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Research paper

Humbert generalized fractional differenced ARMA processes

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A B S T R A C T

In this article, we use the generating functions of the Humbert polynomials to define two types of Humbert generalized fractional differenced ARMA processes. We present stationarity and invertibility conditions for the introduced models. The singularities for the spectral densities of the introduced models are investigated. In particular, Pincherle ARMA, Horadam ARMA and Horadam–Pethe ARMA processes are studied. It is shown that the Pincherle ARMA process has long memory property for $u = 0$. Additionally, we employ the Whittle quasi-likelihood technique to estimate the parameters of the introduced processes. Through this estimation method, we attain results regarding the consistency and normality of the parameter estimators. We also conduct a comprehensive simulation study to validate the performance of the estimation technique for Pincherle ARMA process. Moreover, we apply the Pincherle ARMA process to real-world data, specifically to Spain's 10 years treasury bond yield data, to demonstrate its practical utility in capturing and forecasting market dynamics.

1. Introduction

The study of fractionally differenced time series by Granger and Joyeux (1980) [1] and Hosking in 1981 [2] provided an impetuous to a new research direction in time series modeling. The fractionally differenced time series called the autoregressive fractionally integrated moving average (ARFIMA) model generalizes the autoregressive (AR), moving average (MA) and autoregressive moving average (ARMA) models defined respectively by Yule (1926) [3], Slutsky (1937) [4] and Wold (1938) [5]. Also, the ARFIMA model is an extension of the autoregressive integrated moving average (ARIMA) process defined by Box and Jenkins (1976) [6] to model non-stationary time series by assuming the order of differencing $\nu \in \mathbb{R}$. The fractionally differenced time series is useful to model the data exhibiting long-range dependence (LRD). The data exhibiting LRD behaviors or long memory have a high correlation after a significant lag. For large sample inference for long-memory processes see Giraitis et al. [7]. Anh et al. [8] proposed some continuous time stochastic processes with seasonal long-range dependence and these kinds of long memory processes have spectral pole at non-zero frequencies. In subsequent years, Andel (1986); Gray, Zhang and Woodward (1989, 1994) introduced the concept of Gegenbauer ARMA (GARMA) process. GARMA process also possesses seasonal long-range dependence [9]. The study on

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the usefulness of the Gegenbauer stochastic process is done by Dissanayake et al. [10]. The limit theorems for stationary Gaussian processes and their non-linear transformations with covariance function

$$\rho(h) \simeq \sum_{k=1}^{r} A_k \cos(h\omega_k) h^{-\alpha_k}, \sum_{k=1}^{r} A_k = 1,$$

where $A_k \geq 0, \alpha_k > 0, \omega_k \in [0, \pi), k = 1, \ldots, r$ have been considered in [11]. For seasonal long memory process X_t, the autocorrelation function for lag h denoted by $\rho(h)$ behaves asymptotically as $\rho(h) \simeq \cos(h\omega_0)h^{-\alpha}$ as $h \to \infty$ for some positive $\alpha \in (0, 1)$ and $\omega_0 \in (0, \pi)$ (see [12]). In literature, many tempered distributions and processes are studied using the exponential tempering in the original distribution or process see e.g. and references therein [13–19]. The fractionally integrated process with seasonal components are studied and maximum likelihood estimation is done by Reisen et al. [20]. The parametric spectral density with power-law behavior about a fractional pole at the unknown frequency ω is analyzed and Gaussian estimates and limiting distributional behavior of estimate is studied by Giraitis and Hidalgo [21]. The autoregressive tempered fractionally integrated moving average (ARTFIMA) process is obtained by using exponential tempering in the original ARFIMA process [15]. The ARTFIMA process is semi LRD and has a summable autocovariance function. In ARIMA process the fractional differencing operator $(1 - B)^{\nu}$, $|\nu| < 1$ is considered instead of $(1-B)$, where B is the shift operator. In defining ARTFIMA model the tempered fractional differencing operator $(1-e^{-\lambda}B)^{\nu}$ is used where $\lambda > 0$ is the tempering parameter. The Gegenbauer process uses $(1 - 2uB + B^2)^{\nu}$, $|u| \leq 1$, $|\nu| < \frac{1}{2}$ as a difference operator, which can be written in terms of Gegenbauer polynomials.

In this article, we study Humbert polynomials based time series models. The Gegenbauer and Pincherle polynomials are the particular cases of Humbert polynomials. The Gegenbauer polynomials based time series model, namely GARMA process, is already studied and has been applied in several real world applications emanating from different areas. These processes possess seasonal long memory which helps to capture autocorrelation present in the data, leading to improved forecasting accuracy. We introduce and study two types of Humbert autoregressive fractionally integrated moving average (HARMA) models which are defined by considering Humbert polynomials and obtain the spectral density, stationarity and invertibility conditions of the process. In particular, Pincherle ARMA, Horadam ARMA and Horadam–Pethe ARMA processes are studied. This new class of time series models generalizes the existing models like ARMA, ARIMA, ARFIMA, ARTFIMA and GARMA in several directions. The possible areas of applications of proposed model include sales forecasting in e-commerce industries as it can capture seasonality, trends, and other patterns in historical sales data [22]. Also, the long memory property can capture autocorrelation patterns observed in financial returns, volatility and other indicators. These models can be applied to analyze environmental monitoring data, such as water quality parameters, air pollution levels, and ecosystem dynamics [23]. Further, we also provide the Whittle quasi-likelihood estimation for HARMA processes and applied on simulated data. Also, we applied the Pincherle ARMA model to Spain's 10-year treasury bond yield data.

The rest of the paper is organized as follows. In Section 2, we introduce the Type 1 HARMA (p, ν, u, q) process, where p and q are autoregressive and moving average lags respectively and ν is a differencing parameter. This section includes the study of the stationarity property and spectral density of the introduced process. Also Section 2 includes the study of a particular case of Type 1 HARMA(p, ν, u, q) process by taking $m = 3$, which is Pincherle ARMA (p, ν, u, q) process. Moreover, the spectral density of the Pincherle ARMA (p, ν, u, q) process is obtained and it is shown that for $u = 0$ the model exhibits seasonal long memory property. The Section 3 deals with the Type 2 HARMA process (p, ν, u, q). In this section, the particular cases namely the Horadam ARMA process and the Horadam–Pethe ARMA process are discussed. In Section 4, we provide the Whittle quasi-likelihood method to estimate the parameters of type 1 and type 2 HARMA process and it is shown that the estimators are consistent. The simulation study of Pincherle ARMA and its applications are discussed in Section 5. The last section concludes. Overall, our study contributes to the advancement of time series analysis by introducing novel Humbert generalized fractional differenced ARMA models, investigating their properties, providing parameter estimation techniques, and showcasing their efficacy through simulations and real data applications..

2. Type 1 HARMA(p, ν, u, q) process

In this section, we introduce a new time series model namely type 1 HARMA(p, ν, u, q) process with the help of Humbert polynomials which we call hereafter type 1 Humbert polynomials. For Humbert polynomials and related properties see e.g. [24–26]. A detailed discussion on special functions including Humbert polynomials is given in [25,27].

Definition 2.1 (*Type 1 Humbert Polynomials*)**.** The Humbert polynomials of type 1 $\{\Pi^{\nu}_{n,m}\}_{n=0}^{\infty}$ are defined in terms of generating function as

$$(1 - mut + t^m)^{-\nu} = \sum_{n=0}^{\infty} \Pi^{\nu}_{n,m}(u)t^n, m \in \mathbb{N}, |t| < 1, |u| \leq 1 \text{ and } |\nu| < \frac{1}{2}. \tag{2.1}$$

For the table of main special cases of (2.1), including Gegenbauer, Legendre, Tchebysheff, Pincherle, Kinney polynomials, see Gould (1965) [25]. In above definition, polynomial $\Pi^{\nu}_{n,m}(u)$ is explicitly can be written as follows [24]:

$$\Pi^{\nu}_{n,m}(u) = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-mu)^{n-mk}}{\Gamma((1-\nu-n)+(m-1)k)(n-mk)!k!}, \text{ where } \left\lfloor \frac{n}{m} \right\rfloor \text{ is floor function.}$$

The hypergeometric representation of $\Pi^{\nu}_{n,m}(u)$ is given as follows:

$$\Pi^{\nu}_{n,m}(u) = \frac{(\nu)_n (mu)^n}{n!} {}_mF_{m-1}\left[\frac{-n}{m}, \frac{-n+1}{m}, \dots, \frac{-n-1+m}{n}; \frac{-\nu-n+1}{m-1}, \frac{-\nu-n+2}{m-2}, \dots, \frac{-\nu-n+m-1}{m-1}; \frac{1}{(m-1)^{m-1}u^m}\right].$$

For more properties and results on hypergeometric functions see Srivastava and Manocha (1984) [27]. The type 1 Humbert polynomial satisfies the following recurrence relation

$$(n+1)\Pi^{\nu}_{n+1,m}(u) - mu(n+\nu)\Pi^{\nu}_{n,m}(u) - (n+m\nu-m+1)\Pi^{\nu}_{n-m+1,m}(u) = 0.$$

For $m = 2$ the Humbert polynomials reduces to Gegenbauer polynomials generally denoted as $\{C^{\nu}_n(u)\}^{\infty}_{n=0}$ and for $m = 3$ the polynomials reduce to Pincherle polynomials $\{P^{\nu}_n(u)\}^{\infty}_{n=0}$, see Pincherle (1891) [28]. The generating function of Pincherle polynomials have the following form

$$(1-3ut+t^3)^{-\nu} = \sum_{n=0}^{\infty} P^{\nu}_n(u)t^n,$$

where $P^{\nu}_n(u)$ has the following representation in terms of hypergeometric function [28]

$$P^{\nu}_n(u) = \frac{(\nu)_n(3x)^n}{n!} {}_3F_2\left[\frac{-n}{3}, \frac{-n+1}{3}, \frac{-n+2}{3}; \frac{-n-\nu+1}{2}, \frac{-n-\nu+2}{2}; \frac{-1}{4x^3}\right],$$

where ${}_3F_2(a_1, a_2, a_3; b_1, b_2; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k(a_3)_k}{(b_1)_k(b_2)_k} \frac{x^k}{k!}$ and $(a_1)_k = \frac{\Gamma(a_1+k)}{\Gamma(a_1)}$ see e.g. [29].

Definition 2.2 (*Type 1 HARMA Process*). The type 1 HARMA(p, ν, u, q) process X_t is defined by

$$\Phi(B)(1-muB+B^m)^{\nu}X_t = \Theta(B)\epsilon_t, \tag{2.2}$$

where ϵ_t is Gaussian white noise with variance σ^2, B is the lag operator, $0 \le u \le 2/m$, and $\Phi(B)$, $\Theta(B)$ are stationary AR and invertible MA operators respectively, defined as,

$$\Phi(B) = 1 - \sum_{j=1}^{p} \phi_j B^j, \; \Theta(B) = 1 + \sum_{j=1}^{q} \theta_j B^j, \; \text{ and } B^j(X_t) = X_{t-j}.$$

In next result, the stationarity and invertibility conditions of the type 1 HARMA process are given. Also, the Abel's test which will be used in next theorem is stated below as proposition.

Proposition 2.1 (*Abel's Tests [30]*). *If the series $\sum_{n=0}^{\infty} a_n$ is convergent and $\{b_n\}$ is monotone and bounded sequence then series $\sum_{n=0}^{\infty} a_n b_n$ is also convergent.*

Definition 2.3 (*Asymptotically Equivalent Functions [31]*). The functions f and g are said to be asymptotically equivalent that is, $f(h) \simeq g(h)$ as $h \to \infty$ if $\lim_{h\to\infty} \frac{f(h)}{g(h)} = 1$.

