Cardiff University

PhD Project 2018/2022

Lie Groups and Twisted K-Theory

Samuel Richardson
C1408294

Supervised by:
Dr. Ulrich Pennig

December, 2022

Contents

Abstract

In this thesis we will build to a result that has applications in C*-algebras and twisted K-theory. We aim to understand the behaviour of exponential functors and the cohomology theories their target categories induce. We will use the Weyl map, K-theory, and the suspension-loop adjunction in order to achieve this goal.

We will begin by acquainting ourselves with fibre bundles and a few key theorems and definitions that we will make heavy use of later due to the role fibre bundles play in defining the 0^{th} complex topological K-theory group. We will discuss a few important functors, adjunctions, and characteristic classes. We will also give a description of the cohomology ring of any flag manifold $F_n(\mathbb{C}^k)$ as a quotient ring of the polynomial ring with n generators. We will also begin to understand generalised cohomology theories.

Exponential functors are a particular family of monoidal functors between strict symmetric monoidal categories. We will show that each of these functors induce a family of natural transformations and that the suspension isomorphisms from the 0^{th} degree to the 1^{st} degree commute with the relevant natural transformations.

The source category of an exponential functor is always a category that we will call $\mathscr{C}_{\oplus}$ and the cohomology theory it induces is connective K-theory. We will investigate the effect an exponential functor has on a vector bundle.

We will describe the Weyl map and we will discover that the class of this map in K-theory corresponds to a sum of tensor products of certain formal differences of line bundles with circle components. Finally it will be shown that the class of the Weyl map in our more exotic cohomology theories corresponds to a very similar class where we have instead taken a formal quotient of vector bundles.

Acknowledgements

I would like to extend my deepest thanks to my incredibly knowledgeable and insightful supervisor Dr. Ulrich Pennig for his guidance and patience with me throughout this project, I hope there is opportunity to work with him again wherever the future ends up taking me.

I would like to show my appreciation to the good people of Carmarthenshire County Council, particularly the committee for the Thomas and Elizabeth Williams Scholarship for providing me with the funding to cover my university fees.

Thank you to my family for their love and support, particularly to my parents, my grandmother and my late grandfather for their additional financial support which has allowed me to stay in Cardiff.

I would also like to acknowledge my good friend and housemate Kristian Skoczek, and all my colleagues and friends past and present within the Cardiff GAPT group and mathematics department for putting up with me and for all the interesting conversations and fun experiences we've had over the years.

1 Introduction

Exponential functors, as studied in [29], are monoidal functors from the category of complex inner product vector spaces with unitary isomorphisms equipped with the direct sum to a subcategory of the category of complex inner product vector spaces with unitary isomorphisms equipped with the tensor product. The source and target categories of an exponential functor are both equivalent to strict symmetric monoidal categories so, as we will see, we can construct a cohomology theory from each of these categories and hence investigate the natural transformations induced by exponential functors.

The determinant functor and its powers are examples of exponential functors. The induced maps on classifying spaces correspond to classical twists in twisted K-theory. In this context the power is known as the level. The classical twists, as introduced by Atiyah and Segal [2], correspond to elements in the 3rd cohomology. Indeed, powers of the determinant give rise to a map:

$$\mathrm{SU}(n) \to \mathrm{SU} \simeq BB\mathrm{U}_{\oplus} \to BB\mathrm{U}(1)$$

and since $BB\mathrm{U}(1) \simeq K(\mathbb{Z}, 3)$, an Eilenberg-MacLane space, we therefore obtain a class in $[\mathrm{SU}(n),\ K(\mathbb{Z},\ 3)] \cong H^3(\mathrm{SU}(n);\ \mathbb{Z})$. Twisted K-theory with classical twists and their connection to representations of loop groups have been studied intesively by Freed, Hopkins, and Teleman [14].

We will discuss this family of examples, but our main contribution is that we will also look at more general exponential functors that provide twists with non-trivial contributions from higher cohomology groups. For such a functor F, we obtain a similar map:

$$\mathrm{SU}(n) \to \mathrm{SU} \simeq BB\mathrm{U}_{\oplus} \to BB\mathrm{U}_{\otimes}[\frac{1}{d}]$$

where $d = dim(F(\mathbb{C}))$.

In a different context similar higher twists have also been studied by Teleman [34].

The work of Dadarlat and Pennig [7] provides a link to bundles of C*-algebras. The bundles they investigate are classified up to isomorphism by the same cohomology theories as we achieve here from the target category of an exponential functor. Work by Evans and Pennig [12] consider the twists in this operator-alegbraic setting but lead into the work in this thesis by conjecturing the purely topological angle that we will follow.

We will be investigating the behaviour of a map called the Weyl map, composing it with some useful inclusions, and later with induced maps from exponential functors to determine the classes it represents in our various cohomology theories.

$$\begin{aligned} W : \mathrm{SU}(n)/\mathbb{T} \times \mathbb{T} &\to \mathrm{SU}(n) \\ ([g], Z) &\mapsto gZg^{-1} \end{aligned}$$

The Weyl map and it's connection to twisted K-theory has been studied using the language of gerbes by Becker, Murray, and Stevenson [4]. It is hoped but not known that the Weyl map induces an injection W^*: $h^1_{\otimes}(\mathrm{SU}(n)) \to h^1_{\otimes}(\mathrm{SU}(n).\mathbb{T} \times \mathbb{T})$ in the cohomology theory induced by any target category of an exponential functor.

We will make extensive use of a cohomology theory known as K-theory which has Bott periodicity, that is, for any topological space X and integer n, there is an isomorphism $K^n(X) \cong K^{n+2}(X)$. Therefore, we only need to define two of the abelian groups explicitly: $K^0(X)$ is the Grothendiek completion of the monoid of isomorphism classes of vector bundles over X, and $K^{-1}(X)$ is the quotient of the topological group of invertible matrices with complex valued functions on X as entries, equipped with the equivalence relation $S \sim diag(S, 1)$, by the normal subgroup that is the connected component of the identity. An extensive introduction to K-theory can be found in the book *Complex Topological K-Theory* by Efton Park [28].

We will use the Leray-Hirsch Theorem and the splitting principle in order to discover the nature of the cohomology ring of any finite complex flag manifold $H^*(F_n(\mathbb{C}^k;\, \Bbbk))$ as it is of particular note that $\mathrm{SU}(n)/\mathbb{T}$ is homeomorphic to the complete flag manifold $F_n(\mathbb{C}^n)$; that homeomorphism sends the canonical tautological complex line bundles over $F_n(\mathbb{C}^n)$ to their counterpart canonical complex line bundles over $\mathrm{SU}(n)/\mathbb{T}$, and the Chern classes of these complex line bundles are the generators of the cohomology ring.

The inclusion $\mathrm{SU}(n) \hookrightarrow \mathrm{SU}(\infty) \hookrightarrow \mathrm{U}$ allows us to describe the class of the Weyl map in K-theory thanks to the isomorphism $K^1(X) \cong [X, \mathrm{U}]$. We will show using the Künneth formula isomorphism and the suspension-loop adjunction that there is a representative of $[W]$ that is given by the sum of the complex line bundles over $\mathrm{SU}(n)/\mathbb{T}$ each tensored with a generator of $K^1(\mathbb{S}^1)$ where $\mathbb{S}^1$ is the circle corresponding to each line bundle that exists as a subspace of $\mathbb{T}$.

Armed with the knowledge of where our class exists in K-theory, we will then investigate the effect of exponential functors. We will explain how to construct cohomology theories from strict symmetric monoidal categories and natural transformations of cohomology theories from functors between such categories. Using this method we will show that from the source category of an exponential functor we can construct a cohomology theory known as connective K-theory and from the target category, more exotic cohomology theories that can be described using an integer fixed by the specific exponential functor $d = dim(F(\mathscr{C}))$.

To have a better understanding of the resulting class in our exotic cohomology theories, we can use a ring homomorphism called the Chern character: ch: $K^0(X) \to H^0_{per}(X;\, \mathbb{Q})$ to derive a natural isomorphism of cohomology theories $K^*(X) \otimes \mathbb{Q} \to H^*_{per}(X;\, \mathbb{Q})$ and similar natural transformations of cohomology theories we will call the logarithmic Chern characters $h^*_{\otimes}(X) \otimes \mathbb{Q} \to H^*_{per}(X;\, \mathbb{Q})$. We can then investigate the resulting automorphism in the periodic cohomology with rational coefficients, hopefully resulting in a natural automorphism of cohomology theories induced by the exponential functor.

Chapter 2 will deal with introducing the idea of fibre bundles alongside a handful of examples and related properties that will be of great use to us later. The pullback bundle construction in particular will be a heavy lifter in this project as it is an easy method of constructing fibre bundles over complicated spaces if we have a clear map to the simpler base space of a map we already know to be a fibre bundle.

Chapter 3 will contain an in depth proof of the Leray-Hirsch theorem, a result that allows us to construct a module isomorphism from a fibre bundle provided that certain conditions are satisfied. The Leray-Hirsch isomorphism is a weak analogue of the Künneth formula; it shows that the cohomology ring of the total space and the tensor product of the cohomology rings of the base space and the fibre are isomorphic as cohomology ring of the base space modules. We first show that the map we're constructing is indeed a module homomorphism and then show case by case that it is in fact an isomorphism. The conditions of the Leray-Hirsch theorem impose restrictions on the total space and fibre of a fibre bundle but not on the base space, in our proof we will show we have an isomorphism when our fibre bundle has a CW complex as a base space of varying dimensionality before checking that the base space need not even be a CW complex. We will close the chapter proving that a few familiar fibre bundles satisfy the conditions and thus allow ourselves to describe the cohomology rings of the special unitary groups and flag manifolds at least as modules if nothing more yet.

Chapter 4 will have a more categorical flavour, we will introduce the idea of a classifying space of a group or monoid. We will describe the simplex category Δ, simplicial sets and spaces, and finally two useful constructions known as the nerve of a category and the geometric realisation of a simplicial set/space. We will show that if we have a category with one object and a group as the collection of morphisms, then the geometric realisation of the nerve of our category is a way of constructing the classifying space of the group. We will finish off this chapter by demonstrating that the loop space of a classifying space is homotopy equivalent to the original group.

In chapter 5 we will discuss characteristic classes, these are classes in the cohomology groups of the base space of a vector bundle (a fibre bundle where the fibre is a vector space and the homeomorphisms are fibrewise linear) that act as invariants. If two vector bundles have different characteristic classes then they cannot be equal but the converse is not necessarily true. The Stiefel-Whitney classes are characteristic classes in cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients that we will define axiomatically. The Euler class is a characteristic class in cohomology with integer coefficients but we require that the vector bundles are oriented. The Chern classes are characteristic classes for complex vector bundles in cohomology with integer coefficients; they are built from Euler classes of underlying real vector bundles as complex vector bundles come equipped with a canonical choice of orientation coming from the complex structure. We will use characteristic classes to investigate the cohomology of flag manifolds as a ring and close the chapter with the construction of a very important ring homomorphism called the Chern character.

Chapter 6 is about general cohomology theories. We will discuss the Eilenberg-

Steenrod axioms and a similar set of axioms for reduced cohomology theories. We explore the idea of Ω-spectra and show that given an Ω-spectrum, we can construct our very own (reduced) cohomology theory. Finally, we will discuss the Künneth formula, a result concerning the cohomology rings of product spaces. Not every general cohomology theory admits a Künneth formula but we will show that there is a similar result when one of the spaces in our product space is a torus and that this result is compatible with natural transformations of cohomology theories.

We discuss strict symmetric monoidal categories and exponential functors in chapter 7. We will give some examples of monoidal categories and show that they are strict and symmetric before giving a few examples of exponential functors [29], and then observing how they transform vector bundles into new vector bundles over the same base space. Exponential functors transform the direct sum of vector spaces into a tensor product of vector spaces

We introduce Segal's category (well really it's opposite category) in chapter 8, a very important category for our purposes. Γ-categories are covariant functors from the opposite of Segal's category to the cateogry of pointed categories, and in this chapter we first describe how to construct a Γ-category given a strict symmetric monoidal category and then given a Γ-category, how to construct an Ω-spectrum. This completes the chain and we see that we can construct a cohomology theory from any strict symmetric monoidal category. We follow up this conclusion with a proof that a strong symmetric monoidal functor induces a natural transformation of cohomology theories.

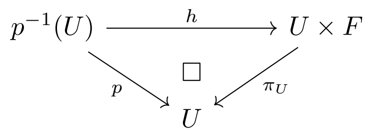
Chapter 9 finally introduces the Weyl map. We see that, after composition with the inclusion of $\mathrm{SU}(n)$ into U, the Weyl map is homotopic to the direct sum of easier to manipulate maps, easier in the sense that each is adjoint to a map that is in turn homotopic to the formal difference of two line bundles. We can put this result together with the Künneth formula for K-theory to express the class of the Weyl map in $K^1(\mathrm{SU}(n)/\mathbb{T})$.

Finally in chapter 10 we describe a family of group homomorphisms that we call the tensor Chern characters as they are built in part using the Chern character we have already met but instead these maps are from the cohomology theories $h^*_\otimes$ constructed using the target categories of exponential functors. We explore the class of the Weyl map after an application of a power of the determinant as an exponential functor. We will discover that not all exponential functors result in natural transformations of cohomology theories but nevertheless, we can describe the class of the Weyl map as it is only dependent on elements in the 0^{th} and 1^{st} degrees and we will finally show that this construction agrees with the natural transformation construction if an exponential functor is also a strong symmetric functor.

2 Fibre Bundles

Let us begin with a few preliminaries. The concept of fibre bundles was first introduced by Steenrod in 1951 [33]. Here we introduce the definition:

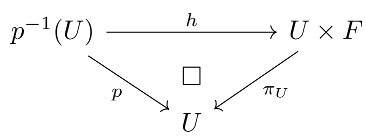
Definition 1. A continuous map p: $E \to B$ is called a **fibre bundle** with **fibre** F if $\forall\ x \in B\ \exists$ an open neighbourhood U of x such that there exists a homeomorphism h: $p^{-1}(U) \to U \times F$ such that:



2.1 Pullback Bundle

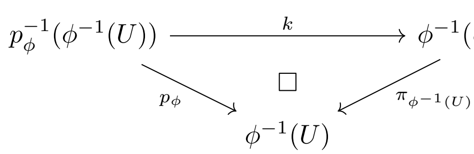
Theorem 1. If p: $E \to X$ is a fibre bundle with fibre F, and ϕ: $Y \to X$ is a continuous map; consider the space $\phi^*E := \{(y,\ e) \in Y \times E : \phi(y) = p(e)\}$, and a map p_ϕ: $\phi^*E \to Y$ defined by $p_\phi(y,\ e) = y$. Then p_ϕ: $\phi^*E \to Y$ is a fibre bundle with fibre F.

Proof. Since p: $E \to X$ is a fibre bundle with fibre F, $\forall\ x \in X\ \exists$ an open neighbourhood U of x such that $\exists$ a homeomorphism h: $p^{-1}(U) \to U \times F$ such that:



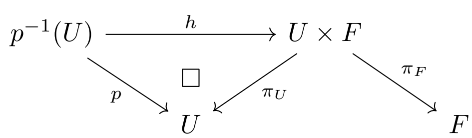
Consider $x = \phi(y)$, since ϕ is continuous, if U is an open neighbourhood of x, then $\phi^{-1}(U)$ is an open neighbourhood of y.

We must show that $\exists$ a homeomorphism k: $p_\phi{}^{-1}(\phi^{-1}(U)) \to \phi^{-1}(U) \times F$ such that:



A point of $p_\phi{}^{-1}(\phi^{-1}(U))$ is of the form $(y,\ e) \in Y{\times}E$ where $p(e) = \phi(y) \in U$. We require that $(\pi_{\phi^{-1}(U)}{\circ}k)(y,\ e) = p_\phi(y,\ e) = y$, to define k we must now determine a map $\pi_F{\circ}k{:}p_\phi{}^{-1}(\phi^{-1}(U)) \to F$.

Since if $\phi(y) \in U$, then $p(e) \in U \implies e \in p^{-1}(U)$ and we have the diagram



the identification $(\pi_F \circ k)(y,\, e) := (\pi_F \circ h)(e)$ is well defined.

It must finally be shown that $k(y,\, e) = (y,\, (\pi_F \circ h)(e))$ is a homeomorphism.

k is continuous because projection maps are continuous and h is continuous since it is a homeomorphism.

Let us design a map l: $\phi^{-1}(U) \times F \to p_\phi{}^{-1}(\phi^{-1}(U))$ to be the inverse of k.

A point in $\phi^{-1}(U) \times F$ is of the form $(u,\, f)$ where $\phi(u) \in U$ and $f \in F$, thus $(\phi(u),\, f) \in U \times F$.

h has a continuous inverse h^{-1}: $U \times F \to p^{-1}(U)$ since it is a homeomorphism, and so $h^{-1}(\phi(u),\, f) \in p^{-1}(U) \subset E$.

Now consider $p(h^{-1}(\phi(u),\, f)) = \pi_U \circ h(h^{-1}(\phi(u),\, f)) = \pi_U(\phi(u),\, f) = \phi(u)$, thus $(u,\, h^{-1}(\phi(u),\, f)) \in \phi^*E$ and since $\phi(u) \in U$, $(u,\, h^{-1}(\phi(u),\, f)) \in p_\phi{}^{-1}(\phi^{-1}(U))$.

Therefore define l by $l(u,\, f) = (u,\, h^{-1}(\phi(u),\, f))$, which is continuous since identities, h^{-1}, and ϕ are continuous. We must check that l is the inverse of k.

It must be shown that $l \circ k = \mathrm{id}_{p_\phi{}^{-1}(\phi^{-1}(U))}$ and $k \circ l = \mathrm{id}_{\phi^{-1}(U) \times F}$.

$$\begin{aligned}
l \circ k(y,e) &= l(y, (\pi_F \circ h)(e)) \\
&= (y, h^{-1}(\phi(y), (\pi_F \circ h)(e))) \\
&= (y, h^{-1}(p(e), (\pi_F \circ h)(e))) \\
&= (y, h^{-1}((\pi_U \circ h)(e), (\pi_F \circ h)(e))) \\
&= (y, h^{-1}(h(e)) \\
&= (y, e) \\
k \circ l(u,f) &= k(u, h^{-1}(\phi(u), f)) \\
&= (u, (\pi_F \circ h)(h^{-1}(\phi(u), f))) \\
&= (u, \pi_F(\phi(u), f)) \\
&= (u, f)
\end{aligned}$$

Therefore k has a continuous inverse in l and is thus a homeomorphism and therefore p_ϕ: $\phi^*E \to Y$ is a fibre bundle with fibre F. △

Definition 2. If p: $E \to X$ is a fibre bundle and ϕ: $Y \to X$ is a continuous map, then the fibre bundle p_ϕ: $\phi^*E \to Y$ is called the **pullback bundle** of p by ϕ.

2.2 Concrete Family of Examples

Consider $\mathrm{SU}(n) = \{X \in \mathbb{M}_{n\times n}(\mathbb{C}) \mid XX^* = X^*X = \mathbb{I}_n,\ det(X) = 1\}$. An element $X \in \mathrm{SU}(n)$ is a matrix of the form:

$$X = \begin{pmatrix} u_{11} + iv_{11} & \dots & u_{1n} + iv_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} + iv_{n1} & \dots & u_{dn} + iv_{nn} \end{pmatrix}$$

where

$$\sum_{i=1}^{n} u_{ij}^2 + v_{ij}^2 = 1\ \forall j,\ \sum_{j=1}^{n} u_{ij}^2 + v_{ij}^2 = 1\ \forall i,\ det(X) = 1$$

Since an element of the m sphere $\mathbb{S}^m$ is of the form $(x_1, ..., x_{m+1})$ where

$$\sum_{i=1}^{m+1} x_i^2 = 1, \text{ and since } \sum_{i=1}^{n} u_{i1}^2 + v_{i1}^2 = 1,$$

$(u_{11}, v_{11}, ..., u_{n1}, v_{n1})$ is an element of $\mathbb{S}^{2n-1}$.

Therefore we can construct a map

$$\begin{aligned} p : \mathrm{SU}(n) &\longrightarrow \mathbb{S}^{2n-1} \\ X &\longmapsto (u_{11}, v_{11}, ..., u_{n1}, v_{n1}) \end{aligned}$$

We will show that p: $\mathrm{SU}(n) \to \mathbb{S}^{2n-1}$ is a fibre bundle with fibre $\mathrm{SU}(n-1)$.

To do so, we must show that for any $x \in \mathbb{S}^{2n-1}$, $\exists$ an open neighbourhood $U \subset \mathbb{S}^{2n-1}$ of x such that $\exists$ a homeomorphism h: $p^{-1}(U) \to U \times \mathrm{SU}(n-1)$ such that

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\quad h \quad} & U \times \mathrm{SU}(n-1) \\ & {\scriptstyle p}\searrow \quad \square \quad \swarrow{\scriptstyle \pi_U} & \\ & U & \end{array}$$

Consider some $x = (x_1, x_2, ..., x_{2n-1}, x_{2n}) \in \mathbb{S}^{2n-1}$. We can always construct the following vector $\mathbf{x} \in \mathbb{C}^n$ and matrix $X \in \mathrm{SU}(n)$:

$$\mathbf{x} = \begin{pmatrix} x_1 + ix_2 \\ \vdots \\ x_{2n-1} + ix_{2n} \end{pmatrix}, \; X = \begin{pmatrix} \mathbf{x} & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{pmatrix}$$

where $\mathbf{x}$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ is an orthonormal basis of $\mathbb{C}^n$. Since $\mathbf{v}_2$, ..., $\mathbf{v}_n$ are columns of a matrix in $\mathrm{SU}(n)$, there also exist corresponding points on the sphere $\mathbb{S}^{2n-1}$ v_2, ..., v_n.

Consider the half $2n$-1 sphere of points in $\mathbb{S}^{2n-1}$ that are strictly on the same side of the hyperplane through the points v_2, ..., v_n as x, this is our neighbourhood U.

In the same way we constructed $\mathbf{x}$ from x, for any $u \in U$ we can construct a vector $\mathbf{u} \in \mathbb{C}^n$. Since u does not lie in the v_2, ..., v_n hyperplane, $\mathbf{u}$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ is a basis of $\mathbb{C}^n$.

We will generate a new orthonormal basis from this basis using the Gram-Schmidt process.

We define the following operations:

$$\langle \mathbf{p}, \mathbf{q} \rangle := \sum_{i=1}^{n} \overline{p}_i \cdot q_i, \quad \mathrm{proj}_{\mathbf{p}}(\mathbf{q}) := \frac{\langle \mathbf{q}, \mathbf{p} \rangle}{\langle \mathbf{p}, \mathbf{p} \rangle} \cdot \mathbf{p}$$

where $\mathbf{p} = (p_1, ..., p_n)$, $\mathbf{q} = (q_1, ..., q_n) \in \mathbb{C}^n$.

Consider the following:

$$\begin{aligned}
\widetilde{\mathbf{w}}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}}(\mathbf{v}_2) \\
\widetilde{\mathbf{w}}_3 &= \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}}(\mathbf{v}_3) - \mathrm{proj}_{\widetilde{\mathbf{w}}_2}(\mathbf{v}_3) \\
&\vdots \\
\widetilde{\mathbf{w}}_n &= \mathbf{v}_n - \mathrm{proj}_{\mathbf{u}}(\mathbf{v}_n) - \mathrm{proj}\widetilde{\mathbf{w}}_2(\mathbf{v}_n) - \ldots - \mathrm{proj}_{\widetilde{\mathbf{w}}_{n-1}}(\mathbf{v}_n)
\end{aligned}$$

Then take

$$\mathbf{w}_i = \frac{\widetilde{\mathbf{w}}_i}{||\widetilde{\mathbf{w}}_i||} \quad \forall i$$

Then $\mathbf{u}$, $\mathbf{w}_2$, ..., $\mathbf{w}_n$ is an orthonormal basis of $\mathbb{C}^n$ and therefore

$$\begin{pmatrix}\mathbf{u} & \mathbf{w}_2 & \ldots & \mathbf{w}_n\end{pmatrix} \in \mathrm{SU}(n)$$

Let us use all this to construct a map:

$$\begin{aligned}
f : U &\longrightarrow \mathrm{SU}(n) \\
u &\longmapsto \begin{pmatrix}\mathbf{u} & \mathbf{w}_2 & \ldots & \mathbf{w}_n\end{pmatrix}
\end{aligned}$$

We would like to find a homeomorphism h: $p^{-1}(U) \to U \times \mathrm{SU}(n\text{-}1)$, in this case we will instead find the continuous inverse first and equivalently prove that it is a homeomorphism instead.

Consider then the map:

$$\begin{aligned}
k : U \times \mathrm{SU}(n-1) &\longrightarrow p^{-1}(U) \\
(u, X) &\longmapsto f(u)\begin{pmatrix}1 & \mathbf{0} \\ \mathbf{0} & X\end{pmatrix}
\end{aligned}$$

where the entries called $\mathbf{0}$ are the n-1 dimensional row and column vectors whose entries are all zero.

k is well defined since the first column of $k(u, X)$ is $\mathbf{u}$ by inspection. It is also clear that k is continuous since the Gram-Schmidt process is continuous.

We would like to find a continuous inverse of k for which we require that $\pi_U \circ k^{-1} = p$.

Let us consider a map:

$$\begin{aligned}
g : \mathrm{SU}(n) &\longrightarrow \mathrm{SU}(n) \\
Y &\longmapsto f(p(Y))^* \cdot Y
\end{aligned}$$

Necessarily from the definitions of p and f, the first row of $f(p(Y))^*$ is the conjugate transpose of the first column of Y and since both matrices are elements of $\mathrm{SU}(n)$, their product will be a matrix of the form:

$$g(Y) = \begin{pmatrix}1 & \mathbf{0} \\ \mathbf{0} & G(Y)\end{pmatrix}$$

Let us now consider the required map:

$$\begin{aligned} h : p^{-1}(U) &\longrightarrow U \times \mathrm{SU}(n-1) \\ Z &\longmapsto (p(Z), G(Z)) \end{aligned}$$

We must finally show that $k \circ h = \mathrm{id}_{p^{\text{-}1}(U)}$ and $h \circ k = \mathrm{id}_{U \times \mathrm{SU}(n\text{-}1)}$

$$\begin{aligned} k \circ h(Z) &= k(p(Z), G(Z)) \\ &= f(p(Z)) \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & G(Z) \end{pmatrix} \\ &= f(p(Z)) f(p(Z))^* \cdot Z \\ &= \mathbb{I}_n \cdot Z \\ &= Z \\ h \circ k(u, X) &= h(f(u) \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & X \end{pmatrix}) \\ &= (p(f(u) \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & X \end{pmatrix}), G(f(u) \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & X \end{pmatrix})) \\ &= (u, G(f(u) \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & X \end{pmatrix})) \\ g(f(u) \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & X \end{pmatrix}) &= f(p(f(u) \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & X \end{pmatrix}))^* \cdot f(u) \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & X \end{pmatrix} \\ &= f(u)^* \cdot f(u) \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & X \end{pmatrix} \\ &= \mathbb{I}_n \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & X \end{pmatrix} \\ &= \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & X \end{pmatrix} \\ \implies G(f(u) \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & X \end{pmatrix}) &= X \\ \implies h \circ k(u, X) &= (u, X) \end{aligned}$$

Therefore h is a homeomorphism with inverse k, and thus p: $\mathrm{SU}(n) \to \mathbb{S}^{2n\text{-}1}$ is a fibre bundle with fibre $\mathrm{SU}(n\text{-}1)$.

2.3 Fibre Bundles from a Lie Group to a Quotient Space

If G is a Lie Group and H is a closed subgroup of G, then we would like to show that the canonical map

$$\begin{aligned} p : G &\to G/H \\ g &\mapsto g \cdot H \end{aligned}$$

is a fibre bundle with fibre H.

Proof. For every point $x \in G/H$ we need to find a neighbourhood $U \subset G/H$ such that $\exists$ a homeomorphism $h\colon\, p^{-1}(U) \to U \times H$ such that

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\quad h \quad} & U \times H \\ & {\scriptstyle p}\searrow \quad \square \quad \swarrow{\scriptstyle \pi_U} & \\ & U & \end{array}$$

Since H is a closed subgroup of G, by the Closed Subgroup Theorem, it is also a Lie group. We call the corresponding Lie algebras of G and H, $\mathfrak{g}$ and $\mathfrak{h}$ respectively, and we may consider the maps $exp_{\mathfrak{g}}\colon\, \mathfrak{g} \to G$ and $exp_{\mathfrak{h}}\colon\, \mathfrak{h} \to H$ along with their derivatives $\mathrm{d}exp_{\mathfrak{g}}|_0 = \mathrm{id}_{\mathfrak{g}}$ and $\mathrm{d}exp_{\mathfrak{h}}|_0 = \mathrm{id}_{\mathfrak{h}}$. Consequently, $\mathfrak{h}$ is a subspace of $\mathfrak{g}$ and we can consider the following short exact sequence:

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \xrightarrow{\quad q \quad} \mathfrak{g}/\mathfrak{h} \longrightarrow 0$$

where q is the clearly surjective canonical map to the quotient.

Every short exact sequence of vector spaces splits so $\exists$ a map $\sigma\colon\, \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ such that $q \circ \sigma = \mathrm{id}_{\mathfrak{g}/\mathfrak{h}}$. Since identities are injective, σ must also be injective.

G/H is not necessarily a group as H is not required to be a normal subgroup, however it does have a left G action: for $g,\, h \in G$, we have $g.p(h) = p(g.h)$.

Let us construct a map $exp_{\mathfrak{g}/\mathfrak{h}} := p \circ exp_{\mathfrak{g}} \circ \sigma\colon\, \mathfrak{g}/\mathfrak{h} \to G/H$.

$$\begin{array}{ccc} G & \xrightarrow{\quad p \quad} & G/H \\ {\scriptstyle exp_{\mathfrak{g}}}\uparrow & \square & \uparrow{\scriptstyle exp_{\mathfrak{g}/\mathfrak{h}}} \\ \mathfrak{g} & \xleftarrow[\quad \sigma \quad]{} & \mathfrak{g}/\mathfrak{h} \end{array}$$

Let 0 be the identity element of G, we can determine the derivative of $exp_{\mathfrak{g}/\mathfrak{h}}$ at $[0]$: $\mathrm{d}exp_{\mathfrak{g}/\mathfrak{h}}|_0 = \mathrm{d}p|_0 \circ \mathrm{d}exp_{\mathfrak{g}}|_0 \circ \mathrm{d}\sigma|_0 = q \circ \mathrm{id}_{\mathfrak{g}} \circ \sigma = q \circ \sigma = \mathrm{id}_{\mathfrak{g}/\mathfrak{h}}$.

Therefore, since $exp_{\mathfrak{g}/\mathfrak{h}}$ is continuously differentiable and has non-zero derivative at $[0]$, by the Inverse Function Theorem, we may construct a neighbourhood U of $[0]$ such that $\exists\ \overline{U} \in \mathfrak{g}/\mathfrak{h}$ for which $exp_{\mathfrak{g}/\mathfrak{h}}|_{\overline{U}}\colon\, \overline{U} \to U = exp_{\mathfrak{g}/\mathfrak{h}}(\overline{U})$ is a diffeomorphism.

Since $exp_{\mathfrak{g}/\mathfrak{h}}$ is a diffeomorphism on $\overline{U}$, let us construct the inverse map $\phi\colon\, U \to \overline{U}$. For any $y \in U$, $\phi(y)$ is such that $exp_{\mathfrak{g}/\mathfrak{h}}(\phi(y)) = y$. Then we can construct a map $\overline{\sigma}\colon\, U \to G$ that makes the following diagram commute:

$$\begin{array}{ccc} U & \xrightarrow{\quad \overline{\sigma} \quad} & G \\ {\scriptstyle \phi}\downarrow & \square & \uparrow{\scriptstyle exp_{\mathfrak{g}}} \\ \overline{U} & \xrightarrow[\quad \sigma \quad]{} & \mathfrak{g} \end{array}$$

So we have

$$\begin{aligned}\overline{\sigma}(y) &= exp_{\mathfrak{g}}(\sigma(\phi(y))) \\ \text{so } p(\overline{\sigma}(y)) &= p(exp_{\mathfrak{g}}(\sigma(\phi(y)))) \\ &= exp_{\mathfrak{g}/\mathfrak{h}}(\phi(y)) \\ &= y\end{aligned}$$

Thus $p \circ \overline{\sigma} = \mathrm{id}_U$

Let $x \in G/H$ with $x = [g]$ for some $g \in G$. Then due to the left G action on G/H we have $x = g.[0]$. We have constructed a neighbourhood U of $[0]$, thus $g.U$ is a neighbourhood of x.

Let us now construct a map $\overline{\sigma}_x\colon\ g.U \to G$

$$\overline{\sigma}_x(y) := g.\overline{\sigma}(g^{-1}.y)$$

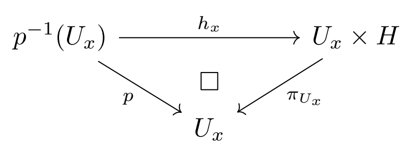
This is well defined since $y \in g.U \implies g^{-1}.y \in U$ so we can apply $\overline{\sigma}$.

Again we have:

$$\begin{aligned}p(\overline{\sigma}_x(y)) &= p(g.\overline{\sigma}(g^{-1}.y)) \\ &= g.p(\overline{\sigma}(g^{-1}.y)) \\ &= g.g^{-1}.y \\ &= y\end{aligned}$$

So $p \circ \overline{\sigma}_x = \mathrm{id}_{g.U}$

Now for every point $x \in G/H$ we have a neighbourhood $U_x = g.U \subset G/H$ where $p(g) = x$. For each neighbourhood let us finally construct a homeomorphism $h_x\colon\ p^{-1}(U_x) \to U_x \times H$ such that:



The inverse of this function is more immediate. Let us define:

$$\begin{aligned}k_x : U_x \times H &\to p^{-1}(U_x) \\ (u,h) &\mapsto \overline{\sigma}_x(u).h\end{aligned}$$

The codomain of k_x is indeed $p^{-1}(U_x)$ since $p(\overline{\sigma}_x(u).h) = p(\overline{\sigma}_x(u)) = u \in U_x$.

As for h_x itself, we define as follows:

$$\begin{aligned}h_x : p^{-1}(U_x) &\to U_x \times H \\ g &\mapsto (p(g), \overline{\sigma}_x(p(g))^{-1}.g)\end{aligned}$$

To show that this too is well defined, we must show that if $h = \overline{\sigma}_x(p(g))^{-1}.g$ then $p(h) = [0] \in G/H$

$$\begin{aligned}
h^{-1} &= g^{-1}.\overline{\sigma}_x(p(g)) \\
p(h^{-1}) &= p(g^{-1}.\overline{\sigma}_x(p(g))) \\
&= g^{-1}.p(\overline{\sigma}_x(p(g)) \\
&= g^{-1}.p(g) \\
&= p(g^{-1}.g) \\
&= p(0) = [0]
\end{aligned}$$

Since H is a group and $p(h^{-1}) = [0]$ implies $h^{-1} \in H$, $h \in H$ and so h_x is well defined.

$(\pi_{U_x} \circ h_x)(u) = p(u)$ so h_x fits into the commutative diagram, and both h_x and k_x are both smooth by construction so to prove that h_x is a homeomorphism, all we must do is prove that k_x is its inverse.

$$\begin{aligned}
(k_x \circ h_x)(g) &= k_x(p(g), \overline{\sigma}_x(p(g))^{-1}.g) \\
&= \overline{\sigma}_x(p(g)).\overline{\sigma}_x(p(g))^{-1}.g \\
&= g \\
(h_x \circ k_x)(u, h) &= h_x(\overline{\sigma}_x(u).h) \\
&= (p(\overline{\sigma}_x(u).h), \overline{\sigma}_x(p(\overline{\sigma}_x(u).h))^{-1}.\overline{\sigma}_x(u).h) \\
&= (p(\overline{\sigma}_x(u)), \overline{\sigma}_x(p(\overline{\sigma}_x(u)))^{-1}.\overline{\sigma}_x(u).h) \\
&= (u, \overline{\sigma}_x(u)^{-1}.\overline{\sigma}_x(u).h) \\
&= (u, h)
\end{aligned}$$

Therefore $k_x \circ h_x = \mathrm{id}_{p^{-1}(U_x)}$ and $h_x \circ k_x = \mathrm{id}_{U_x \times H}$ thus h_x is the required homeomorphism and therefore for a Lie group G and closed subgroup $H \subset G$, the quotient map $p\colon G \to G/H$ is a fibre bundle with fibre H. △

2.4 The Homotopy Extension and Lifting Properties

The definitions and following proofs are described by Hatcher in *Algebraic Topology* [17]

Definition 3. A map $p\colon E \to B$ has the **homotopy lifting property** with respect to a space X if for any given homotopy $h\colon X \times I \to B$ and any given map $H_0\colon X \times \{0\} \to E$ which lifts $h_0 := h|_{X\times\{0\}}$ (i.e. such that $p \circ H_0 = h_0$), there exists a homotopy $H\colon X \times I \to E$ that lifts h (i.e. such that $p \circ H = h$) and $H_0 = H|_{X\times\{0\}}$.

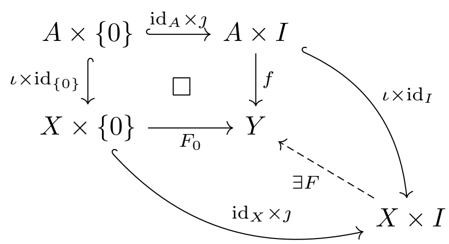
$$\begin{array}{ccc}
X \times \{0\} & \xrightarrow{H_0} & E \\
{\scriptstyle \mathrm{id}_X \times J}\downarrow & \overset{\exists H}{\nearrow} & \downarrow{\scriptstyle p} \\
X \times I & \xrightarrow[h]{} & B
\end{array}$$

A map $p\colon E \to B$ is called a **fibration** if it has the homotopy lifting property with respect to all spaces X.

Lemma 2. If $p\colon E \to B$ is a fibration and $x,y \in B$, then $p^{-1}(x) \simeq p^{-1}(y)$, i.e. we can define a fibre F of a fibration that is unique up to homotopy equivalence.

Lemma 3. Any fibre bundle is a fibration.

Definition 4. Let X be a topological space, and let $\iota\colon A \to X$ be the inclusion of a subspace A into X, the pair (X, A) has the **homotopy extension property** with respect to a space Y if given any homotopy $f\colon A \times I \to Y$ and any map $F_0\colon X \times \{0\} \to Y$ such that $F_0|_{A\times\{0\}} = f_0 := f|_{A\times\{0\}}$, there exists an extension of f to a homotopy $F\colon X \times I \to Y$ such that $F|_{X\times\{0\}} = F_0$ and $F|_{A\times I} = f$.



Equivalently, the pair (X, A) has the homotopy extension property with respect to a space Y if any map $g\colon (X \times \{0\}) \cup (A \times I) \to Y$ can be extended to a map $G\colon X \times I \to Y$ with $G|_{(X\times\{0\})\cup(A\times I)} = g$.

We say that the pair (X, A) has the homotopy extension property if it has the homotopy extension property with respect to all spaces Y, in such cases the inclusion $\iota\colon A \to X$ is called a **cofibration**.

Definition 5. Let X be a topological space, a subspace A of X is called a **retract** of X if there exists a continuous map $r\colon X \to A$ called a **retraction** that is such that $r|_A = \mathrm{id}_A$.

A continuous map $F\colon X \times I \to X$ is called a **deformation retract** of X onto a subspace A if $\forall\ x \in X$ and $a \in A$:

$$F(x,0) = x \qquad\qquad F(x,1) \in A \qquad\qquad F(a,1) = a$$

That is, a deformation retract is a homotopy between the identity map on X and a retraction.

Lemma 4. A pair (X, A) has the homotopy extension property if and only if $(X \times \{0\}) \cup (A \times I)$ is a retract of $X \times I$.

Proof. If $(X \times\{0\}) \cup (A \times I)$ is a retract of $X \times I$ then there exists a retraction $r\colon X \times I \to (X \times \{0\}) \cup (A \times I)$ such that $r|_{(X\times\{0\})\cup(A\times I)} = \mathrm{id}_{(X\times\{0\})\cup(A\times I)}$.

If we have a homotopy $h\colon A \times I \to Y$ and a map $H_0\colon X \times \{0\} \to Y$ such that $H_0|_{A\times\{0\}} = h_0 := h|_{A\times\{0\}}$ we need to find a homotopy $H\colon X \times I \to Y$ such that $H|_{X\times\{0\}} = H_0$ and $H|_{A\times I} = h$.

Since h and H_0 are required to agree on $(X \times\{0\}) \cap (A \times I) = A \times \{0\}$, the map $H_0 \cup h$: $(X \times\{0\}) \cup (A \times I) \to Y$ is well defined, then the composition $(H_0 \cup h) \circ r$: $X \times I \to Y$ is a candidate for H.

$H|_{X\times\{0\}} = ((H_0 \cup h) \circ r)|_{X\times\{0\}} = (H_0 \cup h)|_{r(X\times\{0\})} \circ r|_{X\times\{0\}}$. Since r is a retraction $r|_{(X\times\{0\})\cup(A\times I)} = \mathrm{id}_{(X\times\{0\})\cup(A\times I)}$ so $r|_{X\times\{0\}} = \mathrm{id}_{X\times\{0\}}$ and thus $r(X \times \{0\}) = X \times \{0\}$. Then since $(H_0 \cup h)|_{X\times\{0\}} = H_0$ we achieve the result $H|_{X\times\{0\}} = (H_0 \cup h)|_{X\times\{0\}} \circ \mathrm{id}|_{X\times\{0\}} = H_0$ as required. By an identical argument $H|_{A\times I} = h$. Therefore we have found a suitable homotopy and thus (X, A) has the homotopy extension property with respect to Y, but since Y was arbitrary, (X, A) has the homotopy extension property.

If (X, A) has the homotopy extension property, then for any space Y, if we have a homotopy h: $A \times I \to Y$ and a map H_0: $X \times \{0\} \to Y$ such that $H_0|_{A\times\{0\}} = h|_{A\times\{0\}}$, then there exists a homotopy H: $X \times I \to Y$ such that $H|_{X\times\{0\}} = H_0$ and $H|_{A\times I} = h$.

Let us choose $Y = (X \times \{0\}) \cup (A \times I)$. The inclusions provide a homotopy $\imath$: $A \times I \to Y$ and a map $\jmath$: $X \times \{0\} \to Y$ such that $\jmath|_{A\times\{0\}} = \imath|_{A\times\{0\}}$ therefore there exists a homotopy r: $X \times I \to Y$ such that $r|_{A\times I} = \imath$, $r|_{X\times\{0\}} = \jmath$.

To show that r is a retraction, we must show that $r|_Y = \mathrm{id}_Y$.

$r|_Y = r|_{A\times I} \cup r|_{X\times\{0\}} = \imath \cup \jmath = \mathrm{id}_Y$ since $\imath$ and $\jmath$ agree on the intersection $(X \times \{0\}) \cap (A \times I) = A \times \{0\}$. Therefore $(X \times \{0\}) \cup (A \times I)$ is a retract of $X \times I$.

Therefore a pair (X, A) has the homotopy extension property if and only if $(X \times \{0\}) \cup (A \times I)$ is a retract of $X \times I$. △

Proofs of the following corollary and lemma can also be found in Allen Hatcher's *Algebraic Topology* on page 15 and 16 respectively [17].

Corollary 5. If X is a CW-complex and A is a subcomplex of X, then (X, A) has the homotopy extension property.

Lemma 6. If (X, A) has the homotopy extension property and A is contractible, then the quotient map q: $X \to X/A$ is a homotopy equivalence.

3 The Leray-Hirsch Theorem

There is a result called the Künneth formula that we will properly introduce and make heavy use of later, that relates the cohomology ring of a product space to the cohomology rings of the factor spaces.

The total space of a fibre bundle is locally, but not necessarily globally, the product of two topological spaces, and as such there is a similar but weaker result relating the various cohomology rings that was proved independently by Jean Leray and Guy Hirsch in the 1940s. Here we will flesh out a proof given by Hatcher [17].

Theorem 7. Let $p : E \to B$ be a fibre bundle with fibre F such that for some commutative ring with identity R:

- $H^n(F;\, R)$ is a finitely generated free R-module for all n
- There exist classes $c_j \in H^{k_j}(E;\, R)$ whose restrictions $\iota^*(c_j)$ form a basis for $H^*(F;\, R)$ in each fibre F, where $\iota : F \to E$ is the inclusion.

Then the map

$$\begin{aligned} \Phi : H^*(B;R) \otimes_R H^*(F;R) &\to H^*(E;R) \\ \sum_{ij} b_i \otimes_R \iota^*(c_j) &\mapsto \sum_{ij} p^*(b_i) \smile c_j \end{aligned}$$

is an $H^*(B;\, R)$-module isomorphism.

3.1 Proof of the Leray-Hirsch Theorem

3.1.1 Φ is an $H^*(B;\, R)$-module homomorphism

Proof. It is not too difficult to show that Φ is at least a group homomorphism. Since $H^*(F;\, R)$ is finitely generated in each degree we can write two general elements of $H^*(B;\, R) \otimes_R H^*(F;\, R)$ as:

$$A = \sum_{i=0}^{\infty} \sum_{j_i=0}^{J_i} a_{j_i} \otimes_R \iota^*(c_{j_i}) \quad \text{and} \quad B = \sum_{i=0}^{\infty} \sum_{j_i=0}^{J_i} b_{j_i} \otimes_R \iota^*(c_{j_i})$$

Clearly, we have:

$$\begin{aligned}
\Phi(A+B) &= \Phi(\sum_{i=0}^{\infty}\sum_{j_i=0}^{J_i} a_{j_i}+b_{j_i}\otimes_R \iota^*(c_{j_i})) \\
&= \sum_{i=0}^{\infty}\sum_{j_i=0}^{J_i} p^*(a_{j_i}+b_{j_i})\smile c_{j_i} \\
&= \sum_{i=0}^{\infty}\sum_{j_i=0}^{J_i} p^*(a_{j_i})\smile c_{j_i} + \sum_{i=0}^{\infty}\sum_{j_i=0}^{J_i} p^*(b_{j_i})\smile c_{j_i} \\
\Phi(A)+\Phi(B) &= \sum_{i=0}^{\infty}\sum_{j_i=0}^{J_i} p^*(a_{j_i})\smile c_{j_i} + \sum_{i=0}^{\infty}\sum_{j_i=0}^{J_i} p^*(b_{j_i})\smile c_{j_i}
\end{aligned}$$

To ensure the module structure is maintained, we need to describe the module structure of both rings.

In $H^*(B;\, R)\otimes_R H^*(F;\, R)$ we have:

$$\begin{aligned}
H^*(B;R)\times H^*(B;R)\otimes_R H^*(F;R) &\to H^*(B;R)\otimes_R H^*(F;R) \\
(r, \sum_{ij} b_i\otimes_R \iota^*(c_j)) &\mapsto \sum_{ij} r.b_i\otimes_R \iota^*(c_j)
\end{aligned}$$

In $H^*(E;\, R)$ we have:

$$\begin{aligned}
H^*(B;R)\times H^*(E;R) &\to H^*(E;R) \\
(r,e) &\mapsto p^*(r)\smile e
\end{aligned}$$

Thus, to prove that Φ is a $H^*(B;\, R)$-module homomorphism, we must show that:

$$\begin{aligned}
\Phi(\sum_{ij} r.b_i\otimes_R \iota^*(c_j)) &= p^*(r)\smile \Phi(\sum_{ij} b_i\otimes_R \iota^*(c_j)). \\
\Phi(\sum_{ij} r.b_i\otimes_R \iota^*(c_j)) &= \sum_{ij} p^*(r.b_i)\smile c_j \\
&= \sum_{ij} p^*(r)\smile p^*(b_i)\smile c_j \\
&= p^*(r)\smile \sum_{ij} p^*(b_i)\smile c_j \\
&= p^*(r)\smile \Phi(\sum_{ij} b_i\otimes_R \iota^*(c_j))
\end{aligned}$$

p is a map of spaces and thus p^* is necessarily a ring homomorphism, also since $p^*(r)$ does not depend on i or j and $H^*(E;\, R)$ is a ring, it can be pulled out as a common factor. Since the two sides do indeed agree, Φ is an $H^*(B;\, R)$-module homomorphism. △

Let us now tackle the trickier task of showing that Φ is a bijection and thus an $H^*(B;\, R)$-module isomorphism.

3.1.2 If B is a 0-dimensional CW complex

To begin proving the Leray-Hirsch Theorem, let us consider fibre bundles where the base space is a 0-dimensional CW complex. All such spaces have the form of a set of points equipped with the discrete topology.

We want to show that if the conditions of the theorem hold, then

$$\Phi : H^*(B;R) \otimes_R H^*(F;R) \to H^*(E;R)$$
$$\sum_{ij} b_i \otimes_R \iota^*(c_j) \mapsto \sum_{ij} p^*(b_i) \smile c_j$$

is an isomorphism.

Proof. Since B is a 0-dimensional CW complex and a class in $H^0(X;\, R)$ simply assigns an element of R to each connected component of X, we have the following isomorphism:

$$\prod_B R \to H^0(B;R)$$
$$(r_1, r_2, ...) \mapsto \sum_{i \in B}(\{i\} \mapsto r_i)$$

Since $H^k(B;\, R) = 0\ \forall\ k \neq 0$, we must simply show that $\forall\ k$

$$\Phi^k : H^0(B;R) \otimes_R H^k(F;R) \to H^k(E;R)$$
$$\sum_{ij} b\_i \otimes_R \iota^*(c_j) \mapsto \sum_{ij} p^*(b_i) \smile c_j$$

is an isomorphism.

Since p: $E \to B$ is a fibre bundle, $\forall\ i \in B$, $E_i := p^{\text{-}1}(i)$ is homeomorphic to the fibre F and E is the disjoint union of these fibres.

$$E = \coprod_{i \in B} E_i$$

Since all the fibres are homeomorphic to one another, without loss of generality we may designate $E_1 = F$ and construct homeomorphisms from the other fibres as ψ_i: $E_i \to F$. We must also define the inclusion maps of each fibre into the total space ι_i: $E_i \to E$.

If ${c_j}^{(1)}$ is a basis of $H^*(E_1;\, R) = H^*(F;\, R)$ then ${\psi_i}^*({c_j}^{(1)}) =:\ {c_j}^{(i)}$ is a basis of $H^*(E_i;\, R)$. For classes $c_j \in H^*(E;\, R)$, we require that ${\iota_i}^*(c_j)$ is a basis of $H^*(E_i;\, R)\ \forall\ i \in [n]$, therefore we choose the classes $c_j = ({c_j}^{(1)},\, {c_j}^{(2)},\, ...)$ to satisfy these conditions.

Thus $\forall\ k \in \mathbb{Z}$

$$\begin{aligned} H^k(E;R) &\cong H^k(\coprod_{i\in B} E_i;R) \\ &\cong \prod_{i\in B} H^k(E_i;R) \\ &\cong \prod_{i\in B} H^k(F;R) \end{aligned}$$

We can define the relevant isomorphism as follows:

$$\begin{aligned} H^k(E;R) \to \prod_B H^k(F;R) \\ \phi \mapsto ((\iota_1 \circ \psi_1^{-1})^*(\phi), (\iota_2 \circ \psi_2^{-1})^*(\phi), ...) \end{aligned}$$

Thanks to these isomorphisms, to show that each Φ_k is an isomorphism, we must construct another isomorphism ϕ that ensures that the following diagram commutes:

$$\begin{array}{ccc} H^0(B;R) \otimes_R H^k(F;R) & \xrightarrow{\Phi_k} & H^k(E;R) \\ \cong\uparrow & \square & \downarrow\cong \\ \prod_B R \otimes_R H^k(F;R) & \xrightarrow[\phi]{} & \prod_B H^k(F;R) \end{array}$$

Lemma 8. Let R be a commutative ring with identity, X be a finitely generated free R-module, and B a set. Then there exists an isomorphism

$$\begin{aligned} \phi : \prod_B R \otimes_R X \to \prod_B X \\ \sum_{i,j} r_i \otimes_R x_j \mapsto \sum_{i,j} (r_{i_1} x_j, r_{i_2} x_j, ...) \end{aligned}$$

Proof. : We will show that the following map is both a left and right inverse of ϕ:

$$\begin{aligned} \psi : \prod_B X \to \prod_B R \otimes_R X \\ (x_1, x_2, ...) \mapsto \sum_{i\in B} e_i \otimes_R x_i \end{aligned}$$

where e_i is the vector with 0_R in every entry except the i^{th} which is instead 1_R. The set $\{e_i \mid i \in B\}$ is the standard basis of $\prod_B R$.

Consider a general element of $\prod_B R \otimes_R X$ and apply first ϕ, then ψ:

$$\begin{aligned}
(\psi \circ \phi)(\sum_{i,j} r_i \otimes_R x_j) &= \psi(\phi(\sum_{i,j} r_i \otimes_R x_j)) \\
&= \psi(\sum_{i,j} (r_{i_1} x_j, r_{i_2} x_j, ...)) \\
&= \sum_{i,j} \psi(r_{i_1} x_j, r_{i_2} x_j, ...) \\
&= \sum_{i,j} \sum_{k \in B} e_k \otimes_R r_{i_k} x_j \\
&= \sum_{i,j} \sum_{k \in B} e_k r_{i_k} \otimes_R x_j \\
&= \sum_{i,j} r_i \otimes_R x_j
\end{aligned}$$

Thus $\psi \circ \phi = \mathrm{id}_{\prod_B R \otimes_R X}$.

Now consider a general element of $\prod_B X$ and apply first ψ, then ϕ:

$$\begin{aligned}
(\phi \circ \psi)(x_1, x_2, ...) &= \phi(\psi(x_1, x_2, ...)) \\
&= \phi(\sum_{i \in B} e_i \otimes_R x_i) \\
&= \sum_{i \in B} \phi(e_i \otimes_R x_i) \\
&= \sum_{i \in B} (e_{i_1} x_i, e_{i_2} x_i, ...) \\
&= \sum_{i \in B} (0, ..., x_i, 0, ...) \\
&= (x_1, x_2, ...)
\end{aligned}$$

Thus $\phi \circ \psi = \mathrm{id}_{\prod_B X}$.

Therefore, since ψ is indeed both a left and right inverse of ϕ, both must be isomorphisms. $\triangle$

Since $H^*(F;\, R)$ is a finitely generated free R-module, so must $H^k(F;\, R)$ be a finitely generated free R-module for each level $k \in \mathbb{Z}$. Therefore, in the following diagram, ϕ is an isomorphism by the previous lemma:

$$\begin{array}{ccc}
H^0(B; R) \otimes_R H^k(F; R) & \xrightarrow{\Phi^k} & H^k(E; R) \\
\cong \uparrow & & \downarrow \cong \\
\prod_B R \otimes_R H^k(F; R) & \xrightarrow[\phi]{} & \prod_B H^k(F; R)
\end{array}$$

Finally, this diagram must be shown to commute. Let us introduce names for the two vertical isomorphisms we described earlier: σ: $\prod_B R \to H^0(B;\, R)$ and Ψ: $H^k(E;\, R) \to \prod_B H^k(F;\, R)$.

We therefore need to show that for any element of $\prod_B R \otimes_R H^k(F;\, R)$:

$$\phi(\sum_{i,j} r_i \otimes_R x_j) = \Psi(\Phi_k(\sum_{i,j} \sigma(r_i) \otimes_R x_j))$$

Since $H^k(F;\, R)$ is a finitely generated R-module $\forall\, k$ and we have a basis ${c_j}^{(1)}$ of $H^k(F;\, R)$, we can express any element $x \in H^k(F;\, R)$ in the form:

$$x = \sum_l x_l c_l^{(1)} \text{ for some set of } x_l \in R$$

Now to show the commutativity

$$\phi(\sum_{i,j} r_i \otimes_R x_j) = \sum_{i,j} (r_{i_1} x_j, r_{i_2} x_j, ...)$$

$$\begin{aligned}
\Psi(\Phi_k(\sum_{i,j} \sigma(r_i) \otimes_R x_j)) &= \Psi(\Phi_k(\sum_{i,j} \sum_{m \in B} (\{m\} \mapsto r_{i_m}) \otimes_R \sum_l x_{j_l} c_l^{(1)})) \\
&= \Psi(\Phi_k(\sum_{i,j,l} \sum_{m \in B} (\{m\} \mapsto r_{i_m}) x_{j_l} \otimes_R c_l^{(1)})) \\
&= \Psi(\Phi_k(\sum_{i,j,l} \sum_{m \in B} (\{m\} \mapsto r_{i_m} x_{j_l}) \otimes_R c_l^{(1)})) \\
&= \Psi(\sum_{i,j,l} p^*(\sum_{m \in B} (\{m\} \mapsto r_{i_m} x_{j_l})) \smile c_l) \\
&= \Psi(\sum_{i,j,l} (e \mapsto \{r_{i_m} x_{j_l},\ \text{if } e \in E_m) \smile c_l) \\
&= \sum_{i,j} ((\iota_1 \circ \psi_1^{-1})^*(\sum_l ((e \mapsto \{r_{i_m} x_{j_l},\ \text{if } e \in E_m) \smile c_l)), \\
&\quad , (\iota_2 \circ \psi_2^{-1})^*(\sum_l ((e \mapsto \{r_{i_m} x_{j_l},\ \text{if } e \in E_m) \smile c_l)), ...) \\
&= \sum_{i,j} ((\psi_1^{-1})^*(\sum_l ((e \mapsto r_{i_1} x_{j_l}) \smile c_l^{(1)})), \\
&\quad , (\psi_2^{-1})^*(\sum_l ((e \mapsto r_{i_2} x_{j_l}) \smile c_l^{(2)})), ...) \\
&= \sum_{i,j} (\sum_l ((e \mapsto r_{i_1} x_{j_l}) \smile c_l^{(1)}), \sum_l ((e \mapsto r_{i_2} x_{j_l}) \smile c_l^{(1)}), ...) \\
&= \sum_{i,j} (r_{i_1} \sum_l x_{j_l} c_l^{(1)}, r_{i_2} \sum_l x_{j_l} c_l^{(1)}, ...) \\
&= \sum_{i,j} (r_{i_1} x_j, r_{i_2} x_j, ...) \ \text{ as required}
\end{aligned}$$

Therefore the diagram in question does indeed commute, so since ϕ is an isomorphism, Φ_k must be an isomorphism $\forall\, k$.

