{\mathbb{R}_+} \mathrm{d}p \, \overline{\psi_{\alpha}(p)} i\tilde{f}_{\alpha}(p) = \sum_{\alpha} \int_{\mathbb{R}} \mathrm{d}p \, \overline{\varphi_{\alpha}(p)} i\tilde{f}_{\alpha}(p) = \sum_{\alpha} i \int_{\mathbb{R}} \mathrm{d}p \, \overline{\hat{\varphi}_{\alpha}(p)} f_{\alpha}(p),$$

where $\hat{\varphi}$ denotes the inverse Fourier transform of φ. Since f_{α} has compact support, say on the interval $I = [a, b]$, the last integral then reduces to

$$\int_b^a \mathrm{d}p \, \overline{\hat{\varphi}_{\alpha}(p)} f_{\alpha}(p) = 0,$$

from which we can see that $\hat{\varphi}$ must vanish on the interval $[a, b]$. The inverse Fourier transform of φ (by our conventions) is given by

$$\hat{\varphi}_{\alpha}(x) = \int_{-\infty}^{\infty} \mathrm{d}p \, \varphi_{\alpha}(p) e^{ie^{p}x} = \int_{\mathbb{R}_+} \mathrm{d}p \, \psi_{\alpha}(p) e^{ie^{p}x},$$

which we can analytically extend to the upper complex half plane. The Caley transform $g(z) = \frac{z-i}{z+i}$ biholomorphically maps the upper half plane to the unit disc, and hence the function $g \circ \hat{\varphi}$ is analytic in the unit disc and vanishes on some finite interval. By Liouville's theorem, we conclude that $\hat{\varphi} = 0$ and thus $\psi = 0$. We have reached a contradiction and $F^{+}\big|_{C_0^{\infty}(I)}$ has dense range.

□

From now on we simplify notation by setting $F^{\pm} f =: f^{\pm}$, and describe more analytic properties of both of the transforms defined.

Lemma 4.17.

a) Let $f_\alpha^{a,b}(x) := f_\alpha(e^{-a}(x-b))$ *and* $f_\alpha^j(x) := \overline{f_{\overline{\alpha}}(-x)}$, *then*

$$(f^{a,b})_\alpha^{\pm}(\beta) = e^{\pm ibe^\beta} f_\alpha^{\pm}(\beta + a), \quad (f^j)_\alpha^{\pm}(\beta) = -\overline{f_{\overline{\alpha}}^{\pm}(\beta)}. \tag{4.40}$$

b) Let $f, g \in \mathscr{S}(\mathbb{R}) \otimes \tilde{\mathcal{H}}$ *with* $\operatorname{supp}(f_\alpha) \subset \mathbb{R}_+$, $\operatorname{supp}(g_\alpha) \subset \mathbb{R}_-$ *for all* α. *Then* f_α^+ *and* g_α^- *have bounded analytic extensions to the strip* $S(0,\pi)$ *and* $|f_\alpha^+(\beta + i\alpha)|, |g_\alpha^-(\beta + i\alpha)| \to 0$ *as* $\beta \to \pm\infty$. *The boundary values are given by*

$$f_\alpha^+(\beta + i\pi) = f_\alpha^-(\beta), \quad g_\alpha^-(\beta + i\pi) = g_\alpha^+(\beta), \;\; \beta \in \mathbb{R}.$$

c) Let $\psi \in C_0^\infty(\mathbb{R}) \otimes \tilde{\mathcal{H}}$, *then*

$$(G^{\pm}\psi)_\alpha^{\pm} = \psi_\alpha, \quad (G^{\pm}\psi)_\alpha^{\mp} = 0.$$

Proof. a) We show by calculation:

$$\begin{aligned}
\sqrt{2\pi}(f^{a,b})_\alpha^{\pm}(\beta) &= \pm ie^\beta \int_{\mathbb{R}} \mathrm{d}x \; f_\alpha(e^{-a}(x-b))e^{\pm ixe^\beta} \\
&= \pm ie^\beta \int_{\mathbb{R}} \mathrm{d}x \; e^a f_\alpha(x) e^{\pm i(e^a x + b)e^\beta} \\
&= \pm ie^{\beta + a} e^{\pm ibe^\beta} \int_{\mathbb{R}} \mathrm{d}x \; f_\alpha(x) e^{\pm ixe^{\beta+a}} \\
&= \sqrt{2\pi} e^{\pm ibe^\beta} f_\alpha^{\pm}(\beta + a).
\end{aligned}$$

and

$$\begin{aligned}
\sqrt{2\pi}(f^j)_\alpha^{\pm}(\beta) &= \pm ie^\beta \int_{\mathbb{R}} \mathrm{d}x \; \overline{f_{\overline{\alpha}}(-x)} e^{\pm ixe^\beta} \\
&= \mp ie^\beta \int_\infty^{-\infty} \mathrm{d}x \; \overline{f_{\overline{\alpha}}(x)} e^{\mp ixe^\beta} \\
&= \overline{\mp ie^\beta \int_\infty^{-\infty} \mathrm{d}x \; f_{\overline{\alpha}}(x) e^{\pm ixe^\beta}} \\
&= -\overline{f_{\overline{\alpha}}^{\pm}(\beta)}.
\end{aligned}$$

b) We will prove all statements for f^+ only as the proofs for g^- are analogous by taking $f_\alpha(x) := g_\alpha(-x)$. We can make an appropriate estimate to bound each component function

f_α^+ in the strip:

$$\begin{aligned}
|f_\alpha^+(\beta + i\mu)| &= \left| ie^{\beta+i\mu} \int_{\mathbb{R}} \mathrm{d}x\ f_\beta(x) e^{ixe^{\beta+i\mu}} \right| \\
&= \left| \int_{\mathbb{R}_+} \mathrm{d}x\ f_\alpha(x) \frac{\partial}{\partial x} e^{ixe^{\beta+i\mu}} \right| \\
&= \left| \int_{\mathbb{R}_+} \mathrm{d}x\ \left(\frac{\mathrm{d}}{\mathrm{d}x} f_\alpha(x) \right) e^{ixe^{\beta+i\mu}} \right| \\
&\leq \int_{\mathbb{R}_+} \mathrm{d}x\ |f_\alpha'(x)|\, e^{-xe^\beta \sin(\mu)} \\
&\leq \|f_\alpha'\|_1.
\end{aligned}$$

In the last line, we have used that $x > 0$ and that $0 \leq \mu \leq \pi$. Finally, since each $f_\alpha \in \mathscr{S}(\mathbb{R})$ it certainly has a bounded derivative and hence f_α^+ is bounded in the strip. We also have analyticity of f_α^+ in the strip $S(0,\pi)$ since the transform $f_\alpha \mapsto f_\alpha^+$ amounts to multiplying the Fourier transform of f_α with e^θ. Moreover, the boundary values at $\mu = \pi$ are easy to see by computation.

From this we conclude that since $f_\alpha^+(\beta)$ converges to zero for $\beta \to \pm\infty$ on the two boundaries of the strip and that it is bounded and analytic in the interior, it then follows that $|f_\alpha^+(\beta + i\mu)| \to 0$ as $\beta \to \pm\infty$ for all $\mu \in [0,\pi]$.

c) Let $\psi \in C_0^\infty(\mathbb{R}) \otimes \tilde{\mathcal{H}}$. We compute by substitution

$$\begin{aligned}
(G^\pm\psi)_\alpha^\pm(\beta) &= \frac{e^\beta}{2\pi} \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\xi\ \psi_\alpha(x) e^{\pm i\xi e^\beta} e^{\mp \xi e^x} \\
&= \frac{e^\beta}{2\pi} \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\xi\ \psi_\alpha(x) e^{i\xi(\pm e^\beta \mp e^x)} \\
&= \frac{e^\beta}{2\pi} \int_{\mathbb{R}} \mathrm{d}x\mathrm{d}\xi\ \psi_\alpha(x) \int_{\mathbb{R}} e^{i\xi(\pm e^\beta \mp e^x)} \\
&= e^\beta \int_{\mathbb{R}} \mathrm{d}x\ \psi_\alpha(x) \delta(\pm e^\beta \mp e^x).
\end{aligned}$$

This quantity equals zero for all values of β and x except when $\beta = x$ and so the first relation follows. Similarly for the second relation

$$\begin{aligned}
\left(\hat{\psi}^\pm\right)_\alpha^\mp(\beta) &= -\frac{e^\beta}{2\pi} \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\xi\ \psi_\alpha(x) e^{\mp i\xi e^\beta} e^{\mp \xi e^x} \\
&= -\frac{e^\beta}{2\pi} \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\xi\ \psi_\alpha(x) e^{i\xi(\mp e^\beta \mp e^x)} \\
&= -\frac{e^\beta}{2\pi} \int_{\mathbb{R}} \mathrm{d}x\mathrm{d}\xi\ \psi_\alpha(x) \int_{\mathbb{R}} e^{i\xi(\mp e^\beta \mp e^x)} \\
&= -e^\beta \int_{\mathbb{R}} \mathrm{d}x\ \psi_\alpha(x) \delta(\mp e^\beta \mp e^x) \\
&= \int_{\mathbb{R}} \mathrm{d}x\ \psi_\alpha(x) \delta(x - \beta).
\end{aligned}$$

This equation always is equal to zero for every β and x since it can never happen that e^β is equal to $-e^x$ by positivity of the exponential.

□

We now define two field operators ϕ_S, ϕ'_S on $\mathcal{F}_S(\mathcal{H})$ for $f \in \mathcal{H}$

$$\phi_S(f) = y_S^\dagger(f^+) + y_S(J_1 f^-), \tag{4.41}$$

$$\phi'_S(f) = y_S^\dagger(f^+)' + y_S(J_1 f^-)' \tag{4.42}$$

which are both well-defined by Lemma (4.16) and act covariantly under the symmetry V given that the constituent creation/annihilation operators have the same property.

In Chapter 2 we saw that analogously defined massive fields in two spacetime dimensions were not generally local in the usual sense, and instead, we found that they were wedge-local, taking their localisation regions as wedges in Minkowski space. Similarly, here we find we again have locality only in a specific sense.

Theorem 4.18.

a) The map $f \mapsto \phi_S(f)$ is an operator valued tempered distribution with $\mathcal{D}^S$ contained in the domain of $\phi_S(f)$ for all $f \in \mathscr{S}(\mathbb{R}) \otimes \tilde{\mathcal{H}}$. For $f = f^*$ the operator $\phi_S(f)$ is essentially self-adjoint and has all elements of $\mathcal{D}^S$ as entire analytic vectors.

b) The operator ϕ_S is covariant with respect to the representation V, i.e.

$$V(g_{a,b})\phi_S(f)V(g_{a,b})^* = \phi_S(f^{a,b}).$$

c) The fields ϕ_S and ϕ'_S are relatively half-local in the following sense: For any $a \in \mathbb{R}$, let $f \in \mathscr{S}((a,\infty)) \otimes \tilde{\mathcal{H}}$, $g \in \mathscr{S}((-\infty,a)) \otimes \tilde{\mathcal{H}}$ and $\Psi \in \mathcal{D}_n^S$. Then

$$[\phi_S(f), \phi'_S(g)]\,\Psi = 0 \tag{4.43}$$

d) Let $f, g \in \mathcal{H}$ with $\operatorname{supp}(f_\alpha) \cap \operatorname{supp}(g_\alpha) = \varnothing$ for all α, then ϕ_F is local, i.e.

$$[\phi_F(f), \phi_F(g)] = 0.$$

e) The Fock vacuum Ω_S is cyclic for the field ϕ_S, i.e. for any open interval $I \subset \mathbb{R}$, the subspace

$$\mathcal{D}_I := \operatorname{span}\{\phi_S(f_1)\cdots\phi_S(f_n)\Omega_S \;:\; f_1,\ldots,f_n \in \mathcal{H}, \operatorname{supp}(f_\alpha) \subset I \;\; \forall \alpha, n \in \mathbb{N}\}$$

is dense in $\mathcal{F}_S(\mathcal{H})$.

All of the above statements hold also for ϕ'_S.

Proof. a) The proof for this is identical to the scalar case [BLM11].

b) This is a direct consequence of the linearity of the representation V, the covariance of $y_S, y_S^\dagger$ under V and Lemma (4.17).

c) By applying (4.31)-(4.34) we can see that the commutator between the two fields reduces to

$$[\phi'_S(f), \phi_S(g)]\,\Psi_n = \left[y_S^\dagger(f^+)', y_S(J_1 g^-)\right]\Psi_n + \left[y_S(Jf_1^-)', y_S^\dagger(g^+)\right]\Psi_n = L_n^{f^+,J_1 g^-}\Psi_n + K_n^{J_1 f^-,g^+}\Psi_n.$$

What needs to be shown is that

$$K_n^{J_1 f^-,g^+}(\boldsymbol{\beta})^{\boldsymbol{\alpha}}_{\boldsymbol{\gamma}} = \int \mathrm{d}\beta' f^-_{\overline{\xi}}(\beta') \sum_{\eta_1,\dots,\eta_{n+1}} \delta^{\xi}_{\eta_{n+1}} \delta^{\delta}_{\eta_1} \prod_{l=1}^{n} S^{\alpha_l \eta_{l+1}}_{\eta_l \gamma_l}(\beta' - \beta_l) \cdot g^+_{\delta}(\beta') \tag{4.44}$$

coincides with

$$-L_n^{f^+,J_1 g^-}(\boldsymbol{\beta})^{\boldsymbol{\alpha}}_{\boldsymbol{\gamma}} = \int \mathrm{d}\beta' f^+_{\xi}(\beta') \overline{\sum_{\eta_1,\dots,\eta_{n+1}} \delta^{\xi}_{\eta_{n+1}} \delta^{\delta}_{\eta_1} \prod_{l=1}^{n} S^{\gamma_l \eta_{l+1}}_{\eta_l \alpha_l}(\beta' - \beta_l)} \cdot g^-_{\overline{\delta}}(\beta'). \tag{4.45}$$

Due to the support properties of the components of f and g, by [Lec03, pg. 13] and [MR75] and their Fourier transforms continue to analytic functions in the upper-half complex plane, and moreover the composition of this with the exponential function implies the functions $f^\pm$ and $g^\pm$ are entire analytic in each component. Furthermore, by (4.17) these extensions are bounded in the strip $S(0,\pi)$ and $f^-_{\overline{\alpha_n}}(\beta'+i\mu)$ and $g^+_{\xi}(\beta'+i\mu)$ decay rapidly as $\beta' \to \pm\infty$ in the strip $\mu \in [0,\pi]$ with the boundary values given by $f^-_{\overline{\xi}}(\beta'+i\pi) = f^+_{\overline{\xi}}(\beta')$ and $g^+_{\delta}(\beta'+i\pi) = g^-_{\overline{\delta}}(\beta')$. Applying the analyticity properties of S and also crossing symmetry, we shift the contour of integration from $\mathbb{R}$ to $\mathbb{R}+i\pi$ which now reads

$$\begin{aligned} K_n^{J_1 f^-,g^+}(\boldsymbol{\beta})^{\boldsymbol{\alpha}}_{\boldsymbol{\gamma}} &= \int \mathrm{d}\beta' f^+_{\overline{\xi}}(\beta') \sum_{\eta_1,\dots,\eta_{n+1}} \delta^{\xi}_{\eta_{n+1}} \delta^{\delta}_{\eta_1} \prod_{l=1}^{n} S^{\overline{\eta_l}\alpha_l}_{\gamma_l \overline{\eta_{l+1}}}(\beta_l - \beta') \cdot g^-_{\overline{\delta}}(\beta') \\ &= \int \mathrm{d}\beta' f^+_{\xi}(\beta') \sum_{\eta_1,\dots,\eta_{n+1}} \delta^{\overline{\xi}}_{\overline{\eta}_{n+1}} \delta^{\overline{\delta}}_{\overline{\eta}_1} \prod_{l=1}^{n} S^{\eta_l \alpha_l}_{\gamma_l \eta_{l+1}}(\beta_l - \beta') \cdot g^-_{\overline{\delta}}(\beta'). \end{aligned}$$

By Hermitian analyticity, we finally have that $S^{\eta_l \alpha_l}_{\gamma_l \eta_{l+1}}(\beta_l - \beta') = \overline{S^{\gamma_l \eta_{l+1}}_{\eta_l \alpha_l}(\beta' - \beta_l)}$ and the proof is finished.

d) By standard commutation relations we have that

$$
\begin{aligned}
[\phi_F(f), \phi_F(g)] &= \langle \overline{f^-}, g^+ \rangle_{\mathcal{H}} - \langle \overline{g^-}, f^+ \rangle_{\mathcal{H}} \\
&= \sum_{\alpha} \int_{\mathbb{R}} \mathrm{d}\theta \; f_\alpha^-(\theta) g_\alpha^+(\theta) \; - \int_{\mathbb{R}} \mathrm{d}\theta \; g_\alpha^-(\theta) f_\alpha^+(\theta) \\
&= \sum_{\alpha} \int_{\mathbb{R}} \mathrm{d}x \mathrm{d}y \mathrm{d}\theta \; \frac{e^{2\theta}}{2\pi} \int_{\mathbb{R}^2} f_\alpha(x) g_\alpha(y) \left(e^{ie^\theta (y-x)} - e^{-ie^\theta (y-x)} \right) \\
&= \frac{1}{2\pi} \sum_{\alpha} \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}y \; f_\alpha(x) g_\alpha(y) \int_0^\infty \mathrm{d}p \; p \left(e^{ip(y-x)} - e^{-ip(y-x)} \right) \\
&= \frac{1}{2\pi} \sum_{\alpha} \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}y \; f_\alpha(x) g_\alpha(y) \int_0^\infty \mathrm{d}p \; p e^{ip(y-x)} - p e^{-ip(y-x)} \\
&= \frac{1}{2\pi} \sum_{\alpha} \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}y \; f_\alpha(x) g_\alpha(y) \int_{\mathbb{R}} \mathrm{d}p \; p e^{ip(y-x)} \\
&= \sum_{\alpha} \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}y \; f_\alpha(x) g_\alpha(y) (-i\delta'(y-x)) \\
&= -i \sum_{\alpha} \int_{\mathbb{R}} \mathrm{d}x \mathrm{d}y \; f_\alpha(x) g_\alpha(x) \mathrm{d}x.
\end{aligned}
$$