Theorem 2.1. *Let $\{X_t\}$ be the type 1 HARMA(p, ν, u, q) process defined in (2.2) and all roots of $\Phi(B) = 0$ and $\Theta(B) = 0$ lie outside the unit circle then the HARMA(p, ν, u, q) process is stationary and invertible for $|\nu| < 1/2$ and $0 \le u \le 2/m$.*

Proof. Using (2.2), one can write

$$X_t = \frac{\Theta(B)}{\Phi(B)}(1-muB+B^m)^{-\nu}\epsilon_t, \; \text{ where } \frac{\Theta(B)}{\Phi(B)} = \sum_{j=0}^{\infty} \psi_j B^j.$$

Further,

$$(1-muB+B^m)^{-\nu} = \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} \left(muB - B^m\right)^n.$$

Then (2.2) can be written as

$$X_t = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \psi_j \frac{(\nu)_n}{n!}(muB-B^m)^n \epsilon_{t-j-n}$$

$$= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\psi_j \frac{(\nu)_n}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}(mu)^{n-r}B^{mr}\epsilon_{t-n}$$
$$= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\psi_j \frac{(\nu)_n}{n!}(mu-1)^n\epsilon_{t-n-mj}.$$

The variance of the process X_t is given by

$$\mathrm{Var}(X_t) = \sigma^2\sum_{j=0}^{\infty}\psi_j^2\sum_{n=0}^{\infty}\left(\frac{(\nu)_n}{n!}\right)^2(mu-1)^{2n} = \sigma^2\sum_{j=0}^{\infty}\psi_j^2\sum_{n=0}^{\infty}\left(\frac{\Gamma(\nu+n)}{\Gamma(\nu)\Gamma(n+1)}\right)^2(mu-1)^{2n}.$$

Let $a_n = (mu-1)^{2n}$ and $\{b_n\} = \left(\frac{\Gamma(\nu+n)}{\Gamma(\nu)\Gamma(n+1)}\right)^2$, then using Abel's test $\sum_{n=0}^{\infty} a_n$ converges for $0 < u < \frac{2}{m}$ and using Stirling's approximation, for large n, $b_n \simeq \frac{n^{2\nu-2}}{(\Gamma(\nu))^2}$, which implies that the sequence is bounded for $\nu < \frac{1}{2}$. We can write $b_n = \binom{\nu+n-1}{n}$ and it is known that $\binom{n}{x}$ is decreasing for $x \geq \lfloor\frac{n}{2}\rfloor$ this implies that $\{b_n\}$ is decreasing for $\nu \leq 1$. This indicates that the sequence is bounded and monotone for $\nu < 1/2$. Also, $\sum_{j=0}^{\infty}\psi_j^2$ is convergent, hence the $\mathrm{Var}(X_t)$ converges for the defined range. Similarly to prove the invertibility condition we define the process (2.2) as

$$\epsilon_t = \pi(B)X_t,$$

where $\pi(B) = \frac{\Phi(B)}{\Theta(B)}(1-muB+B^m)^{\nu}$ and again using the same argument discussed above the $\pi(z)$ will converge for $-\frac{1}{2} < \nu < 1$ and $0 < u < \frac{2}{m}$. For $u = 0$ and $u = \frac{2}{m}$the variance can be defined as follows

$$\mathrm{Var}(X_t) = \sigma^2\sum_{n=0}^{\infty}\left(\frac{\Gamma(\nu+n)}{\Gamma(\nu)\Gamma(n+1)}\right)^2$$
$$= \sigma^2\sum_{n=0}^{N}\left(\frac{\Gamma(\nu+n)}{\Gamma(\nu)\Gamma(n+1)}\right)^2 + \sum_{n=N+1}^{\infty}\left(\frac{\Gamma(\nu+n)}{\Gamma(\nu)\Gamma(n+1)}\right)^2.$$

In the above equation, the first summation is finite and the terms inside the second summation behave like $\frac{n^{2\nu-2}}{\Gamma(\nu)^2}$ for large n and it is bounded for $\nu < \frac{1}{2}$. Hence, the HARMA process is stationary and invertible for $|\nu| < 1/2$ and $0 \leq u \leq \frac{2}{m}$. □

Theorem 2.2. *For a type 1* HARMA(p,ν,u,q) *process defined in* (2.2), *under the assumptions of theorem* 2.1 *the spectral density takes the following form*

$$f_x(\omega) = \frac{\sigma^2}{2\pi}\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2+m^2u^2-2mu(\cos(\omega)+\cos((1-m)\omega))+2\cos(m\omega))^{-\nu},$$

where $z = e^{-\iota\omega},\ \omega \in (-\pi,\pi)$.

Proof. Rewrite (2.2) as follows

$$X_t = \Psi(B)\epsilon_t,$$

where $\Psi(B) = \frac{\Theta(B)}{\Phi(B)}(1-muB+B^m)^{-\nu}$. Then using the definition of the spectral density of linear process, we have

$$f_x(\omega) = |\Psi(z)|^2 f_\epsilon(\omega), \tag{2.3}$$

where $z = e^{-\iota\omega}$ and $f_\epsilon(\omega)$ is spectral density of the innovation term. The spectral density of the innovation process ϵ_t is $\sigma^2/2\pi$. Then (2.3) becomes,

$$f_x(\omega) = \frac{\sigma^2}{2\pi}|\Psi(z)|^2 = \frac{\sigma^2}{2\pi}\frac{|\Theta(z)|^2}{|\Phi(z)|^2}\left|1-mue^{-\iota\omega}+e^{-m\iota\omega}\right|^{-2\nu}$$
$$= \frac{\sigma^2|\Theta(e^{-\iota\omega})|^2\left|1-mue^{-\iota\omega}+e^{-m\iota\omega}\right|^{-2\nu}}{2\pi|\Phi(e^{-\iota\omega})|^2}.$$

Here, $\left|1-mue^{-\iota\omega}+e^{-m\iota\omega}\right|^{-2\nu} = (2+m^2u^2-2mu(\cos(\omega)+\cos((1-m)\omega))+2\cos(m\omega))^{-\nu}$ and the spectral density takes the following form

$$f_x(\omega) = \frac{\sigma^2}{2\pi}\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2+m^2u^2-2mu(\cos(\omega)+\cos((1-m)\omega))+2\cos(m\omega))^{-\nu}. \quad \square \tag{2.4}$$

Definition 2.4 (*Singular Point [31]*)**.** The point $\omega = \omega_0$ is said to be singular point of function f if at $\omega = \omega_0$, f fails to be analytic, that is $f(\omega_0) = \infty$.

Next, the definition of seasonal or cyclic long-memory is given, which is characterized by having a spectral pole at a frequency $\kappa \in \mathbb{R}$ different from 0, see, e.g., [8,9].

Definition 2.5 (*Seasonal Long Memory*)**.** The stationary time series $\{X_t\}$ is said to have seasonal long memory if there exist $\omega_0 \in \mathbb{R}$ and $\alpha \in (0, 1)$ such that

$$\rho(h) \simeq h^{-\alpha}\cos(h\omega_0), \text{ as } h \to \infty,$$

and $\cos(h\omega_0) \neq 1$.

Theorem 2.3. *Let $\{X_t\}$ be the stationary type 1* HARMA(p, ν, u, q) *process and all the assumptions of theorem 2.1 hold then the spectral density of* HARMA(p, ν, u, q) $\{X_t\}$ *has singular spectrum*

(a) at $u = 0$ and $\omega = \frac{4n\pi \pm \pi}{m}$ for $-\frac{m\pm 1}{4} < n < \frac{m\mp 1}{4}$;
(b) at $u = \frac{2}{m}(-1)^n \cos(\frac{4n\pi}{m-2})$ and $\omega = \pm\frac{2n\pi}{m-2}$ for $m \neq 2$ and $-\frac{(m-2)}{4} < n < \frac{(m-2)}{4}$;
(c) at $\omega = \cos^{-1}(u)$ for $m = 2$.

Proof. From (2.4), the spectral density of the process $\{X_t\}$ is

$$f_x(\omega) = \frac{\sigma^2}{2\pi}\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2 + m^2u^2 + 2\cos(m\omega) - 2mu(\cos(\omega) + \cos((m-1)\omega)))^{-\nu}, \text{ where } z = e^{-\iota\omega}.$$

We consider the denominator and find the zeros as follows,

$$\begin{aligned}
&2 + m^2u^2 + 2\cos(m\omega) - 2mu(\cos(\omega) + \cos((m-1)\omega)) \\
&= 2 + 2\cos(m\omega) + [mu - \{\cos(\omega) + \cos((m-1)\omega)\}]^2 - [\cos(\omega) + \cos((m-1)\omega)]^2 \\
&= 4\cos^2\left[\frac{m\omega}{2}\right] + [mu - \{\cos(\omega) + \cos((m-1)\omega)\}]^2 - 4\cos^2\left[\frac{m\omega}{2}\right]\cos^2\left[\frac{(m-2)\omega}{2}\right] \\
&= 4\cos^2\left[\frac{m\omega}{2}\right]\sin^2\left[\frac{(m-2)\omega}{2}\right] + \left[mu - 2\cos\left(\frac{m\omega}{2}\right)\cos\left(\frac{(2-m)\omega}{2}\right)\right]^2.
\end{aligned}$$

We have the following two cases.

(a) The first term, $4\cos^2\left[\frac{m\omega}{2}\right]\sin^2\left[\frac{(m-2)\omega}{2}\right] \geq 0$ for all m and $-\pi < \omega < \pi$.

$$\begin{aligned}
&4\cos^2\left[\frac{m\omega}{2}\right]\sin^2\left[\frac{(m-2)\omega}{2}\right] = 0 \\
&\qquad \text{if } \cos^2\left(\frac{m\omega}{2}\right) = 0 \text{ or } \sin^2\left(\frac{(m-2)\omega}{2}\right) = 0 \text{ or both} \\
&\qquad \Rightarrow \cos\left(\frac{m\omega}{2}\right) = 0 \text{ for } \omega_1 = (4n \pm 1)\frac{\pi}{m}, \text{ for all } m \in \mathbb{N} \text{ and} \\
&\qquad\qquad n = 0, \pm 1, \pm 2, \ldots.
\end{aligned}$$

We find the condition of singularity by solving the second term, $\left[mu - 2\cos\left(\frac{m\omega}{2}\right)\cos\left(\frac{(2-m)\omega}{2}\right)\right]^2$ at ω_1, which yields $u = 0$.
Also, the singular point $\omega_1 \in (-\pi, \pi)$ for $-\frac{m\pm 1}{4} < n < \frac{m\mp 1}{4}$.
Therefore, the type 1 HARMA(p, ν, u, q) process $\{X_t\}$ will have singular points for $u = 0$ and $\omega_1 = (4n \pm 1)\frac{\pi}{m}$ for all m and $-\frac{m\pm 1}{4} < n < \frac{m\mp 1}{4}$.
This proves the part (a).