Therefore Φ is an isomorphism, and the Leray-Hirsch theorem holds for fibre bundles with 0-dimensional CW complex base spaces. △

3.1.3 If B is a finite-dimensional CW complex

To prove that the theorem holds for all finite-dimensional CW complex base spaces we will use an induction argument.

Let us assume that the Leray-Hirsch theorem holds for (n-1)-dimensional CW complex base spaces.

Let B be an n-dimensional CW complex and consider $B' \subset B$ to be the subspace of B obtained by removing a single point from the interior of each of the n-cells of B.

We will also let $E' := p^{-1}(B')$

Proof. We must show that the following diagram, where R is understood to be the coefficient ring, commutes and that the vertical maps are all isomorphisms:

$$\begin{array}{ccccccccc} ... \longrightarrow & H^*(B,B')\otimes H^*(F) & \longrightarrow & H^*(B)\otimes H^*(F) & \longrightarrow & H^*(B')\otimes H^*(F) & \longrightarrow ... \\ & \downarrow \Phi_{B,B'} & & \downarrow \Phi & & \downarrow \Phi_{B'} & \\ ... \longrightarrow & H^*(E,E') & \longrightarrow & H^*(E) & \longrightarrow & H^*(E') & \longrightarrow ... \end{array}$$

Since every pair of topological spaces (X, Y) with $Y \subset X$ generates a long exact sequence of cohomology groups

$$... \longrightarrow H^*(X,Y) \longrightarrow H^*(X) \longrightarrow H^*(Y) \longrightarrow ...$$

the pairs (B, B') and (E, E') generate long exact sequences and the bottom row of our diagram is an exact sequence automatically. Also, since $H^k(F;\ R)$ is a finitely generated free R-module $\forall\ k \in \mathbb{Z}$, the functor (-) $\otimes_R H^*(F;\ R)$ is exact, thus the top row of our diagram is an exact sequence too.

Let us consider the space $B^{n\text{-}1}$, the n-1 skeleton of B. It is clear to see that B' is homotopy equivalent to $B^{n\text{-}1}$ since removing a single point from each n cell allows us to continuously retract each to its boundary.

Homotopy equivalence in spaces induces isomorphisms in their cohomology groups. To show that $\Phi_{B'}$ is an isomorphism, we need to find a fibre bundle from a total space X that is homotopic to E' onto $B^{n\text{-}1}$ with fibre F; by assumption, the Leray-Hirsch theorem will hold for this fibre bundle

$$\begin{array}{ccc} H^*(B')\otimes H^*(F) & \xrightarrow{\cong} & H^*(B^{n-1})\otimes H^*(F) \\ \Phi_{B'}\downarrow & \square & \downarrow\cong \\ H^*(E') & \xrightarrow[\cong]{} & H^*(X) \end{array}$$

From this diagram, if we can prove the existence of X, then clearly $\Phi_{B'}$ must be an isomorphism.

Definition 6. A space X is said to be ***k*-connected** if $\forall\ i \leq k$, $\pi_i(X) = 0$.

For a space X, and subspace A of X, the pair (X, A) is said to be k-connected if $\forall\ i \leq k$, $\pi_i(X,\ A) = 0$. By the long exact sequence of homotopy groups induced by the short exact sequence of the pair, this implies that (X, A) is k-connected iff $\pi_i(A) \cong \pi_i(X)$ $\forall$ $i \leq k$-1 and ι^*: $\pi_k(A) \to \pi_k(X)$, the map induced by the inclusion, is a surjection.

Lemma 9. Consider a fibre bundle p: $E \to B$ and a subspace B' $\subset B$ such that (B, B') is k-connected, then (E, p^{-1}(B')) is also k-connected.

Proof. Consider any map g_0: (D^i, ∂D^i) $\to$ (E, p^{-1}(B')), with $i \leq k$, a map from the i-disc to E that maps the boundary totally into p^{-1}(B').

Since (B, B') is k-connected, there must exist a homotopy between any two maps (D^i, ∂D^i) $\to$ (B, B') where $i \leq k$, thus consider the maps $f_0 = pg_0$ and f_1, any map such that $Im(f_1) \subseteq B$'. Let f be the homotopy between f_0 and f_1.

We thus achieve the following diagram:

$$\begin{array}{ccc} (D^i, \partial D^i) \times \{0\} & \xrightarrow{g_0} & (E, p^{-1}(B')) \\ \big\downarrow & \square & \big\downarrow p \\ (D^i, \partial D^i) \times I & \xrightarrow[f]{} & (B, B') \end{array}$$

Since p: $E \to B$ is a fibre bundle, it is also a fibration and thus satisfies the homotopy lifting property. We therefore know that there exists a homotopy g such that the following diagram commutes:

$$\begin{array}{ccc} (D^i, \partial D^i) \times \{0\} & \xrightarrow{g_0} & (E, p^{-1}(B')) \\ \big\downarrow & \nearrow^{g} & \big\downarrow p \\ (D^i, \partial D^i) \times I & \xrightarrow[f]{} & (B, B') \end{array}$$

Thus we may consider g_1 such that $f_1 = pg_1$, therefore, since $Im(f_1) \subseteq B$', $Im(g_1) \subseteq p^{-1}$(B').

Since such a g exists for any choice of g_0 and $i \leq k$, if (B, B') is k-connected, (E, p^{-1}(B')) must also be k-connected. △

Since B' and B^{n-1} are homotopy equivalent, (B', B^{n-1}) is k-connected $\forall$ k, thus by our lemma, (E', p^{-1}(B^{n-1})) must be k-connected $\forall$ k, therefore p^{-1}(B^{n-1}) is homotopy equivalent to E'. Additionally, since p: $E \to B$ is a fibre bundle with fibre F, then so are the maps p: E' $\to B$', and p: p^{-1}(B^{n-1}) $\to B^{n-1}$.

We have therefore found a space that allows our diagram to commute:

$$\begin{array}{ccc} H^*(B') \otimes H^*(F) & \xrightarrow{\cong} & H^*(B^{n-1}) \otimes H^*(F) \\ \Phi_{B'} \big\downarrow & \square & \big\downarrow \cong \\ H^*(E') & \xrightarrow[\cong]{} & H^*(p^{-1}(B^{n-1})) \end{array}$$

Therefore $\Phi_{B'}$ is an isomorphism.

We removed a set of points (read 0-dimensional CW complex) from B to construct B'. Let the points themselves be denoted x_α, and the n-cells from which they were removed, e_α respectively.

Since p: $E \to B$ is a fibre bundle, for each x_α $\exists$ a neighbourhood $U_\alpha \subset e_\alpha$ such that $\exists$ a homeomorphism h_α: $p^{-1}(U_\alpha) \to U_\alpha \times F$ such that:

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \xrightarrow{h_\alpha} & U_\alpha \times F \\ & {\scriptstyle p}\searrow \quad \square \quad \swarrow{\scriptstyle \pi_{U\_\alpha}} & \\ & U_\alpha & \end{array}$$

Let U_α' $\subset U_\alpha$ be the subspace constructed by removing x_α from U_α.

Let U and U' be defined as follows:

$$U = \bigcup_\alpha U_\alpha \qquad\qquad U' = \bigcup_\alpha U'_\alpha$$

Now let us consider a space $K = B \backslash U$. It is clear that $K \subset B$' $\subset B$ and that the closure of K is contained within the interior of B' since each U_α is a neighbourhood of the corresponding missing point.

Therefore, since $B\backslash K = U$ and B'$\backslash K = U$', by excision, the inclusion map ι: (U, U') $\to$ (B, B') induces isomorphisms ι^*: $H^k(B, B'; R) \to H^k(U, U'; R)$ $\forall\ k \in \mathbb{Z}$.

It is also clear that $p^{-1}(K) \subset p^{-1}(B') = E' \subset p^{-1}(B) = E$ and thus the inclusion as described and the fibre bundle induce a similar collection of isomorphisms ι'^*: $H^k(E, E'; R) \to H^k(p^{-1}(U), p^{-1}(U'); R)$ induced by the inclusion ι': $(p^{-1}(U), p^{-1}(U')) \to (E, E')$.

To prove that $\Phi_{B,B'}$ is an isomorphism, we must show, in the following diagram, that $\Phi_{U,U'}$ is an isomorphism

$$\begin{array}{ccc} H^*(B,B') \otimes H^*(F) & \xrightarrow{\cong} & H^*(U,U') \otimes H^*(F) \\ \downarrow{\scriptstyle \Phi_{B,B'}} & \square & \downarrow{\scriptstyle \Phi_{U,U'}} \\ H^*(E,E') & \xrightarrow[\cong]{} & H^*(p^{-1}(U),p^{-1}(U')) \end{array}$$

To do so, consider the following diagram:

$$\begin{array}{ccccccccc} ... & \longrightarrow & H^*(U') \otimes H^*(F) & \longrightarrow & H^*(U,U') \otimes H^*(F) & \longrightarrow & H^*(U) \otimes H^*(F) & \longrightarrow & ... \\ & & \downarrow{\scriptstyle \Phi_{U'}} & & \downarrow{\scriptstyle \Phi_{U,U'}} & & \downarrow{\scriptstyle \Phi_U} & & \\ ... & \longrightarrow & H^*(p^{-1}(U')) & \longrightarrow & H^*(p^{-1}(U),p^{-1}(U')) & \longrightarrow & H^*(p^{-1}(U)) & \longrightarrow & ... \end{array}$$

The rows are exact by the induced long exact sequences of the pairs (U, U') and $(p^{-1}(U), p^{-1}(U'))$ and the fact that the functor (-) $\otimes_R H^*(F; R)$ is exact.

Let us denote by N the set of n-cells of B. Since each U_α is a neighbourhood wholly contained within a single corresponding n-cell e_α, it must be the case that each U_α is isomorphic to the n-disc D^n.

Since every disc D^n is homotopy equivalent to a single point, U must be homotopy equivalent to a set of distinct points labelled by N.

We must also consider the fact that a disc without a single point in it's interior is homotopy equivalent to its boundary thus each U_α' is homotopy equivalent to $\partial D^n = \mathbb{S}^{n\text{-}1}$, thus U' is homotopy equivalent to the disjoint union of $|N|$ n-1 spheres, we will call this space $N\mathbb{S}^{n\text{-}1}$

$$N\mathbb{S}^{n-1} := \coprod_\alpha \mathbb{S}^{n-1}$$

As in the proof that $\Phi_{B'}$ is an isomorphism, we can construct spaces X and Y such that $p\colon X \to N$ and $p\colon Y \to N\mathbb{S}^{n\text{-}1}$ are fibre bundles with fibre F and so the following diagrams commute:

$$\begin{array}{ccc} H^*(U) \otimes H^*(F) & \xrightarrow{\cong} & H^*(N) \otimes H^*(F) \\ \Phi_U \big\downarrow & \square & \big\downarrow \Phi_N \\ H^*(p^{-1}(U)) & \xrightarrow[\cong]{} & H^*(X) \end{array}$$

$$\begin{array}{ccc} H^*(U') \otimes H^*(F) & \xrightarrow{\cong} & H^*(N\mathbb{S}^{n-1}) \otimes H^*(F) \\ \Phi_{U'} \big\downarrow & \square & \big\downarrow \Phi_{N\mathbb{S}^{n-1}} \\ H^*(p^{-1}(U')) & \xrightarrow[\cong]{} & H^*(Y) \end{array}$$

Therefore since we have proven that the Leray-Hirsch theorem holds for the case of 0-dimensional CW complex base spaces, Φ_N is an isomorphism, and since $N\mathbb{S}^{n\text{-}1}$ is an (n-1)-dimensional CW complex, $\Phi_{N\mathbb{S}^{n\text{-}1}}$ is an isomorpism by assumption.

Therefore, Φ_U and $\Phi_{U'}$ are also isomorphisms.

Finally, it must be shown that all the diagrams are commutative. The following are commutative by the naturality of the cup product:

$$\begin{array}{ccccc} H^*(B, B') \otimes H^*(F) & \longrightarrow & H^*(B) \otimes H^*(F) & \longrightarrow & H^*(B') \otimes H^*(F) \\ \big\downarrow \Phi_{B,B'} & & \big\downarrow \Phi_B & & \big\downarrow \Phi_{B'} \\ H^*(E, E') & \longrightarrow & H^*(E) & \longrightarrow & H^*(E') \end{array}$$

$$\begin{array}{ccccc} H^*(U, U') \otimes H^*(F) & \longrightarrow & H^*(U) \otimes H^*(F) & \longrightarrow & H^*(U') \otimes H^*(F) \\ \big\downarrow \Phi_{U,U'} & & \big\downarrow \Phi_U & & \big\downarrow \Phi_{U'} \\ H^*(p^{-1}(U), p^{-1}(U')) & \longrightarrow & H^*(p^{-1}(U)) & \longrightarrow & H^*(p^{-1}(U')) \end{array}$$

We must finally show that the following diagrams commute:

$$\begin{array}{ccc} H^*(B') \otimes H^*(F) & \xrightarrow{\delta\otimes \mathrm{id}} & H^*(B,B') \otimes H^*(F) \\ \Phi_{B'}\downarrow & & \downarrow\Phi_{B,B'} \\ H^*(E') & \xrightarrow[\delta]{} & H^*(E,E') \end{array}$$

$$\begin{array}{ccc} H^*(U') \otimes H^*(F) & \xrightarrow{\delta\otimes \mathrm{id}} & H^*(U,U') \otimes H^*(F) \\ \Phi_{U'}\downarrow & & \downarrow\Phi_{U,U'} \\ H^*(p^{-1}(U')) & \xrightarrow[\delta]{} & H^*(p^{-1}(U),p^{-1}(U')) \end{array}$$

Since the two squares are identical in structure, we only need to check once.

Let us choose the first and consider a general element of H*(B') ⊗ H*(F) and map it into H*(E, E') in both directions:

$$\begin{aligned} \Phi_{B,B'}((\delta\otimes\mathrm{id})(\sum_{i,j} b_i\otimes\iota^*(c_j))) &= \Phi_{B,B'}(\sum_{i,j}\delta(b_i)\otimes\iota^*(c_j)) \\ &= \sum_{i,j} p^*(\delta(b_i))\smile c_j \\ \delta(\Phi_{B'}(\sum_{i,j} b_i\otimes\iota^*(c_j))) &= \delta(\sum_{i,j} p^*(b_i)\smile c_j) \\ &= \sum_{i,j}\delta(p^*(b_i)\smile c_j) \\ &= \sum_{i,j}\delta(p^*(b_i))\smile c_j,\ \text{since } \delta(c_j)=0 \end{aligned}$$

Hence, since $\delta \circ p^* = p^* \circ \delta$, the squares must commute.

Thus by the Five Lemma $\Phi_{U,U'}$ is an isomorphism, thus $\Phi_{B,\,B'}$ is an isomorphism, and finally, again by the Five Lemma, Φ is an isomorphism.

Therefore the Leray-Hirsch Theorem is true for fibre bundles with finite dimensional CW complex base spaces. △

3.1.4 If *B* is an infinite-dimensional CW complex

The argument when our base space is an infinite dimensional CW complex is not too difficult since we have already laid most of the ground work.

Proof. Let B be an infinite-dimensional CW complex and let B^n be the n-skeleton of B for any finite n.

Clearly, (B, B^n) will be n-connected, thus by our lemma, (E, $p^{-1}(B^n)$) must also be n-connected. Let us examine the following diagram:

$$\begin{array}{ccc} H^*(B;R)\otimes_R H^*(F;R) & \longrightarrow & H^*(B^n;R)\otimes_R H^*(F;R) \\ {\scriptstyle\Phi}\downarrow & \square & \downarrow{\scriptstyle\Phi_{B^n}} \\ H^*(E;R) & \longrightarrow & H^*(p^{-1}(B^n);R) \end{array}$$

Since B^n is an n-dimensional CW subcomplex of B, $p\colon p^{\text{-}1}(B^n) \to B^n$ will be a fibre bundle with fibre F, satisfying the conditions of the Leray-Hirsch theorem, and with a finite-dimensional CW complex base space; therefore, Φ_{B^n} is an isomorphism.

The n-connectedness of $(B,\, B^n)$ and $(E,\, p^{\text{-}1}(B^n))$ ensure that for all $k \leq n$ the horizontal maps are isomorphisms in the following diagram:

$$\begin{array}{ccc} H^{k-*}(B;R)\otimes_R H^*(F;R) & \xrightarrow{\cong} & H^{k-*}(B^n;R)\otimes_R H^*(F;R) \\ {\scriptstyle\Phi_k}\downarrow & \square & \downarrow{\scriptstyle\Phi_{B^n k}} \\ H^k(E;R) & \xrightarrow[\cong]{} & H^k(p^{-1}(B^n);R) \end{array}$$

Since a graded module homomorphism is an isomorphism if and only if on each level it restricts to an abelian group isomorphism and Φ_{B^n} is an isomorphism of $H^*(B^n;\, R)$-modules, $\Phi_{B^n k}$ must be an isomorphism of abelian groups, therefore so too must Φ_k.

Since n was arbitrary, we can deduce that Φ_k is an isomorphism of abelian groups for arbitrary $k \in \mathbb{N}$, and therefore we achieve the result that Φ is an isomorphism of $H^*(B;\, R)$-modules. △

3.1.5 If *B* is not a CW complex

Definition 7. A map $f\colon X \to Y$ is called a **weak homotopy equivalence** if the induced maps $f_*\colon \pi_n(X,\, x_0) \to \pi_n(Y, f(x_0))$ are isomorphisms for all $n \geq 0$ and choices of basepoint $x_0 \in X$. A weak homotopy equivalence is all you need to ensure an isomorphism of cohomology rings.

For a topological space X and a CW complex Z, a weak homotopy equivalence $f\colon Z \to X$ is called a **CW approximation** to X.

Lemma 10. For any topological space X, a CW complex Z can be constructed such that there exists a CW approximation $f\colon Z \to X$

A proof of this statement can be attributed to Hatcher [17].

Now will show that if a fibre bundle $p\colon E \to B$ satisfies the conditions of the Leray-Hirsch theorem but B is not a CW complex, even still, the map Φ is an isomorphism.

Proof. Even though B is not a CW complex, there exists a CW approximation $f\colon A \to B$ and we can construct the pullback bundle that makes the following a commutative diagram:

$$\begin{array}{ccc} f^*(E) & \xrightarrow{\pi_E} & E \\ {\scriptstyle p^*}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & B \end{array}$$

$p^*\colon f^*(E) \to A$ is also a fibre bundle with fibre F. Since this diagram commutes by construction, it will easy to show that Φ is an isomorphism if we can prove that p^* satisfies the conditions of the Leray-Hirsch theorem.

Since fibre bundles induce long exact sequences of homotopy groups we can examine the connection between these two fibre bundles in terms of these sequences:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \pi_n(F) & \longrightarrow & \pi_n(f^*(E)) & \longrightarrow & \pi_n(A) & \longrightarrow & \pi_{n-1}(F) & \longrightarrow & \dots \\ & & \big\downarrow\cong & & \big\downarrow & & \big\downarrow\cong & & \big\downarrow\cong & & \\ \dots & \longrightarrow & \pi_n(F) & \longrightarrow & \pi_n(E) & \longrightarrow & \pi_n(B) & \longrightarrow & \pi_{n-1}(F) & \longrightarrow & \dots \end{array}$$

Since the fibres are the same and f is a CW approximation, only every third vertical map is not automatically an isomorphism. The Five Lemma takes care of those maps and ensures that every vertical map is an isomorphism! Thus the map $\pi_E\colon f^*(E) \to E$ in the pullback diagram induces isomorphisms on all homotopy groups thus it it is a weak homotopy equivalence and must induce an isomorphism in cohomology.

Since p^* and p have the same fibre and p satisifies the conditions of the Leray-Hirsch theorem, $p^*\colon f^*(E) \to A$ is a fibre bundle with fibre F which for some ring with identity R, $H^n(F;\, R)$ is a finitely generated free R-module for all n.

Additionally, the classes $c_j \in H^{k_j}(E;\, R)$ pull back to classes $\pi_E^{*\text{-}1}(c_j) \in H^{k_j}(f^*(E);\, R)$ and thus by the commutativity of the following diagram we have classes that restrict by the inclusion of the fibre into $f^*(E)$ to a basis of the cohomology ring for each fibre.

$$\begin{array}{ccc} H^*(E;R) & \xrightarrow{\pi_E^*} & H^*(f^*(E);R) \\ {\scriptstyle \iota^*}\big\downarrow & \square & \big\downarrow{\scriptstyle \iota_f^*} \\ H^*(F;R) & \xrightarrow[\cong]{} & H^*(F;R) \end{array}$$

By the commutativity of the pull back diagram, functoriality of the tensor product with $H^*(F;\, R)$, and naturality of the cup product, the following diagram commutes:

$$\begin{array}{ccc} H^*(B;R) \otimes_R H^*(F;R) & \xrightarrow{f^* \otimes_R \mathrm{id}} & H^*(A;R) \otimes_R H^*(F;R) \\ {\scriptstyle \Phi}\big\downarrow & \square & \big\downarrow{\scriptstyle \Phi_A} \\ H^*(E;R) & \xrightarrow[\pi_E^*]{} & H^*(f^*(E);R) \end{array}$$

The weak homotopy equivalences ensure that the two horizontal maps are isomorphisms and now we have seen that p^* is a fibre bundle satisfying the conditions of the Leray-Hirsch theorem and has a CW complex as a base space, thus Φ_A is an isomorphism. Therefore the commutativity of the diagram ensures that Φ is also an isomorphism.

Thus, provided the conditions hold, the Leray-Hirsch map is an isomorphism regardless of what kind of topological space we have as a base space. △

3.2 Applications of the Leray-Hirsch Theorem

3.2.1 Cohomology of the Special Unitary Groups

Definition 8. The **exterior algebra** on a finite set of generators x_i with coefficients in a ring R, denoted $\Lambda_R[x_1, ..., x_N]$, where each x_i is in degree d_i, is a ring with an addition operation naturally derived from the ring R, and a multiplication operation defined by the multiplication in R together with the following property:

$$x_p \wedge x_q = (-1)^{d_p d_q}(x_q \wedge x_p)$$

where x_i is in degree d_i.

Theorem 11. The cohomology ring of the groups $\mathrm{SU}(n)$ with coefficients in $\mathbb{Z}$ for $n \geq 2$ is the exterior algebra on n-1 generators x_i each in a different odd degree i, with $3 \leq i \leq 2n\text{-}1$:

$$H^*(\mathrm{SU}(n); \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[x_3, ..., x_{2n\text{-}1}]$$

Proof. We will attempt to prove this using an induction argument.

$\mathrm{SU}(1) \cong 0$ and since we know there is a fibre bundle p: $\mathrm{SU}(2) \to \mathbb{S}^3$ with fibre $\mathrm{SU}(1)$, we know that $\mathrm{SU}(2) \cong \mathbb{S}^3$. Hence $H^*(\mathrm{SU}(2);\ \mathbb{Z}) \cong H^*(\mathbb{S}^3;\ \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[x_3]$ and thus the base case holds.

Let us assume that $H^*(\mathrm{SU}(n\text{-}1);\ \mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_3,\ ...,\ x_{2n\text{-}3}]$ and hopefully conclude that $H^*(\mathrm{SU}(n);\ \mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_3,\ ...,\ x_{2n\text{-}1}]$.

In an earlier section we determined that the map p: $\mathrm{SU}(n) \to \mathbb{S}^{2n\text{-}1}$ defined by $p(X) = (u_{11},\ v_{11},\ ...,\ u_{n1},\ v_{n1})$ where $u_{j1} + \boldsymbol{i} v_{j1}$ is the j,1 entry in the matrix X, is a fibre bundle with fibre $\mathrm{SU}(n\text{-}1)$.

In order to find the cohomology rings of the groups $\mathrm{SU}(n)$ with coefficients in $\mathbb{Z}$ we would like to show that this fibre bundle satisfies the conditions of the Leray-Hirsch Theorem.

Since a fibre bundle induces a short exact sequence, we can consider the long exact sequence of homotopy groups associated with the fibre bundle:

$$... \longrightarrow \pi_{k+1}(\mathbb{S}^{2n-1}) \longrightarrow \pi_k(\mathrm{SU}(n-1)) \longrightarrow \pi_k(\mathrm{SU}(n)) \longrightarrow \pi_k(\mathbb{S}^{2n-1}) \longrightarrow ...$$

Since $\pi_k(\mathbb{S}^n) = 0$ when $k < n$, the long exact sequence above implies that $\pi_k(\mathrm{SU}(n\text{-}1)) \cong \pi_k(\mathrm{SU}(n))\ \forall\ k < 2n\text{-}2$ and $\pi_{2n-2}(\mathrm{SU}(n\text{-}1)) \to \pi_{2n-2}(\mathrm{SU}(n))$ is an epimorphism. Therefore the pair $(\mathrm{SU}(n),\ \mathrm{SU}(n\text{-}1))$ is $2n$-2 connected.

An application of the Hurewicz theorem therefore tells us that $H^k(\mathrm{SU}(n); \mathbb{Z}) \to H^k(\mathrm{SU}(n\text{-}1); \mathbb{Z})$ is at least an epimorphism for $k \leq 2n\text{-}3$, and by our assumption that $H^*(\mathrm{SU}(n\text{-}1); \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[x_3, ..., x_{2n\text{-}3}]$, there must be classes $c_3, ..., c_{2n\text{-}3} \in H^*(\mathrm{SU}(n); \mathbb{Z})$ that these epimorphisms send to the classes corresponding to the generators $x_3, ..., x_{2n\text{-}3}$ respectively.

A basis of $H^*(\mathrm{SU}(n\text{-}1); \mathbb{Z})$ is deduced by a basis of $\Lambda_{\mathbb{Z}}[x_3, ..., x_{2n\text{-}3}]$ which is given by forming the wedge product of distinct x_is. The cup products of the corresponding c_i classes in $H^*(\mathrm{SU}(n); \mathbb{Z})$ will restrict to this basis.

Therefore, by assumption, the cohomology groups of the fibre are finitely generated free $\mathbb{Z}$-modules, and there exist classes in the cohomology ring of the total space whose restrictions form a basis of the cohomology ring of each fibre.

Thus we may apply the Leray-Hirsch Theorem and so,

$$\begin{aligned} H^*(\mathrm{SU}(n);\mathbb{Z}) &\cong H^*(\mathbb{S}^{2n-1};\mathbb{Z}) \otimes H^*(\mathrm{SU}(n-1);\mathbb{Z}) \\ &\cong \Lambda_{\mathbb{Z}}[x_{2n-1}] \otimes \Lambda_{\mathbb{Z}}[x_3, ..., x_{2n-3}] \end{aligned}$$

Finally, we must show that there is an isomorphism

$$\lambda : \Lambda_{\mathbb{Z}}[x_{2n-1}] \otimes \Lambda_{\mathbb{Z}}[x_3, ..., x_{2n-3}] \to \Lambda_{\mathbb{Z}}[x_3, ..., x_{2n-1}]$$

Since $1 \otimes p(x_3, ..., x_{2n-3}) + x_{2n-1} \otimes q(x_3, ..., x_{2n-3})$ is a generic element of $\Lambda_{\mathbb{Z}}[x_{2n-1}] \otimes \Lambda_{\mathbb{Z}}[x_3, ..., x_{2n-3}]$, we will show by a short argument that λ can be defined by:

$$\begin{aligned} \lambda(1 \otimes p(x_3, ..., x_{2n-3}) + x_{2n-1} \otimes q(x_3, ..., x_{2n-3})) = p(x_3, ..., x_{2n-3}) \\ + x_{2n-1} \wedge q(x_3, ..., x_{2n-3}) \end{aligned}$$

We are looking for an isomorphism of rings, and so we only need to show that λ is both injective and surjective.

λ is injective if $Ker(\lambda) = \{0\}$.

$$\begin{aligned} \lambda(1 \otimes p(x_3, ..., x_{2n-3}) + x_{2n-1} \otimes q(x_3, ..., x_{2n-3})) &= 0 \\ p(x_3, ..., x_{2n-3}) + x_{2n-1} \wedge q(x_3, ..., x_{2n-3}) &= 0 \\ \implies p(x_3, ..., x_{2n-3}) = 0 \text{ and } x_{2n-1} \wedge q(x_3, ..., x_{2n-3}) &= 0 \\ \implies p(x_3, ..., x_{2n-3}) = 0 \text{ and } q(x_3, ..., x_{2n-3}) &= 0 \\ \implies 1 \otimes p(x_3, ..., x_{2n-3}) + x_{2n-1} \otimes q(x_3, ..., x_{2n-3}) &= 1 \otimes 0 + x_{2n-1} \otimes 0 \\ &= 0 \end{aligned}$$

Thus $Ker(\lambda) = \{0\}$ and so λ is indeed injective.

Let us now show that λ is surjective.

Consider $T(x_3, ..., x_{2n\text{-}1})$, a generic element of $\Lambda_{\mathbb{Z}}[x_3, ..., x_{2n-1}]$. We can describe $T(x_3, ..., x_{2n\text{-}1})$ in the following manner:

$$T(x_3, ..., x_{2n-1}) = \sum_{(i_3,...,i_{2n-1}) \in B} t_{(i_3,...,i_{2n-1})} \bigwedge_{j=1}^{n-1} x_{2j+1}^{i_{2j+1}}$$

for some given $t_{(i_3,...,i_{2n-1})} \in \mathbb{Z}$, and where B is the set of tuples $(i_3, ..., i_{2n\text{-}1})$ where each i_{k} is either 0 or 1.

Let us make the following denotations:

$$T_0(x_3, ..., x_{2n-3}) = \sum_{(i_3,...,i_{2n-3},0)\in B} t_{(i_3,...,i_{2n-3},0)} \bigwedge_{j=1}^{n-2} x_{2j+1}^{i_{2j+1}}$$

$$T_1(x_3, ..., x_{2n-3}) = \sum_{(i_3,...,i_{2n-3},1)\in B} t_{(i_3,...,i_{2n-3},1)} \bigwedge_{j=1}^{n-2} x_{2j+1}^{i_{2j+1}}$$

Clearly,

$$T(x_3, ..., x_{2n-1}) = T_0(x_3, ..., x_{2n-3}) + x_{2n-1} \wedge T_1(x_3, ..., x_{2n-3})$$

and thus $\lambda(1 \otimes T_0(x_3, ..., x_{2n-3}) + x_{2n-1} \otimes T_1(x_3, ..., x_{2n-3})) = T(x_3, ..., x_{2n-1})$, so λ is surjective.

Therefore λ is an isomorphism and we achieve the result:

$$H^*(\mathrm{SU}(n); \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[x_3, ..., x_{2n-1}]$$

△

3.2.2 Cohomology of Grassmannian Manifolds

Definition 9. A **Grassmannian** of an n-dimensional vector space V, denoted $Gr_k(\mathrm{V})$ is the set of all linear k-dimensional vector subspaces of V.

An ***n*-flag** in $\mathbb{C}^k$ is an n-tuple of orthogonal 1 dimensional vector spaces in $\mathbb{C}^k$ or equivalently, a sequence of vector subspaces $\mathrm{V}_1 \subset ... \subset \mathrm{V}_n \subseteq \mathbb{C}^k$ where V_i has dimension i.

The set of all n-flags in $\mathbb{C}^k$ forms a topological space denoted $F_n(\mathbb{C}^k)$ which is a subspace of the product of n copies of $\mathbb{CP}^{k\text{-}1}$.

There is a natural fibre bundle p: $F_n(\mathbb{C}^k) \to Gr_n(\mathbb{C}^k)$ where p maps each flag $\mathrm{V}_1 \subset ... \subset \mathrm{V}_n \subseteq \mathbb{C}^k$ to the subspace V_n. $p^{\text{-}1}(\mathbb{C}^n)$ is the set of n-flags where $\mathrm{V}_n = \mathbb{C}^n$ which is also the space $F_n(\mathbb{C}^n)$. Thus $F_n(\mathbb{C}^n)$ is the fibre.

We would like to determine the cohomology structure of the Grassmannians $Gr_n(\mathbb{C}^\infty)$ using the fibre bundle p: $F_n(\mathbb{C}^\infty) \to Gr_n(\mathbb{C}^\infty)$ with fibre $F_n(\mathbb{C}^n)$. Clearly, in order to do this we must in turn determine the cohomology structure of $F_n(\mathbb{C}^\infty)$.

As a result, we need to consider the map q: $F_n(\mathbb{C}^\infty) \to F_{n\text{-}1}(\mathbb{C}^\infty)$ which maps each flag $\mathrm{V}_1 \subset ... \subset \mathrm{V}_{n\text{-}1} \subset \mathrm{V}_n \subseteq \mathbb{C}^\infty$ to the flag $\mathrm{V}_1 \subset ... \subset \mathrm{V}_{n\text{-}1} \subseteq \mathbb{C}^\infty$. We see from the fact that $q^{\text{-}1}(\mathbb{C}^1 \subset ... \subset \mathbb{C}^{n\text{-}1} \subseteq \mathbb{C}^\infty)$ is the set of flags of the form $\mathbb{C}^1 \subset ... \subset \mathbb{C}^{n\text{-}1} \subset \mathrm{V}_n \subseteq \mathbb{C}^\infty$ and that choosing a suitable V_n is equivalent to choosing a line that is orthogonal to $\mathbb{C}^{n\text{-}1}$ in $\mathbb{C}^\infty$ which in turn is equivalent to choosing a line in $\mathbb{C}^\infty$, that since q is a fibre bundle, the fibre is $\mathbb{CP}^\infty$.

Since $F_1(\mathbb{C}^\infty) = \mathbb{CP}^\infty$, we know that $H^*(F_1(\mathbb{C}^\infty);\, \mathbb{Z}) = \mathbb{Z}[x]$ where x is in degree 2. Then if q satisfies the conditions of the Leray-Hirsch Theorem then

$$H^*(F_n(\mathbb{C}^\infty);\mathbb{Z}) \cong \bigotimes_{i=1}^{n} \mathbb{Z}[x^{(i)}] = \mathbb{Z}[x^{(1)}, ..., x^{(n)}]$$

where each $x^{(i)}$ is in degree 2. We also need to know the cohomology of the groups $F_n(\mathbb{C}^k)$, which we can determine from the similar map q: $F_n(\mathbb{C}^k) \to F_{n\text{-}1}(\mathbb{C}^k)$ which maps each flag $\mathrm{V}_1 \subset ... \subset \mathrm{V}_{n\text{-}1} \subseteq \mathrm{V}_n \subseteq \mathbb{C}^k$ to the flag $\mathrm{V}_1 \subset ... \subset \mathrm{V}_{n\text{-}1} \subseteq \mathbb{C}^k$.

Again, we see that $q^{\text{-}1}(\mathbb{C}^1 \subset ... \subset \mathbb{C}^{n\text{-}1} \subseteq \mathbb{C}^k)$ is the set of flags of the form $\mathbb{C}^1 \subset ... \subset \mathbb{C}^{n\text{-}1} \subset \mathrm{V}_n \subseteq \mathbb{C}^k$ and that choosing a suitable V_n is equivalent to choosing a line that is orthogonal to $\mathbb{C}^{n\text{-}1}$ in $\mathbb{C}^k$ which in turn is equivalent to choosing a line in $\mathbb{C}^{k\text{-}n+1}$, therefore since q is a fibre bundle, the fibre is $\mathbb{CP}^{k\text{-}n}$.

We know that $H^*(\mathbb{CP}^k;\, \mathbb{Z}) = \mathbb{Z}[x]/(x^{k+1})$ where x is in degree 2, and since $F_1(\mathbb{C}^k) = \mathbb{CP}^{k\text{-}1}$, if q is a fibre bundle, and satisfies the conditions of the Leray-Hirsch Theorem, we have:

$$\begin{aligned} H^*(F_n(\mathbb{C}^k);\mathbb{Z}) &\cong \bigotimes_{i=1}^{n} \mathbb{Z}[x^{(i)}]/((x^{(i)})^{k-i+1}) \\ &\cong \mathbb{Z}[x^{(1)}, ..., x^{(n)}]/((x^{(1)})^k, ..., (x^{(n)})^{k-n+1}) \end{aligned}$$

is an isomorphism of abelian groups.

We will be investigating the cohomology of these spaces again in a later section with the aim of achieving a description of the cohomology ring as a ring instead of just as modules.

4 Classifying Spaces and Adjunctions

4.1 Simplicial Sets and Spaces

Definition 10. The **simplex category $\mathbf{\Delta}$**, as described by Grothendieck [15], is the category whose objects are ordered sets of the form $[n] = \{0, 1, ..., n\}$ for $n \in \mathbb{N}$ and morphisms are order preserving maps $f\colon [n] \to [m]$, i.e. maps such that $i < j$ in $[n] \implies f(i) \leq f(j)$ in $[m]$.

The monomorphisms $\delta^{n,0}, ..., \delta^{n,n}\colon [n\text{-}1] \to [n]$ such that $(\delta^{n,i})^{-1}(i) = \emptyset$ and the epimorphisms $\sigma^{n,0}, ..., \sigma^{n,n}\colon [n{+}1] \to [n]$ such that $|(\sigma^{n,i})^{-1}(i)| = 2$ together generate every morphism in $\mathbf{\Delta}$.

Originally defined by Eilenberg and Zilber [11], a **simplicial set** is a contravariant functor $X\colon \mathbf{\Delta} \to \mathbf{Set}$ and a **simplicial space** is a contravariant functor $Y\colon \mathbf{\Delta} \to \mathbf{Top}$. The images of the monomorphisms $\delta^{n,i}$ are called the **face maps** and denoted $d_{n,i}$ and the images of the epimorphisms $\sigma^{n,i}$ are called the **degeneracy maps** and denoted $s_{n,i}$.

In addition, there is a covariant functor $\mathbf{\Delta} \to \mathbf{Top}$ that sends each object $[n]$ to the standard n-simplex:

$$\Delta^n = \{(x_0, ..., x_n) \mid 0 \leq x_i \leq 1, \sum_{i=0}^{n} x_i = 1\} \subset \mathbb{R}^{n+1}$$

The images of the monomorphisms $\delta^{n,i}$ are called the **coface maps** and denoted $d^{n,i}$ and the images of the epimorphisms $\sigma^{n,i}$ are called the **codegeneracy maps** and denoted $s^{n,i}$.

$$\begin{aligned} d^{n,i} &: \Delta^{n-1} \to \Delta^n \\ (x_0, ..., x_{n-1}) &\mapsto (x_0, ..., x_{i-1}, 0, x_i, ..., x_{n-1}) \\ s^{n,i} &: \Delta^{n+1} \to \Delta^n \\ (x_0, ..., x_{n+1}) &\mapsto (x_0, ..., x_{i-1}, x_i + x_{i+1}, x_{i+2}, ..., x_{n+1}) \end{aligned}$$

As described by Segal in *Categories and Cohomology Theories* [32], the **geometric realisation** of a simplicial set or space X is defined as the following space:

$$|X| = (\coprod_{i=0}^{\infty} X([i]) \times \Delta^i)/\sim$$

where for $(x, p) \in X[n] \times \Delta^n$, $(y, q) \in X[n\text{-}1] \times \Delta^{n\text{-}1}$ we have $(x, p) \sim (y, q)$ if:

$$\begin{aligned} d_{n,i}(x) = y &\text{ and } d^{n,i}(q) = p \\ \text{or } s_{n-1,i}(y) = x &\text{ and } s^{n-1,i}(p) = q \end{aligned}$$

There is a related notion called the **fat geometric realisation** of a simplicial set or space X defined as:

$$||X|| = (\coprod_{i=0}^{\infty} X([i]) \times \Delta^i)/\sim$$

where in this case, $(x,\ p) \sim (y,\ q)$ just if $d_{n,i}(x) = y$ and $d^{n,i}(q) = p$. i.e. the geometric realisation is the quotient of the fat geometric realisation by the equivalence relation induced by the degeneracy and codegeneracy maps.

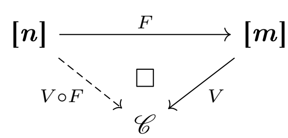
Definition 11. Consider each object $[n]$ of $\mathbf{\Delta}$ as a category in its own right $\boldsymbol{[n]}$. The collection of objects of $\boldsymbol{[n]}$ is the set $[n]$ and the collections of morphisms of $\boldsymbol{[n]}$ are sets with sizes given by:

$$|\mathrm{Hom}_{\boldsymbol{[n]}}(i,j)| = \begin{cases} 1, & i \leq j \\ 0, & else \end{cases}$$

In addition, for every morphism $f\colon\ [n] \to [m]$ in $\mathbf{\Delta}$ we can determine a functor $F\colon\ \boldsymbol{[n]} \to \boldsymbol{[m]}$ that sends each object i of $\boldsymbol{[n]}$ to the object $f(i)$ in $\boldsymbol{[m]}$ and each morphism $\phi \in \mathrm{Hom}_{\boldsymbol{[n]}}(i,j)$ to the morphism $\psi \in \mathrm{Hom}_{\boldsymbol{[m]}}(f(i),f(j))$ which necessarily exists since f is order preserving.

Let $\mathscr{C}$ be a small category. By definition each $\boldsymbol{[n]}$ is a small category and so we may consider the collection of morphisms $\mathrm{Hom}_{\mathbf{Cat}}(\boldsymbol{[n]},\ \mathscr{C})$ in the category of small categories. This collection is itself a set since **Cat** is a locally small category.

Now for each morphism $f\colon\ [n] \to [m]$ in $\mathbf{\Delta}$ we can define a mapping of sets $\mathrm{Hom}_{\mathbf{Cat}}(\boldsymbol{[m]},\ \mathscr{C}) \to \mathrm{Hom}_{\mathbf{Cat}}(\boldsymbol{[n]},\ \mathscr{C})$ by precomposition with the induced functor $F\colon\ \boldsymbol{[n]} \to \boldsymbol{[m]}$ as in the following commutative diagram in **Cat**:



The **nerve** of $\mathscr{C}$ is the simplicial set $N(\mathscr{C})\colon\ \mathbf{\Delta} \to \mathbf{Set}$ that sends each object $[n]$ of $\mathbf{\Delta}$ to the set $\mathrm{Hom}_{\mathbf{Cat}}(\boldsymbol{[n]},\ \mathscr{C})$. This is consistent with the definition by Segal in *Classifying Spaces and Spectral Sequences* [31]

4.2 Classifying Spaces

Definition 12. Let G be a topological group. Following the definition by Steenrod [33], a **principal G-bundle** is a fibre bundle $p\colon\ E \to B$ together with a continuous right action $\alpha\colon\ E \times G \to E$ such that $\forall\ x \in B$, if $y \in p^{-1}(x)$, then $\alpha(y,g) \in p^{-1}(x)\ \forall\ g \in G$. Furthermore, $\forall\ x \in B,\ y \in p^{-1}(x)$, the map $G \to p^{-1}(x)$ given by $g \mapsto \alpha(y,g)$ is a homeomorphism.

Lemma 12. Let G be a topological group and let $p\colon\ E \to B$ be a principal G-bundle such that E is weakly contractible i.e. E is path connected and $\forall\ i$ $\pi_i(E) = 0$. Then for all CW compleces X, the map $\phi\colon\ [X,\ B] \to \mathcal{P}_G(X)$ from the set of homotopy classes of maps from X to B to the set of isomorphism classes of principal G-bundles over X, given by $\phi([f]) = f^*E$ is a bijection.

A proof of this lemma is provided by Theorem 7.4 in lecture notes by Mitchell [27].

This lemma implies that any other principal G-bundle can be written as a pullback of such a fibre bundle, we call a principal G-bundle with contractible total space a **universal G-bundle**. The base space is called a **classifying space of G**, and is denoted BG. It can be seen that for a given G, BG is unique up to homotopy equivalence.

4.2.1 A Candidate for BG

Definition 13. As discussed by Kelly in *Basic Concepts of Enriched Category Theory* [21], a **topologically enriched category** is a small category where the collections of morphisms are topological spaces and the composition of morphisms is continuous.

A **topological category** is a topologically enriched category where the collection of objects is also a topological space.

Any topologically enriched category can be considered as a topological category in a simple manner. If we have a topologically enriched category $\mathscr{X}$' with a set of objects X, then there is a topological category $\mathscr{X}$ with X equipped with the discrete topology as the topological space of objects and spaces of morphisms for each pair of objects inherited from $\mathscr{X}$'.

Theorem 13. For a topological group G consider the topological category $\mathscr{G}$ of one object $*_{\mathscr{G}}$ with morphisms the elements of G. Composition of morphisms is defined by composition in the group G. Then the geometric realisation of the nerve of $\mathscr{G}$, $|N(\mathscr{G})|$ is a classifying space of G.

Proof. In order to show for a topological group G and the topological category $\mathscr{G}$ of one object $*_{\mathscr{G}}$ with morphisms the elements of G, that $|N(\mathscr{G})|$ is a candidate for BG, we must show that it is the base space of a universal G-bundle.

It is natural to consider a similar topological category to $\mathscr{G}$ that we will call $\mathscr{H}$ that has as a collection of objects the topological group G and exactly one morphism from every object to every other object. Composition of morphisms is defined in the following manner: for $\phi \in \mathrm{Hom}_{\mathscr{H}}(g_1,g_2)$, $\psi \in \mathrm{Hom}_{\mathscr{H}}(g_2,g_3)$, $\psi \circ \phi$ is defined to be the unique morphism in $\mathrm{Hom}_{\mathscr{H}}(g_1,g_3)$.

Lemma 14. If $F,G\colon \mathscr{C} \to \mathscr{D}$ are two functors and $\eta\colon F \to G$ is a natural transformation, then there is a homotopy between the two continuous maps $|N(F)|,|N(G)|\colon |N(\mathscr{C})| \to |N(\mathscr{D})|$

Proof. Recall the category **[1]** that has as objects the set $\{0,1\}$ and three morphisms $\mathrm{id}_0 \in \mathrm{Hom}_{\mathbf{[1]}}(0,0)$, $\mathrm{id}_1 \in \mathrm{Hom}_{\mathbf{[1]}}(1,1)$, and $\phi \in \mathrm{Hom}_{\mathbf{[1]}}(0,1)$.

Consider a new functor $H\colon \mathscr{C} \times \mathbf{[1]} \to \mathscr{D}$. H maps pairs of objects in the following manner:

$$H(X,N) = \begin{cases} F(X), & N = 0 \\ G(X), & N = 1 \end{cases}$$

and, since the natural transformation $\eta\colon F \to G$ ensures that for any morphism $f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$ the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\big\downarrow & \square & \big\downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\eta_Y]{} & G(Y) \end{array}$$

H maps pairs of morphisms in the following manner: for $f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$

$$H(f,\xi) = \begin{cases} F(f), & \xi = \mathrm{id}_0 \\ G(f), & \xi = \mathrm{id}_1 \\ \eta_Y \circ F(f) = G(f) \circ \eta_X, & \xi = \phi \end{cases}$$

Now let us consider the continuous map $|N(H)|\colon |N(\mathscr{C} \times \mathbf{[1]})| \to |N(\mathscr{D})|$. It is known that the nerve and geometric realisation respect products and so this is the map $|N(H)|\colon |N(\mathscr{C})| \times |N(\mathbf{[1]})| \to |N(\mathscr{D})|$.

It can be shown that $|N(\mathbf{[1]})| \cong [0,1]$ the unit interval and since the end points of the interval correspond to the two objects of $\mathbf{[1]}$, we must have that $H(X,0) = F(X)$ implies $|N(H)|(x,0) = |N(F)|(x)$ and likewise $H(X,1) = G(X)$ implies $|N(H)|(x,1) = |N(G)|(x)$ therefore $|N(H)|\colon |N(\mathscr{C})| \times [0,1] \to |N(\mathscr{D})|$ is a homotopy from $|N(F)|$ to $|N(G)|$. △

Lemma 15. If $\mathscr{C}$ is a category with a terminal object, then $|N(\mathscr{C})|$ is contractible.

Proof. The category $\mathbf{[0]}$ has a single object 0 and a single morphism id_0. Let $\mathscr{C}$ be any category with a terminal object T, that is, for every object S of $\mathscr{C}$, there is exactly one morphism in the set $\mathrm{Hom}_{\mathscr{C}}(S,T)$.

Let $F\colon \mathbf{[0]} \to \mathscr{C}$ be the functor such that $F(0) = T$ and $F(\mathrm{id}_0) = \mathrm{id}_T$ and let $G\colon \mathscr{C} \to \mathbf{[0]}$ be the functor such that $G(X) = 0$ for all objects X of $\mathscr{C}$ and $G(\phi) = \mathrm{id}_0$ for all morphisms ϕ of $\mathscr{C}$.

Let us consider the composition of these two functors in both ways possible. $GF\colon \mathbf{[0]} \to \mathbf{[0]}$ sends the object 0 to itself, and the morphism id_0 also to itself thus GF is equal to the identity functor on $\mathbf{[0]}$, $\mathrm{id}_{\mathbf{[0]}}$. $FG\colon \mathscr{C} \to \mathscr{C}$ sends every object X of $\mathscr{C}$ to the terminal object T and every morphism ϕ of $\mathscr{C}$ to id_T.

There is a natural transformation $\tau\colon \mathrm{id}_{\mathscr{C}} \to FG$ that has as components the morphisms $\tau_X\colon \mathrm{id}_{\mathscr{C}}(X) \to FG(X)$ in $\mathrm{Hom}_{\mathscr{C}}(X,T)$ which are by definition unique for each object X. τ is indeed a natural transformation as $\forall$ objects X, Y of $\mathscr{C}$, and morphisms $\phi \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$, the following diagram clearly commutes:

$$\begin{array}{ccc} \mathrm{id}_{\mathscr{C}}(X) & \xrightarrow{\tau_X} & FG(X) \\ {\scriptstyle \mathrm{id}_{\mathscr{C}}(\phi)}\big\downarrow & \square & \big\downarrow{\scriptstyle FG(\phi)} \\ \mathrm{id}_{\mathscr{C}}(Y) & \xrightarrow[\tau_Y]{} & FG(Y) \end{array} \quad = \quad \begin{array}{ccc} X & \xrightarrow{\tau_X} & T \\ {\scriptstyle \phi}\big\downarrow & \square & \big\downarrow{\scriptstyle \mathrm{id}_T} \\ Y & \xrightarrow[\tau_Y]{} & T \end{array}$$

Therefore since we have two functors $\mathrm{id}_{\mathscr{C}}, FG \colon \mathscr{C} \to \mathscr{C}$ and a natural transformation $\tau \colon \mathrm{id}_{\mathscr{C}} \to FG$, by the previous lemma we know that there must exist a homotopy $H \colon |N(\mathscr{C})| \times [0,1] \to |N(\mathscr{C})|$ where $H(x,0) = |N(\mathrm{id}_{\mathscr{C}})|(x) = \mathrm{id}_{|N(\mathscr{C})|}(x) = x$ and $H(x,1) = |N(FG)|(x) = |N(F)|(|N(G)|(x))$, since $|N(\mathbf{[0]})|$ is a single point, $|N(G)|$ is the map to just that point, and $|N(F)|$ is the inclusion of that point onto t, the vertex of $|N(\mathscr{C})|$ corresponding to the terminal object T. Thus, $H(x,1) = |N(F)|(|N(G)|(x)) = t$, and therefore the identity is homotopic to the map to a single point and thus $|N(\mathscr{C})|$ is contractible. △

Since the category $\mathscr{H}$ has exactly one morphism from each object to every object, every object of $\mathscr{H}$ must be a terminal object, and thus $|N(\mathscr{H})|$ must be contractible.

$|N(\mathscr{H})|$ is our candidate for the total space of the universal G-bundle that has $|N(\mathscr{G})|$ as a base space. We must show that there is a G-action on the space $|N(\mathscr{H})|$, and that the quotient of this space by the action is $|N(\mathscr{G})|$.

We can describe a G-action on the category $\mathscr{H}$ which consists of:

- a G-action on the group of objects $\alpha \colon \mathrm{ob}\mathscr{H} \times G \to \mathrm{ob}\mathscr{H}$ which sends the pair (h, g) of an object labelled with the group element h and a group element g, to the object labelled hg,
- a G-action on the collection of morphisms $\alpha \colon \mathrm{mor}\mathscr{H} \times G \to \mathrm{mor}\mathscr{H}$ which sends the pair (ϕ, g) of a morphism $\phi \in \mathrm{Hom}_{\mathscr{H}}(h_1,h_2)$ and a group element g to the unique morphism $\phi g \in \mathrm{Hom}_{\mathscr{H}}(h_1 g, h_2 g)$

Let us examine the simplicial spaces $N(\mathscr{G})$ and $N(\mathscr{H})$. We will be determining whether the G-action on $\mathscr{H}$ induces a G-action on $N(\mathscr{H})([n])$ with the desired quotient for each object $[n]$ of Δ.

$N(\mathscr{H})([n]) = \mathrm{Hom}_{\mathbf{Cat}}(\boldsymbol{[n]}, \mathscr{H})$ is equivalent to G^{n+1} since each object of $\boldsymbol{[n]}$ can be sent to any of the elements of $\mathrm{ob}\mathscr{H} = G$ and the image of each morphism will be automatically determined since there is a unique morphism in $\mathscr{H}$ between any two objects.

$N(\mathscr{G})([n]) = \mathrm{Hom}_{\mathbf{Cat}}(\boldsymbol{[n]}, \mathscr{G})$ is equivalent to G^n, since each object $\boldsymbol{[n]}$ is necessarily sent to the object $*_{\mathscr{G}}$, an element of $\mathrm{Hom}_{\mathbf{Cat}}(\boldsymbol{[n]}, \mathscr{G})$ is uniquely determined by the images of the morphisms $\phi_i \colon i\text{-}1 \to i$ for $1 \leq i \leq n$, each of which can be any of the elements of $\mathrm{Hom}_{\mathscr{G}}(*_{\mathscr{G}}, *_{\mathscr{G}}) = G$.

We can define a G-action on $N(\mathscr{H})([n])$ derived from the G-action on $\mathscr{H}$:

$$\begin{aligned} \alpha_n : N(\mathscr{H})([n]) \times G &\to N(\mathscr{H})([n]) \\ ((h_0, ..., h_n), g) &\mapsto (h_0 g, ..., h_n g) \end{aligned}$$

To define the quotient of $N(\mathscr{H})([n])$ by the G-action, we first define an equivalence relation $\sim_G$ in the following manner:

$$(h_0, ..., h_n) \sim_G \alpha_n((h_0, ..., h_n), g) \ \forall \ g \in G$$

It is easy to show that this is indeed an equivalence relation as it is reflexive:

$$\begin{aligned} (h_0, ..., h_n) &\sim_G \alpha_n((h_0, ..., h_n), 1_G) \\ &= (h_0, ..., h_n) \end{aligned}$$

symmetric:

$$\begin{aligned}
(h_0, ..., h_n) &\sim_G (k_0, ..., k_n) \\
\implies (k_0, ..., k_n) &= \alpha_n((h_0, ..., h_n), g) \\
&= (h_0 g, ..., h_n g) \text{ for some } g \in G \\
\text{but also } (k_0, ..., k_n) &\sim_G \alpha_n((k_0, ..., k_n), g^{-1}) \\
&= (k_0 g^{-1}, ..., k_n g^{-1}) = (h_0 g g^{-1}, ..., h_n g g^{-1}) \\
\implies (k_0, ..., k_n) &\sim_G (h_0, ..., h_n)
\end{aligned}$$

and transitive:

$$\begin{aligned}
(h_0, ..., h_n) &\sim_G (k_0, ..., k_n) \\
\text{and } (k_0, ..., k_n) &\sim_G (l_0, ..., l_n) \\
\implies (k_0, ..., k_n) &= \alpha_n((h_0, ..., h_n), g_1) \\
&= (h_0 g_1, ..., h_n g_1) \\
\text{and } (l_0, ..., l_n) &= \alpha_n((k_0, ..., k_n), g_2) \\
&= (k_0 g_2, ..., k_n g_2) \text{ for some } g_1, g_2 \in G \\
\implies (l_0, ..., l_n) &= (h_0 g_1 g_2, ..., h_n g_1 g_2) \\
&= \alpha_n((h_0, ..., h_n), g_1 g_2) \\
\implies (h_0, ..., h_n) &\sim_G (l_0, ..., l_n)
\end{aligned}$$

We now examine the quotient space that we denote by

$$N(\mathscr{H})([n])/G := N(\mathscr{H})([n])/{\sim_G}$$

Points in this space are classes of the form $[(h_0, ..., h_n)]$, but due to the equivalence relation we can always choose $(1_G, h_1 h_0^{-1}, ..., h_n h_0^{-1})$ to be the representative of each class and we see that $N(\mathscr{H})([n])/G \cong G^n$.