We can see that this disappears if $\mathrm{supp}(f_\alpha) \cap \mathrm{supp}(g_\alpha) = \varnothing$.

e) Let $\mathcal{P}_S(I)$ denote the algebra generated by all polynomials in the field $\phi_S(f)$ with $\mathrm{supp}(f_\alpha) \subset I$ for all α. We look to apply a Reeh Schlieder type argument and assume that $\mathcal{P}_S(I)\Omega_S$ is *not* dense in $\mathcal{H}$ and so Ω_S is not cyclic for $\mathcal{P}_S(I)$. By positivity of its generator, we can consider translations $V^\tau(z)$ for z in the upper half-plane, so define a function

$$h(x) := \langle u, V^\tau(z) \phi_S(f) \Omega_S \rangle_{\mathcal{H}},$$

which is holomorphic in the upper half-plane. We can then also define a function for some $u \in \mathcal{H}$ by

$$g(z) := \overline{h(\overline{z})},$$

to be holomorphic on the lower half-plane and also $g(x) = f(x)$ for $x \in \mathbb{R}$, i.e. the two complex values functions agree on the real line. Let $\psi \in \mathcal{F}_S(\mathcal{H})$ be such that

$$\langle \psi, V^\tau(z) \phi_S(f) \Omega \rangle = 0,$$

for all $\phi_S(f) \in \mathcal{P}_S(I)$. In particular, if we take the interval $I_0 \subset I$ such that $I_0 + \varepsilon = I$ for some small positive ε. Then for all $\phi_S(f_0) \in \mathcal{P}_S(I_0)$ and $x \leq \varepsilon$

$$h(x) = \langle \psi, V^\tau(x) \phi_S(f_0) \Omega_S \rangle = \langle \psi, V^\tau(x) \phi_S(f_0) V^\tau(x)^* \Omega \rangle = 0.$$

We can see from the consideration made previously that there is a holomorphic function in the upper half complex plane and also another in the lower half complex plane which coincide on the real line, vanish on an interval on that line and so must vanish everywhere by the Edge-Of-The-Wedge theorem [Rud71]. Hence we can conclude that the above actually holds for arbitrary translations $V^{\tau}(x), x \in \mathbb{R}$ so it must be that $\langle \psi, \phi_S(f)\Omega_S\rangle = 0$ for $\mathrm{supp}(f_\alpha) = \mathbb{R}$ for all α, but since $\mathcal{P}_S(\mathbb{R})\Omega_S$ is dense in $\mathcal{F}_S(\mathcal{H})$ we have a contradiction and thus $\mathcal{P}_S(I)\Omega_S$ is dense in $\mathcal{F}_S(\mathcal{H})$ if and only if $\mathcal{P}_S(\mathbb{R})\Omega_S$ is dense.

□

With the definition of these fields and an understanding of their locality properties, we now look to the algebraic formulation and in particular the von Neumann algebras generated by the exponentiated version of these fields. To this end, we denote the self-adjoint closures of $\phi_S(f)$ and $\phi'_S(f)$ (for $f = f^*$) by the same symbols and define the following

$$\mathcal{A}_S(\mathbb{R}_+) := \{e^{i\phi_S(f)} \;:\; f = f^* \in \mathscr{S}(\mathbb{R}_+) \otimes \tilde{\mathcal{H}}\}''$$

$$\mathcal{A}_S(\mathbb{R}_-) := \{e^{i\phi'_S(f)} \;:\; f = f^* \in \mathscr{S}(\mathbb{R}_-) \otimes \tilde{\mathcal{H}}\}''.$$

Proposition 4.19. *Let $S \in \mathcal{S}_{\lim}(\mathcal{H})$, then the algebras $\mathcal{A}_S(\mathbb{R}_+), \mathcal{A}_S(\mathbb{R}_-)$ have the following properties:*

a) For $a \geq 0, b \in \mathbb{R}$

$$V(g_{a,b})\mathcal{A}_S(\mathbb{R}_+)V(g_{a,b})^* \subset \mathcal{A}_S(\mathbb{R}_+).$$

b) The vacuum vector Ω_S is cyclic and separating for $\mathcal{A}_S(\mathbb{R}_+)$.

Proof. a) The algebra $\mathcal{A}_S(\mathbb{R}_+)$ is generated by the elements $e^{i\phi_S(f)}$ (for $f = f^* \in \mathscr{S}(\mathbb{R}_+)$) which by linearity of the representation V and Theorem (4.18) transforms as $V(g_{a,b})e^{i\phi_S(f)}V(g_{a,b})^* = e^{i\phi_S(f^{a,b})}$. Since it is clear that $f^{a,b} \in \mathscr{S}(\mathbb{R}_+) \otimes \tilde{\mathcal{H}}$ the result follows.

b) As we have taken the self-adjoint closure of $\phi_S(f)$ the cyclicity of Ω_S for $\mathcal{A}_S(\mathbb{R}_+)$ follows from standard Reeh-Schlieder type arguments (as in, for example, [BY90]) together with the cyclicity of Ω_S for $\phi_S(f)$ given by Theorem (4.18) (e).

Given the commutation relations between $\phi_S(f)$ and $\phi'_S(g)$ as in Theorem (4.18) (c) for $f \in \mathscr{S}(\mathbb{R}_+) \otimes \tilde{\mathcal{H}}$ and $g \in \mathscr{S}(\mathbb{R}_-) \otimes \tilde{\mathcal{H}}$, hence the unitaries $e^{i\phi_| S(f)}$, $e^{i\phi_{S;}(g)}$ commute for the same f, g implying that $\mathcal{A}_S(\mathbb{R}_-) \subset \mathcal{A}_S(\mathbb{R}_+)'$. Since Ω_S is cyclic for $\mathcal{A}_S(\mathbb{R}_-)$ by the same arguments as above, it is then separating for $\mathcal{A}_S(\mathbb{R}_+)$

□

4.4 Local Operators

Here we look into the question of the size of the local algebras $\mathcal{A}(\mathcal{I})$ for an interval $\mathcal{I} \subset \mathbb{R}$. It will become increasingly clear that the size of $\mathcal{A}(\mathcal{I})$ depends on the limit values of S.

One may wonder for which cases these interval algebras are trivial (where only multiples of the identity exist), if they contain a richer spectrum of observables, or even if they are isomorphic to interval algebras present in a free field theory.

The local algebras $\mathcal{A}_S(I)$ are analogous to the algebras generated by double cones constructed in Chapter 2, however with the geometry altered in this setting, we consider the algebras on intervals instead. However, we define them in a similar way:

$$\mathcal{A}_S(a,b) := V^\tau(a)\mathcal{A}_S(\mathbb{R}_+)V^\tau(-a) \cap V^\tau(b)\mathcal{A}_S(\mathbb{R}_-)V^\tau(-b). \tag{4.46}$$

For more general subsets, we construct by additivity. To assess the size of these algebras, we look to derive potential obstructions to the existence of local operators and analyse the results.

Firstly, we recall the notion of a partial trace of finite dimensional operators - an operation prevalent in quantum mechanics (and particularly quantum computing) where it is used to calculate reduced density matrices in multi-partite systems, thereby formulating a method of decoherence in quantum measurements. We refer the interested reader to [Maz16, Par12, NC00] for more in-depth discussions on the partial trace and its applications.

Definition 4.20. Let $\tilde{\mathcal{H}}, \tilde{\mathcal{K}}$ be finite dimensional Hilbert spaces and $A \in \mathcal{B}(\tilde{\mathcal{H}}), B \in \mathcal{B}(\tilde{\mathcal{K}})$. Then we define the left partial trace $\mathrm{pt}_L : \mathcal{B}(\tilde{\mathcal{H}} \otimes \tilde{\mathcal{K}}) \to \mathcal{B}(\tilde{\mathcal{K}})$ as the unique linear map satisfying

$$\mathrm{pt}_L(A \otimes B) = \mathrm{Tr}(A)B,$$

or similarly the right partial trace $\mathrm{pt}_R : \mathcal{B}(\tilde{\mathcal{H}} \otimes \tilde{\mathcal{K}}) \to \mathcal{B}(\tilde{\mathcal{H}})$ as

$$\mathrm{pt}_R(A \otimes B) = \mathrm{Tr}(B)A.$$

A subscript of "L" and "R" simply indicates which tensor slot we apply the trace operation to. In other literature, it is common to see a subscript of the space involved, such as $\mathrm{pt}_{\tilde{\mathcal{H}}}$, but we will be concerning ourselves with tensor products of a single Hilbert space this would lead to ambiguity.

For arbitrary tensor powers of finite dimensional Hilbert spaces, it may be unclear how to extend this idea - many possibilities exist, all of which give rise to other well-defined operations. We fix our conventions with the following.

Definition 4.21. Let $\tilde{\mathcal{H}}$ be a finite dimensional Hilbert space, $A_i \in \mathcal{B}(\tilde{\mathcal{H}})$ for $i \in \{1, \dots, n\}$ and some $n \in \mathbb{N}$. Then we define the partial trace $\mathrm{pt}_L : \mathcal{B}(\tilde{\mathcal{H}}^{\otimes n}) \to \mathcal{B}(\tilde{\mathcal{H}}^{\otimes (n-1)})$ as the linear extension

$$\mathrm{pt}_L \left(\bigotimes_{i=1}^n A_i \right) = \mathrm{Tr}(A_1) \bigotimes_{i=2}^n A_i$$

and similarly $\mathrm{pt}_R : \mathcal{B}(\tilde{\mathcal{H}}^{\otimes n}) \to \mathcal{B}(\tilde{\mathcal{H}}^{\otimes (n-1)})$:

$$\mathrm{pt}_R \left(\bigotimes_{i=1}^n A_i \right) = \mathrm{Tr}(A_n) \bigotimes_{i=1}^{n-1} A_i.$$

We furthermore define a helpful map for future discussions.

Definition 4.22. The sesquilinear map $m : \mathcal{H} \times \mathcal{H} \to M(d_{\tilde{\mathcal{H}}}, \mathbb{C})$ (the space of $d_{\tilde{\mathcal{H}}} \times d_{\tilde{\mathcal{H}}}$ complex valued matrices) is defined component-wise by

$$(m(f, g))^\alpha_\beta = \langle f_\alpha, g_\beta \rangle_{L^2(\mathbb{R})}.$$

Some simple properties of this map can be spotted from the definition - for example, $\mathrm{Tr}(m(f,g)) = \langle f, g \rangle$ and $m(f,g)$ is self-adjoint if $f = g$. It is also possible to choose vector-valued functions f, g to give specific $m(f,g)$ - for example, if we take $f = g$ such that $\{f_\alpha\}$ is an orthonormal set then $m(f,f) = 1$, the identity matrix.

With this map, we define an operator we will later show produces an obstruction to the existence of local observable in the interval algebras $\mathcal{A}_S(I)$ - an operator which is dependent on the limits $S_\pm$.

Definition 4.23. Let $S_- \in \mathcal{R}_{\lim}(\tilde{\mathcal{H}})$ and $f, g \in \mathcal{H}$. Then we define an operator $S_n^{f,g}$ on $\tilde{\mathcal{H}}^{\otimes n}$ ($n \in \mathbb{N}_0$)

$$S_n^{f,g} = \mathrm{pt}_L \Big[\left(m(f,g) \otimes 1^{\otimes n} \right) \rho_F^{n+1}(\sigma_{n+1}^{-1}) \rho_{S_-}^{n+1}(\sigma_{n+1}) \Big]. \tag{4.47}$$

With $S_n^{f,g}$ defined for all n, we take its direct sum over all n:

$$S^{f,g} := \bigoplus_{n \geq 0} 1_{L^2(\mathbb{R})}^{\otimes n} \otimes S_n^{f,g}.$$

Lemma 4.24. *Let $\psi_1, \psi_2 \in \mathscr{S}(\mathbb{R}) \otimes \tilde{\mathcal{H}}$ and $n \in \mathbb{N}_0$. The following operators converge to zero in the weak operator topology as $\lambda \to \infty$.*

$$P_{n-2} y_S(V_1(0,\lambda)\psi_1) J y_S(V_1(0,\lambda)\psi_2) P_n \tag{4.48}$$

$$P_{n+2} y_S^{\dagger}(V_1(0,\lambda)\psi_1) J y_S^{\dagger}(V_1(0,\lambda)\psi_2) P_n \tag{4.49}$$

$$P_n y_S^{\dagger}(V_1(0,\lambda)\psi_1) J y_S(V_1(0,\lambda)\psi_2) P_n \tag{4.50}$$

$$\left[y_S(V_1(0,\lambda)\psi_1), y_S^{\dagger}(V_1(0,\lambda)\psi_2)'\right] - S^{\psi_1,\psi_2}. \tag{4.51}$$

Proof. Let $\Psi_n \in \mathcal{H}_n \cap \mathscr{S}(\mathbb{R}^n) \otimes \tilde{\mathcal{H}}^{\otimes n}$ then using the definition of the annihilation operator we find that for $k = 1, 2$, and $|\boldsymbol{\alpha}| = n-1$,

$$\begin{aligned}
\|y_S(V_1(0,\lambda)\psi_k)\Psi_n\|^2 &= \langle y_S(V_1(0,\lambda)\psi_k)\Psi_n, y_S(V_1(0,\lambda)\psi_k)\Psi_n\rangle \\
&= n \sum_{\alpha_0,\gamma_0} \left\langle \int \mathrm{d}\beta_0 \overline{\psi_k^{\alpha_0}(\beta_0+\lambda)} \Psi_n^{\alpha_0\boldsymbol{\alpha}}(\beta_0,\boldsymbol{\beta}), \int \mathrm{d}\beta_0' \overline{\psi_k^{\gamma_0\boldsymbol{\alpha}}(\beta_0'+\lambda)} \Psi_n^{\gamma_0\boldsymbol{\alpha}}(\beta_0',\boldsymbol{\beta})\right\rangle \\
&= n \sum_{\alpha_0,\gamma_0,\boldsymbol{\alpha}} \int \mathrm{d}\beta_0 \mathrm{d}\beta_0' d\boldsymbol{\beta} \psi_k^{\alpha_0}(\beta_0+\lambda) \overline{\psi_k^{\gamma_0}(\beta_0'+\lambda) \Psi_n^{\alpha_0\boldsymbol{\alpha}}(\beta_0,\boldsymbol{\beta})} \Psi_n^{\gamma_0\boldsymbol{\alpha}}(\beta_0',\boldsymbol{\beta}).
\end{aligned}$$

Pointwise, the integral goes to zero as $\lambda \to \infty$ due to the properties of the Schwartz class functions. Moreover, since all functions are of this class, we may apply dominated convergence (for every index) to show that $y_S(V_1(0,\lambda)\psi_k)\Psi_n \to 0$ in the Hilbert space norm. In addition, we always have the norm bound $\|y_S(V_1(0,\lambda)\psi_k)P_n\| \leq \sqrt{n}\|\psi_k\|$ independent of λ, hence $y_S(V_1(0,\lambda)\psi_k)P_n$ tends to 0 as $\lambda \to \infty$ in the strong operator topology. This together with $\|J\| = 1$ implies that (4.48) does indeed vanish in the strong operator topology. (4.49) differs from the adjoint of (4.48) by only redefinitions of the involved functions, and so we can immediately conclude that it vanishes in the weak operator topology. To show that (4.50) vanishes in the weak operator topology we look to the scalar product

$$\langle y_S^{\dagger}(V_1(0,\lambda)\psi_1) J y_S(V_1(0,\lambda)\psi_2) P_n\Psi_n,\ \Phi_n\rangle = \langle J y_S(V_1(0,\lambda)\psi_2) P_n\Psi_n,\ y_S(V_1(0,\lambda)\psi_k) P_n\Phi_n\rangle$$

for $\Psi_n \in \mathcal{H}_n \cap \mathscr{S}(\mathbb{R}^n) \otimes \tilde{\mathcal{H}}$ which converges to 0 as $\lambda \to \infty$ on account of $y_S(V_1(0,\lambda)\psi_k)P_n$ vanishing strongly.

For the final operator, we note that we know the value of the commutator from (4.34) so we expand the scalar product for $\Phi_n, \Psi_n \in \mathcal{H}_n \cap \mathscr{S}(\mathbb{R}^n) \otimes \tilde{\mathcal{H}}^{\otimes n}$

$$\begin{aligned}
&\langle \Phi_n, \left(\left[y_S\left(V_1(\lambda,0)\psi_1\right), y_S^{\dagger}\left(V_1(\lambda,0)\psi_2\right)'\right] - S_n^{\psi_1,\psi_2}\right)\Psi_n\rangle \\
&\quad = \sum_{\boldsymbol{\alpha}} \int d^n\boldsymbol{\beta} \mathrm{d}\beta' \overline{\Phi_n^{\boldsymbol{\alpha}}(\boldsymbol{\beta})} \left(\psi_2^{\gamma}(\beta'+\lambda) \overline{S_{n+1}^{\sigma_{n+1}}(\boldsymbol{\beta},\beta')_{\delta\boldsymbol{\alpha}}^{\boldsymbol{\eta}\gamma} \psi_1^{\delta}(\beta'+\lambda)} - S_n^{\psi_1,\psi_2}\right) \Psi_n^{\boldsymbol{\eta}}(\boldsymbol{\beta}) \\
&\quad = \sum_{\boldsymbol{\alpha}} \int d^n\boldsymbol{\beta} \mathrm{d}\beta' \overline{\Phi_n^{\boldsymbol{\alpha}}(\boldsymbol{\beta})} \left(\psi_2^{\gamma}(\beta') \overline{S_{n+1}^{\sigma_{n+1}}(\boldsymbol{\beta},\beta'-\lambda)_{\delta\boldsymbol{\alpha}}^{\boldsymbol{\eta}\gamma} \psi_1^{\delta}(\beta')} - S_n^{\psi_1,\psi_2}\right) \Psi_n^{\boldsymbol{\eta}}(\boldsymbol{\beta})
\end{aligned}$$

where we have used a substitution in the functions' arguments.