(b) Again the term

$$4\cos^2\left(\frac{m\omega}{2}\right)\sin^2\left(\frac{(m-2)\omega}{2}\right) = 0$$

when

$$\begin{aligned}
&\sin^2\left(\frac{(m-2)\omega}{2}\right) = 0 \\
&\sin\left(\frac{(m-2)\omega}{2}\right) = 0 \text{ for } \omega_2 = \frac{\pm 2n\pi}{m-2} \text{ for all } m \in \mathbb{N}-\{2\}, \qquad \text{and } n = 0, \pm 1, \pm 2, \ldots.
\end{aligned}$$

At ω_2, the second term will become zero if and only if,

$$\left[mu - \left(2\cos\left(\frac{m\omega_2}{2}\right)\cos\left(\frac{(2-m)\omega_2}{2}\right)\right)\right]^2 = 0$$

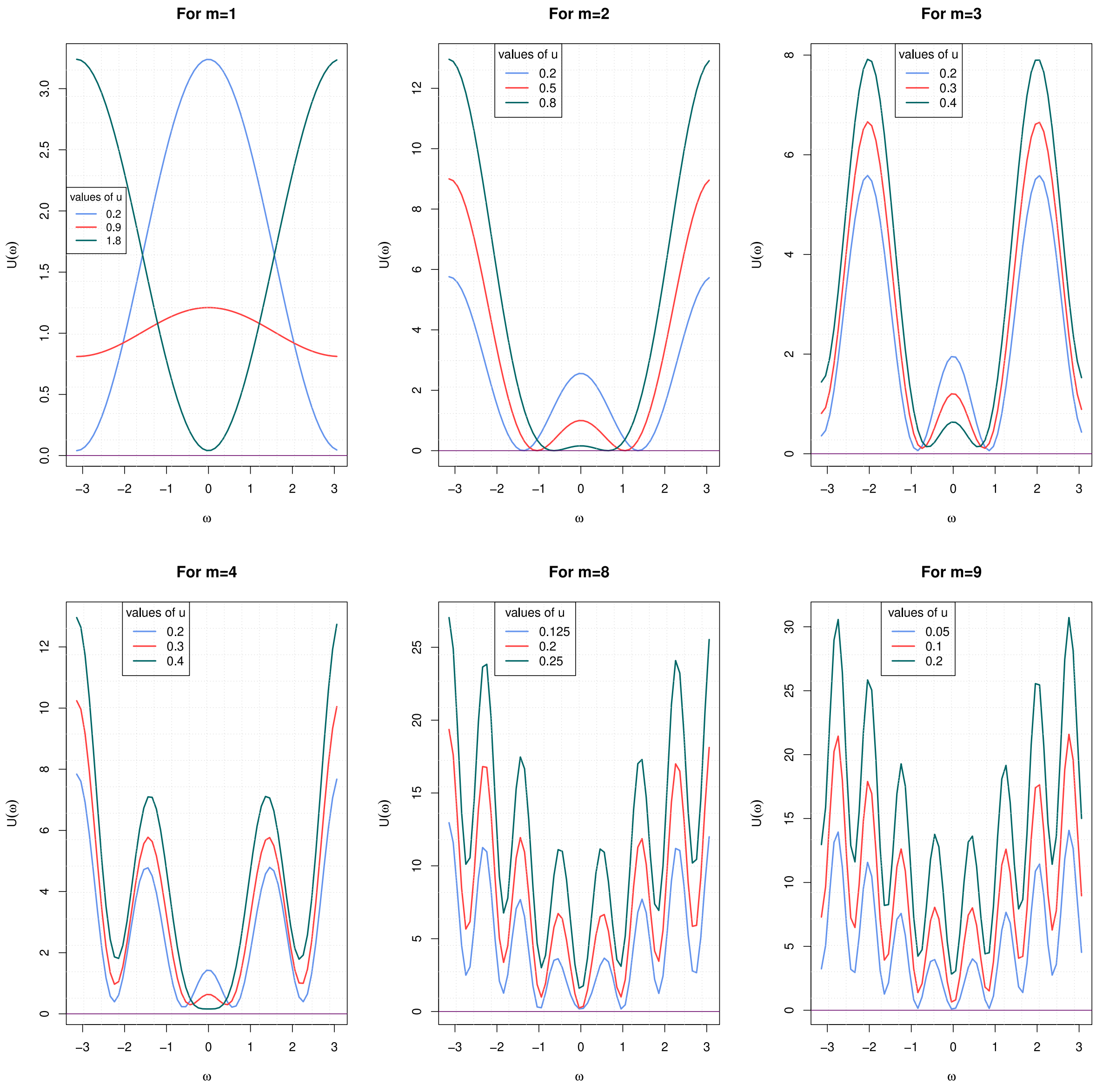


Fig. 1. Plot of the function $U(\omega)$ for different values of $m \in \{1, 2, 3, 4, 8, 9\}$ and $0 \le u \le 2/m$.

$$\Rightarrow mu - 2(-1)^n \cos\left(\pm\frac{4n\pi}{m-2}\right) = 0$$
$$\Rightarrow u = \frac{2(-1)^n}{m} \cos\left(\frac{4n\pi}{m-2}\right).$$

This proves the part (b).

(c) In (2.4) let $U(\omega) = (2 + m^2u^2 - 2mu(\cos(\omega) + \cos((1-m)\omega)) + 2\cos(m\omega))$. For different values of $m = 1, 2, 3, 4$ and $0 \le u < 2/m$ in Fig. 1, we observe that the function $U(\omega)$ does not attain 0 for $\omega \in (-\pi, \pi)$. This signifies that the spectral density defined by (2.4) has no singularity for $m = 1, 3, 4$ and $0 \le u < 2/m$. For $m = 2$, the spectral density is unbounded since $U(\omega)$ takes value 0 at $\omega = \cos^{-1}(u)$. Therefore, we conclude that for $m = 2$ the singularities are at $\omega = \cos^{-1}(u)$. □

In Fig. 1, observe the behavior of function $U(\omega)$ for $\omega \in (-\pi, \pi)$ and for different values of m and u. For $m = 2$, the function U touches the x-axis for all values of u. Further, for $m = 1$ it touches the x-axis only for $u = 0$. For other cases see Theorem 2.3.

Definition 2.6 (*Slowly Varying Function [32]*). A function $b(\omega)$ is said to be slowly varying at ω_0 if for $\delta > 0$, $(\omega-\omega_0)^{\delta}b(\omega)$ is increasing and $(\omega-\omega_0)^{-\delta}b(\omega)$ is decreasing in some right-hand neighborhood of ω_0. Also, $(\omega-\omega_0)^{\delta}b(\omega)$ is decreasing and $(\omega-\omega_0)^{-\delta}b(\omega)$ is increasing in some left-hand neighborhood of ω_0.

We need the following lemma which is given in [32] to prove our next result.

Lemma 2.1 (*Gray et al. [32]*). *Let $R(\tau) = \int_0^{\pi} P(\omega)d\omega$ where τ is an integer and $P(\omega)$ is spectral density. Suppose $P(\omega)$ can be expressed as*

$$P(\omega) = b(\omega)|\omega-\omega_0|^{-\beta} \tag{2.5}$$

with $0 < \beta < \frac{1}{2}$ and $\omega_0 \in (0,\pi)$. Further, suppose that $b(w)$ is non-negative and of bounded variation in $(0,\omega_0-\epsilon)\cup(\omega_0+\epsilon,\pi)$ for $\epsilon > 0$. Also suppose that $b(\omega)$ is slowly varying at ω_0, then as $\tau \to \infty$

$$R(\tau) \simeq \tau^{2\beta-1}cos(\tau\omega_0).$$

Theorem 2.4. *The stationary type 1 HARMA$(p,\nu,0,q)$ process has seasonal long memory for $0 < \nu < 1/2$.*

Proof. The spectral density of type 1 HARMA$(0,\nu,u,0)$ process is given by

$$f_x(\omega) = \frac{\sigma^2}{2\pi}(2+m^2u^2-2mu(\cos(\omega)+\cos((1-m)\omega))+2\cos(m\omega))^{-\nu}.$$

For $u = 0$ the spectral density has the form

$$f_x(\omega) = \frac{\sigma^2}{2\pi}(2+2\cos(m\omega)). \tag{2.6}$$

Also, the spectral density is unbounded at $\omega_0 = (4n\pm 1)\frac{\pi}{m}, -\frac{m\pm1}{4} < n < \frac{m\mp1}{4}$. which implies that the covariance is not absolutely summable for $u = 0$ at frequency ω_0. To prove the process is seasonal long memory we use Lemma 2.1 defined by Gray et al. [32]. Now (2.6) can be rewritten as

$$f_x(\omega) = \frac{\sigma^2(2+2\cos(m\omega))^{-\nu}|\omega-\omega_0|^{-2\nu}}{2\pi|\omega-\omega_0|^{-2\nu}}.$$

Comparing the above equation with (2.5)

$$b(\omega) = \frac{\sigma^2(2\cos(m\omega)+2)^{-\nu}}{2\pi|\omega-\omega_0|^{-2\nu}}.$$

Now to show $b(\omega)$ is slowly varying at ω_0, consider the case $\omega > \omega_0$ and for $\delta > 0$ define,

$$l(\omega) = b(\omega)(\omega-\omega_0)^{\delta} = \frac{\sigma^2}{2\pi}(2+2\cos(m\omega))^{-\nu}(\omega-\omega_0)^{\delta+2\nu}$$

and

$$l'(\omega) = \frac{\sigma^2}{2\pi}(\omega-\omega_0)^{\delta+2\nu-1}(2+2\cos(m\omega))^{-\nu-1}((\delta+2\nu)(2+2\cos(m\omega))+2\nu m\sin(m\omega)(\omega-\omega_0)).$$

For $\omega > \omega_0$ the terms $(\omega-\omega_0)^{\delta+2\nu-1}$, $(\delta+2\nu)$ and $(2+2\cos(mw))$ are positive. It can be easily shown that

$$\lim_{\omega\to\omega_0}(2\nu m\sin(m\omega)(\omega-\omega_0)+(\delta+2\nu)(2+2\cos(m\omega))) > 0.$$

Thus in some right hand neighborhood of ω_0, i.e. for $\omega \to \omega_0^+$, $l'(\omega) > 0$ and $(\omega-\omega_0)^{\delta}b(\omega)$ is increasing and similarly $(\omega-\omega_0)^{-\delta}b(\omega)$ is decreasing when $\omega \to \omega_0^+$. Similarly, it can be easily shown that for $\omega < \omega_0$, $(\omega-\omega_0)^{\delta}b(\omega)$ is decreasing and $(\omega-\omega_0)^{-\delta}b(\omega)$ is increasing in some left hand neighborhood of ω_0. Thus the function is slowly varying at ω_0.

Also, it can be easily verified that the function $b(w)$ has bounded derivative in $(0,\omega_0-\epsilon)\cup(\omega_0+\epsilon,\pi)$, hence it is of bounded variation in $(0,\omega_0-\epsilon)\cup(\omega_0+\epsilon,\pi)$.

Using the above two results and Lemma 2.1 the autocorrelation function $R(h)$ of the type 1 Humbert ARMA process takes the following asymptotic form

$$R(h) \simeq h^{2\nu-1}\cos(h\omega_0), \text{ as } h \to \infty. \tag{2.7}$$

The result (2.7) implies that the process is seasonal long memory for $0 < \nu < 1/2$. □

2.1. Pincherle ARMA (p,ν,u,q) process

This section deals with the special case of the type 1 HARMA process for $m = 3$. The Pincherle polynomials are polynomials introduced by Pincherle (1891) [28]. The Pincherle polynomials were generalized to Humbert Polynomials by Humbert (1920) [24].

Definition 2.7 (*Pincherle Polynomials*)**.** The Pincherle polynomials $P_n^{\nu}(u)$ are defined as the coefficient of t in the expansion of $(1-3ut+t^n)^{-\nu}$. The Pincherle polynomials are defined by taking $m=3$ in type 1 Humbert polynomials that is $P_n^{\nu}(u)=\Pi_{n,3}^{\nu}(u)$. Also, the generating function relation for Pincherle polynomials is given by

$$(1-3ut+t^n)^{-\nu}=\sum_{n=0}^{\infty}P_n(u)t^n,\ \ |t|<1,\ |\nu|<1/2,\ \ |u|\le 1.$$

The polynomials satisfy the following difference equation [33]

$$(n+1)P_{n+1}^{\nu}(u)-3u(n+\nu)P_n^{\nu}(u)+(n+3\nu-2)P_{n-2}^{\nu}(u)=0.$$

The coefficient of Pincherle polynomials can be written as $P_0^{\nu}(u)=1$, $P_1^{\nu}(u)=3\nu$, $P_2^{\nu}(u)=9\nu(\nu+1)u^2/2$ and the *nth* coefficient takes the form [33]

$$\frac{\Gamma(n+\nu)\Gamma(1/3)\Gamma(2/3)}{\Gamma(\nu)\Gamma((n+1)/3)\Gamma((n+2)/3)\Gamma((n+3)/3)}.$$

Definition 2.8 (*Pincherle ARMA Process*)**.** The Pincherle ARMA (p,ν,u,q) process is defined by taking $m=3$ in type 1 HARMA process defined in (2.2) and the process has the form defined below

$$\Phi(B)(1-3uB+B^3)^{\nu}X_t=\Theta(B)\epsilon_t, \tag{2.8}$$

where ϵ_t is Gaussian white noise with variance σ^2, $0\le u\le 2/3$ and B, $\Phi(B)$ and $\Theta(B)$ are lag, stationary AR and invertible MA operators respectively defined in definition 2.2.