There is in fact a homeomorphism between $N(\mathscr{H})([n])/G$ and $N(\mathscr{G})([n])$ given by the map:

$$\begin{aligned}
\phi : N(\mathscr{H}([n])/G &\to N(\mathscr{G})([n]) \\
[(h_0, ..., h_n)] &\mapsto (h_1 h_0^{-1}, ..., h_i h_{i-1}^{-1}, ..., h_n h_{n-1}^{-1})
\end{aligned}$$

which is continuous since composition and inversion are required to be in a topological group, with a similarly continuous map:

$$\begin{aligned}
\psi : N(\mathscr{G})([n]) &\to N(\mathscr{H})([n])/G \\
(g_1, ..., g_n) &\mapsto [(1_G, g_1, g_2 g_1, ..., g_n ... g_1)]
\end{aligned}$$

which can easily be shown to be the inverse of ϕ:

$$\begin{aligned}\psi(\phi([(h_0,...,h_n)])) &= \psi((h_1h_0^{-1},...,h_nh_{n-1}^{-1})) \\ &= [(1_G, h_1h_0^{-1}, h_2h_1^{-1}h_1h_0^{-1},...,h_nh_{n-1}^{-1}...h_1h_0^{-1})] \\ &= [(1_G, h_1h_0^{-1},...,h_nh_0^{-1})] \\ &= [(h_0,...,h_n)] \\ \phi(\psi((g_1,...,g_n))) &= \phi([(1_G, g_1, g_2g_1,...,g_n...g_1)]) \\ &= (g_1 1_G^{-1}, g_2g_1g_1^{-1},...,g_n...g_1(g_{n-1}...g_1)^{-1}) \\ &= (g_1, g_2g_1g_1^{-1},...,g_n...g_1g_1^{-1}...g_{n-1}^{-1}) \\ &= (g_1,...,g_n)\end{aligned}$$

While we now have a homeomorphism on every level, before we can say we have a homeomorphism of simplicial spaces we need to verify that our constructions are compatible with the face and degeneracy maps.

In the following diagrams, q denotes the quotient maps $N(\mathscr{H})([m]) \to N(\mathscr{H})([m])/G$, ϕ denotes the homeomorphisms $N(\mathscr{H})([m])/G \to N(\mathscr{G})([m])$, and $d_{n,i}$, $s_{n,i}$ denote the face and degeneracy maps respectively in the appropriate simplicial spaces. In the diagrams it is clear which of these maps is meant in each case but in the proof following they will be appropriately indexed.

$$\begin{array}{ccc} N(\mathscr{H})([n]) & \xrightarrow{\phi\circ q} & N(\mathscr{G})([n]) \\ {\scriptstyle d_{n,i}}\downarrow & \square & \downarrow{\scriptstyle d_{n,i}} \\ N(\mathscr{H})([n-1]) & \underset{\phi\circ q}{\longrightarrow} & N(\mathscr{G})([n-1]) \end{array} \qquad \begin{array}{ccc} N(\mathscr{H})([n]) & \xrightarrow{\phi\circ q} & N(\mathscr{G})([n]) \\ {\scriptstyle s_{n,i}}\downarrow & \square & \downarrow{\scriptstyle s_{n,i}} \\ N(\mathscr{H})([n+1]) & \underset{\phi\circ q}{\longrightarrow} & N(\mathscr{G})([n+1]) \end{array}$$

Let us first describe what the face and degeneracy maps do in our simplicial spaces:

$$\begin{aligned} d_{n,i}^{\mathscr{H}} &: N(\mathscr{H})([n]) \to N(\mathscr{H})([n-1]) \\ (h_0,...,h_n) &\mapsto (h_0,...,h_{i-1},h_{i+1},...,h_n) \\ d_{n,i}^{\mathscr{G}} &: N(\mathscr{G})([n]) \to N(\mathscr{G})([n-1]) \\ (g_1,...,g_n) &\mapsto \begin{cases} (g_2,...,g_n),\ i=0 \\ (g_1,...,g_{i-1},g_{i+1}g_i,g_{i+2},...,g_n),\ 1\le i\le n-1 \\ (g_1,...,g_{n-1}),\ i=n \end{cases} \\ s_{n,i}^{\mathscr{H}} &: N(\mathscr{H})([n]) \to N(\mathscr{H})([n+1]) \\ (h_0,...,h_n) &\mapsto (h_0,...,h_i,h_i,...,h_n) \\ s_{n,i}^{\mathscr{G}} &: N(\mathscr{G})([n]) \to N(\mathscr{G})([n+1]) \\ (g_1,...,g_n) &\mapsto (g_1,...,g_i,1_G,g_{i+1},...,g_n) \end{aligned}$$

Now to check that the diagrams commute as we would like:

$$
\begin{aligned}
(d^{\mathscr{G}}_{n,i} \circ (\phi_n \circ q_n))(h_0, ..., h_n) &= (d^{\mathscr{G}}_{n,i} \circ \phi_n)([(h_0, ..., h_n)]) \\
&= d^{\mathscr{G}}_{n,i}(h_1 h_0^{-1}, ..., h_n h_{n-1}^{-1}) \\
&= \begin{cases} (h_2 h_1^{-1}, ..., h_n h_{n-1}^{-1}),\ i = 0 \\ (h_1 h_0^{-1}, ..., h_{i+1} h_{i-1}^{-1}, ..., h_n h_{n-1}^{-1}),\ 1 \leq i \leq n-1 \\ (h_1 h_0^{-1}, ..., h_{n-1} h_{n-2}^{-1}),\ i = n \end{cases} \\
((\phi_{n-1} \circ q_{n-1}) \circ d^{\mathscr{H}}_{n,i})(h_0, ..., h_n) &= (\phi_{n-1} \circ q_{n-1})(h_0, ..., h_{i-1}, h_{i+1}, ..., h_n) \\
&= \phi_{n-1}([(h_0, ..., h_{i-1}, h_{i+1}, ..., h_n)]) \\
&= \begin{cases} (h_2 h_1^{-1}, ..., h_n h_{n-1}^{-1}),\ i = 0 \\ (h_1 h_0^{-1}, ..., h_{i+1} h_{i-1}^{-1}, ..., h_n h_{n-1}^{-1}),\ 1 \leq i \leq n-1 \\ (h_1 h_0^{-1}, ..., h_{n-1} h_{n-2}^{-1}),\ i = n \end{cases} \\
(s^{\mathscr{G}}_{n,i} \circ (\phi_n \circ q_n))(h_0, ..., h_n) &= (s^{\mathscr{G}}_{n,i} \circ \phi_n)([(h_0, ..., h_n)]) \\
&= s^{\mathscr{G}}_{n,i}(h_1 h_0^{-1}, ..., h_n h_{n-1}^{-1}) \\
&= (h_1 h_0^{-1}, ..., h_i h_{i-1}^{-1}, 1_G, h_{i+1} h_i^{-1}, ..., h_n h_{n-1}^{-1}) \\
((\phi_{n+1} \circ q_{n+1}) \circ s^{\mathscr{H}}_{n,i})(h_0, ..., h_n) &= (\phi_{n+1} \circ q_{n+1})(h_0, ..., h_i, h_i, ..., h_n) \\
&= \phi_{n+1}([(h_0, ..., h_i, h_i, ..., h_n)]) \\
&= (h_1 h_0^{-1}, ..., h_i h_{i-1}^{-1}, 1_G, h_{i+1} h_i^{-1}, ..., h_n h_{n-1}^{-1})
\end{aligned}
$$

We see that the diagrams both commute therefore the composition $\phi \circ q$ with components $\phi_k \circ q_k$ is a simplicial map.

This also ensures that the intermediary $N(\mathscr{H})/G := Im(q)$ is a simplicial space and the fact that $\phi_n\colon N(\mathscr{H})([n])/G \to N(\mathscr{G})([n])$ is a homeomorphism for each n ensures that $||N(\mathscr{H})/G||$ is at least homotopy equivalent to $||N(\mathscr{G})||$ as proved by Segal [32]. Additionally, if the inclusion group homomorphism $\iota\colon 1 \to G$ is a cofibration then more helpfully we have that the induced map $\phi^*\colon |N(\mathscr{H})/G| \to |N(\mathscr{G})|$ is also a homotopy equivalence.

We can define a G-action on $|N(\mathscr{H})|$ via the G-actions we have defined on each level of the nerve:

$$
\begin{aligned}
\alpha : |N(\mathscr{H})| \times G &\to |N(\mathscr{H})| \\
([((h_0, ..., h_n), t)], g) &\mapsto [(\alpha_n((h_0, ..., h_n), g), t)]
\end{aligned}
$$

It is the quotient of this action that must be, at least up to homotopy equivalence, a principal G-bundle over $|N(\mathscr{G})|$ in order for $|N(\mathscr{G})|$ to be a valid model for BG.

The G-action quotient and the geometric realisation are quotients that do not interfere with each other, the G-action is a quotient on each $N(\mathscr{H})([i])$ with no relations between levels or to the simplices while the geometric realisation only has relations between consecutive levels $N(\mathscr{H})([i]) \times \Delta^i$ with no relations within any individual level, therefore:

$$|N(\mathscr{H})|/G \cong |N(\mathscr{H})/G| \simeq |N(\mathscr{G})|$$

and we see that the quotient of the contractible space $|N(\mathscr{H})|$ by the G-action we have defined is, up to homotopy equivalence, $|N(\mathscr{G})|$.

The local triviality of the map $\phi^* \circ q$: $|N(\mathscr{H})| \to |N(\mathscr{G})|$ is another consequence of the group homomorphism ι: $1 \to G$ being a cofibration as proved by Segal [32].

Thus $|N(\mathscr{G})|$ is the base space of a principal G-bundle, and therefore a model for the classifying space BG. △

4.3 The Suspension-Loop Adjunction

The following very important functors and the adjunction between them are defined here using the definitions provided in *Topology a Categorical Approach* [6]

Definition 14. For two pointed topological spaces $(X,\ x_0)$, $(Y,\ y_0)$, the **smash product** $X \wedge Y$ is defined as $(X \times Y)/\sim$ where $\sim$ is the equivalence relation:

$$(x, y_0) \sim (x', y_0)\ \forall\ x, x' \in X$$
$$(x_0, y) \sim (x_0, y')\ \forall\ y, y' \in Y$$

$X \wedge Y$ is a pointed topological space with base point $[(x_0,\ \mathrm{y}_0)]$.

For a pointed topological space $(X,\ x_0)$, the **reduced suspension** ΣX is defined to be the smash product $X \wedge \mathbb{S}^1$. The mapping Σ: **Top*** $\to$ **Top*** can be shown to be a covariant functor.

Definition 15. For two pointed topological spaces $(X,\ x_0)$, $(Y,\ y_0)$, consider the set $\mathrm{Hom}_{\mathbf{Top*}}(X,\ Y)$ (and it is a set as **Top*** is a locally small category since it is a subcategory of **Set**). We can equip $\mathrm{Hom}_{\mathbf{Top*}}(X,\ Y)$ with the **compact-open topology** defined to be the coarsest topology containing all sets of the form:

$$S(K, U) := \{f \in \mathrm{Hom}_{\mathbf{Top*}}(X, Y) \mid f(K) \subseteq U\}$$

where $K \subset X$ is compact and $U \subset Y$ is open. Additionally we can define the base point to be the map $X \to Y$ that sends all of X to the base point y_0.

For a pointed topological space $(X,\ x_0)$, the **loop space** ΩX is defined to be the space $\mathrm{Hom}_{\mathbf{Top*}}(\mathbb{S}^1,\ X)$. The mapping Ω: **Top*** $\to$ **Top*** can also be shown to be a covariant functor.

Lemma 16. There is an adjunction $\Sigma \dashv \Omega$

Proof. We would like to show that for all pointed topological spaces, X and Y, we have an isomorphism that is natural in both arguments:

$$\Theta_{X,Y} : \mathrm{Hom}_{\mathbf{Top*}}(\Sigma X, Y) \to \mathrm{Hom}_{\mathbf{Top*}}(X, \Omega Y)$$

Helpfully, if $(Z,\ z_0)$ is locally compact and Hausdorff, and $\mathrm{Hom}_{\mathbf{Top*}}(Z,\ Y)$ is equipped with the compact-open topology, then for any other pointed topological space $(X,\ x_0)$ there is a bijection

$$\theta_{Z,X,Y} : \mathrm{Hom}_{\mathbf{Top*}}(X \wedge Z, Y) \to \mathrm{Hom}_{\mathbf{Top*}}(X, \mathrm{Hom}_{\mathbf{Top*}}(Z, Y))$$

that sends a morphism g: $X \wedge Z \to Y$ to the morphism that sends each $x \in X$ to g evaluated at x, $g([(x, \text{-})])$: $Z \to Y$. [6]

Since $\mathbb{S}^1$ is compact Hausdorff, taking $Z = \mathbb{S}^1$ we see that $\theta_{\mathbb{S}^1, X, Y} = \Theta_{X,Y}$ is an isomorphism. We must now check that this isomorphism is natural.

Let ϕ: $X \to X$', ψ: $Y \to Y$' be continuous maps of pointed topological spaces. We must show that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{Top*}}(\Sigma X', Y) & \xrightarrow{\Theta_{X',Y}} & \mathrm{Hom}_{\mathbf{Top*}}(X', \Omega Y) \\
{\scriptstyle \mathrm{Hom}_{\mathbf{Top*}}(\Sigma\phi, Y)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}_{\mathbf{Top*}}(\phi, \Omega Y)} \\
\mathrm{Hom}_{\mathbf{Top*}}(\Sigma X, Y) & \xrightarrow{\Theta_{X,Y}} & \mathrm{Hom}_{\mathbf{Top*}}(X, \Omega Y) \\
{\scriptstyle \mathrm{Hom}_{\mathbf{Top*}}(\Sigma X, \psi)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}_{\mathbf{Top*}}(X, \Omega\psi)} \\
\mathrm{Hom}_{\mathbf{Top*}}(\Sigma X, Y') & \xrightarrow{\Theta_{X,Y'}} & \mathrm{Hom}_{\mathbf{Top*}}(X, \Omega Y')
\end{array}$$

We therefore need to show that for any continuous map $f \in \mathrm{Hom}_{\mathbf{Top*}}(\Sigma X', Y)$ we have $(\mathrm{Hom}_{\mathbf{Top*}}(\phi, \Omega Y) \circ \Theta_{X',Y})(f) = (\Theta_{X,Y} \circ \mathrm{Hom}_{\mathbf{Top*}}(\Sigma\phi, Y))(f)$, and for any continuous map $g \in \mathrm{Hom}_{\mathbf{Top*}}(\Sigma X, Y)$ we similarly have another equality $(\mathrm{Hom}_{\mathbf{Top*}}(X, \Omega\psi) \circ \Theta_{X,Y})(g) = (\Theta_{X,Y'} \circ \mathrm{Hom}_{\mathbf{Top*}}(\Sigma X, \psi))(g)$.

$$\begin{aligned}
f :&\ \Sigma X' \to Y \\
\mathrm{Hom}_{\mathbf{Top*}}(\Sigma\phi, Y)(f) :&\ \ \Sigma X \to Y \\
= f \circ \Sigma\phi :&\ \ \Sigma X \to Y \\
\Sigma\phi :&\ \ \Sigma X \to \Sigma X' \\
&\ (x, z) \mapsto (\phi(x), z) \\
\implies f \circ \Sigma\phi :&\ \ \Sigma X \to Y \\
&\ (x, z) \mapsto f(\phi(x), z) \\
(\Theta_{X,Y} \circ \mathrm{Hom}_{\mathbf{Top*}}(\Sigma\phi, Y))(f) :&\quad X \to \Omega Y \\
= \Theta_{X,Y}(f \circ \Sigma\phi) :&\quad X \to \Omega Y \\
&\quad x \mapsto (f \circ \Sigma\phi)(x, -) \\
= &\quad x \mapsto f(\phi(x), -) \\
\Theta_{X',Y}(f) :&\quad X' \to \Omega Y \\
&\quad x' \mapsto f(x', -) \\
(\mathrm{Hom}_{\mathbf{Top*}}(\phi, \Omega Y) \circ \Theta_{X',Y})(f) :&\quad X \to \Omega Y \\
= \Theta_{X',Y}(f) \circ \phi :&\quad X \to \Omega Y \\
&\quad x \mapsto f(\phi(x), -)
\end{aligned}$$

Thus $(\text{Hom}_{\textbf{Top*}}(\phi,\Omega Y)\circ\ \Theta_{X',Y})(f) = (\Theta_{X,Y}\circ\text{Hom}_{\textbf{Top*}}(\Sigma\phi, Y))(f)$,

$$\begin{aligned}
g &: \Sigma X \to Y \\
\text{Hom}_{\textbf{Top*}}(\Sigma X, \psi)(g) &: \Sigma X \to Y' \\
= \psi \circ g &: \Sigma X \to Y' \\
(\Theta_{X,Y'} \circ \text{Hom}_{\textbf{Top*}}(\Sigma X, \psi))(g) &: \quad X \to \Omega Y' \\
= \Theta_{X,Y'}(\psi \circ g) &: \quad X \to \Omega Y' \\
&\quad\ \ x \mapsto (\psi \circ g)(x, -) \\
= &\quad\ \ x \mapsto \psi \circ g(x, -) \\
\Theta_{X,Y}(g) &: \quad X \to \Omega Y \\
&\quad\ \ x \mapsto g(x, -) \\
(\text{Hom}_{\textbf{Top*}}(X, \Omega\psi) \circ \Theta_{X,Y})(g) &: \quad X \to \Omega Y' \\
= \Omega\psi \circ \Theta_{X,Y}(g) &: \quad X \to \Omega Y' \\
&\quad\ \ x \mapsto \Omega\psi(g(x, -)) \\
\Omega\psi &: \Omega Y \to \Omega Y' \\
&\quad\ \ \omega \mapsto \psi \circ \omega \\
\implies \Omega\psi \circ \Theta_{X,Y}(g) &: \quad X \to \Omega Y' \\
&\quad\ \ x \mapsto \psi \circ g(x, -)
\end{aligned}$$

Thus $(\text{Hom}_{\textbf{Top*}}(X,\Omega\psi)\circ\ \Theta_{X,Y})(g) = (\Theta_{X,Y'}\circ\text{Hom}_{\textbf{Top*}}(\Sigma X,\psi))(g)$, and therefore $\Theta_{X,Y}$ is natural in both arguments and the adjunction $\Sigma \dashv \Omega$ follows. △

4.4 The Homotopy Fibre

Definition 16. Let f: $A \to B$ be a continuous function of topological spaces.
The **mapping path space** is defined to be the space:

$$E_f := \{(a, p) \mid a \in A, p : [0, 1] \to B \text{ such that } p(0) = f(a)\}$$

Lemma 17. The map π: $E_f \to B$ given by $\pi(a,\ p) := p(1)$ is a fibration.

Proof. Given a topological space X, a homotopy h: $X \times I \to B$, and a map H_0: $X \times \{0\} \to E_f$ such that $\pi \circ H_0 = h|_{X\times\{0\}}$, we need to show the existence of a map H: $X \times I \to E_f$ such that $\pi \circ H = h$ and $H|_{X\times\{0\}} = H_0$.

Let π_A: $E_f \to A$ and π_{B^I}: $E_f \to B^I$ be the projections of the mapping path space onto the first and second coordinates respectively, and let us define $a_x := (\pi_A \circ H_0)(x)$ and $p_x := (\pi_{B^I} \circ H_0)(x)$, i.e. $H_0(x) = (a_x,\ p_x)$. Since $H_0(x) \in E_f$ and $\pi \circ H_0 = h|_{X\times\{0\}}$, p_x is a path from $f(a_x)$ to $h(x,\ 0)$.

In describing the homotopy H, we require that $H(x,\ 0) = H_0(x) = (a_x,\ p_x)$ and $\pi(H(x,\ t)) = h(x,\ t)$, i.e. the path $\pi_{B^I}(H(x,\ t))$ needs to end at $h(x,\ t)$ and needs to be p_x when $t = 0$. Consider the homotopy:

$$H(x,t) := (a_x, s \mapsto \begin{cases} p_x(\frac{s}{1-\frac{t}{2}}),\ 0 \leq s \leq 1 - \frac{t}{2} \\ h(x, 2(s - (1 - \frac{t}{2}))),\ 1 - \frac{t}{2} < s \leq 1 \end{cases} \quad)$$

We can easily verify that this indeed is a well defined map and that our two requirements hold:

$$\begin{aligned}
\text{at } s = 1 - \frac{t}{2}: \qquad p_x(\frac{1-\frac{t}{2}}{1-\frac{t}{2}}) &= p_x(1) \\
&= \pi(H_0(x)) \\
h(x, 2(1-\frac{t}{2} - (1-\frac{t}{2}))) &= h(x,0) \\
&= \pi(H_0(x)) \\
\text{at } s = 1: \quad \pi_{B^I}(H(x,t))(1) &= h(x, 2(1-(1-\frac{t}{2}))) \\
&= h(x, 2(\frac{t}{2})) \\
&= h(x,t) \\
\text{at } t = 0: \quad \pi_{B^I}(H(x,0))(s) &= \begin{cases} p_x(\frac{s}{1-\frac{0}{2}}), \ 0 \leq s \leq 1 - \frac{0}{2} \\ h(x, 2(s-(1-\frac{0}{2}))), \ 1 - \frac{0}{2} < s \leq 1 \end{cases} \\
&= p_x(s)
\end{aligned}$$

Thus H is a homotopy that lifts our given homotopy h and agrees with our given map H_0. Since both were arbitrary, $\pi\colon E_f \to B$ is a fibration. △

Definition 17. If E_f is the mapping path space of a map $f\colon A \to B$, then the fibre of the fibration $\pi\colon E_f \to B$ is called the **homotopy fibre** of f and is denoted F_f; particularly, it is unique up to homotopy equivalence.

Lemma 18. If $f\colon * \to B$ is a map from the one point space to any topological space B, then the homotopy fibre of f is homotopy equivalent to ΩB.

Proof. For any continuous function of topological spaces $f\colon A \to B$, the homotopy fibre can be written as:

$$F_f \simeq \{(a,p) \in E_f \mid \pi(a,p) = p(1) = b_0 \text{ for some } b_0 \in B\}$$

up to homotopy equivalence.

Since any choice of b_0 yields a space homotopy equivalent to F_f, let us consider the space when $b_0 = p(0) = f(a)$; there every path $p\colon [0,\ 1] \to B$ is actually a loop $p\colon \mathbb{S}^1 \to B$ since $p(0) = p(1)$!

For a general topological space A there may be many choices for $a \in A$ such that $(a,\ p) \in F_f$. However for $A = *$ the one point space, we naturally only have once choice for $a \in *$ so for $f\colon * \to B$ we have:

$$\begin{aligned}
E_f &= \{(a,p) \mid a \in *,\ p : [0,1] \to B \text{ such that } p(0) = f(a)\} \\
&= \{p : [0,1] \to B \text{ such that } p(0) = f(a)\} \\
\implies F_f &\simeq \{p : \mathbb{S}^1 \to B \text{ such that } p(0) = p(1) = f(a)\} \\
&= \{p : (\mathbb{S}^1, 0) \to (B, f(a))\} = \Omega B
\end{aligned}$$

△

Theorem 19. Let G be a topological group and H be an admissible subgroup of G, that is, the quotient map $G \to G/H$ is a principal H-bundle. Then the homotopy fibre of the map of classifying spaces $B\iota\colon BH \to BG$ induced by the inclusion is weakly equivalent to G/H, i.e. $\pi_k(F_{B\iota}) \cong \pi_k(G/H) \ \forall\ k \in \mathbb{N}$.

A proof for this statement is given by Theorem 11.3 in the notes by Mitchell [27].

For H equal to the trivial group, this theorem together with the previous lemma boil down to the following statement:

Corollary 20. For any topological group G, there is a homotopy equivalence $\Omega BG \simeq G$.

5 Characteristic Classes and $\mathbf{K^0}$

A vector bundle p: $E \to B$ is a fibre bundle where each fibre $p^{-1}(b)$ is a fixed vector space V, and for the open neighbourhoods U of the points $x \in B$ the homeomorphisms h_U: $U \times$ V $\to p^{-1}(U)$ are fibrewise linear:

i.e. $h_U(x, \mathbf{v}) + h_U(x, \mathbf{w}) = h_U(x, \mathbf{v}+\mathbf{w})$ and $h_U(x, \lambda\mathbf{v}) = \lambda h_U(x, \mathbf{v})$ for any λ in the underlying field of V.

The following definition and proofs of the non-trivial statements within it can be found in *Complex Topological K-Theory* by Efton Park [28].

Definition 18. The set of isomorphism classes of complex vector bundles over a topological space X is denoted $Vect(X)$. $Vect(X)$ is an abelian monoid when equipped with the direct sum of vector bundles as a binary operation.

The **Grothendieck completion** of an abelain monoid A is the abelian group $(A \times A)/\sim$ where $(a, b) \sim (a', b')$ if $a + b' + c = a' + b + c$ for some $c \in A$. A class $[(a, b)] \in (A \times A)/\sim$ is denoted a - b and called a **formal difference**.

The Grothendieck completion of $Vect(X)$ is an abelian group denoted $K^0(X)$ called the **0th K-theory group** of X.

The abelian group $K^0(X)$ can be seen to be a ring when equipped with the tensor product of vector bundles as a binary operation.

The following definition for characteristic classes, along with the definitions of specific characteristic classes found in the next section, all adhere to the definitions provided by Milnor and Stasheff [26]

Definition 19. A **characteristic class** of a vector bundle p: $E \to B$ is some class $\phi(E) \in H^*(B; G)$ such that for any continuous map f: $B' \to B$, we have $\phi(f^*E) = f^*(\phi(E))$ where f^* is the pullback of E to B' by f on the left hand side, and the induced map in cohomology on the right hand side.

Lemma 21. If p: $E \to B$ and q: $E' \to B$ are two vector bundles such that there is an isomorphsim f: $E' \to E$, then if $\phi(E)$ is a characteristic class of p, then $\phi(E') = \phi(E)$, i.e. characteristic classes are unique up to isomorphism and thus define maps ϕ: $Vect(B) \to H^*(B; G)$, which by the definition of characteristic classes, are the components of a natural transformation ϕ: $Vect \to H^*(\text{-}; G)$.

Many characteristic classes define more than just maps and thus a natural transformation between functors from **Top** to **Set**.

As we will see, we can construct characteristic classes that define monoid homomorphisms and thus natural transformations $K^0 \to H^*(\text{-}; G)$ as functors **Top** $\to$ **Grp**, with the goal for now being to construct a characteristic class that defines ring homomorphisms and thus a natural transformation $K^0 \to H^*(\text{-}; G)$ as functors **Top** $\to$ **Rng**.

5.1 Stiefel-Whitney Classes

Definition 20. Consider an n-dimensional real vector bundle p: $E \to B$ (i.e. where the fibre is $\mathbb{R}^n$ for some $n \in \mathbb{N}$). The **Stiefel-Whitney classes** are a sequence of classes $w_i(E) \in H^i(B;\ \mathbb{Z}/2\mathbb{Z})$ for $i \in \mathbb{Z}$ subject to the following axioms:

- $$w_0(E) = 1 \in H^0(B; \mathbb{Z}/2\mathbb{Z}),$$
  and $w_i(E) = 0 \in H^i(B; \mathbb{Z}/2\mathbb{Z})\ \forall i > n$,
- If p': E' $\to B$' is a n-dimensional real vector bundle and f: $B \to B$' is covered by a bundle map g: $E \to E$' (i.e. g maps each fibre of p isomorphically onto a fibre of p'), then $w_i(E) = f^*(w_i(E'))= w_i(f^*(E'))$,
- If q: E' $\to B$ is an m-dimensional real vector bundle over the same base space as p, then
  $$w_k(E \oplus E') = \sum_{i=0}^{k} w_i(E) \smile w_{k-i}(E')$$
- For the canonical line bundle over the circle γ_1^1: $M \to \mathbb{S}^1$, the Stiefel-Whitney class $w_1(M) \neq 0 \in H^1(\mathbb{S}^1;\ \mathbb{Z}/2\mathbb{Z})$

We can define the **total Stiefel-Whitney class** of an n-dimensional real vector bundle p: $E \to B$ to be the class $w(E) \in H^*(B;\ \mathbb{Z}/2\mathbb{Z})$ as follows:

$$w(E) := \sum_{i=0}^{\infty} w_i(E) = 1 + w_1(E) + ... + w_n(E) \text{ by the first axiom}$$

Together with the third axiom, the total Stiefel-Whitney class permits us to write the following relation for any two real vector bundles over the same base space p: $E \to B$ and q: E' $\to B$:

$$w(E \oplus E') = w(E)w(E')$$

5.2 The Euler Class

Definition 21. Let V be a real vector space of dimension $n > 0$. An **orientation of V** is an equivalence class of bases of V. Two bases $\mathbf{v}_1$, ..., $\mathbf{v}_n$ and $\mathbf{w}_1$, ..., $\mathbf{w}_n$ are said to be equivalent if the matrix A satisfying the equation

$$(\mathbf{v}_1, ..., \mathbf{v}_n)A = (\mathbf{w}_1, ..., \mathbf{w}_n)$$

has positive determinant.

A linear map k: V $\to$ V is said to be **orientation preserving** if $\mathbf{v}_1$, ..., $\mathbf{v}_n$ and $k(\mathbf{v}_1)$, ..., $k(\mathbf{v}_n)$ are in the same orientation of V $\forall$ bases $\mathbf{v}_1$, ..., $\mathbf{v}_n$ of V.

Let p: $E \to B$ be an n-dimensional real vector bundle with fibre V. An **orientation of *p*** is a function which assigns an orientation to each fibre V such that for every point $x \in B$, there exists a neighbourhood of x, U and a homeomorphism h: $U \times \mathbb{R}^n \to p^{-1}(U)$ so that for each fibre $\mathrm{V} = p^{-1}(u)$ over U, the homomorphisms ψ_u: $x \mapsto h(u, x)$ from $\mathbb{R}^n$ to $\mathrm{V} \cong \mathbb{R}^n$ are orientation preserving for all $u \in U$. If we can construct an orientation of a vector bundle p, then we say that p is **orientable**. An orientable vector bundle equipped with an orientation is called **oriented**.

Let $\mathrm{V}_0 := \mathrm{V}\backslash\{0\}$. The choice of an orientation of V corresponds to a choice of generator μ_{V} of the singular homology group $H_n(\mathrm{V}, \mathrm{V}_0; \mathbb{Z})$. We can use μ_{V} to obtain a generator u_{V} of the cohomology group $H^n(\mathrm{V}, \mathrm{V}_0; \mathbb{Z})$ using the inner product equation $<u_{\mathrm{V}}, \mu_{\mathrm{V}}> = 1$.

Theorem 22. If p: $E \to B$ is an oriented n-dimensional real vector bundle, and z: $B \to E$ is the zero section to define $E_0 = E\backslash z(B)$ then $H^i(E, E_0; \mathbb{Z})$ is trivial for $i < n$ and $H^n(E, E_0; \mathbb{Z})$ has exactly one cohomology class u, that we call the **orientation class**, such that under the induced map of the inclusion ι: $\mathrm{V} \to E$, the restriction of u to $\iota^*(u) \in H^n(\mathrm{V}, \mathrm{V}_0; \mathbb{Z})$ is equal to the already chosen generator u_{V} for every fibre.

Furthermore Ψ: $H^k(E; \mathbb{Z}) \to H^{k+n}(E, E_0; \mathbb{Z})$ defined by $\Psi(x) = x \smile u$ is an isomorphism for all k.

A proof of this theorem can be found in section 10 of *Characteristic Classes* by Milnor and Stasheff. [26]

The inclusion of the pair $(E, \emptyset) \to (E, E_0)$ gives rise to the restriction homomorphism $(\text{-})|_E$: $H^*(E, E_0; \mathbb{Z}) \to H^*(E; \mathbb{Z})$. Notice that since the fibre of p is a vector space and vector spaces are contractible to a point, E and B are homotopy equivalent and thus p^*: $H^n(B; \mathbb{Z}) \to H^n(E; \mathbb{Z})$ is an isomorphism.

Definition 22. The **Euler class**, named for it's relation to the Euler characteristic, of an oriented n-dimensional real vector bundle p: $E \to B$ is the cohomology class $e(E) \in H^n(B; \mathbb{Z})$ such that $p^*(e(E)) = u|_E$, where u is the orientation class of the vector bundle.

Lemma 23. For two oriented n-dimensional real vector bundles p: $E \to B$ and p': E' $\to B$', if a map f: $B \to B$' is covered by an orientation preserving bundle map g: $E \to E$', then:

$$e(E) = f^*(e(E'))$$

Lemma 24. For any two oriented real vector bundles over the same base space p: $E \to B$, q: E' $\to B$ we have the following relation:

$$e(E \oplus E') = e(E) \smile e(E')$$

Lemma 25. The ring homomorphism q: $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ given by $q(x) = x + 2\mathbb{Z}$ induces a ring homomorphism q^*: $H^*(B; \mathbb{Z}) \to H^*(B; \mathbb{Z}/2\mathbb{Z})$. For any oriented n-dimensional real vector bundle p: $E \to B$, it can be shown that:

$$q^*(e(E)) = w_n(E)$$

Proofs of these lemmas are given in the oriented bundles and the Euler class chapters also in Milnor and Stasheff's book *Characteristic Classes* [26].

5.3 Chern Classes

Let p: $E \to B$ be an n-dimensional complex vector bundle i.e. the fibre is $\mathbb{C}^n$. Let us denote by $p_{\mathbb{R}}$: $E \to B$, the underlying $2n$-dimensional real vector bundle.

Lemma 26. If p is a complex vector bundle, then $p_{\mathbb{R}}$ has a canonical choice of orientation.

Proof. Let us choose a complex basis of each fibre of p over B, $(\mathbf{v}_1, ..., \mathbf{v}_n)$, then after ignoring the complex structure, $(\mathbf{v}_1, i\mathbf{v}_1, ..., \mathbf{v}_n, i\mathbf{v}_n)$ is a basis for each fibre of $p_{\mathbb{R}}$ over B. △

As a result, the Euler class $e(E) \in H^{2n}(B; \mathbb{Z})$ is well defined.

Let z: $B \to E$ be the zero section, and then let $E_0 = E \backslash z(B)$. A point in E_0 is given by any non-zero vector in E, $\mathbf{v} \in p^{-1}(b)$. Given a Hermitian metric on each fibre V, we can define a space E' whose points are of the form $(\mathbf{v}, \mathbf{w})$ where $\mathbf{w}$ is orthogonal to $\mathbf{v}$ in the fibre $p^{-1}(b)$.

Then the projection map p': E' $\to E_0$ given by $p'(\mathbf{v}, \mathbf{w}) = (\mathbf{v})$ is an $(n$-1)-dimensional complex vector bundle. Clearly, this process iterates.

Lemma 27. Let p: $E \to B$ be an oriented n-dimensional real vector bundle, and let z: $B \to E$ be the zero section, $E_0 = E \backslash z(B)$, and p_0: $E_0 \to B$ be the restriction of p to E_0, then the following sequence is exact and is called the **Gysin sequence**:

$$... \longrightarrow H^{i-n}(B;\mathbb{Z}) \xrightarrow{\smile e(B)} H^i(B;\mathbb{Z}) \xrightarrow{p_0^*} H^i(E_0;\mathbb{Z}) \xrightarrow{\xi} H^{i-n+1}(B;\mathbb{Z}) \xrightarrow{\smile e(B)} ...$$

The map ξ is derived in part from the map coming from the Snake Lemma in the long exact sequence in cohomology of the pair (E, E_0).

Thus, if p: $E \to B$ is an n-dimensional complex vector bundle, $p_{\mathbb{R}}$ admits an exact Gysin sequence:

$$... \longrightarrow H^{i-2n}(B;\mathbb{Z}) \xrightarrow{\smile e(B)} H^i(B;\mathbb{Z}) \xrightarrow{(p_{\mathbb{R}})_0^*} H^i(E_0;\mathbb{Z}) \longrightarrow H^{i-2n+1}(B;\mathbb{Z}) \xrightarrow{\smile e(B)} ...$$

For $i < 2n$-1, clearly $H^{i\text{-}2n}(B; \mathbb{Z})$ and $H^{i\text{-}2n+1}(B; \mathbb{Z})$ are both zero, and so the map $(p_{\mathbb{R}})_0^*$: $H^i(B; \mathbb{Z}) \to H^i(E_0; \mathbb{Z})$ is an isomorphism for $i < 2n$-1.

Definition 23. The **Chern classes** of an n-dimensional complex vector bundle p: $E \to B$ are a collection of cohomolgy classes $c_i(E) \in H^{2i}(B; \mathbb{Z})$ defined as follows:

$$\begin{aligned} c_i(E) &= 0 && \forall i > n \\ c_n(E) &= e(E) \\ c_i(E) &= ((p_{\mathbb{R}})_0^*)^{-1}(c_i(E')) && \forall i < n \end{aligned}$$

Specifically $c_{n\text{-}1}(E) = ((p_{\mathbb{R}})_0^*)^{-1}(c_{n\text{-}1}(E')) = ((p_{\mathbb{R}})_0^*)^{-1}(e(E'))$ since p': E' $\to E_0$ is an $(n$-1)-dimensional complex vector bundle, and thus since the process of reducing the dimension iterates, we see that the Chern classes are a series of Euler classes of related complex vector bundles.

As with the Stiefel-Whitney classes, we may define the **total Chern class** of an n-dimensional complex vector bundle $p\colon E \to B$ to be the class $c(E) \in H^*(B;\mathbb{Z})$ given by:

$$c(E) := \sum_{i=0}^{\infty} c_i(E) = 1 + c_1(E) + ... + c_n(E)$$

Proofs of the following lemmas can be found in *Characteristic Classes* by Milnor and Stasheff [26].

Lemma 28. Let $p\colon E \to B$ and $p'\colon E' \to B'$ are two n-dimensional complex vector bundles, if a map $f\colon B \to B'$ is covered by a bundle map $g\colon E \to E'$, then:

$$c(E) = f^*(c(E'))$$

Lemma 29. Similarly to the other characteristic classes discussed, for any two complex vector bundles over the same base space $p\colon E \to B$, $q\colon E' \to B$ we have the following relation:

$$c(E \oplus E') = c(E)c(E')$$

Corollary 30. If a complex vector bundle $p\colon E \to B$ decomposes into the direct sum of complex line bundles i.e. $E = L_1 \oplus ... \oplus L_n$ and $p = l_1 \oplus ... \oplus l_n$ where each $l_i\colon L_i \to B$ is a vector bundle with fibre $\mathbb{C}$, then

$$c(E) = \prod_{i=1}^{n} c(L_i)$$
$$\sum_{i=0}^{n} c_i(E) = \prod_{i=1}^{n} (1 + c_1(L_i))$$

5.4 Chern Classes and Flag Manifolds

The Chern classes of some certain vector bundles will be very useful to know.

Lemma 31. Let us consider the tautological line bundle over complex projective space $\mathbb{CP}^k$, that is, the 1 dimensional vector bundle γ^1 that has total space $E^1\mathbb{CP}^k := \{(Span(\mathbf{v}), \mathbf{w}) \mid \mathbf{v} \in \mathbb{C}^{k+1}\backslash\{0\}, \mathbf{w} = \lambda\mathbf{v} \text{ for } \lambda \in \mathbb{C}\}$ and is given by:

$$\gamma^1 : E^1\mathbb{CP}^k \to \mathbb{CP}^k$$
$$(Span(\mathbf{v}), \mathbf{w}) \mapsto Span(\mathbf{v})$$

The Chern class $c_1(E^1\mathbb{CP}^k)$ is the generator of the ring $H^*(\mathbb{CP}^k;\mathbb{Z})$.

Proof. Since γ^1 is a complex line bundle, the underlying real plane bundle $\gamma^1_{\mathbb{R}}$ has a canonical choice of orientation and we may construct a Gysin sequence:

$$... \xrightarrow{\gamma^{1*}_{\mathbb{R}}} H^{i+1}(E_0;\mathbb{Z}) \longrightarrow H^i(\mathbb{CP}^k;\mathbb{Z}) \xrightarrow{\smile c_1(E)} H^{i+2}(\mathbb{CP}^k;\mathbb{Z}) \xrightarrow{\gamma^{1*}_{\mathbb{R}}} H^{i+2}(E_0;\mathbb{Z}) \longrightarrow ...$$

where we write $E = E^1\mathbb{CP}^k$ for brevity and so E_0 as E without the zero section leaves us with $E_0 = \{(Span(\mathbf{v}), \mathbf{w}) \mid \mathbf{v} \in \mathbb{C}^{k+1}\backslash\{0\}, \mathbf{w} = \lambda\mathbf{v} \text{ for } \lambda \in \mathbb{C}\backslash\{0\}\}$.

There is a clear isomorphism between E_0 and $\mathbb{C}^{k+1}\backslash\{0\}$ given by

$$\phi : \mathbb{C}^{k+1}\backslash\{0\} \to E_0$$
$$\mathbf{w} \mapsto (Span(\mathbf{w}), \mathbf{w})$$

Since additionally, $\mathbb{C}^n\backslash\{0\}$ is homotopy equivalent to the sphere $\mathbb{S}^{2n-1}$, we have that E_0 is homotopy equivalent to $\mathbb{S}^{2k+1}$ and so $H^i(E_0;\,\mathbb{Z}) \cong H^i(\mathbb{S}^{2k+1};\,\mathbb{Z})$ for all $i \in \mathbb{Z}$; in particular, $H^i(E_0;\,\mathbb{Z}) \cong 0$ for $i \notin \{0,\, 2k+1\}$ and so

$$\smile c_1(E) : H^i(\mathbb{CP}^k;\mathbb{Z}) \to H^{i+2}(\mathbb{CP}^k;\mathbb{Z})$$
$$x \mapsto x \smile c_1(E)$$

is an isomorphism for $i \leq$ -3, $0 \leq i \leq 2k$-2, $i \geq 2k+1$. With the knowledge that $H^i(X;\,\Bbbk) \cong 0$ for $i < 0$, and a closer examination of the remaining sections of the Gysin sequence, we will determine the nature of $H^*(\mathbb{CP}^k;\,\mathbb{Z})$ as a ring.

From the following section of the Gysin sequence, since $H^0(\mathbb{S}^{2k+1};\,\mathbb{Z}) \cong \mathbb{Z}$, we can determine the nature of $H^0(\mathbb{CP}^k;\,\mathbb{Z})$ for $k \in \mathbb{N}$

$$0 \longrightarrow H^{-2}(\mathbb{CP}^k;\mathbb{Z}) \xrightarrow{\smile c_1(E)} H^0(\mathbb{CP}^k;\mathbb{Z}) \longrightarrow \mathbb{Z} \longrightarrow H^{-1}(\mathbb{CP}^k;\mathbb{Z})$$

Since $H^i(X;\,\Bbbk) \cong 0$ for $i < 0$, this leaves us with the map $H^0(\mathbb{CP}^k;\,\mathbb{Z}) \to \mathbb{Z}$ which must be an isomorphism $\forall\ k \in \mathbb{N}$.

From successive isomorphisms we have $\mathbb{Z} \cong H^0(\mathbb{CP}^k;\,\mathbb{Z}) \cong H^{2i}(\mathbb{CP}^k;\,\mathbb{Z})$ for $0 \leq i \leq k$. Since the only automorphisms on $\mathbb{Z}$ are $\mathrm{id}_{\mathbb{Z}}$ and -$\mathrm{id}_{\mathbb{Z}}$, generators of these groups must be mapped by the isomorphisms to generators. Therefore since $1 \smile c_1(E) = c_1(E)$, by iteration we have that $c_1(E)^i$ is a generator of $H^{2i}(\mathbb{CP}^k;\,\mathbb{Z})$ for $0 \leq i \leq k$.

Next let us inspect the following section of the Gysin sequence:

$$H^0(\mathbb{CP}^k;\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} H^{-1}(\mathbb{CP}^k;\mathbb{Z}) \xrightarrow{\smile c_1(E)} H^1(\mathbb{CP}^k;\mathbb{Z}) \longrightarrow H^1(\mathbb{S}^{2k+1};\mathbb{Z})$$

Since in exact sequences, a zero map must follow an isomorphism and an injection must follow a zero map, since we have $H^{-1}(\mathbb{CP}^k;\,\mathbb{Z}) \cong 0$, necessarily $H^{2i-1}(\mathbb{CP}^k;\,\mathbb{Z}) \cong 0$ for $0 \leq i \leq k$.

Since $\mathbb{CP}^k$ is a $2k$-dimensional CW-complex, $H^i(\mathbb{CP}^k;\,\mathbb{Z}) \cong 0$ for $i \geq 2k+1$, we now have a full description of the cohomology ring of complex projective space:

$$H^*(\mathbb{CP}^k;\mathbb{Z}) = \mathbb{Z}[c_1(E^1\mathbb{CP}^k)]/(c_1(E^1\mathbb{CP}^k)^{k+1})$$

△

The process of constructing line bundles in the following theorem is known as the **splitting principle** [24].

Theorem 32. Let $p\colon E \to X$ be an n-dimensional vector bundle over a paracompact space X. There exists a space Y and a map $q\colon Y \to X$ such that the induced map $q^*\colon H^*(X;\mathbb{Z}) \to H^*(Y;\mathbb{Z})$ is injective, and the pullback bundle $q^*(p)\colon q^*(E) \to Y$ decomposes into the direct sum of n line bundles.

Proof. To prove this theorem we will show that there exists a space Y_1 and a map $q_1\colon Y_1 \to X$ such that the induced map $q_1^*\colon H^*(X;\mathbb{Z}) \to H^*(Y_1;\mathbb{Z})$ is injective, and the pullback bundle $q_1^*(p)\colon q_1^*(E) \to Y_1$ decomposes into the direct sum of a line bundle and an $(n\text{-}1)$-dimensional vector bundle, i.e. there exists a line bundle $l_1\colon L_1 \to Y_1$ and an $(n\text{-}1)$-dimensional vector bundle $p_{n\text{-}1}\colon Q_1 \to Y_1$ such that $q_1^*(E) = L_1 \oplus Q_1$ and $q_1^*(p) = l_1 \oplus p_{n\text{-}1}$.

We will be able to iteratively apply this process to the resulting $(n\text{-}k)$-dimensional vector bundles $p_{n\text{-}k}\colon Q_k \to Y_k$ to construct spaces Y_{k+1} and maps $q_{k+1}\colon Y_{k+1} \to Y_k$ such that the induced maps $q_{k+1}^*\colon H^*(Y_k;\mathbb{Z}) \to H^*(Y_{k+1};\mathbb{Z})$ are injective and the pullback bundle $q_{k+1}^*(q_k)\colon q_{k+1}^*(Q_k) \to Y_{k+1}$ decomposes into the direct sum of a line bundle $l_{k+1}\colon L_{k+1} \to Y_{k+1}$ and an $(n\text{-}(k{+}1))$-dimensional vector bundle $p_{n\text{-}(k+1)}\colon Q_{k+1} \to Y_{k+1}$.

If the base case is true, then the space $Y = Y_n$ and the map $q = (q_1 \circ \ldots \circ q_n)\colon Y \to X$ are such that the induced map $q^*\colon H^*(X;\mathbb{Z}) \to H^*(Y;\mathbb{Z})$ is injective, and the pullback bundle $q^*(p)\colon q^*(E) \to Y$ decomposes into the direct sum of n line bundles, namely

$$q^*(E) = \bigoplus_{i=1}^{n} \tilde{L}_i$$
$$\text{and } q^*(p) = \bigoplus_{i=1}^{n} \tilde{l}_i$$

where $\tilde{l}_i\colon \tilde{L}_i \to Y$ is the pullback of $l_i\colon L_i \to Y_i$ by the map $(q_{i+1} \circ \ldots \circ q_n)$ for each i.

Let $p\colon E \to X$ be an n-dimensional vector bundle and, if $z\colon X \to E$ is the zero section, i.e. $z(x) = 0_x \in \mathrm{V}_x \subset E$, let us consider $P(E) := E \backslash z(X)/\sim$ where $\mathbf{v} \sim \mathbf{w}$ iff $\mathbf{v} = \lambda \mathbf{w}$ for some $\lambda \in \mathbb{C}\backslash\{0\}$. The map $p_0\colon P(E) \to X$ given by $p_0([\mathbf{v}]) = p(\mathbf{v})$ is well defined since any other representative of $[\mathbf{v}]$ must be in the same vector space as $\mathbf{v}$ and thus in the same fibre. The map p_0 is a fibre bundle, choosing points $[\mathbf{v}] \in P(E)$ such that $p_0([\mathbf{v}]) = x$ for some $x \in X$ is equivalent to choosing lines in the vector space V_x, i.e. $p_0^{-1}(x) \cong F_1(\mathrm{V}_x) \cong \mathbb{CP}^{n-1}$ since p is an n-dimensional vector bundle.

Let us produce the following pullback diagram:

$$\begin{array}{ccc} p_0^*(E) & \longrightarrow & E \\ {\scriptstyle \pi}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ P(E) & \xrightarrow[p_0]{} & X \end{array}$$

$p_0^*(E) := \{([\mathbf{v}], \mathbf{w}) \in P(E) \times E \mid \mathbf{v}, \mathbf{w} \in \mathrm{V}_x$ for some $x \in X\}$ and the map $\pi\colon p_0^*(E) \to P(E)$ is simply the projection map to the $P(E)$ coordinate. We may decompose this n-dimensional vector bundle into the direct sum of a line bundle and an (n-1)-dimensional vector bundle in the following manner:

Let us define $L := \{([\mathbf{v}], \mathbf{w}) \in P(E) \times E \mid \mathbf{w} \in [\mathbf{v}]\}$ and $Q := \{([\mathbf{v}], \mathbf{w}) \in P(E) \times E \mid \mathbf{v} \perp \mathbf{w}\}$, clearly $p_0^*(E) = L \oplus Q$. Additionally, we have projection maps to the $P(E)$ coordinate $\pi_L\colon L \to P(E)$, $\pi_Q\colon Q \to P(E)$ so that $\pi = \pi_L \oplus \pi_Q$, therefore the pullback vector bundle π decomposes into a line bundle and an (n-1)-dimensional vector bundle. △

Lemma 33. For an n-dimensional vector bundle $p\colon E \to X$ using the splitting principle we can construct the flag bundle $\pi\colon p_0^*(E) \to P(E)$ and decompose it into the direct sum of the line bundle $\pi_L\colon L \to P(E)$ and the (n-1)-dimensional vector bundle $\pi_Q\colon Q \to P(E)$. Then we have the following isomorphism in the category of rings:

$$\begin{aligned} H^*(P(E)); \mathbb{Z}) &\cong H^*(X; \mathbb{Z})[c_1(L), c_1(Q), ..., c_{n-1}(Q)]/(c(L)c(Q) - c(E)) \\ &\cong H^*(X; \mathbb{Z})[c_1(L)]/(\sum_{j=0}^{n}(-1)^j p_0^*(c_{n-j}(E))c_1(L)^j) \end{aligned}$$

Proof. Since we already know that $H^k(\mathbb{CP}^{n-1};\, \mathbb{Z})$ is a finitely generated free $\mathbb{Z}$-module for all k and since in the following pullback diagram:

$$\begin{array}{ccc} \iota_x(L) & \longrightarrow & L \\ {\scriptstyle\gamma_x}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle\pi_L} \\ F_1(\mathrm{V}_x) & \xrightarrow[\iota_x]{} & P(E) \end{array}$$

where $\iota_x\colon F_1(\mathrm{V}_x) \to P(E)$ is the inclusion of the fibre over $x \in X$ into the total space, $\gamma_x\colon \iota_x^*(L) \to F_1(\mathrm{V}_x)$ is the tautological bundle over $F_1(\mathrm{V}_x)$, and thus the restrictions $\iota_x^*(c_1(L)^i)$, where $c_1(L) \in H^2(P(E);\, \mathbb{Z})$, form a basis for $H^*(F_1(\mathrm{V}_x);\, \mathbb{Z}) \cong H^*(\mathbb{CP}^{n-1};\, \mathbb{Z})$ for each $x \in X$. The conditions are met and so we may apply the Leray-Hirsch Theorem to the fibre bundle $p_0\colon P(E) \to X$ that has fibre $F_1(\mathrm{V}_x) \cong \mathbb{CP}^{n-1}$, and achieve the following isomorphism:

$$\begin{aligned} H^*(P(E); \mathbb{Z}) &\cong H^*(X; \mathbb{Z}) \otimes H^*(\mathbb{CP}^{n-1}; \mathbb{Z}) \\ &\cong H^*(X; \mathbb{Z})[x]/(x^n) \end{aligned}$$

in the category of $H^*(X;\, \mathbb{Z})$-modules where x is the Chern class of the tautological line bundle over $\mathbb{CP}^{n-1}$. Thus, every element of $H^*(P(E);\, \mathbb{Z})$ can be written in the form:

$$\sum_{i=0}^{n-1} a_i c_1(L)^i$$

where $a_i \in H^*(X; \mathbb{Z})$ and for $b \in H^*(X; \mathbb{Z})$ we have:

$$b \cdot \sum_{i=0}^{n-1} a_i c_1(L)^i = \sum_{i=0}^{n-1} (b \cdot a_i) c_1(L)^i$$

However, while we cannot have a linearly independent $c_1(L)^n$ term, it is not necessarily true that $c_1(L)^n = 0$ since the Leray-Hirsch Theorem does not give us an isomorphism of rings.

To eliminate the $c_1(L)^n$ term and describe the multiplication structure of $H^*(P(E); \mathbb{Z})$ as a ring, we instead turn to the relation $c(L)c(Q) = p_0^*(c(E))$ derived from the facts that $L \oplus Q = p_0^*(E)$ and p_0 is covered by the bundle map $p_0^*(E) \to E$, the projection to the E coordinate.

The ring $H^*(P(E); \mathbb{Z})$ is therefore generated by $c_1(L)$ along with the set $\{c_1(Q), ..., c_{n\text{-}1}(Q)\}$ (though these elements may not be independent), together with the condition that $c(L)c(Q)$ - $p_0^*(c(E)) = 0$. We therefore may write the ring in the following manner:

$$H^*(P(E); \mathbb{Z}) \cong H^*(X; \mathbb{Z})[c_1(L), c_1(Q), ..., c_{n-1}(Q)]/(c(L)c(Q) - p_0^*(c(E)))$$

Since the element $p_0^*(c(E))$ is some fixed element of $H^*(P(E); \mathbb{Z})$, using the relation we can determine $c(Q)$ explicitly in terms of powers of $c_1(L)$ by comparing like degrees:

$$\begin{aligned}
(1 + c_1(L))(1 + c_1(Q) + ... + c_{n-1}(Q)) &= 1 + p_0^*(c_1(E)) + ... + p_0^*(c_n(E)) \\
1 + c_1(Q) + ... + c_{n-1}(Q) + \\
c_1(L) + c_1(Q)c_1(L) + ... + c_{n-1}(Q)c_1(L) &= 1 + p_0^*(c_1(E)) + ... + p_0^*(c_n(E)) \\
\implies c_i(Q) + c_{i-1}(Q)c_1(L) &= p_0^*(c_i(E)), \text{ for } 1 \leq i \leq n \\
\implies \text{ recursively for } 1 \leq i \leq n, \ c_i(Q) &= \sum_{j=0}^{i} (-1)^j p_0^*(c_{i-j}(E)) c_1(L)^j \\
\text{so since } c_n(Q) = 0, \ 0 &= \sum_{j=0}^{n} (-1)^j p_0^*(c_{n-j}(E)) c_1(L)^j
\end{aligned}$$

So to write our ring $H^*(P(E); \mathbb{Z})$ with an independent generating set, we can say:

$$H^*(P(E); \mathbb{Z}) \cong H^*(X; \mathbb{Z})[c_1(L)]/(\sum_{j=0}^{n} (-1)^j p_0^*(c_{n-j}(E)) c_1(L)^j)$$

△

Corollary 34. Let each x_i be the Chern class of the tautological line bundle γ_i: $L_i \to F_n(\mathbb{C}^k)$ where $L_i := \{((\mathrm{V}_1 \subset ... \subset \mathrm{V}_n \subseteq \mathbb{C}^k), \mathbf{v}) \mid \mathbf{v} \in \mathrm{V}_i, \mathbf{v} \perp \mathbf{w} \ \forall \ \mathbf{w} \in \mathrm{V}_{i\text{-}1}\}$, and γ_i is the projection to the flag coordinate, and, abusing notation, let $c(Q_n)$ denote simultaneously the total Chern class and the set of Chern classes of the (k-n)-dimensional vector bundle $Q_n \to F_n(\mathbb{C}^k)$ such that $L_1 \oplus ... \oplus L_n \oplus Q_n = \mathbb{C}^k$. Finally, let $P(z)$ be the set of partitions of z into n non-negative integers, elements of $P(z)$ are tuples $(r_1, ..., r_n)$ such that

$$\text{for } 1 \leq i \leq n,\ r_i \geq 0 \text{ and } \sum_{i=1}^{n} r_i = z$$

Then the cohomology ring of a finite dimensional flag manifold $F_n(\mathbb{C}^k)$ is given by:

$$\begin{aligned}
H^*(F_n(\mathbb{C}^k); \mathbb{Z}) &\cong \mathbb{Z}[x_1, ..., x_n, c(Q_n)]/((\prod_{i=1}^{n}(1 + x_i))c(Q_n) - 1) \\
&\cong \mathbb{Z}[x_1, ..., x_n, c(Q_n)]/(\sum_{i=1}^{n} \sum_{m=k-n-i+1}^{k-n} \sigma_i(x_1, ..., x_n)c_m(Q_n)) \\
&\cong \mathbb{Z}[x_1, ..., x_n, c(Q_n)]/(\sum_{m=k-n+1}^{k} \sum_{i=0}^{k-m} \sigma_{n-i}(x_1, ..., x_n)c_{m-n+i}(Q_n)) \\
&\cong \mathbb{Z}[x_1, ..., x_n]/ \\
&\quad (\sum_{m=k-n+1}^{k} \sum_{i=0}^{k-m} (-1)^{m-n+i}\sigma_{n-i}(x_1, ..., x_n) \sum_{\substack{(r_1,...,r_n) \\ \in P(m-n+i)}} \prod_{j=1}^{n} x_j^{r_j})
\end{aligned}$$

Proof. Consider the trivial k-bundle over a single point $\mathbb{C}^k \to *$. The space $P(\mathbb{C}^k) = \mathbb{C}^k \backslash \{0\}/\sim$ where $\mathbf{v} \sim \mathbf{w}$ iff $\mathbf{v} = \lambda\mathbf{w}$ for some $\lambda \in \mathbb{C}\backslash\{0\}$ is clearly equivalent to the space $F_1(\mathbb{C}^k)$. We will prove this relation with a proof by induction.