We again apply dominated convergence for the limit $\lambda \to \infty$; as argued previously, the functions Φ_n, ψ_1, ψ_2 are of Schwartz class and so have bounded norm. The tensor $S_{n+1}^{\sigma_{n+1}}(\boldsymbol{\beta}, \beta'-$

λ) is unitary and so have unit norm. Similarly $S_n^{\psi_1,\psi_2}$ is a partial trace of products of unitary representations of $\mathfrak{S}_n$ and a finite scalar product and so clearly has bounded norm, also. With the requirements of dominated convergence satisfied, what remains to be shown is that

$$\lim_{\lambda\to\infty}\int \mathrm{d}\beta' \psi_2^\gamma(\beta')\overline{S_{n+1}^{\sigma_{n+1}}(\boldsymbol{\beta},\beta'-\lambda)_{\delta\boldsymbol{\zeta}}^{\boldsymbol{\eta}\gamma}\psi_1^\delta(\beta')}\Psi_n^{\boldsymbol{\zeta}}(\boldsymbol{\beta}) = (S_n^{\psi_1,\psi_2}\Psi_n)^{\boldsymbol{\zeta}}(\boldsymbol{\beta}).$$

Evaluating the limit by dominated convergence and expanding the left hand side according to (4.37) gives

$$\begin{aligned}\int \mathrm{d}\beta' \psi_2^\gamma(\beta')&\overline{\rho_{S_-}^{n+1}(\sigma_{n+1})_{\delta\boldsymbol{\zeta}}^{\boldsymbol{\eta}\gamma}\psi_1^\delta(\beta')}\Psi_n^{\boldsymbol{\eta}}(\boldsymbol{\beta})\\ &= \int \mathrm{d}\beta' \psi_2^\gamma(\beta')\overline{\sum_{\xi_1,\dots,\xi_{n+1}}\delta_{\xi_{n+1}}^\gamma\delta_{\xi_1}^\delta\prod_{l=1}^n (S_-)_{\xi_l\zeta_l}^{\eta_l\xi_{l+1}}}\,\overline{\psi_1^\delta(\beta')}\Psi_n^{\boldsymbol{\eta}}(\boldsymbol{\beta}).\end{aligned}\tag{4.52}$$

We can read off here the expression for $n=0$ easily:

$$S_0^{\psi_1,\psi_2}\Omega_S = \langle\psi_1,\psi_2\rangle\Omega_S.$$

For $n>0$, by (4.8) iii) we can simplify slightly by applying $\overline{(S_-)_{\xi_l\zeta_l}^{\eta_l\xi_{l+1}}} = (S_-)_{\xi_l\eta_l}^{\zeta_l\xi_{l+1}}$:

$$\begin{aligned}\overline{\rho_{S_-}^{n+1}(\sigma_{n+1})_{\delta\boldsymbol{\zeta}}^{\boldsymbol{\eta}\gamma}} &= \overline{\sum_{\xi_1,\dots,\xi_{n+1}}\delta_{\xi_{n+1}}^\gamma\delta_{\xi_1}^\delta\prod_{l=1}^n (S_-)_{\xi_l\zeta_l}^{\eta_l\xi_{l+1}}}\\ &= \sum_{\xi_1,\dots,\xi_{n+1}}\delta_{\xi_{n+1}}^\gamma\delta_{\xi_1}^\delta\prod_{l=1}^n (S_-)_{\xi_l\eta_l}^{\zeta_l\xi_{l+1}}\\ &= \rho_{S_-}^{n+1}(\sigma_{n+1})_{\delta\boldsymbol{\eta}}^{\boldsymbol{\zeta}\gamma}\end{aligned}$$

With this cosmetic simplification in mind, we illustrate the action of $S_n^{\psi_1,\psi_2}$ to realise it coincides with that of (4.52). Let $\Psi_{n+1}\in\mathcal{H}_{n+1}\cap\mathscr{S}(\mathbb{R}^{n+1})\otimes\tilde{\mathcal{H}}^{\otimes n+1}$ then

$$\begin{aligned}\Big((m(\psi_1,\psi_2)\otimes 1^{\otimes n})\,&\rho_F^{n+1}(\sigma_{n+1}^{-1})\rho_{S_-}^{n+1}(\sigma_{n+1})\Psi_{n+1}\Big)^{\boldsymbol{\alpha}}(\boldsymbol{\beta})\\ &= \langle\psi_1^{\alpha_1},\psi_2^\tau\rangle_{L^2(\mathbb{R})}\Big(\rho_F^{n+1}(\sigma_{n+1}^{-1})\rho_{S_-}^{n+1}(\sigma_{n+1})\Psi_{n+1}\Big)^{\tau\alpha_2\cdots\alpha_{n+1}}(\boldsymbol{\beta})\\ &= \langle\psi_1^{\alpha_1},\psi_2^\tau\rangle_{L^2(\mathbb{R})}\Big(\rho_{S_-}^{n+1}(\sigma_{n+1})\Psi_{n+1}\Big)^{\alpha_2\cdots\alpha_{n+1}\tau}(\boldsymbol{\beta})\\ &= \langle\psi_1^{\alpha_1},\psi_2^\tau\rangle_{L^2(\mathbb{R})}\rho_{S_-}^{n+1}(\sigma_{n+1})_{\gamma_1\cdots\gamma_{n+1}}^{\alpha_2\cdots\alpha_n\tau}\Psi_{n+1}^{\boldsymbol{\gamma}}(\boldsymbol{\beta}).\end{aligned}$$

The partial trace introduces a $\delta_{\gamma_1}^{\alpha_1}$ term, and (4.37) gives the full expression for $(S_-)_{n+1}^{\sigma_{n+1}}(\boldsymbol{\beta},\beta') = \rho_{S_-}^{n+1}(\sigma_{n+1})$ to coincide with (4.52). $\square$

Proposition 4.25. *Let $A\in\mathcal{A}(I)$ for some bounded interval $I=(a,b)$, $S\in\mathcal{S}_{\lim}(\tilde{\mathcal{H}})$, then $[A,S^{J_1g^-,g'^+}]=0$ for any $g\in\mathscr{S}(b,\infty)\otimes\tilde{\mathcal{H}}$, $g'\in\mathscr{S}(-\infty,a)\otimes\tilde{\mathcal{H}}$.*

Proof. Set $f^{[\prime]} = (g^{[\prime]})^{0,\lambda}$ for $\lambda \geq 0$. Then the closed field operator $\phi_S(f)$ is affiliated with $V^\tau(b)\mathcal{A}_S(\mathbb{R}_+)V^\tau(b)^*$ and $\phi'(f')$ with $V^\tau(a)\mathcal{A}_S(\mathbb{R}_-)V^\tau(a)^*$. This implies that their product $\phi_S(f)\phi'_S(f')$ must commute with

$$\mathcal{A}_S(I) = V^\tau(a)\mathcal{A}_S(\mathbb{R}_+)V^\tau(a)^* \cap V^\tau(b)\mathcal{A}_S(\mathbb{R}_-)V^\tau(b)^*.$$

We have

$$\begin{aligned}\phi_S(f)\phi'_S(f') = {} & y_S^\dagger(f^+)y_S^\dagger(f'^+)' + y_S^\dagger(f^+)y_S(Jf'^-)' \\ & + y_S^\dagger(f'^+)'y_S(J_1f^-) + y_S(J_1f^-)z_S(Jf'^-)' \\ & + \left[y_S(J_1f^-), y_S^\dagger(f'^+)'\right].\end{aligned}$$

By (4.24) we know that the first four terms vanish in the weak operator topology, and we are left with $S^{J_1g^-,g'^+}$ as claimed. □

This gives us some insight into the size of these interval algebras, though the problem is still not transparent. The above commutation relation between algebra elements and the obstruction operator is a necessary condition, but not a sufficient one and in addition, $S_n^{f,g}$ may be an extremely complicated operator - though its action is purely on (the tensor powers of) the finite dimensional space $\tilde{\mathcal{H}}$. We can, of course, immediately construct operators that commute with $S_n^{f,g}$. For example, let $A_n \in \mathcal{B}(L^2(\mathbb{R}^{\otimes n})$ for all $n > 0$ and take $A_0 = 1_{L^2(\mathbb{R})}$, then the operator

$$A = \bigoplus_{n\geq 0} A_n \otimes 1_{\tilde{\mathcal{H}}}^{\otimes n}$$

commutes with $S_n^{f,g}$.

We can also consider simple examples of $S_n^{f,g}$ - the case of $S_- = F$ (as is the case for the $O(N)$ σ-models, for example) the obstruction reduces to $S_n^{f,g} = \langle f, g\rangle \cdot 1_{\mathcal{H}}$ for all n. This is analogous to the findings in [BLM11] for the free Bose case where such an obstruction is no longer present. In [BLM11] it is found to always be the case that $S_+ = S_-$ in the scalar case, but for $d \geq 2$ this is no longer necessarily the case by the definition of S and we so far have seen a possible obstruction involving S_-, but S_+ has not played a role. It would be natural to wonder if there exists a similar obstruction governed by the opposite limit. Part (i) of (4.8) offers help in this direction and we insert this relation into (4.52) for $\Psi_n \in \mathcal{H}_n \cap \mathscr{S}(\mathbb{R}^n) \otimes \tilde{\mathcal{H}}^{\otimes n}$

$$\begin{aligned}&\int \mathrm{d}\beta' \psi_2^\gamma(\beta') \overline{\sum_{\xi_1,\dots,\xi_{n+1}} \delta^\gamma_{\xi_{n+1}} \delta^\delta_{\xi_1} \prod_{l=1}^n (S_-)^{\eta_l \xi_{l+1}}_{\xi_l \zeta_l} \overline{\psi_1^\delta(\beta')} \Psi_n^{\boldsymbol{\eta}}(\boldsymbol{\beta})} \\ &\quad = \int \mathrm{d}\beta' \psi_2^\gamma(\beta') \sum_{\xi_1,\dots,\xi_{n+1}} \delta^\gamma_{\xi_{n+1}} \delta^\delta_{\xi_1} \prod_{l=1}^n (S_+)^{\xi_l\zeta_l}_{\eta_l \xi_{l+1}} \overline{\psi_1^\delta(\beta')} \Psi_n^{\boldsymbol{\eta}}(\boldsymbol{\beta}).\end{aligned} \tag{4.53}$$

Definition 4.26. Let $S_+ \in \mathcal{R}_{\lim}(\tilde{\mathcal{H}})$, with $f, g \in \mathcal{H}$. For $n \in \mathbb{N}_0$ we define the operator $\tilde{S}_n^{f,g}$ on $\tilde{\mathcal{H}}^{\otimes n}$:

$$\tilde{S}_n^{f,g}\xi_n = \mathrm{pt}_R\Big[\left(1^{\otimes n} \otimes m(\overline{g}, \overline{f})\right) \rho_F^{n+1}(\sigma_{n+1})\rho_{S_+}^{n+1}(\sigma_{n+1}^{-1})\Big]\xi_n. \tag{4.54}$$

We briefly compare $\tilde{S}_n^{f,g}$ to $S_n^{f,g}$ to point out the subtle differences - firstly we take a partial trace in a different tensor slot, which is reflected in the change of position of m in the formula. Moreover, the arguments of m undergo an exchange in position and gain an extra conjugation. Finally, the arguments of both ρ_F^{n+1} and $\rho_{S_+}^{n+1}$ are inverted.

Analogously to $S^{f,g}$ we take the direct sum of $\tilde{S}_n^{f,g}$ over all n

$$\tilde{S}^{f,g} := \bigoplus_{n\geq 0} 1_{L^2(\mathbb{R})}^{\otimes n} \otimes \tilde{S}_n^{f,g}$$

and we now show that $S_n^{f,g}$ does indeed coincide with the expected (4.53).

Corollary 4.27. *Let $S_+ \in \mathcal{R}_{\lim}(\tilde{\mathcal{H}})$, $f, g \in L^2(\mathbb{R}) \otimes \tilde{\mathcal{H}}$ then*

$$\left(\tilde{S}_n^{f,g}\Psi_n\right)^{\alpha}(\boldsymbol{\beta}) = \int \mathrm{d}\beta' g^{\gamma}(\beta') \sum_{\xi_1,\dots,\xi_{n+1}} \delta^{\gamma}_{\xi_{n+1}} \delta^{\delta}_{\xi_1} \prod_{l=1}^{n} (S_+)^{\xi_l \zeta_{l+1}}_{\eta_l \xi_{l+1}} \overline{f^{\delta}(\beta')} \Psi_n^{\boldsymbol{\eta}}(\boldsymbol{\beta}), \qquad \Psi_n \in \mathcal{H}_n\,.$$

Proof. To show this, we will analyse the operator (4.54), in particular the quantity within the partial trace. To this end, let $\Psi_n \in \mathcal{H}_n$, then

$$\begin{aligned}
&\Big[\left(1^{\otimes n} \otimes m(f,g)\right) \rho_F^{n+1}(\sigma_{n+1})\rho_{S_+}^{n+1}(\sigma_{n+1}^{-1})\Psi_{n+1}\Big]^{\boldsymbol{\alpha}}(\boldsymbol{\beta}) \\
&\qquad = \langle \overline{g^{\alpha_{n+1}}}, \overline{f^{\delta}} \rangle_{L^2(\mathbb{R})} \Big[\rho_F^{n+1}(\sigma_{n+1})\rho_{S_+}^{n+1}(\sigma_{n+1}^{-1})\Psi_{n+1}\Big]^{\alpha_1\dots\alpha_n\delta}(\boldsymbol{\beta}) \\
&\qquad = \langle \overline{g^{\alpha_{n+1}}}, \overline{f^{\delta}} \rangle_{L^2(\mathbb{R})} \Big[\rho_{S_+}^{n+1}(\sigma_{n+1}^{-1})\Psi_{n+1}\Big]^{\delta\alpha_1\dots\alpha_n}(\boldsymbol{\beta}) \\
&\qquad = \langle \overline{g^{\alpha_{n+1}}}, \overline{f^{\delta}} \rangle_{L^2(\mathbb{R})} \rho_{S_+}^{n+1}(\sigma_{n+1}^{-1})^{\delta\alpha_1\dots\alpha_n}_{\eta_1\dots\eta_n\gamma} \Psi_{n+1}^{\eta_1\dots\eta_n\gamma}(\boldsymbol{\beta}).
\end{aligned}$$

Taking the partial trace in the right-most tensor slot introduces a $\delta_{\gamma}^{\alpha_{n+1}}$ term which proves the assertion. □

As mentioned briefly before, these obstruction operators provide a necessary condition on the size of the interval algebras $\mathcal{A}_S(I), I \subset \mathbb{R}$, but the commutativity condition is not entirely transparent for the even the simplest cases and would prove incalculable in generality.

In some cases of S_-, the obstruction operator $S^{f,g}$ has a much simpler form.

Proposition 4.28. *Let $A, B \in \mathcal{B}(\tilde{\mathcal{H}})$ be commuting involutive, orthogonal matrices. Then $R := F(A \otimes B)$ is a unitary R-matrix and $R^* = R^c$.*

Proof. We begin by showing that R satisfies the Yang-Baxter equations which reads

$$(R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R). \tag{4.55}$$

Focusing attention first on the left hand side, we can rewrite it as

$$\begin{aligned}(R \otimes 1)(1 \otimes R)(R \otimes 1) &= (F(A \otimes B) \otimes 1)(1 \otimes F(A \otimes B))(F(A \otimes B) \otimes 1) \\ &= (F \otimes 1)(1 \otimes F)(F \otimes 1)(A \otimes B \otimes 1)(1 \otimes A \otimes B)(A \otimes 1 \otimes B)(A \otimes B \otimes 1) \\ &= (F \otimes 1)(1 \otimes F)(F \otimes 1)(A^2 \otimes AB \otimes B^2).\end{aligned}$$

Repeating the same procedure for the right hand side we have the similar expression

$$(1 \otimes R)(R \otimes 1)(1 \otimes R) = (1 \otimes F)(F \otimes 1)(1 \otimes F)(A^2 \otimes BA \otimes B^2).$$

It is well known that the tensor flip solves the Yang-Baxter equation in any dimension, and given that it is involutive (4.55) simplifies to

$$A^2 \otimes AB \otimes B^2 = A^2 \otimes BA \otimes B^2$$

which is satisfied by the commutativity condition between A and B.

Unitarity of R follows from A, B being orthogonal and F being unitary and self-adjoint.

From Proposition (4.8) the condition $R^* = R^c$ is equivalent to $R^l = R^r = R$ which we can compute with matrix operations more explicitly. Indeed

$$R^l = F(A^* \otimes \overline{B}) = F(A \otimes B),$$

since A is involutive and orthogonal (implying it is self-adjoint) and B being real-valued due to orthogonality. In an identical fashion, we find also $R^r = R$.

□

The above shows that given certain conditions on A, B the matrix $F(A \otimes B)$ is a feasible limit of a rapidity-dependent S-matrix. We define the subset $\mathcal{R}'_{\lim}(\mathcal{K}) \subset \mathcal{R}_{\lim}(\mathcal{K})$ to be the elements $R \in \mathcal{R}_{\lim}(\mathcal{K})$ which can be written as $R = F(A \otimes B)$ for commuting involutive, orthogonal matrices $A, B \in \mathcal{B}(\mathcal{K})$.