Theorem 2.5. *Let $\{X_t\}$ be the Pincherle* ARMA(p,ν,u,q) *process defined in* (2.8) *and all roots of $\Phi(B)=0$ and $\Theta(B)=0$ lie outside the unit circle then the Pincherle* ARMA(p,ν,u,q) *process is stationary and invertible for $|\nu|<1/2$ and $0\le u\le 2/3$.*

Proof. The proof can be easily done by taking $m=3$ in the proof of theorem 2.1. □

Theorem 2.6. *The stationary Pincherle HARMA$(p,\nu,0,q)$ process has seasonal long memory for $0<\nu<1/2$ at $\omega_0=\pi/3$.*

Proof. According to theorem 2.3 the spectral density of the Pincherle ARMA process has singularity at $u=0$ for $\omega_0=\pi/3$. Also, similar to the proof of theorem 2.4 the autocovariance function of Pincherle ARMA process $\gamma(h)$ has the asymptotic form $R(h)\simeq h^{2\nu-1}\cos(h\omega_0)$. This proves that the process has a seasonal long memory for $0<\nu<1/2$ at $\omega_0=\pi/3$. □

Theorem 2.7. *For a Pincherle ARMA(p,ν,u,q) process defined in* (2.8)*, the spectral density takes the following form*

$$f_x(\omega)=\frac{\sigma^2}{2\pi}\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(8\cos^3(\omega)-12u\cos^2(\omega)-C\cos(\omega)+D)^{-\nu},$$

where $z=e^{-\iota\omega}$, $C=6+6u$, and $D=2+6u+9u^2$.

Proof. Taking $m=3$ in (2.4) gives us the desired spectral density. □

Theorem 2.8. *The autocovariance function for the Pincherle ARMA process takes the following form*

$$\gamma(h)=\sigma^2\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\psi_j\psi_{j+h}P_n^{\nu}(u)P_{n+h}^{\nu}(u).$$

Proof. For lag h the autocovariance of the process $\{X_t\}$ and $\{X_{t+h}\}$ using the (2.8) is given by

$$\mathrm{Cov}(X_tX_{t+h})=\mathrm{E}[X_tX_{t+h}],$$

where X_t can be written as

$$X_t=\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\psi_jP_n^{\nu}(u)\epsilon_{t-j-n}$$

and

$$\mathrm{E}[X_tX_{t+h}]=\sigma^2\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\psi_j\psi_{j+h}P_n^{\nu}(u)P_{n+h}^{\nu}(u).\quad\square$$

3. Type 2 HARMA(p, ν, u, q) process

Milovovic and Dordevic (1987) [26] considered the following generalization of Gegenbauer polynomials, which we call type 2 Humbert polynomials and are used to define the type 2 HARMA process.

Definition 3.1 (*Type 2 Humbert Polynomials*)**.** The type 2 Humbert polynomials are defined by considering the polynomials $Q^{\nu}_{n,m}(u)$ defined by the following generating function

$$(1-2ut+t^m)^{-\nu}=\sum_{n=0}^{\infty}Q^{\nu}_{n,m}(u)t^n,\ \ |t|<1,\ \ |\nu|<1/2,\ \ |u|\le 1. \tag{3.1}$$

Here $Q^{\nu}_{n,m}(u)=\Pi^{\nu}_{n,m}(\frac{2u}{m})$ (see (2.1)).

The explicit form of the polynomials $Q^{\nu}_{n,m}(u)$ is defined by

$$Q^{\nu}_{n,m}(u)=\sum_{k=0}^{[\frac{n}{m}]}(-1)^k\frac{(\nu)_{(n-(m-1)k)}}{k!(n-mk)!}(2u)^{n-mk},$$

where $\nu_0=1$ and $(\nu)_n=\nu(\nu+1)\cdots(\nu+n-1)$.

Definition 3.2 (*Type 2 HARMA Process*)**.** The type 2 HARMA process is defined by using the above-defined generation function as follows

$$\Phi(B)(1-2uB+B^m)^{\nu}X_t=\Theta(B)\epsilon_t, \tag{3.2}$$

where ϵ_t is Gaussian white noise with variance σ^2, $0\le u<1$, and B, $\Phi(B)$, $\Theta(B)$ are lag, stationary AR and invertible MA operators respectively defined in definition 2.2.

For $m=2$ the above polynomials in (3.1) is Gegenbauer polynomials and $Q^{\nu}_{n,2}(u)=C^{\nu}_n(u)$. Also, for $m=3$ the polynomials in (3.1) are known as Horadam–Pethe polynomials and for $m=1$ they are known as Horadam polynomials, see Gould (1965) [25], Horadam (1985) [34] and Horadam and Pethe (1981) [35].

Theorem 3.1. *Let $\{X_t\}$ be the type 2* HARMA(p, ν, u, q) *process and all roots of $\Phi(B)=0$ and $\Theta(B)=0$ lies outside the unit circle then the* HARMA(p, ν, u, q) *process is stationary and invertible for $|\nu|<1/2$ and $0\le u\le 1$.*

Proof. The process is stationary and invertible for $|\nu|<1/2$ and $0\le u\le 1$ can be easily proved using the proof for the stationarity of type 1 HARMA process defined in 2.1. □

Theorem 3.2. *For a type 2 Humbert ARMA(p, ν, u, q) process defined in (3.2), under the assumptions of theorem 3.1 the spectral density takes the following form*

$$f_x(\omega)=\frac{\sigma^2}{2\pi}\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2+4u^2-4u(\cos(\omega)+\cos((1-m)\omega))+2\cos(m\omega))^{-\nu}, \tag{3.3}$$

where $z=e^{-\iota\omega}$.

Proof. Rewrite (3.2) as follows

$$X_t=\Psi(B)\epsilon_t,$$

where $\Psi(B)=\frac{\Theta(B)}{\Phi(B)}\Delta^{\nu}$ and $\Delta^{\nu}=(1-2uz+z^m)^{-\nu}$. The spectral density of the innovation process ϵ_t is given by $\sigma^2/2\pi$, which implies

$$f_x(\omega)=\frac{\sigma^2}{2\pi}|\Psi(z)|^2=\frac{\sigma^2}{2\pi}\frac{|\Theta(z)|^2}{|\Phi(z)|^2}\left|1-2uz+z^m\right|^{-2\nu},$$

where $z=e^{-\iota\omega}$. Furthermore,

$$\left|1-2ue^{-\iota\omega}+e^{-m\iota\omega}\right|^{-2\nu}=(2+4u^2-4u(\cos(\omega)+\cos((1-m)\omega))+2\cos(m\omega))^{-\nu},$$

and the spectral density takes the following form

$$f_x(\omega)=\frac{\sigma^2}{2\pi}\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2+4u^2-4u(\cos(\omega)+\cos((1-m)\omega))+2\cos(m\omega))^{-\nu}.\quad \square$$

Theorem 3.3. *Under the assumption of theorem 3.1 let $\{X_t\}$ be the type 2* HARMA(p, ν, u, q) *process then the spectral density of* HARMA(p, ν, u, q) *process has singularities*

(a) at $u = 0$ *and* $\omega = \frac{4n\pi \pm \pi}{m}$ *for* $-\frac{m\pm1}{4} < n < \frac{m\mp1}{4}$.
(b) at $u = (-1)^n \cos(\frac{4n\pi}{m-2})$ *and* $\omega = \pm\frac{2n\pi}{m-2}$ *for* $m \neq 2$ *and* $-\frac{(m-2)}{4} < n < \frac{(m-2)}{4}$.

Proof. The spectral density of the type 2 HARMA process is

$$f_x(\omega) = \frac{\sigma^2}{2\pi}\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2 + 4u^2 - 4u(\cos(\omega) + \cos((1-m)\omega)) + 2\cos(m\omega))^{-\nu}.$$

Similar to the proof in Theorem 2.3, we find the zeros by writing the denominator as follows

$$2 + 4u^2 - 4u(\cos(\omega) + \cos((1-m)\omega)) + 2\cos(m\omega) =$$
$$4\cos^2\left[\frac{m\omega}{2}\right]\sin^2\left[\frac{(m-2)\omega}{2}\right] + \left[2u - 2\cos\left(\frac{m\omega}{2}\right)\cos\left(\frac{(2-m)\omega}{2}\right)\right]^2 \tag{3.4}$$

The proof of part *(a)* is the same as the part *(a)* of theorem 2.3. To prove the part *(b)* the term $4\cos^2\left[\frac{m\omega}{2}\right]\sin^2\left[\frac{(m-2)\omega}{2}\right] = 0$ at $\omega_0 = \frac{\pm 2n\pi}{m-2}$. For this ω_0 the second term of (3.4) is zero for $u = (-1)^n\cos(\frac{4n\pi}{m-2})$ for all $m \in \mathbb{N}-\{2\}$ and $-\frac{(m-2)}{4} < n < \frac{(m-2)}{4}$. □

The particular cases of the type 2 Horadam ARMA process is discussed as follows:

3.1. Horadam ARMA(p, ν, u, q) process

Definition 3.3 (*Horadam Polynomials*)**.** In (3.1) by taking $m = 1$ the reduced polynomials are known as Horadam polynomials. The Horadam polynomials is defined as the coefficient of t in the expansion of $(1-2ut+t)$ and the generating function relation is given as follows

$$(1 - 2ut + t)^{-\nu} = \sum_{n=0}^{\infty} Q^{\nu}_{n,1}(u)t^n,\ |t| < 1,\ |\nu| < 1/2,\ |u| \leq 1.$$

Definition 3.4 (*The Horadam ARMA Process*)**.** The time series process defined using the generating function of Horadam polynomials are defined by the Horadam ARMA process, which is a special case of type2 HARMA process for m=1 and the process takes the following form

$$\Phi(B)(1 - 2uB + B)^{\nu}X_t = \Theta(B)\epsilon_t, \tag{3.5}$$

where ϵ_t is Gaussian white noise with variance σ^2, $0 \leq u \leq 1$, and B, $\Phi(B)$, $\Theta(B)$ are lag, stationary AR and invertible MA operators respectively defined in definition 2.2.

Theorem 3.4. *For a Horadam ARMA(p, ν, u, q) process defined in (3.5), the spectral density takes the following form*

$$f_x(\omega) = \frac{\sigma^2}{2\pi}\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2 + 4u^2 - 4u - 4u\cos(\omega) + 2\cos(\omega))^{-\nu},\ z = e^{-\iota\omega}. \tag{3.6}$$

Proof. This can be easily proved by taking $m = 1$ in the spectral density of type 2 HARMA process defined in (3.3). □

3.2. Horadam–Pethe ARMA(p, ν, u, q) process

Taking $m = 3$ in (3.1) the reduced form of the polynomials is known as Horadam–Pethe polynomials and the corresponding time series defined using the generating function of Horadam–Pethe polynomials is known as Horadam–Pethe ARMA process defined as follows

$$\Phi(B)(1 - 2uB + B^3)^{\nu}X_t = \Theta(B)\epsilon_t, \tag{3.7}$$

where $(1 - 2uB + B^3)^{-\nu} = \sum_{n=0}^{\infty} Q^{\nu}_{n,3}(u)t^n$.

Theorem 3.5. *Under the assumptions of theorem 3.1 for a Horadam–Pethe ARMA(p, ν, u, q) process defined in (3.7), the spectral density takes the following form*

$$f_x(\omega) = \frac{\sigma^2}{2\pi}\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2 + 4u^2 - 4u(\cos(\omega) + \cos(2\omega)) + 2\cos(3\omega))^{-\nu},$$

where $z = e^{-\iota\omega}$.