In the following pullback diagram $L_1 := \{(\mathrm{V}_1, \mathbf{v}) \in F_1(\mathbb{C}^k) \times \mathbb{C}^k \mid \mathbf{v} \in \mathrm{V}_1\}$, $Q_1 := \{(\mathrm{V}_1, \mathbf{w}) \in F_1(\mathbb{C}^k) \times \mathbb{C}^k \mid \mathbf{v} \perp \mathbf{w} \ \forall\ \mathbf{v} \in \mathrm{V}_1\}$, and f_1: $F_1(\mathbb{C}^k) \to *$ sends every 1-flag to the 0-flag in $\mathbb{C}^k$.

$$\begin{array}{ccc}
L_1 \oplus Q_1 & \longrightarrow & \mathbb{C}^k \\
\downarrow & \lrcorner & \downarrow \\
F_1(\mathbb{C}^k) & \xrightarrow[f_1]{} & *
\end{array}$$

Since $\mathbb{C}^k \to *$ is the trivial bundle, $c(\mathbb{C}^k) = 1$, and thus $f_1^*(c(\mathbb{C}^k)) = 1$. Let us write $x_1 = c_1(L_1)$, from our previous theorem we have:

$$\begin{aligned}
H^*(F_1(\mathbb{C}^k); \mathbb{Z}) &\cong H^*(*; \mathbb{Z})[x_1]/(\sum_{j=0}^{k}(-1)^j f_1^*(c_{k-j}(\mathbb{C}^k))x_1^j) \\
&\cong \mathbb{Z}[x_1]/((-1)^k x_1^k) \\
&\cong \mathbb{Z}[x_1]/(x_1^k)
\end{aligned}$$

From the congruence $F_1(\mathbb{C}^k) \cong \mathbb{CP}^{k-1}$ we know that $H^*(F_1(\mathbb{C}^k);\ \mathbb{Z}) \cong \mathbb{Z}[x]/(x^k)$ where x is the first Chern class of the tautological line bundle over $F_1(\mathbb{C}^k)$. These two methods agree, and to prove the base case holds, let us show that our formula yields the same result:

$$\begin{aligned} H^*(F_1(\mathbb{C}^k);\mathbb{Z}) &\cong \mathbb{Z}[x_1]/(\sum_{m=k}^{k}\sum_{i=0}^{0}(-1)^{m-1+i}\sigma_{1-i}(x_1)x_1^{m-1+i}) \\ &\cong \mathbb{Z}[x_1]/((-1)^{k-1}x_1^k) \\ &\cong \mathbb{Z}[x_1]/(x_1^k) \end{aligned}$$

Thus, since all these methods agree, our base case holds.

To proceed we will show that if $Q_n = \{((\mathrm{V}_1 \subset ... \subset \mathrm{V}_n), \mathbf{w}) \in F_n(\mathbb{C}^k) \times \mathbb{C}^k \mid \mathbf{v} \perp \mathbf{w} \ \forall\ \mathbf{v} \in \mathrm{V}_n\}$, is the complement to the direct sum of the tautological line bundles over $F_n(\mathbb{C}^k)$ then $P(Q_n) \cong F_{n+1}(\mathbb{C}^k)$.

Let us denote by V a flag $(\mathrm{V}_1 \subset ... \subset \mathrm{V}_n \subseteq \mathbb{C}^k) \in F_n(\mathbb{C}^k)$. $P(Q_n) = \{(\mathrm{V}, \mathbf{w}) \in F_n(\mathbb{C}^k) \times \mathbb{C}^k\backslash\{0 \mid \mathbf{v} \perp \mathbf{w} \ \forall\ \mathbf{v} \in \mathrm{V}_n\}/\sim$ where $(\mathrm{V}, \mathbf{w}) \sim (\mathrm{V'}, \mathbf{w'})$ iff $\mathrm{V} = \mathrm{V'}$ and $\mathbf{w} = \lambda\mathbf{w'}$ for some $\lambda \in \mathbb{C}\backslash\{0\}$. Therefore equivalently, $P(Q_n) = \{(\mathrm{V}, \mathrm{W}) \in F_n(\mathbb{C}^k) \times F_1(\mathbb{C}^k)$ such that $\mathrm{V}_n \perp \mathrm{W}\}$, or additionally equivalently, since V_n and W must be perpendicular, let $\mathrm{V}_{n+1} = \mathrm{V}_n \oplus \mathrm{W}$, then $P(Q_n) = \{\mathrm{V}_1 \subset ... \subset \mathrm{V}_n \subset \mathrm{V}_{n+1} \subseteq \mathbb{C}^k\} = F_{n+1}(\mathbb{C}^k)$.

Next we would like to show that the following formula for the Chern classes of Q_n hold in terms of the Chern classes of the tautological line bundles over $F_n(\mathbb{C}^k)$, let $P(i)$ be the set of partitions of i into n non-negative integers, and $x_j = c_1(L_j)$ where $L_j = \{(\mathrm{V}, \mathbf{w}) \in F_n(\mathbb{C}^k) \times \mathbb{C}^k \mid \mathbf{w} \in \mathrm{V}_j, \mathbf{v} \perp \mathbf{w} \ \forall\ \mathbf{v} \in \mathrm{V}_{j\text{-}1}\}$:

$$c_i(Q_n) = (-1)^i \sum_{(r_1,\ldots,r_n)\in P(i)} \prod_{j=1}^{n} x_j^{r_j}$$

$$c(Q_n) = \sum_{i=0}^{k-n}(-1)^i \sum_{(r_1,\ldots,r_n)\in P(i)} \prod_{j=1}^{n} x_j^{r_j}$$

We will show that the tautological line bundles over a flag manifold $F_{n\text{-}1}(\mathbb{C}^k)$ can be pulled back by the map f_n: $F_n(\mathbb{C}^k) \to F_{n\text{-}1}(\mathbb{C}^k)$ where $f_n(\mathrm{V}_1 \subset ... \subset \mathrm{V}_n \subseteq \mathbb{C}^k) = (\mathrm{V}_1 \subset ... \subset \mathrm{V}_{n\text{-}1} \subset \mathbb{C}^k)$ to the corresponding tautological line bundles over $F_n(\mathbb{C}^k)$. This formula will then be shown to hold via an induction argument.

Let $L_j := \{(\mathrm{V}, \mathbf{w}) \in F_{n\text{-}1}(\mathbb{C}^k) \times \mathbb{C}^k \mid \mathbf{w} \in \mathrm{V}_j, \mathbf{v} \perp \mathbf{w} \ \forall\ \mathbf{v} \in \mathrm{V}_{j\text{-}1}\}$ be the total space of the tautological line bundle γ_j: $L_j \to F_{n\text{-}1}(\mathbb{C}^k)$ and construct the following pullback diagram:

$$\begin{array}{ccc} f_n^*(L_j) & \longrightarrow & L_j \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \gamma_j} \\ F_n(\mathbb{C}^k) & \xrightarrow[f_n]{} & F_{n-1}(\mathbb{C}^k) \end{array}$$

$$\begin{aligned}
f_n^*(L_j) &= \{((\mathrm{V}, \mathbf{w}), \mathrm{V}') \in L_j \times F_n(\mathbb{C}^k) \mid \gamma_j(\mathrm{V}, \mathbf{w}) = f_n(\mathrm{V}')\} \\
&= \{((\mathrm{V}, \mathbf{w}), \mathrm{V}') \in L_j \times F_n(\mathbb{C}^k) \mid \mathrm{V} = f_n(\mathrm{V}')\} \\
&= \{((\mathrm{V}, \mathbf{w}), \mathrm{V}' \in L_j \times F_n(\mathbb{C}^k) \mid \mathbf{w} \in \mathrm{V}_j, \mathbf{v} \perp \mathbf{w} \\
&\quad \forall\ \mathrm{V}_{j-1}, \mathrm{V}_i = \mathrm{V}_i' \ \forall\ 1 \leq i \leq n-1\} \\
&= \{(\mathrm{V}, \mathbf{w}) \in F_n(\mathbb{C}^k) \times \mathbb{C}^k \mid \mathbf{w} \in \mathrm{V}_j, \mathbf{v} \perp \mathbf{w} \\
&\quad \forall\ \mathrm{V}_{j-1}, \mathrm{V}_i = \mathrm{V}_i' \ \forall\ 1 \leq i \leq n-1\}
\end{aligned}$$

Thus $f_n^*(L_j)$ is the total space of the j^{th} tautological line bundle over $F_n(\mathbb{C}^k)$, therefore $c_1(f_n^*(L_j)) = f_n^*(c_1(L_j))$ will be a generator of the ring $H^*(F_n(\mathbb{C}^k);\mathbb{Z})$ if we are able to show that the Chern classes of the tautological line bundles are.

The base case for our relation is easy. Since we require that $c(L_1)c(Q_1) = c(\mathbb{C}^k)$, we can use the recursive formula we found to determine the Chern classes of Q_1. For $1 \leq i \leq k$-1 we have:

$$\begin{aligned}
c_i(Q_1) &= \sum_{j=0}^{i} (-1)^j f_1^*(c_{i-j}(\mathbb{C}^k)) x_1^j \\
&= (-1)^i x_1^i \\
\implies c(Q_1) &= \sum_{i=0}^{k-1} (-1)^i x_1^i
\end{aligned}$$

Our formula for $n = 1$ requires us to partition i into 1 non negative integer. Clearly, there is only one 1-tuple (namely (i)) that will partition i and thus:

$$\begin{aligned}
c_i(Q_1) &= \sum_{i=0}^{k-1} (-1)^i \sum_{(r_1) \in P(i)} \prod_{j=1}^{1} x_1^{r_1} \\
&= \sum_{i=0}^{k-1} (-1)^i \sum_{(r_1) \in P(i)} x_1^{r_1} \\
&= \sum_{i=0}^{k-1} (-1)^i x_1^i
\end{aligned}$$

exactly as required.

Now, let us assume our formula works for Q_n and reach the conclusion that it must also therefore hold for Q_{n+1}.

Let us denote by x_j the Chern classes of the tautological line bundles over $F_{n+1}(\mathbb{C}^k)$, we have already seen that for $1 \leq j \leq n$, $x_j = f_{n+1}^*(y_j)$ where the y_j are the Chern classes of the tautological line bundles over $F_n(\mathbb{C}^k)$. We have also showed that the following pullback diagram is valid:

$$\begin{array}{ccc} L_{n+1} \oplus Q_{n+1} & \longrightarrow & Q_n \\ \downarrow & \lrcorner & \downarrow \\ F_{n+1}(\mathbb{C}^k) & \xrightarrow[f_{n+1}]{} & F_n(\mathbb{C}^k) \end{array}$$

Thus we may apply our lemma to achieve:

$$\begin{aligned} c_i(Q_{n+1}) &= \sum_{j=0}^{i} (-1)^j f_{n+1}^*(c_{i-j}(Q_n)) x_{n+1}^j \\ &= \sum_{j=0}^{i} (-1)^j ((-1)^{i-j} \sum_{(r_1,\ldots,r_n)\in P(i-j)} \prod_{l=1}^{n} x_l^{r_l}) x_{n+1}^j \text{ by assumption} \\ &= \sum_{j=0}^{i} (-1)^i (\sum_{(r_1,\ldots,r_n,j)\in P(i)} \prod_{l=1}^{n} x_l^{r_l}) x_{n+1}^j \\ &= (-1)^i \sum_{j=0}^{i} (\sum_{(r_1,\ldots,r_n,j)\in P(i)} (\prod_{l=1}^{n} x_l^{r_l}) x_{n+1}^j) \\ &= (-1)^i \sum_{(r_1,\ldots,r_{n+1})\in P(i)} \prod_{l=1}^{n+1} x_l^{r_l} \end{aligned}$$

exactly as required, thus our formula holds for all n.

The final push is to assume the ring $H^*(F_n(\mathbb{C}^k);\mathbb{Z})$ behaves as we expect and thus conclude that the ring $H^*(F_{n+1}(\mathbb{C}^k);\mathbb{Z})$ follows suit.

We have done all the work required to say that:

$$H^*(F_{n+1}(\mathbb{C}^k);\mathbb{Z}) \cong H^*(F_n(\mathbb{C}^k);\mathbb{Z})[x_{n+1}, c(Q_{n+1})]/((1+x_{n+1})c(Q_{n+1}) - c(Q_n))$$

By our assumption:

$$H^*(F_n(\mathbb{C}^k);\mathbb{Z}) \cong \mathbb{Z}[x_1,...,x_n,c(Q_n)]/((\prod_{i=1}^{n}(1+x_i))c(Q_n) - 1)$$

Therefore:

$$\begin{aligned}
H^*(F_{n+1}(\mathbb{C}^k);\mathbb{Z}) &\cong (\mathbb{Z}[x_1, ..., x_n, c(Q_n)]/ \\
&\quad ((\prod_{i=1}^{n}(1+x_i))c(Q_n) - 1))[x_{n+1}, c(Q_{n+1})]/ \\
&\quad ((1+x_{n+1})c(Q_{n+1}) - c(Q_n)) \\
&\cong \mathbb{Z}[x_1, ..., x_n, c(Q_n), x_{n+1}, c(Q_{n+1})]/ \\
&\quad ((\prod_{i=1}^{n}(1+x_i))c(Q_n) - 1, (1+x_{n+1})c(Q_{n+1}) - c(Q_n)) \\
&\cong \mathbb{Z}[x_1, ..., x_{n+1}, c(Q_{n+1})]/((\prod_{i=1}^{n}(1+x_i))(1+x_{n+1})c(Q_{n+1}) - 1) \\
&\cong \mathbb{Z}[x_1, ..., x_{n+1}, c(Q_{n+1})]/((\prod_{i=1}^{n+1}(1+x_i))c(Q_{n+1}) - 1)
\end{aligned}$$

Thus our corollary holds for all flag manifolds $F_n(\mathbb{C}^k)$.

Since $c(Q_n)$ is a polynomial in x_1, ..., x_n we are able to rewrite the cohomology ring explicitly in terms of x_1, ..., x_n by determining the nature of the ideal:

$$\begin{aligned}
(\prod_{i=1}^{n}(1+x_i))c(Q_n) &= 1 \\
(\sum_{i=0}^{n}\sigma_i(x_1, ..., x_n))c(Q_n) &= 1 \\
1 + c_1(Q_n) + ... + c_{k-n}(Q_n) + & \\
\sigma_1(x_1, ..., x_n) + \sigma_1(x_1, ..., x_n)c_1(Q_n) + ... + \sigma_1(x_1, ..., x_n)c_{k-n}(Q_n) + & \\
... & \\
\sigma_n(x_1, ..., x_n) + \sigma_n(x_1, ..., x_n)c_1(Q_n) + ... + \sigma_n(x_1, ..., x_n)c_{k-n}(Q_n) &= 1
\end{aligned}$$

To reduce the number of terms, let us show the following:

$$\sum_{j=0}^{i}\sigma_j(x_1, ..., x_n)c_{i-j}(Q_n) = 0 \text{ for } 1 \le i \le k-n.$$

Let $X = x_{p_1}^{r_1} ... x_{p_j}^{r_j}$ be a general term of our sum where the p_q are distinct elements of the set $\{1, ..., n\}$, and we have $r_1 + ... + r_j = i$ with $r_q > 0$. Necessarily $j \le i$ and $j \le n$.

To show the equality of our formula we must show that the coefficient of X in our sum is 0. This will be done by examining the coefficients of X in each of the summands $\sigma_k(x_1, ..., x_n)c_{i\text{-}k}(Q_n)$ and we discover that:

the coefficient of X in $c_i(Q_n)$ is $(-1)^i$,

$$\text{the coefficient of } X \text{ in } \sigma_1(x_1, ..., x_n)c_{i\text{-}1}(Q_n) \text{ is } (\text{-}1)^{i\text{-}1}j,$$
$$...$$
$$\text{the coefficient of } X \text{ in } \sigma_k(x_1, ..., x_n)c_{i\text{-}k}(Q_n) \text{ is } (\text{-}1)^{i\text{-}k}\tbinom{j}{k},$$
$$...$$
$$\text{the coefficient of } X \text{ in } \sigma_j(x_1, ..., x_n)c_{i\text{-}j}(Q_n) \text{ is } (\text{-}1)^{i\text{-}j}\tbinom{j}{j} = (\text{-}1)^{i\text{-}j}.$$

The j^{th} term is reached since $j \leq i$ and $j \leq n$.

Thus we must have that the coefficient of X in our full sum is:

$$\begin{aligned}\sum_{k=1}^{j}(-1)^{i-k}\binom{j}{k} &= (-1)^i\sum_{k=1}^{j}(-1)^k\binom{j}{k}\\ &= (-1)^i(0) = 0\end{aligned}$$

since the alternating sum of binomial coefficients is zero, therefore our sum works too! Let us return to our ideal now that these terms have vanished:

$$\begin{aligned}&\sigma_1(x_1, ..., x_n)c_{k-n}(Q_n)+\\ &\sigma_2(x_1, ..., x_n)c_{k-n-1}(Q_n) + \sigma_2(x_1, ..., x_n)c_{k-n}(Q_n)\\ &...\\ &\sigma_n(x_1, ..., x_n)c_{k-2n}(Q_n) + ... + \sigma_n(x_1, ..., x_n)c_{k-n+1}(Q_n) = 0\end{aligned}$$

We can group the summands along the rows first to achieve:

$$\sum_{i=1}^{n}\sum_{m=k-n-i+1}^{k-n}\sigma_i(x_1, ..., x_n)c_m(Q_n) = 0$$

Or to better demonstrate which polynomial vanishes in each degree we can group the summands down the columns first to achieve:

$$\sum_{m=k-n+1}^{k}\sum_{i=0}^{k-m}\sigma_{n-i}(x_1, ..., x_n)c_{m-n+i}(Q_n) = 0$$

and finally, substituting our formula for the Chern classes of Q_n, we achieve:

$$\sum_{m=k-n+1}^{k}\sum_{i=0}^{k-m}(-1)^{m-n+i}\sigma_{n-i}(x_1, ..., x_n)\sum_{(r_1,...,r_n)\in P(m-n+i)}\prod_{j=1}^{n}x_j^{r_j} = 0$$

Therefore, in the ring $H^*(F_n(\mathbb{C}^k); \mathbb{Z})$, the ideal can be given by any of these formulae. △

Corollary 35. The cohomology ring of a total flag manifold is given by:

$$H^*(F_n(\mathbb{C}^n); \mathbb{Z}) \cong \mathbb{Z}[x_1, ..., x_n]/(\sum_{i=1}^{n}\sigma_i(x_1, ..., x_n))$$

This result is consistent with the work of Borel [5].

Proof. Let us examine the ideal for $k = n$:

$$\sum_{m=1}^{n}\sum_{i=0}^{n-m}(-1)^{m-n+i}\sigma_{n-i}(x_1,...,x_n)\sum_{(r_1,...,r_n)\in P(m-n+i)}\prod_{j=1}^{n}x_j^{r_j}=0$$

Since for our permitted values m - $n + i < 0$ unless $m + i = n$ and $P(t) = \emptyset$ for $t < 0$, this ideal simplifies:

$$\sum_{m=1}^{n}\sigma_m(x_1,...,x_n)\sum_{(r_1,...,r_n)\in P(0)}\prod_{j=1}^{n}x_j^{r_j}=0$$

and since (0, ..., 0) is the only n-tuple in $P(0)$, we achieve:

$$\sum_{m=1}^{n}\sigma_m(x_1,...,x_n)=0$$

Therefore,

$$H^*(F_n(\mathbb{C}^n);\mathbb{Z})\cong\mathbb{Z}[x_1,...,x_n]/(\sum_{i=1}^{n}\sigma_i(x_1,...,x_n))$$

as required. △

5.5 The Chern Character

To define a useful ring homomorphism we must first evoke some definitions and results from Macdonald's *Symmetric Functions and Hall Polynomials* [23]

Definition 24. Consider the set of polynomials in n indeterminates t_1, ..., t_n with integer coefficients $\mathbb{Z}[t_1, ..., t_n]$. A **symmetric polynomial** is a polynomial $f(t_1, ..., t_n) \in \mathbb{Z}[t_1, ..., t_n]$ that is invariant under all permutations of t_1, ..., t_n, that is, for all elements $\psi \in S(n)$ the symmetric group acting on the set $\{1, ..., n\}$, we have $f(t_1, ..., t_n) = f(t_{\psi(1)}, ..., t_{\psi(n)})$. Let us denote by $\mathcal{S}_{\mathbb{Z}}[t_1, ..., t_n]$ the set of symmetric polynomials in n indeterminates with coefficients in $\mathbb{Z}$.

It is known that $\mathcal{S}_{\mathbb{Z}}[t_1, ..., t_n]$ is a subring of $\mathbb{Z}[t_1, ..., t_n]$ and indeed a polynomial ring in its own right in n algebraically independent generators:

$$\mathcal{S}_{\mathbb{Z}}[t_1, ..., t_n] = \mathbb{Z}[\sigma_1, ..., \sigma_n]$$

where σ_k is the k^{th} **elementary symmetric polynomial** in n indeterminates.

Let us define a set of equivalence relations $\sim_k$ on $S(n)$.

$\psi \sim_k \phi$ iff $\psi(1) = \phi(1)$, and ..., and $\psi(k) = \phi(k)$.

The elementary symmetric polynomials are defined as follows:

$$\sigma_k(t_1, ..., t_n) := \sum_{[\psi] \in S(n)/\sim_k} \prod_{i=1}^{k} t_{\psi(i)}$$

A key feature of the elementary symmetric polynomials is the property:

$$1 + \sum_{i=1}^{n} \sigma_i = \prod_{i=1}^{n} (1 + t_i)$$

Let each t_i have degree 1, we denote by $\mathcal{S}^k_{\mathbb{Z}}[t_1, ..., t_n]$ the additive subgroup of $\mathcal{S}_{\mathbb{Z}}[t_1, ..., t_n]$ of all k-dimensional symmetric polynomials.

Let $I = (i_1, ..., i_r)$ be a partition of $k \in \mathbb{N}$, i.e. $\sum_{j=1}^{r} i_j = k$ and without loss of generality for our purposes, let $k \geq i_1 \geq ... \geq i_r > 0$. Let $P(k)$ be the set of partitions of k.

For any $n \geq k$ and any partition I of k we define the following symmetric polynomial in n indeterminates:

$$s_I(\sigma_1, ..., \sigma_k) = s_{i_1,...,i_r}(\sigma_1, ..., \sigma_k) := \sum_{[\psi] \in S(n)/\sim_r} \prod_{j=1}^{r} t_{\psi(j)}^{i_j}$$

It is clear that s_I is a polynomial in only σ_1, ..., σ_k as any further elementary symmetric polynomials are in too high a degree to produce a k-dimensional polynomial by addition or multiplication.

For $n < k$ it is still possible to define s_I, but the result is only polynomial in σ_1, ..., σ_n as symmetric polynomials in higher degrees are also polynomial in σ_1, ..., σ_n. Also, for $I = (i_1, ..., i_r)$, $s_I(\sigma_1, ..., \sigma_n) = 0$ if $n < r$ as we run out of indeterminates.

Lemma 36. $\mathcal{S}^k_{\mathbb{Z}}[t_1,\ ...,\ t_n]$ has as an additive basis the set:

$$\{s_I(\sigma_1, ..., \sigma_k) \mid I \in P(k)\}$$

Now to introduce the reason these constructions are useful.

Lemma 37. If $p\colon E \to B$ is a complex vector bundle. For any partition I of $k \in \mathbb{N}$, we can define the symmetric polynomial $s_I(c(E)) := s_I(c_1(E),\ ...,\ c_n(E))$.

Then $s_I(c(E)) = s_I(\sigma_1(t_1,\ ...,\ t_n),\ ...,\ \sigma_n(t_1,\ ...,\ t_n))$ for some set of indeterminates $t_1,\ ...,\ t_n$.

Proof. For a complex vector bundle $p\colon E \to B$ that decomposes into the direct sum of n complex line bundles $l_i\colon L_i \to B$ we have

$$c(E) = \prod_{i=1}^{n} c(L_i)$$
$$\sum_{i=0}^{n} c_i(E) = \prod_{i=1}^{n}(1 + c_1(L_i))$$

and by comparison we have

$$c_i(E) = \sigma_i(c_1(L_1), ..., c_1(L_n))$$

So for any partition I of some $k \in \mathbb{N}$ we are permitted to write:

$$\begin{aligned} s_I(c(E)) &:= s_I(c_1(E), ..., c_k(E)) \\ &= s_I(\sigma_1(c_1(L_1), ..., c_1(L_n)), ..., \sigma_k(c_1(L_1), ..., c_1(L_n))). \end{aligned}$$

If, however, $p\colon E \to B$ does not decompose into the direct sum of line bundles, we are still able to define $s_I(c(E))$ via the splitting principle. Below we will construct the flag bundles and associated maps and classes to ensure the relation still holds.

As we had when we defined the Chern classes, let $z\colon B \to E$ be the zero section and $E_0 = E \backslash z(B)$. Let us consider the space $P_1(E) = E_0/\!\sim$ where $\sim$ is the equivalence relation given by $\mathbf{v} \sim \mathbf{w}$ iff $\mathbf{v} = \lambda\mathbf{w}$ for some $\lambda \in \mathbb{C}$. There is a map $f_1\colon P_1(E) \to B$ given by $f_1([\mathbf{v}]) = p(\mathbf{v})$. This map is well defined as $\lambda\mathbf{v}$ is necessarily in the same fibre as $\mathbf{v}$ $\forall\ \lambda \in \mathbb{C}$.

We may consider the pullback of E by f_1:

$$\begin{array}{ccc} f_1^*(E) & \longrightarrow & E \\ {\scriptstyle q_1}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p} \\ P_1(E) & \xrightarrow[f_1]{} & B \end{array}$$

where $f_1{}^*(E) = \{([\mathbf{v}],\ \mathbf{w}) \in P_1(E) \times E \mid p(\mathbf{v}) = p(\mathbf{w})\}$ and $q_1([\mathbf{v}],\ \mathbf{w}) = [\mathbf{v}]$.

Let $L_1 = \{([\mathbf{v}],\ \mathbf{w}) \in f_1{}^*(E) \mid \mathbf{w} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{C}\}$ and $l_1 = q_1|_{L_1}$, then $l_1\colon L_1 \to P_1(E)$ is a line bundle. Let E_1 be the orthogonal complement of

L_1 in $f_1{}^*(E)$ and $p_1 = q_1|_{E_1}$. q_1: $f_1{}^*(E) \to P_1(E)$ thus decomposes into the direct sum of the vector bundles l_1: $L_1 \to P_1(E)$ and p_1: $E_1 \to P_1(E)$.

Since the total Chern class is natural, we know that:

$$\begin{aligned} f_1^*(c(E)) &= c(f_1^*(E)) \\ &= c(L_1 \oplus E_1) \\ &= c(L_1)c(E_1) \end{aligned}$$

This process iterates.

At each following step we have a vector bundle p_k: $E_k \to P_k(E)$, we consider another zero section z_k: $P_k(E) \to E_k$ and define $(E_k)_0 = E_k \backslash z_k(P_k(E))$ and $P_{k+1}(E) = (E_k)_0/{\sim}$ where $\sim$ is the equivalence relation given by ([$\mathbf{v}_1$, ..., $\mathbf{v}_k$], $\mathbf{w}$) $\sim$ ([$\mathbf{v}_1$', ..., $\mathbf{v}_k$'], $\mathbf{w}$') iff [$\mathbf{v}_1$, ..., $\mathbf{v}_k$] = [$\mathbf{v}_1$', ..., $\mathbf{v}_k$'] and $\mathbf{w}' = \lambda\mathbf{w}$ for some $\lambda \in \mathbb{C}$.

We have maps f_{k+1}: $P_{k+1}(E) \to P_k(E)$ and thus we may construct the pullback of E_k by f_{k+1} resulting in a vector bundle q_{k+1}: $f_{k+1}{}^*(E_k) \to P_{k+1}(E)$ that decomposes into the direct sum of the line bundle l_{k+1}: $L_{k+1} \to P_{k+1}(E)$ and the vector bundle p_{k+1}: $E_{k+1} \to P_{k+1}(E)$. Therefore giving us the formula:

$$f_{k+1}^*(c(E_k)) = c(L_{k+1})c(E_{k+1})$$

If p: $E \to B$ is an n-dimensional complex vector bundle, this process terminates as the vector bundle q_n: $f_n{}^*(E_{n\text{-}1}) \to P_n(E)$ is equal to the line bundle l_n: $L_n \to P_n(E)$.

Since each l_i: $L_i \to P_i(E)$ is a line bundle there is only once Chern class of note for each $1 \leq i \leq n$: $c_1(L_i) \in H^2(P_i(E); \mathbb{Z})$.

This results in a family of equations:

$$\begin{aligned} f_1^*(c(E)) &= (1 + c_1(L_1))c(E_1) \\ f_2^*(c(E_1)) &= (1 + c_1(L_2))c(E_2) \\ &\vdots \\ f_{n-1}^*(c(E_{n-2})) &= (1 + c_1(L_{n-1}))c(E_{n-1}) \\ f_n^*(c(E_{n-1})) &= 1 + c_1(L_n) \end{aligned}$$

In addition, since f_1: $P_1(E) \to B$ and the maps f_i: $P_i(E) \to P_{i\text{-}1}(E)$ for $2 \leq i \leq n$ are surjections, the induced maps $f_1{}^*$: $H^*(B; \mathbb{Z}) \to H^*(P_1(E); \mathbb{Z})$ and $f_i{}^*$: $H^*(P_{i\text{-}1}(E); \mathbb{Z}) \to H^*(P_i(E); \mathbb{Z})$ for $2 \leq i \leq n$ must be injections.

Therefore we may rewrite our family of equations:

$$\begin{aligned} c(E) &= f_1^{*\text{-}1}((1 + c_1(L_1))c(E_1)) \\ c(E_1) &= f_2^{*\text{-}1}((1 + c_1(L_2))c(E_2)) \\ &\vdots \\ c(E_{n-2}) &= {f_{n-1}^*}^{\text{-}1}((1 + c_1(L_{n-1}))c(E_{n-1})) \\ c(E_{n-1}) &= f_n^{*\text{-}1}(1 + c_1(L_n)) \end{aligned}$$

and so, by making all the substitutions, the family may be written as a single equation. However, since each ${f_k}^*$ is a ring homomorphism, we may first simplify even further:

For $1 \leq i \leq n$ let us denote by $\chi_i(E)$ the class in $H^2(B; \mathbb{Z})$ such that:

$$\chi_i(E) = (f_i^* \circ ... \circ f_1^*)^{-1}(c_1(L_i))$$

Then:

$$c(E) = (1 + \chi_1(E))...(1 + \chi_n(E))$$

And so, for an n-dimensional non-decomposable complex vector bundle p: $E \to B$ and any partition I of some $k \in \mathbb{N}$ we write:

$$\begin{aligned} s_I(c(E)) &:= s_I(c_1(E), ..., c_k(E)) \\ &= s_I(\sigma_1(\chi_1(E), ..., \chi_n(E)), ..., \sigma_k(\chi_1(E), ..., \chi_n(E))) \end{aligned}$$

Note that $s_I(c(E))$ is a polynomial in the Chern classes $c_1(E)$, ..., $c_k(E)$ so in practice it will not regularly be necessary to compute the classes $\chi_i(E)$, they are just necessary to show that the Chern classes are elementary symmetric polynomials in some set of n indeterminates.

By the splitting principle, if p: $E \to B$ is an n-dimensional vector bundle, we can construct a space Y and a map ϕ: $Y \to B$ such that the pullback bundle $\phi^*(p)$: $\phi^*(E) \to Y$ can be decomposed into the direct sum of line bundles:

$$\phi^*(E) = \bigoplus_{i=1}^{n} L'_i$$

The line bundles l_i': L_i' $\to Y$ can be ordered in such a way that $c_1(L_i') = \phi^*(\chi_i(E))$ and since ϕ^* is required to be an injection, we have that $\chi_i(E) = (\phi^*)^{-1}(c_1(L_i'))$

We can verify that $c_i(\phi^*(E)) = \phi^*(c_i(E))$.

$$\begin{aligned} c_i(\phi^*(E)) &= c_i(\bigoplus_{j=1}^{n} L'_j) \\ &= \sigma_i(c_1(L'_1), ..., c_1(L'_n)) \\ \phi^*(c_i(E)) &= \phi^*(\sigma_i(\chi_1(E), ..., \chi_n(E))) \\ &= \sigma_i(\phi^*(\chi_1(E)), ..., \phi^*(\chi_n(E))) \\ &= \sigma_i(\phi^*((\phi^*)^{-1}(c_1(L'_1))), ..., \phi^*((\phi^*)^{-1}(c_1(L'_1)))) \\ &= \sigma_i(c_1(L'_1), ..., c_1(L'_n)) \end{aligned}$$

and so for any partition I of $k \in \mathbb{N}$, $s_I(c(E)) = (\phi^*)^{-1}(s_I(c(\phi^* E)))$. △

Lemma 38. If $I = (i_1, ..., i_r)$ is a partition of $k \in \mathbb{N}$ and p: $E \to B$, p': E' $\to B$ are two complex vector bundles over the same space then:

$$s_I(c(E \oplus E')) = \sum_{j=0}^{r} s_{i_1,...,i_j}(c(E)) + s_{i_{j+1},...,i_r}(c(E'))$$

Definition 25. For an n-dimensional complex vector bundle $p\colon E \to B$, the **Chern character** $ch(E)$ is defined to be the following:

$$ch(E) := n + \sum_{k=1}^{\infty} \frac{s_k(c(E))}{k!} \in H^*(B;\mathbb{Q})$$

Notice that if $p\colon E \to B$ is a line bundle we have:

$$ch(E) = exp(c_1(E))$$

The Chern character can been shown to have some beneficial properties that the total Chern class lacks.

Theorem 39. Since the total Chern class is a characteristic class, and the Chern classes all lie in even degree, the Chern character $ch\colon \mathit{Vect}(B) \to H^{even}(B;\mathbb{Z})$ is well defined where:

$$H^{even}(B;\mathbb{Q}) := \bigoplus_{k\in 2\mathbb{Z}} H^k(B;\mathbb{Q})$$

If $p\colon E \to B$ and $q\colon F \to B$ are vector bundles over B, then

$$ch(E \oplus F) = ch(E) + ch(F)$$
$$\text{and } ch(E \otimes F) = ch(E)ch(F)$$

therefore the Chern character extends to a ring homomorphism $ch\colon K^0(B) \to H^*(B;\mathbb{Q})$ and thus a natural transformation $K^0 \to H^*(\text{-};\mathbb{Q})$ as functors **Top** $\to$ **Rng**

Proof. It is not too tricky to show that the Chern character satisfies the first property. For $p\colon E \to B$ and $q\colon F \to B$, two complex vector bundles over the same space B, we have:

$$\begin{aligned}
ch(E \oplus F) &= rank(E \oplus F) + \sum_{k=1}^{\infty} \frac{s_k(c(E \oplus F))}{k!} \\
&= rank(E) + rank(F) + \sum_{k=1}^{\infty} \frac{s_k(c(E)) + s_k(c(F))}{k!} \\
&= rank(E) + \sum_{k=1}^{\infty} \frac{s_k(c(E))}{k!} + rank(F) + \sum_{k=1}^{\infty} \frac{s_k(c(F))}{k!} \\
&= ch(E) + ch(F)
\end{aligned}$$

Thus the direct sum of vector bundles becomes addition of their Chern characters.

The set of vector bundles over a base space B together with the direct sum is a monoid. Group completion of this monoid ensures that the Chern character is a homomorphism of abelian groups.

To prove that the multiplication operation is preserved, we will first consider when the two complex vector bundles can both be decomposed into the direct sum of line bundles. This decomposition will help as the tensor product distributes over the direct sum.

To achieve the result we want we will have to determine the Chern classes of the tensor product of vector bundles.

Lemma 40. For two complex line bundles l: $L \to B$ and l': L' $\to B$ over the same base space, we have:

$$\begin{aligned} c_0(L \otimes L') &= 1, \\ c_1(L \otimes L') &= c_1(L) + c_1(L'), \\ c_i(L \otimes L') &= 0, \quad \text{for } i \neq 0, 1. \end{aligned}$$

Let p: $E \to B$ and q: $F \to B$ be complex vector bundles over the same space such that there exist complex line bundles l_i: $L_i \to B$ for $1 \leq i \leq n$ and l_j': L_j' $\to B$ for $1 \leq j \leq m$ where $p = l_1 \oplus ... \oplus l_n$, $q = l_1$' $\oplus ... \oplus l_m$', $E = L_1 \oplus ... \oplus L_n$, and $F = L_1$' $\oplus ... \oplus L_m$'

$$\begin{aligned} ch(E \otimes F) &= ch((L_1 \oplus ... \oplus L_n) \otimes (L'_1 \oplus ... \oplus L'_m)) \\ &= ch(\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} (L_i \otimes L'_j)) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} ch(L_i \otimes L_j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} exp(c_1(L_i \otimes L_j)) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} exp(c_1(L_i) + c_1(L_j)) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} exp(c_1(L_i))exp(c_1(L_j)) \\ &= (\sum_{i=1}^{n} exp(c_1(L_i)))(\sum_{j=1}^{m} exp(c_1(L_j))) \\ &= (\sum_{i=1}^{n} ch(L_i))(\sum_{j=1}^{m} ch(L_j)) \\ &= ch(\bigoplus_{i=1}^{n} L_i)ch(\bigoplus_{j=1}^{m} L_j) \\ &= ch(E)ch(F) \end{aligned}$$

Thus the Chern character of the tensor product of two vector bundles that can be decomposed into the direct sum of line bundles is indeed equal to the product of their Chern characters.

If however, $p\colon E \to B$ and $q\colon F \to B$ are two vector bundles that cannot be decomposed into the direct sum of line bundles, we will need to implement the splitting principle.

Let Y and Z be the spaces and $\phi\colon Y \to B$ and $\psi\colon Z \to B$ be the maps such that the pullback bundles $\phi^*(B) \to Y$ and $\psi^*(B) \to Z$ decompose into the direct sum of line bundles.

We can define the vector bundle $p \otimes q\colon E \otimes F \to B$ and a continuous map $\phi \otimes \psi\colon Y \otimes Z \to B$ to construct a pullback bundle:

$$\begin{array}{ccc} (\phi \otimes \psi)^*(E \otimes F) & \longrightarrow & E \otimes F \\ {\scriptstyle (\phi\otimes\psi)^*(p\otimes q)}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p\otimes q} \\ Y \otimes Z & \xrightarrow[\phi\otimes\psi]{} & B \end{array}$$

Since the tensor product is compatible with the pullback construction, here the pullback bundle is equivalent to the tensor product of the two pullback bundles $\phi^*(p) \otimes \psi^*(q)\colon \phi^*(E) \otimes \psi^*(F) \to Y \otimes Z$.

The pullback bundles $\phi^*(p)$ and $\psi^*(q)$ both decompose into the direct sum of line bundles so $ch(\phi^*(E) \otimes \psi^*(F)) = ch(\phi^*(E))ch(\psi^*(F))$ as we have already seen.

Therefore:

$$\begin{aligned} ch(E \otimes F) &= rank(E \otimes F) + \sum_{k=1}^{\infty} \frac{s_k(c(E \otimes F)}{k!} \\ &= rank(\phi^*(E) \otimes \psi^*(F)) + \sum_{k=1}^{\infty} \frac{(\phi^* \otimes \phi^*)^{-1}(s_k(c(\phi^*(E) \otimes \psi^*(F))))}{k!} \\ &= (\phi^* \otimes \phi^*)^{-1}(ch(\phi^*(E) \otimes \psi^*(F))) \\ &= (\phi^* \otimes \phi^*)^{-1}(ch(\phi^*(E))ch(\psi^*(F))) \\ &= (\phi^*)^{-1}(ch(\phi^*(E)))(\psi^*)^{-1}(ch(\psi^*(F))) \\ &= ch(E)ch(F) \end{aligned}$$

Thus the tensor product of any pair of vector bundles becomes multiplication of the Chern characters.

These two properties together can be extended to ensure that the Chern character is a ring homomorphism:

$$ch : K^0(B) \to H^{even}(B; \mathbb{Q})$$

△

6 General Cohomology Theories and the Chern Character

6.1 General Cohomology Theories

Definition 26. A **cohomology theory** is a sequence of contravariant functors from the category of pairs of CW-complexes to the category of abelian groups h^k: **CWpair** $\to$ **AbGrp** together with natural transformations between consecutive functors with components d: $h^k(A) \to h^{k+1}(X,\, A)$ called the **boundary homomorphisms** where $h^k(A) := h^k(A,\, \emptyset)$ such that:

- If f, g: $(X,\, A) \to (Y,\, B)$ are homotopic maps, then the maps induced by the functors f^*,g^*: $h^k(Y,\, B) \to h^k(X,\, A)$ are equal for each k

- For any pair of CW-complexes $(X,\, A)$, the inclusions i: $A \to X$ and j: $(X,\, \emptyset) \to (X,\, A)$ induce a long exact sequence:

$$... \longrightarrow h^{k-1}(A) \xrightarrow{d} h^k(X, A) \xrightarrow{j^*} h^k(X) \xrightarrow{i^*} h^k(A) \xrightarrow{d} h^{k+1}(X, A) \longrightarrow ...$$

- If A and B are two subcompleces of X such that $X = A \cup B$, then the inclusion i: $(A,\, A \cap B) \to (X,\, B)$ induces an isomorphism for each functor i^*: $h^k(X,\, B) \to h^k(A,\, A \cap B)$

- If we have a pair of CW-complexes $(X_\alpha,\, A_\alpha)$ for each α in some set I and let $(X,\, A) = \coprod_{\alpha \in I}(X_\alpha,\, A_\alpha)$, then the inclusions i_α: $(X_\alpha,\, A_\alpha) \to (X,\, A)$ induce an isomorphism for each k

$$i^* : h^k(X, A) \to \prod_{\alpha \in I} h^k(X_\alpha, A_\alpha)$$

If, in addition, a cohomology theory h^* is such that $h^k(pt) = 0 \;\forall\; k \neq 0$, we say that h^* is an **ordinary cohomology theory**, if it does not satisfy this property, we call it an **extraordinary cohomology theory**.

These are the Eilenberg-Steenrod axioms as defined by Eilenberg and Steenrod in *Foundations of Algebraic Topology* [10].

For ordinary cohomology theories and many (at least all those we will discuss) extraordinary cohomology theories, we can define a functor also denoted h^*: **CWpair** $\to$ **Rng** by equipping the graded abelian group

$$h^*(X, A) := \bigoplus_{k \in \mathbb{Z}} h^k(X, A)$$

with a multiplication structure derived in part by the contravariance of the functors h^k and the diagonal map Δ: $X \to X \times X$, where $\Delta(x) = (x,\, x)$. [17]

Let us continue by defining a couple of extraordinary cohomology theories built using some familiar abelian groups.

Definition 27. Consider a compact Hausdorff space X and let $C(X)$ be the set of all complex valued functions on X. Let $\mathrm{GL}(n,\ C(X))$ be the topological group of invertible $n\times n$ matrices with entries in $C(X)$ and let $\mathrm{GL}(n,\ C(X))_0$ be the connected component of $\mathbb{I}_n$ in $\mathrm{GL}(n,\ C(X))$ where 1 and 0 are the constant maps to 1 and 0 in $\mathbb{C}$ respectively.

We define the following groups:

$$\mathrm{GL}(C(X)) = (\bigcup_{n\in\mathbb{N}} \mathrm{GL}(n, C(X)))/\sim$$
$$\text{and}\quad \mathrm{GL}(C(X))_0 = (\bigcup_{n\in\mathbb{N}} \mathrm{GL}(n, C(X))_0)/\sim$$

where $\sim$ is the equivalence relation defined by $A \sim diag(A,\ 1)$.

$\mathrm{GL}(C(X))_0$ is a normal subgroup of $\mathrm{GL}(C(X))$ and so we may define a group called the **-1$^{\text{st}}$ K-theory group**:

$$K^{-1}(X) := \mathrm{GL}(C(X))/\mathrm{GL}(C(X))_0$$

Together with our previous construction of the 0^{th} K-theory group, and a property called **Bott-periodicity** that ensures that $K^n(X) \cong K^{n+2}(X)\ \forall\ n \in \mathbb{Z}$, we are able to define an extraordinary cohomology theory called **complex K-theory** using the functors K^n.

There are many non-trivial statements in this definition including but not limited to: each mapping K^n defines a functor **CWpair** $\to$ **AbGrp**, we have Bott-periodicity, that we have boundary homomorphisms, and thus that we have an extraordinary cohomology theory. Proofs of all of these statements can be found in *Complex Topological K-Theory* by Efton Park [28].

Definition 28. Using ordinary cohomology with coefficients in G for some group G as a base, we can construct an extraordinary cohomology theory we will call **periodic cohomology** with coefficients in G:

$$H^i_{per}(X;G) := \bigoplus_{k\in 2\mathbb{Z}} H^{i+k}(X;G)$$

Clearly there is an isomorphism $H^i_{per}(X;G) \cong H^{i+2}_{per}(X;G)\ \forall\ i$.

We often write $H^{even}(X;\ G) = H^0_{per}(X;G)$ and we may define $H^{odd}(X;\ G) = H^{-1}_{per}(X;G)$.

6.2 Behaviour of the Chern Character

We have defined the Chern character for some space X as a ring homomorphism ch: $K^0(X) \to H^{even}(X;\ \mathbb{Q})$, we would like to extend this to a natural transformation ch: $K^* \to H^*_{per}(\text{-};\ \mathbb{Q})$.

Bott periodicity gives us an isomorphism $K^n(X) \cong K^0(X)$ for even integers $n \in 2\mathbb{Z}$, so we may immediately construct an abelian group homomorphism we may call ch:

$$\begin{array}{ccc} K^n(X) & \overset{ch}{\dashrightarrow} & H^n_{per}(X;\mathbb{Q}) \\ \cong\big\downarrow & \square & \big\uparrow\cong \\ K^0(X) & \underset{ch}{\longrightarrow} & H^{even}(X;\mathbb{Q}) \end{array}$$

The isomorphisms $K^{-1}(X) \cong K^0(X \times \mathbb{R})$ and $H^{n-1}(X;\,\mathbb{Q}) \cong H^n(X \times \mathbb{R};\,\mathbb{Q})$ ensure the existence of an abelian group homomorphism we can call the odd Chern character:

$$\begin{array}{ccc} K^{-1}(X) & \overset{ch}{\dashrightarrow} & H^{odd}(X;\mathbb{Q}) \\ \cong\big\downarrow & \square & \big\uparrow\cong \\ K^0(X \times \mathbb{R}) & \underset{ch}{\longrightarrow} & H^{even}(X \times \mathbb{R};\mathbb{Q}) \end{array}$$

and Bott periodicity once again gives us an isomorphism $K^n(X) \cong K^{-1}(X)$ for odd integers $n \in -1 + 2\mathbb{Z}$ and the final set of abelian group homomorphisms can be constructed:

$$\begin{array}{ccc} K^n(X) & \overset{ch}{\dashrightarrow} & H^n_{per}(X;\mathbb{Q}) \\ \cong\big\downarrow & \square & \big\uparrow\cong \\ K^{-1}(X) & \underset{ch}{\longrightarrow} & H^{odd}(X;\mathbb{Q}) \end{array}$$

Since K theory and cohomology are functors and all of these isomorphisms are functorial, for any map $\phi\colon X \to Y$ and $n \in \mathbb{Z}$, the appropriate Chern characters ensure the following diagram commutes:

$$\begin{array}{ccc} K^n(Y) & \overset{ch}{\longrightarrow} & H^n_{per}(Y;\mathbb{Q}) \\ \phi^*\big\downarrow & \square & \big\downarrow\phi^* \\ K^n(X) & \underset{ch}{\longrightarrow} & H^n_{per}(X;\mathbb{Q}) \end{array}$$

The following definitions are provided by Hilton [18].

Definition 29. Let h^* be a cohomology theory, the corresponding **reduced cohomology theory** $\tilde{h}^*$ is a sequence of contravariant functors from the category of pointed topological spaces (at least CW-complexes) to the category of abelian groups $\tilde{h}$: **Top*** $\to$ **AbGrp**. If $i\colon\, pt \to X$ is the inclusion of a point to the basepoint of a CW-complex X, then $\tilde{h}^k(X)$ is the kernel of the map induced by the cohomology theory for each k, i.e. the following sequence is exact $\forall\ k$:

$$0 \longrightarrow \tilde{h}^k(X) \longrightarrow h^k(X) \overset{i^*}{\longrightarrow} h^k(pt)$$

The **coefficient ring** of a cohomology theory defined to be the graded ring $\breve{h}^*$ where each $\breve{h}^k := \tilde{h}^k(\mathbb{S}^0)$.

Let us examine the coefficient rings of some familiar cohomology theories.

Firstly, cohomology with rational coefficients $H^*(\text{-};\, \mathbb{Q})$ unsurprisingly has coefficient ring $\breve{H}^* \cong \mathbb{Q}$. This is true since $H^0(pt;\, \mathbb{Q}) \cong \mathbb{Q}$, $H^0(\mathbb{S}^0;\, \mathbb{Q}) \cong \mathbb{Q} \times \mathbb{Q}$ and the map induced by $i\colon\ pt \to \mathbb{S}^0$ is the projection onto one of the entries $i^*(p,\, q) = p$. All other groups in the coefficient ring are trivial since for $k \neq 0$, $H^k(\mathbb{S}^0;\, \mathbb{Q}) \cong 0$.

It is also easy to determine the coefficient ring for complex K-theory. We already know $K^0(\mathbb{S}^0) \cong \mathbb{Z}\times\mathbb{Z}$, $K^0(pt) \cong \mathbb{Z}$ and the map induced by the inclusion $i\colon\ pt \to \mathbb{S}^0$ is again the projection onto one of the entries $i^*(p,\, q) = p$ so we have $\tilde{K}^0(\mathbb{S}^0) \cong \mathbb{Z}$. The other K-theory group is easy, $\tilde{K}^{\text{-}1}(\mathbb{S}^0) \cong 0$ since $K^{\text{-}1}(\mathbb{S}^0)$ $\cong 0$.

Bott periodicity ensures that $K^n(X) \cong K^{n\text{-}2}(X)$, a property maintained in reduced K-theory, thus as a graded abelian group, the coefficient ring of complex K-theory is as follows:

$$\breve{K}^n \cong \begin{cases} \mathbb{Z}, & n \in 2\mathbb{Z} \\ 0, & n \in 1 + 2\mathbb{Z} \end{cases}$$

Since K-theory admits a cup product, as a graded ring, the coefficient ring is:

$$\breve{K}^* = \mathbb{Z}[x, x^{-1}]$$

where x is a generator in degree 2.

The other cohomology theory currently of note is periodic cohomology with rational coefficients $H^*_{per}(\text{-};\, \mathbb{Q})$. As we saw before, $\tilde{H}^0(\mathbb{S}^0;\, \mathbb{Q}) \cong \mathbb{Q}$ and for all other $n \neq 0$, $\tilde{H}^n(\mathbb{S}^0;\, \mathbb{Q}) \cong 0$. Since relative periodic cohomology is the kernel of a group homomorphism we have:

$$\tilde{H}^n_{per}(X;\mathbb{Q}) = \bigoplus_{k\in 2\mathbb{Z}} \tilde{H}^{n+k}(X;\mathbb{Q})$$

Thus as a graded abelian group the coefficient ring of periodic cohomology is:

$$\breve{H}^n_{per} \cong \begin{cases} \mathbb{Q}, & n \in 2\mathbb{Z} \\ 0, & n \in 1 + 2\mathbb{Z} \end{cases}$$

and again, thanks to the cup product, as a ring:

$$\breve{H}^*_{per} = \mathbb{Q}[y, y^{-1}]$$

where y is a generator in degree 2.

Lemma 41. If k^* and h^* are cohomology theories, and $\breve{h}^*$ is a graded vector space over $\mathbb{Q}$, then any homomorphism of graded rings $\breve{\eta}\colon\ \breve{k}^* \to \breve{h}^*$ has a unique extension to a natural transformation $\eta\colon\ k^* \to h^*$.

Proven by Hilton. Theorem 3.22 [18]

Thus in order to understand the Chern character as a natural transformation, since $\breve{H}^*_{per}$ is a graded vector space over $\mathbb{Q}$, we need only understand the homomorphism of graded rings obtained by restricting the Chern character to the coefficient rings which will naturally extend to the full natural transformation.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \breve{K}^n & \hookrightarrow & K^n(\mathbb{S}^0) & \xrightarrow{i^*} & K^n(pt) \\ & & \Big\downarrow{\scriptstyle \breve{ch}} & \square & \Big\downarrow{\scriptstyle ch} & \square & \Big\downarrow{\scriptstyle ch} \\ 0 & \longrightarrow & \breve{H}^n_{per} & \hookrightarrow & H^n_{per}(\mathbb{S}^0;\mathbb{Q}) & \xrightarrow[i^*]{} & H^n_{per}(pt;\mathbb{Q}) \end{array}$$

6.3 Axiomatic Reduced Cohomology Theories

There is an alternative axiomatic definition for reduced cohomology theories; often we prefer not to work with pairs of CW-complexes, but only with pointed topological spaces. Again we defer to definitions provided in *General Cohomology Theory and K-theory* by Hilton [18]

Definition 30. For any based continuous map $f\colon X \to Y$ let us define C_f the **mapping cone** of f as follows:

$$C_f := ((X \times [0,1]) \sqcup Y)/\sim$$

where $\sim$ is the equivalence relation defined as follows:

$$\begin{aligned} (x,0) &\sim (x',0) \\ (x,1) &\sim f(x) \\ \text{and } (x_0,t) &\sim (x_0,t') \end{aligned}$$

Definition 31. A **reduced cohomology theory** is a sequence of contravariant functors from the category of pointed topological spaces to the category of abelian groups h^k: **Top*** $\to$ **AbGrp** together with a sequence of natural transformations with components $\sigma^n_X\colon h^n(X) \to h^{n+1}(\Sigma X)$ such that:

- If f, $g\colon X \to Y$ are homotopic based continuous maps, then the maps induced by the functors f^*, $g^*\colon h^k(Y) \to h^k(X)$ are equal for each k
- The natural transformations $\sigma^n\colon h^n \to h^{n+1}\Sigma$ are all natural isomorphisms.
- for any based continuous map $f\colon X \to Y$, the sequence

  $$X \xrightarrow{f} Y \xrightarrow{i} C_f$$

  where $i\colon Y \to C_f$ is the inclusion of Y into the mapping cone of f, induces an exact sequence for each k:

  $$h^k(X) \xleftarrow{f^*} h^k(Y) \xleftarrow{i^*} h^k(C_f)$$

It can, but won't be shown that the definition of a cohomology theory described earlier is equivalent to this definition of a reduced cohomology theory when restricted to pointed topological spaces.