Proposition 4.29. *Let* $F(A \otimes B) =: S_- \in \mathcal{R}'_{\lim}(\mathcal{K})$ *then*

$$S_n^{f,g} = \mathrm{Tr}\,(m(f,g)A)\, A^{\otimes n-1} \otimes B^n \quad \text{for all} \quad f, g \in L^2(\mathbb{R}) \otimes \mathcal{K}, n \geq 1.$$

Proof. The result follows from the expression (4.47) once we show that

$$\rho_F^{n+1}(\sigma_{n+1}^{-1})\rho_{S_-}^{n+1}(\sigma_{n+1}) = A^{\otimes n} \otimes B^n \tag{4.56}$$

and we proceed by induction on n. For $n = 0$, $\sigma_1 = \mathrm{id}$ and the statement is trivial. We assume (4.56) holds for some $n = k$ and approach the case of $n = k + 1$.

The representations ρ_F, ρ_{S_-} decompose according to the following expression

$$\rho_F^j(\sigma_{i,k+2}) = 1^{\otimes i-1} \otimes \rho_F^{j-i}(\sigma_{k+2-i}),$$

and similarly for ρ_{S_-}. For $i < k + 2$ yielding the simplification

$$\begin{aligned}
\rho_F^{k+2}(\sigma_{k+2}^{-1})\rho_{S_-}^{k+2}(\sigma_{k+2}) &= \rho_F^{k+2}(\sigma_{2,k+2}^{-1}\tau_1)\rho_{S_-}^{k+2}(\tau_1\sigma_{2,k+2}) \\
&= \rho_F^{k+2}(\sigma_{2,k+2}^{-1})(F \otimes 1^{\otimes k})(F(A \otimes B) \otimes 1^{\otimes n})\rho_{S_-}^{k+2}(\sigma_{2,k+2}) \\
&= \left(1 \otimes \rho_F^{k+1}(\sigma_{k+1}^{-1})\right)\left(A \otimes B \otimes 1^{\otimes n}\right)\left(1 \otimes \rho_{S_-}^{k+1}(\sigma_{k+1})\right) \\
&= (A \otimes 1^{\otimes n} \otimes B)\left(1 \otimes \rho_F^{k+1}(\sigma_{k+1}^{-1})\rho_{S_-}^{k+1}(\sigma_{k+1})\right) \\
&= (A \otimes 1^{\otimes n} \otimes B)(1 \otimes A^{\otimes n} \otimes B^n) \\
&= A^{\otimes n+1} \otimes B^{n+1}
\end{aligned}$$

and so it follows for all n. Inserting this into (4.47) we now reach the assertion.

□

So far, this description has been heavily dependent on general test functions, whereas the operator $S_n^{f,g}$ itself is a multiplication operator acting on the finite dimensional Hilbert space $\tilde{\mathcal{H}}$. For a more general expression, one may formulate in a function-independent manner which motivates the following preparatory result.

Lemma 4.30. *Let $g \in \mathscr{S}(-\infty, -1)$ be non-negative. Then*

$$\langle \overline{g^+}, g^+ \rangle_{L^2(\mathbb{R})} \neq 0.$$

Proof. Initially, we expand the scalar product and express g^+ in terms of g by (4.38):

$$\begin{aligned}
2\pi\langle \overline{g^+}, g^+ \rangle_{L^2(\mathbb{R})} &= -\int_{\mathbb{R}} e^{2\beta} \int_{-\infty}^{-1} \int_{-\infty}^{-1} \mathrm{d}x\mathrm{d}y\mathrm{d}\beta \; g(x)g(y)e^{ie^\beta(x+y)} \\
&= -\int_{-\infty}^{-1} \int_{-\infty}^{-1} \mathrm{d}x\mathrm{d}y \; g(x)g(y) \int_{\mathbb{R}} \mathrm{d}\beta \; e^{2\beta}e^{ie^\beta(x+y)}.
\end{aligned}$$

All functions appearing in the integrand are measurable, hence we may invoke Fubini's Theorem followed by the substitution $p = e^\beta$. Prior to continuing calculations, recall the definition of the Heaviside function

$$H(p) = \begin{cases} 1, & p \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

whose Fourier transform is well-documented (see, for example [Bel19]) and has a distributional expression:

$$\tilde{H}(\xi) = \pi\delta(\xi) - \frac{1}{i\xi}$$

where $\frac{1}{i\xi}$ is understood as the principle value.

Now

$$\begin{aligned} 2\pi\langle\overline{g^+}, g^+\rangle_{L^2(\mathbb{R})} &= -\int_{-\infty}^{-1}\int_{-\infty}^{-1} g(x)g(y)\int_0^\infty pe^{ip(x+y)}\mathrm{d}p\mathrm{d}xdy \\ &= -\int_{-\infty}^{-1}\int_{-\infty}^{-1} g(x)g(y)\int_{-\infty}^\infty H(p)pe^{ip(x+y)}\mathrm{d}p\mathrm{d}xdy \\ &= -\int_{-\infty}^{-1}\int_{-\infty}^{-1} g(x)g(y)\frac{\mathrm{d}}{d(x+y)}\left(\pi\delta(x+y) - \frac{1}{i(x+y)}.\right)\mathrm{d}xdy \end{aligned}$$

Given the support of f, the contribution given by the delta distribution vanishes leaving just

$$\langle\overline{g^+}, g^+\rangle_{L^2(\mathbb{R})} = -\frac{i}{2\pi}\int_1^\infty\int_1^\infty g(x)g(y)\frac{1}{(x+y)^2}\mathrm{d}xdy.$$

The kernel appearing is always positive, and g is a positive function thus $\langle\overline{g^+}, g^+\rangle_{L^2(\mathbb{R})} \neq 0$.

$\square$

Lemma 4.31. *Let $M \in \mathcal{B}(\tilde{\mathcal{H}})$ and $S_- \in \mathcal{R}_{\lim}(\tilde{\mathcal{H}})$ then the operator $X(M) : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ defined by*

$$\begin{aligned} X(M) &= \bigoplus_{n\geq 0} X_n(M), \\ X_0(M) &= 0, \\ X_n(M) &= \mathrm{pt}_L\left[(M \otimes 1^{\otimes n})\rho_F^{n+1}(\sigma_{n+1}^{-1})\rho_{S_-}^{n+1}(\sigma_{n+1})\right] - \mathrm{Tr}(M)1. \end{aligned} \tag{4.57}$$

is an element of $\mathcal{A}_S(I)'$.

Proof. Fix $I = (-1,1)$ without loss of generality, and let $f \in \mathscr{S}(-\infty,-1)\otimes\tilde{\mathcal{H}}$, $g \in \mathscr{S}(1,\infty)\otimes\tilde{\mathcal{H}}$. Then the operator $X_n(M)$ differs from that of $S_n^{J_1 f^-, g^+}$ by a $-\mathrm{Tr}(M)1$ term, and the labelled matrix M. Note that we can write $X_0(M) = \mathrm{Tr}(M)1 - \mathrm{Tr}(M)1$ so the previous observation is true for $n \geq 0$.

Let the functions f, g now be real-valued, positive functions such that $f_\nu(-x) = g_\mu(x)$ for fixed $\nu, \mu \in \{1, \ldots d_{\tilde{\mathcal{H}}}\}$ and all other components zero. Now $m(J_1 f^-, g^+) = \langle\overline{g_\mu^+}, g_\mu^+\rangle_{L^2(\mathbb{R})}E_\mu^\nu$, where E_μ^ν has only the ν,μ-th entry equal to one, and everywhere else zero. Suitable normalisation of g_μ can be chosen such that $\langle\overline{g_\mu^+}, g_\mu^+\rangle_{L^2(\mathbb{R})} = 1$ and such $m(J_1 f^-, g^+)$ then form an orthonormal basis for $\mathcal{B}(\tilde{\mathcal{H}})$. Linearity of the trace and the closure of $\mathcal{A}_S(I)'$ under linear combinations implies that we may write any $M \in \mathcal{B}(\tilde{\mathcal{H}})$ as a linear combination of suitably chosen $m(J_1 f^-, g^+)$ and hence $X(M) \in \mathcal{A}_S(I)'$ for $M \in \mathcal{B}(\tilde{\mathcal{H}})$.

□

This now gives us the freedom to consider $X(M)$ as a potential obstruction to local operators for any $M \in \mathcal{B}(\tilde{\mathcal{H}})$ which needs no reference to analyticity properties of functions.

A cyclic vacuum vector is the ideal situation we would find ourselves in for a local algebra. With the definition of $X(M)$ we find a restriction on potential values of $S_\pm$ for this to be the case.

Theorem 4.32. *Let $S \in \mathcal{S}_{\lim}(\tilde{\mathcal{H}})$ and $I \subset \mathbb{R}$, then if Ω_S is cyclic for $\mathcal{A}_S(I)$, $S_\pm = F$.*

Proof. Let Ω_S be cyclic for $\mathcal{A}_S(I)$, then in particular it is separating for $\mathcal{A}_S(I)'$. By definition $X(M)\Omega_S = 0$ for any M, and hence $X_n(M)$ must vanish for all n. Let $S_- \in \mathcal{R}_{\lim}(\tilde{\mathcal{H}})$, and choose $M = E^\nu_\mu$ for any $\nu, \mu \in \{1, \dots, d_{\tilde{\mathcal{H}}}\}$ then

$$X_1(M)^\alpha_\beta = E^a_b (FS_-)^{\mu\alpha}_{\nu\beta} - \delta^\nu_\mu \delta^\alpha_\beta = 0$$

We read off that $(S_-)^{\alpha\nu}_{\mu\beta} = \delta^\nu_\mu \delta^\alpha_\beta$ which is precisely $S_- = F$.

□

With Ω_S cyclic for $\mathcal{A}_S(I)$ it is known that $\mathcal{A}_S(I)$ is isomorphic to a unique hyperfinite Type III_1 factor for any I [BL04, BDF87]. This is one possible case for $\mathcal{A}_S(I)$, and is the case where we find the largest possible algebra. The opposite extreme is when $\mathcal{A}_S(I)$ contains only multiples of the identity which is now the direction of focus.

In the abstract algebraic setting, the notion of obstructions to local observables can be described by projections onto certain subspaces. Analysis of such projections is an equivalent problem to analysis of obstruction operators. We can consider the intersection of all commutants of interval algebras $\mathcal{A}_S(I)'$ and define what we will refer to as an *algebra at infinity*:

$$\mathcal{A}^\infty_S := \bigcap_{\mathcal{I} \in \mathbb{R}} \mathcal{A}_S(I)'.$$

We describe this in much more detail in the following Chapter.

Chapter 5

Deformations of Chiral Theories and Trivial Inclusions

In the previous chapter, we alluded to the possibility of a trivial local algebra - that is, an algebra where the only local observables for the model that exist are multiples of the identity. Though this case is pathological for a physically relevant quantum field theory, the problem is a relevant one in other contexts. In particular, this idea is deeply rooted in the analysis of half-sided modular inclusions (which we will formally define in the next section) in which context they are referred to as a *singular inclusion*.

The first example of a singular inclusion was found recently by [LTU19] in the context of free probability. However, we present here the findings as in [LS22] using simpler tools in the context of field theory deformations, and in particular warped convolutions. We will speak of a Hilbert space $\mathcal{H}$ (for now remaining general) with a (vacuum) vector Ω.

5.1 Half-Sided Modular Inclusions

We begin by defining a half-sided modular inclusion in its own right together with the notion of a Borchers triple, before aligning it with our understanding of the chiral models we have previously constructed. The results presented here are some of those that have been published in the joint paper [LS22] with G. Lechner.

We begin with preliminary definitions in our abstract setting, in particular, we use the shorthand notation

$$\sigma_t(\cdot) := \Delta^{it}(\cdot)\Delta^{-it}$$

where Δ is the modular operator associated with a von Neumann algebra $\mathcal{M}$ (for more on Tomita-Takesaki theory, see Appendix D).

Definition 5.1. For von Neumann algebras $\mathcal{M}, \mathcal{N}$ on $\mathcal{H}$ with Ω a cyclic and separating vector, the inclusion $\mathcal{N} \subset \mathcal{M}$ is called *half-sided modular* if $\sigma_t(\mathcal{N}) \subseteq \mathcal{N}$ for all $t \leq 0$.

Definition 5.2. A one-dimensional Borchers triple $(\mathcal{M}, V^\tau, \Omega)$ consists of a von Neumann algebra $\mathcal{M}$, a strongly continuous, unitary one-parameter group V^τ with positive generator such that

$$V^\tau(x)\,\mathcal{M}\,V^\tau(x)^{-1} \subset \mathcal{M}, \quad \text{for} \quad x \geq 0$$

and a vector Ω which is cyclic and separating for $\mathcal{M}$ and invariant under V^τ, that is $V^\tau(x)\Omega = \Omega$ for all $x \in \mathbb{R}$.

We recall that for a one-dimensional Borchers triple (M, V^τ, Ω) Borchers theorem [Bor92] says that

$$\Delta^{it}V^{tau}(x)\Delta^{-it} = V^\tau(e^{-2\pi}x), \quad JV^\tau(x)J = V^\tau(-x), \quad t, x \in \mathbb{R}$$

where J, Δ are the modular data of $\mathcal{M}$ (see Appendix D). Thus V^τ extends to an anti-unitary representation of the affine group.

To link with previous discussions in Chapter 3, we can realise the von Neumann algebra $\mathcal{M}$ as the half-line algebra $\mathcal{A}_S(\mathbb{R}_+)$, the unitary group V^τ as the restriction of the representation V of the affine group to the translations, and the vector Ω as the vacuum Ω_S. We then realise that in this model we have a Borchers triple, with an example of a half-sided modular inclusion given by $\mathcal{A}_S(1, \infty) \subset \mathcal{A}_S(\mathbb{R}_+)$. The results in this Chapter will be described in the former context of a more abstract algebraic structure, but as we can see they are still applicable to the specific quantum field theoretic setting we have previously analysed. Moreover, we will be restricting our setting by tacitly choosing the finite dimensional component $\tilde{\mathcal{H}} = \mathbb{C}$ (the scalar case) and dropping any explicit dependence on an underlying S-matrix S.

Similarly to (4.46) we define

$$\mathcal{A}(a, b) := V^\tau(a)\,\mathcal{M}\,V^\tau(a)^{-1} \cap V^\tau(b)\,\mathcal{M}'\,V^\tau(b)^{-1} \tag{5.1}$$

and call the mapping $\mathbb{R} \supset I \mapsto \mathcal{A}(I) \in \mathcal{B}(\mathcal{H})$ the "local net associated with $(\mathcal{M}, V, \Omega)$". It is clear from the definition that Ω is separating for $\mathcal{A}(a, b)$, however, cyclicity is not inherited and hence we have no immediate notion of size.

We define the "local subspace"

$$\mathcal{H}_{\text{loc}} := \overline{\mathcal{A}(I)\Omega} \subset \mathcal{H}, \quad I \subset \mathbb{R}$$

which is the smallest space on which we have cyclicity of Ω by definition.

Lemma 5.3. [LS22] *The subspace $\mathcal{H}_{\text{loc}}$ is independent of the choice of I.*

As discussed in Chapter 2 for the analogous case for massive integral models and the von Neumann algebras (2.20), there are three possibilities we consider for the size of the algebras $\mathcal{A}(I)$ and we describe these in this context by way of the projection P_{loc} onto the local subspace $\mathcal{H}_{\text{loc}}$ and the projection P_Ω onto $\mathbb{C}\Omega$:

1) "The Standard Case": $P_{\text{loc}} = 1$,

2) "The Intermediate Case": $P_\Omega \lneqq P_{\text{loc}} \lneqq 1$,

3) "The Singular Case": $P_{\text{loc}} = P_\Omega$.

By (4.46) and Lemma (5.3) we can see that the algebra $\mathcal{A}(0,1)$ and the relative commutant $\mathcal{N}' \cap \mathcal{M}$ coincide, and so the three cases above can also be described in terms of the commutant $\mathcal{N}' \cap \mathcal{M}$. In particular, these descriptions coincide with those for cases 1)–3) listed in Chapter 2 when $\mathcal{A}_S(\mathcal{O})$ is replaced by $\mathcal{N}' \cap \mathcal{M}$.

In the previous Chapter, we derived an expression (4.57) for an obstruction operator $X(M)$ taking arguments in the space of complex valued matrices. By calculating the short distance scaling limit of a certain product of field operators, it was shown that all local observables must commute with such an operator, and in an abstract algebraic setting we describe this by the notion of an "algebra at infinity" [BR87]. This may be defined in a number of ways, but we choose the following.

Definition 5.4. Let $\mathcal{N} \subset \mathcal{M}$ be a half-sided modular inclusion and $I \mapsto \mathcal{A}(I)$ $(I \subset \mathbb{R})$ the local net associated to it. The *algebra at infinity* is then the von Neumann algebra

$$\mathcal{A}_\infty := \bigcap_{I \subset \mathbb{R}} \mathcal{A}(I)' \tag{5.2}$$

The relevance of $\mathcal{A}_\infty$ comes into play when one considers its elements, and its relative size to $\mathcal{A}(I)$. Any element $X \in \mathcal{A}_\infty$ can be interpreted as an obstruction to local observables, and it may then be concluded that the operator $X(M)$ defined in (4.57) belongs to the algebra at infinity. As we look to analyse the singular case, we note the following result.

Proposition 5.5. *Let $\mathcal{N} \subset \mathcal{M}$ be a half-sided modular inclusion and its algebra at infinity be $\mathcal{A}_\infty$. Then the following statements are equivalent:*

i) The inclusion $\mathcal{N} \subset \mathcal{M}$ is singular, i.e. $P_{\mathrm{loc}} = P_\Omega$.

ii) The algebra at infinity contains all bounded linear operators on $\mathcal{H}$, i.e. $\mathcal{A}_\infty = \mathcal{B}(\mathcal{H})$.

iii) $P_\Omega \in \mathcal{A}_\infty$.