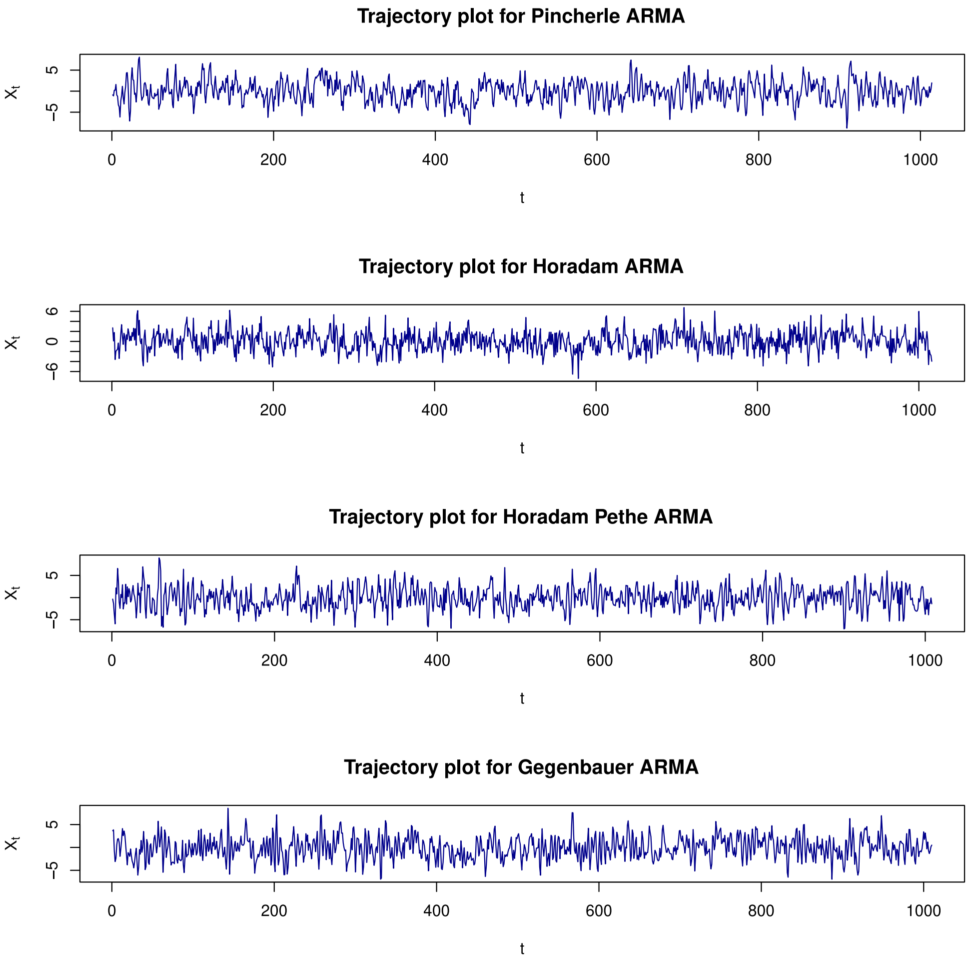


Fig. 2. Trajectory plots for Pincherle, Horadam, Horadam–Pethe and Gegenbauer ARMA processes for $p = 1, q = 0, \nu = 0.3$ and $u = 0.1$.

Remark 3.1. Taking $m = 2$ the polynomials in (3.1) reduced to Gegenbauer polynomials and $Q^{\nu}_{n,2}(u) = C^{\nu}_{n}(u)$. Moreover, the corresponding time series using the generating function of Gegenbauer polynomials namely the Gegenbauer Autoregressive Moving Average (GARMA) process is studied by Gray and Zhand in 1989 (see [32]).

Remark 3.2. The stationarity and invertibility condition for Horadam ARMA and Horadam–Pethe ARMA process is the same as the type 2 HARMA process, which is the process is stationary and invertible if all roots of $\Phi(B) = 0$ and $\Theta(B) = 0$ lies outside the unit circle and $|\nu| < 1/2$ and $0 \leq u < 1$.

The time series plots for simulated Pincherle, Horadam, Horadam–Pethe and Gegenbauer ARMA processes are given in Fig. 2. We simulated time series of size 1000 from each process. All these series have in theory infinite differencing terms. We consider only finite terms by truncating the binomial expansions of the different shift operators. For Pincherle ARMA process the relation defined in (2.8) is used, that is

$$X_t = \frac{\Theta(B)}{\Phi(B)}(1 - 3uB + B^3)^{-\nu}\epsilon_t. \tag{3.8}$$

The series $Z_t = (1 - 3uB + B^3)^{-\nu}\epsilon_t$ is generated using the simulated innovation series $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$. Further, we approximate Z_t by considering first 4 terms in the binomial expansion of $(1 - 3uB + B^3)^{-\nu}$, which is

$$\begin{aligned} Z_t = (1 - 3uB + B^3)^{-\nu}\epsilon_t &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^j\frac{(\nu)_n}{n!}\binom{n}{j}(3u)^{n-j}B^{2j+n}\epsilon_t \\ &\approx \sum_{n=0}^{4}\sum_{j=0}^{n}(-1)^j\frac{(\nu)_n}{n!}\binom{n}{j}(3u)^{n-j}\epsilon_{t-n-2j}. \end{aligned}$$

Now by generating the series Z_t Eq. (3.8) takes the following form

$$X_t = \frac{\Theta(B)}{\Phi(B)} Z_t,$$

which is nothing but the ARMA process which is simulated using the "nloptr" library in R by passing the Z_t as innovation series. Using the same approach, we simulate the Horadam, Gegenbauer and Horadam Pethe ARMA processes by taking the binomial expansion of $(1 - 2uB + B^m)^{-\nu}$, for $m = 1, 2$ and 3 respectively.

4. Parameter estimation

In this section, we introduce the Whittle quasi-likelihood estimation method for the type 1 and type 2 Humbert ARMA Processes. The Whittle quasi-likelihood technique leverages the empirical spectral density and theoretical spectral density to estimate the model parameters of the time series X_t, where $t \in \{0, 1, \ldots, n\}$ and n denotes the sample size. The estimation process involves minimizing the likelihood function. Consider the set of harmonic frequencies $\omega_j, j = 0, 1, \ldots, n/2$. These frequencies are selected to define the empirical spectral density, which plays a crucial role in the Whittle quasi-likelihood estimation. The empirical spectral density provides a representation of the distribution of frequencies in the time series data. The estimation process starts by calculating the empirical spectral density. This involves computing the periodogram, which is a commonly used estimator of the spectral density expressed as follows

$$I_x(\omega_j) = \frac{1}{2\pi}\{R(0) + \sum_{s=1}^{n-1} R(s)\cos(s\omega_j)\}, \;\; \omega_j = \frac{2\pi j}{n}, \; j = 0, 1, \ldots, n/2, \tag{4.9}$$

where $R(s) = \frac{1}{n}\sum_{i=1}^{n-s}(X_i - \bar{X})(X_{i+s} - \bar{X}), \;\; s = 0, 1, \ldots, (n-1)$, is the sample autocovariance function with sample mean $\bar{X}$. The Whittle quasi-likelihood estimation method aims to find the model parameters that minimize the discrepancy between the empirical and theoretical spectral densities that is $I_x(\omega_j)$ and $f_x(\omega_j)$ respectively. This is achieved by optimizing the likelihood function

$$W_n(\theta) = \sum_{j=1}^{n}\left(\frac{I_x(\omega_j)}{f_x(\omega_j)} + \log(f_x(\omega_j))\right)$$

where θ represents unknown parameters $\theta = (\nu, u)$, which is a row vector for both type 1 and type 2 HARMA processes. We estimate the parameters by minimizing the likelihood function $W_n(\theta)$ with respect to unknown parameter θ.

Pincherle ARMA Process: Let us assume $S = \{\nu, u : |\nu| < 1/2, 0 \le u \le 2/3\}$ and $S_0 \subset S$ is a compact set. From theorem 2.7, the spectral density of Pincherle ARMA process is

$$f_x(\omega) = \frac{\sigma^2}{2\pi}\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(8\cos^3(\omega) - 12u\cos^2(\omega) - C\cos(\omega) + D)^{-\nu},$$

where $z = e^{-\iota\omega}, \;\; C = 6 + 6u,$ and $D = 2 + 6u + 9u^2$ and empirical spectral density can be calculated using (4.9). The estimate of θ is given by

$$\hat{\theta}_n = \underset{\theta}{\operatorname{argmin}}\, W_n(\theta), \;\; \theta \in S_0.$$

Horadam and Horadam–Pethe ARMA Process: To estimate the parameters of Horadam and Horadam–Pethe ARMA processes we assume $S' = \{\nu, \; u : |\nu| < 1/2, \;\; 0 \le u \le 1\}$ and $S'_0 \subset S'$ is a compact set. From theorem 3.2 the spectral density for the Horadam process takes the following form

$$f_x(\omega) = \frac{\sigma^2}{2\pi}\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2 + 4u^2 - 4u - 4u\cos(\omega) + 2\cos(\omega))^{-\nu}, \;\; z = e^{-\iota\omega}.$$

From theorem 3.5 the spectral density for the Horadam–Pethe ARMA process has the following form

$$f_x(\omega) = \frac{\sigma^2}{2\pi}\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2 + 4u^2 - 4u(\cos(\omega) + \cos(2\omega)) + 2\cos(3\omega))^{-\nu}.$$

The estimate of θ is obtained by minimizing the likelihood

$$\hat{\theta}_n = \underset{\theta}{\operatorname{argmin}}\, W_n(\theta), \;\; \theta \in S'_0.$$

Theorem 4.1. *Assume the conditions of theorem 2.5 holds, then the Whittle quasi-likelihood estimators for the Pincherle ARMA process are consistent. That is,* $\lim_{n\to\infty}\hat{\theta}_n = \theta$ *a.s.*

Proof. To prove the consistency of the Whittle quasi-likelihood method we use the result defined by Hannan (see theorem 1 in [36]). Assume that the parameters lies in the compact space S_0. We can rewrite the spectral density of the Pincherle ARMA process defined in theorem 2.7 as, $f_x(\omega) = \frac{\sigma^2}{2\pi}K(\omega)$ and $K(\omega)$ is given as follows

$$K(\omega) = \frac{|\Theta(z)|^2}{|\Phi(z)|^2}(8\cos^3(\omega) - 12u\cos^2(\omega) - C\cos(\omega) + D)^{-\nu},$$

where $z = e^{-\iota\omega},\ C = 6+6u,$ and $D = 2+6u+9u^2$. First, we prove that the time series defined in (2.8) can be represented as $X_t = \sum_{k=0}^{\infty} a_k\epsilon_{n-k}$, where $\sum_{k=1}^{\infty} a_k^2 < \infty$ and $a_0 = 1$. The (2.8) can be stated as

$$\begin{aligned} X_t &= \left(\sum_{j=0}^{\infty}\psi_j B^j\right)\left(\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(-1)^r\frac{\Gamma(\nu+n)}{\Gamma(n+1)\Gamma n}\binom{n}{r}(3\upsilon)^{n-r}B^{n+2r}\right)\epsilon_t \\ &= \left(\sum_{j=0}^{\infty}\psi_j B^j\right)\left(\sum_{i=0}^{\infty}\rho_i B^i\right)\epsilon_t, \end{aligned}$$

where

$$\rho_i = \frac{\Gamma(\nu+i)(3u)^i}{\Gamma(\nu)\Gamma(i+1)} - \frac{\Gamma(\nu+i-2)(3u)^{i-3}}{\Gamma(\nu)\Gamma(2)\Gamma(i-2)}. \tag{4.10}$$

In the following way, X_t can be reformulated

$$X_t = \sum_{k=0}^{\infty} a_k B^k\epsilon_t = \sum_{k=0}^{\infty} a_k\epsilon_{t-k},$$

where $a_k = \sum_{s=0}^{k}\psi_{k-s}\rho_s$ and $a_0 = 1$. To prove $\sum_{k=0}^{\infty} a_k^2 < \infty$ we can show $\sum_{k=0}^{\infty}|a_k| < \infty$. The operators $\Theta(B)$ and $\Phi(B)$ can be characterized as stationary autoregressive and invertible moving average operators, respectively. Their corresponding polynomial representations $\frac{\Theta(z)}{\Phi(z)} = \sum_{j=0}^{\infty}\psi_j z^j$, where the series $\sum_{j=0}^{\infty}|\psi_j| < \infty$ for $|z| \le 1+\epsilon$. Consequently, we can deduce that the absolute values of the coefficients ψ_j decrease exponentially with increasing j, bounded by the inequality $|\psi_j| < C(1+\epsilon)^{-j}$, where C represents a constant. Also, in (4.10) for large i using Stirling's approximation ρ_i can be approximated as follows

$$\rho_i \sim i^{\nu-1}(3u)^i - (i-2)^{\nu-1}(3u)^{i-3},$$

which clearly indicates that the $\sum_{i=0}^{\infty}|\rho_i| < \infty$ for $|\nu| < 1/2$ and $0 \le u \le 2/3$. Further,

$$\sum_{k=0}^{\infty}|a_k| \le \sum_{k=0}^{\infty}\sum_{s=0}^{k}|\psi_{k-s}||\rho_s| = \sum_{s=0}^{\infty}\sum_{k=s}^{\infty}|\psi_{k-s}||\rho_s| = \sum_{s=0}^{\infty}\sum_{r=0}^{\infty}|\psi_r||\rho_s|. \tag{4.11}$$

Since, we have proved that $\sum_{i=0}^{\infty}|\rho_i|$ is finite which implies $\sum_{k=0}^{\infty}|a_k| < \infty$.