Definition 32. As first introduced by Adams [1], a family Y of pointed topological spaces Y_n together with a family of based continuous maps $g_n\colon Y_n \to \Omega Y_{n+1}$ is called an **Ω-spectrum** if g_n is a homotopy equivalence for all n.

Lemma 42. If Y is an Ω-spectrum, then there exists a reduced cohomology theory h_Y^* consisting of contravariant functors $h_Y^k\colon \mathbf{Top^*} \to \mathbf{AbGrp}$ defined by a mapping of objects: $\forall\ X \in ob(\mathbf{Top^*})$

$$h_Y^k(X) := [X, Y_k]$$

and mapping of morphisms: $\forall\ f \in \mathrm{Hom}_{\mathbf{Top^*}}(X,\ Z)$

$$\begin{aligned} h_Y^k(f) : h_Y^k(Z) &\to h_Y^k(X) \\ [\phi] &\mapsto [\phi \circ f] \end{aligned}$$

together with natural transformations $\sigma^n\colon\ h_Y^n \to h_Y^{n+1}\Sigma$ with components:

$$\begin{aligned} \sigma_X^n : h_Y^n(X) &\to h_Y^{n+1}(\Sigma X) \\ [\phi] &\mapsto [\Theta^{-1}_{X,Y_{n+1}}(g_n \circ \phi)] \end{aligned}$$

where $\Theta_{X,Y}\colon\ \mathrm{Hom}_{\mathbf{Top^*}}(\Sigma X,\ Y) \to \mathrm{Hom}_{\mathbf{Top^*}}(X,\ \Omega Y)$ are the component isomorphisms of the suspension-loop adjunction $\Sigma \dashv \Omega$.

Proof. Clearly each h_Y^k is a contravariant functor, as for every $\phi\colon\ X \to Y_k$ we have $h_Y^k(\mathrm{id}_X)[\phi] = [\phi \circ \mathrm{id}_X] = [\phi]$ and thus $h_Y^k(\mathrm{id}_X) = \mathrm{id}_{h_Y^k(X)}$.

To show that each σ^n is a natural transformation, we must show that for any $f\colon\ X \to Z$ the following diagram commutes:

$$\begin{array}{ccc} h_Y^n(Z) & \xrightarrow{\sigma_Z^n} & h_Y^{n+1}(\Sigma Z) \\ {\scriptstyle h_Y^n(f)}\downarrow & \square & \downarrow{\scriptstyle h_Y^{n+1}(\Sigma f)} \\ h_Y^n(X) & \xrightarrow[\sigma_X^n]{} & h_Y^{n+1}(\Sigma X) \end{array}$$

That is, for any $[\phi] \in h_Y^n(Z)$ we must show that:

$$(h_Y^{n+1}(\Sigma f)(\sigma_Z^n[\phi]) = (\sigma_X^n(h_Y^n(f)[\phi])).$$

$$\begin{aligned} (h_Y^{n+1}(\Sigma f)(\sigma_Z^n[\phi]) &= [\Theta^{-1}_{Z,Y_{n+1}}(g_n \circ \phi) \circ \Sigma f] \\ \text{and } (\sigma_X^n(h_Y^n(f)[\phi])) &= [\Theta^{-1}_{X,Y_{n+1}}(g_n \circ \phi \circ f)]. \end{aligned}$$

Let $g_n \circ \phi\colon\ Z \to \Omega Y_{n+1}$ be given by $(g_n \circ \phi)(z) = \zeta_z\colon\ \mathbb{S}^1 \to Y_{n+1}$ then $\Theta^{-1}_{X,Y_{n+1}}(g_n \circ \phi)\colon \Sigma Z \to Y_{n+1}$ is the mapping $(z,\ t) \mapsto \zeta_z(t)$.

Let us examine the behaviour of the two maps from ΣX to Y_{n+1}.

$$\begin{aligned}(\Theta^{-1}_{Z,Y_{n+1}}(g_n \circ \phi) \circ \Sigma f)(x,t) &= \Theta^{-1}_{Z,Y_{n+1}}(g_n \circ \phi)(f(x),t) \\ &= \zeta_{f(x)}(t) \\ (\Theta_{X,Y_{n+1}}(g_n \circ \phi \circ f))(x,t) &= (\Theta_{X,Y_{n+1}}(g_n \circ \phi))(f(x),t) \\ &= \zeta_{f(x)}(t)\end{aligned}$$

Therefore the two maps are equal and thus trivially homotopic and every σ^n is a natural transformation.

It will be shown that the functors h^k_Y together with the natural transformations σ_n satisfy the three axioms for a reduced cohomology theory.

Let f, g: $X \to Z$ be based continuous maps such that there exists a homotopy H: $X \times [0,1] \to Z$ where $H(x_0,\ t) = z_0$, $H(x,\ 0) = f(x)$, and $H(x,\ 1) = g(x)$.

Let ϕ: $Z \to Y_k$ be a representative of the class $[\phi] \in h^k_Y(Z)$, it must be shown that $f^*[\phi] = [\phi \circ f]$ and $g^*[\phi] = [\phi \circ g]$ are the same class in $h^k_Y(X)$, that is we must show there exists a homotopy between $\phi \circ f$ and $\phi \circ g$.

$\phi \circ H$: $X \times [0,1] \to Y_k$ is exactly such a homotopy.

$$\begin{aligned}(\phi \circ H)(x,0) &= \phi(f(x)) \\ &= (\phi \circ f)(x)\end{aligned} \qquad\qquad \begin{aligned}(\phi \circ H)(x,1) &= \phi(g(x)) \\ &= (\phi \circ g)(x)\end{aligned}$$

Thus, $f^*[\phi] = g^*[\phi]$ and since $[\phi]$ was arbitrary, f^* and g^* are equal maps.

Secondly, it must be shown that σ^n: $h^n_Y \to h^{n+1}_Y\Sigma$ is a natural isomorphism for all $n \in \mathbb{Z}$, that is, every component morphism σ^n_X: $h^n_Y(X) \to h^{n+1}_Y(\Sigma X)$ must be shown to be an isomorphism.

For any class $[\phi] \in h^n_Y(X)$, we have $\sigma^n_X[\phi] = [\Theta^{-1}_{X,Y_{n+1}}(g_n \circ \phi)]$, we would like to find a mapping τ^n_X: $h^{n+1}_Y(\Sigma X) \to h^n_Y(X)$ so that $\sigma^n \circ \tau^n = \mathrm{id}_{h^{n+1}_Y\Sigma}$ and $\tau^n \circ \sigma^n = \mathrm{id}_{h^n_Y}$.

In order to define τ^n_x, we will again use the component isomorphisms of the suspension loop adjunction $\Theta_{X,Y}$: $\mathrm{Hom}_{\mathbf{Top^*}}(\Sigma X,\ Y) \to \mathrm{Hom}_{\mathbf{Top^*}}(X,\ \Omega Y)$.

Also, since the maps g_n: $Y_n \to \Omega Y_{n+1}$ are homotopy equivalences, there exist maps called their homotopy inverses g_n': $\Omega Y_{n+1} \to Y_n$ such that $g_n \circ g_n$' is homotopic to $\mathrm{id}_{\Omega Y_{n+1}}$ and g_n'$\circ g_n$ is homotopic to id_{Y_n}.

Let $[\psi] \in h^{n+1}_Y(\Sigma X)$, we define $\tau^n_X[\psi] := [g_n\text{'}\circ\Theta_{X,Y_{n+1}}(\psi)]$

For all $[\phi] \in h^n_Y(X)$:

$$\begin{aligned}\tau^n_X(\sigma^n_X[\phi]) &= \tau^n_X[\Theta^{-1}_{X,Y_{n+1}}(g_n \circ \phi)] \\ &= [g'_n \circ \Theta_{X,Y_{n+1}}(\Theta^{-1}_{X,Y_{n+1}}(g_n \circ \phi))] \\ &= [g'_n \circ g_n \circ \phi] \\ &= [\phi]\end{aligned}$$

since $g_n \circ g'_n$ is homotopic to the identity, and for all $[\psi] \in h_Y^{n+1}(\Sigma X)$

$$\begin{aligned}\sigma_X^n(\tau_X^n[\psi]) &= \sigma_X^n[g'_n \circ \Theta_{X,Y_{n+1}}(\psi)]\\ &= [\Theta^{-1}_{X,Y_{n+1}}(g_n \circ g'_n \circ \Theta_{X,Y_{n+1}}(\psi))]\\ &= [\Theta^{-1}_{X,Y_{n+1}}(\Theta_{X,Y_{n+1}}(\psi))]\\ &= [\psi]\end{aligned}$$

since $g'_n \circ g_n$ is homotopic to the identity.

Therefore each σ_X^n is an isomorphism, and thus each σ^n is a natural isomorphism.

Finally, we must show that if $f\colon X \to Z$ is a based continuous map and $i\colon Z \to C_f$ is the inclusion of Z into the mapping cone of f, then the sequence:

$$h_Y^k(C_f) \xrightarrow{i^*} h_Y^k(Z) \xrightarrow{f^*} h_Y^k(X)$$

is exact for all $k \in \mathbb{Z}$, that is $\mathrm{Im}(i^*) = \mathrm{Ker}(f^*)$.

To show that $\mathrm{Im}(i^*) \subset \mathrm{Ker}(f^*)$ we must show that if $[\psi] \in h_Y^k(Z)$ is such that $\exists\, [\phi] \in h_Y^k(C_f)$ with $i^*[\phi] = [\psi]$, then $f^*[\psi] = [0] \in h_Y^k(X)$, or equivalently, $\forall\, [\phi] \in h_Y^k(C_f)$, we have $f^*(i^*[\phi]) = [0] \in h_Y^k(X)$. By the definition of these maps then, it must be shown that for all based continuous maps $\phi\colon C_f \to Y_k$, $\phi \circ i \circ f\colon X \to Y_k$ is homotopic to the mapping $x \mapsto y_0$ that sends all of X to the base point of Y_k.

There is a homotopy $H\colon X \times [0,1] \to C_f$ given by $H(x, t) = [x, t]$. It is seen that $H(x, 0) = [x_0, z_0]$ the base point of C_f, and $H(x, 1) = [f(x)] = i \circ f(x)$.

$\phi \circ H\colon X \times [0,1] \to Y_k$ therefore is a homotopy from the mapping $x \mapsto y_0$ to $\phi \circ i \circ f$ as $\phi \circ H(x, 0) = \phi[x_0, z_0] = y_0$ since ϕ is based continuous, and $\phi \circ H(x, 1) = \phi \circ i \circ f(x)$.

Thus $\mathrm{Im}(i^*) \subset \mathrm{Ker}(f^*)$.

To show that $\mathrm{Ker}(f^*) \subset \mathrm{Im}(i^*)$ we must show that if $[\psi] \in h_Y^k(Z)$ is such that $f^*[\psi] = [0] \in h_Y^k(X)$, then $\exists[\phi] \in h_Y^k(C_f)$ such that $i^*[\phi] = [\psi]$.

$f^*[\psi] = [0]$ means that there exists some homotopy $H\colon X \times [0,1] \to Y_k$ with $H(x, 0) = y_0$, $H(x, 1) = (\psi \circ f)(x)$, and $H(x_0, t) = y_0$.

Let us describe a map $\phi\colon C_f \to Y_k$:

$$\begin{aligned}\phi[x,t] &= H(x,t)\\ \phi[z] &= \psi(z)\end{aligned}$$

for $(x,t) \in X \times [0,1] \subset C_f$, and $z \in Z \subset C_f$.

To check that ϕ is well defined, we must show that the different representatives of each class of C_f are mapped identically:

$$\begin{aligned}\phi[x,0] &= H(x,0) & \phi[x,1] &= H(x,1) & \phi[x_0,t] &= H(x_0,t)\\ &= y_0 & &= (\psi \circ f)(x) & &= y_0\\ &= H(x',0) & &= \psi(f(x)) & &= H(x_0,t')\\ &= \phi[x',0] & &= \phi[f(x)] & &= \phi[x_0,t']\end{aligned}$$

And since $\phi \circ i = \psi$ by definition of the inclusion, we have found a map that exactly maps to any ψ in the kernel of f^*. Since this is equality, taking the homotopy classes clearly gives us $i^*[\phi] = [\psi]$, and thus $[\psi] \in \mathrm{Im}(i^*)$.

Therefore $\mathrm{Ker}(f^*) \subset \mathrm{Im}(i^*)$ and thus $\mathrm{Ker}(f^*) = \mathrm{Im}(i^*)$.

Therefore, from any Ω-spectrum, a reduced cohomology theory can be constructed. △

6.4 Künneth formula and Products with Tori

Definition 33. A cohomology theory h^* has a property called the **Künneth formula** over a ring R if for any topological spaces X, Y where $h^n(Y)$ is a finitely generated free R-module for all n, for all k, we have an isomorphism of abelian groups:

$$h^k(X \times Y) \cong \bigoplus_{i+j=k} h^i(X) \otimes_R h^j(Y)$$

and we have an isomorphism of rings:

$$h^*(X \times Y) \cong h^*(X) \otimes_R h^*(Y)$$

Ordinary cohomology with coefficients in a field $\Bbbk$ has the Künneth formula over $\Bbbk$ [17] and K-theory has the Künneth formula over $\mathbb{Z}$ [28]. Thanks to Bott-periodicity, the Künneth formula group isomorphisms for K-theory are as follows:

$$K^0(X \times Y) \cong (K^0(X) \otimes K^0(Y)) \oplus (K^1(X) \otimes K^1(Y))$$
$$K^1(X \times Y) \cong (K^0(X) \otimes K^1(Y)) \oplus (K^1(X) \otimes K^0(Y))$$

As we have seen, the Leray-Hirsch Theorem is a generalisation of the Künneth formula in ordinary cohomology for total spaces of fibre bundles, i.e. spaces that are locally but not necessarily globally a product space, the trade off being that we achieve only a module isomoprhism instead of a ring isomorphism.

Many general cohomology theories do not admit a Künneth formula but we will see that there is a family of product spaces where the ring induced by any general cohomology theory can be decomposed into a tensor product.

Lemma 43. Let h^* be a general cohomology theory and X be a CW-complex.

Then $h^*(X \times \mathbb{S}^1) \cong h^*(X) \otimes \Lambda^*_{\mathbb{Z}}[z]$ where z has degree 1.

Furthermore, $h^*(X \times \mathbb{T}^n) \cong h^*(X) \otimes \Lambda^*_{\mathbb{Z}}[z_1, ..., z_n]$ where each z_i has degree 1.

Proof. Let D^1 and d^1 denote closed and open respectively connected proper subsets of $\mathbb{S}^1$.

We will first examine the long exact sequence of the pair $(X \times \mathbb{S}^1, X \times D^1)$:

$$\begin{array}{l} ... \to h^k(X \times \mathbb{S}^1, X \times D^1) \longrightarrow h^k(X \times \mathbb{S}^1) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad h^{k+1}(X \times D^1) \to h^{k+1}(X \times \mathbb{S}^1, X \times D^1) \to ... \end{array}$$

Since D^1 is a closed and connected proper subset of $\mathbb{S}^1$, it is homeomorphic to a closed interval which in turn is homotopy equivalent to a single point. The resulting homotopy equivalence allows us to simplify our long exact sequence:

$$\begin{array}{ccc} ... \to h^k(X \times \mathbb{S}^1, X \times \{1\}) \to h^k(X \times \mathbb{S}^1) & & \\ \downarrow & & \\ h^k(X) \longrightarrow h^{k+1}(X \times \mathbb{S}^1, X \times \{1\}) \to ... & & \end{array}$$

The maps $h^k(X \times \mathbb{S}^1) \to h^k(X)$ are induced by the inclusion $\iota\colon X \to X \times \mathbb{S}^1$. ι has a clear left inverse in the projection map $\pi_X\colon X \times \mathbb{S}^1 \to X$, and so, since each h^k is a contravariant functor:

$$\begin{aligned} \pi_X \circ \iota &= \mathrm{id}_X \\ \implies \quad (\pi_X \circ \iota)^* &= \mathrm{id}_X^* \\ \iota^* \circ \pi_X^* &= \mathrm{id}_{h^k(X)} \end{aligned}$$

Since $f \circ g$ being a surjection implies that f is a surjection and since identities clearly are, ι^* is a surjection, thus the maps $h^k(X) \to h^{k+1}(X \times \mathbb{S}^1,\, X \times D^1)$ must be zero maps and we obtain a sequence of split short exact sequences:

$$0 \longrightarrow h^k(X \times \mathbb{S}^1, X \times D^1) \longrightarrow h^k(X \times \mathbb{S}^1) \xrightarrow{\iota^*} h^k(X) \longrightarrow 0$$

The split gives us the isomorphism:

$$h^k(X \times \mathbb{S}^1) \cong h^k(X \times \mathbb{S}^1, X \times D^1) \oplus h^k(X)$$

Let us try and find a friendlier way of expressing the relative term.

If $d^1 \subset D^1$, then let us excise $X \times d^1 \subset X \times D^1 \subset X \times \mathbb{S}^1$ and obtain the following isomorphism in every degree:

$$\begin{aligned} h^k(X \times \mathbb{S}^1, X \times D^1) &\cong h^k(X \times \mathbb{S}^1 \backslash d^1, X \times D^1 \backslash d^1) \\ &\cong h^k(X \times D^1, X \times \mathbb{S}^0) \end{aligned}$$

We may also construct a long exact sequence using this pair:

$$\begin{array}{ccc} ... \to h^k(X \times D^1, X \times \mathbb{S}^0) \to h^k(X \times D^1) & & \\ \downarrow & & \\ h^k(X \times \mathbb{S}^0) \to h^{k+1}(X \times D^1, X \times \mathbb{S}^0) \to ... & & \end{array}$$

For the same reason as before, $h^k(X \times D^1) \cong h^k(X)$. The map $h^k(X \times D^1) \to h^k(X \times \mathbb{S}^0)$ is induced in part by the inclusion $\jmath\colon \mathbb{S}^0 \to D^1$, additionally, we have a map $h^k(X \times \mathbb{S}^0) \to h^k(X)$ induced by the inclusion $\iota_X\colon X \to X \times$

$\mathbb{S}^0$. If the following diagram commutes then our long exact sequence will break down into split small exact sequences:

$$\begin{array}{ccc} h^k(X \times D^1) & \xrightarrow{\quad\cong\quad} & h^k(X) \\ & {\scriptstyle (\mathrm{id}_X \times \jmath)^*} \searrow \quad \nearrow {\scriptstyle \iota_X^*} & \\ & h^k(X \times \mathbb{S}^0) & \end{array}$$

The composition $(\mathrm{id}_X \times \jmath) \circ \iota_X$ is equal to the inclusion of X into $X \times D^1$. Composing or each side with the projection map $X \times D^1 \to X$ yields a map at least homotopic to the identity on each space so this is the map that induces our isomorphism, thus the diagram necessarily commutes.

Since $f \circ g$ being an injection implies that g is an injection and since identities clearly are, $(\mathrm{id}_X \times \jmath)^*$ is an injection, thus the maps $h^k(X \times D^1, X \times \mathbb{S}^0) \to h^k(X \times D^1)$ must be zero maps and we obtain a sequence of split short exact sequences:

$$0 \longrightarrow h^k(X \times D^1) \longrightarrow h^k(X \times \mathbb{S}^0) \longrightarrow h^{k+1}(X \times D^1, X \times \mathbb{S}^0) \longrightarrow 0$$

In the following diagram, since h^* is a cohomology theory, the first two vertical maps must be isomorphisms, we would like to find a map ϕ such that the left square will commute, then since the rows are exact, the right square will commute and $h^k(X \times D^1, X \times \mathbb{S}^0)$ will be isomorphic to the cokernel of ϕ by the five lemma:

$$\begin{array}{ccccccccc} 0 \longrightarrow & h^k(X \times D^1) & \longrightarrow & h^k(X \times \mathbb{S}^0) & \longrightarrow & h^{k+1}(X \times D^1, X \times \mathbb{S}^0) & \longrightarrow 0 \\ & \cong \downarrow & \square & \downarrow \cong & \square & \downarrow \cong & \\ 0 \longrightarrow & h^k(X) & \xrightarrow[\phi]{} & h^k(X) \oplus h^k(X) & \longrightarrow & Coker(\phi) & \longrightarrow 0 \end{array}$$

The central isomorphism is due to the fact that $X \times \mathbb{S}^0 \cong X \coprod X$, if we define the maps on spaces as follows:

$$\begin{aligned} \iota_1 : X &\to X \times \mathbb{S}^0 \\ x &\mapsto (x, 1) \end{aligned} \qquad\qquad \begin{aligned} \iota_{-1} : X &\to X \times \mathbb{S}^0 \\ x &\mapsto (x, -1) \end{aligned}$$

Then the isomorphism $h^k(X \times \mathbb{S}^0) \to h^k(X) \oplus h^k(X)$ is the map I^* which is induced by the inclusions i.e. $I^*(\xi) = (\iota_1^*(\xi), \iota_{-1}^*(\xi))$.

For $\xi = (\mathrm{id}_X \times \jmath)^*(\tilde{\xi})$, we will have $\iota_1^*(\xi) = \iota_{-1}^*(\xi)$, therefore the sensible choice for ϕ in our diagram is the diagonal map, i.e. $\phi(x) = (x, x)$.

The map $h^k(X) \oplus h^k(X) \to Coker(\phi)$ will be the difference map in order to keep the bottom row exact, therefore $Coker(\phi) \cong h^k(X)$ and we achieve the result that $h^{k+1}(X \times D^1, X \times \mathbb{S}^0) \cong h^k(X)$.

Returning to our first split exact sequence then, we can see the result of all these isomorphisms:

$$\begin{aligned}h^k(X \times \mathbb{S}^1) &\cong h^k(X \times \mathbb{S}^1, X \times D^1) \oplus h^k(X)\\ &\cong h^k(X \times D^1, X \times \mathbb{S}^0) \oplus h^k(X)\\ &\cong h^{k-1}(X) \oplus h^k(X)\end{aligned}$$

This is the case in every degree k, but we can make this neater and describe the whole graded abelian group at once. If z is some degree 1 object, then we can consider the exterior algebra $\Lambda_{\mathbb{Z}}[z]$, which as a graded abelian group, has one copy of $\mathbb{Z}$ in degree 0, one copy of $\mathbb{Z}$ in degree 1, and is trivial otherwise.

We can safely tensor with a single copy of $\mathbb{Z}$ and see that:

$$\begin{aligned}h^k(X \times \mathbb{S}^1) &\cong h^{k-1}(X) \otimes \mathbb{Z} \oplus h^k(X) \otimes \mathbb{Z}\\ &\cong h^{k-1}(X) \otimes \Lambda^1_{\mathbb{Z}}[z] \oplus h^k(X) \otimes \Lambda^0_{\mathbb{Z}}[z]\\ &\cong \bigoplus_{i=0}^{\infty} h^{k-i}(X) \otimes \Lambda^i_{\mathbb{Z}}[z]\end{aligned}$$

Therefore

$$h^*(X \times \mathbb{S}^1) \cong h^*(X) \otimes \Lambda^*_{\mathbb{Z}}[z]$$

Since $\mathbb{T}^n$ is the product of n copies of $\mathbb{S}^1$, the graded abelian group $h^*(X \times \mathbb{T}^n)$ can similarly be described:

$$\begin{aligned}h^*(X \times \mathbb{T}^n) &\cong h^*(X \times \mathbb{T}^{n-1} \times \mathbb{S}^1)\\ &\cong h^*(X \times \mathbb{T}^{n-1}) \otimes \Lambda^*_{\mathbb{Z}}[z_1]\\ &\dots\\ &\cong h^*(X) \bigotimes_{i=1}^{n} \Lambda^*_{\mathbb{Z}}[z_i]\\ &\cong h^*(X) \otimes \Lambda^*_{\mathbb{Z}}[z_1, ..., z_n]\end{aligned}$$

where each z_i is in degree 1. △

Naturally, by the homeomorphism $pt \times X \cong X$, we have for any cohomology theory that $h^*(\mathbb{T}^n) \cong h^*(pt) \otimes \Lambda^*_{\mathbb{Z}}[z_1, ..., z_n]$.

Corollary 44. Let h^* be a general cohomology theory and let $\tilde{h}^*$ be the corresponding reduced cohomology theory defined by $\tilde{h}^*(X) := h^*(X, pt)$, then for any CW-complex X:

$$\tilde{h}^*(X \times \mathbb{T}^n) \cong \tilde{h}^*(X) \otimes \Lambda^0_{\mathbb{Z}}[z_1, ..., z_n] \oplus h^*(X) \otimes \tilde{\Lambda}^*_{\mathbb{Z}}[z_1, ..., z_n]$$

where we define $\tilde{\Lambda}^*_{\mathbb{Z}}[z_1, ..., z_n] := \Lambda^*_{\mathbb{Z}}[z_1, ..., z_n] \backslash \Lambda^0_{\mathbb{Z}}[z_1, ..., z_n]$ and each z_i is in degree 1.

Proof. If we have a pointed topological space X with basepoint x_0 and D^1 a closed, connected subset of $\mathbb{S}^1$ containing a point z_0 we take to be the base point of $\mathbb{S}^1$, then we may say that $(x_0,\, z_0)$ is the basepoint of $X \times \mathbb{S}^1$ and we have a triple of pointed topological spaces $(x_0,\, z_0) \subset X \times D^1 \subset X \times \mathbb{S}^1$ and so, as shown by Hatcher [17], we can construct a long exact sequence:

$$\begin{array}{l} ... \to h^k(X \times \mathbb{S}^1, X \times D^1) \longrightarrow \tilde{h}^k(X \times \mathbb{S}^1) \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad \tilde{h}^k(X \times D^1) \to h^{k-1}(X \times \mathbb{S}^1, X \times D^1) \to ... \end{array}$$

where $\tilde{h}(X) = h(X,\, pt)$.

Again, $X \times D^1$ is homotopy equivalent to X and an identical argument to previously ensures that $\tilde{h}^k(X \times \mathbb{S}^1) \to \tilde{h}^k(X \times D^1)$ is a surjection, thus this long exact sequence splits and for each k:

$$0 \longrightarrow h^k(X \times \mathbb{S}^1, X \times D^1) \longrightarrow \tilde{h}^k(X \times \mathbb{S}^1) \xrightarrow{\iota^*} \tilde{h}^k(X \times D^1) \longrightarrow 0$$

and the split gives us an isomorphism:

$$\tilde{h}^k(X \times \mathbb{S}^1) \cong \tilde{h}^k(X) \oplus h^k(X \times \mathbb{S}^1, X \times D^1)$$

Additionally, we have already determined that $h^k(X \times \mathbb{S}^1,\, X \times D^1) \cong h^{k\text{-}1}(X)$, and so our isomorphism simplifies again:

$$\tilde{h}^k(X \times \mathbb{S}^1) \cong \tilde{h}^k(X) \oplus h^{k-1}(X)$$

Once again we can write our isomorphism in terms of exterior algebras:

$$\begin{aligned} \tilde{h}^k(X \times \mathbb{S}^1) &\cong \tilde{h}^k(X) \otimes \mathbb{Z} \oplus h^{k-1}(X) \otimes \mathbb{Z} \\ &= \tilde{h}^k(X) \otimes \Lambda^0_{\mathbb{Z}}[z] \oplus h^{k-1}(X) \otimes \tilde{\Lambda}^*_{\mathbb{Z}}[z] \end{aligned}$$

where z is an element of degree 1.

Thus we have

$$\tilde{h}^*(X \times \mathbb{S}^1) \cong \tilde{h}^*(X) \otimes \Lambda^0_{\mathbb{Z}}[z] \oplus h^*(X) \otimes \tilde{\Lambda}^*_{\mathbb{Z}}[z]$$

We need to make quick note of a ring isomorphism by describing its effect on simple elements, let $p(z_1,\, ...,\, z_{n\text{-}1}) = b_1 z_1 + b_2 z_2 + ... + b_{1,\, ...,\, n\text{-}1} z_1 \wedge ... \wedge z_n$ be a general element of $\tilde{\Lambda}^*_{\mathbb{Z}}[z_1,\, ...,\, z_{n\text{-}1}]$:

$$\begin{aligned} \lambda : \qquad \tilde{\Lambda}^*_{\mathbb{Z}}[z_n] \otimes \Lambda^0_{\mathbb{Z}}[z_1, ..., z_{n-1}] &\to \tilde{\Lambda}^*_{\mathbb{Z}}[z_1, ..., z_n] \\ \oplus \Lambda^*_{\mathbb{Z}}[z_n] \otimes \tilde{\Lambda}^*_{\mathbb{Z}}[z_1, ..., z_{n-1}] & \\ a_n z_n \otimes a_0 \oplus (b_0 + b_n z_n) \otimes p(z_1, ..., z_{n-1}) &\mapsto a_0 a_n z_n + b_0 p(z_1, ..., z_{n-1}) \\ &\quad + b_n z_n \wedge p(z_1, ..., z_{n-1}) \end{aligned}$$

After which we can investigate products with an arbitrary torus:

$$
\begin{aligned}
\tilde{h}^*(X \times \mathbb{T}^n) &\cong \tilde{h}^*(X \times \mathbb{T}^{n-1}) \otimes \Lambda^0_{\mathbb{Z}}[z_1] \oplus h^*(X \times \mathbb{T}^{n-1}) \otimes \tilde{\Lambda}^*_{\mathbb{Z}}[z_1] \\
&\cong (\tilde{h}^*(X \times \mathbb{T}^{n-2}) \otimes \Lambda^0_{\mathbb{Z}}[z_2] \oplus h^*(X \times \mathbb{T}^{n-2}) \otimes \tilde{\Lambda}^*_{\mathbb{Z}}[z_2]) \otimes \Lambda^0_{\mathbb{Z}}[z_1] \\
&\quad \oplus (h^*(X \times \mathbb{T}^{n-2}) \otimes \Lambda^*_{\mathbb{Z}}[z_2]) \otimes \tilde{\Lambda}^*_{\mathbb{Z}}[z_1] \\
&\cong \tilde{h}^*(X \times \mathbb{T}^{n-2}) \otimes \Lambda^0_{\mathbb{Z}}[z_1, z_2] \\
&\quad \oplus h^*(X \times \mathbb{T}^{n-2}) \otimes (\tilde{\Lambda}^*_{\mathbb{Z}}[z_2] \otimes \Lambda^0_{\mathbb{Z}}[z_1] \oplus \Lambda^*_{\mathbb{Z}}[z_2] \otimes \tilde{\Lambda}^*_{\mathbb{Z}}[z_1]) \\
&\cong \tilde{h}^*(X \times \mathbb{T}^{n-2}) \otimes \Lambda^0_{\mathbb{Z}}[z_1, z_2] \oplus h^*(X \times \mathbb{T}^{n-2}) \otimes \tilde{\Lambda}^*_{\mathbb{Z}}[z_1, z_2] \\
&... \\
&\cong \tilde{h}^*(X) \otimes \Lambda^0_{\mathbb{Z}}[z_1, ..., z_n] \oplus h^*(X) \otimes \tilde{\Lambda}^*_{\mathbb{Z}}[z_1, ..., z_n]
\end{aligned}
$$

where each z_i is in degree 1. △

With the homeomorphism $\mathbb{T}^n \cong pt \times \mathbb{T}^n$ and the obvious isomorphism $\tilde{h}^*(pt) \cong 0$, we achieve the result $\tilde{h}^*(\mathbb{T}^n) \cong h^*(pt) \otimes \tilde{\Lambda}^*_{\mathbb{Z}}[z_1, ..., z_n]$.

6.4.1 Compatibility with a Natural Transformation

Definition 34. A **natural transformation of cohomology theories** is a natural transformation η: $h^* \to k^*$ which in turn consists of a family of natural transformations η^n: $h^n \to k^n$ such that the required components commute with the boundary maps i.e. for each $n \in \mathbb{Z}$, pairs of topological spaces (X, A) and (Y, B), and map of pairs ϕ: $(X, A) \to (Y, B)$:

$$
\begin{array}{ccc}
h^n(Y,B) & \xrightarrow{\eta^n_{Y,B}} & k^n(Y,B) \\
\downarrow{\scriptstyle \phi^*} & \square & \downarrow{\scriptstyle \phi^*} \\
h^n(X,A) & \xrightarrow[\eta^n_{X,A}]{} & k^n(X,A)
\end{array}
\qquad\qquad
\begin{array}{ccc}
h^n(A) & \xrightarrow{\eta^n_A} & k^n(A) \\
\downarrow{\scriptstyle d_h} & \square & \downarrow{\scriptstyle d_k} \\
h^{n+1}(X,A) & \xrightarrow[\eta^{n+1}_{X,A}]{} & k^{n+1}(X,A)
\end{array}
$$

Lemma 45. For any general cohomology theories h^* and k^*, natural transformation of cohomology theories η: $h^* \to k^*$, and topological space X, the cohomology of the product with a torus construction is compatible with the natural transformation:

$$
\begin{array}{ccc}
h^*(X \times \mathbb{T}^n) & \xrightarrow{\eta_{X \times \mathbb{T}^n}} & k^*(X \times \mathbb{T}^n) \\
\cong\downarrow & \square & \downarrow\cong \\
h^*(X) \otimes \Lambda^*_{\mathbb{Z}}[z_1, ..., z_n] & \xrightarrow[\eta_X \otimes \mathrm{id}]{} & k^*(X) \otimes \Lambda^*_{\mathbb{Z}}[z_1, ..., z_n]
\end{array}
$$

Proof. The homeomorphism $X \times \mathbb{T}^n \cong X \times \mathbb{T}^{n-1} \times \mathbb{S}^1$ together with the isomorphism $\Lambda^*_{\mathbb{Z}}[z_1, ..., z_{n\text{-}1}] \otimes \Lambda^*_{\mathbb{Z}}[z_n] \cong \Lambda^*_{\mathbb{Z}}[z_1, ..., z_n]$ ensure that we can work up to higher dimensional torii inductively once we have shown that the following diagram commutes:

$$\begin{array}{ccc} h^*(X \times \mathbb{S}^1) & \xrightarrow{\eta_{X \times \mathbb{S}^1}} & k^*(X \times \mathbb{S}^1) \\ \cong\downarrow & \square & \downarrow\cong \\ h^*(X) \otimes \Lambda^*_{\mathbb{Z}}[z] & \xrightarrow[\eta_X \otimes \mathrm{id}]{} & k^*(X) \otimes \Lambda^*_{\mathbb{Z}}[z] \end{array}$$

This diagram will commute if in turn, for each n:

$$\begin{array}{ccc} h^n(X \times \mathbb{S}^1) & \xrightarrow{\eta^n_{X \times \mathbb{S}^1}} & k^n(X \times \mathbb{S}^1) \\ \cong\downarrow & \square & \downarrow\cong \\ h^n(X) \oplus h^{n-1}(X) & \xrightarrow[\eta^n_X \oplus \eta^{n-1}_X]{} & k^n(X) \oplus k^{n-1}(X) \end{array}$$

We will show that this is the case by examining some exact sequences.

The pair $(X \times \mathbb{S}^1,\ X \times D^1)$ induces a long exact sequence in both h^* and k^*, the natural transformation of cohomology theories gives us corresponding components to connect the two sequences:

$$\begin{array}{ccccccccc} \ldots \to & h^n(X \times \mathbb{S}^1, X \times D^1) & \to & h^n(X \times \mathbb{S}^1) & \to & h^n(X \times D^1) & \to & h^{n+1}(X \times \mathbb{S}^1, X \times D^1) & \to \ldots \\ & \eta^n_{X \times \mathbb{S}^1, X \times D^1}\downarrow & \square & \eta^n_{X \times \mathbb{S}^1}\downarrow & \square & \eta^n_{X \times D^1}\downarrow & \square & \downarrow\eta^{n+1}_{X \times \mathbb{S}^1, X \times D^1} & \\ \ldots \to & k^n(X \times \mathbb{S}^1, X \times D^1) & \to & k^n(X \times \mathbb{S}^1) & \to & k^n(X \times D^1) & \to & k^{n+1}(X \times \mathbb{S}^1, X \times D^1) & \to \ldots \end{array}$$

η being a natural transformation of cohomology theories ensures that every square in this diagram commutes. The two squares contained within any given degree commute as the horizontal maps are induced by the two inclusion maps of pairs $(X \times \mathbb{S}^1, \emptyset) \hookrightarrow (X \times \mathbb{S}^1,\ X \times D^1)$ and $(X \times D^1, \emptyset) \hookrightarrow (X \times \mathbb{S}^1, \emptyset)$ and the vertical maps are components of a natural transformation, and the squares between degrees commute as a natural transformation of cohomology theories requires the relevant components to commute with the boundary maps.

The splitting of these long exact sequences comes from a map on spaces which together with the fact that $X \times D^1$ is homeomorphic to X allows us to construct a sequence of commutative diagrams by ensuring that every square we need to construct will still commute by the nature of a natural transformation of cohomology theories, the excision isomorphism too comes from a homeomorphism and a inclusion of pairs and so for each n:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & h^n(X \times D^1, X \times \mathbb{S}^0) & \longrightarrow & h^n(X \times \mathbb{S}^1) & \longrightarrow & h^n(X) & \longrightarrow & 0 \\ & & \eta^n_{X \times D^1, X \times \mathbb{S}^0}\downarrow & \square & \eta^n_{X \times \mathbb{S}^1}\downarrow & \square & \downarrow\eta^n_X & & \\ 0 & \longrightarrow & k^n(X \times D^1, X \times \mathbb{S}^0) & \longrightarrow & k^n(X \times \mathbb{S}^1) & \longrightarrow & k^n(X) & \longrightarrow & 0 \end{array}$$

which ensures that the following diagram is commutative:

$$\begin{array}{ccc} h^n(X \times \mathbb{S}^1) & \xrightarrow{\cong} & h^n(X) \oplus h^n(X \times D^1, X \times \mathbb{S}^0) \\ {\scriptstyle \eta^n_{X\times\mathbb{S}^1}}\downarrow & \Box & \downarrow{\scriptstyle \eta^n_X \oplus \eta^n_{X\times D^1, X\times\mathbb{S}^0}} \\ k^n(X \times \mathbb{S}^1) & \xrightarrow[\cong]{} & k^n(X) \oplus k^n(X \times D^1, X \times \mathbb{S}^0) \end{array}$$

Thus all we must do is show that the following diagram holds:

$$\begin{array}{ccc} h^n(X \times D^1, X \times \mathbb{S}^0) & \xrightarrow{\cong} & h^{n-1}(X) \\ {\scriptstyle \eta^n_{X\times D^1, X\times\mathbb{S}^0}}\downarrow & \Box & \downarrow{\scriptstyle \eta^{n-1}_X} \\ k^n(X \times D^1, X \times \mathbb{S}^0) & \xrightarrow[\cong]{} & k^{n-1}(X) \end{array}$$

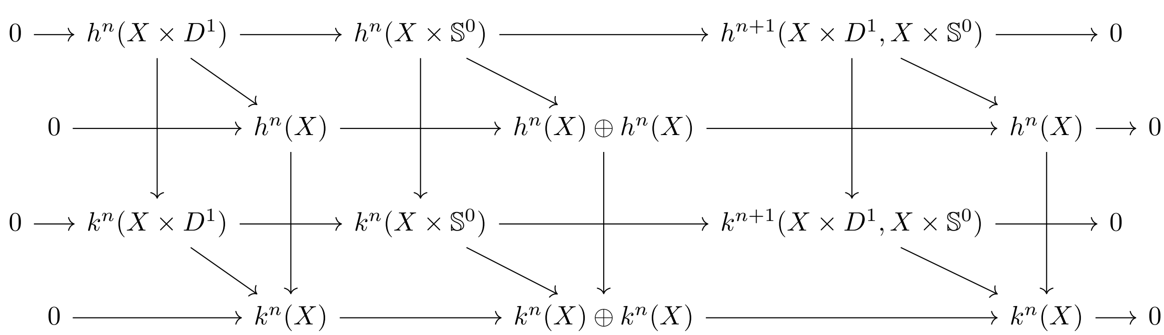
We can start by investigating the long exact sequences of the pair $(X \times D^1, X \times \mathbb{S}^0)$, again connected by components of our natural transformation of cohomology theories:

$$\begin{array}{ccccccccc} ... \to & h^n(X \times D^1, X \times \mathbb{S}^0) & \to & h^n(X \times D^1) & \to & h^n(X \times \mathbb{S}^0) & \to & h^{n+1}(X \times D^1, X \times \mathbb{S}^0) & \to ... \\ & {\scriptstyle \eta^n_{X\times D^1, X\times\mathbb{S}^0}}\downarrow & \Box & {\scriptstyle \eta^n_{X\times D^1}}\downarrow & \Box & {\scriptstyle \eta^n_{X\times\mathbb{S}^0}}\downarrow & \Box & \downarrow{\scriptstyle \eta^{n+1}_{X\times D^1, X\times\mathbb{S}^0}} & \\ ... \to & k^n(X \times D^1, X \times \mathbb{S}^0) & \to & k^n(X \times D^1) & \to & k^n(X \times \mathbb{S}^0) & \to & k^{n+1}(X \times D^1, X \times \mathbb{S}^0) & \to ... \end{array}$$

Once again the split is induced by a map of spaces and so we achieve a sequence of commutative diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & h^n(X \times D^1) & \longrightarrow & h^n(X \times \mathbb{S}^0) & \longrightarrow & h^{n+1}(X \times D^1, X \times \mathbb{S}^0) & \longrightarrow & 0 \\ & & {\scriptstyle \eta^n_{X\times D^1}}\downarrow & \Box & {\scriptstyle \eta^n_{X\times\mathbb{S}^0}}\downarrow & \Box & \downarrow{\scriptstyle \eta^{n+1}_{X\times D^1, X\times\mathbb{S}^0}} & & \\ 0 & \longrightarrow & k^n(X \times D^1) & \longrightarrow & k^n(X \times \mathbb{S}^0) & \longrightarrow & k^{n+1}(X \times D^1, X \times \mathbb{S}^0) & \longrightarrow & 0 \end{array}$$

Again, homeomorphisms will generate commutative squares so additionally, the additivity isomoprhism will make a commutative square and, with an extra bit of homological algebra and diagram chasing, every square in the following diagram will commute:



and the diagram we require is the rightmost vertical parallelogram.

Therefore all our previous diagrams do indeed commute and we achieve the result that:

$$\begin{array}{ccc} h^*(X \times \mathbb{T}^n) & \xrightarrow{\eta_{X\times\mathbb{T}^n}} & k^*(X \times \mathbb{T}^n) \\ \cong\downarrow & \square & \downarrow\cong \\ h^*(X) \otimes \Lambda^*_{\mathbb{Z}}[z_1, ..., z_n] & \xrightarrow[\eta_X \otimes \mathrm{id}]{} & k^*(X) \otimes \Lambda^*_{\mathbb{Z}}[z_1, ..., z_n] \end{array}$$

△

7 Monoidal Categories and Exponential Functors

7.1 Monoidal Categories

Monoidal categories were first introduced under different names independently by Bénabou and MacLane. The following definition is derived from the work of Kelly [20] who paired down the coherence conditions of MacLane's definition to just the pentagon and triangle identities.

Definition 35. A **monoidal category** is a category $\mathscr{C}$ equipped with a monoidal structure which consists of:

- a bifunctor $\circ\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ called the monoidal product
- an object I called the unit object
- natural isomorphisms α, λ and ρ called the associator, and the left and right unitors respectively which together express that up to isomorphism:
  - $\circ$ is associative, α has components $\alpha_{A,B,C}\colon A{\circ}(B{\circ}C) \to (A{\circ}B){\circ}C$
  - $\circ$ has I as a left and right identity, λ and ρ have respective components $\lambda_A\colon I{\circ}A \to A$, and $\rho_A\colon A{\circ}I \to A$
- coherence conditions on α, λ and ρ requiring that the following diagrams commute:

$$\begin{array}{ccccc}
A \circ (B \circ (C \circ D)) & \xrightarrow{\alpha_{A,B,C\circ D}} & (A \circ B) \circ (C \circ D) & \xrightarrow{\alpha_{A\circ B,C,D}} & ((A \circ B) \circ C) \circ D \\
\downarrow{\scriptstyle \mathrm{id}_A \circ \alpha_{B,C,D}} & & \square & & \uparrow{\scriptstyle \alpha_{A,B,C} \circ \mathrm{id}_D} \\
A \circ ((B \circ C) \circ D) & & \xrightarrow[\alpha_{A,B\circ C,D}]{} & & (A \circ (B \circ C)) \circ D
\end{array}$$

$$\begin{array}{ccc}
A \circ (I \circ B) & \xrightarrow{\alpha_{A,I,B}} & (A \circ I) \circ B \\
{\scriptstyle \mathrm{id}_A \circ \lambda_B} \searrow & \square & \swarrow {\scriptstyle \rho_A \circ \mathrm{id}_B} \\
 & A \circ B &
\end{array}$$

A **strict monoidal category** is a monoidal category where the associator and unitors are identities.

The coherence conditions are satisfied automatically in a strict monoidal category.

7.1.1 Examples of Monoidal Categories

Consider the category $\mathscr{C}_\oplus$ that has as objects the set $\mathbb{N}$ and collections of morphisms as follows:

$$\mathrm{Hom}_{\mathscr{C}_\oplus}(n, m) = \begin{cases} \mathrm{U}(n), & n = m \\ \emptyset, & else \end{cases}$$

Where $\mathrm{U}(0)$ is defined to be the trivial group.

We may equip this category with a monoidal structure consisting of a bifunctor $\oplus\colon \mathscr{C}_\oplus \times \mathscr{C}_\oplus \to \mathscr{C}_\oplus$ that maps pairs of objects $(n,\, m)$ to the object $n{+}m$, and maps pairs of morphisms $(A,\, B)$ where $A \in \mathrm{U}(n)$, $B \in \mathrm{U}(m)$ to the morphism $diag(A,\, B) \in \mathrm{U}(n{+}m)$. 0 is the unit object, and the associator and unitors are identities.

Now consider the category $\mathscr{C}_\otimes$ that has as objects the set $\mathbb{N}$ and collections of morphisms as follows:

$$\mathrm{Hom}_{\mathscr{C}_\otimes}(n,m) = \begin{cases} \mathrm{U}((\mathbb{C}^d)^{\otimes n}), & n = m \\ \emptyset, & else \end{cases}$$

We may equip this category with a monoidal structure consisting of a bifunctor $\otimes\colon \mathscr{C}_\otimes \times \mathscr{C}_\otimes \to \mathscr{C}_\otimes$ that maps pairs of objects $(n,\, m)$ again to the object $n{+}m$, and maps pairs of morphisms $(A,\, B)$ where $A \in \mathrm{U}((\mathbb{C}^d)^{\otimes n})$, $B \in \mathrm{U}((\mathbb{C}^d)^{\otimes m})$ to the morphism $A{\otimes}B \in \mathrm{U}((\mathbb{C}^d)^{\otimes(n+m)})$. Again 0 is the unit object, and the associator and unitors are identities.

7.1.2 Geometric Realisation of the Nerve of Monoidal Categories

As with any category we can construct a topological space $|N(\mathscr{C})|$ for any monoidal category $\mathscr{C}$, the difference here is that the monoidal structure on $\mathscr{C}$ imbues $|N(\mathscr{C})|$ with the structure of a monoid. Then, since $|N(\mathscr{C})|$ is a monoid, there exists a category with one object and collection of morphisms $|N(\mathscr{C})|$ with composition of morphisms defined as in the monoid, and the geometric realisation of the nerve of *that* category can be taken and will be a model for the classifying space of $|N(\mathscr{C})|$.

Let us attempt this on the monoidal categories we have already described.

For $i \in \mathbb{N}$, let $\mathscr{C}_{\oplus i}$ be the category with one object and collection of morphisms $\mathrm{U}(i)$. By inspection we can see that $\mathscr{C}_\oplus$ is the coproduct of all of these categories:

$$\mathscr{C}_\oplus = \coprod_{i=0}^{\infty} \mathscr{C}_{\oplus i}, \text{ and so } |N(\mathscr{C}_\oplus)| = |N(\coprod_{i=0}^{\infty} \mathscr{C}_{\oplus i})| = \coprod_{i=0}^{\infty} |N(\mathscr{C}_{\oplus i})|$$

Since each $\mathscr{C}_{\oplus i}$ is a category with one object and morphism space $\mathrm{U}(i)$, we know that $|N(\mathscr{C}_{\oplus i})| = B(\mathrm{U}(i))$, the classifying space of the topological group $\mathrm{U}(i)$. Therefore:

$$|N(\mathscr{C}_\oplus)| = \coprod_{i=0}^{\infty} B\mathrm{U}(i)$$

Since we have the structure of a moniod on $|N(\mathscr{C}_\oplus)|$, we can again construct it's classifying space:

$$B(|N(\mathscr{C}_\oplus)|) = B(\coprod_{i=0}^{\infty} B\mathrm{U}(i))$$

We would like to construct a space X such that $\pi_i(X) \cong \pi_i(B(|N(\mathscr{C}_\oplus)|)$ for $i \geq 2$, and $\pi_1(X) = 0$

There is a monoid homomorphism:

$$\omega : \coprod_{i=0}^{\infty} B\mathrm{U}(i) \to \mathbb{N}$$

given by $\omega(E) = n$ for $E \in B\mathrm{U}(n)$, since constructing classifying spaces is a functor, this map induces a group homomorphism:

$$B\omega : B(\coprod_{i=0}^{\infty} B\mathrm{U}(i)) \to B(\mathbb{N})$$

The homotopy fibre of this map $B\omega$ will be denoted $BB\mathrm{U}_\oplus$.

Lemma 46. $BB\mathrm{U}_\oplus$ is such that $\pi_i(BB\mathrm{U}_\oplus) \cong \pi_i(B(|N(\mathscr{C}_\oplus)|)$ for $i \geq 2$ and $\pi_1(BB\mathrm{U}_\oplus) = 0$

Proof. The group homomorphism $B\omega$ induces a fibration $\tilde{B\omega}$: $EB\mathrm{U}_\oplus \to B\mathbb{N}$ with fibre $BB\mathrm{U}_\oplus$ where

$$EB\mathrm{U}_\oplus := \{(a,p) \mid a \in B(\coprod_{i=0}^{\infty} B\mathrm{U}(i)),\ p : [0,1] \to B(\mathbb{N}) \text{ such that } p(0) = B\omega(a)\}$$

$$BB\mathrm{U}_\oplus \simeq \{(a,p) \in EB\mathrm{U}_\oplus \mid \tilde{B\omega}(a,p) = p(1) = b_0 \text{ for some } b_0 \in B(\mathbb{N})\}$$

Naturally there is an inclusion ι: $BB\mathrm{U}_\oplus \to EB\mathrm{U}_\oplus$, but there is also an inclusion $\jmath$: $\Omega B(\mathbb{N}) \to BB\mathrm{U}_\oplus$ since $\Omega B(\mathbb{N}) = \{(a, p) \in BB\mathrm{U}_\oplus \mid B\omega(a) = p(0) = p(1) = b_0$ where b_0 is the basepoint of $B(\mathbb{N})\}$

Together with $\tilde{B\omega}$, ι and $\jmath$ induce a long exact sequence of topological spaces which in turn induces a long exact sequence of homotopy groups:

$$... \to \pi_{k+1}(B(\mathbb{N})) \to \pi_k(BB\mathrm{U}_\oplus) \to \pi_k(EB\mathrm{U}_\oplus) \to \pi_k(B(\mathbb{N})) \to \pi_{k-1}(BB\mathrm{U}_\oplus) \to ...$$

There is a canonical injection:

$$\phi : B(\coprod_{i=0}^{\infty} B\mathrm{U}(i)) \to EB\mathrm{U}_\oplus$$
$$a \mapsto (a, c_{B\omega(a)})$$

where $c_{B\omega(a)}$ is the constant path at $B\omega(a)$.

There is a deformation retract of $EB\mathrm{U}_\oplus$ onto $Im(\phi)$ given by:

$$H : EB\mathrm{U}_\oplus \times I \to EB\mathrm{U}_\oplus$$
$$((a,p),t) \mapsto (a, p_t)$$

where p_t: $I \to B(\mathbb{N})$ is given by $p_t(s) = p(s(1 - t))$.

H is indeed a homotopy retract since $H((a,\ p),\ 0) = (a,\ p_0) = (a,\ p)$ as $p_0(s) = p(s)\ \forall\ s \in I$, $H((a,\ p),\ 1) = (a,\ p_1) \in Im(\phi)$ as $p_1(s) = p(0)\ \forall\ s \in I$ so $p_1 = c_{B\omega(a)}$, and for $(a,\ c_{B\omega(a)}) \in Im(\phi)$, $H((a,\ c_{B\omega(a)}),\ 1) = (a,\ c_{B\omega(a)_1})$ $= (a,\ c_{B\omega(a)})$ as $c_{B\omega(a)_1}(s) = c_{B\omega(a)}(0) = B\omega(a) = c_{B\omega(a)}(s)\ \forall\ s \in I$.

Thus since a homotopy retract is a particular case of homotopy equivalence, $EB\mathrm{U}_\oplus$ is homotopy equivalent to $Im(\phi)$ and since ϕ is an injection, $Im(\phi)$ is isomorphic to $B(|N(\mathscr{C}_\oplus)|)$, therefore $\pi_k(B(|N(\mathscr{C}_\oplus)|)) \cong \pi_k(EB\mathrm{U}_\oplus)\ \forall\ k$.

Additionally, we know

$$\begin{aligned} \pi_k(B(\mathbb{N})) &= [\mathbb{S}^k, B(\mathbb{N})] \\ &= [\mathbb{S}^{k-1}, \Omega B(\mathbb{N})] \\ &= [\mathbb{S}^{k-1}, \mathbb{Z}] \\ &= \pi_{k-1}(\mathbb{Z}) \\ &= \begin{cases} \mathbb{Z}, & k = 1 \\ 0, & else \end{cases} \end{aligned}$$

Therefore $\pi_k(BB\mathrm{U}_\oplus) \cong \pi_k(B(|N(\mathscr{C}_\oplus)|))$ for $k \geq 2$ and we are left with only a small section of the long exact sequence to untangle:

$$0 \to \pi_1(BB\mathrm{U}_\oplus) \to \pi_1(B(|N(\mathscr{C}_\oplus)|)) \to \pi_1(B(\mathbb{N})) \to \pi_0(BB\mathrm{U}_\oplus) \to \pi_0(B(|N(\mathscr{C}_\oplus)|)) \to 0$$

The map $\pi_1(B\omega)$: $\pi_1(B(|N(\mathscr{C}_\oplus)|)) \to \pi_1(B(\mathbb{N}))$ is an isomorphism as it fits into the following commutative diagram:

$$\begin{array}{ccccccc} \pi_1(B(|N(\mathscr{C}_\oplus)|)) & \xrightarrow{\cong} & \pi_0(\Omega B(|N(\mathscr{C}_\oplus)|)) & \xrightarrow{\cong} & \pi_0(B\mathrm{U} \times \mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z} \\ {\scriptstyle \pi_1(B\omega)}\downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{Z}}} \\ \pi_1(B(\mathbb{N})) & \xrightarrow[\cong]{} & \pi_0(\Omega B(\mathbb{N})) & \xrightarrow[\cong]{} & \pi_0(\mathbb{Z}) & \xrightarrow[\cong]{} & \mathbb{Z} \end{array}$$

Therefore, $\pi_1(BB\mathrm{U}_\oplus) \cong 0$ and $\pi_0(BB\mathrm{U}_\oplus) \cong \pi_1(B(|N(\mathscr{C}_\oplus)|)) \cong 0$.

Thus $BB\mathrm{U}_\oplus$ is exactly as we require. $\triangle$

Definition 36. Let X_i for $i \in \mathbb{N}$ be objects in some category $\mathscr{C}$ and $f_i \in \mathrm{Hom}_{\mathscr{C}}(X_i,\ X_{i+1})$ be a sequence of composable morphisms.

If $\mathscr{C}$ is a category where the objects are topological spaces perhaps with additional algebraic structure, and the morphisms are continuous homomorphisms, the **colimit** of the sequence $X_0 \to X_1 \to \ldots\ \ X_n \to \ldots$ is the object of $\mathscr{C}$ $colim(X_0 \to X_1 \to \ldots\ X_n \to \ldots)$ defined as follows:

$$colim(X_0 \to X_1 \to \ldots \to X_n \to \ldots) := (\coprod_{i=0}^{\infty} X_i)/\sim$$

where for $p \in X_i$, $q \in X_j$, $p \sim q$ if $f_{k\text{-}1}(\ldots(f_i(p))\ldots) = f_{k\text{-}1}(\ldots(f_j(q))\ldots) \in X_k$ for some $k \geq i,j$.

the **homotopy colimit** of $X_0 \to X_1 \to ... \ X_n \to ...$ is the object *hocolim*$(X_0 \to X_1 \to ... \ X_n \to ...)$ defined as follows:

$$hocolim(X_0 \to X_1 \to ... \to X_n \to ...) := (\coprod_{i=0}^{\infty}(X_i \times [i, i+1]))/\sim$$

where for $x \in X_i$, $(x, i{+}1) \sim (f_i(x), i{+}1)$.