Proof. i)⇒ii) In the singular case the interval algebras are trivial (that is, they consist only of constant multiples of the identity), and hence $\mathcal{A}(I)' = (\mathbb{C}1_{\mathcal{H}})' = \mathcal{B}(\mathcal{H})$. By definition then it is clear that $\mathcal{A}_\infty = \mathcal{B}(\mathcal{H})$.

ii)⇒iii): Given that all orthogonal projections are naturally bounded, this is trivial.

iii)⇒i): As previously mentioned the relative commutant $\mathcal{N}' \cap \mathcal{M}$ and the algebra $\mathcal{A}_S(I)$ coincide, so let $A \in \mathcal{A}_S(I)$. Moreover, given that $P_\Omega \in \mathcal{A}_\infty$, then $P_\Omega \in \mathcal{A}(0,1)'$ and we calculate

$$A\Omega = AP_\Omega\Omega = P_\Omega A\Omega = \omega(A)\Omega.$$

The vector Ω is separating for $\mathcal{A}(I)$ a priori, so we conclude that $A = \omega(A)1_{\mathcal{H}}$ which implies i) since all local observables are trivial. □

The procedure to derive the obstruction operator $S^{f,g}$ considered previously can be reformulated to suit this more abstract setting. In general, we begin by taking an operator $A \in \mathcal{N} \vee J\mathcal{N}J$ which by definition is localised in the region $(-\infty,-1] \cup [1,\infty)$. Similar to the short distance scaling limits formerly implemented, we scale by modular action to operators $\sigma_t(A)$ which are then localised in the region $(-\infty,-e^{2\pi t}] \cup [e^{-2\pi t},\infty)$ and then consider the limit $t \to -\infty$, resulting in elements of $\mathcal{A}_\infty$.

Before proceeding we note first that any strongly continuous one-parameter group W whose generator has a purely absolutely continuous spectrum satisfies $\operatorname*{w-lim}_{x\to\pm\infty} W(x) = 0$ [Yaf92, Page 30], which implies

$$\operatorname*{w-lim}_{x\to\pm\infty} V^\tau(x) = P_\Omega. \tag{5.3}$$

The von Neumann algebra $\mathcal{M}$ is also type III_1 [Wie93], which via the Connes' characterisation implies that the spectrum of $\log(\Delta)$ is purely absolutely continuous up to an eigenvalue 0 with eigenspace $\mathbb{C}\Omega$ which further implies that

$$\operatorname*{w-lim}_{x\to\pm\infty}\Delta^{it} = P_\Omega \tag{5.4}$$

by the same arguments as the previous assertion. We now collect some results on these weak limits and the operators in $\mathcal{A}_\infty$.

Lemma 5.6. *Let $A \in \mathcal{N} \vee J\mathcal{N}J$ and $L \in \mathcal{B}(\mathcal{H})$ be such that*

$$\operatorname*{w-lim}_{t\to-\infty} \sigma_t(A) = L.$$

Then:

i) $L \in \mathcal{A}_\infty$,

ii) $[L, \Delta^{it}] = 0$ *for all* $t \in \mathbb{R}$,

iii) $L\Omega = \omega(A)\Omega$.

Proof. i) For a fixed $t \in \mathbb{R}$ the shifted operator $\sigma_t(A)$ is an element of the shifted algebra $\sigma_t(\mathcal{N} \vee J\mathcal{N}J)$, but the latter is no more than

$$V(e^{-2\pi t})\,\mathcal{M}\,V(e^{-2\pi t})^* \vee V(-e^{-2\pi t})\,\mathcal{M}'\,V(-e^{-2\pi t})^* =: \mathcal{M}_t\,.$$

Clearly we have $\mathcal{M}_t \subset \mathcal{M}_s$ for all $t < s$ so the limit $t \to -\infty$ is equivalent to $\bigcap_{t<0} \mathcal{M}_t = \mathcal{A}_\infty$ which implies $L \in \mathcal{A}_\infty$.

ii) By assumption $\operatorname*{w-lim}_{t\to-\infty}(\sigma_t(A)) = L$, and so for fixed $s \in \mathbb{R}$ it is also true that $\sigma_{t+s}(A) \to L$ in the weak sense as $t \to -\infty$. But by definition $\sigma_{t+s}(A) = \Delta^{is}\sigma_t(A)\Delta^{-is} \to \sigma_s(L)$ as $t \to -\infty$ from which we conclude that $\sigma_s(L) = L$.

iii) Taking into account the assumption of the weak limit of $\sigma_t(A)$ and the invariance of Ω under V^τ we see that $\Delta^{it}A\Omega = \Delta^{it}A\Delta^{-it}\Omega = \sigma_t(A)\Omega \to L\Omega$ weakly as $t \to -\infty$. In parallel, the application of (5.4) implies that $\Delta^{-t}A\Omega \to P_\Omega A\Omega = \omega(A)\Omega$. The separating property of Ω being inherited by intersections gives us the required conclusion that $L\Omega = \omega(A)\Omega$. □

Having now discussed the algebra at infinity and analysed properties of its elements, we now move on to describe our framework for constructing an example of a singular inclusion. This construction will be similar to those considered previously in chiral field theory models, however, we will restrict ourselves to the scalar case, but the extension to more complicated settings could be possible in future investigations.

5.2 Model Deformations and Warped Convolutions

In this section we present our description of a deformation of a half-sided modular inclusion, and in particular that of a warped convolution [BLS10, BS08]. To do so, we first describe the previous data of a Borchers' triple in terms of a standard subspace before moving to the Hilbert space representation which is linked closely to the chiral models in the previous Chapter.

Definition 5.7. Let $\mathcal{H}$ be a Hilbert space, then a standard pair (V_1^τ, H) over $\mathcal{H}$ is both

i) a closed real standard subspace $H \subset \mathcal{H}$. That is, H is cyclic for $\mathcal{H}$: $H + iH$ is dense in $\mathcal{H}$, and H is separating for $\mathcal{H}$: $H \cap iH = \{0\}$.

ii) a strongly continuous one-parameter group $V_1^\tau(x)$ with positive generator P, with $\ker(P) = \{0\}$ such that $V_1^\tau(x)H \subset H$ for all $x \geq 0$,

A standard pair may give rise to a one-dimensional Borchers triple by second quantisation. To see this we consider the standard Bose Fock space $\mathcal{F}(\mathcal{H})$ over $\mathcal{H}$ on which we have a vacuum vector Ω. The Weyl unitaries $W(h), h \in H$ generate a von Neumann algebra $\mathcal{M}(H)$ by double commutant

$$\mathcal{M}(H) := \{W(h) \ : \ h \in H\}''$$

which aligns with our previous definition of $\mathcal{A}_S(\mathbb{R}_+)$ where the exponentiated field operators play the role of Weyl operators. This algebra, together with the second quantised translational operators V^τ and the vacuum Ω form a one-dimensional Borchers triple $(\mathcal{M}(H), V^\tau, \Omega)$. Furthermore, we may construct modular data J, Δ for $(\mathcal{M}(H), V^\tau, \Omega)$ by taking the second quantisation of the modular data J_1, Δ_1 of H which is defined in the usual way by polar decomposition of the operator $S_1 : H + iH \mapsto H + iH, S_1(h + ih) = h - ih$.

If the one-parameter groups $V_1^\tau(x), \Delta^{it}$ are irreducible, then the standard pair $(V_1^\tau(x), H)$ is also called irreducible and it is unique up to a unitary equivalence. To align with our previous constructions, we may present this data in the following way [LL15].

We take a specific Hilbert space in rapidity space, and note the explicit action of the one-parameter group $V_1^\tau(x)$ as translations in position space acting by multiplication in rapidity space similar to (4.16):

$$\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}\theta), \quad (V_1^\tau(x)\psi)(\theta) = e^{ixe^\theta}\psi(\theta).$$

In this case, the standard subspace H has the explicit form

$$H = \{\psi \in \mathbb{H}^2(S(0,\pi)) \ : \ \overline{\psi(\theta + i\pi)} = \psi(\theta) \text{ a.e.}\}$$

where

$$\mathbb{H}^2(S(0,\pi)) = \left\{\varphi : S(0,\pi) \to \mathbb{C} \text{ analytic} \mid \sup_{0<\lambda<\pi} \int_{\mathbb{R}} \mathrm{d}\theta \, |\varphi(\theta + i\lambda)|^2 < \infty \right\}$$

then an $\psi \in H$ are boundary values of some $\varphi \in \mathbb{H}^2(S(0,\pi))$.

Recalling the definitions of the creation/annihilation and field operators in the massive case (2.7b) in Chapter 2 setting $S = F$ and taking $\tilde{\mathcal{H}} = \mathbb{C}$, we introduce a deformation parameter κ and restrict to the scalar setting. That is, we consider the annihilation operator z_κ as a linear map on the space of finite particle number $\mathcal{D} \subset \mathcal{F}(\mathcal{H})$ defined explicitly as

$$(z_\kappa(\varphi)\Psi)_n(\boldsymbol{\theta}) = \sqrt{n+1} \int_{\mathbb{R}} \mathrm{d}\theta \, \overline{\varphi(\theta)} \prod_{j=1}^{n} e^{i\kappa \sinh(\theta - \theta_j)} \Psi_n(\theta, \boldsymbol{\theta}) \tag{5.5}$$

for $\varphi \in \mathcal{H}$, and similarly the creation operator $z_\kappa^\dagger$ which here coincides with the adjoint of z_κ. Moreover, we define the field operator

$$\phi_\kappa(\xi) = z_\kappa^\dagger(\xi) + z_\kappa(J_1\xi), \quad \xi \in H + iH.$$

Similarly to Theorem (4.18), the field ϕ_κ is essentially self-adjoint and we denote its closure by the same symbol. Moreover, ϕ_κ transforms covariently under the symmetry V and the conjugation operator acts by $J\phi_\kappa(\xi)J = \phi_{-\kappa}(J_1\xi)$. For $\kappa \geq 0$ we have $[e^{i\phi_\kappa(\xi)}, e^{i\phi_{-\kappa}(\xi')}] = 0$ for $\xi \in H, \xi' \in H'$ and finally Ω is cyclic for the polynomial algebra generated by ϕ_κ.

We note the following proposition from [LS22].

Proposition 5.8. *Let $\kappa \geq 0$ and*

$$\mathcal{M}_\kappa := \left\{e^{i\phi_\kappa(h)} \ : \ h \in H\right\}'' \subset \mathcal{B}(\mathcal{H}) \tag{5.6}$$

Then $(\mathcal{M}_\kappa, V, \Omega)$ is a one-dimensional Borchers triple with unique vacuum vector Ω on the Bose Fock space $\mathcal{F}(\mathcal{H})$. For $\kappa = 0$ we have $\mathcal{M}_0 = \mathcal{M}(H)$, the second quantisation of the irreducible standard pair (V_1^τ, H).

Following this we will use similar notation of a subscript κ meaning the "deformed version" throughout - particularly the subalgebra $\mathcal{N}_\kappa = T(1)\,\mathcal{M}_\kappa\, T(-1)$, the algebra at infinity $\mathcal{A}_{\kappa,\infty}$ etc.

Having described a one-dimensional Borchers triple depending on a deformation parameter we require more to apply a suitable deformation scheme, in particular a warped convolution [BS08]. In this direction we now look to construct a two-dimensional Borchers triple, which can be viewed as a one-dimensional Borchers triple [LST13, BT13].

Analogously to the one-dimensional case, a two-dimensional Borchers triple $(\mathcal{M}, T, \Omega)$ consists of a von Neumann algebra $\mathcal{M}$ over a Hilbert space $\mathcal{H}$ with standard vector Ω and a two-parameter unitary representation T of the translation group which leaves Ω invariant. The algebra $\mathcal{M}$ acts under translations as

$$T(x,y)\,\mathcal{M}\,T(x,y)^* \subset \mathcal{M}$$

for $x \geq 0, y \leq 0$ and both one-parameter groups $T(x,0)$ and $T(0,y)$ have positive generators. Geometrically, the parameters x, y are light-ray coordinates of general vectors $\xi \in \mathbb{R}^2$ which decomposes as $x = \frac{1}{2}(\xi_0 + \xi_1)$ and $y = \frac{1}{2}(\xi_0 - \xi_1)$ where ξ_0 is the temporal coordinate, and ξ_1 the spatial.

Restricting T to a one-parameter group by choosing $V^\tau(x) := T(x,0)$ gives rise to a one-dimensional Borchers triple, and similarly one may also construct a two-dimensional Borchers triple from a one-dimensional one by taking the simple definition $T(x,y) := V^\tau(x)$.

Lemma 5.9. [LS22]

i) The unitary $\mathbb{R}^2$-representation on $L^2(\mathbb{R}, \mathrm{d}\theta)$ given by

$$(T_1(\xi)\psi)(\theta) = e^{ip(\theta)\cdot\xi}\psi(\theta), \quad \xi \in \mathbb{R}^2, \quad p(\theta) := (\cosh(\theta), \sinh(\theta)) \tag{5.7}$$

has positive energy and satisfies

$$T_1(\xi)H \subset H, \quad \xi \in W_R.$$

ii) The second quantisation T of T_1, the von Neumann algebra $\mathcal{M}(H)$ and the Fock vacuum Ω form a two-dimensional Borchers triple.

The representation T and the modular operator $\Delta^{it}, t \in \mathbb{R}$ together form a representation of the Poincaré group on two dimensional Minkowski space, precisely the representation used in Chapter 2.

Within the context of a two-dimensional Borchers triple, we are now able to consider a more specific deformation procedure, in particular, that of a warped convolution which was first introduced in [BS08]. We concern ourselves with smooth operators which are defined as operators $A \in \mathcal{B}(\mathcal{H})$ such that the map $x \mapsto T(x)AT(-x)$ is smooth in the strong topology. Similarly, we say a vector $\psi \in \mathcal{H}$ is smooth if $x \mapsto T(x)\psi$ in norm.

We label by Q_κ a 2×2 matrix that is antisymmetric with respect to the Minkowski scalar product, conditions which fix the form to

$$Q_\kappa = \begin{pmatrix} 0 & \kappa \\ \kappa & 0 \end{pmatrix}, \quad \kappa \in \mathbb{R}$$

and then take elements of the von Neumann algebra

$$\mathcal{M}_\kappa = \{A_\kappa \ : \ A \in \mathcal{M} \ \text{smooth}\}''$$

to be the oscillatory integrals

$$A_\kappa \Psi := \frac{1}{(2\pi)^2} \int \mathrm{d}x\mathrm{d}y \; e^{-iy\cdot x} T(Q_\kappa y) A T(-Q_\kappa y) T(x) \Psi. \tag{5.8}$$

It is straightforward to immediately notice that $A_0 = A$: With $\kappa = 0$ the translations $T(Q_0 y) = T(-Q_0 y) = T(0) = 1_{\mathcal{H}}$ by definition of Q_κ and the remaining integral we calculate using (5.7).

$$A_0\Psi = \frac{1}{(2\pi)^2} \int \mathrm{d}x\mathrm{d}y \; e^{-iy\cdot x} A T(x) \Psi = \frac{1}{(2\pi)^2} \int \mathrm{d}x\mathrm{d}y \; e^{-iy\cdot x} A e^{ip(\theta)\cdot x} \Psi = A\Psi \frac{1}{(2\pi)^2} \int \mathrm{d}x\mathrm{d}y \; e^{i(p(\theta)-y)\cdot x}$$

which then equals $A\Psi$ after we note the remaining integral evaluates to 1 by applying

$$\int \mathrm{d}x \; e^{iyx} = 2\pi\delta(y)$$

where δ is the dirac delta function.

A second simple property is $A_\kappa \Omega = A\Omega$ for all $\kappa \in \mathbb{R}$. Noting the translational invariance of the vacuum Ω (5.8) reduces to

$$A_\kappa \Omega = \frac{1}{(2\pi)^2} \int \mathrm{d}x\mathrm{d}y \; e^{-iy\cdot x} T(Q_\kappa y) A\Omega = \frac{1}{(2\pi)^2} \int \mathrm{d}x\mathrm{d}y \; e^{-iy\cdot x} e^{ip(\theta)\cdot Q_\kappa y} A\Omega$$

and the explicit form of Q_κ and the Minkowski scalar product implies we may rewrite the second exponential as $e^{iy\cdot Q_\kappa p(\theta)}$ and the assertion follows with similar arguments to the previous property. For further properties, we refer the reader to [BLS10].