Next we show $\frac{1}{a+K(\omega)}$ is continuous for $\omega \in (-\pi, \pi)$ and for all $a > 0$. Observe that

$$\frac{1}{a+K(\omega)} = \frac{\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(8\cos^3(\omega) - 12u\cos^2(\omega) - C\cos(\omega) + D)^{\nu}}{a\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(8\cos^3(\omega) - 12u\cos^2(\omega) - C\cos(\omega) + D)^{\nu} + 1}.$$

It is easy to see that $(8\cos^3(\omega) - 12u\cos^2(\omega) - C\cos(\omega) + D)^{\nu}$ is continuous for $0 < \nu < 1/2$. Hence, $\frac{1}{a+K(\omega)}$ is continuous $\omega \in (-\pi, \pi)$, $|\nu| < 1/2$ and $0 \le u \le 2/3$ for all $a > 0$. Also, for $\theta \in S_0$ and $\omega \in (-\pi, \pi)$, the spectral density $f_x(\omega)$ is uniquely defined. Therefore, we conclude the Whittle quasi-likelihood estimators for the Pincherle ARMA process are consistent. □

Theorem 4.2. *Assuming the conditions of theorem 3.1 holds, then the Whittle quasi-likelihood estimators for the Horadam ARMA process are consistent. That is,* $\lim_{n\to\infty}\hat{\theta}_n = \theta$ *a.s.*

Proof. Assume that the parameter vector lies in the compact space S_0'. Similar to the previous theorem taking $\frac{\Theta(B)}{\Phi(B)} = \sum_{j=0}^{\infty}(\psi_j B^j)$ and $(1-2uB+B) = \sum_{k=0}^{\infty}\sum_{r=0}^{k}(-1)^r\frac{\Gamma(\nu+k)}{\Gamma(\nu)\Gamma(k+1)}(2u)^{k-r}B^k$ using (3.7) the process X_t can be expressed as follows

$$X_t = \frac{\Theta(B)}{\Phi(B)}(1-2uB+B)\epsilon_t = \sum_{j=0}^{\infty}(\psi_j B^j)\sum_{n=0}^{\infty}\sum_{r=0}^{n}(-1)^r\frac{\Gamma(\nu+n)}{\Gamma(\nu)\Gamma(n+1)}(2u)^{n-r}B^n\epsilon_t$$

$$= \sum_{j=0}^{\infty}(\psi_j B^j)\sum_{n=0}^{\infty}(\beta_n B^n),$$

where $\beta_n = \sum_{r=0}^{n}(-1)^r \frac{\Gamma(\nu+n)}{\Gamma(\nu)\Gamma(n+1)}\binom{n}{r}(2u)^{n-r}$. Further, the X_t can have following representation

$$X_t = \sum_{j=0}^{\infty} a_k \epsilon_{t-k},$$

where $a_k = \sum_{s=0}^{k}\psi_{k-s}\beta_s$. Similar to the previous theorem using Stirling's approximation, it can be easily proved that $\sum_{k=0}^{\infty}|a_k| < \infty$ and $a_0 = 1$ for $|u| \le 1$ and $|\nu| < 1/2$. Also, the spectral density for the Horadam ARMA process using (3.6) can be written as

$$f_x(\omega) = \frac{\sigma^2}{2\pi}K(\omega),$$

where $K(\omega) = \frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2+4u^2-4u-4u\cos(\omega)+2\cos(\omega))^{-\nu},\ z = e^{-\iota\omega}$. To prove that $\frac{1}{K(\omega)+a}$ is continuous for $a > 0$ we have

$$\frac{1}{K(\omega)+a} = \frac{\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2+4u^2-4u-4u\cos(\omega)+2\cos(\omega))^{\nu}}{a\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2+4u^2-4u-4u\cos(\omega)+2\cos(\omega))^{\nu}+1},$$

here $(2+4u^2-4u-4u\cos(\omega)+2\cos(\omega))^{\nu}$ is continuous for $0 < \nu < 1/2$ and $|u| < 1$ implying $\frac{1}{K(\omega)+a}$ is continuous for $|\nu| < 1/2$ and $|u| \le 1$. Moreover, the parameter space $\theta \in S_0'$ defines the spectral density uniquely. These conditions satisfy the results given by Hannan [36] and hence prove the consistency. □

Theorem 4.3. *Assuming the conditions of theorem 3.1 holds, then the Whittle quasi-likelihood estimators for the Horadam–Pethe ARMA process are consistent. That is,* $\lim_{n\to\infty}\hat{\theta}_n = \theta$ *a.s.*

Proof. Assume that the parameter vector lie in the compact space S_0'. The Horadam–Pethe ARMA process X_t has the following moving average representation

$$X_t = \left(\sum_{j=0}^{\infty}\psi_j B^j\right)\left(\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(-1)^r\frac{\Gamma(\nu+n)}{\Gamma(n+1)\Gamma n}\binom{n}{r}(2u)^{n-r}B^{n+2r}\right)\epsilon_t = \left(\sum_{j=0}^{\infty}\psi_j B^j\right)\left(\sum_{i=0}^{\infty}\zeta_i B^i\right)\epsilon_t,$$

where

$$\zeta_i = \frac{\Gamma(\nu+i)}{\Gamma(\nu)\Gamma(i+1)}(2u)^i - \frac{\Gamma(\nu+i-2)(2u)^{i-3}}{\Gamma(\nu)\Gamma(2)\Gamma(i-2)}. \tag{4.12}$$

We can rewrite X_t as follows

$$X_t = \sum_{k=0}^{\infty}a_k B^k\epsilon_t = \sum_{k=0}^{\infty}a_k\epsilon_{t-k},$$

where $a_k = \sum_{s=0}^{k}\psi_{k-s}\zeta_s$. The proof for $\sum_{k=0}^{\infty}|a_k| < \infty$ is similar to theorem 4.1. Also, it can be proved that $\frac{1}{K(\omega)+a}$ is continuous for $a > 0$, here it can be expressed as

$$\frac{1}{K(\omega)+a} = \frac{\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2+4u^2-4u(\cos(\omega)+\cos(2\omega))+2\cos(3\omega))^{\nu}}{a\frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2+4u^2-4u(\cos(\omega)+\cos(2\omega))+2\cos(3\omega))^{\nu}+1}.$$

Again using the same argument as theorem 4.1 it can be proved that $\frac{1}{K(\omega)+a}$ is continuous for $a > 0$ and the parameter vector θ defines the spectral density uniquely. These all conditions satisfy the results defined by Hannan (1973) [36] hence prove the consistency of the Whittle quasi-likelihood estimators. □

Remark 4.1. In order to establish the normality of the estimators, we exclude the scenario where the spectral density is unbounded that is at u = 0 and u = 1. For a detailed examination of the unbounded spectral density case at these points, refer to the paper by Fox and Taqqu [37]. In the next results, $\frac{\partial \log K(\omega)}{\partial\theta}$ is 2×1 column vector which represents the derivative of $\log K(\omega)$ with respect to both the parameters ν and u and $\left\{\frac{\partial \log K(\omega)}{\partial\theta}\right\}\left\{\frac{\partial \log K(\omega)}{\partial\theta}\right\}'$ will be a 2×2 matrix, where $\left\{\frac{\partial \log K(\omega)}{\partial\theta}\right\}'$ represents the transpose of a 2×1 column vector.

Theorem 4.4. *Let the Whittle quasi-likelihood estimate for the Pincherle ARMA process be defined as follows*

$$\hat{\theta}_n = \underset{\theta}{\operatorname{argmin}}\ W_n(\theta),\ \ \theta \in \Omega_0,$$

where $\Omega_0 \subset \Omega = \{\nu, u : |\nu| < 1/2, 0 < u < 2/3\}$ is a compact set then for the Whittle quasi-likelihood estimators for Pincherle ARMA process the $n^{-1/2}(\hat{\theta}_n - \theta) \sim \mathcal{N}(0, W^{-1})$, where W represents the variance–covariance matrix having the following form

$$W = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left\{\frac{\partial \log K(\omega)}{\partial\theta}\right\}\left\{\frac{\partial \log K(\omega)}{\partial\theta}\right\}' d\omega.$$

Proof. Using the results defined by Hannan (1973) (see theorem 2 in [36]) we need to verify the following conditions to check the asymptotic normality of the parameters.

(a) $K(\omega) > 0$ for all $\omega \in (-\pi, \pi)$ and $\theta \in \Omega_0$.
(b) $K(\omega)$ twice differentiable of parameters ν and u.
(c) The time series defined in (2.8) can be written as $X_t = \sum_{k=0}^{\infty} a_k\epsilon_{t-k},\ \sum_{k=0}^{\infty} a_k < \infty$ and $a_0 = 1$.

The condition (a) can be proved by rewriting $K(\omega)$ as follows

$$\begin{aligned}
K(\omega) &= \frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2 + 9u^2 + 2\cos(3\omega) - 6u(\cos(\omega) + \cos(2\omega)))^{-\nu}\\
&= \frac{|\Theta(z)|^2}{|\Phi(z)|^2}((2 + 2\cos(3\omega) + [3u - \{\cos(\omega) + \cos(2\omega)\}]^2) - [\cos(\omega) + \cos(2\omega)]^2)^{-\nu}\\
&= \frac{|\Theta(z)|^2}{|\Phi(z)|^2}\left(4\cos^2\left[\frac{3\omega}{2}\right] + [3u - \{\cos(\omega) + \cos(2\omega)\}]^2 - 4\cos^2\left[\frac{3\omega}{2}\right]\cos^2\left[\frac{\omega}{2}\right]\right)^{-\nu}\\
&= \frac{|\Theta(z)|^2}{|\Phi(z)|^2}\left(4\cos^2\left[\frac{3\omega}{2}\right]\sin^2\left[\frac{\omega}{2}\right] + \left[3u - 2\cos\left(\frac{3\omega}{2}\right)\cos\left(\frac{\omega}{2}\right)\right]^2\right)^{-\nu}.
\end{aligned}$$

This indicates that $K(\omega) > 0$ for $0 < u < 2/3$ and $|\nu| < 1/2$. The condition (b) can be easily verified as the function does not have any singularity for $0 < u < 2/3$ hence continuous and differentiable. Moreover, the condition (c) is proven in the theorem 4.1. Thus the Whittle quasi-likelihood estimates for $|\nu| < 1/2$ and $0 < u < 2/3$ are asymptotically normal. □

Theorem 4.5. *Let the Whittle quasi-likelihood estimate for the Horadam ARMA process be defined as follows*

$$\hat{\theta}_n = \underset{\theta}{\operatorname{argmin}}\ W_n(\theta), \theta \in \Omega_0',$$

where $\Omega_0' \subset \Omega' = \{\nu, u : |\nu| < 1/2, 0 < u < 1\}$ is a compact set then for the Whittle quasi-likelihood estimators for Horadam ARMA process the $n^{-1/2}(\hat{\theta}_n - \theta) \sim \mathcal{N}(0, W^{-1})$, where W represents the variance–covariance matrix having the following form

$$W = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left\{\frac{\partial \log K(\omega)}{\partial\theta}\right\}\left\{\frac{\partial \log K(\omega)}{\partial\theta}\right\}' d\omega.$$