Lemma 47. If $X_0 \to X_1 \to ... \to X_n \to ...$ is a sequence of topological spaces potentially with additional algebraic structures and appropriate continuous homomorphisms, then *hocolim*$(X_0 \to X_1 \to ... \to X_n \to ...)$ and *hocolim*$(X_1 \to X_2 \to ... \to X_n \to ...)$ are homotopy equivalent.

Proof. Let us make definitions $H_0 :=$ *hocolim*$(X_0 \to X_1 \to ... \to X_n \to ...)$ and $H_1 :=$ *hocolim*$(X_1 \to X_2 \to ... \to X_n \to ...)$. We need to find continuous maps $f\colon H_0 \to H_1$ and $g\colon H_1 \to H_0$ such that $g \circ f$ is homotopic to id_{H_0} and $f \circ g$ is homotopic to id_{H_1}.

If we take H_0 as it is and simply translate H_1 by one unit interval we may write:

$$H_0 = (\coprod_{i=0}^{\infty}(X_i \times [i, i+1]))/\sim \qquad H_1 \cong (\coprod_{i=1}^{\infty}(X_i \times [i, i+1]))/\sim$$

where for $x \in X_i$, $(x, i{+}1) \sim (f_i(x), i{+}1)$ for all i.

Let $g\colon H_1 \to H_0$ simply be the inclusion $[x, t] \mapsto [x, t]$ and let $f\colon H_0 \to H_1$ be defined as follows:

$$f([x,t]) := \begin{cases} [f_0(x), 1], & x \in X_0, \\ [x, t], & else \end{cases}$$

f is continuous as $[x, 1] = [f_0(x), 1]$ for $x \in X_0$ by the equivalence relation.

$(f \circ g)([x,t]) = f([x,t]) = [x,t]$ since $x \notin X_0$, therefore $f \circ g$ is equal to id_{H_1} let alone homotopy equivalent.

The composition $g \circ f$ however, maps as follows:

$$(g \circ f)([x,t]) := \begin{cases} [f_0(x), 1], & x \in X_0, \\ [x, t], & else \end{cases}$$

We need to describe a homotopy between id_{H_0} and $g \circ f$. Let $H\colon H_0 \times [0,1] \to H_0$ be defined as follows:

$$H([x,t], s) := \begin{cases} [x, t + (1-t)s], & x \in X_0, \\ [x, t], & else \end{cases}$$

Then $H([x,t],0) = [x,t]$ and $H([x,t],1) = (g \circ f)([x,t])$ since $[x,1] = [f_0(x),1]$ for $x \in X_0$, therefore H is a homotopy from id_{H_0} to $g \circ f$, and thus H_0 and H_1 are homotopy equivalent. △

Corollary 48. If $X_0 \to X_1 \to ... \to X_n \to ...$ is a sequence of topological spaces, then $hocolim(X_0 \to X_1 \to ... \to X_n \to ...)$ and $hocolim(X_k \to X_{k+1} \to ... \to X_n \to ...)$ are homotopy equivalent for all $k \in \mathbb{N}$.

Proof. Let $H_i := hocolim(X_i \to X_{i+1} \to ... \to X_n \to ...)$. We have already proven that H_0 is homotopy equivalent to H_1. By a simple relabelling $Y_j = X_{i+j}$ and $g_{\mathrm{j}} = f_{i+j}$ for $j \in \mathbb{N}$, $Y_0 \to Y_1 \to ... \to Y_n \to ...$ is a sequence of algebraic structures and appropriate homomorphisms and therefore we know that $H_i = hocolim(Y_0 \to Y_1 \to ... \to Y_n \to ...)$ is homotopy equivalent to $H_{i+1} = hocolim(Y_1 \to Y_2 \to ... \to Y_n \to ...)$ for all $i \in \mathbb{N}$.

Therefore, via the homotopy equivalences to all homotopy colimits in between, H_0 is homotopy equivalent to H_k for all $k \in \mathbb{N}$. △

Lemma 49. If $X_0 \to X_1 \to ... \to X_n \to ...$ is a sequence of CW-complexes and continuous maps, such that $f_i(X_i)$ is a subcomplex of X_{i+1} for all i, then $hocolim(X_0 \to X_1 \to ... \to X_n \to ...)$ is homotopy equivalent to $colim(X_0 \to X_1 \to ... \to X_n \to ...)$.

Proof. Let us write $H := hocolim(X_0 \to X_1 \to ... \to X_n \to ...)$ and $C := colim(X_0 \to X_1 \to ... \to X_n \to ...)$. We again need to find two continuous maps $f\colon H \to C$ and $g\colon C \to H$ where $g \circ f$ is homotopic to id_H and $f \circ g$ is homotopic to id_C.

There is a simple map $f\colon H \to C$ where $f([x,t]) = [x]$. f is well defined since $[x, i{+}1] = [f_i(x), i{+}1]$ for every $x \in X_i$ in H, but we clearly have $[x] = [f_i(x)]$ in C so f is indeed well defined.

A map $g\colon C \to H$ is trickier to produce. △

Consider the inclusion maps $\iota\colon \mathrm{U}(n) \hookrightarrow \mathrm{U}(n{+}1)$ given by $\iota(X) = diag(X, 1)$ and then the sequence $\mathrm{U}(0) \hookrightarrow \mathrm{U}(1) \hookrightarrow ... \hookrightarrow \mathrm{U}(n) \hookrightarrow ...$

Let $\mathrm{U} := colim(\mathrm{U}(0) \hookrightarrow \mathrm{U}(1) \hookrightarrow ... \hookrightarrow \mathrm{U}(n) \hookrightarrow ...)$. Since each $\mathrm{U}(n)$ is a group and the colimit is a construction by use of homomorphisms, U is also a group and thus $B(\mathrm{U})$ can be constructed.

The inclusion maps $\iota_n\colon \mathrm{U}(n) \to \mathrm{U}(n{+}1)$ will induce maps between classifying spaces $B(\iota_n)\colon B\mathrm{U}(n) \to B\mathrm{U}(n{+}1)$ since B: **Mon** → **Top** is a covariant functor.

Lemma 50. $B\mathrm{U} = colim(B\mathrm{U}(0) \to B\mathrm{U}(1) \to ... \to B\mathrm{U}(n) \to ...)$

Proof. △

We can therefore describe a monoid homomorphism:

$$\coprod_{n=0}^{\infty} B(\iota_n) : \coprod_{n=0}^{\infty} B\mathrm{U}(n) \to \coprod_{n=0}^{\infty} B\mathrm{U}(n)$$

and from repeated application of this homomorphism we obtain a sequence and thus we can construct that sequence's homotopy colimit H:

$$H := hocolim(\coprod_{n=0}^{\infty} B\mathrm{U}(n) \to \coprod_{n=0}^{\infty} B\mathrm{U}(n) \to ...) = (\coprod_{i=0}^{\infty}((\coprod_{n=0}^{\infty} B\mathrm{U}(n))\times[i, i{+}1]))/\sim$$

where for $x \in \coprod_{j=0}^{\infty} B\mathrm{U}(j)$, $(x,\, i{+}1) \sim (\coprod_{j=0}^{\infty} B(\iota_j)(x),\, i{+}1)$, or equivalently, for $y \in B\mathrm{U}(n)$, $(y,\, i{+}1) \sim (B(\iota_n)(y),\, i{+}1)$ for all $i \in \mathbb{N}$.

The nature of the disjoint union allows us to rearrange H:

$$
\begin{aligned}
H &= (\coprod_{i=0}^{\infty}((\coprod_{n=0}^{\infty} B\mathrm{U}(n)) \times [i, i+1]))/\sim \\
&= (\coprod_{m=0}^{\infty}(\coprod_{j=0}^{\infty} B\mathrm{U}(j) \times [m+j, m+j+1]) \sqcup \coprod_{n=1}^{\infty}(\coprod_{i=0}^{\infty} B\mathrm{U}(n+i) \times [i, i+1]))/\sim \\
&= \coprod_{m=0}^{\infty} \mathit{hocolim}(B\mathrm{U}(0) \to B\mathrm{U}(1) \to ...) \sqcup \coprod_{n=1}^{\infty} \mathit{hocolim}(B\mathrm{U}(n) \to B\mathrm{U}(n+1) \to ...) \\
&\simeq \coprod_{n \in \mathbb{Z}} \mathit{hocolim}(B\mathrm{U}(0) \to B\mathrm{U}(1) \to ...) \\
&\simeq \coprod_{n \in \mathbb{Z}} \mathit{colim}(B\mathrm{U}(0) \to B\mathrm{U}(1) \to ...) \simeq \coprod_{n \in \mathbb{Z}} B\mathrm{U} = B\mathrm{U} \times \mathbb{Z}
\end{aligned}
$$

By the Group Completion Theorem [25]:

$$B(\coprod_{i=0}^{\infty} B\mathrm{U}(i)) \simeq B(B\mathrm{U} \times \mathbb{Z})$$

and by Harris [16]:

$$B(\coprod_{i=0}^{\infty} B\mathrm{U}(i)) \simeq \mathrm{U}$$

Now for $i \in \mathbb{N}$, let $\mathscr{C}_{\otimes i}$ be the category with one object and collection of morphisms $\mathrm{U}((\mathbb{C}^d)^{\otimes i})$.

As before, $\mathscr{C}_{\otimes}$ is the disjoint union of all of these categories and so:

$$|N(\mathscr{C}_{\otimes})| = |N(\coprod_{i=0}^{\infty} \mathscr{C}_{\otimes i})| = \coprod_{i=0}^{\infty} |N(\mathscr{C}_{\otimes i})| = \coprod_{i=0}^{\infty} B\mathrm{U}((\mathbb{C}^d)^{\otimes i})$$

The monoidal structure of $\mathscr{C}_{\otimes}$ induces a moniod structure on $|N(\mathscr{C}_{\otimes})|$ and thus we may construct $B(|N(\mathscr{C}_{\otimes})|)$

As in the case with $\mathscr{C}_{\oplus}$, we would like to find a space X such that $\pi_i(X) \cong \pi_i(B(|N(\mathscr{C}_{\otimes})|))$ for $i \geq 2$, and $\pi_1(X) = 0$.

Luckily, the same process as before works identically! There is a monoid homomorphism:

$$\psi : \coprod_{i=0}^{\infty} B\mathrm{U}((\mathbb{C}^d)^{\otimes i}) \to \mathbb{N}$$

given by $\psi(E) = n$ for $E \in B\mathrm{U}((\mathbb{C}^d)^{\otimes n})$, and again, the classifying space construction functor induces a group homomorphism:

$$B\psi : B(B\mathrm{U}((\mathbb{C}^d)^{\otimes i})) \to B(\mathbb{N})$$

The homotopy fibre of this map will be denoted $BB\mathrm{U}_{\otimes}[\frac{1}{d}]$, and has the desired properties as the codomain of $B\psi$ is $B(\mathbb{N})$ exactly as was the case when we defined $BB\mathrm{U}_{\oplus}$.

For some fixed integer d, consider the map $.d\colon \mathbb{Z} \to \mathbb{Z}$ given by $z \mapsto z.d$ and then the sequence $\mathbb{Z} \to \mathbb{Z} \to ... \to \mathbb{Z} \to ...$

$$colim(\mathbb{Z} \to \mathbb{Z} \to ... \to \mathbb{Z} \to ...) \cong \{ \frac{p}{q} \mid p \in \mathbb{Z},\ q = d^k,\ k \in \mathbb{N}\} = \mathbb{Z}[\frac{1}{d}] \subset \mathbb{Q}$$

Similarly, for some fixed integer d, let us consider the map (-) $\otimes\, \mathbb{I}_d$: $\mathrm{U}((\mathbb{C}^d)^{\otimes i}) \to \mathrm{U}((\mathbb{C}^d)^{\otimes (i+1)})$ given by $A \mapsto A \otimes \mathbb{I}_d$ the Kronecker product, and then the sequence $\mathrm{U}((\mathbb{C}^d)^{\otimes 0}) \to \mathrm{U}((\mathbb{C}^d)^{\otimes 1}) \to ... \to ...\ \mathrm{U}((\mathbb{C}^d)^{\otimes n}) \to ...$

We will make the following definition:

$$\mathrm{U}[\frac{1}{d}] := colim(\mathrm{U}((\mathbb{C}^d)^{\otimes 0}) \to \mathrm{U}((\mathbb{C}^d)^{\otimes 1}) \to ... \to ...\mathrm{U}((\mathbb{C}^d)^{\otimes n}) \to ...)$$

Again, since each $\mathrm{U}((\mathbb{C}^d)^{\otimes n})$ is a group and the colimit uses homomorphisms in its construction, $\mathrm{U}[\frac{1}{d}]$ is also a group and $B\mathrm{U}[\frac{1}{d}]$ can be constructed.

Analogously to the case of $\mathscr{C}_{\oplus}$, by the Group Completion Theorem [25]:

$$B(|N(\mathscr{C}_{\otimes})|) = B(\coprod_{i=0}^{\infty} B\mathrm{U}((\mathbb{C}^d)^{\otimes i})) \simeq B(B\mathrm{U}[\frac{1}{d}] \times \mathbb{Z})$$

and by Harris [16]:

$$B(\coprod_{i=0}^{\infty} B\mathrm{U}((\mathbb{C}^d)^{\otimes i})) \simeq \mathrm{U}[\frac{1}{d}]$$

7.1.3 Strict Symmetric Monoidal Categories

Definition 37. A **braided monoidal category**, as introduced by Joyal and Street [19], is a monoidal category $\mathscr{C}$ together with a natural isomorphism β called the braiding with components $\beta_{A,B}\colon A \circ B \to B \circ A$ satisfying a further pair of coherence conditions:

$$\begin{array}{ccccc}
A \circ (B \circ C) & \xrightarrow{\alpha_{A,B,C}} & (A \circ B) \circ C & \xrightarrow{\beta_{A\circ B,C}} & C \circ (A \circ B) \\
\downarrow{\scriptstyle \mathrm{id}_A \circ \beta_{B,C}} & & \square & & \downarrow{\scriptstyle \alpha_{C,A,B}} \\
A \circ (C \circ B) & \xrightarrow[\alpha_{A,C,B}]{} & (A \circ C) \circ B & \xrightarrow[\beta_{A,C} \circ \mathrm{id}_B]{} & (C \circ A) \circ B
\end{array}$$

$$\begin{array}{ccccc}
(A \circ B) \circ C & \xrightarrow{\alpha^{-1}_{A,B,C}} & A \circ (B \circ C) & \xrightarrow{\beta_{A,B\circ C}} & (B \circ C) \circ A \\
\downarrow{\scriptstyle \beta_{A,B} \circ \mathrm{id}_C} & & \square & & \downarrow{\scriptstyle \alpha^{-1}_{B,C,A}} \\
(B \circ A) \circ C & \xrightarrow[\alpha^{-1}_{B,A,C}]{} & B \circ (A \circ C) & \xrightarrow[\mathrm{id}_B \circ \beta_{A,C}]{} & B \circ (C \circ A)
\end{array}$$

A **symmetric monoidal category** is a braided monoidal category such that the braiding satisfies $\beta_{A,B} \cdot \beta_{B,A} = \mathrm{id}_{A \circ B}$ for all objects A, B. In this case, the two additional coherence conditions imply one another.

A **strict symmetric monoidal category** is a symmetric monoidal category where the associator and unitors are identities, that is a symmetric monoidal category that is also a strict monoidal category.

We have already seen that $\mathscr{C}_{\oplus}$ and $\mathscr{C}_{\otimes}$ are strict monoidal categories, to show they are strict symmetric monoidal categories all we must do is describe the braidings.

For each pair of objects n, $m \in \mathrm{ob}\mathscr{C}_{\oplus}$ we must find a unitary linear transformation $\beta_{n,m} \in \mathrm{Hom}_{\mathscr{C}_{\oplus}}(n \oplus m, m \oplus n)$ such that $\beta_{n,m} \circ \beta_{m,n} = \mathrm{id}_{n \oplus m}$, and for every pair of morphisms $A \in \mathrm{U}(n)$, $B \in \mathrm{U}(m)$, the following diagram commutes:

$$\begin{array}{ccc} n \oplus m & \xrightarrow{\beta_{n,m}} & m \oplus n \\ {\scriptstyle A \oplus B}\downarrow & \square & \downarrow{\scriptstyle B \oplus A} \\ n \oplus m & \xrightarrow[\beta_{n,m}]{} & m \oplus n \end{array}$$

For any morphism $X \in \mathrm{U}(n+m)$ let us define:

$$\beta_{n,m}(X) := \begin{pmatrix} \mathbf{0} & \mathbb{I}_m \\ \mathbb{I}_n & \mathbf{0} \end{pmatrix} X \begin{pmatrix} \mathbf{0} & \mathbb{I}_n \\ \mathbb{I}_m & \mathbf{0} \end{pmatrix}$$

where $\mathbf{0}$ is the $n \times m$ or $m \times n$ zero matrix as appropriate.

For $A \in \mathrm{U}(n)$, $B \in \mathrm{U}(m)$, this results in the following equation:

$$\beta_{n,m}(diag(A, B)) = diag(B, A)$$

For each pair of objects n, $m \in \mathrm{ob}\mathscr{C}_{\otimes}$ we must find a unitary linear transformation $\beta_{n,m} \in \mathrm{Hom}_{\mathscr{C}_{\otimes}}(n \otimes m, m \otimes n)$ such that $\beta_{n,m} \circ \beta_{m,n} = \mathrm{id}_{n \otimes m}$, and for every pair of morphisms $A \in \mathrm{U}((\mathbb{C}^d)^{\otimes n})$, $B \in \mathrm{U}((\mathbb{C}^d)^{\otimes m})$, the following diagram commutes:

$$\begin{array}{ccc} n \otimes m & \xrightarrow{\beta_{n,m}} & m \otimes n \\ {\scriptstyle A \otimes B}\downarrow & \square & \downarrow{\scriptstyle B \otimes A} \\ n \otimes m & \xrightarrow[\beta_{n,m}]{} & m \otimes n \end{array}$$

In this case $\beta_{n,m}$ is a unitary linear transformation $\beta_{n,m}\colon (\mathbb{C}^d)^{\otimes(n+m)} \to (\mathbb{C}^d)^{\otimes(n+m)}$ defined as follows for $\mathbf{v}_i$, $\mathbf{w}_j \in \mathbb{C}^d$ for $1 \leq i \leq n$, $1 \leq j \leq m$:

$$\beta_{n,m}(\mathbf{v}_1 \otimes ... \otimes ...\mathbf{v}_n \otimes \mathbf{w}_1 \otimes ... \otimes \mathbf{w}_m) = \mathbf{w}_1 \otimes ... \otimes ...\mathbf{w}_m \otimes \mathbf{v}_1 \otimes ... \otimes \mathbf{v}_n$$

7.2 Exponential Functors

Definition 38. Let $\mathscr{C}$ and $\mathscr{D}$ be monoidal categories. A **monoidal functor** as described by MacLane [22] is a functor $F\colon \mathscr{C} \to \mathscr{D}$ together with a natural transformation ϕ with components $\phi_{A,B}\colon\ FA \circ_{\mathscr{D}} FB \to F(A \circ_{\mathscr{C}} B)$ and a morphism $\psi\colon I_{\mathscr{D}} \to F(I_{\mathscr{C}})$ which are such that the following diagrams commute:

$$
\begin{array}{ccc}
FA \circ_{\mathscr{D}} (FB \circ_{\mathscr{D}} FC) & \xrightarrow{\alpha_{FA,FB,FC}} & (FA \circ_{\mathscr{D}} FB) \circ_{\mathscr{D}} FC \\
{\scriptstyle \mathrm{id}_{FA} \circ_{\mathscr{D}} \phi_{B,C}} \downarrow & & \downarrow {\scriptstyle \phi_{A,B} \circ_{\mathscr{D}} \mathrm{id}_{FC}} \\
FA \circ_{\mathscr{D}} F(B \circ_{\mathscr{C}} C) & \square & F(A \circ_{\mathscr{C}} B) \circ_{\mathscr{D}} FC \\
{\scriptstyle \phi_{A,B \circ_{\mathscr{C}} C}} \downarrow & & \downarrow {\scriptstyle \phi_{A \circ_{\mathscr{C}} B, C}} \\
F(A \circ_{\mathscr{C}} (B \circ_{\mathscr{C}} C)) & \xrightarrow[F\alpha_{A,B,C}]{} & F((A \circ_{\mathscr{C}} B) \circ_{\mathscr{C}} C)
\end{array}
$$

$$
\begin{array}{ccc}
FA \circ_{\mathscr{D}} I_{\mathscr{D}} & \xrightarrow{\mathrm{id}_{FA} \circ_{\mathscr{D}} \psi} & FA \circ_{\mathscr{D}} F(I_{\mathscr{C}}) \\
{\scriptstyle \rho_{FA}} \downarrow & \square & \downarrow {\scriptstyle \phi_{A, I_{\mathscr{C}}}} \\
FA & \xleftarrow[F\rho_A]{} & F(A \circ_{\mathscr{C}} I_{\mathscr{C}})
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
I_{\mathscr{D}} \circ_{\mathscr{D}} FB & \xrightarrow{\psi \circ_{\mathscr{D}} \mathrm{id}_{FB}} & F(I_{\mathscr{C}}) \circ_{\mathscr{D}} FB \\
{\scriptstyle \lambda_{FB}} \downarrow & \square & \downarrow {\scriptstyle \phi_{I_{\mathscr{D}}, B}} \\
FB & \xleftarrow[F\lambda_B]{} & F(I_{\mathscr{C}} \circ_{\mathscr{C}} B)
\end{array}
$$

A monoidal functor is called a **strong monoidal functor** if the components of ϕ and ψ are all isomorphisms, or a **strict monoidal functor** if the components of ϕ and ψ are all the appropriate identity morphisms. An exponential functor where the components of ϕ and ψ are not all isomorphisms is called a **lax monoidal functor**

Definition 39. Let $\mathbf{Vect}_{\Bbbk}$ be the category whose objects are vector spaces over the field $\Bbbk$ and morphisms are linear maps. This category will be donoted **Vect** when the field $\Bbbk$ is understood or arbitrary.

An **exponential functor** [29] is a monoidal functor $F\colon \mathbf{Vect} \to \mathbf{Vect}$ that preserves adjoints (i.e. $F(A^*) = F(A)^*$) such that for any vector spaces V and W, $\phi_{\mathrm{V,W}}\colon\ F(\mathrm{V}\oplus\mathrm{W}) \to F(\mathrm{V})\otimes F(\mathrm{W})$ is a natural isomorphism.

7.2.1 The Determinant

Let $\mathbf{Vect}_{\mathbb{C}}^{\mathrm{iso}}$ be the category with complex vector spaces as objects and linear automorphisms as the only morphisms.

As each morphism of $\mathbf{Vect}_{\mathbb{C}}^{\mathrm{iso}}$ can be given as a square matrix, let us determine whether the determinant is an exponential functor.

Let $det\colon\ \mathbf{Vect}_{\mathbb{C}}^{\mathrm{iso}} \to \mathbf{Vect}_{\mathbb{C}}^{\mathrm{iso}}$ map every object to the vector space $\mathbb{C}$ and every morphism A to $det(A)$, then for any two vector spaces V and W we have $det(\mathrm{V}\oplus\mathrm{W}) = \mathbb{C}$ and $det(\mathrm{V})\otimes det(\mathrm{W}) = \mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$ so the required isomorphism has the potential to exist. On morphisms, since $A\oplus B = diag(A,B)$ and $det(diag(A,B)) = det(A)det(B)$, the isomorphism is given as $\phi\colon\ \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ where $\phi(\alpha \otimes \beta) = \alpha\beta$, thus the determinant is an exponential functor.

The determinant gives us a clue towards another family of exponential functors.

Any $n \times n$ square matrix A is a linear transformation A: V $\rightarrow$ V for some n dimensional vector space V with basis $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$, and thus induces a map $\Lambda^n(A)$: $\Lambda^n(\mathrm{V}) \rightarrow \Lambda^n(\mathrm{V})$ given by $\Lambda^n(A)(\mathbf{v}_1 \wedge ... \wedge \mathbf{v}_n) = A\mathbf{v}_1 \wedge ... \wedge A\mathbf{v}_n$. Since $\Lambda^n(\mathrm{V})$ is a 1 dimensional vector space, $A\mathbf{v}_1 \wedge ... \wedge A\mathbf{v}_n = \lambda . \mathbf{v}_1 \wedge ... \wedge \mathbf{v}_n$ for some constant λ and it so happens that $\lambda = det(A)$.

7.2.2 The Direct Sum of Exterior Powers

For any two vector spaces V and W, there is an isomorphism for each $k \in \mathbb{N}$

$$\lambda^k : \Lambda^k(\mathrm{V} \oplus \mathrm{W}) \rightarrow \bigoplus_{i+j=k} \Lambda^i(\mathrm{V}) \otimes \Lambda^j(\mathrm{W})$$

Each λ^k can be shown to be an isomorphism by considering what happens on the basis elements. If V has basis $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ and W has basis $\{\mathbf{w}_1, ..., \mathbf{w}_m\}$ then V$\oplus$W has basis $\{(\mathbf{v}_1,0), ..., (\mathbf{v}_n,0), (0,\mathbf{w}_1), ..., (0,\mathbf{w}_m)\}$, $\Lambda^i(\mathrm{V})$ has basis elements of the form $\mathbf{v}_{p_1} \wedge ... \wedge \mathbf{v}_{p_i}$ where $p_x < p_y$ for $x < y$, $\Lambda^j(\mathrm{W})$ has basis elements of the form $\mathbf{w}_{q_1} \wedge ... \wedge \mathbf{w}_{q_j}$ where $q_x < q_y$ for $x < y$ and thus $\Lambda^k(\mathrm{V} \oplus \mathrm{W})$ has basis elements of the form $(\mathbf{v}_{p_1},0) \wedge ... \wedge (\mathbf{v}_{p_i},0) \wedge (0,\mathbf{w}_{q_1}) \wedge ... \wedge (0,\mathbf{w}_{q_j})$ where $i+j=k$, and $p_x < p_y$ & $q_x < q_y$ for $x < y$, and $\bigoplus_{i+j=k} \Lambda^i(\mathrm{V}) \otimes \Lambda^j(\mathrm{W})$ has basis elements of the form $(0, ..., \mathbf{v}_{p_1} \wedge ... \wedge \mathbf{v}_{p_i} \otimes \mathbf{w}_{q_1} \wedge ... \wedge \mathbf{w}_{q_j}, ..., 0)$.

Clearly if λ^k maps each basis element $(\mathbf{v}_{p_1},0) \wedge ... \wedge (\mathbf{v}_{p_i},0) \wedge (0,\mathbf{w}_{q_1}) \wedge ... \wedge (0,\mathbf{w}_{q_j})$ to the basis element $(0, ..., \mathbf{v}_{p_1} \wedge ... \wedge \mathbf{v}_{p_i} \otimes \mathbf{w}_{q_1} \wedge ... \wedge \mathbf{w}_{q_j}, ..., 0)$, then each λ^k must be an isomorphism as only $(0,0) \wedge ... \wedge (0,0)$ can map to $(0, ..., 0)$ and, as we will demonstrate, the domain and codomain are vector spaces of the same dimension.

$dim(\mathrm{V}) = n$ and $dim(\mathrm{W}) = m$ so $dim(\mathrm{V} \oplus \mathrm{W}) = n+m$ and

$$dim(\Lambda^k(\mathrm{V} \oplus \mathrm{W})) = \binom{n+m}{k}$$

$$dim(\bigoplus_{i+j=k} \Lambda^i(\mathrm{V}) \otimes \Lambda^j(\mathrm{W})) = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}$$

$$\text{and } \binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}$$

which is a relabelling of the Chu-Vandermonde identity.

However, we have not yet described an exponential functor as $\Lambda^k(\mathrm{V} \oplus \mathrm{W})$ is not isomorphic to $\Lambda^k(\mathrm{V}) \otimes \Lambda^k(\mathrm{W})$.

Instead let us consider the functor F: **Vect** $\rightarrow$ **Vect** defined as follows:

$$F(\mathrm{V}) = \bigoplus_{k=0}^{\infty} \Lambda^k(\mathrm{V})$$

$F(\mathrm{V})$ is finite dimensional as $\Lambda^k(\mathrm{V}) = 0$ for $k > dim(\mathrm{V})$.

We must show that for any vector spaces V and W we have a natural isomorphism $\phi_{\mathrm{V,W}}$ $F(\mathrm{V}\oplus\mathrm{W}) \to F(\mathrm{V})\otimes F(\mathrm{W})$.

$$\begin{aligned}
F(\mathrm{V}) \otimes F(\mathrm{W}) &= \bigoplus_{i=0}^{\infty} \Lambda^i(\mathrm{V}) \otimes \bigoplus_{j=0}^{\infty} \Lambda^j(\mathrm{W}) \\
&= \bigoplus_{i=0}^{\infty} \bigoplus_{j=0}^{\infty} \Lambda^i(\mathrm{V}) \otimes \Lambda^j(\mathrm{W}) \\
&= \bigoplus_{k=0}^{\infty} \bigoplus_{i+j=k} \Lambda^i(\mathrm{V}) \otimes \Lambda^j(\mathrm{W}) \\
&\cong \bigoplus_{k=0}^{\infty} \Lambda^k(\mathrm{V} \oplus \mathrm{W}), \text{ by the isomorphisms } \lambda^k \\
&= F(\mathrm{V} \oplus \mathrm{W})
\end{aligned}$$

7.2.3 Modification to the Direct Sum of Exterior Powers

Consider the functor G: **Vect** $\to$ **Vect** defined as follows for some fixed d:

$$G(\mathrm{V}) = \bigoplus_{k=0}^{\infty} (\mathbb{C}^d)^{\otimes k} \otimes \Lambda^k(\mathrm{V})$$

Again, we can show that for any two vector spaces V and W, there is a natural isomorphism $\psi_{\mathrm{V,W}}$: $G(\mathrm{V}\oplus\mathrm{W}) \to G(\mathrm{V})\otimes G(\mathrm{W})$

$$\begin{aligned}
G(\mathrm{V}) \otimes G(\mathrm{W}) &= (\bigoplus_{i=0}^{\infty} (\mathbb{C}^d)^{\otimes i} \otimes \Lambda^i(\mathrm{V})) \otimes (\bigoplus_{j=0}^{\infty} (\mathbb{C}^d)^{\otimes j} \otimes \Lambda^j(\mathrm{W})) \\
&= \bigoplus_{i=0}^{\infty} \bigoplus_{j=0}^{\infty} (\mathbb{C}^d)^{\otimes i} \otimes \Lambda^i(\mathrm{V}) \otimes (\mathbb{C}^d)^{\otimes j} \otimes \Lambda^j(\mathrm{W}) \\
&\cong \bigoplus_{k=0}^{\infty} \bigoplus_{i+j=k} (\mathbb{C}^d)^{\otimes (i+j)} \otimes \Lambda^i(\mathrm{V}) \otimes \Lambda^j(\mathrm{W}) \\
&= \bigoplus_{k=0}^{\infty} (\mathbb{C}^d)^{\otimes k} \otimes \bigoplus_{i+j=k} \Lambda^i(\mathrm{V}) \otimes \Lambda^j(\mathrm{W}) \\
&\cong \bigoplus_{k=0}^{\infty} (\mathbb{C}^d)^{\otimes k} \otimes \Lambda^k(\mathrm{V} \oplus \mathrm{W}), \text{ by the isomorphisms } \lambda^k \\
&= G(\mathrm{V} \oplus \mathrm{W})
\end{aligned}$$

7.3 The Effect of Exponential Functors on Vector Bundles

Let p: $E \to X$ be a vector bundle with fibre V and let F: $\mathscr{C}_\oplus \to \mathscr{C}_\otimes$ be an exponential functor between the categories $\mathscr{C}_\oplus$ and $\mathscr{C}_\otimes$ as described previously.

Let us write $B\mathscr{C}_\oplus = |N(\mathscr{C}_\oplus)|$ and $B\mathscr{C}_\otimes = |N(\mathscr{C}_\otimes)|$, then $[X, B\mathscr{C}_\oplus]$, the set of homotopy classes of maps from X to $B\mathscr{C}_\oplus$ is in bijection with the set of isomorphism classes of vector bundles over X, and similarly, $[X, B\mathscr{C}_\otimes]$ is in bijection with the set of isomorphism classes of rank d^n vector bundles over X for some n and fixed $d \in \mathbb{N}$.

Since the nerve and geometric realisation act functorially, F induces a map BF: $B\mathscr{C}_\oplus \to B\mathscr{C}_\otimes$ which in turn induces a map $BF\circ$ -: $[X, B\mathscr{C}_\oplus] \to [X, B\mathscr{C}_\otimes]$ and thus for any vector bundle over X we can construct another vector bundle over X by means of any exponential functor. Essentially, for any vector bundle p: $E \to X$, this map sends each fibre $p^{-1}(x)$ for $x \in X$, to $F(p^{-1}(x))$. Let us see how the total space E is transformed by this map.

Let the vector bundle p: $E \to X$ have fibre V. For each $x \in X$, $\exists$ an open neighbourhood $U \subset X$ of x such that $\exists$ a homeomorphism ϕ: $E|_U \to U \times$ V such that:

$$\begin{array}{ccc} E|_U & \xrightarrow{\phi} & U \times \mathrm{V} \\ & {\scriptstyle p}\searrow \;\; \square \;\; \swarrow{\scriptstyle \pi_U} & \\ & U & \end{array}$$

Let I be an indexing subset of X so that the open neighbourhoods as described above U_i for $i \in I \subset X$ form a cover of X, let ϕ_i be the corresponding homeomorphism for each i.

The map π below is a clear trivial vector bundle which can clearly be transformed by the exponential functor F:

$$\pi : \coprod_{i\in I}(U_i \times \mathrm{V}) \to \coprod_{i\in I} U_i$$

There are also equivalence relations to compare π with p:

$$\begin{array}{ccc} \coprod_{i\in I}(U_i \times \mathrm{V}) & \xrightarrow{\sim_E} & E \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ \coprod_{i\in I} U_i & \xrightarrow[\sim_X]{} & X \end{array}$$

where, for (i, x), $(j, y) \in \coprod_{i\in I} U_i$, $(i, x) \sim_X (j, y)$ iff $x = y$, and for $(i, x, \mathbf{v})$, $(j, y, \mathbf{w}) \in \coprod_{i\in I}(U_i \times \mathrm{V})$, $(i, x, \mathbf{v}) \sim_E (j, y, \mathbf{w})$ iff $x = y$ and $(\phi_j\circ\phi_i{}^{-1})(x, \mathbf{v}) = (x, \mathbf{w})$.

$\phi_j\circ\phi_i{}^{-1}$: $(U_i \cap U_j) \times \mathrm{V} \to (U_i \cap U_j) \times \mathrm{V}$ is a homeomorphism such that $(\phi_j\circ\phi_i{}^{-1})(x, \mathbf{v}) = (x, \phi_{ji}(x)\mathbf{v})$ where ϕ_{ji}: $U_i \cap U_j \to \mathrm{U(V)}$. It is clear that this homeomorphism can be written in this form as the ϕ homeomorphisms take our point $(x, \mathbf{v})$ into and out of the fibre over x and both must respect projection to the first component. In addition, since this is a homeomorphism, there is a unitary matrix based on x, $\phi_{ji}(x)$ that under this homeomorphism, linearly transforms the fibre V.

It is clear that $\sim_X$ is an equivalence relation since equality of the first entries is an equivalence relation. However, we must check that $\sim_E$ is an equivalence relation too.

Reflexivity: $(i,\, x,\, \mathbf{v}) \sim_E (i,\, x,\, \mathbf{v})$ iff $x = x$ and $(\phi_i \circ \phi_i{}^{-1})(x,\, \mathbf{v}) = (x,\, \mathbf{v})$. These conditions hold as ϕ_i is a homeomorphism and thus composing with it's inverse is equal to the appropriate identity. We have $\phi_{ii} = \mathbb{I}_{rank(\mathrm{V})}$.

Transitivity: if $(i,\, x,\, \mathbf{v}) \sim_E (j,\, y,\, \mathbf{w})$ and $(j,\, y,\, \mathbf{w}) \sim_E (k,\, z,\, \mathbf{u})$ then $(i,\, x,\, \mathbf{v}) \sim_E (k,\, z,\, \mathbf{u})$, true iff $x = z$ and $(\phi_k \circ \phi_i{}^{-1})(x,\, \mathbf{v}) = (z,\, \mathbf{u})$.

We know that $x = y$ and $y = z$ so $x = z$ since equality is an equivalence relation. We also know that $(\phi_j \circ \phi_i{}^{-1})(x,\, \mathbf{v}) = (y,\, \mathbf{w})$ and $(\phi_k \circ \phi_j{}^{-1})(y,\, \mathbf{w}) = (z,\, \mathbf{u})$ so $(\phi_k \circ \phi_j{}^{-1})((\phi_j \circ \phi_i{}^{-1})(x,\, \mathbf{v})) = (z,\, \mathbf{u})$ thus $(\phi_k \circ \phi_j{}^{-1} \circ \phi_j \circ \phi_i{}^{-1})(x,\, \mathbf{v}) = (z,\, \mathbf{u})$ and since ϕ_j is a homeomorphism we have $(\phi_k \circ \phi_i{}^{-1})(x,\, \mathbf{v}) = (z,\, \mathbf{u})$ as required. We have $\phi_{kj}\phi_{ji} = \phi_{ki}$.

Symmetry: if $(i,\, x,\, \mathbf{v}) \sim_E (j,\, y,\, \mathbf{w})$ then $(j,\, y,\, \mathbf{w}) \sim_E (i,\, x,\, \mathbf{v})$, true iff $y = x$ and $(\phi_i \circ \phi_j{}^{-1})(y,\, \mathbf{w}) = (x,\, \mathbf{v})$.

Clearly if $x = y$ then $y = x$ since equality is an equivalence relation, the second condition follows from reflexivity and transitivity. We would like to show that $(\phi_j \circ \phi_i{}^{-1})((\phi_i \circ \phi_j{}^{-1})(x,\, \mathbf{v}) = (x,\, \mathbf{v})$ and so:

$$\begin{aligned}(\phi_j \circ \phi_i{}^{-1})((\phi_i \circ \phi_j{}^{-1})(x, \mathbf{v})) &= (\phi_j \circ \phi_i{}^{-1})(x, \phi_{ij}\mathbf{v}) \\ &= (x, \phi_{ji}\phi_{ij}\mathbf{v})\end{aligned}$$

by transitivity, we have $\phi_{ji}\phi_{ij} = \phi_{jj}$ and by reflexivity $\phi_{jj} = \mathbb{I}_{rank(\mathrm{V})}$ and we achieve the required result.

Let us finally apply our exponential functor F and use this information to construct a new total space for a vector bundle over X.

We can imediately apply F to π to obtain a vector bundle

$$F(\pi) : \coprod_{i \in I}(U_i \times F(\mathrm{V})) \to \coprod_{i \in I} U_i$$

Lets see if we can't apply an equivalence relation on this vector bundle to obtain a vector bundle over X we can call $F(p)$.

For brevity, let us write $\psi_{ij} := F(\phi_{ij})$

For $(i,\, x,\, \mathbf{v}),\, (j,\, y,\, \mathbf{w}) \in \coprod_{i \in I}(U_i \times F(\mathrm{V}))$ we will construct an equivalence relation $\sim$ where $(i,\, x,\, \mathbf{v}) \sim (j,\, y,\, \mathbf{w})$ iff $x = y$ and $\psi_{ji}(x)\mathbf{v} = \mathbf{w}$.

The functoriality of F ensures that $\sim$ is an equivalence relation:

Reflexivity: $(i,\, x,\, \mathbf{v}) \sim (i,\, x,\, \mathbf{v})$ as $x = x$ and $\psi_{ii}(x)\mathbf{v} = F(\phi_{ii})(x)\mathbf{v} = F(\mathbb{I}_{rank(\mathrm{V})})\mathbf{v} = \mathbb{I}_{rank(F(\mathrm{V}))}\mathbf{v} = \mathbf{v}$.

Transitivity: if $(i,\, x,\, \mathbf{v}) \sim (j,\, y,\, \mathbf{w})$ and $(j,\, y,\, \mathbf{w}) \sim (k,\, z,\, \mathbf{u})$ then $x = y = z \implies x = z$ since equality is an equivalence relation and $\psi_{kj(x)}(\psi_{ji}(x)\mathbf{v}) = \mathbf{u}$ and $\psi_{kj(x)}(\psi_{ji}(x)\mathbf{v}) = (\psi_{kj(x)} \circ \psi_{ji}(x))\mathbf{v} = (F(\phi_{kj}(x)) \circ F(\phi_{ji}(x))\mathbf{v} = F(\phi_{kj(x)} \circ \phi_{ji}(x))\mathbf{v} = F(\phi_{ki}(x))\mathbf{v} = \psi_{ki}(x)\mathbf{v}$ and so $\psi_{ki}(x)\mathbf{v} = \mathbf{u}$ so $(i,\, x,\, \mathbf{v}) \sim (k,\, z,\, \mathbf{u})$ as required.

Symmetry: if $(i,\, x,\, \mathbf{v}) \sim (j,\, y,\, \mathbf{w})$ then $(j,\, y,\, \mathbf{w}) \sim (i,\, x,\, \mathbf{v})$ as $x = y \implies y = x$ and since $\psi_{ji}(x)\mathbf{v} = \mathbf{w}$, $\psi_{ij}(x)\circ\psi_{ji}(x)\mathbf{v} = \psi_{ii}(x)\mathbf{v}$ by transitivity and $\psi_{ii}(x)\mathbf{v} = \mathbb{I}_{rank(F(\mathrm{V}))}\mathbf{v} = \mathbf{v}$ as required by reflexivity.

Then we have a vector bundle over X with fibre $F(\mathrm{V})$ we can call $F(p)$:

$$F(p) : (\coprod_{i \in I}(U_i \times F(\mathrm{V})))/\sim \to X$$

8 Segal's Category and Γ-spaces

The aim of this chapter is to ensure that we can construct cohomology theories from strict symmetric monoidal categories. We need to enlist the help of a strange but useful category introduced, unsurprisingly, by Segal in *Categories and Cohomology Theories* [32].

Definition 40. Segal's category is a category $\mathbf{\Gamma}$, but for our purposes it will be more beneficial to describe the opposite category $\mathbf{\Gamma}^{\mathrm{op}}$ whose objects are pointed sets of the form $[n]^* = \{0, 1, ..., n\}$ for $n \in \mathbb{N}$ where 0 is the basepoint and morphisms are pointed maps $f\colon [n]^* \to [m]^*$ such that $f(0) = 0$.

A **Γ-set** is a covariant functor $X\colon \mathbf{\Gamma}^{\mathrm{op}} \to \mathbf{Set^*}$ such that $X([0]^*)$ is isomorphic to $[0]^*$ and $X([n]^*)$ is isomorphic to $X([1]^*)^n$, a **Γ-space** is a covariant functor $Y\colon \mathbf{\Gamma}^{\mathrm{op}} \to \mathbf{Top^*}$ such that $Y([0]^*)$ is homotopy equivalent to Δ^0 and $Y([n]^*)$ is homotopy equivalent to $Y([1]^*)^n$, and a **Γ-category** is a covariant functor $Z\colon \mathbf{\Gamma}^{\mathrm{op}} \to \mathbf{Cat^*}$ such that $Z([0]^*)$ is equivalent to **[0]** and $Z([n]^*)$ is equivalent to $Z([1]^*)^n$.

In addition there is a contravariant functor $\mathbf{\Delta} \to \mathbf{\Gamma}^{\mathrm{op}}$ that sends each object $[n]$ to $[n]^*$ as one would expect, and sends each order preserving map $f\colon [n] \to [m]$, to the pointed map $f^*\colon [m]^* \to [n]^*$ defined as follows:

$$f^*(i) := \begin{cases} \min\{j \in [n] \mid f(j) = i\}, & f^{-1}(i) \neq \emptyset \\ 0, & else \end{cases}$$

f^* is pointed as either $f(0) = 0$, in which case $0 \in f^{-1}(0)$ and must be the minimum, or $f(0) \geq 0$, and thus $f(i) \geq 0$ since f is order preserving, and thus $f^{-1}(0) = \emptyset$, therefore we must have $f^*(0) = 0$ in both cases.

The morphisms of $\mathbf{\Delta}$ are generated by the monomorphisms $\delta^{n,i}$ and the epimorphisms $\sigma^{n,i}$, we can easily describe the images of these morphisms by our contravariant functor $\mathbf{\Delta} \to \mathbf{\Gamma}^{\mathrm{op}}$.

The monomorphisms $\delta^{n,i}\colon [n\text{-}1] \to [n]$ such that $(\delta^{n,i})^{-1}(i) = \emptyset$ are sent to pointed maps $\delta_{n,i}\colon [n]^* \to [n\text{-}1]^*$ given by:

$$\delta_{n,i}(j) = \begin{cases} j, & j < i \\ 0, & j = i \\ j-1, & j > i \end{cases}$$

The epimorphisms $\sigma^{n,i}\colon [n{+}1] \to [n]$ such that $|(\sigma^{n,i})^{-1}(i)| = 2$ are sent to pointed maps $\sigma_{n,i}\colon [n]^* \to [n{+}1]^*$ given by:

$$\sigma_{n,i}(j) = \begin{cases} j, & j \leq i \\ j+1, & j > i \end{cases}$$

If X is an established Γ-set, Γ-space or Γ-category, we will say **the simplicial set**, **space**, or **category** $\boldsymbol{X}$, to mean the functor X precomposed with this functor $\mathbf{\Delta} \to \mathbf{\Gamma}^{\mathrm{op}}$.

Lemma 51. If X is a Γ-space, then the functors $X_n^{'}$ defined as $X_n^{'}([m]^*) = X([m]^* \vee [n]^*)$ and BX defined as $BX([n]^*) = |X_n^{'}|$ are also Γ-spaces.

Proof. It must be shown that $X_n^{'}([0]^*)$ is homotopy equivalent to Δ^0 and $X_n^{'}([m]^*)$ is homotopy equivalent to $X_n^{'}([1]^*)^m \ \forall\ n \in \mathbb{N}$.

$[n]^* \vee [m]^* = ([n]^* \times [m]^*)/\sim$ where $(i, 0) \sim (0, 0) \sim (0, j)$, for all $i \in [n]^*$, $j \in [m]^*$ therefore $[n]^* \vee [m]^* \cong [nm]^*$

For all $n \in \mathbb{N}$ $X_n^{'}([0]^*) = X([0]^* \vee [n]^*) \cong X([0]^*) \simeq \Delta^0$ since X is a Γ-space thus the first requirement holds.

$X_n^{'}([m]^*) = X([m]^* \vee [n]^*) \cong X([mn]^*) \simeq X([1]^*)^{mn}$ since X is a Γ-space, and $X_n^{'}([1]^*)^m = X([1]^* \vee [n]^*)^m \cong X([n]^*)^m \simeq (X([1]^*)^n)^m = X([1]^*)^{mn}$, therefore the second requirement holds and thus $X_n^{'}$ is a Γ-space $\forall\ n \in \mathbb{N}$.

Now for BX. $BX([0]^*) = |X_0^{'}|$ where $X_0^{'}$ is the simplicial space $X_0^{'}$. We must first investigate the Γ-space $X_0^{'}$. For $n \in \mathbb{N}$, $X_0^{'}([n]^*) = X([n]^* \vee [0]^*) \cong X([0]^*) \simeq \Delta^0$. For any $f\colon [n]^* \to [m]^*$, we have:

$$\begin{array}{ccc}
X_0^{'}([n]^*) & \xrightarrow{X_0^{'}(f)} & X_0^{'}([m]^*) \\
\cong\downarrow & \square & \downarrow\cong \\
X([0]^*) & \xrightarrow[X(\mathrm{id}_{[0]^*})]{} & X([0]^*) \\
\simeq\downarrow & \square & \downarrow\simeq \\
\Delta^0 & \xrightarrow[\mathrm{id}_{\Delta^0}]{} & \Delta^0
\end{array}$$

That is, each $X_0^{'}(f)$ is homotopic to id_{Δ^0}

As a simplicial space $X_0^{'}$ also sends each object $[n]$ of $\mathbf{\Delta}$ to $X([0]^*) \simeq \Delta^0$ and each morphism $f\colon [n] \to [m]$ to a map homotopic to id_{Δ^0}

$$\begin{aligned}
|X_0^{'}| &= (\coprod_{i=0}^{\infty} X_0^{'}([i]) \times \Delta^i)/\sim \\
&\cong (\coprod_{i=0}^{\infty} X([0]^*) \times \Delta^i)/\sim \\
&\simeq (\coprod_{i=0}^{\infty} \Delta^0 \times \Delta^0)/\sim \\
&\simeq \Delta^0 \times \Delta^0 \\
&\cong \Delta^0
\end{aligned}$$

Thus $BX([0]^*) = |X_0^{'}| \simeq \Delta^0$ as required

$BX([n]^*) = |X_n^{'}|$ where $X_n^{'}$ is the simplicial space $X_n^{'}$, we would like to show that this space is homotopy equivalent to $BX([1]^*)^n = |X_1^{'}|^n$ where $X_1^{'}$ is the simplicial space $X_1^{'}$ [13] △

8.1 Constructing a Γ-category from a Strict Symmetric Monoidal Category

Definition 41. Based on notes by Freed [13], if $\mathscr{C}$ is a strict symmetric monoidal category, let us construct a covariant functor $\mathscr{C}^\Gamma\colon \mathbf{\Gamma}^{\mathrm{op}} \to \mathbf{Cat}^*$, where $\mathscr{C}^\Gamma([n]^*)$ is defined to be the category with objects (c, ρ) where $c\colon \mathcal{P}([n]^*)^* \to \mathrm{ob}\mathscr{C}$ is an assignment to an object of $\mathscr{C}$ for every pointed subset of $[n]^*$ and ρ is a natural transformation with components that are isomorphisms $\rho_{P,Q}\colon\ c(P) \otimes c(Q) \to c(P \vee Q)$ for $P,\ Q \subseteq [n]^*$ such that $P \cap Q = \{0\}$, and morphisms $f\colon\ (c,\ \rho) \to (c',\ \rho')$ are a set of morphisms $\{f_R \in \mathrm{Hom}_{\mathscr{C}}(c(R),\ c'(R)) \mid R \subseteq [n]^*\}$ such that for $P \cap Q = \{0\}$ the following diagram commutes:

$$\begin{array}{ccc} c(P) \otimes c(Q) & \xrightarrow{\rho_{P,Q}} & c(P \vee Q) \\ {\scriptstyle f_P \otimes f_Q}\downarrow & \square & \downarrow{\scriptstyle f_{P \vee Q}} \\ c'(P) \otimes c'(Q) & \xrightarrow[\rho'_{P,Q}]{} & c'(P \vee Q) \end{array}$$

we further require that for all objects $(c,\ \rho)$ of any of the categories $\mathscr{C}^\Gamma([n]^*)$:

- $c(\{0\}) = I$, the unit object of $\mathscr{C}$,
- the isomorphism $\rho_{\{0\},P}\colon\ c(\{0\}) \otimes c(P) \to c(\{0\} \vee P)$ must be equal to $\mathrm{id}_{c(P)}$ for all pointed subsets $P \subseteq [n]^*$
- for $P,\ Q,\ R \subseteq [n]^*$ such that $P \cap Q \cap R = \{0\}$, the following diagrams must commute:

$$\begin{array}{ccc} c(P) \otimes c(Q) & \xrightarrow{\beta_{c(P),c(Q)}} & c(Q) \otimes c(P) \\ & {\scriptstyle \rho_{P,Q}}\searrow \quad \square \quad \swarrow{\scriptstyle \rho_{Q,P}} & \\ & c(P \vee Q) & \end{array}$$

$$\begin{array}{ccc} c(P) \otimes c(Q) \otimes c(R) & \xrightarrow{\rho_{P,Q} \otimes \mathrm{id}_{c(R)}} & c(P \vee Q) \otimes c(R) \\ {\scriptstyle \mathrm{id}_{c(P)} \otimes \rho_{Q,R}}\downarrow & \square & \downarrow{\scriptstyle \rho_{P \vee Q, R}} \\ c(P) \otimes c(Q \vee R) & \xrightarrow[\rho_{P, Q \vee R}]{} & c(P \vee Q \vee R) \end{array}$$

If $\mathrm{Hom}_{\mathscr{C}}(I,\ I)$ is not a group under composition of morphisms, then we further need to require that for all morphisms f in any of the categories $\mathscr{C}^\Gamma([n]^*)$, $f_{\{0\}} = \mathrm{id}_I$. If $\mathrm{Hom}_{\mathscr{C}}(I,\ I)$ is a group under composition of morphisms, then this condition is met automatically since we require $f_{\{0\}}^{\otimes 2} = f_{\{0\}} \circ f_{\{0\}} = f_{\{0\}}$ and if $f_{\{0\}}$ is an isomorphism then we have:

$$\begin{aligned} f_{\{0\}} \circ f_{\{0\}} \circ f_{\{0\}}^{-1} &= f_{\{0\}} \circ f_{\{0\}}^{-1} \\ f_{\{0\}} &= \mathrm{id}_I \end{aligned}$$

The identity morphism of an object $(c,\ \rho)$ in $\mathscr{C}^\Gamma([n]^*)$ is the morphism $\mathrm{id}_{(c,\rho)} = \{\mathrm{id}_{c(P)} \in \mathrm{Hom}_{\mathscr{C}}(c(P),\ c(P)) \mid P \subseteq [n]^*\}$ where every component is the identity on it's respecive object.

Composition of morphisms is defined piecewise on components. If $f \in \mathrm{Hom}_{\mathscr{C}^\Gamma([n]^*)}((c,\ \rho),\ (c',\ \rho'))$ and $g \in \mathrm{Hom}_{\mathscr{C}^\Gamma([n]^*)}((c',\ \rho'),\ (c'',\ \rho''))$ then $g \circ f = \{g_P \circ f_P\colon\ c(P) \to c''(P) \mid P \subseteq [n]^*\}$.

Since $\mathscr{C}$ is a category, composing with an identity morphism on either side yields the morphism we started with, composing several morphisms with the relevant appropriate morphisms at once will behave exactly the same way. For any morphism $f\colon\ (c,\ \rho) \to (c',\rho')$, $f \circ \mathrm{id}_{(c,\ \rho)} = f = \mathrm{id}_{(c',\rho')} \circ f$ as $f_P \circ \mathrm{id}_{c(P)} = f_P = \mathrm{id}_{c'(P)} \circ f_P$ for each $P \subseteq [n]^*$.

Similarly, appropriate composition of morphisms in $\mathscr{C}$ is associative so simultaneously composing morphisms piecewise is also associative. For any $f\colon\ (c,\ \rho) \to (c',\ \rho')$, $g\colon\ (c',\ \rho') \to (c'',\ \rho'')$, and $h\colon\ (c'',\ \rho'') \to (c''',\ \rho''')$, $(h \circ g) \circ f = h \circ (g \circ f)$ as $(h_P \circ g_P) \circ f_P = h_P \circ (g_P \circ f_P)$ for all $P \subseteq [n]^*$.

Ensuring that this structure does in fact define a Γ-category will be tantamount in proving that strict symmetric monoidal categories induce cohomology theories.

The following theorem is a fleshing out of the work by Freed [13]

Theorem 52. If $\mathscr{C}$ is a strict symmetric monoidal category, then $\mathscr{C}^\Gamma$ is a Γ-category and $|N(\mathscr{C}^\Gamma)|$ is a Γ-space.

Thus we may construct a reduced cohomology theory $h^*_{\mathscr{C}}$ with a sequence of functors $h^k_{\mathscr{C}}\colon$ **Top*** $\to$ **AbGrp** defined as follows since $\mathscr{C}^\Gamma([1]^*) \cong \mathscr{C}$:

$$h^k_{\mathscr{C}}(X) := \begin{cases} [X, \Omega^{(-k+1)}B|N(\mathscr{C})|], & k \leq 0 \\ [X, B^{(k)}|N(\mathscr{C})|], & k > 0 \end{cases}$$

Proof. To show that $\mathscr{C}^\Gamma$ is a Γ-category, we need to show that $\mathscr{C}^\Gamma([0]^*) \cong \mathbf{[0]}$, and to show that the reduced cohomology is as we say it is, we need to show that $\mathscr{C}^\Gamma([1]^*) \cong \mathscr{C}$.

To show either equivalence, we must first investigate the objects and morphisms in the categories $\mathscr{C}^\Gamma([0]^*)$ and $\mathscr{C}^\Gamma([1]^*)$.

Objects in $\mathscr{C}^\Gamma([0]^*)$ are pairs $(c,\ \rho)$ where $c\colon\ \mathcal{P}([0]^*)^* \to \mathrm{ob}\mathscr{C}$ is required to satisfy $c(\{0\}) = I$, and ρ is a natural transformation with only one component since there is only one pair $P,\ Q \subseteq [0]^*$ with $P \cap Q = \{0\}$ and we require $\rho_{\{0\},\{0\}} = \mathrm{id}_{c(\{0\})} = \mathrm{id}_I$.