With this definition in hand, we now turn back to the context of Borchers triples and construct data in a more concrete way. In particular, for a triple $(\mathcal{M}, T, \Omega)$ we define the deformed von Neumann algebra

$$\mathcal{M}_\kappa := \{A_\kappa \ : \ A \in \mathcal{M} \ \text{is smooth}\}''. \tag{5.9}$$

Proposition 5.10. *Let $(\mathcal{M}, T, \Omega)$ be a Borchers triple, then Ω is standard (cyclic and separating) for $\mathcal{M}_\kappa$ for $\kappa \geq 0$.*

Proof. Since $A_\kappa \Omega = A\Omega$ and Ω is cyclic for $\mathcal{M}$, cyclicity for $\mathcal{M}_\kappa$ follows immediately. For the separating property, let $A \in \mathcal{M}$ and $A' \in \mathcal{M}'$, then for all $x \in W_R$ and $y \in W_L$ we have $[T(x)AT(x)^*, T(y)A'T(y)^*] = 0$. The positivity of the spectrum of T and the assumption that $\kappa \geq 0$ implies that $Q_\kappa \text{spectrum}(T) \subset W_R$ and thus by [BLS10, Prop 2.10] it follows that $[A_\kappa, A'_{-\kappa}] = 0$ from which we can conclude that $(\mathcal{M}')_{-\kappa} \subset (\mathcal{M}_\kappa)'$. Cyclicity of Ω for $\mathcal{M}'$ together with $A_\kappa \Omega = A\Omega$ implies that Ω is also cyclic for $(\mathcal{M}')_{-\kappa}$ and thus since $(\mathcal{M}')_{-\kappa} \subset (\mathcal{M}_\kappa)'$ it is also cyclic for the latter showing that Ω is separating for its commutant $\mathcal{M}_\kappa$. $\square$

Noticing that this deformation procedure does not affect the group T, it is clear from Proposition (5.9) that the triple $(\mathcal{M}_\kappa, T, \Omega)$ is a Borchers triple. Moreover, the deformed operators z_κ initially defined independently of the deformation procedure (5.8) is the result of deforming the operator z_0:

$$\begin{aligned}
(z_0(\varphi)_\kappa \Psi)_n(\boldsymbol{\theta}) &= \frac{\sqrt{n+1}}{(2\pi)^2} \int \mathrm{d}x\mathrm{d}x\mathrm{d}\theta' e^{-y\cdot x} e^{i\sum_{j=1}^n p(\theta_j)\cdot Q_\kappa y} \overline{\varphi(\theta')} e^{-i(p(\theta')\cdot Q_\kappa y + \sum_{j=1}^n p(\theta_j)\cdot Q_\kappa y)} e^{i\sum_{j=1}^n p(\theta_j)\cdot x} \Psi_n(\theta', \boldsymbol{\theta}) \\
&= \frac{1}{2\pi} \int \mathrm{d}x\mathrm{d}\theta' \sum_{j=1}^n \delta(p(\theta_j) - y) e^{-ip(\theta')\cdot Q_\kappa y} \overline{\varphi(\theta')} \Psi_n(\theta', \boldsymbol{\theta}) \\
&= \int \mathrm{d}\theta' \prod_{j=1}^n e^{-ip(\theta')\cdot Q_\kappa p(\theta_j)} \overline{\varphi(\theta')} \Psi_n(\theta', \boldsymbol{\theta}) \\
&= \int \mathrm{d}\theta' \prod_{j=1}^n e^{i\kappa \sinh(\theta' - \theta_j)} \overline{\varphi(\theta')} \Psi_n(\theta', \boldsymbol{\theta}) \\
&= (z_k(\varphi)\Psi)_n(\boldsymbol{\theta})
\end{aligned}$$

where we have used that

$$p(\theta') \cdot Q_\kappa p(\theta_j) = \kappa \sinh(\theta_j)\cosh(\theta') - \kappa \sinh(\theta')\cosh(\theta_j) = -\kappa \sinh(\theta' - \theta_j)$$

and setting $m = 1$ (the mass is of no importance to our current discussions).

In analogy to the one-dimensional setting, we can generate elements of the standard subspace H via test functions $\mathscr{S}(\mathbb{R}^2) \ni f \mapsto f^\pm \in \mathcal{H}$:

$$f^\pm(\theta) = \int_{\mathbb{R}^2} \mathrm{d}x \; f(x) e^{\pm ip(\theta)\cdot x}.$$

We then define the field operator ϕ_0 as

$$\phi_0(f) := z_0^\dagger(f^+) + z_0(\overline{f^-})$$

which is affiliated with the algebra $\mathcal{M}$ and moreover if interpreted as the free scalar field of unit mass on two-dimensional Minkowski space. The deformed version $\phi_\kappa(f)$ for $\kappa > 0$ is then affiliated with the algebra $\mathcal{M}_\kappa$ - the quantum field theory built from such data we then consider as being restricted to the chiral light ray, a context we examined for closely in Chapter 4.

5.3 Deformed and Singular Inclusions

We now look to proving the main result of this Chapter which we state here, and the proof will come in a number of parts.

Theorem 5.11. *Let* $(\mathcal{M}_\kappa, T, \Omega)$ *be the Borchers triple as defined in* (5.6) *for* $\kappa \geq 0$. *Then for* $\kappa = 0$, *the inclusion is standard (case 1), and for any* $\kappa > 0$ *the inclusion is singular (case 3).*

As a preliminary step, we consider some properties of the field ϕ_κ to facilitate further calculations (more can be found in [GL08]). We make use of the kernel theorem by taking vectors $\Psi(F)$ for $F \in \mathscr{S}((\mathbb{R}^2)^n)$ and taking their definitions to be the linear and continuous extensions of a product of free scalar fields acting on the vacuum, that is

$$\Psi(f_1 \otimes \cdots \otimes f_n) := \phi_0(f_1)\cdots\phi_0(f_n)\Omega \tag{5.10}$$

for $f_1, \ldots, f_n \in \mathscr{S}(\mathbb{R}^2)$.

On these vectors, we can more easily write down the explicit action of the deformed fields, and in particular keep track of all multiplicative factors that arise as in (5.5). Explicitly, we write the action of ϕ_κ as [Sol08, GL08]

$$\psi_\kappa(g)\Psi(F) = \Psi(g \otimes_\kappa F), \quad g \in \mathscr{S}(\mathbb{R}^2), F \in \mathscr{S}(\mathbb{R}^{2n}), \tag{5.11}$$

where the deformed tensor product $\otimes_\kappa$ is a variation of the Moyal tensor product [GL08] accounting for every necessary exponential factor between momenta. For $F \in \mathscr{S}(\mathbb{R}^{2n}), G \in \mathscr{S}(\mathbb{R}^{2m})$ we define it in momentum space as

$$(\widetilde{G \otimes_\kappa F})(p_1, \ldots, p_m; q_1, \ldots, q_n) = e^{i \sum_{j=1}^m p_l \cdot Q_\kappa \sum_{k=1}^n q_k} \tilde{G}(p_1, \ldots, p_m)\tilde{F}(q_1, \ldots, q_n). \tag{5.12}$$

It is clear from the above definition that the deformed tensor product $\otimes_\kappa$ is invariant under Poincaré group transformations - translations follow immediately, whereas boosts are clear once it is noted that a matrix boost transformation of parameter λ

$$\lambda \mapsto \begin{pmatrix} \cosh(\lambda) & \sinh(\lambda) \\ \sinh(\lambda) & \cosh(\lambda) \end{pmatrix}$$

commutes with any Q_κ. This in particular, together with the translational invariance of the vacuum state, implies that

$$\omega(F \otimes_\kappa G) = \omega(F \otimes G).$$

By Wick's theorem [Wic50], the vacuum state is described in terms of the n-point functions of the theory, and the reconstruction theorem illustrates how all information of a model is encoded in such functions. In particular, we have

$$\langle \Omega, \Psi(F) \rangle = \langle \Psi(F^*), \Omega \rangle = W_n(F), \quad F \mathscr{S}(\mathbb{R}^{2n}),$$

where $F^*(x_1, \ldots, x_n) = \overline{F(x_n, \ldots, x_1)}$ and the n-point functions $W_n \in \mathscr{S}(\mathbb{R}^{2n})$ can be written in terms of a product of two-point functions in momentum space:

$$\tilde{W}_n(p_1, \ldots, p_n) = \begin{cases} 0, & n \text{ odd}, \\ \sum_{\lambda,\mu} \prod_{k=1}^{n/2} \tilde{W}_2(p_{\lambda_k}, p_{\mu_k}), & n \text{ even} \end{cases} \tag{5.13}$$

where the two-point functions are given by

$$\tilde{W}_2(p, q) = \frac{1}{\varepsilon(p^1)} \delta(p^0 - \varepsilon(p^1))\, \delta(p + q), \qquad \varepsilon(p^1) = \sqrt{(p^1)^2 + 1} \tag{5.14}$$

for $p = (p_0, p_1), q = (q_0, q_1)$. The above sum $\sum_{(\lambda,\mu)}$ runs over all partitions (λ, μ) of $\{1, \ldots, n\}$ which is split into $n/2$ disjoint pairs (λ_k, μ_k) for $k = 1, \ldots, n/2$ and $\lambda_k < \mu_k$. These partitions are referred to here as "contractions" as a reference to the delta distributions present in (5.14) contracting momenta.

We now deal with these n-point functions in a specific weak limit.

Theorem 5.12. [LS22]

Let $\kappa \neq 0$, X be a polynomial in the field operators $\phi_\kappa(f)$ $(f \in \mathscr{S}(\mathbb{R}^2))$ and Y' a polynomial in the field operators $\phi_{-\kappa}(g)$ $(g \in \mathscr{S}(\mathbb{R}^2))$. Then, for any vectors Ψ, Ψ' of finite particle number

$$\lim_{t \to \pm\infty} \langle \Psi', \sigma_t(XY')\Psi \rangle = \langle \Psi', (\omega(XY')P_\Omega + \omega(X)\omega(Y')P_\Omega^\perp)\, \Psi \rangle. \tag{5.15}$$

Proof. The equation (5.15) is linear in both X and Y' on both sides, and hence it is sufficient to consider monomials in the field, that is we may choose X and Y' to be of the form

$$X = \phi_\kappa(f_1) \cdots \phi_\kappa(f_n), \quad Y' = \phi_{-\kappa}(g_1) \cdots \phi_{-\kappa}(g_m),$$

for any $n, m \in \mathbb{N}_0$ and functions $f_1, \dots, f_n, g_1, \dots, g_m \in C_0^\infty(\mathbb{R})$. Via the same arguments as Lemma (3.10) one can show that vectors of the form (5.10) form a dense subspace of the n-particle space $\mathcal{H}^{\otimes n}$, and hence it is sufficient to choose

$$\Psi' = \Psi(l^*), \quad \Psi = \Psi(r)$$

for $l \in \mathscr{S}(\mathbb{R}^{2a}), r \in \mathscr{S}(\mathbb{R}^{2b})$ and arbitrary $a, b \in \mathbb{N}_0$.

To expand out and simplify the left hand side of (5.12) we first introduce the following compact notation

$$f^{\kappa,t} := \Lambda_t^* f_1 \otimes_\kappa \cdots \otimes_\kappa \Lambda_t^* f_n, \quad g^{-\kappa,t} := \Lambda_t^* g_1 \otimes_{-\kappa} \cdots \otimes_{-\kappa} \Lambda_t^* g_m$$

which with (5.11) allows us to calculate

$$\begin{aligned}\langle \Psi', \sigma_t(XY')\Psi\rangle &= \langle \Psi(l^*), \phi_\kappa(\Lambda_t^* f_1)\cdots\phi_\kappa(\Lambda_t^* f_n)\phi_{-\kappa}(\Lambda_t^* g_1)\cdots\phi_{-\kappa}(\Lambda_t^* g_m)\Psi(r)\rangle \\ &= W_{n+m+a+b}(l \otimes (f^{\kappa,t} \otimes_\kappa (g^{-\kappa,t} \otimes_{-\kappa} r))) \\ &= W_{n+m+a+b}((l \otimes_\kappa f^{\kappa,t}) \otimes (g^{-\kappa,t} \otimes_{-\kappa} r))).\end{aligned} \tag{5.16}$$

By the definition of W in (5.13), we can immediately see that this vanishes when $N := n+m+a+b$ is odd, and the right hand side may also vanish for some choices of n, m, a, b: for example, for $m = 0$, then the right hand side reduces to

$$\langle \Psi', \Omega\rangle\langle \Omega, \Psi\rangle\omega(X)$$

which is non-zero if all of n, a, b are even which is not the case if N is odd.

To more easily keep track of momenta and their affiliated functions in further calculations, we use the shorthand notation for their sums:

$$p(l) = \sum_{k=1}^{a} p_k, \quad p(f) = \sum_{k=a+1}^{a+n} p_k, \quad p(g) = \sum_{k=a+n+1}^{a+n+m} p_k, \quad p(r) = \sum_{k=a+n+m+1}^{N} p_k. \tag{5.17}$$

Implementing now (5.13) and its explicit form in terms of a sum over contractions in momentum space, we reformulate (5.16) for the case of even N

$$\langle \Psi, \sigma_t(XY')\Psi'\rangle = \sum_{(\lambda,\mu)} W_{(\lambda,\mu)}(t)$$

where

$$\begin{aligned}W_{(\lambda,\mu)}(t) := \int_{\mathbb{R}^{2N}} \mathrm{d}p\ \tilde{l}(p_1, \dots, p_a)\widetilde{f^\kappa}(\Lambda_t p_{a+1}, \dots, \Lambda_t p_{a+n})\widetilde{g^{-\kappa}}(\Lambda_t p_{a+n+1}, \dots, \Lambda_t p_{a+n+m}) \\ \times \tilde{r}(p_{a+n+m+1}, \dots, p_N)\ e^{ip(l)\cdot Q_\kappa p(f)} e^{-ip(q)\cdot Q_\kappa p(r)} \prod_{k=1}^{N/2} \tilde{W}_2(-p_{\lambda_k}, -p_{\mu_k}).\end{aligned} \tag{5.18}$$

We can immediately see that the integrand above vanishes pointwise in the limit $t \to \pm\infty$ due to the compact support properties of f and g together with the non-divergence of the remaining factors. The dependence on t in the integrals differs between partitions (λ, μ), and in order to apply a dominated convergence argument we must analyse these individually and so we introduce specific index sets

$$\begin{aligned}
\mathcal{I}(l) &:= \{1, \dots, a\}, \\
\mathcal{I}(f) &:= \{a+1, \dots, a+n\}, \\
\mathcal{I}(g) &:= \{a+n+1, \dots, a+n+m\}, \\
\mathcal{I}(r) &:= \{a+n+m+1, \dots, N\}
\end{aligned}$$

which correspond to the indices on the momenta of $\tilde{l}, \widetilde{f^\kappa}, \widetilde{g^{-\kappa}}$ and $\tilde{r}$, respectively. We now list our criterion for four types of contractions distinguished using the above index sets.

(I) A contraction (λ, μ) is of type (I) if there exists a $k \in \{1, \dots, N/2\}$ such that either of the indices λ_k, μ_k lie in the union $\mathcal{I}(l) \cup \mathcal{I}(r)$, but not both. In this case, a variable of either $\widetilde{f^\kappa}, \widetilde{g^{-\kappa}}$ is contracted with a variable of either $\tilde{l}, \tilde{r}$.

(II) A contraction (λ, μ) is of type (II) if it is not of type (I) and for all $k \in \{1, \dots, N/2\}$ the indices λ_k, μ_k are both elements of either $\mathcal{I}(f)$ or $\mathcal{I}(g)$. In this case we call the contraction (λ_k, μ_k) "f-internal" or "g-internal", respectively.

(III) A contraction (λ, μ) is of type (III) if it is not of type (I) or (II) and if there exists a $k \in \{1, \dots, N/2\}$ such that $\lambda_k \in \mathcal{I}(l)$ and $\mu_k \in \mathcal{I}(r)$. In this case, the contraction (λ, μ) contracts between $\widetilde{f^\kappa}$ and $\widetilde{g^{-\kappa}}$, and also between $\tilde{l}, \tilde{r}$, but there are no contractions between deformed functions $\widetilde{f^\kappa}, \widetilde{g^{-\kappa}}$ and undeformed functions $\tilde{l}, \tilde{r}$ as in type (I).

(IV) A contraction (λ, μ) is of type (IV) if it is not of type (I), (II) or (III). In this remaining case, (λ, μ) contracts variables between $\widetilde{f^\kappa}$ and $\widetilde{g^{-\kappa}}$, but all variables of $\tilde{l}$ are contracted with each other, and similarly for $\tilde{r}$.

These four cases exhaust all possibilities for contractions.

(I): For this case we wish to show that

$$\lim_{t \to \pm\infty} W_{(\lambda,\mu)}(t) = 0, \quad (\lambda, \mu) \text{ of type (I).} \tag{5.19}$$

The expression for the two-point functions in momentum space contain a delta distribution, namely $\tilde{W}_2(p,q) = \varepsilon(p^1)^{-1}\delta(p^0 - \varepsilon(p^1))\delta(p+q)$ - the second delta is responsible for the case-specific contraction which set $p_{\lambda_k} = -p_{\mu_k}$, while the first restricts p_{μ_k} to the upper-mass shell. This latter transformation means that the momentum $p_{\mu_k} = (\varepsilon(p^1_{\mu_k}, p^1_{\mu_k})$ from which we may make the substitution $\sinh(\theta_k) := p^1_{\mu_k}, \frac{\mathrm{d}p^1_{\mu_k}}{\mathrm{d}\theta_k}$ into rapidity space. Here the boosts Λ_t act as translations in θ_k and hence by applying the triangle inequality we may write

$$|W_{(\lambda,\mu)}(t)| \leq \int_{\mathbb{R}^N} \mathrm{d}\theta_{\lambda_k}\mathrm{d}\theta_{\mu_k}\ F(\boldsymbol{\theta}, \boldsymbol{\theta}' - t, \boldsymbol{\theta}'' - t, \boldsymbol{\theta}''') \prod_{k=1}^{N/2} \delta(\theta_{\lambda_k} - \theta_{\mu_k}) \tag{5.20}$$

where $F \in \mathscr{S}(\mathbb{R}^N)$ and the rapidities $\boldsymbol{\theta} = (\theta_1, \ldots \theta_a)$ (and similarly for $\boldsymbol{\theta}', \boldsymbol{\theta}'', \boldsymbol{\theta}'''$) are the transformed variables of $\tilde{l}$ (and $\widetilde{f^\kappa}, \widetilde{g^{-\kappa}}, \tilde{r}$), respectively) and also the notation $\boldsymbol{\theta}' - t := (\theta'_1, \ldots, \theta'_n - t)$ has been introduced.

With F being a Schwartz function, it a priori comes equipped with a bound formed by seminorms allowing us to write

$$|W_{(\lambda,\mu)}(t)| \leq C \int_{\mathbb{R}^{N/2}} d\boldsymbol{\theta}\ \prod_{\alpha}(1+\theta_a^2)^{-1}(1+(\theta_\alpha - t^2)^{-1}\prod_{\beta}(1+\theta_\beta^2)^{-2}$$

where the first product is a result of the contraction (λ_k, μ_k) of type (I), linking a t-dependent variable with a t-independent one. The second product is the remaining contributions from the contractions (λ_k, μ_k) which link two variables that are both either t-dependent or t-independent (for the former, a change of variables can leave the resulting term completely t-independent). Clearly, this is sufficient to employ Lebesgue's dominated convergence and (5.19) holds.