Proof. This can be proved again using the conditions defined in theorem 4.4. The $K(\omega)$ for the Horadam ARMA process can be written as follows

$$\begin{aligned}
K(\omega) &= \frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2 + 4u^2 - 4u - 4u\cos(\omega) + 2\cos(\omega))^{-\nu}\\
&= \frac{|\Theta(z)|^2}{|\Phi(z)|^2}\left(4\cos^2\left[\frac{\omega}{2}\right]\sin^2\left[\frac{\omega}{2}\right] + \left[2u - 2\cos^2\left(\frac{\omega}{2}\right)\right]^2\right)^{-\nu},
\end{aligned}$$

From this expression, it is evident that $K(\omega)$ is strictly greater than zero and that $K(\omega)$ is twice differentiable for all values of $\theta \in \Omega_0'$. Furthermore, the moving average representation for the Horadam process is provided in theorem 4.2. By establishing this moving average representation, we have successfully proven the desired result. □

Theorem 4.6. *Let the Whittle quasi-likelihood estimate for the Horadam–Pethe ARMA process be defined as follows*

$$\hat{\theta}_n = \underset{\theta}{\operatorname{argmin}}\ W_n(\theta), \theta \in \Omega_0',$$

where $\Omega_0' \subset \Omega' = \{\nu, u : |\nu| < 1/2, 0 < u < 1\}$ is a compact set then for the Whittle quasi-likelihood estimators for Horadam–Pethe ARMA process the $n^{-1/2}(\hat{\theta}_n - \theta) \sim \mathcal{N}(0, W^{-1})$, where W represents the variance–covariance matrix having *the following form*

$$W = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left\{\frac{\partial \log K(\omega)}{\partial\theta}\right\}\left\{\frac{\partial \log K(\omega)}{\partial\theta}\right\}' d\omega.$$

Proof. The $K(\omega)$ for the Horadam–Pethe ARMA process can be written as follows

$$\begin{aligned} K(\omega) &= \frac{|\Theta(z)|^2}{|\Phi(z)|^2}(2+4u^2-4u(\cos(\omega)+\cos(2\omega))+2\cos(3\omega))^{-\nu} \\ &= \frac{|\Theta(z)|^2}{|\Phi(z)|^2}\left(4\cos^2\left[\frac{3\omega}{2}\right]\sin^2\left[\frac{\omega}{2}\right]+\left[2u-2\cos\left(\frac{3\omega}{2}\right)\cos\left(\frac{\omega}{2}\right)\right]^2\right)^{-\nu}. \end{aligned}$$

This expression indicates that $K(\omega)$ is greater than zero and possesses two continuous derivatives for all values of $\theta \in \Omega_0'$. Furthermore, theorem 4.3 provides the moving average representation for the Horadam–Pethe ARMA process. Thus, by establishing the aforementioned moving average representation and considering the expression for $K(\omega)$, the desired result has been successfully demonstrated. □

5. Simulation study for Pincherle ARMA process and its application

In order to evaluate the efficacy of the parameter estimation techniques introduced, we employ simulated data. The simulation study serves as a valuable tool in the evaluation of parameter estimation techniques. It enables us to empirically examine the accuracy and reliability of the estimation method by comparing the estimated parameters to the actual parameters obtained from synthetic data. The use of simulated data provides several advantages for performance assessment. Firstly, it allows us to create controlled experiments where the true parameters are known, facilitating a direct comparison. Secondly, simulations provide the flexibility to generate data with specific properties or characteristics, allowing us to investigate the behavior of estimation techniques under different scenarios. Finally, by repeating the simulation process multiple times, we can obtain statistical measures of performance, such as average estimation error, providing a more comprehensive evaluation. By conducting a simulation study, we can gather empirical evidence that sheds light on the effectiveness of these statistical techniques. Through the utilization of appropriate simulation methods, we generate a synthetic time series based on an initial set of parameters. Subsequently, we apply the defined parameter estimation techniques to the simulated series, aiming to estimate the underlying parameters.

By comparing the estimated parameters with the actual parameters used in the simulation, we can assess the performance of the applied techniques. This comparison serves as a fundamental metric to evaluate the accuracy and reliability of the estimation methods. If the estimated parameters closely align with the actual parameters, it indicates that the techniques effectively capture the underlying characteristics of the data. On the other hand, significant discrepancies between the estimated and actual parameters may indicate limitations or potential areas for improvement in the estimation techniques.

The data from the Pincherle ARMA process is simulated by first simulating the i.i.d. innovations $\epsilon_t \sim \mathcal{N}(0,\sigma^2)$. The simulation and estimation study is done using R. Now we use the relation defined in (2.8) as,

$$X_t = \frac{\Theta(B)}{\Phi(B)}(1-3uB+B^3)^{-\nu}\epsilon_t. \tag{5.13}$$

The series $Z_t = (1-3uB+B^3)^{-\nu}\epsilon_t$ is generated using the simulated innovation series ϵ_t in (5.13). The generation of the series is done by taking the binomial expansion of $(1-3uB+B^3)^{-\nu}$ up to 4 terms, which is given as follows

$$\begin{aligned} Z_t = (1-3uB+B^3)^{-\nu}\epsilon_t &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^j\frac{(\nu)_n}{n!}\binom{n}{j}(3u)^{n-j}B^{2j+n}\epsilon_t \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^j\frac{(\nu)_n}{n!}\binom{n}{j}(3u)^{n-j}\epsilon_{t-n-2j}. \end{aligned}$$

Now by generating the series Z_t the (5.13) takes the following form

$$X_t = \frac{\Theta(B)}{\Phi(B)}Z_t,$$

which is nothing but the ARMA process which is simulated using the inbuilt R library by passing the Z_t as innovation series. Further to check the effectiveness of the model the parameter estimation is done on the simulated series. By assuming two different combinations of the initial set of parameters that is $u = 0.2$, $\nu = 0.4$ and $u = 0.1$, $\nu = 0.3$ the series Z_t is generated and the two series Pincherle ARMA(1, 0.2, 0.15, 0) and Pincherle ARMA(1, 0.3, 0.1, 0) is generated using the above-defined procedure. The results of parameter estimation using the Whittle quasi-likelihood approach are summarized in Table 1.

Table 1
Actual and estimated parameter values for two different choices of parameters estimated by the Pincherle ARMA process using the Whittle quasi-likelihood approach.

	Actual	Estimated
Case 1	$\nu = 0.2$, $u = 0.15$	$\hat{\nu} = 0.21$, $\hat{u} = 0.13$
Case 2	$\nu = 0.3$, $u = 0.1$	$\hat{\nu} = 0.32$, $\hat{u} = 0.07$

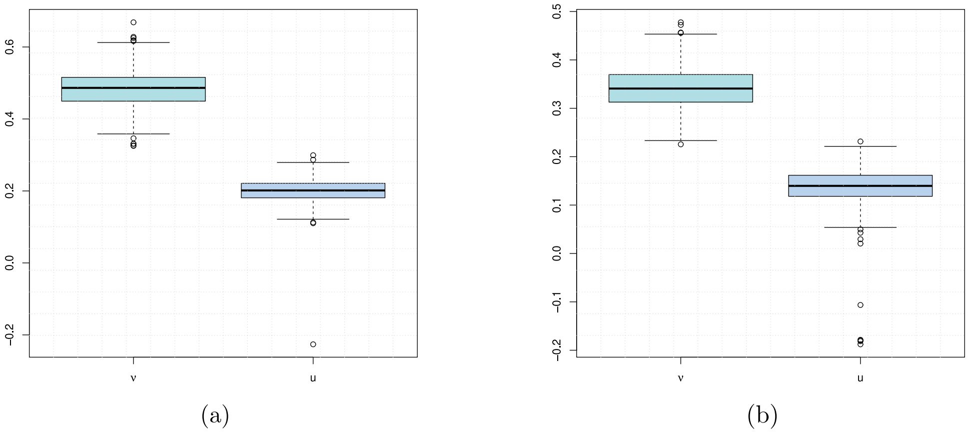


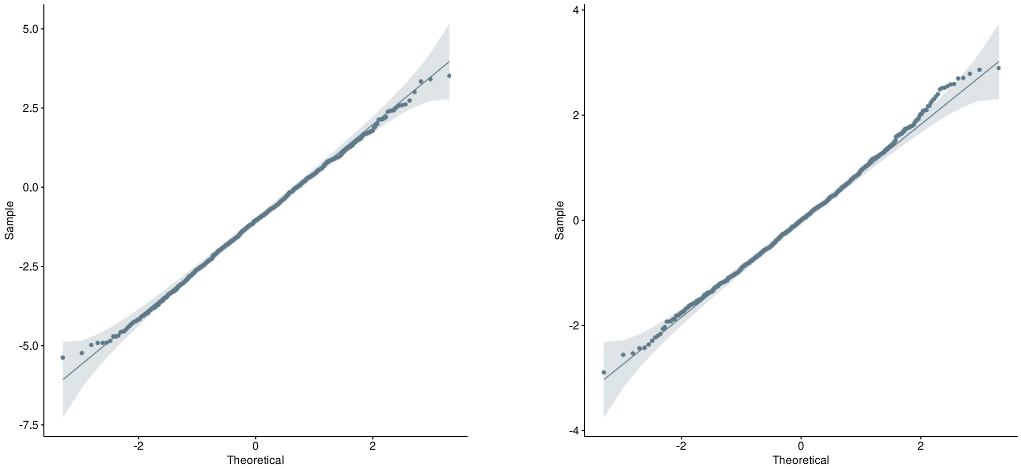
Fig. 3. Box plot of parameters using 1000 samples for $\nu = 0.45$ and $u = 0.2$ (left) and for $\nu = 0.35$ and $u = 0.1$ (right) based on Whittle quasi-likelihood approach.

From the above Table 1 it is clearly shown that the estimate of ν and u from the Whittle quasi-likelihood approach is good. In order to assess the effectiveness of the Whittle quasi-likelihood technique based on empirical spectral density, we construct box plots for different parameters. To create these box plots, we perform a simulation of 1000 series, assuming fixed values for the parameters $\nu = 0.4$ and $u = 0.1$. Each simulated series consists of 1000 observations. Using the Whittle quasi-likelihood estimation method, we estimate the parameters ν and u from each simulated series. By repeating this process for all 1000 simulated series, we obtain a distribution of estimated parameters for each parameter. we construct box plots. Each box plot represents the variability and central tendency of the estimated parameters across the 1000 simulations. The box plot displays the median value, the interquartile range (IQR), and any potential outliers for each parameter. The box plots provide a comprehensive visual representation of the estimated parameters' variability and the overall performance of the Whittle quasi-likelihood technique. Fig. 3 displays the box plots for the estimated parameters obtained from each simulation, allowing for a visual assessment of their distribution and variability.

To assess the asymptotic normality of the estimates, we performed a comprehensive simulation analysis, which consists of 1000 datasets. Each dataset consisted of a sequence of length 1000, with the parameter values set at $\nu = 0.45$ and $\nu = 0.2$. The estimated parameters were denoted as $\hat{\nu}$ and $\hat{u}$. In order to visualize the results, we constructed Q-Q plots for the standardized differences, namely $\sqrt{n}(\hat{\nu} - \nu)$ and $\sqrt{n}(\hat{u} - u)$. These plots provide a graphical representation of the comparison between the observed quantiles and the theoretical quantiles of the standard normal distribution. The Q-Q plots for $\sqrt{n}(\hat{d} - d)$ and $\sqrt{n}(\hat{u} - u)$ are depicted in Fig. 4.

Furthermore, we demonstrate the normality of the estimated parameters by conducting the Shapiro–Wilk normality test. The resulting p-values for both parameters, ν and u are found to be greater than 0.05. This indicates that the variables $\sqrt{n}(\hat{d} - d)$ and $\sqrt{n}(\hat{u} - u)$ follow a normal distribution.