Thus since there are no other pointed subsets of $[0]^*$, c is required to be the map to the unit object, and ρ must have a single component that is required to be the identity, the category $\mathscr{C}^\Gamma([0]^*)$ has only one object $(c_I,\ \rho_{\mathrm{id}_I})$.

Since there is only one object, we only need to examine one collection of morphisms $\mathrm{Hom}_{\mathscr{C}^\Gamma([0]^*)}((c_I,\ \rho_{\mathrm{id}_I}),\ (c_I,\ \rho_{\mathrm{id}_I}))$. Morphisms in $\mathscr{C}^\Gamma([0]^*)$ are defined to be sets of morphisms $\{f_R \in \mathrm{Hom}_{\mathscr{C}}(c_I(R),\ c_I(R)) \mid R \subseteq [0]^*\}$ such that the following diagram commutes for $P \cap Q = \{0\}$:

$$\begin{array}{ccc} c_I(P) \otimes c_I(Q) & \xrightarrow{\rho_{P,Q}} & c_I(P \vee Q) \\ {\scriptstyle f_P \otimes f_Q}\big\downarrow & \square & \big\downarrow{\scriptstyle f_{P \vee Q}} \\ c_I(P) \otimes c_I(Q) & \xrightarrow[\rho_{P,Q}]{} & c_I(P \vee Q) \end{array}$$

and since there is only one pointed subset of $[0]^*$, the morphisms $f \in \mathrm{Hom}_{\mathscr{C}^\Gamma([0]^*)}((c_I, \rho_{\mathrm{id}_I}), (c_I, \rho_{\mathrm{id}_I}))$ are sets containing one morphism $f_{\{0\}} \in \mathrm{Hom}_{\mathscr{C}}(I, I)$ such that:

$$\begin{array}{ccc} I \otimes I & \xrightarrow{\mathrm{id}_I} & I \\ {\scriptstyle f_{\{0\}} \otimes f_{\{0\}}}\big\downarrow & \square & \big\downarrow{\scriptstyle f_{\{0\}}} \\ I \otimes I & \xrightarrow[\mathrm{id}_I]{} & I \end{array}$$

By hook (if every morphism in $\mathrm{Hom}_{\mathscr{C}}(I, I)$ is an isomorphism) or by crook (forcing it through the definition) $f_{\{0\}} = \mathrm{id}_I$ for any morphism, so the category $\mathscr{C}^\Gamma([0]^*)$ has only a single morphism $\mathrm{id}_{(c_I,\, \rho_{\mathrm{id}_I})}$ where $\mathrm{id}_{(c_I,\, \rho_{\mathrm{id}_I}),\, \{0\}} = \mathrm{id}_I$.

Since both $\mathscr{C}^\Gamma([0]^*)$ and **[0]** are cateogries with only one object and only one morphism, any functor one could construct between them must be full, faithful, and essentially surjective on objects, and thus $\mathscr{C}^\Gamma([0]^*) \cong$ **[0]**.

Now to examine $\mathscr{C}^\Gamma([1]^*)$. Again objects are pairs (c, ρ) where $c\colon \mathcal{P}([1]^*)^* \to \mathrm{ob}\mathscr{C}$ and ρ is a natural transformation, but now we have a bit more freedom. $c(\{0\})$ is still required to be I but we now have another pointed subset of $[1]^*$ and we can let $c(\{0, 1\})$ be any object we like in $\mathscr{C}$, ρ however is still fixed as we require $\rho_{\{0\},\{0\}} = \mathrm{id}_I$, and $\rho_{\{0\},\{0,1\}} = \rho_{\{0,1\},\{0\}} = \mathrm{id}_{c(\{0,1\})}$. Let $(c_X, \rho_{\mathrm{id}})$ denote the object where $c(\{0, 1\}) = X$, we can see that the objects of $\mathscr{C}^\Gamma([1]^*)$ are all of the form $(c_X, \rho_{\mathrm{id}})$ for $X \in \mathrm{ob}\mathscr{C}$.

Morphisms $f \in \mathrm{Hom}_{\mathscr{C}^\Gamma([1]^*)}((c_X, \rho_{\mathrm{id}}), (c_Y, \rho_{\mathrm{id}}))$ must consist of a pair of morphisms $(f_{\{0\}}, f_{\{0,1\}})$ where $f_{\{0\}} \in \mathrm{Hom}_{\mathscr{C}}(I, I)$, and $f_{\{0,1\}} \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ such that the following diagrams all commute:

$$\begin{array}{ccc} I \otimes I & \xrightarrow{\mathrm{id}_I} & I \\ {\scriptstyle f_{\{0\}} \otimes f_{\{0\}}}\big\downarrow & \square & \big\downarrow{\scriptstyle f_{\{0\}}} \\ I \otimes I & \xrightarrow[\mathrm{id}_I]{} & I \end{array} \qquad \begin{array}{ccc} I \otimes X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle f_{\{0\}} \otimes f_{\{0,1\}}}\big\downarrow & \square & \big\downarrow{\scriptstyle f_{\{0,1\}}} \\ I \otimes Y & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array} \qquad \begin{array}{ccc} X \otimes I & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle f_{\{0,1\}} \otimes f_{\{0\}}}\big\downarrow & \square & \big\downarrow{\scriptstyle f_{\{0,1\}}} \\ Y \otimes I & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array}$$

As we have already seen, either the commutativity of the first diagram or an additional requirement ensures that for every morphism f in the category $\mathscr{C}^\Gamma([1]^*)$, $f_{\{0\}} = \mathrm{id}_I$. Since $\mathscr{C}$ is a strict symmetric monoidal category, the unitors are identities and so $\mathrm{id}_I \otimes \phi = \phi \otimes \mathrm{id}_I = \phi$ for any morphism ϕ in the category $\mathscr{C}$. For any morphism $\phi \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$, let (ϕ) denote the morphism in $\mathrm{Hom}_{\mathscr{C}^\Gamma([1]^*)}((c_X, \rho_{\mathrm{id}}), (c_Y, \rho_{\mathrm{id}}))$ where $(\phi)_{\{0,1\}} = \phi$. Again, we can see that all morphisms in $\mathscr{C}^\Gamma([1]^*)$ are of the form (ϕ) for some morphism ϕ in $\mathscr{C}$. To ensure that $\mathscr{C}^\Gamma([1]^*)$ is a category, the identities are given by $\mathrm{id}_{(c_X,\, \rho_{\mathrm{id}})} = (\mathrm{id}_X)$

and the composition law is given by $g \circ f = (g_{\{0\}} \circ f_{\{0\}},\, g_{\{0,1\}} \circ f_{\{0,1\}}) = (\mathrm{id_I},\, g_{\{0,1\}} \circ f_{\{0,1\}}) = (g_{\{0,1\}} \circ f_{\{0,1\}})$

To show the equivalence $\mathscr{C}^\Gamma([1]^*) \cong \mathscr{C}$ we need to construct a functor that is full, faithful, and essentially surjective on objects. This is not very difficult as we can use the following functor:

$$\begin{aligned} E : \mathscr{C}^\Gamma([1]^*) &\to \mathscr{C} \\ (c, \rho) &\mapsto c(\{0,1\}) \\ f &\mapsto f_{\{0,1\}} \end{aligned}$$

Clearly, for any object $X \in \mathrm{ob}\mathscr{C}$, the object $(c_X,\, \rho_{\mathrm{id}}) \in \mathrm{ob}\mathscr{C}^\Gamma([1]^*)$ is such that $E((c_X,\, \rho_{\mathrm{id}})) = X$ so E is essentially surjective on objects.

Now to show that the map E^*: $\mathrm{Hom}_{\mathscr{C}^\Gamma([1]^*)}((c,\, \rho),\, (c',\, \rho')) \to \mathrm{Hom}_{\mathscr{C}}(E((c,\, \rho)),\, E((c',\, \rho')))$ is injective and surjective for all objects $(c,\, \rho),\, (c',\, \rho') \in \mathrm{ob}(\mathscr{C}^\Gamma([1]^*))$.

Surjectivity is easy. For any morphism $\phi \in \mathrm{Hom}_{\mathscr{C}}(c(\{0,1\}),\, c'(\{0,1\}))$, clearly the morphism $(\phi) \in \mathrm{Hom}_{\mathscr{C}^\Gamma([1]^*)}((c,\, \rho),\, (c',\, \rho'))$ is such that $E((\phi)) = \phi$ as $E((\phi)) = (\phi)_{\{0,1\}} = \phi$. Since the objects were arbitrary, this map is surjective for any objects and our functor is full.

For injectivity we need to show that if $E(f) = E(g)$ then $f = g$. If $E(f) = E(g)$ then $f_{\{0,1\}} = g_{\{0,1\}} = \psi$ let's say, then $f = (\mathrm{id}_I,\, \psi) = g$, thus our map is injective and thus our functor is faithful.

Therefore $\mathscr{C}^\Gamma([1]^*) \cong \mathscr{C}$.

To show that $\mathscr{C}^\Gamma$ is a Γ-category, all that is left to show is that $\mathscr{C}^\Gamma([n]^*) \cong (\mathscr{C}^\Gamma([1]^*))^n \cong \mathscr{C}^n$ for each n.

Objects (c, ρ) of the category $\mathscr{C}^\Gamma([n]^*)$ can be determined up to isomorphism by the images under c of the pointed subsets $\{0,\, 1\}, \ldots, \{0,\, n\} \subseteq [n]^*$. The coherence conditions ensure that, for any two objects $(c,\, \rho),\, (c',\, \rho')$ are such that $c(\{0,\, i\}) = c'(\{0,\, i\})\ \forall\ i \in \{1, \ldots, n\}$, if $P = \{0,\, p_1, \ldots, p_k\} \subseteq [n]^*$, we have:

$$\begin{aligned} c(P) &\cong c(\{0,p_1\}) \otimes \ldots \otimes \ldots c(\{0,p_k\}) \\ &\cong c'(\{0,p_1\}) \otimes \ldots \otimes \ldots c'(\{0,p_k\}) \\ &\cong c'(P) \end{aligned}$$

A morphism $f \in \mathrm{Hom}_{\mathscr{C}^\Gamma([n]^*)}((c,\, \rho),\, (c',\, \rho'))$ is a set $\{f_R\colon\ c(R) \to c'(R) \mid R \subseteq [n]^*\}$, and due to the required coherence conditions, unlike objects, f is uniquely determined by the maps $f_{\{0,1\}}, \ldots, f_{\{0,n\}}$ as for $P = \{0,\, p_1, \ldots, p_k\}$, f_P will have to be the morphism in $\mathscr{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} c(\{0,p_1\}) \otimes \ldots \otimes c(\{0,p_k\}) & \xrightarrow{\cong} & c(P) \\ {\scriptstyle f_{\{0,p_1\}} \otimes \ldots \otimes f_{\{0,p_k\}}} \big\downarrow & \square & \big\downarrow {\scriptstyle f_P} \\ c'(\{0,p_1\}) \otimes \ldots \otimes c'(\{0,p_k\}) & \xrightarrow[\cong]{} & c'(P) \end{array}$$

Let us construct a functor:

$$
\begin{aligned}
E : \mathscr{C}^{\Gamma}([n]^*) &\to \mathscr{C}^n \\
(c, \rho) &\mapsto (c(\{0,1\}), ..., c(\{0,n\})) \\
f &\mapsto (f_{\{0,1\}}, ..., f_{\{0,n\}})
\end{aligned}
$$

It's easy to show that E is full and faithful. Let (c, ρ) and (c', ρ') be any two objects in $\mathscr{C}^{\Gamma}([n]^*)$. We need to show that the map E^*: $\mathrm{Hom}_{\mathscr{C}^{\Gamma}([n]^*)}((c, \rho), (c', \rho')) \to \mathrm{Hom}_{\mathscr{C}^n}(E((c, \rho)), E((c', \rho')))$ is both injective and surjective.

If f, $g \in \mathrm{Hom}_{\mathscr{C}^{\Gamma}([n]^*)}((c, \rho), (c', \rho'))$ are such that $E(f) = E(g)$, then ($f_{\{0,1\}}$, ..., $f_{\{0,n\}}$) = ($g_{\{0,1\}}$, ..., $g_{\{0,n\}}$), therefore $f_{\{0,i\}} = g_{\{0,i\}}$ for all $i \in \{1, ..., n\}$, and since we have seen that a morphism is uniquely determined by exactly these maps, $f = g$ and thus the map E^* is injective and the functor E is faithful.

Surjectivity is even easier, if $(\phi_1, ..., \phi_n) \in \mathrm{Hom}_{\mathscr{C}^n}(E((c, \rho)), E((c', \rho')))$, then $\phi_i \in \mathrm{Hom}_{\mathscr{C}}(c(\{0, i\}), c'(\{0, i\}))$ for each i, and so there exists a morphism $\phi \in \mathrm{Hom}_{\mathscr{C}^{\Gamma}([n]^*)}((c, \rho), (c', \rho'))$ where $\phi_{\{0,i\}} = \phi_i$ for each i and we see that $E(\phi) = (\phi_1, ..., \phi_n)$, thus E^* is surjective and E is full.

Finally we can easily show that E is essentailly surjective on objects. For any object $(X_1, ..., X_n) \in \mathrm{ob}(\mathscr{C}^n)$, let ($c$, ρ) be any object in $\mathscr{C}^{\Gamma}([n]^*)$ where $c(\{0, i\}) = X_i$ for each i, then $E((c, \rho)) = (X_1, ..., X_n)$ and thus E is, better than essentially, exactly surjective on objects. Notice that (c, ρ) is not necessarily unique, this is a good example of an equivalence where the mapping of objects is not injective.

Since E is full, faithful, and essentially surjective on objects, we have an equivalence $\mathscr{C}^{\Gamma}([n]^*) \cong \mathscr{C}^n \cong \mathscr{C}^{\Gamma}([1]^*)^n$.

Finally we need to show that $\mathscr{C}^{\Gamma}$: $\mathbf{\Gamma}^{\mathrm{op}} \to \mathbf{Cat^*}$ is a covariant functor, we already have an appropriate mapping of objects, we just need to construct a functor between appropriate cateogries for each morphism in $\mathbf{\Gamma}^{\mathrm{op}}$.

A morphism in $\mathbf{\Gamma}^{\mathrm{op}}$ is a pointed map between pointed sets f: $[n]^* \to [m]^*$, that is a map of sets with no conditions but for one: that $f(0) = 0$.

Let us define a mapping of objects and morphisms $\mathscr{C}^{\Gamma}(f)$ and then prove that it defines a functor:

$$
\begin{aligned}
\mathscr{C}^{\Gamma}(f) : \mathscr{C}^{\Gamma}([n]^*) &\to \mathscr{C}^{\Gamma}([m]^*) \\
(c, \rho) &\mapsto (\mathscr{C}^{\Gamma}(f)c, \mathscr{C}^{\Gamma}(f)\rho) \\
\phi &\mapsto \mathscr{C}^{\Gamma}(f)\phi
\end{aligned}
$$

Let us define a mapping of power sets:

$$
\begin{aligned}
f^* : \mathcal{P}([m]^*)^* &\to \mathcal{P}([n]^*)^* \\
P &\mapsto \{i \in [n]^* \mid f(i) \in P \backslash \{0\}\} \cup \{0\}
\end{aligned}
$$

By definition, for all pointed subsets $P \subseteq [m]^*$, $0 \in f^*(P) \subseteq [n]^*$ and thus this map is well defined. Then for any pointed sets P, $Q \subseteq [m]^*$ we define:

$$\begin{aligned} \mathscr{C}^{\Gamma}(f)c : \mathcal{P}([m]^*)^* &\to \mathscr{C} \\ P &\mapsto c(f^*(P)) \\ \mathscr{C}^{\Gamma}(f)\rho_{P,Q} &= \rho_{f^*(P),f^*(Q)} \\ \mathscr{C}^{\Gamma}(f)\phi_P &= \phi_{f^*(P)} \end{aligned}$$

It is not too difficult show that if (c, ρ) is an object of $\mathscr{C}^{\Gamma}([n]^*)$ then $(\mathscr{C}^{\Gamma}(f)c, \mathscr{C}^{\Gamma}(f)\rho)$ is an object of $\mathscr{C}^{\Gamma}([m]^*)$.

$f^*(\{0\}) = \{0\}$ so $\mathscr{C}^{\Gamma}(f)c(\{0\}) = c(\{0\}) = I$.

For any pointed subsets P, $Q \subseteq [n]^*$ with $P \cap Q = \{0\}$, $\rho_{P,Q}$ an isomorphism and $\rho_{\{0\},P} = \mathrm{id}_{c(P)}$, so if we can show that for any pointed subsets R, $S \subseteq [m]^*$ with $R \cap S = \{0\}$, that $f^*(R) \cap f^*(S) = \{0\}$, then necessarily, $\mathscr{C}^{\Gamma}(f)\rho_{R,S} = \rho_{f^*(R),f^*(S)}$ is well defined and is is an isomorphism and $\mathscr{C}^{\Gamma}(f)\rho_{\{0\},R} = \rho_{f^*(\{0\}),f^*(R)} = \rho_{\{0\},f^*(R)} = \mathrm{id}_{c(f^*(R))}$.

Let R, $S \subseteq [m]^*$ with $R \cap S = \{0\}$. $f^*(R) = \{i \in [n]^* \mid f(i) \in R\backslash\{0\}\} \cup \{0\}$. Since $R\backslash\{0\} \cap S\backslash\{0\} = \emptyset$, $\{i \in [n]^* \mid f(i) \in R\backslash\{0\}\} \cap \{i \in [n]^* \mid f(i) \in S\backslash\{0\}\} = \emptyset$ since f is a mapping of sets, no $i \in [n]^*$ exists such that $f(i) \in R\backslash\{0\}$ and $f(i) \in S\backslash\{0\}$, thus since we add 0 into both, $f^*(R) \cap f^*(S) = \{0\}$ as required.

Since (c, ρ) is an object of $\mathscr{C}^{\Gamma}([n]^*)$, the coherence diagrams will commute for any pointed subsets of $[n]^*$, so since for any pointed subset $P \subseteq [m]^*$, $f^*(P)$ is a pointed subset of $[n]^*$, for any pointed subsets P, Q, $R \subseteq [m]^*$ with $P \cap Q \cap R = \{0\}$, the following diagrams are required to commute:

$$\begin{array}{ccc} c(f^*(P)) \otimes c(f^*(Q)) & \xrightarrow{\beta_{c(f^*(P)),c(f^*(Q))}} & c(f^*(Q)) \otimes c(f^*(P)) \\ & \searrow^{\rho_{f^*(P),f^*(Q)}} \quad \square \quad \swarrow^{\rho_{f^*(Q),f^*(P)}} & \\ & c(f^*(P) \vee f^*(Q)) & \end{array}$$

$$\begin{array}{ccc} c(f^*(P)) \otimes c(f^*(Q)) \otimes c(f^*(R)) & \xrightarrow{\rho_{f^*(P),f^*(Q)} \otimes \mathrm{id}_{c(f^*(R))}} & c(f^*(P) \vee f^*(Q)) \otimes c(f^*(R)) \\ \downarrow{\scriptstyle \mathrm{id}_{c(f^*(P))} \otimes \rho_{f^*(Q),f^*(R)}} & \square & \downarrow{\scriptstyle \rho_{f^*(P) \vee f^*(Q),f^*(R)}} \\ c(f^*(P)) \otimes c(f^*(Q) \vee f^*(R)) & \xrightarrow[\rho_{f^*(P),f^*(Q) \vee f^*(R)}]{} & c(f^*(P) \vee f^*(Q) \vee f^*(R)) \end{array}$$

Since $\mathscr{C}^{\Gamma}(f)c(P) = c(f^*(P))$ and $\mathscr{C}^{\Gamma}(f)\rho_{P,Q} = \rho_{f^*(P),f^*(Q)}$ for any pointed subsets P, $Q \subseteq [m]^*$ with $P \cap Q = \{0\}$, these are almost exactly the diagrams we need to commute, once we show that $\mathscr{C}^{\Gamma}(f)c(P \vee Q) = c(f^*(P) \vee f^*(Q))$ for any pointed subsets P, $Q \subseteq [m]^*$ with $P \cap Q = \{0\}$ we will be done.

$\mathscr{C}^\Gamma(f)c(P \vee Q) = c(f^*(P \vee Q))$. Since $P \cap Q = \{0\}$,

$$\begin{aligned} P \vee Q &= P\backslash\{0\} \cup Q\backslash\{0\} \cup \{0\} \\ \implies f^*(P \vee Q) &= f^*(P\backslash\{0\} \cup Q\backslash\{0\} \cup \{0\}) \\ &= \{i \in [n]^* \mid f(i) \in P\backslash\{0\} \cup Q\backslash\{0\}\} \cup \{0\} \\ &= \{i \in [n]^* \mid f(i) \in P\backslash\{0\}\} \cup \{i \in [n]^* \mid f(i) \in Q\backslash\{0\}\} \cup \{0\} \\ &= f^*(P) \vee f^*(Q) \end{aligned}$$

since $P \cap Q = \{0\} \implies f^*(P) \cap f^*(Q) = \{0\}$.

Therefore $(\mathscr{C}^\Gamma(f)c, \mathscr{C}^\Gamma(f)\rho)$ is indeed an object in $\mathscr{C}^\Gamma([m]^*)$.

We need to also check that we have a valid mapping of morphisms. If ϕ: $(c, \rho) \to (c', \rho')$ is a morphism in $\mathscr{C}^\Gamma([n]^*)$ we need to show that $\mathscr{C}^\Gamma(f)\phi$: $(\mathscr{C}^\Gamma(f)c, \mathscr{C}^\Gamma(f)\rho) \to (\mathscr{C}^\Gamma(f)c', \mathscr{C}^\Gamma(f)\rho')$ is a morphism in $\mathscr{C}^\Gamma([m]^*)$ for any morphism f: $[n]^* \to [m]^*$ in $\mathbf{\Gamma}^{\mathrm{op}}$.

$\phi = \{\phi_P\colon c(P) \to c'(P) \mid P \subseteq [n]^*\}$ such that $\phi_{\{0\}} = \mathrm{id}_I$ and for pointed subsets P, $Q \subseteq [n]^*$ with $P \cap Q = \{0\}$:

$$\begin{array}{ccc} c(P) \otimes c(Q) & \xrightarrow{\rho_{P,Q}} & c(P \vee Q) \\ {\scriptstyle \phi_P \otimes \phi_Q}\downarrow & \square & \downarrow{\scriptstyle \phi_{P \vee Q}} \\ c'(P) \otimes c'(Q) & \xrightarrow[\rho'_{P,Q}]{} & c'(P \vee Q) \end{array}$$

$\mathscr{C}^\Gamma(f)\phi_P = \phi_{f^*(P)}$ so $\mathscr{C}^\Gamma(f)\phi_P\colon c(f^*(P)) \to c'(f^*(P))$ and since $\mathscr{C}^\Gamma(f)c(P)$ is defined to be $c(f^*(P))$, therefore $\mathscr{C}^\Gamma(f)\phi_P\colon \mathscr{C}^\Gamma(f)c(P) \to \mathscr{C}^\Gamma(f)c'(P)$ as required and we can see that the following diagram commutes:

$$\begin{array}{ccc} c(f^*(P)) \otimes c(f^*(Q)) & \xrightarrow{\rho_{f^*(P),f^*(Q)}} & c(f^*(P) \vee f^*(Q)) \\ {\scriptstyle \phi_{f^*(P)} \otimes \phi_{f^*(Q)}}\downarrow & \square & \downarrow{\scriptstyle \phi_{f^*(P) \vee f^*(Q)}} \\ c'(f^*(P)) \otimes c'(f^*(Q)) & \xrightarrow[\rho'_{f^*(P),f^*(Q)}]{} & c'(f^*(P) \vee f^*(Q)) \end{array}$$

We have already checked that $f^*(P) \vee f^*(Q) = f^*(P \vee Q)$ so the diagram we require does in fact commute. Since we have also checked that $f^*(\{0\}) = \{0\}$, $\mathscr{C}^\Gamma(f)\phi_{\{0\}} = \phi_{\{0\}} = \mathrm{id}_I$, $\mathscr{C}^\Gamma(f)\phi$ is indeed a valid functor in $\mathscr{C}^\Gamma([m]^*)$.

The identity morphisms of $\mathscr{C}^\Gamma([n]^*)$ are the morphisms $\mathrm{id}_{(c,\rho)}$ where $\mathrm{id}_{(c,\rho)P} = \mathrm{id}_{c(P)}$. $\mathscr{C}^\Gamma(f)\mathrm{id}_{(c,\rho),P} = \mathrm{id}_{(c,\rho),f^*(P)} = \mathrm{id}_{c(f^*(P))} = \mathrm{id}_{\mathscr{C}^\Gamma(f)c(P)}$ for any pointed subset $P \subseteq [m]^*$ therefore $\mathscr{C}^\Gamma(f)$ preserves identities.

Finally we must show that if $\phi \in \mathrm{Hom}_{\mathscr{C}^\Gamma([n]^*)}((c_1, \rho_1), (c_2, \rho_2))$ and $\psi \in \mathrm{Hom}_{\mathscr{C}^\Gamma([n]^*)}((c_2, \rho_2), (c_3, \rho_3))$, then $\mathscr{C}^\Gamma(f)(\psi \circ \phi) = \mathscr{C}^\Gamma(f)\psi \circ \mathscr{C}^\Gamma(f)\phi$. For every pointed subset $P \subseteq [n]^*$ and $Q \subseteq [m]^*$

$$\begin{aligned} (\psi \circ \phi)_P &= \psi_P \circ \phi_P \\ \mathscr{C}^\Gamma(f)(\psi \circ \phi)_Q &= (\psi \circ \phi)_{f^*(Q)} \\ &= \psi_{f^*(Q)} \circ \phi_{f^*(Q)} \\ &= \mathscr{C}^\Gamma(f)\psi_{f^*(Q)} \circ \mathscr{C}^\Gamma(f)\phi_{f^*(Q)} \end{aligned}$$

Therefore $\mathscr{C}^\Gamma(f)$ preserves composition, and is thus a functor as expected.

To ensure $\mathscr{C}^\Gamma$ is a functor, we need to check that the mapping $\mathscr{C}^\Gamma\colon \mathbf{\Gamma}^{\text{op}} \to$ **Cat*** also preserves identities and composition.

The morphism $\text{id}_{[n]^*}\colon [n]^* \to [n]^*$ sends each element of $[n]^*$ to itself, we need to show that $\mathscr{C}^\Gamma(\text{id}_{[n]^*})$ is the identity functor on the category $\mathscr{C}^\Gamma([n]^*)$.

To do so we need to show that for any object (c, ρ) and any morphism ϕ in $\mathscr{C}^\Gamma([n]^*)$, $\mathscr{C}^\Gamma(\text{id}_{[n]^*})(c, \rho) = (c, \rho)$ and $\mathscr{C}^\Gamma(\text{id}_{[n]^*})\phi = \phi$, most of the work is done already, we simply need to show that for any pointed subset $P \subseteq [n]^*$, $\text{id}_{[n]^*}{}^*(P) = P$.

$$\begin{aligned}\text{id}^*_{[n]^*}(P) &= \{i \in [n]^* \mid \text{id}_{[n]^*}(i) \in P\backslash\{0\}\} \cup \{0\}\\ &= \{i \in [n]^* \mid i \in P\backslash\{0\}\} \cup \{0\}\\ &= P\backslash\{0\} \cup \{0\} = P\end{aligned}$$

Therefore $\mathscr{C}^\Gamma$ preserves identities.

Finally, if $f\colon [m]^* \to [n]^*$ and $g\colon [n]^* \to [y]^*$ we want to show the functors $\mathscr{C}^\Gamma(g \circ f)$ and $\mathscr{C}^\Gamma(g) \circ \mathscr{C}^\Gamma(f)$ are equal. Therefore we need to show that for any object (c, ρ) and any morphism ϕ in $\mathscr{C}^\Gamma([m]^*)$, $\mathscr{C}^\Gamma(g \circ f)(c, \rho) = (\mathscr{C}^\Gamma(g) \circ \mathscr{C}^\Gamma(f))(c, \rho)$ and $\mathscr{C}^\Gamma(g \circ f)\phi = (\mathscr{C}^\Gamma(g) \circ \mathscr{C}^\Gamma(f))\phi$. Similarly to with the identities, this boils down to checking that for any pointed subset $P \subseteq [y]^*$, $(g \circ f)^*(P) = (f^* \circ g^*)(P)$.

$$\begin{aligned}(g \circ f)^*(P) &= \{i \in [m]^* \mid (g \circ f)(i) \in P\backslash\{0\}\} \cup \{0\}\\ &= \{i \in [m]^* \mid g(f(i)) \in P\backslash\{0\}\} \cup \{0\}\\ (f^* \circ g^*)(P) &= f^*(g^*(P))\\ &= f^*(\{i \in [n]^* \mid g(i) \in P\backslash\{0\}\} \cup \{0\})\\ &= \{j \in [m]^* \mid f(j) \in \{i \in [n]^* \mid g(i) \in P\backslash\{0\}\}\} \cup \{0\}\\ &= \{i \in [m]^* \mid g(f(i)) \in P\backslash\{0\}\} \cup \{0\}\end{aligned}$$

Therefore $\mathscr{C}^\Gamma$ is a covariant functor and therefore we have checked everything to ensure that $\mathscr{C}^\Gamma$ is indeed a Γ-category.

We want to check that $|N(\mathscr{C}^\Gamma)|$ is a Γ-space. Since the nerve and geometric realisation are both functors, constructing a Γ-space from a Γ-category is easy simply by passing each cateogry through these two functors.

We need to show that:

$$\begin{aligned}|N(\mathscr{C}^\Gamma)|([0]^*) &\simeq \Delta^0\\ \text{and}\quad |N(\mathscr{C}^\Gamma)|([n]^*) &\simeq (|N(\mathscr{C}^\Gamma)|([1]^*))^n\end{aligned}$$

The equivalences we have shown make short work of this:

$$\begin{aligned}|N(\mathscr{C}^\Gamma)|([0]^*) &= |N(\mathscr{C}^\Gamma([0]^*))| \\ &\simeq |N(\mathbf{[0]})| = \Delta^0 \\ |N(\mathscr{C}^\Gamma)|([n]^*) &= |N(\mathscr{C}^\Gamma([n]^*))| \\ &\simeq |N(\mathscr{C}^n)| \\ &\simeq |N((\mathscr{C}^\Gamma([1]^*))^n)| \\ &\simeq |N(\mathscr{C}^\Gamma([1]^*))|^n\end{aligned}$$

△

8.2 Cohomology Theories Constructed from Strict Symmetric Monoidal Categories

We have seen that the categories $\mathscr{C}_\oplus$ and $\mathscr{C}_\otimes$ are strict symmetric monoidal categories. We shall now determine the nature of the cohomology theories one may construct from them.

Let us denote by $h^*_\oplus := h^*_{\mathscr{C}_\oplus}$ the reduced cohomology theory constructed from $\mathscr{C}_\oplus$ and $h^*_\otimes := h^*_{\mathscr{C}_\otimes}$ the reduced cohomology theory constructed from $\mathscr{C}_\otimes$.

$$h^k_\oplus(X) := \begin{cases}[X, \Omega^{(-k+1)}B|N(\mathscr{C}_\oplus)|], & k \leq 0 \\ [X, B^{(k)}|N(\mathscr{C}_\oplus)|], & k > 0\end{cases}$$

Since $B|N\mathscr{C}^*_\oplus| \simeq B(B\mathrm{U} \times \mathbb{Z})$, $h^*_\oplus$ is very similar to K-theory

$$K^n(X) = \begin{cases}[X, B\mathrm{U} \times \mathbb{Z}], & n \in 2\mathbb{Z} \\ [X, \mathrm{U}], & n \in 1 + 2\mathbb{Z}\end{cases}$$

Since $\Omega(B\mathrm{U} \times \mathbb{Z}) = \Omega(B\mathrm{U}) \simeq \mathrm{U}$, $\Omega^{(2)}(B\mathrm{U}) \simeq B\mathrm{U} \times \mathbb{Z}$, and $\Omega^{(2)}(\mathrm{U}) \simeq \mathrm{U}$ by Bott periodicity, we have:

$$h^k_\oplus(X) := \begin{cases}\tilde{K}^k(X), & k \leq 0 \\ [X, B^{(k)}(B\mathrm{U} \times \mathbb{Z})], & k > 0\end{cases}$$

We must also investigate the nature of $B^{(k)}(B\mathrm{U} \times \mathbb{Z})$.

One way is to notice the effects of the classifying space construction on homotopy groups.

From Bott periodicity and the fact that $B\mathrm{U}$ is a connected space, we know that:

$$\pi_i(B\mathrm{U} \times \mathbb{Z}) = \begin{cases}\mathbb{Z}, & i \in 2\mathbb{Z}, i \geq 0 \\ 0, & else\end{cases}$$

$$\pi_i(\mathrm{U}) = \begin{cases}\mathbb{Z}, & i \in 2\mathbb{Z} + 1, i \geq 0 \\ 0, & else\end{cases}$$

For a topological group X it is easy to see that:

$$\begin{aligned}\pi_i(BX) &= [\mathbb{S}^i, BX] \\ &= [\mathbb{S}^{i-1}, \Omega BX] \\ &= [\mathbb{S}^{i-1}, X] \\ &= \pi_{i-1}(X)\end{aligned}$$

and thus we see that:

$$\pi_i(B^{(k)}(B\mathrm{U}\times\mathbb{Z})) = \begin{cases}\mathbb{Z}, & i \in 2\mathbb{Z}+k, i \geq k \\ 0, & else\end{cases}$$

In particular, this shows us that $B(B\mathrm{U}\times\mathbb{Z}) \simeq \mathrm{U}$, so additionally and critically $h^1_\oplus(X) = K^1(X)$.

Since the spaces in this spectrum are the same as in K-theory but for increasingly many trivial homotopy groups, this cohomology theory is often called connective K-theory and is sometimes denoted cK^* :

$$cK^k(X) := h^k_\oplus(X) := \begin{cases}\tilde{K}^k(X), & k \leq 1 \\ [X, B^{(k-1)}\mathrm{U}], & k > 1\end{cases}$$

The cohomology theory $h^*_\otimes$ is much more exotic. For starters it is entirely dependent on the integer d so really we have a family of related categories and cohomology theories with context informing us which incarnation of $\mathscr{C}_\otimes$ we are dealing with. In all cases, by our definition, we have:

$$h^k_\otimes(X) := \begin{cases}[X, \Omega^{(-k+1)}B|N(\mathscr{C}_\otimes)|], & k \leq 0 \\ [X, B^{(k)}|N(\mathscr{C}_\otimes)|], & k > 0\end{cases}$$

We can at least examine the case when $d = 1$. In the category $\mathscr{C}_\otimes$ there are no morphisms between different objects and otherwise we see that we have $\mathrm{Hom}_{\mathscr{C}_\otimes}(i,\, i) = \mathrm{U}((\mathbb{C}^d)^{\otimes i}) = \mathrm{U}((\mathbb{C})^{\otimes i}) = \mathrm{U}(1)$ in the case $d = 1$.

Lemma 53. Let $\mathscr{N}$ be the category with a set of objects $\mathrm{ob}(\mathscr{N}) = \mathbb{N}$ and just the identity morphisms from each object to itself, and let $\mathscr{U}(1)$ be the category with one object and morphisms $\mathrm{Hom}_{\mathscr{U}(1)}(*,\, *) = \mathrm{U}(1)$ as a topological group.

Then, when $d = 1$, $\mathscr{C}_\otimes$ is equivalent to the product category $\mathscr{N} \times \mathscr{U}(1)$ as strict symmetric monoidal categories.

Proof. First we need to describe the monoidal structure on $\mathscr{N} \times \mathscr{U}(1)$.

$$\begin{aligned}\otimes : (\mathscr{N}\times\mathscr{U}(1)) \times (\mathscr{N}\times\mathscr{U}(1)) &\to \mathscr{N}\times\mathscr{U}(1) \\ ((n,*),(m,*)) &\mapsto (n+m,*) \\ ((\mathrm{id}_n,u),(\mathrm{id}_m,v)) &\mapsto (\mathrm{id}_{n+m},uv)\end{aligned}$$

The unit object is $(0, *)$, the unitors are identities since 0 is the additive identity of the abelian monoid $\mathbb{N}$, and the associator is an identity since $\mathrm{U}(1)$ is a group.

We can also introduce a braiding that must be symmetric since U(1) is an abelian group.

To show we have an equivalence of cateogries, we must describe a functor F: $\mathscr{N} \times \mathscr{U}(1) \to \mathscr{C}_\otimes$ that is full, faithful, and essentially surjective on objects. Let us consider the following functor and then show that it satisfies these properties:

$$\begin{aligned} F : \mathscr{N} \times \mathscr{U}(1) &\to \mathscr{C}_\otimes \\ (n, *) &\mapsto n \\ (\mathrm{id}_n, u) &\mapsto u \in \mathrm{Hom}_{\mathscr{C}_\otimes}(n, n) \end{aligned}$$

More than just essentially surjective on objects, F is a bijection of objects! For each object $n \in \mathrm{ob}\mathscr{C}_\otimes$ we can see that the object $(n, *) \in \mathrm{ob}(\mathscr{N} \times \mathscr{U}(1))$ is such that $F((n, *)) = n$, an equality instead of merely an isomorphism.

Now we need to examine the morphisms of the category $\mathscr{N} \times \mathscr{U}(1)$. Since there are no morphisms between different objects in $\mathscr{N}$ we see that:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{N} \times \mathscr{U}(1)}((n, *), (m, *)) &= \mathrm{Hom}_{\mathscr{N}}(n, m) \times \mathrm{Hom}_{\mathscr{U}(1)}(*, *) \\ &= \begin{cases} \{(\mathrm{id}_n, u) \mid u \in \mathrm{U}(1)\}, & n = m \\ \emptyset, & n \neq m \end{cases} \end{aligned}$$

To show that F is both full and faithful, we need to show that the following map is both injective and surjective for all pairs of of objects $(n, *)$, $(m, *)$:

$$\begin{aligned} F : \mathrm{Hom}_{\mathscr{N} \times \mathscr{U}(1)}((n, *), (m, *)) &\to \mathrm{Hom}_{\mathscr{C}_\otimes}(F((n, *)), F((m, *))) \\ \phi &\mapsto F(\phi) \end{aligned}$$

From our morphism investigation, it is clear that we need to check two cases: when the two objects are the same, and when the two objects are different.

Let's deal with the case where the objects are different first. Let $n \neq m$, then:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{N} \times \mathscr{U}(1)}((n, *), (m, *)) &\cong \emptyset \\ \mathrm{Hom}_{\mathscr{C}_\otimes}(F((n, *)), F((m, *))) &\cong \mathrm{Hom}_{\mathscr{C}_\otimes}(n, m) \\ &\cong \emptyset \end{aligned}$$

and since any map from the empty set to itself is necessarily a bijection, our map certainly is in this case.

Now for the case when the objects are the same:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{N} \times \mathscr{U}(1)}((n, *), (n, *)) &\cong \mathrm{U}(1) \\ \mathrm{Hom}_{\mathscr{C}_\otimes}(F((n, *)), F((n, *))) &\cong \mathrm{Hom}_{\mathscr{C}_\otimes}(n, n) \\ &\cong \mathrm{U}(1) \\ F : \mathrm{Hom}_{\mathscr{N} \times \mathscr{U}(1)}((n, *), (n, *)) &\to \mathrm{Hom}_{\mathscr{C}_\otimes}(n, n) \\ (\mathrm{id}_n, u) &\mapsto u \end{aligned}$$

Now we will show that this map is injective and surjective.

We need to show that if $F((\mathrm{id}_n,\ u)) = F((\mathrm{id}_n,\ v))$ then $(\mathrm{id}_n,\ u) = (\mathrm{id}_n,\ v)$.

$$\begin{aligned} F((\mathrm{id}_n, u)) &= F((\mathrm{id}_n, v)) \\ \implies u &= v \\ \implies (\mathrm{id}_n, u) &= (\mathrm{id}_n, v) \end{aligned}$$

Thus the map F is injective in this case and thus the functor F is faithful.

Finally we need to show that for all morphisms $u \in \mathrm{Hom}_{\mathscr{C}_\otimes}(n,\ n)$ there is a morphism $\phi \in \mathrm{Hom}_{\mathscr{N}\ \times\ \mathscr{U}(1)}((n,\ *),\ (n,\ *))$ such that $F(\phi) = u$. Clearly $F((\mathrm{id}_n,\ u)) = u$ and thus the map F is surjective in this case and thus the functor F is full.

Since the functor F is full, faithful, and essentially surjective on objects, $\mathscr{C}_\otimes$ and $\mathscr{N}\ \times\ \mathscr{U}(1)$ are equivalent as categories.

To show that the categories are equivalent as strict symmetric monoidal categories we need to show that the monoidal product and the braiding both commute with the functor F. Let's look at the monoidal product first and show that the following diagram commutes in **Cat**:

$$\begin{array}{ccc} (\mathscr{N} \times \mathscr{U}(1)) \times (\mathscr{N} \times \mathscr{U}(1)) & \xrightarrow{\otimes} & \mathscr{N} \times \mathscr{U}(1) \\ {\scriptstyle F\times F}\downarrow & \square & \downarrow{\scriptstyle F} \\ \mathscr{C}_\otimes \times \mathscr{C}_\otimes & \xrightarrow[\otimes]{} & \mathscr{C}_\otimes \end{array}$$

We will consider two general objects $(n,\ *)$, $(m,\ *)$ and two general morphisms $(\mathrm{id}_n,\ u)$, $(\mathrm{id}_m,\ v)$ in the category $\mathscr{N}\ \times\ \mathscr{U}(1)$, now to investigate the commutativity of the diagram:

$$\begin{aligned} F(\otimes((n,*),(m,*))) &= F((n,*) \otimes (m,*)) \\ &= F((n+m,*)) \\ &= n+m \\ \otimes((F\times F)((n,*),(m,*))) &= \otimes(F((n,*)),F((m,*))) \\ &= n \otimes m \\ &= n+m \\ F(\otimes((\mathrm{id}_n,u),(\mathrm{id}_m,v))) &= F((\mathrm{id}_n,u) \otimes (\mathrm{id}_m,v)) \\ &= F((\mathrm{id}_{n+m},uv)) \\ &= uv \in \mathrm{Hom}_{\mathscr{C}_\otimes}(n+m,n+m) \\ \otimes((F\times F)((\mathrm{id}_n,u),(\mathrm{id}_m,v))) &= \otimes(F((\mathrm{id}_n,u)),F((\mathrm{id}_m,v))) \\ &= (u \in \mathrm{Hom}_{\mathscr{C}_\otimes}(n,n)) \otimes (v \in \mathrm{Hom}_{\mathscr{C}_\otimes}(m,m)) \\ &= uv \in \mathrm{Hom}_{\mathscr{C}_\otimes}(n+m,n+m) \end{aligned}$$

Thus we have shown the commutativity on objects and morphisms and thus the two categories are equivalent as monoidal categories.

To show that the braiding is also preserved, we need to show that for any two objects $(n, *)$, $(m, *) \in \text{ob}(\mathscr{N} \times \mathscr{U}(1))$ the following diagram commutes in $\mathscr{C}_\otimes$:

$$\begin{array}{ccc} F((n,*)\otimes(m,*)) & \xrightarrow{\cong} & F((n,*))\otimes F((m,*)) \\ {\scriptstyle F(\beta_{(n,*),(m,*)})}\big\downarrow & \square & \big\downarrow{\scriptstyle \beta_{F((n,*)),F((m,*))}} \\ F((m,*)\otimes(n,*)) & \xrightarrow[\cong]{} & F((m,*))\otimes F((n,*)) \end{array}$$

That is, since all four of these objects are the object $n + m$ in $\mathscr{C}_\otimes$, we want to show that as morphisms in $\text{Hom}_{\mathscr{C}_\otimes}(n + m,\ n + m)$, $F(\beta_{(n,*),(m,*)})$ and $\beta_{F((n,*)),F((m,*))}$ are equal. Let $(\text{id}_n,\ u)$ and $(\text{id}_m,\ v)$ be two morphisms in $\mathscr{N} \times \mathscr{U}(1)$:

$$\begin{aligned} F(\beta_{(n,*),(m,*)})(F((\text{id}_n,u)\otimes(\text{id}_m,v))) &= F((\text{id}_m,v)\otimes(\text{id}_n,u)) \\ &= F((\text{id}_{m+n},vu)) \\ &= vu \in \text{Hom}_{\mathscr{C}_\otimes}(m+n,m+n) \\ \beta_{F((n,*)),F((m,*))}(F((\text{id}_n,u)\otimes(\text{id}_m,v))) &= \beta_{n,m}(F((\text{id}_n,u))\otimes F((\text{id}_m,v))) \\ &= \beta_{n,m}((u\in\text{Hom}_{\mathscr{C}_\otimes}(n,n))\otimes(v\in\text{Hom}_{\mathscr{C}_\otimes}(m,m))) \\ &= (v\in\text{Hom}_{\mathscr{C}_\otimes}(m,m))\otimes(u\in\text{Hom}_{\mathscr{C}_\otimes}(n,n)) \\ &= vu \in \text{Hom}_{\mathscr{C}_\otimes}(m+n,m+n) \end{aligned}$$

Thus the braiding is preserved and we achieve the result that when $d = 1$, $\mathscr{C}_\otimes$ and $\mathscr{N} \times \mathscr{U}(1)$ are equivalent as strict symmetric monoidal categories. △

This equivalence allows us to see that:

$$\begin{aligned} |N(\mathscr{C}_\otimes)| &= |N(\mathscr{N}\times\mathscr{U}(1))| \\ &= |N(\mathscr{N})| \times |N(\mathscr{U}(1)| \\ &= \mathbb{N}\times B\text{U}(1) \\ \implies \quad \Omega B|N(\mathscr{C}_\otimes)| &\simeq B\text{U}(1)\times\mathbb{Z} \end{aligned}$$

Definition 42. An **Eilenberg-MacLane space**, introduced by Eilenberg and MacLane [9], is a topological space $K(G\ i)$ for a given discrete group G and integer $i \in \mathbb{Z}$, such that for any topological space X,

$$[X,\ K(G,\ i)] \cong H^i(X;\ G)$$

Lemma 54. If A is an abelian group equipped with the discrete topology, then clearly $A \simeq K(A,\ 0)$, the Eilenberg-MacLane space in degree 0 with A coefficients.

Additionally, and not trivially, the classifying spaces B^nA all exist, are abelian monoids, and $B^nA \simeq K(A,\ n)$.

Using this Eilenberg-MacLane space lemma and how well the classifying space construction works with products ensures that the strict symmetric monoidal category $\mathscr{C}_\otimes$ when $d = 1$ generates a surprisingly well behaved cohomology theory:

$$h^k_\otimes(X) = \tilde{H}^k(X;\mathbb{Z}) \times \tilde{H}^{k+2}(X;\mathbb{Z})$$

8.3 Functors between Strict Symmetric Monoidal Categories

There are many ways to define functors between strict symmetric monoidal categories but not all of them will induce natural transformations of cohomology theories. Following Baez's definition [3]

Definition 43. If $\mathscr{C}$ and $\mathscr{D}$ are strict symmetric monoidal categories and a functor $F\colon \mathscr{C} \to \mathscr{D}$ together with a natural transformation $\phi\colon \otimes_{\mathscr{D}} \circ (F, F) \to F \circ \otimes_{\mathscr{C}}$ and a morphism $\psi\colon I_{\mathscr{D}} \to F(I_{\mathscr{C}})$ is a monoidal functor, then we call F together with ϕ and ψ a **symmetric monoidal functor** if for all objects A, $B \in \mathrm{ob}(\mathscr{C})$ the following diagram commutes:

$$\begin{array}{ccc} FA \circ_{\mathscr{D}} FB & \xrightarrow{\phi_{A,B}} & F(A \circ_{\mathscr{C}} B) \\ {\scriptstyle \beta^{\mathscr{D}}_{FA,FB}}\big\downarrow & \square & \big\downarrow{\scriptstyle F(\beta^{\mathscr{C}}_{A,B})} \\ FB \circ_{\mathscr{D}} FA & \xrightarrow[\phi_{B,A}]{} & F(B \circ_{\mathscr{C}} A) \end{array}$$

A symmetric monoidal functor is called a **strong symmetric monoidal functor** if the components of ϕ and ψ are all isomorphisms.

Theorem 55. A monoidal functor induces a natural transformation of cohomology theories $\tau_F\colon h^*_{\mathscr{C}} \to h^*_{\mathscr{D}}$ if it is a strong symmetric monoidal functor.

Proof. Let $F\colon \mathscr{C} \to \mathscr{D}$ together with a natural isomorphism $\phi\colon \otimes_{\mathscr{D}} \circ (F, F) \to F \circ \otimes_{\mathscr{C}}$ and an isomorphism $\psi\colon I_{\mathscr{D}} \to F(I_{\mathscr{C}})$ be a symmetric monoidal functor. To show that a natural transformation of cohomology theories is induced we must first show that this data induces a morphism in the category of Γ-cateogries.

We would like to construct a sequence of functors $F^*_n\colon \mathscr{C}^\Gamma([n]^*) \to \mathscr{D}^\Gamma([n]^*)$ that are induced by our strong symmetric monoidal functor.

To construct the functors F^*_0 and F^*_1 we don't even need our strong monoidal functor to be symmetric.

Since $\mathscr{C}^\Gamma([0]^*)$ and $\mathscr{D}^\Gamma([0]^*)$ are both categories with a single object consisting of a map that sends the pointed subset of $[0]^*$ to their respective unit objects and a natural transformation whose only component is the identity on their respective unit objects, and a single morphism which also consists of only a single morphism that is the identity on their respective unit object, if (c, ρ) is our object in $\mathscr{C}^\Gamma([0]^*)$, we need $F^*_0((c, \rho))$ to be our object and $F^*_0(\mathrm{id}_{(c,\, \rho)})$ to be our morphism in $\mathscr{D}^\Gamma([0]^*)$. Let us construct our functor:

$$\begin{aligned} F^*_0 : \mathscr{C}^\Gamma([0]^*) &\to \mathscr{D}^\Gamma([0]^*) \\ (c, \rho) &\mapsto ((\psi^{-1} \circ F)(c), (\psi^{-1} \circ F)(\rho)) \\ f &\mapsto (\psi^{-1} \circ F)(f) \end{aligned}$$

$(\psi^{-1} \circ \text{-})\colon \mathscr{D} \to \mathscr{D}$ here denotes a partially defined mapping of categories where:

$$\begin{aligned}(\psi^{-1} \circ -) : \mathscr{D} &\to \mathscr{D} \\ X &\mapsto \psi^{-1}(X) \\ f &\mapsto \psi^{-1} \circ f \circ \psi\end{aligned}$$

This mapping is only defined on the object $F(I_{\mathscr{C}})$ in $\mathscr{D}$ and $\psi^{-1}(F(I_{\mathscr{C}}))$. Since ψ is an isomorphism, the mapping of morphisms is defined as the morphism in $\mathrm{Hom}_{\mathscr{D}}(I_{\mathscr{D}}, I_{\mathscr{D}})$ that ensures the following diagram commutes:

$$\begin{array}{ccc} F(I_{\mathscr{C}}) & \xrightarrow{\psi^{-1}} & I_{\mathscr{D}} \\ {\scriptstyle f}\big\downarrow & \square & \big\downarrow{\scriptstyle (\psi^{-1}\circ -)(f)} \\ F(I_{\mathscr{C}}) & \xrightarrow[\psi^{-1}]{} & I_{\mathscr{D}} \end{array}$$

Since $c(\{0\}) = I_{\mathscr{C}}$, $\rho_{\{0\},\{0\}} = \mathrm{id}_{I_{\mathscr{C}}}$, and $f_{\{0\}} = \mathrm{id}_{I_{\mathscr{C}}}$, we see that, when π_c and π_ρ are projections onto the first and second entry of an object respectively:

$$\begin{aligned} F_0^*((c,\rho)) &= ((\psi^{-1} \circ F)(c), (\psi^{-1} \circ F)(\rho)) \\ \implies \pi_c(F_0^*((c,\rho)))(\{0\}) &= (\psi^{-1} \circ F)(c(\{0\})) \\ &= (\psi^{-1} \circ F)(I_{\mathscr{C}}) \\ &= (\psi^{-1} \circ -)(F(I_{\mathscr{C}})) \\ &= I_{\mathscr{D}}, \pi_\rho(F_0^*((c,\rho)))_{\{0\},\{0\}} \qquad = (\psi^{-1} \circ F)(\rho_{\{0\},\{0\}}) \\ &= (\psi^{-1} \circ F)(\mathrm{id}_{I_{\mathscr{C}}}) \\ &= (\psi^{-1} \circ -)(\mathrm{id}_{F(I_{\mathscr{C}})}) \\ &= \psi^{-1} \circ \mathrm{id}_{F(I_{\mathscr{C}})} \circ \psi \\ &= \psi^{-1} \circ \psi = \mathrm{id}_{I_{\mathscr{D}}} \\ \text{and } F_0^*(f) &= (\psi^{-1} \circ F)(f) \\ \implies (\psi^{-1} \circ F)(f)_{\{0\}} &= (\psi^{-1} \circ F)(f_{\{0\}}) \\ &= (\psi^{-1} \circ F)(\mathrm{id}_{I_{\mathscr{C}}}) \\ &= \mathrm{id}_{I_{\mathscr{D}}} \text{ as above} \end{aligned}$$

Thus $F_0^*\colon \mathscr{C}^\Gamma([0]^*) \to \mathscr{D}^\Gamma([0]^*)$ is indeed a functor as the identity and composition conditions automatically hold since these are cateogries with only one object and one morphism.

On the next level our cateogries are no longer as simple as a single object and a single morphism. We have seen that all objects of $\mathscr{C}^\Gamma([1]^*)$ and $\mathscr{D}^\Gamma([1]^*)$ consist of pairs (c, ρ) where $c(\{0\})$ is required to be the relevant unit object and all the components of ρ are required to be the relevant identities; to construct a functor $F_1^*\colon \mathscr{C}^\Gamma([1]^*) \to \mathscr{D}^\Gamma([1]^*)$ induced by our strong monoidal functor, we will have to check that all of this is maintained.

$$\begin{aligned} F_1^* : \mathscr{C}^\Gamma([1]^*) &\to \mathscr{D}^\Gamma([1]^*) \\ (c,\rho) &\mapsto (F_1^* c, F_1^* \rho) \\ f &\mapsto F_1^* f \end{aligned}$$

We need to describe these mappings only in terms of our strong monoidal functor so let us do exactly that, let P, Q be pointed subsets of $[1]^*$ with $P \cap Q = \{0\}$:

$$\begin{aligned} F_1^* c : \mathcal{P}([1]^*)^* &\to \mathrm{ob}\mathscr{D} \\ P &\mapsto \begin{cases} (\psi^{-1} \circ F)(c(P)), & P = \{0\} \\ F(c(P)), & else \end{cases} \\ F_1^* \rho_{P,Q} &= \begin{cases} (\psi^{-1} \circ F)(\rho_{P,Q}), & P = Q = \{0\} \\ F(\rho_{P,Q}), & else \end{cases} \\ F_1^* f_P &= \begin{cases} (\psi^{-1} \circ F)(f_P), & P = \{0\} \\ F(f_P), & else \end{cases} \end{aligned}$$

We have already seen that $(\psi^{-1} \circ F)(I_{\mathscr{C}}) = I_{\mathscr{D}}$ and $(\psi^{-1} \circ F)(\mathrm{id}_{I_{\mathscr{C}}}) = \mathrm{id}_{I_{\mathscr{D}}}$ and we can also easily see that $F_1^*(\mathrm{id}_{c(\{0,1\})} = F(\mathrm{id}_{c(\{0,1\})}) = \mathrm{id}_{F(c(\{0,1\}))}$ since F is a functor, therefore for any object (c, ρ) of $\mathscr{C}^\Gamma([1]^*)$, $F_1^*((c, \rho))$ is an object in $\mathscr{D}^\Gamma([1]^*)$.

These facts also show that $F_1^*(\mathrm{id}_{(c,\ \rho)}) = \mathrm{id}_{F_1^*((c,\ \rho))}$ and together with the fact that F is a functor ensures that composition of morphisms is preserved, we can say that F_1^*: $\mathscr{C}^\Gamma([1]^*) \to \mathscr{D}^\Gamma([1]^*)$ is indeed a functor that is induced by our strong monoidal functor.

On higher levels, again we have more to concern ourselves with. For objects $(c,\ \rho)$ of a category $\mathscr{C}^\Gamma([n]^*)$ for $n \geq 2$, it will not be true in general that all the components of ρ are the relevant identities. We would like the following to be a functor:

$$\begin{aligned} F_n^* : \mathscr{C}^\Gamma([n]^*) &\to \mathscr{D}^\Gamma([n]^*) \\ (c,\rho) &\mapsto (F_n^* c, F_n^* \rho) \\ f &\mapsto F_n^* f \end{aligned}$$

We will describe these mappings and then show that F_n^* is indeed a functor.