(II): For this case we show that $W_{(\lambda,\mu)}(t)$ is independent of t. By definition, the variables of $\widetilde{f^\kappa}$ are contracted amongst themselves and similarly for $\widetilde{g^{-\kappa}}$, whereas the variables of $\tilde{l}, \tilde{r}$ are contracted in any combination and do not matter to this argument.

The delta distribution $\delta(p+q)$ in the formula for the two-point functions $\tilde{W}_2$ imply that two variables, say p_{λ_k}, p_{μ_k}, of $\widetilde{f^\kappa}$ are then related by $p_{\lambda_k} = -p_{\mu_k}$ (and similarly for $\widetilde{g^{-\kappa}}$). Since only variables of $\widetilde{f^\kappa}$ are contracted with other variables of $\widetilde{f^\kappa}$, we conclude that the sums of momenta $p(f), p(g)$ vanish and the exponential factors in the integrand of (5.18) drop out.

Given the description of contractions in type (II), the product of two point functions $\prod_{k=1}^{N/2} \tilde{W}_2(p_{\lambda_k}, p_{\mu_k})$ can be split between a distinct product of contractions for the three functions $\tilde{l}\cdot\tilde{r}, \widetilde{f^\kappa}$ and $\widetilde{g^{-\kappa}}$ giving

$$W_{(\lambda,\mu)}(t) = \int dp_{\lambda_k} dp_{\mu_k}\ \tilde{l}(p_1,\dots,p_a)\tilde{r}(p_{a+n+m+1},\dots,p_N) \prod_{k,\{\lambda_k,\mu_k\}\in\mathcal{I}(l)\cup\mathcal{I}(r)} \tilde{W}_2(-p_{\lambda_k},-p_{\mu_k})$$
$$\times \int \mathrm{d}p_{\lambda_k} dp_{\mu_k}\ \widetilde{f^\kappa}(p_{a+1},\dots,p_{a+n}) \prod_{k,\{\lambda_k,\mu_k\}\subset\mathcal{I}(f)} \tilde{W}_2(-p_{\lambda_k},-p_{\mu_k})$$
$$\times \int \mathrm{d}p_{\lambda_k} dp_{\mu_k}\ \widetilde{g^\kappa}(p_{a+n+1},\dots,p_{a+n+m}) \prod_{k,\{\lambda_k,\mu_k\}\subset\mathcal{I}(g)} \tilde{W}_2(-p_{\lambda_k},-p_{\mu_k})$$

The above expression indicates that $W_{(\lambda,\mu)}(t)$ for the case of type (II) contractions reduces to the product of three sums - the first of these sums is of the contractions over $\mathcal{I}(l)\cup\mathcal{I}(r)$. By Wick's theorem [Wic50] this coincides with $\omega(\phi_0(l)\phi_0(r)) = \langle\Psi(l^*),\Psi(r)\rangle$. Similarly, the second sum is of the contractions over $\mathcal{I}(f)$ which coincide with $\omega(\phi_\kappa(f_1)\cdots\phi_\kappa(f_n)) = \omega(X)$, and also analogously for the final sum over $\mathcal{I}(g)$ which coincides with $\omega(Y')$.

Thus we arrive at the compact expression

$$\sum_{(\lambda,\mu)\ \text{type (II)}} W_{(\lambda,\mu)}(t) = \omega(X)\omega(Y')\langle\Psi',\Psi\rangle \tag{5.21}$$

(III): As in case (III) we claim that

$$\lim_{t\to\pm\infty} W_{(\lambda,\mu)}(t) = 0, \quad (\lambda,\mu) \text{ of type (III)}. \tag{5.22}$$

Similarly to case (II) we may remove the t-dependence of $\widetilde{f^\kappa}$ and $\widetilde{g^{-\kappa}}$ by making the substitution $\Lambda_t^* p_k \mapsto p_k$ - as a result the exponential factors gain t-dependence. Moreover, since the variables $p_k\in\mathcal{I}(f)\cup\mathcal{I}(g)$ are contracted amongst each other, the support of the delta distributions imply that we have $p(f)+p(g)=0$, and also $p(l)+p(r)=0$. With the antisymmetry of Q_κ, we calculate the resulting exponential factors:

$$e^{ip(l)\cdot Q_\kappa p(f)}e^{-ip(q)\cdot Q_\kappa p(r)} = e^{ip(l)\cdot Q_\kappa\Lambda_{-t}p(f)}e^{-i\Lambda_{-t}p(g)\cdot Q_\kappa p(r)} = e^{2ip(l)\cdot Q_\kappa\Lambda_{-t}p(f)}.$$

As in the argument for case (I) we make the appropriate substitutions to the rapidity formulation obtaining

$$W_{(\lambda,\mu)}(t) = \int \mathrm{d}\theta\mathrm{d}\theta' L\left(\theta_1,\dots,\theta_{\frac{a+b}{2}}\right) F\left(\theta_1',\dots,\theta'_{\frac{n+m}{2}}\right)\prod_{j,k} e^{2i\kappa\sinh(\theta_j-\theta_k'+t)}. \tag{5.23}$$

The functions $L\in\mathscr{S}(\mathbb{R}^{\frac{a+b}{2}}), F\in\mathscr{S}(\mathbb{R}^{\frac{n+m}{2}})$ are concatenated versions of the functions $\tilde{l}\otimes\tilde{r}$ and $\widetilde{f^\kappa},\widetilde{g^{-\kappa}}$, respectively, in the rapidity parameterisation and accounting for the support of the delta distributions in the two-point functions.

The remaining exponential factor arises from noting that any f-internal contractions (λ_k, μ_k) mean that the $p_{\lambda_k} + p_{\mu_k}$ contributions in the sum $p(f)$ vanish, and similarly for $p(l)$. For the remaining contractions that are specifically *not* f- or l-internal, we relabel their sums $p(f)', p(l)'$.

The previous calculation of $p(\theta) \cdot Q_\kappa \Lambda_{-t} p(\theta') = \kappa \sinh(\theta - \theta' - t)$ the product sums over j, k for the remaining contractions.

Thus far, the arguments are the same for both cases (III) and (IV), but we now distinguish the two by noting that for case (III) the product over j is non-empty and we claim that

$$\lim_{t \to \pm\infty} W_{(\lambda,\mu)}(t) = 0, \quad (\lambda, \mu) \text{ of type (III)}. \tag{5.24}$$

Fixing a j in the product in (5.23) we rewrite the integral, apply integration by parts and estimate via the triangle inequality

$$\begin{aligned} \left|W_{(\lambda,\mu)}(t)\right| &= \left| \int \mathrm{d}\theta \mathrm{d}\theta' \; \frac{L\left(\theta_1, \ldots, \theta_{\frac{a+b}{2}}\right) F\left(\theta'_1, \ldots, \theta'_{\frac{n+m}{2}}\right)}{\sum_k 2i\kappa \cosh(\theta_j - \theta'_k + t)} \frac{\partial}{\partial \theta_j} \prod_{j,k} e^{2i\kappa \sinh(\theta_j - \theta'_k + t)} \right| \\ &\leq \int \mathrm{d}\theta \mathrm{d}\theta' \; \left| F(\theta'_1, \ldots, \theta'_{\frac{n+m}{2}}) \frac{\partial}{\partial \theta_j} \frac{L\left(\theta_1, \ldots, \theta_{\frac{a+b}{2}}\right)}{\sum_k 2i\kappa \cosh(\theta_j - \theta'_k + t)} \right|. \end{aligned}$$

Applying the product rule for the derivative, and taking the estimate

$$\left| \frac{\partial}{\partial \theta_j} \frac{L\left(\theta_1, \ldots, \theta_{\frac{a+b}{2}}\right)}{\sum_k 2i\kappa \cosh(\theta_j - \theta'_k + t)} \right| \leq \frac{1}{\cosh(\theta_j - \theta_k + t)}$$

it is clear that the integrand vanishes pointwise for $t \to \pm\infty$. Furthermore, with the lower bound $\cosh(x) \geq 1$ the assumptions for dominated convergence are satisfied and we have the limit as claimed.

(IV): Following the arguments for case (III) to (5.23), we remark that the product over j, k drops out considering there are no contractions between $\tilde{l}$ and $\tilde{r}$ in this case. The remaining integrand is t-independent and as in case (II) we split the result into three sums of distinct contractions: The sum of contractions over $\tilde{l}$ clearly results in

$$\omega(\phi_0(l_1) \cdots \phi_0(l_a)) = \langle \Psi(l^*), \Omega \rangle = \langle \Psi', \Omega \rangle.$$

In analogy, the sum of contractions over $\tilde{r}$ gives $\langle \Omega, \Psi(r) \rangle = \langle \Psi, \Omega \rangle$.

The final sum is over contractions between $\widetilde{f^\kappa}$ and $\widetilde{g^{-\kappa}}$, but in particular not those that are f- or g-internal. Thus, by accounting for the missing contractions we have

$$\begin{aligned}\omega(\phi_\kappa(f_1)\cdots\phi_\kappa(f_n)\phi_{-\kappa}(g_1)\cdots\phi_{-\kappa}(g_m)) - \omega(\phi_\kappa(f_1)\cdots\phi_\kappa(f_n))\cdot\omega(\phi_{-\kappa}(g_1)\cdots\phi_{-\kappa}(g_m))\\ = \omega(XY') - \omega(X)\omega(Y'),\end{aligned}$$

then summing over all type (IV) contractions

$$\sum_{(\lambda,\mu)\ \text{type (IV)}} W_{(\lambda,\mu)}(t) = (\omega(XY') - \omega(X)\omega(Y'))\langle\Psi,\Omega\rangle\langle\Omega,\Psi'\rangle.$$

With each case accounted for, we sum over them all to arrive at the assertion

$$\begin{aligned}\lim_{t\to\pm\infty}\langle\Psi',\sigma_t(XY')\Psi\rangle &= \langle\Psi',((\omega(XY') - \omega(X)\omega(Y))P_\Omega + \omega(X)\omega(Y')1_{\mathcal{H}})\Psi\rangle\\ &= \langle\Psi',(\omega(XY')P_\Omega + \omega(X)\omega(Y')P_\Omega^\perp)\Psi\rangle.\end{aligned}$$

□

We remark that though we employed functions f and g as arguments for the scaled field operators, their support properties were irrelevant and not mentioned - in the context of scaling limits of integrable models in Chapter 3, the derivation of the obstruction operator paid particular heed to the support of the function arguments involved. Furthermore, the previous construct employed the one-sided limit $t \to -\infty$ which corresponds to scaling points $x \in \mathbb{R}\backslash\{0\}$ to ∞ and the support of the function arguments in the same direction. The opposite limit $t \to \infty$, also calculated in the above result taking points x to 0, is not one that has a geometric sense but for the abstract setting provides additional information.

Next, we extend this result to arbitrary elements of the von Neumann algebras $\mathcal{M}_\kappa, \mathcal{M}_\kappa'$.

Theorem 5.13. *Let the Borchers triple* $\mathcal{M}_\kappa, T, \Omega)$ $(\kappa > 0)$ *be defined as in* (5.6). *Then for any* $A \in \mathcal{M}_\kappa, B \in \mathcal{M}_{-\kappa}'$

$$\operatorname*{w-lim}_{t\to\pm\infty} \sigma_t(AB) = \omega(AB)P_\Omega + \omega(A)\omega(B)P_\Omega^\perp. \tag{5.25}$$

Proof. We will write $\Psi_A = A\Omega$ and $P_{AB} = \omega(AB)P_\Omega + \omega(A)\omega(B)P_\Omega^\perp$ as shorthand notation, and similarly for other operators. We consider vectors of the form $\Psi' = \Psi_{L^*}, \Psi = \Psi_R$ where L and R are closed operators affiliated to the left and right algebras $\mathcal{M}_\kappa'$ and $\mathcal{M}_\kappa$, respectively. The domains $\mathcal{D}(L), \mathcal{D}(R)$ of R and L, respectively, are such that $\Omega \in \mathcal{D}(L^*) \cap \mathcal{D}(R)$. The algebras $\mathcal{M}_\kappa$ and $\mathcal{M}_\kappa'$ are stable under the action of the modular group, meaning that $\sigma_t(A^*)\Omega \in \mathcal{D}(L^*), \sigma(B) \in \mathcal{D}(R)$ and

$$L^*\sigma_t(A^*)\Omega = \sigma_t(A^*)L^*\Omega, \quad R\sigma_t(B)\Omega = \sigma_t(B)R\Omega.$$

Consequently, we may simplify the scalar product appearing in the limit on the left hand side of (5.25)

$$\begin{aligned}\langle\Psi_{L^*},\sigma_t(AB)\Psi_R\rangle &= \langle\Psi_{L^*},\sigma_t(A)\sigma_t(B)\Psi_R\rangle\\ &= \langle\sigma_t(A^*)\Psi_{L^*},\sigma_t(B)\Psi_R\rangle\\ &= \langle L^*\sigma_t(A^*)\Omega, R\sigma_t(B)\Omega\rangle\\ &= \langle\sigma_{-t}(L^*)\Psi_{A^*},\sigma_{-t}(R)\Psi_B\rangle.\end{aligned}$$

Comparing the second and fourth lines in the above we see a symmetry between A and L, and also between B and R, one which can be replicated for the right hand side of (5.25) in scalar product

$$\begin{aligned}\langle\Psi_{L^*},P_{AB}\Psi_R\rangle &= \omega(AB)\omega(L)\omega(R)+\omega(A)\omega(B)\omega(LR)-\omega(A)\omega(B)\omega(L)\omega(R)\\ &= \langle\Psi_{A^*},P_{LR}\Psi_B\rangle.\end{aligned}$$

Then

$$\langle\sigma_t(A^*)\Psi_{L^*},\sigma_t(B)\Psi_R\rangle-\langle\Psi_{L^*},P_{AB}\Psi_R\rangle = \langle\sigma_{-t}(L^*)\Psi_{A^*},\sigma_{-t}(R)\Psi_B\rangle-\langle\Psi_{A^*},P_{LR}\Psi_B\rangle. \quad (5.26)$$

The implied symmetry in the above formula suggests that if the limit (5.25) holds for $t\to-\infty$ then it also holds for $t\to\infty$ and vice versa.

We will first show that the right hand side of (5.26) vanishes as $t\to\pm\infty$ for specific choices of L,R,A and B. In particular, we take L and R to be arbitrary polynomials of field operators ϕ_κ as in Theorem (5.12), each taking functions as arguments (smearing) with supports on the left and right, hence the operators L and R are affiliated with the algebras $\mathcal{M}'_\kappa$ and $\mathcal{M}_\kappa$ respectively. We also choose A,B to be bounded and smooth, hence $A\in\mathcal{M}^\infty_\kappa, B\in\mathcal{M}'^\infty_\kappa$, which will generalise Theorem (5.12) to vectors Ψ,Ψ' that are not necessarily of finite particle number but instead $\Psi'=\Psi_{A^*},\Psi=\Psi_B$. Moreover, denote by $Q_n := P_1\oplus P_2\oplus\cdots\oplus P_n$ the orthogonal projection onto the particle space of at most n in the Fock space $\mathcal{F}(\mathcal{H})$. Given the chosen form of L, the complement $Q_n^\perp$ leaves the domain of L^* invariant - this is the result of a field operator only changes the particle number by one, and hence a finite polynomial of such operators alters the particle number by a finite amount. Hence there exists an $m\in\mathbb{N}$ such that for all $n\in\mathbb{N},t\in\mathbb{R}$

$$\begin{aligned}&|\langle\sigma_t(L^*)\Psi_{A^*},\sigma_t(R)\Psi_B\rangle-\langle\Psi_{A^*},P_{LR}\Psi_B\rangle|\\ &\leq|\langle Q_n\Psi_{A^*},\sigma_t(LR-P_{LR})\Psi_B\rangle|+|\langle Q_n^\perp\Psi_{A^*},(\sigma_t(LR-P_{LR})\Psi_B\rangle|\\ &=|\langle Q_n\Psi_{A^*},\sigma_t(LR-P_{LR})Q_{n+m}\Psi_B\rangle|+|\langle Q_n^\perp\Psi_{A^*},\sigma_t(LR-P_{LR})Q_{n-m}^\perp\Psi_B\rangle|\\ &\leq|\langle Q_n\Psi_{A^*},\sigma_t(LR-P_{LR})Q_{n+m}\Psi_B\rangle|+\|\sigma_t(L^*)Q_n^\perp\Psi_{A^*}\|\|\sigma_t(R)Q_{n-m}^\perp\Psi_B\|+\|Q_n^\perp\Psi_{A^*}\|\|P_{LR}\|\|Q_{n-m}^\perp\Psi_B\|.\end{aligned}$$

With the presence of the projection Q_n, Q_{n+m} in the first term, the vectors are of finite particle number and thus this term vanishes in the limit by Theorem (5.12). Clearly the complements $Q_n^\perp, Q_{n-m}^\perp \to 0$ strongly in the limit $n \to \infty$ and as an immediate consequence the third term above vanishes in this limit.

All that remains is to show that the two norms $\|\sigma_t(L^*)Q_n^\perp \Psi_{A^*}\|, \|\sigma_t(R)Q_{n-m}^\perp \Psi_B\|$ vanish uniformly in $t \in \mathbb{R}$ for $n \to \infty$. Both A, B are smooth, and hence the vectors Ψ_{A^*}, Ψ_B are elements of $\bigcap_{k\geq 0} \mathcal{D}(P_0^k)$ (the intersection over all k of the domain of the k-th power of the generator of the time translations P_0 - the second quantisation of $\frac{1}{2}(P + P^{-1}) \geq 1$). The particle number operator N is the second quantisation of the identity operator and it then follows that $P_0 \geq N$ which in turn implies $\Psi_{A^*}, \Psi_B \in \bigcap_{k\geq 0} \mathcal{D}(N^k)$.