Real Data Application: We conduct an analysis using the Pincherle ARMA model on the daily percentage yield data of Spain's 10-year treasury bond. The data covers the period from October 7th, 2011 to June 7th, 2018. The Pincherle ARMA model is compared with other existing models, namely autoregressive integrated moving average (ARIMA), autoregressive fractionally integrated moving average (ARFIMA), autoregressive tempered fractionally integrated moving average (ARTFIMA), and Gegenbauer autoregressive moving average (GARMA). The daily percentage yield is a commonly used metric in financial markets to measure the return on investment for fixed-income securities, such as government bonds. It represents the change in the bond's yield, expressed as a percentage, from one day to the next. This yield data is of particular interest to investors, traders, and policymakers as it provides insights into the performance and market dynamics of long-term government debt. By analyzing this dataset, we can gain valuable insights into the behavior and patterns of Spain's 10-year treasury bond yield over the given time frame. The objective of applying various models, including the Pincherle ARMA model, is to accurately capture and forecast the future movements and trends in the bond yield, thereby assisting in decision-making processes related to investment strategies, risk management, and financial planning. The trajectory plot for the introduced dataset is given in Fig. 5



(a) (b)

Fig. 4. Q-Q plots using 1000 samples for $\sqrt{n}(\hat{\nu} - \nu)$ (left) and $\sqrt{n}(\hat{u} - u)$ (right)

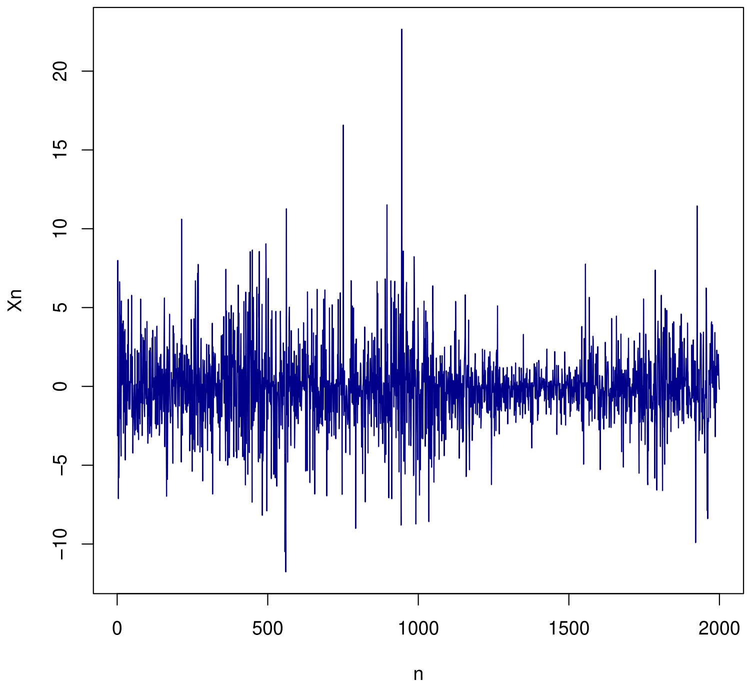


Fig. 5. Trajectory plot for Spain's 10-year treasury yield series dataset.

To evaluate the accuracy of these models, we utilize two common metrics: the root mean square error (RMSE) and the mean absolute error (MAE). These metrics provide measures of the deviation between the predicted values and the actual values of the bond yield data.

Table 2 presents the results of the accuracy assessment for each model. From the table, we observe that the RMSE of the Pincherle ARMA process is lower than that of the other models. This indicates that the Pincherle ARMA model performs better in terms of capturing the overall variability in the bond yield data. However, it is worth noting that the MAE for the Pincherle ARMA process is slightly higher compared to the ARFIMA and ARTFIMA models. The MAE represents the average magnitude of the errors made by the models, regardless of their direction. In this case, the slightly higher MAE suggests that the Pincherle ARMA model may have a slightly larger average error in its predictions when compared to the ARFIMA and ARTFIMA models.

Table 2
The goodness-of-fit measures of different models using RMSE and MAE metrics.

Models	RMSE	MAE
ARIMA	2.588	1.972
ARFIMA	2.476	1.831
ARTFIMA	2.463	1.801
GARMA	2.584	1.962
Pincherle ARMA	2.406	1.840

Based on these findings, we can conclude that the Pincherle ARMA process demonstrates good accuracy in predicting Spain's 10-year treasury bond yield when compared to the other models evaluated in this study. This suggests that the Pincherle ARMA model can be a valuable tool in other fields where accurate forecasting is required. It may be particularly useful in financial applications where predicting bond yields is crucial for investment decisions and risk management

6. Conclusions

We study the general Humbert polynomials based autoregressive moving average called here HARMA (p, ν, u, q) time series models. Initially, type 1 HARMA (p, ν, u, q) process defined in (2.2) and its stationarity and invertibility conditions are derived. We also compute the spectral density of the above process. For $m = 3$ in (2.8), we focus on a particular case Pincherle ARMA (p, ν, u, q) process, by obtaining the spectral density and also prove that for $u = 0$ and $0 < \nu < 1/2$, the process also exhibits seasonal long memory property. In the subsequent section, we study similar properties of particular cases of type 2 HARMA (p, ν, u, q) process defined in (3.2) for $m = 1$ and $m = 3$ named as Horadam ARMA process and Horadam–Pethe ARMA process respectively. We also provide the Whittle quasi-likelihood estimation method to estimate the parameters of the HARMA process. We also provide the results for the consistency and normality of the estimators. The simulation study on 1000 series each of length 1000 is performed and real data of Spain's 10-year treasury bond daily percentage yield is used to show the application of the Pincherle ARMA model.

Further, we believe that the proposed time series models will be helpful in the modeling of real-world data from other fields. Also, the estimation techniques for example, minimum contrast estimation [38,39] will be applied for the discussed models. This technique estimates the parameters by minimizing the spectral density and empirical spectral density of the process. Maximum likelihood estimation is the particular case of minimum contrast estimation. Apart from this, Pincherle, Horadam and Horadam–Pethe random fields will be interest of study on the line of Gegenbauer random fields [40]. Moreover, one can study the tempered versions of Humbert, Pincherle, Horadam and Horadam–Pethe ARMA processes similar to Sabzikar et al. [15]. Due to the presence of singularities in the spectral density of the introduced models, the estimation techniques become more challenging to implement. We may need to employ advanced numerical methods, such as numerical optimization algorithms or robust estimation techniques, to address the challenges posed by the singularities. However, it is important to note that the introduced processes in this study assume constant volatility. Consequently, these models may not be suitable for capturing heteroscedastic data that exhibit persistence in the conditional variance of the innovation term. In future research, an interesting avenue to explore would be extending the concept of the Humbert processes to incorporate volatility modeling, specifically by developing a Humbert-GARCH process.

Declaration of competing interest

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Data availability

Data will be made available on request.

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References

[1] Granger CWJ, Joyeux R. An introduction to long-memory time series models and fractional differencing. J Time Series Anal 1980;1:15–29.
[2] Hosking JRM. Fractional differencing. Biometrika 1981;68(1):165–76.
[3] Yule GU. Why do we sometimes get nonsense-correlations between time series? A study in sampling and the nature of time series. J R Stat Soc 1926;89:1–64.
[4] Slutsky E. The sommation of random causes as the source of cyclic processes. Econometrics 1937;5:105–46.
[5] Wold H. A study in the analysis of stationary time series. Stockholm: Almgrist & Wiksell; 1938.
[6] Box GEP, Jenkins GM. Time series analysis: forecasting and control. San Francisco: Holden-Day; 1976.
[7] Giraitis L, Koul HL, Surgailis D. Large sample inference for long memory processes. London: Imperial College Press; 2012.
[8] Anh VV, Knopova VP, Leonenko NN. Continuous-time stochastic processes with cyclical long-range dependence. Aust N Z J Stat 2004;46(2):275–96.
[9] Castro TB, Rachinger H. Aggregation of seasonal long-memory processes. Econ Stat 2021;17:95–106.
[10] Dissanayake GS, Peiris MS, Proietti T. Fractionally differenced gegenbauer processes with long memory: a review. Statist Sci 2018;33:413–26.
[11] Ivanov AV, Leonenko N, Ruiz-Medina MD, Savich IN. Limit theorems for weighted nonlinear transformations of Gaussian stationary processes with singular spectra. Ann Probab 2013;41(2):1088–114.
[12] Chung C-F. Estimating a generalized long memory process. J Appl Econometr 1996;73(1):237–59.
[13] Kumar A, Vellaisamy P. Inverse tempered stable subordinators. Statist Probab Lett 2014;103:134–41.
[14] Rosinski J. Tempering stable processes. Stoch Process Appl 2007;117:677–707.
[15] Sabzikar F, McLeod AI, Meerschaert MM. Parameter estimation for ARTFIMA time series. J Statist Plann Inference 2019;200:129–45.
[16] Torricelli L, Cerioli A. Tempered positive Linnik processes and their representations. Electron J Stat 2022;16(2):6313–47.
[17] Zheng M, Karniadakis GE. Numerical methods for SPDEs with tempered stable processes. SIAM J Sci Comput 2015;37(3).
[18] Grabchak M. Tempered stable distributions: stochastic models for multiscale processes. New York: Springer; 2016.
[19] Baeumer B, Meerschaert MM. Tempered stable Lévy motion and transient super-diffusion. J Comput Appl Math 2010;233:2438–48.
[20] Reisen VA, Rodrigues AL, Palma W. Estimation of seasonal fractionally integrated processes. Comput Statist Data Anal 2006;50(2):568–82.
[21] Giraitis L, Hidalgo J, Robinson PM. Gaussian estimation of parametric spectral density with unknown pole. Ann Statist 2001;29(4):987–1023.
[22] Thawonmas R, Damrongthammasakul S. Sales forecasting in fashion retail using ARIMA models. Int J Adv Comput Sci Appl 2019;10(1):77–81.
[23] Can M, Gucuyener E, Bostanci A. Time series analysis of atmospheric pollutants using ARMA models. Fresenius Environ Bull 2019;28(8):6107–15.
[24] Humbert P. Some extensions of Pincherle's polynomials. Proc Edinb Math Soc 1920;39(1):21–4.
[25] Gould HW. Inverse series relations and other expansions involving Humbert polynomials. Duke Math J 1965;32(4):697–711.
[26] Milovanovic GV, Djordjevic GB. On some properties of Humbert's polynomials. Fibonacci Quart 1987;25:356–60.
[27] Srivastava H, Manocha H. Treatise on generating functions. New York: John Wiley & Sons; 1984.
[28] Pincherle S. Una nuova estensione delle funzioni sferiche. In: Memorie Della Accademia Reale Di Bologna (in Italian), I. 1891, p. 337–69.
[29] Abramowitz M, Stegun IA. In: Handbook of mathematical functions with formulas, graphs and mathematical tables, New York: Dover; 1992.
[30] Bartle Robert G, Donald R Sherbert. Introduction to real analysis. Vol. 2, New York: John Wiley; 2000.
[31] Gamelin TW. Complex analysis. New York: Springer; 2001.
[32] Gray HL, Zhang NF, Woodward WA. On generalized fractional processes. J Time Series Anal 1989;10:233–57.
[33] Baker B. On the relation between Pincherie's polynomials and the hypergeometric function. Proc Edinb Math Soc 1921;39(4):58–62.
[34] Horadam AF. Gegenbauer polynomials revisited. Fibonacci Quart 1985;23(4):294–9.
[35] Horadam AF, Pethe S. Polynomials associated with Gegenbauer polynomials. Fibonacci Quart 1981;19(5):393–8.
[36] Hannan EJ. The asymptotic theory of linear time series models. J Appl Probab 1973;10(01):130–45.
[37] Fox R, Taqqu MS. Large sample properties of parameter estimates for strongly dependent stationary Gaussian time series. Ann Statist 1986;14(2):517–32.
[38] Anh VV, Leonenko NN, Sakhno LM. On a class of minimum contrast estimators for fractional stochastic processes and fields. J Statist Plann Inference 2004;123:161–85.
[39] Anh VV, Leonenko NN, Sakhno LM. Minimum contrast estimation of random processes based on information of second and third orders. J Statist Plann Inference 2007;137(4):1302–31.
[40] Espejo RM, Leonenko N, Ruiz-Medina MD. Gegenbauer random fields. Random Oper Stoch Equ 2014;22(1):1–16.