For each field operators $\phi_{\pm\kappa}(f), f \in \mathscr{S}(\mathbb{R}^2)$ we have that the norm $\|\phi_{\pm\kappa}(f)N^{-1/2}\|$ is finite [Lec03, Lec12] and so there exists a $k \in \mathbb{N}$ such that $L^* N^{-k}$ and RN^{-k} are bounded. In addition, it is clear that $Q_n^\perp$ and Δ^{it} commute with N for all $n \in \mathbb{N}, t \in \mathbb{R}$ (given that neither affects the particle number), choosing a large enough k we may make the estimate

$$\|\sigma_t(L^*)Q_n^\perp \Psi_{A^*}\| = \|L^* N^{-k} \Delta^{-it} Q_n^\perp N^k \Psi_{A^*}\| \leq \|L^* N^{-k}\| \|Q_n^\perp N^k \Psi_{A^*}\|$$

which vanishes as $n \to \infty$ uniformly in t as required. The second norm $\|\sigma_t(R)Q_{n-m}^\perp \Psi_B\|$ can be estimated in an analogous way.

Noting that Ψ_{A^*}, Ψ_B range over dense subspaces of $\mathcal{F}(\mathcal{H})$ we may apply similar Reeh-Schlieder arguments to those we have applied in previous results, together with the fact that $\sigma_t(AB - P_{AB})$ is uniformly bounded in norm for all $t \in \mathbb{R}$ to conclude the weak limit $\sigma_t(AB) \to P_{AB}$ for smooth A and B.

To generalise further we again consider smooth A, B but now an $L \in \mathcal{M}_\kappa', R \in \mathcal{M}_\kappa$ which are bounded but not necessarily smooth. We know that the left hand side of (5.26) goes to zero in the limit $t \to \pm\infty$ and thus the desired limit $\langle \Psi_{A^*}, \sigma_{-t}(LR - P_{LR})\Psi_B \rangle \to 0$ holds too. Now, the smooth algebras $\mathcal{M}_\kappa'^\infty \subset \mathcal{M}_\kappa', \mathcal{M}_\kappa^\infty \subset \mathcal{M}_\kappa$ are strongly dense, they have Ω as a cyclic vector and we conclude that the limit also holds for arbitrary vectors in the left and right hand sides of the scalar product. □

The singularity of the inclusion $\mathcal{N}_\kappa' \cap \mathcal{M}_\kappa$ then follows from the following Lemma.

Lemma 5.14. [LS22]

Let $\mathcal{H}$ be a Hilbert space of dimensional $\dim(\mathcal{H}) > 1$ *and $\mathcal{N} \subset \mathcal{M}$ a half-sided modular inclusion on $\mathcal{H}$. Then $\omega(\cdot) = \langle \Omega, \cdot \Omega \rangle$ is not a product state on $\mathcal{N} \vee J\mathcal{N}J$. That is, $\omega(AB) \neq$*

$\omega(A) \cdot \omega(B)$ *for* $A, B \in \mathcal{N} \vee J\mathcal{N}J$.

Proof. Assume that ω is a product state and let $A \in \mathcal{N}, B \in J\mathcal{N}J$. Then $T(x)AT(x)^{-1} \in \mathcal{N}$ for all $x > 0$ also. Thus

$$\langle A^*\Omega, T(-x)B\Omega \rangle = \omega(T(x)AT(-x)B) = \omega(A)\omega(B) = \langle A^*\Omega, P_\Omega B\Omega \rangle$$

for $x > 0$. Since Ω is cyclic, this implies that $T(-x) = P_\Omega$ which is only possible if $P_\Omega = 1$, i.e. $\mathcal{F}(\mathcal{H}) = \mathbb{C}\Omega$ is one dimensional which is a contradiction. □

Finally, we return to the proof of our main result.

Proof of Theorem (5.11). The case of $\kappa = 0$ follows immediately from the second quantisation of $\mathcal{M} = \mathcal{M}(H)$. For $\kappa > 0$, let $A \in \mathcal{N}_\kappa, B \in J\mathcal{N}_\kappa J$, then by Theorem (5.13) and Lemma (5.6) part (i)

$$P_{AB} = (\omega(AB) - \omega(A)\omega(B))P_\Omega + \omega(A)\omega(B)1_\mathcal{H} \in \mathcal{A}_{\kappa,\infty}.$$

We have freedom (by virtue of Lemma (5.14)) to choose A, B such that $\omega(AB) - \omega(A)\omega(B) \neq 0$, and then

$$S = \frac{AB - \omega(A)\omega(B)1_\mathcal{H}}{\omega(AB) - \omega(A)\omega(B)} \in \mathcal{N}_\kappa \vee J\mathcal{N}_\kappa J$$

is such that $\sigma_t(S) \to P_\Omega$ weakly as $t \to -\infty$. This weak limit lies in $\mathcal{A}_{\kappa,\infty}$ by Lemma (5.6) which implies that the inclusion $\mathcal{N}_\kappa \subset \mathcal{M}_\kappa$ by Proposition (5.5). □

Chapter 6

Conclusion

In the present work, we have covered a number of topics in the area of quantum field theory, and in particular, paying close attention to our understanding of the structure of models with particular starting data. Having already a firm grasp on general models and how they are realised as a representation of abstract ZF algebras, we have formulated and illustrated a method whereby (for favourable, simple conditions) we may write the underlying data of a model in a much simpler fashion.

Having already a great many resources for the description of the Bose/Fermi models in the existing literature, our methodology presents a natural process to realise potentially more complicated systems as a tensor product of these more well-understood ones. Though the conditions under which this is possible are fairly limited in the general case, it seems it is most relevant on the subject of short distance scaling limits of integrable models.

As in the scalar case [BLM11], the scaling limit results in the unscaled fields splitting into a tensor product of chiral fields on the real line - the interplay of these one-dimensional models is governed by the limit at the infinities of the S-matrix governing the unscaled model. On the one hand, this limit may fall under the umbrella of constant S-matrix examples where our previous discussions become relevant, in which case we may pass through our discussed processes to read the scaling limit in a more transparent way. On the other hand, these chiral models are still able to be described and examples of half-line local fields constructed within them.

As in the massive case, we realise that not much can be said about the size of the resulting local algebras in the general case, however, we can at least provide a sufficient condition for operators to lie in these algebras by way of a commutation condition with an obstruction operator. Due to the involved nature of this operator, it is difficult to draw more conclusions from this data alone, though it goes some way to highlight the vastly more complicated nature of this setting in comparison to the scalar case.

In the quantum field theoretic setting, the most favourable result is to have a very large local algebra containing many physically relevant observables. The opposite extreme of a trivial algebra is less desired and actually pathological. In the context of half-sided modular inclusions and von Neumann algebras however it is a most prevalent question of the existence of such objects. Though only a single example came through recently in [LTU19], we have shown here many more examples as in [LS22] arising via our motivation from one-dimensional chiral models. By applying a deformation procedure on models constructed from two-dimensional Borchers triples it was shown that the result actually produces a singular inclusion in a specific limit. It is also clear that half-sided modular inclusions can vary in a discontinuous manner with deformations, perhaps motivating further discussions with deformation procedures. Though this result does not provide constructive examples of quantum field theories, it might inform us of methods that exclude the singular case.

The investigations we have undertaken may be viewed in a disjoint manner with each Chapter providing further insights into areas of quantum field theory with problems in their own right, or with each moving forward in the understanding of scaling limits of quantum fields and their resulting models from varying standpoints and contexts. A full comprehension of such an idea would be most enlightening for answers to questions surrounding QFT in general, such as asymptotic freedom of models, but the results outlined here provide headway in this direction.

Appendix A

Minkowski Space Geometry

We work in 1+1 dimensional Minkowski space as an analogue of the real plane $\mathbb{R}^2$ endowed with the Minkowski inner product with signature $(+,-)$ meaning the scalar product of coordinates $x = (x_0, x_1), y = (y_0, y_1)$ is given by $x \cdot y = x_0 y_0 - x_1 y_1$.

The plane may be separated into disjoint regions described by the inner product, in particular, a point $x \in \mathbb{R}^2$ is called

- **timelike** if $x \cdot x > 0$
- **lightlike** if $x \cdot x = 0$
- **spacelike** if $x \cdot x < 0$.

This inner product is invariant under transformations of the Poincaré group $\mathcal{P}$ generated by translations τ_a acting as $x \mapsto x + a$ for $a \in \mathbb{R}^2$ and the boost transformations

$$\Lambda(\lambda) : x \mapsto \begin{pmatrix} \cosh(\lambda) & \sinh(\lambda) \\ \sinh(\lambda) & \cosh(\lambda) \end{pmatrix} x, \quad \lambda \in \mathbb{R},$$

and the reflections $j : x \mapsto (-x_0, x_1)$ and $-j : x \mapsto (x_0, -x_1)$.

The Poincaré group may be split into a number of subgroups, the first of mention being the *proper* Poincaré group $\mathcal{P}_+$ which is generated by the same translations and boosts, but also the total reflection $-1 : x \mapsto -x$.

The final subgroup we mention is the one used throughout this work on the subject of massive integrable models - the *proper orthochronous* Poincaré group generated by the translations and boosts only.

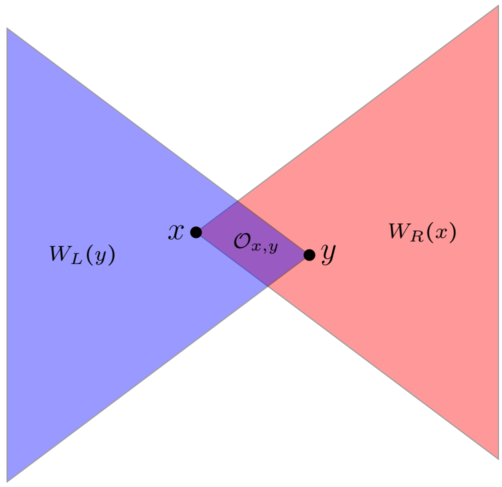

Figure A.1: The overlap of the wedges $W_L(y)$ and $W_R(x)$ describe the double cone region $\mathcal{O}_{x,y}$.

We move on to the description of relevant regions in this spacetime and recall our definition of a right wedge as

$$W_R = \{x \in \mathbb{R}^2 \ : \ x_1 > |x_0|\}$$

and the left wedge $W_L = -W_R$. The set of all wedges is denoted by $\mathcal{W}$.

The wedges W_L, W_R are invariant under the action of the boost transformations since the eigenvectors of $\Lambda(\lambda)$ are lightlike.

A *double cone* $\mathcal{O}_{x,y}$ is the intersection of two overlapping wedges centred at $x, y \in \mathbb{R}^2$, being defined as a set intersection as

$$\mathcal{O}_{x,y} = (W_R + x) \cap (W_L + y)$$

which is non-empty for $x - y \in W_L$ and $x_1 < y_1$. Geometrically, this is illustrated in Figure (A.1).

Appendix B

Gauge Groups and Internal Symmetries

The model described in Chapter 2 is described on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^2) \otimes \tilde{\mathcal{H}}$ where $\tilde{\mathcal{H}}$ is some finite dimensional Hilbert space. To further enrich the physical interpretation of this model, one may consider a compact Lie group G as the global gauge group and identify charges of particles with equivalences classes q of irreducible representations of G as in [Lec07].

We consider a set of a finite number of possible charges labelled by $\mathcal{Q}$ and to account for antiparticles whose charges are given by $\overline{q}$ according to some conjugacy class, we assume this conjugation leaves $\mathcal{Q}$ invariant.

In addition to the symmetry described by the Poincaré group as in Chapter 2, we have the additional "internal" symmetry $\tilde{W}_1$ as a representation of the group G on $\tilde{\mathcal{H}}$. This extends to $\mathcal{H}$ in the usual way by trivial action on the $L^2(\mathbb{R}^2)$ component

$$W_1(g) = 1_{L^2(\mathbb{R}^2)} \otimes \tilde{W}_1.$$

This additional structure is mainly cosmetic to the mathematics, but physically is most relevant to the structure of particle interactions allowing for a richer spectrum of particles under description. This finally gives rise to an additional property in the definition of an S-matrix in definition (2.1) in this setting which is specific to the model under a chosen gauge. It is referred to as *gauge invariance*:

$$[S(\theta), W_1(g) \otimes W_1(g)] = 0, \quad \text{for all } g \in G, \theta \in \mathbb{R}.$$

For more on this subject we refer the reader to [Lec07, AL17] and references therein.

An example of a chosen gauge may be the orthogonal matrices of order N as in the $O(N)$ σ-models, or a non-abelian gauge theory taking a non-commutative group as a gauge as is the case for Yang-Mills theory.

Appendix C

Algebraic Quantum Field Theory

We now recall the framework for algebraic quantum field theory before applying what we have described above and discussing its implications for this particular axiomatic paradigm. A model in this context is described in terms of an algebra of local observables $\mathcal{A}$ over a Hilbert Space $\mathcal{H}$. This algebra $\mathcal{A}$ contains all subalgebras $\mathcal{A}(\mathcal{O})$ which is the algebra of all observables localised in the region $\mathcal{O} \subset \mathbb{R}^2$ of Minkowski space. It is usual to take $\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ to be a von Neumann algebra, that is a $*$-subalgebra closed in the weak topology.

Algebraic quantum field theory is then concerned with the net generated by the map

$$\mathbb{R}^2 \supset \mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$$

and the isotony axiom:

$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \text{ for } \mathcal{O}_1 \subset \mathcal{O}_2.$$

Relativity implies that two events that are space-like separated cannot interfere with each other, a property known as causality. This physical phenomenon must be reflected in the mathematical framework, and this is implemented by requiring that observables that are localised in regions spacelike separated from each other must commute. That is

$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)' \qquad \mathcal{O}_1 \subset \mathcal{O}_2'.$$

The stronger condition of Haag duality [BW76] is known to hold in the specific case of $\mathcal{O}$ a wedge region such as W_R and the net is generated by Wightman fields [Mun01]:

$$\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})'.$$

Similar to what we have seen already we demand that there exists a representation U of the proper orthochronus Poincare group $\mathcal{P}_+^\uparrow$ such that:

$$U(g)\mathcal{A}(\mathcal{O})U(g)^* = \mathcal{A}(g\mathcal{O}), \text{ for } g \in \mathcal{P}_+^\uparrow.$$

The generators P^μ of the translation group $U(x) = e^{ix_\mu P^\mu}$ (the subgroup of $\mathcal{P}_+^\uparrow$) are interpreted as the energy and momentum operators of the theory, and we demand that the joint spectrum of P^μ be contained in the closed forward light cone $V^+ := \{(p_0, p_1) \in \mathbb{R}^2 \ : \ p_0 \leq |p_1|\}$ of Minkowski space - this is known as the condition of positive energy.

As we have already described, in the algebraic setup we must also have a vector that acts as a representation of the physical vacuum, denoted previously as $\Omega \in \mathcal{H}$ having zero energy and momentum. The vector Ω is invariant under U and is unique up to a constant scalar multiple. It is also cyclic and separating for the local algebras, that is

$$\overline{\mathcal{A}(\mathcal{O})\Omega} = \mathcal{H}, \qquad \mathcal{O} \subset \mathbb{R}^2 \text{ open}$$

i.e. cyclic, and $A\Omega = 0$ for $A \in \mathcal{A}(\mathcal{O})$ if and only if $A = 0$, i.e. separating. This is known in quantum field theory as the Reeh-Schlieder property.

If the net $\mathcal{A} : \mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ has all the properties (except perhaps Haag duality) it is known as a local net on $\mathbb{R}^2$.

Appendix D

Tomita-Takesaki Modular Theory

A well known area of operator algebra with applications to mathematical physics is Tomita-Takesaki modular theory [Tak79, Tak03]. We briefly outline the general theory and describe the data appearing here.

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ containing a cyclic and separating vector Ω. We define the operator S_1 on $\mathcal{H}$ as

$$S_1 A\Omega = A^*\Omega$$

for all $A \in \mathcal{M}$. This operator can be extended to a closed and anti-linear operator S on a dense subset of $\mathcal{H}$ which takes a polar decomposition

$$S = \Delta^{-1/2} J = J\Delta^{1/2}.$$

The *modular operator* Δ is unique, positive and self-adjoint and the *modular conjugation* J is unique, involutive and anti-unitary. We say that these operators are associated with the pair $(\mathcal{M}, \Omega)$. Further to this, one may extend the operator Δ by complex powers Δ^{it} such that the latter is a unitary operator for all $t \in \mathbb{R}$ and then $\{\Delta^{it} \ : \ t \in \mathbb{R}\}$ forms a strongly continuous one-parameter group.

This leads to the following well-known result.

Theorem D.1. *Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω. Then $J\Omega = \Omega = \Delta\Omega$ and the following holds*

$$J\,\mathcal{M}\,J = \mathcal{M}', \quad \Delta^{it}\,\mathcal{M}\,\Delta^{-it} = \mathcal{M}$$

where $\mathcal{M}'$ is the commutant of $\mathcal{M}$.

By way of naturally defined operators, we can observe a simple relationship between the elements of a von Neumann algebra and its commutant, proving invaluable in the discussion of local algebras in a quantum field theoretic context.

To touch with quantum field theory and the previous data described in Chapter 2, it can be shown in favourable conditions that the Bisognano-Wichmann theorem [BW76, Mun01]. For the von Neumann algebra (as in (2.19)) and vacuum vector pair $(\mathcal{A}_S(W_R), \Omega_S)$ and modular operator Δ associated with this pair we have

$$\Delta^{it} = U(0, 2\pi t), \quad t \in \mathbb{R}.$$

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