Let P and Q be pointed subsets of $[n]^*$ with $P \cap Q = \{0\}$:

$$F_n^* c : \mathcal{P}([n]^*)^* \to \mathrm{ob}\mathscr{D}$$

$$P \mapsto \begin{cases} (\psi^{-1} \circ F)(c(P)), & P = \{0\} \\ F(c(P)), & else \end{cases}$$

$$F_n^* \rho_{P,Q} = \begin{cases} (\psi^{-1} \circ F)(\rho_{P,Q}), & P = \{0\} \text{ and } Q = \{0\} \\ F(\rho_{P,Q}), & P = \{0\} \text{ or } Q = \{0\} \\ F(\rho_{P,Q}) \circ \phi_{F(c(P)),F(c(Q))}, & else \end{cases}$$

$$F_n^* f_P = \begin{cases} (\psi^{-1} \circ F)(f_P), & P = \{0\} \\ F(f_P), & else \end{cases}$$

First, we need to show that for any object (c, ρ) in $\mathscr{C}^\Gamma([n]^*)$, $F_n^*((c, \rho))$ is in fact an object in $\mathscr{D}^\Gamma([n]^*)$.

Since, as we have already seen, $(\psi^{-1} \circ F)(I_\mathscr{C}) = I_\mathscr{D}$ and $(\psi^{-1} \circ F)(\mathrm{id}_{I_\mathscr{C}}) = \mathrm{id}_{I_\mathscr{D}}$, $F_n^* c(\{0\}) = (\psi^{-1} \circ F)(I_\mathscr{C}) = I_\mathscr{D}$ and $F_n^*\ \rho_{\{0\},\{0\}} = (\psi^{-1} \circ F)(\mathrm{id}_{I_\mathscr{C}}) = \mathrm{id}_{I_\mathscr{D}}$ as required.

For any pointed subset $P \subseteq [n]^*$ where $P \neq \{0\}$, since F is a functor, $F_n^*\ \rho_{\{0\},P} = F(\rho_{\{0\},P}) = F(\mathrm{id}_{c(P)}) = \mathrm{id}_{F(c(P))} = \mathrm{id}_{F_n^* c(P)}$ also as required.

We also require that $F_n^*\ \rho_{P,Q}$ is an isomorphism for any pointed subsets P, $Q \in [n]^*$ with $P \cap Q = \{0\}$. If either P or Q is identically $\{0\}$, then we have already checked that this component is the identity, and thus an isomorphism but if neither P nor Q are $\{0\}$ then $F_n^*\ \rho_{P,Q} = F(\rho_{P,Q}) \circ \phi_{F(c(P)),F(c(Q))}$. Since ρ is a natural isomorphism and F is a functor, $F(\rho_{P,Q})$ is an isomorphism and since our monoidal functor is strong, $\phi_{P,Q}$ is also an isomorphism, therefore $F_n^*\ \rho$ is a natural isomorphism.

There are also several diagrams that must be shown to commute. Notice that in a strict symmetric monoidal category, since the unitors are identities, the components of the braiding where at least one of the indices are the unit must be equal to the identity on the other index, i.e. $\beta_{I,X} = \beta_{X,I} = \mathrm{id}_X$ for any object X in a strict symmetric monoidal category with unit object I and braiding β.

For any pair of pointed subsets P, $Q \subseteq [n]^*$ with $P \cap Q = \{0\}$, we need to show that the following diagram commutes:

$$\begin{array}{ccc} F_n^* c(P) \otimes_\mathscr{D} F_n^* c(Q) & \xrightarrow{\beta^\mathscr{D}_{F_n^* c(P), F_n^* c(Q)}} & F_n^* c(Q) \otimes_\mathscr{D} F_n^* c(P) \\ & \square & \\ {\scriptstyle F_n^* \rho_{P,Q}} \searrow & & \swarrow {\scriptstyle F_n^* \rho_{Q,P}} \\ & F_n^* c(P \vee Q) & \end{array}$$

There are three different cases we need to examine.

If $P = Q = \{0\}$, then $P \vee Q = \{0\}$ and since $F_n^* c(\{0\}) = I_\mathscr{D}$, $I_\mathscr{D} \otimes_\mathscr{D} I_\mathscr{D} = I_\mathscr{D}$, $F_n^*\ \rho_{\{0\},\{0\}} = \mathrm{id}_{I_\mathscr{D}}$, and $\beta^\mathscr{D}_{I_\mathscr{D},I_\mathscr{D}} = \mathrm{id}_{I_\mathscr{D}}$ as we have seen, our diagram simply becomes a triangle with $I_\mathscr{D}$ at each vertex, and $\mathrm{id}_{I_\mathscr{D}}$ on each edge which clearly commutes:

$$\begin{array}{ccc} I_{\mathscr{D}} & \xrightarrow{\mathrm{id}_{I_{\mathscr{D}}}} & I_{\mathscr{D}} \\ & \searrow^{\mathrm{id}_{I_{\mathscr{D}}}} \quad \square \quad \swarrow^{\mathrm{id}_{I_{\mathscr{D}}}} & \\ & I_{\mathscr{D}} & \end{array}$$

The next case is when exactly one of P or Q is equal to $\{0\}$. Since $\beta_{X,Y} \circ \beta_{Y,X} = \mathrm{id}_{X \otimes Y}$ in any strict symmetric monoidal category, without loss of generality, we can assume that $P \neq \{0\}$ and $Q = \{0\}$. Then $P \vee Q = P$, and since $F_n^* c(\{0\}) = I_{\mathscr{D}}$, $I_{\mathscr{D}} \otimes_{\mathscr{D}} X = X \otimes_{\mathscr{D}} I_{\mathscr{D}} = X$, and $\beta^{\mathscr{D}}_{I_{\mathscr{D}},X} = \beta^{\mathscr{D}}_{X,I_{\mathscr{D}}} = \mathrm{id}_X$ our diagram simplifies, and since $\rho_{P,\{0\}} = \rho_{\{0\},P} = \mathrm{id}_{c(P)}$ and F is a functor, $F(\mathrm{id}_{c(P)}) = \mathrm{id}_{F(c(P))} = \mathrm{id}_{F_n^* c(P)}$ and so $F_n^*\ \rho_{P,\{0\}} = F(\rho_{P,\{0\}}) = F(\mathrm{id}_{c(P)}) = \mathrm{id}_{F_n^* c(P)}$ and $F_n^*\ \rho_{\{0\},P} = F(\rho_{\{0\},P}) = F(\mathrm{id}_{c(P)}) = \mathrm{id}_{F_n^* c(P)}$ and again our diagram is a triangle with the same object at each vertex and the identity morphism of that object on each edge, again this clearly commutes:

$$\begin{array}{ccc} F(c(P)) & \xrightarrow{\mathrm{id}_{F(c(P))}} & F(c(P)) \\ & \searrow^{\mathrm{id}_{F(c(P))}} \quad \square \quad \swarrow^{\mathrm{id}_{F(c(P))}} & \\ & F(c(P)) & \end{array}$$

Finally we have the case when neither P nor Q are equal to $\{0\}$. $F_n^* c(R) = F(c(R))$ for all pointed subsets $R \subseteq [n]^*$ with $R \neq \{0\}$. Since (c, ρ) is an object in $\mathscr{C}^{\Gamma}([n]^*)$ and F is a functor, the following diagram necessarily commutes in $\mathscr{D}$:

$$\begin{array}{ccc} F(c(P) \otimes_{\mathscr{C}} c(Q)) & \xrightarrow{F(\beta^{\mathscr{C}}_{c(P),c(Q)})} & F(c(Q) \otimes_{\mathscr{C}} c(P)) \\ & \searrow^{F(\rho_{P,Q})} \quad \square \quad \swarrow^{F(\rho_{Q,P})} & \\ & F(c(P \vee Q)) & \end{array}$$

Since our monoidal functor is a symmetric monoidal functor, we also have for any objects X, $Y \in \mathrm{ob}\mathscr{C}$:

$$\begin{array}{ccc} F(X) \otimes_{\mathscr{D}} F(Y) & \xrightarrow{\beta^{\mathscr{D}}_{F(X),F(Y)}} & F(Y) \otimes_{\mathscr{D}} F(X) \\ \downarrow \phi_{X,Y} & \square & \downarrow \phi_{Y,X} \\ F(X \otimes_{\mathscr{C}} Y) & \xrightarrow[F(\beta^{\mathscr{C}}_{X,Y})]{} & F(Y \otimes_{\mathscr{C}} X) \end{array}$$

Combining these two diagrams in the case where $X = c(P)$ and $Y = c(Q)$ we see that the following diagram commutes:

$$\begin{array}{ccc} F(c(P)) \otimes_{\mathscr{D}} F(c(Q)) & \xrightarrow{\beta^{\mathscr{D}}_{F(c(P)),F(c(Q))}} & F(c(Q)) \otimes_{\mathscr{D}} F(c(P)) \\ \downarrow{\scriptstyle \phi_{c(P),c(Q)}} & \square & \downarrow{\scriptstyle \phi_{c(Q),c(P)}} \\ F(c(P) \otimes_{\mathscr{C}} c(Q)) & \xrightarrow{F(\beta^{\mathscr{C}}_{c(P),c(Q)})} & F(c(Q) \otimes_{\mathscr{C}} c(P)) \\ {\scriptstyle F(\rho_{P,Q})} \searrow & \square & \swarrow {\scriptstyle F(\rho_{Q,P})} \\ & F(c(P \vee Q)) & \end{array}$$

and therefore:

$$\begin{array}{ccc} F(c(P)) \otimes_{\mathscr{D}} F(c(Q))) & \xrightarrow{\beta^{\mathscr{D}}_{F(c(P)),F(c(Q))}} & F(c(Q)) \otimes_{\mathscr{D}} F(c(P)) \\ {\scriptstyle F(\rho_{P,Q}) \circ \phi_{c(P),c(Q)}} \searrow & \square & \swarrow {\scriptstyle F(\rho_{Q,P}) \circ \phi_{c(Q),c(P)}} \\ & F(c(P \vee Q)) & \end{array}$$

and since $F_n^* \ \rho_{P,Q} = F(\rho_{P,Q}) \circ \phi_{c(P),c(Q)}$ when neither P nor Q are $\{0\}$, this is the required diagram and thus in all cases, this diagram commutes.

We also need to show for any pointed subsets P, Q, $R \subseteq [n]^*$, with $P \cap Q \cap R = \{0\}$ that the following diagram commutes:

$$\begin{array}{ccc} F_n^* c(P) \otimes_{\mathscr{D}} F_n^* c(Q) \otimes_{\mathscr{D}} F_n^* c(R) & \xrightarrow{F_n^* \rho_{P,Q} \otimes_{\mathscr{D}} \mathrm{id}_{F_n^* c(R)}} & F_n^* c(P \vee Q) \otimes_{\mathscr{D}} F_n^* c(R) \\ \downarrow{\scriptstyle \mathrm{id}_{F_n^* cP} \otimes_{\mathscr{D}} F_n^* \rho_{Q,R}} & \square & \downarrow{\scriptstyle F_n^* \rho_{P \vee Q,R}} \\ F_n^* c(P) \otimes_{\mathscr{D}} F_n^* c(Q \vee R) & \xrightarrow[F_n^* \rho_{P,Q \vee R}]{} & F_n^* c(P \vee Q \vee R) \end{array}$$

Again there are several cases we need to examine.

When $P = Q = R = \{0\}$, since $F_n^* c(\{0\}) = I_{\mathscr{D}}$, $F_n^* \ \rho_{\{0\},\{0\}} = \mathrm{id}_{I_{\mathscr{D}}}$, $I_{\mathscr{D}} \otimes_{\mathscr{D}} I_{\mathscr{D}} = I_{\mathscr{D}}$ and $\mathrm{id}_{I_{\mathscr{D}}} \otimes_{\mathscr{D}} \mathrm{id}_{I_{\mathscr{D}}} = \mathrm{id}_{I_{\mathscr{D}}}$, the diagram boils down to a square with $I_{\mathscr{D}}$ at each vertex and $\mathrm{id}_{I_{\mathscr{D}}}$ on each edge which clearly commutes:

$$\begin{array}{ccc} I_{\mathscr{D}} & \xrightarrow{\mathrm{id}_{I_{\mathscr{D}}}} & I_{\mathscr{D}} \\ \downarrow{\scriptstyle \mathrm{id}_{I_{\mathscr{D}}}} & \square & \downarrow{\scriptstyle \mathrm{id}_{I_{\mathscr{D}}}} \\ I_{\mathscr{D}} & \xrightarrow[\mathrm{id}_{I_{\mathscr{D}}}]{} & I_{\mathscr{D}} \end{array}$$

If exactly two of P, Q, and R are equal to $\{0\}$, then a similar situation arises. Since $F_n^* c(S) = F(c(S))$, and $F_n^* \ \rho_{\{0\},S} = F_n^* \ \rho_{S,\{0\}} = \mathrm{is}_{F(c(S))}$ for any pointed subset $S \subseteq [n]^*$ with $S \neq \{0\}$, $X \otimes_{\mathscr{D}} I_{\mathscr{D}} = I_{\mathscr{D}} \otimes_{\mathscr{D}} X = X$ for any object X of $\mathscr{D}$ and $\mathrm{id}_{I_{\mathscr{D}}} \otimes_{\mathscr{D}} \xi = \xi \otimes_{\mathscr{D}} I_{\mathscr{D}} = \xi$ for any morphism ξ of $\mathscr{D}$, the three different cases all end up boiling down to a diagram consisting of a square with the same object at each vertex and the relevant identity morphism on each edge, all of which clearly commute:

$$
\begin{array}{ccc}
F(c(P)) & \xrightarrow{\mathrm{id}_{F(c(P))}} & F(c(P)) \\
{\scriptstyle \mathrm{id}_{F(c(P))}}\downarrow & \Box & \downarrow{\scriptstyle \mathrm{id}_{F(c(P))}} \\
F(c(P)) & \xrightarrow[\mathrm{id}_{F(c(P))}]{} & F(c(P))
\end{array}
\qquad
\begin{array}{ccc}
F(c(Q)) & \xrightarrow{\mathrm{id}_{F(c(Q))}} & F(c(Q)) \\
{\scriptstyle \mathrm{id}_{F(c(Q))}}\downarrow & \Box & \downarrow{\scriptstyle \mathrm{id}_{F(c(Q))}} \\
F(c(Q)) & \xrightarrow[\mathrm{id}_{F(c(Q))}]{} & F(c(Q))
\end{array}
\qquad
\begin{array}{ccc}
F(c(R)) & \xrightarrow{\mathrm{id}_{F(c(R))}} & F(c(R)) \\
{\scriptstyle \mathrm{id}_{F(c(R))}}\downarrow & \Box & \downarrow{\scriptstyle \mathrm{id}_{F(c(R))}} \\
F(c(R)) & \xrightarrow[\mathrm{id}_{F(c(R))}]{} & F(c(R))
\end{array}
$$

The next set of cases is when exactly one of P, Q, or R is $\{0\}$, again the unit object of $\mathscr{D}$ and it's identity morphism occur in each diagram but the result is subtly different depending on which of P, Q, or R is $\{0\}$.

We also use the fact that $\rho_{S,\{0\}} = \rho_{\{0\},S} = \mathrm{id}_{c(S)}$ and that F is a functor to achieve the following diagrams which all clearly commute:

if $R = \{0\}$

$$
\begin{array}{ccc}
F_n^*c(P) \otimes_{\mathscr{D}} F_n^*c(Q) & \xrightarrow{F_n^*\rho_{P,Q}} & F_n^*c(P \vee Q) \\
{\scriptstyle \mathrm{id}_{F_n^*c(P)} \otimes_{\mathscr{D}} \mathrm{id}_{F_n^*c(Q)}}\downarrow & \Box & \downarrow{\scriptstyle \mathrm{id}_{F_n^*c(P \vee Q)}} \\
F_n^*c(P) \otimes_{\mathscr{D}} F_n^*c(Q) & \xrightarrow[F_n^*\rho_{P,Q}]{} & F_n^*c(P \vee Q)
\end{array}
$$

if $Q = \{0\}$

$$
\begin{array}{ccc}
F_n^*c(P) \otimes_{\mathscr{D}} F_n^*c(R) & \xrightarrow{\mathrm{id}_{F_n^*c(P)} \otimes_{\mathscr{D}} \mathrm{id}_{F_n^*c(R)}} & F_n^*c(P) \otimes_{\mathscr{D}} F_n^*c(R) \\
{\scriptstyle \mathrm{id}_{F_n^*c(P)} \otimes_{\mathscr{D}} \mathrm{id}_{F_n^*c(R)}}\downarrow & \Box & \downarrow{\scriptstyle F_n^*\rho_{P,R}} \\
F_n^*c(P) \otimes_{\mathscr{D}} F_n^*c(R) & \xrightarrow[F_n^*\rho_{P,R}]{} & F_n^*c(P \vee R)
\end{array}
$$

if $P = \{0\}$

$$
\begin{array}{ccc}
F_n^*c(Q) \otimes_{\mathscr{D}} F_n^*c(R) & \xrightarrow{\mathrm{id}_{F_n^*c(Q)} \otimes_{\mathscr{D}} \mathrm{id}_{F_n^*c(R)}} & F_n^*c(Q) \otimes_{\mathscr{D}} F_n^*c(R) \\
{\scriptstyle F_n^*\rho_{Q,R}}\downarrow & \Box & \downarrow{\scriptstyle F_n^*\rho_{Q,R}} \\
F_n^*c(Q \vee R) & \xrightarrow[\mathrm{id}_{F_n^*c(Q \vee R)}]{} & F_n^*c(Q \vee R)
\end{array}
$$

Finally, we need tackle the case when none of P, Q, or R are $\{0\}$.

Since (c, ρ) is an object in $\mathscr{C}^\Gamma([n]^*)$ and F is a functor we know that:

$$
\begin{array}{ccc}
F(c(P) \otimes_{\mathscr{C}} c(Q) \otimes_{\mathscr{C}} c(R)) & \xrightarrow{F(\rho_{P,Q} \otimes_{\mathscr{C}} \mathrm{id}_{c(R)})} & F(c(P \vee Q) \otimes_{\mathscr{C}} c(R)) \\
{\scriptstyle F(\mathrm{id}_{c(P)} \otimes_{\mathscr{C}} \rho_{Q,R})}\downarrow & \Box & \downarrow{\scriptstyle F(\rho_{P \vee Q,R})} \\
F(c(P) \otimes_{\mathscr{C}} c(Q \vee R)) & \xrightarrow[F(\rho_{P,Q \vee R})]{} & F(c(P \vee Q \vee R))
\end{array}
$$

Since $\mathscr{D}$ is a strict monoidal category, the associator is the identity and since F together with ϕ and ψ is a monoidal functor the following diagram also commutes:

$$
\begin{array}{ccc}
F(c(P)) \otimes_{\mathscr{D}} F(c(Q)) \otimes_{\mathscr{D}} F(c(R)) & \xrightarrow{\phi_{c(P),c(Q)} \otimes_{\mathscr{D}} \mathrm{id}_{F(c(R))}} & F(c(P) \otimes_{\mathscr{C}} c(Q)) \otimes_{\mathscr{D}} F(c(R)) \\
{\scriptstyle \mathrm{id}_{F(c(P))} \otimes_{\mathscr{D}} \phi_{c(Q),c(R)}}\downarrow & \Box & \downarrow{\scriptstyle \phi_{c(P) \otimes_{\mathscr{C}} c(Q),c(R)}} \\
F(c(P)) \otimes_{\mathscr{D}} F(c(Q) \otimes_{\mathscr{C}} c(R)) & \xrightarrow[\phi_{c(P),c(Q) \otimes_{\mathscr{C}} c(R)}]{} & F(c(P) \otimes_{\mathscr{C}} c(Q) \otimes_{\mathscr{C}} c(R))
\end{array}
$$

Recalling the diagram that we want to show is a commutative diagram and the fact that $F_n^*\,\rho_{S,T} = F(\rho_{S,T}) \circ \phi_{c(S),c(T)}$ for any pair of pointed subsets S, $T \subseteq [n]^*$ with $S \cap T = \{0\}$ and $S \neq \{0\} \neq T$, if we can show that all the squares in the following diagram commute then we are done:

$$
\begin{array}{ccccc}
F(c(P)) \otimes F(c(Q)) \otimes_{\mathscr{D}} F(c(R)) & \xrightarrow{\phi_{c(P),c(Q)} \otimes_{\mathscr{D}} \mathrm{id}_{F(c(R))}} & F(c(P) \otimes_{\mathscr{C}} c(Q)) \otimes_{\mathscr{D}} F(c(R)) & \xrightarrow{F(\rho_{P,Q}) \otimes_{\mathscr{D}} \mathrm{id}_{F(c(R))}} & F(c(P \vee Q)) \otimes_{\mathscr{D}} F(c(R)) \\
{\scriptstyle \mathrm{id}_{F(c(P))} \otimes_{\mathscr{D}} \phi_{c(Q),c(R)}} \Big\downarrow & \square & \Big\downarrow {\scriptstyle \phi_{c(P) \otimes_{\mathscr{C}} c(Q),c(R)}} & & \Big\downarrow {\scriptstyle \phi_{c(P \vee Q),c(R)}} \\
F(c(P)) \otimes_{\mathscr{D}} F(c(Q) \otimes_{\mathscr{C}} c(R)) & \xrightarrow[\phi_{c(P),c(Q) \otimes_{\mathscr{C}} c(R)}]{} & F(c(P) \otimes_{\mathscr{C}} c(Q) \otimes_{\mathscr{C}} c(R)) & \xrightarrow{F(\rho_{P,Q} \otimes_{\mathscr{C}} \mathrm{id}_{c(R)})} & F(c(P \vee Q) \otimes_{\mathscr{C}} c(R)) \\
{\scriptstyle \mathrm{id}_{F(c(P))} \otimes_{\mathscr{D}} F(\rho_{Q,R})} \Big\downarrow & & {\scriptstyle F(\mathrm{id}_{c(P)} \otimes_{\mathscr{C}} \rho_{Q,R})} \Big\downarrow & \square & \Big\downarrow {\scriptstyle F(\rho_{P \vee Q,R})} \\
F(c(P)) \otimes_{\mathscr{D}} F(c(Q \vee R)) & \xrightarrow[\phi_{c(P),c(Q \vee R)}]{} & F(c(P) \otimes_{\mathscr{C}} c(Q \vee R)) & \xrightarrow[F(\rho_{P,Q \vee R})]{} & F(c(P \vee Q \vee R))
\end{array}
$$

Since $\phi\colon \otimes_{\mathscr{D}} \circ (F, F) \to F \circ \otimes_{\mathscr{C}}$ is a natural transformation, for any morphisms $f\colon X \to Y$ and $g\colon X' \to Y'$ the following diagram commutes:

$$
\begin{array}{ccc}
F(X) \otimes_{\mathscr{D}} F(X') & \xrightarrow{F(f) \otimes_{\mathscr{D}} F(f')} & F(Y) \otimes_{\mathscr{D}} F(Y') \\
{\scriptstyle \phi_{X,X'}} \Big\downarrow & \square & \Big\downarrow {\scriptstyle \phi_{Y,Y'}} \\
F(X \otimes_{\mathscr{C}} X') & \xrightarrow[F(f \otimes_{\mathscr{C}} f')]{} & F(Y \otimes_{\mathscr{C}} Y')
\end{array}
$$

The top right square and bottom left square in our diagram are both this commutative diagram for the morphisms $\rho_{P,Q}$ and $\mathrm{id}_{c(R)}$, and $\mathrm{id}_{c(P)}$ and $\rho_{Q,R}$ respectively. Thus all four squares in the diagram commute and we achieve the desired commutative diagram:

$$
\begin{array}{ccc}
F_n^* c(P) \otimes_{\mathscr{D}} F_n^* c(Q) \otimes_{\mathscr{D}} F_n^* c(R) & \xrightarrow{F_n^* \rho_{P,Q} \otimes_{\mathscr{D}} \mathrm{id}_{F_n^* c(R)}} & F_n^* c(P \vee Q) \otimes_{\mathscr{D}} F_n^* c(R) \\
{\scriptstyle \mathrm{id}_{F_n^* cP} \otimes_{\mathscr{D}} F_n^* \rho_{Q,R}} \Big\downarrow & \square & \Big\downarrow {\scriptstyle F_n^* \rho_{P \vee Q,R}} \\
F_n^* c(P) \otimes_{\mathscr{D}} F_n^* c(Q \vee R) & \xrightarrow[F_n^* \rho_{P,Q \vee R}]{} & F_n^* c(P \vee Q \vee R)
\end{array}
$$

Since we have now shown that all the desired conditions are preserved, we know that if (c, ρ) is an object in $\mathscr{C}^\Gamma([n]^*)$, then $F_n^*((c, \rho))$ is an object in $\mathscr{D}^\Gamma([n]^*)$.

Now to show that if $f\colon (c, \rho) \to (c', \rho')$ is a morphism in $\mathscr{C}^\Gamma([n]^*)$, then $F_n^* f\colon F_n^*((c, \rho)) \to F_n^*((c', \rho'))$ is a morphism in $\mathscr{D}^\Gamma([n]^*)$.

$f_{\{0\}}$ is required to be $\mathrm{id}_{I_{\mathscr{C}}}$ for any morphism in $\mathscr{C}^\Gamma([n]^*)$. We need to show that $F_n^* f_{\{0\}}$ is the identity on $I_{\mathscr{D}}$. Since $F_n^* f_{\{0\}} = (\psi^{-1} \circ F)(\mathrm{id}_{I_{\mathscr{C}}})$ clearly $F_n^* f_{\{0\}}\colon I_{\mathscr{D}} \to I_{\mathscr{D}}$ as $F_n^*(I_{\mathscr{C}}) = I_{\mathscr{D}}$, additionally the following diagram must commute:

$$
\begin{array}{ccc}
F(I_{\mathscr{C}}) & \xrightarrow{\psi^{-1}} & I_{\mathscr{D}} \\
{\scriptstyle \mathrm{id}_{I_{\mathscr{C}}}} \Big\downarrow & \square & \Big\downarrow {\scriptstyle F_n^* f_{\{0\}}} \\
F(I_{\mathscr{C}}) & \xrightarrow[\psi^{-1}]{} & I_{\mathscr{D}}
\end{array}
$$

Therefore $F_n^* f_{\{0\}} = \psi^{-1} \circ \mathrm{id}_{I_{\mathscr{C}}} \circ \psi = \psi^{-1} \circ \psi = \mathrm{id}_{I_{\mathscr{D}}}$.

For any other component of a morphism in $\mathscr{C}^\Gamma([n]^*)$, $f_P\colon c(P) \to c'(P)$ for $P \subseteq [n]^*$ with $P \neq \{0\}$, since $F_n^* c(P) = F(c(P))$, $F_n^* f_P\colon F(c(P)) \to F(c'(P))$ as required.

We need to check that the following diagram commutes for any P, $Q \subseteq [n]^*$ with $P \cap Q = \{0\}$:

$$\begin{array}{ccc} F_n^* c(P) \otimes_{\mathscr{D}} F_n^* c(Q) & \xrightarrow{F_n^* \rho_{P,Q}} & F_n^* c(P) \vee Q) \\ {\scriptstyle F_n^* f_P \otimes_{\mathscr{D}} F_n^* f_Q}\downarrow & \square & \downarrow{\scriptstyle F_n^* f_{P \vee Q}} \\ F_n^* c'(P) \otimes_{\mathscr{D}} F_n^* c'(Q) & \xrightarrow[F_n^* \rho'_{P,Q}]{} & F_n^* c'(P \vee Q) \end{array}$$

For $P = Q = \{0\}$, clearly all of the vertices are $I_{\mathscr{D}}$ and all of the edges are $\mathrm{id}_{I_{\mathscr{D}}}$ so in this case the diagram clearly commutes. If only one of P and Q is equal to $\{0\}$, since the unitors are identities, without loss of generality we can say $Q = \{0\}$ and $P \neq \{0\}$, then this diagram instead has $F_n^* c(P)$ at each vertex and $\mathrm{id}_{F_n^* c(P)}$ on each edge, another diagram that clearly commutes.

If neither P nor Q is equal to $\{0\}$ then since F is a functor we know the following diagram commutes:

$$\begin{array}{ccc} F(c(P) \otimes_{\mathscr{C}} c(Q)) & \xrightarrow{F(\rho_{P,Q})} & F(c(P \vee Q)) \\ {\scriptstyle F(f_P \otimes_{\mathscr{C}} f_Q)}\downarrow & \square & \downarrow{\scriptstyle F(f_{P \vee Q})} \\ F(c'(P) \otimes_{\mathscr{C}} c'(Q)) & \xrightarrow[F(\rho'_{P,Q})]{} & F(c'(P \vee Q)) \end{array}$$

and since F together with ϕ and ψ is a monoidal functor, we have another commutative diagram:

$$\begin{array}{ccc} F(c(P)) \otimes_{\mathscr{D}} F(c(Q)) & \xrightarrow{\phi_{c(P),c(Q)}} & F(c(P) \otimes_{\mathscr{C}} c(Q)) \\ {\scriptstyle F(f_P) \otimes_{\mathscr{D}} F(f_Q)}\downarrow & \square & \downarrow{\scriptstyle F(f_P \otimes_{\mathscr{C}} f_Q)} \\ F(c'(P)) \otimes_{\mathscr{D}} F(c'(Q)) & \xrightarrow[\phi_{c'(P),c'(Q)}]{} & F(c'(P) \otimes_{\mathscr{C}} c'(Q)) \end{array}$$

Therefore:

$$\begin{array}{ccc} F(c(P)) \otimes_{\mathscr{D}} F(c(Q)) & \xrightarrow{F(\rho_{P,Q}) \circ \phi_{c(P),c(Q)}} & F(c(P \vee Q)) \\ {\scriptstyle F(f_P) \otimes_{\mathscr{D}} F(f_Q)}\downarrow & \square & \downarrow{\scriptstyle F(f_{P \vee Q})} \\ F(c'(P)) \otimes_{\mathscr{D}} F(c'(Q)) & \xrightarrow[F(\rho_{P,Q}) \circ \phi_{c'(P),c'(Q)}]{} & F(c'(P \vee Q)) \end{array}$$

and since $F_n^*\ \rho_{P,Q} = F(\rho_{P,Q}) \circ \phi_{c(P),c(Q)}$ and $F_n^* f_P = F(f_P)$ when neither P nor Q is $\{0\}$, this is exactly the diagram we needed to commute.

Therefore F_n^* is a valid mapping of morphisms.

Finally it needs to be shown that F_n^* preserves both the identities and composition.

If $\mathrm{id}_{(c,\rho)} = \{\mathrm{id}_{c(P)} \mid P \subseteq [n]^*\}$ then $F_n^*\mathrm{id}_{(c,\rho)}$ needs to be shown to be equal to $\mathrm{id}_{(F_n^* c, F_n^* \rho)}$. We have already seen that $F_n^* f_{\{0\}} = \mathrm{id}_{I_\mathscr{D}}$ for any morphism in $\mathscr{C}^\Gamma([n]^*)$ and for an other pointed subset $P \subseteq [n]^*$, $F_n^* f_P = F(f(P))$ so since $\mathrm{id}_{(c,\rho),P} = \mathrm{id}_{c(P)}$ and F is a functor, $F_n^*\mathrm{id}_{(c,\rho)} = \{\mathrm{id}_{I_\mathscr{D}}\} \cup \{\mathrm{id}_{F(c(P))} \mid P \subseteq [n]^*$, $P \neq \{0\}\} = \{\mathrm{id}_{F_n^* c(P)} \mid P \subseteq [n]^*\}$ as required.

Lastly, if f: $(c, \rho) \to (c', \rho')$ and g: $(c', \rho') \to (c'', \rho'')$ are morphisms in $\mathscr{C}^\Gamma([n]^*)$ then we need to show that $F_n^*(g \circ f) = F_n^* g \circ F_n^* f$.

Again this works component-wise, and since $f_{\{0\}} = g_{\{0\}} = \mathrm{id}_{I_\mathscr{D}}$ is required, $F_n^*(g \circ f)_{\{0\}} = (F_n^* g \circ F_n^* f)_{\{0\}} = \mathrm{id}_{I_\mathscr{D}}$. For every pointed subset $P \neq \{0\}$, we simply have $F(g_P \circ f_P) = F(g_P) \circ F(f_P)$ which holds since F is a functor.

Therefore, F_n^*: $\mathscr{C}^\Gamma([n]^*) \to \mathscr{D}^\Gamma([n]^*)$ is a functor for every $n \in \mathbb{N}$.

To achieve a Γ-category homomorphism we need to show that one final condition is satisfied.

For any morphism in $\mathbf{\Gamma}^{\mathrm{op}}$, i.e. a pointed map ξ: $[n]^* \to [m]^*$, we require the following diagram to commute in **Cat***:

$$\begin{array}{ccc} \mathscr{C}^\Gamma([n]^*) & \xrightarrow{\mathscr{C}^\Gamma(\xi)} & \mathscr{C}^\Gamma([m]^*) \\ {\scriptstyle F_n^*}\downarrow & \square & \downarrow{\scriptstyle F_m^*} \\ \mathscr{D}^\Gamma([n]^*) & \xrightarrow[\mathscr{D}^\Gamma(\xi)]{} & \mathscr{D}^\Gamma([m]^*) \end{array}$$

Since we have shown that all of these maps are indeed functors, all that needs to be shown is that the mapping of objects and morphisms agree.

Let (c, ρ) be an object of $\mathscr{C}^\Gamma([n]^*)$, then for pointed subsets P, $Q \subseteq [m]^*$ with $P \cap Q = \{0\}$:

$$\begin{aligned} (F_m^* \circ \mathscr{C}^\Gamma(\xi))((c,\rho)) &= F_m^*(\mathscr{C}^\Gamma(\xi)((c,\rho))) \\ &= (F_m^*(\mathscr{C}^\Gamma(\xi)c), F_m^*(\mathscr{C}^\Gamma(\xi)\rho)) \\ F_m^*(\mathscr{C}^\Gamma(\xi)c)(P) &= F_m^*(c(\xi^*(P))) \\ &= \begin{cases} I_\mathscr{D}, & \xi^*(P) = \{0\} \\ F(c(\xi^*(P))), & else \end{cases} \\ F_m^*(\mathscr{C}^\Gamma(\xi)\rho)_{P,Q} &= F_m^*(\mathscr{C}^\Gamma \xi_{P,Q}) \\ &= F_m^*(\rho_{\xi^*(P),\xi^*(Q)}) \\ &= \begin{cases} \mathrm{id}_{I_\mathscr{D}}, \ \xi^*(P) = \{0\} \text{ and } \xi^*(Q) = \{0\} \\ F(\rho_{\xi^*(P),\xi^*(Q)}), \ \xi^*(P) = \{0\} \text{ or } \xi^*(Q) = \{0\} \\ F(\rho_{\xi^*(P),\xi^*(Q)}) \circ \phi_{F(c(\xi^*(P))),F(c(\xi^*(Q)))}, \ else \end{cases} \end{aligned}$$

$$
\begin{aligned}
(\mathscr{D}^\Gamma(\xi)\circ F_n^*)((c,\rho)) &= \mathscr{D}^\Gamma(\xi)(F_n^*((c,\rho)))\\
&= (\mathscr{D}^\Gamma(\xi)(F_n^* c), \mathscr{D}^\Gamma(\xi)(F_n^*\rho))\\
(\mathscr{D}^\Gamma(\xi)(F_n^* c))(P) &= F_n^* c(\xi^*(P))\\
&= \begin{cases} I_{\mathscr{D}}, & \xi^*(P)=\{0\}\\ F(c(\xi^*(P))), & else \end{cases}\\
(\mathscr{D}^\Gamma(\xi)(F_n^*\rho))_{P,Q} &= F_n^*\rho_{\xi^*(P),\xi^*(Q)}\\
&= \begin{cases} \mathrm{id}_{I_{\mathscr{D}}},\ \xi^*(P)=\{0\} \text{ and } \xi^*(Q)=\{0\}\\ F(\rho_{\xi^*(P),\xi^*(Q)}),\ \xi^*(P)=\{0\} \text{ or } \xi^*(Q)=\{0\}\\ F(\rho_{\xi^*(P),\xi^*(Q)})\circ\phi_{F(c(\xi^*(P))),F(c(\xi^*(Q)))},\ else \end{cases}
\end{aligned}
$$

Thus the mapping of objects commutes and all that is left is to show the same for morphisms. Let f: $(c,\ \rho) \to (c',\ \rho')$ be a morphism in $\mathscr{C}^\Gamma([n]^*)$, then for any pointed subset $P \subseteq [m]^*$:

$$
\begin{aligned}
(F_m^*\circ\mathscr{C}^\Gamma(\xi))f_P &= F_m^*(\mathscr{C}^\Gamma(\xi)f_P)\\
&= F_m^* f_{\xi^*(P)}\\
&= \begin{cases} \mathrm{id}_{I_{\mathscr{D}}}, & \xi^*(P)=\{0\}\\ F(f_{\xi^*(P)}), & else \end{cases}\\
(\mathscr{D}^\Gamma(\xi)(F_n^*))f_P &= F_n^* f_{\xi^*(P)}\\
&= \begin{cases} \mathrm{id}_{I_{\mathscr{D}}}, & \xi^*(P)=\{0\}\\ F(f_{\xi^*(P)}), & else \end{cases}
\end{aligned}
$$

Thus the mapping of morphisms also commutes and thus the diagram commutes in **Cat***.

Therefore F^*: $\mathscr{C}^\Gamma \to \mathscr{D}^\Gamma$ is a Γ-category homomorphism. △

9 The Weyl Map

For any given exponential functor $F\colon \mathscr{C}_\oplus \to \mathscr{C}_\otimes$ we do not automatically achieve a natural transformation of cohomology theories, but since every map involved in defining the cohomology groups in each degree is functorial, we do achieve a sequence of natural transformations with components:

$$\begin{aligned} F_X^{*k} : h_\oplus^k(X) &\to h_\otimes^k(X) \\ [\phi] &\mapsto [F^* \circ \phi] \end{aligned}$$

where

$$F^* = \begin{cases} \Omega^{(-k+1)} B|N(F)|, & k \leq 0 \\ B^{(k)}|N(F)|, & k > 0 \end{cases}$$

Ultimately, it is this sequence of natural transformations that we would like to better understand.

Since the other maps are simply multiple applications of either the loop functor or classifying space construction to the map F^*: $B|N(\mathscr{C}_\oplus)| \to B|N(\mathscr{C}_\otimes)|$, our focus will be on this map and it's corresponding natural transformation F^{*1}: $K^1 \to h_\otimes^1$

The inclusions $\mathrm{SU}(n) \hookrightarrow \mathrm{SU}(\infty) \hookrightarrow \mathrm{U}$ induce a class in $K^1(\mathrm{SU}(n)) \cong h_\oplus^1(\mathrm{SU}(n))$, and we can postcompose these inclusions with the map F^*: U $\to B|N(\mathscr{C}_\otimes)|$ to induce a non-trivial class in $h_\otimes^1(\mathrm{SU}(n))$.

Since we have already investigated the effect exponential functors have on vector bundles, it will be much easier to determine the nature of these classes if we are able to precompose the inclusions with a map that plays well with the suspension isomorphism, so as to involve the 0th K-theory group whose elements are all formal differences of vector bundles. At which point, since F^{*1} is a natural transformation, determining the nature of the corresponding class in our exotic cohomology theory will hopefully be as easy as applying our exponential functor to those vector bundles while turning the formal difference into a formal quotient.

We will use a map known as the Weyl map, identically defined as in the work by Becker, Murray, and Stevenson [4] that will hopefully help us achieve this goal.

Definition 44. Consider the group $\mathrm{SU}(n)$. Diagonal matrices within this group are of the form $\mathrm{diag}(z_1, ..., z_n)$ where $|z_i| = 1 \ \forall\ i$ and $z_1 \cdot ... \cdot z_n = 1$. Since the set of elements $z \in \mathbb{C}$ such that $|z| = 1$ is isomorphic to $\mathbb{S}^1$, the set of diagonal matrices in $\mathrm{SU}(n)$ is isomorphic to $\mathbb{T}^{n\text{-}1}$, the product of n-1 copies of $\mathbb{S}^1$. We call this subgroup the **maximal torus** of $\mathrm{SU}(n)$ and will denote it $\mathbb{T}$. We can form the quotient space $\mathrm{SU}(n)/\mathbb{T}$ by the equivalence relation: $g \sim h$ if $g = hZ$ for some $Z \in \mathbb{T}$, and thus we can define the **Weyl map** as follows:

$$\begin{aligned} W : \mathrm{SU}(n)/\mathbb{T} \times \mathbb{T} &\to \mathrm{SU}(n) \\ ([g], Z) &\mapsto gZg^{-1} \end{aligned}$$

This map is well defined since for any other representative h of the class $[g]$, we have $h = gT$ for some $T \in \mathbb{T}$ and thus $hZh^{-1} = gTZT^{-1}g^{-1} = gZg^{-1}$ since elements of $\mathbb{T}$ commute with each other. We would like to determine how the map induced by the Weyl map in cohomology with rational coefficients behaves:

$$W^* : H^*(\mathrm{SU}(n); \mathbb{Q}) \to H^*(\mathrm{SU}(n)/\mathbb{T}; \mathbb{Q}) \otimes H^*(\mathbb{T}; \mathbb{Q})$$

We have seen that $H^*(\mathrm{SU}(n); \mathbb{Q}) = \Lambda_{\mathbb{Q}}[x_3, ..., x_{2n-1}]$ where each x_i is in degree i, and since $\mathbb{T}$ is isomorphic to the product of n-1 copies of $\mathbb{S}^1$, we can say that $H^*(\mathbb{T}; \mathbb{Q}) = \Lambda_{\mathbb{Q}}[z^{(1)}, ..., z^{(n-1)}]$ where each $z^{(j)}$ is in degree 1.

Lemma 56. The space $\mathrm{SU}(n)/\mathbb{T}$ is homeomorphic to $F_n(\mathbb{C}^n)$

Proof. Let us consider general elements $A \in \mathrm{SU}(n)$ and $Z \in \mathbb{T}$.

$$A = [\mathbf{v}_1|...|\mathbf{v}_n] \text{ where } \mathbf{v}_i \perp \mathbf{v}_j \text{ for } i \neq j,$$
$$Z = diag(z_1, ..., z_{n-1}, \overline{z_1...z_{n-1}}) \text{ where } z_i \in \mathbb{S}^1 \subset \mathbb{C}\ \forall\ i$$

The class of A, $[A] \in \mathrm{SU}(n)/\mathbb{T}$ is the set $\{AZ \in \mathrm{SU}(n) \mid Z \in \mathbb{T}\}$

Let us consider a map Ψ: $\mathrm{SU}(n)/\mathbb{T} \to F_n(\mathbb{C}^n)$ defined as follows:

$$\Psi([\mathbf{v}_1|...|\mathbf{v}_n]) = (Span(\mathbf{v}_1) \subset Span(\mathbf{v}_1, \mathbf{v}_2) \subset ... \subset Span(\mathbf{v}_1, ..., \mathbf{v}_{n-1}) \subset \mathbb{C}^n)$$

This map is well defined since all representatives of the class of $[\mathbf{v}_1| \ldots | \mathbf{v}_n]$ are of the form $[z_1\mathbf{v}_1| \ldots | z_{n-1}\mathbf{v}_{n-1}| \overline{z_1...z_n}\ \mathbf{v}_n]$ where each $z_i \in \mathbb{S}^1 \subset \mathbb{C}$

We would like to show that Ψ is a homeomorphism. We therefore must show that Ψ is a continuous bijection, that $\mathrm{SU}(n)/\mathbb{T}$ is compact, and that $F_n(\mathbb{C}^n)$ is Hausdorff.

$\mathrm{SU}(n)/\mathbb{T}$ is compact since quotients of compact spaces are compact and $\mathrm{SU}(n)$ is compact.

$F_n(\mathbb{C}^n)$ is Hausdorff as it inherits the property as it is a subspace of the Hausdorff space $(\mathbb{CP}^{n-1})^{n-1}$.

Let us consider two matrices $A = [\mathbf{v}_1| \ldots |\mathbf{v}_n]$ and $B = [\mathbf{w}_1| \ldots |\mathbf{w}_n]$. We would like to show that if $\Psi([A]) = \Psi([B])$, then $[A] = [B]$.

$$\begin{aligned}
\Psi([A]) &= \Psi([B]) \\
(Span(\mathbf{v}_1) \subset ... \subset Span(\mathbf{v}_1, ..., \mathbf{v}_{n-1}) \subset \mathbb{C}^n) &= (Span(\mathbf{w}_1) \subset ... \subset Span(\mathbf{w}_1, ..., \mathbf{w}_{n-1}) \subset \mathbb{C}^n) \\
\implies Span(\mathbf{v}_1, ..., \mathbf{v}_i) &= Span(\mathbf{w}_1, ..., \mathbf{w}_i)\ \forall\ i \\
\text{Since } \mathbf{v}_i \perp \mathbf{v}_j &\text{ and } \mathbf{w}_i \perp \mathbf{w}_j\ \forall\ i \neq j, \\
\implies Span(\mathbf{v}_i) &= Span(\mathbf{w}_i)\ \forall\ i \\
\implies \mathbf{w}_i &= \lambda_i \mathbf{v}_i \text{ for some } \lambda_i \in \mathbb{C}\ \forall\ i \\
\text{Since } |\mathbf{v}_i| = 1 = |\mathbf{w}_i|\ \forall\ i,\ \implies \mathbf{w}_i &= z_i \mathbf{v}_i \text{ for some } z_i \in \mathbb{S}^1 \subset \mathbb{C}\ \forall\ i \\
\text{Therefore } B &= [z_1\mathbf{v}_1|...|z_n\mathbf{v}_n] \\
\text{Since } det(B) = 1, \text{ we must have } z_n &= \overline{z_1...z_{n-1}} \text{ and so} \\
B &= AZ \text{ for some } Z \in \mathbb{T} \\
\text{Then } [A] &= [B]
\end{aligned}$$

Therefore Ψ is injective.

Let us consider the map Φ: $F_n(\mathbb{C}^n) \to \mathrm{SU}(n)$ defined in the following manner:

$$\Phi(\mathrm{V}_1 \subset ... \subset \mathrm{V}_{n-1} \subset \mathrm{V}_n = \mathbb{C}^n) = [\mathbf{v}_1|...|\mathbf{v}_n]$$

where $\mathbf{v}_i$ is a vector in V_i such that $|\mathbf{v}_i| = 1$ and $\mathbf{v}_i \perp \mathbf{w}\ \forall\ \mathbf{w} \in \mathrm{V}_{i\text{-}1}$.

Necessarily, the class of $[\mathbf{v}_1|\ ...\ |\mathbf{v}_n]$ in $\mathrm{SU}(n)/\mathbb{T}$ will be mapped by Ψ to the flag $(\mathrm{V}_1 \subset ... \subset \mathrm{V}_{n\text{-}1} \subset \mathbb{C}^n)$. Since this construction works for any flag in $F_n(\mathbb{C}^n)$, Ψ must be surjective, and thus bijective.

In order to show that Ψ is continuous, let us consider its composition with the inclusion into $(\mathbb{CP}^{n\text{-}1})^{n\text{-}1}$ and the following diagram:

$$\begin{array}{ccc}
\mathrm{SU}(n) & \xrightarrow{\ \Xi\ } & (\mathbb{C}^n \backslash \{0\})^{n-1} \\
{\scriptstyle q_1}\big\downarrow & \square & \big\downarrow{\scriptstyle q_2} \\
\mathrm{SU}(n)/\mathbb{T} & \xrightarrow[\iota\circ\Psi]{} & (\mathbb{CP}^{n-1})^{n-1}
\end{array}$$

Where q_1 and q_2 are quotient maps and Ξ is defined as followed in order to make the diagram commute:

$$\begin{aligned}
\Xi : \mathrm{SU}(n) &\to (\mathbb{C}^n \backslash \{0\})^{n-1} \\
A &\mapsto (A\mathbf{e}_1, ..., A\mathbf{e}_{n-1})
\end{aligned}$$

where each $\mathbf{e}_i$ is the i^{th} unit vector in $\mathbb{C}^n$. In essence, Ξ maps an $n \times n$ matrix to the n-1 tuple of its first n-1 columns.

The diagram commutes, as for a general matrix $A = [\mathbf{v}_1|\ ...\ |\mathbf{v}_n] \in \mathrm{SU}(n)$:

$$\begin{aligned}
(q_2 \circ \Xi)([\mathbf{v}_1|...|\mathbf{v}_n]) &= q_2((\mathbf{v}_1, ..., \mathbf{v}_{n-1})) \\
&= (Span(\mathbf{v}_1), ..., Span(\mathbf{v}_{n-1})) \\
(\iota \circ \Psi \circ q_1)([\mathbf{v}_1|...|\mathbf{v}_n]) &= (\iota \circ \Psi)([[\mathbf{v}_1|...|\mathbf{v}_n]]) \\
&= \iota((Span(\mathbf{v}_1) \subset ... \subset Span(\mathbf{v}_1, ..., \mathbf{v}_{n-1}) \subset \mathbb{C}^n)) \\
&= (Span(\mathbf{v}_1), ..., Span(\mathbf{v}_{n-1}))
\end{aligned}$$

Let us finally consider an open set $U \in (\mathbb{CP}^{n\text{-}1})^{n\text{-}1}$. The quotient topology dictates that a set in the quotient is open if and only if its preimage by the quotient is open, therefore $q_2{}^{\text{-}1}(U)$ is open in $(\mathbb{C}^n \backslash \{0\})^{n\text{-}1}$. Ξ as a map is linear as it is a series of matrix multiplications, therefore Ξ is continuous and so since $q_2{}^{\text{-}1}(U)$ is open, $\Xi^{-1}(q_2{}^{\text{-}1}(U))$ is open in $\mathrm{SU}(n)$. Since q_1 is a quotient map and $\Xi^{-1}(q_2{}^{\text{-}1}(U))$ is open, necessarily, $q_1(\Xi^{-1}(q_2{}^{\text{-}1}(U)))$ is open in $\mathrm{SU}(n)/\mathbb{T}$, and since the diagram commutes, $q_1(\Xi^{-1}(q_2{}^{\text{-}1}(U))) = (\iota \circ \Psi)^{\text{-}1}(U)$.

Therefore $U \in (\mathbb{CP}^{n-1})^{n\text{-}1}$ is open $\implies$ $(\iota \circ \Psi)^{\text{-}1}(U) \in \mathrm{SU}(n)/\mathbb{T}$ is open and thus $\iota \circ \Psi$ is continuous. Since the inclusion is continuous, we therefore know that Ψ must be continuous.

Since Ψ is a continuous bijection from a compact space to a Hausdorff space, Ψ is a homeomorphism and thus $\mathrm{SU}(n)/\mathbb{T} \cong F_n(\mathbb{C}^n)$. $\triangle$

Hence $H^*(\mathrm{SU}(n)/\mathbb{T}; \mathbb{Q}) = H^*(F_n(\mathbb{C}^n); \mathbb{Q})$

The group $\mathbb{T}$ has elements of the form $diag(z_1, ..., z_{n\text{-}1}, \overline{z_1...z_{n\text{-}1}})$. Let $\mathbb{T}_i$ be the subgroup of $\mathbb{T}$ whose elements are of the form $diag(\mathbb{I}_{i\text{-}1}, z, \mathbb{I}_{n\text{-}i\text{-}1}, \bar{z})$ for some $z \in \mathbb{S}^1 \subset \mathbb{C}$. There is a clear isomorphism from $\mathbb{T}_1 \times ... \times \mathbb{T}_{n\text{-}1} \to \mathbb{T}$ given by $(Z_1, ..., Z_{n\text{-}1}) \mapsto Z_1...Z_{n\text{-}1}$, and so we can write any $Z \in \mathbb{T}$ as a product $Z_1...Z_{n\text{-}1}$ where each $Z_i \in \mathbb{T}_i \subset \mathbb{T}$.

Let $\hat{W}$: $\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T} \to \mathrm{SU}(\infty)$ be the composition of the Weyl map with the the inclusion of $\mathrm{SU}(n)$ into $\mathrm{SU}(\infty)$ and let W_∞: $\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T} \to \mathrm{SU}(\infty)$ be given by:

$$W_\infty : ([g], Z_1...Z_{n-1}) \mapsto diag(gZ_1g^{-1}, ..., gZ_{n-1}g^{-1}, 1, ...)$$

Lemma 57. There exists a homotopy between the maps $\hat{W}$ and W_∞.

Proof. Let us define a matrix to assist us in our quest:

$$\mathrm{Rot}(t) = \begin{bmatrix} cos(t\pi/2)\mathbb{I}_n & -sin(t\pi/2)\mathbb{I}_n \\ sin(t\pi/2)\mathbb{I}_n & cos(t\pi/2)\mathbb{I}_n \end{bmatrix}$$

Since each $n \times n$ block is a diagonal matrix, the blocks commute with one another and so

$$\begin{aligned} det(\mathrm{Rot}(t)) &= det(cos(t\pi/2)\mathbb{I}_n \cos(t\pi/2)\mathbb{I}_n - (-sin(t\pi/2)\mathbb{I}_n)sin(t\pi/2)\mathbb{I}_n) \\ &= det(cos^2(t\pi/2)\mathbb{I}_n + sin^2(t\pi/2)\mathbb{I}_n) \\ &= det((cos^2(t\pi/2) + sin^2(t\pi/2))\mathbb{I}_n) \\ &= det(\mathbb{I}_n) = 1 \end{aligned}$$

$$\mathrm{Rot}(t)^{-1} = \begin{bmatrix} cos(t\pi/2)\mathbb{I}_n & sin(t\pi/2)\mathbb{I}_n \\ -sin(t\pi/2)\mathbb{I}_n & cos(t\pi/2)\mathbb{I}_n \end{bmatrix} = \mathrm{Rot}(t)^*$$

Thus $\mathrm{Rot}(t)$ is an element of $\mathrm{SU}(2n)$ $\forall$ t.

This matrix is useful as:

$$\mathrm{Rot}(0) = \begin{bmatrix} cos(0)\mathbb{I}_n & -sin(0)\mathbb{I}_n \\ sin(0)\mathbb{I}_n & cos(0)\mathbb{I}_n \end{bmatrix} = \begin{bmatrix} \mathbb{I}_n & 0 \\ 0 & \mathbb{I}_n \end{bmatrix} = \mathbb{I}_{2n} \qquad \mathrm{Rot}(1) = \begin{bmatrix} cos(\pi/2)\mathbb{I}_n & -sin(\pi/2)\mathbb{I}_n \\ sin(\pi/2)\mathbb{I}_n & cos(\pi/2)\mathbb{I}_n \end{bmatrix} = \begin{bmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{bmatrix}$$

Let $\mathrm{Rot}_i(t) := diag(\mathbb{I}_{(i-1)n}, \mathrm{Rot}(t), 1, ...) \in \mathrm{SU}(\infty)$. $\mathrm{Rot}_i(t)$ has inverse $\mathrm{Rot}_i^{-1}(t) = diag(\mathbb{I}_{(i-1)n}, \mathrm{Rot}(t)^{-1}, 1, ...)$ We can now use these matrices to construct the required homotopy:

$$\begin{aligned} H([g], Z, t) = \{&diag(gZ_1g^{-1}, ..., gZ_ig^{-1}, 1, ...)\mathrm{Rot}_i((n-2)t - i + 1) \\ &diag(\mathbb{I}_{(i-1)n}, gZ_{i+1}g^{-1}...gZ_{n-1}g^{-1}, 1, ...)\mathrm{Rot}_i^{-1}((n-2)t - i + 1), \\ &\frac{i-1}{n-2} \leq t \leq \frac{i}{n-2} \text{ for } 1 \leq i \leq n-2 \end{aligned}$$

We can see that for $t = 0$, we require $i = 1$ and so

$$\begin{aligned}
H([g], Z, 0) &= diag(gZ_1g^{-1}, 1, ...)\mathrm{Rot}_1(0)diag(gZ_2g^{-1}...gZ_{n-1}g^{-1}, 1, ...)\mathrm{Rot}_1^{-1}(0) \\
&= diag(gZ_1g^{-1}, 1, ...)diag(\mathrm{Rot}(0), 1, ...)diag(gZ_2g^{-1}...gZ_{n-1}g^{-1}, 1, ...)diag(\mathrm{Rot}(0)^{-1}, 1, ...) \\
&= diag(gZ_1g^{-1}, 1, ...)diag(\mathbb{I}_{2n}, 1, ...)diag(gZ_2g^{-1}...gZ_{n-1}g^{-1}, 1, ...)diag(\mathbb{I}_{2n}, 1, ...) \\
&= diag(gZ_1g^{-1}, 1, ...)diag(gZ_2g^{-1}...gZ_{n-1}g^{-1}, 1, ...) \\
&= diag(gZ_1g^{-1}...gZ_{n-1}g^{-1}, 1, ...) = \hat{W}([g], Z)
\end{aligned}$$

and for $t = 1$, we require $i = n$ - 2, so

$$\begin{aligned}
H([g], Z, 1) &= diag(gZ_1g^{-1}, ..., gZ_{n-2}g^{-1}, 1, ...)\mathrm{Rot}_{n-2}(1)diag(\mathbb{I}_{(n-3)n}, gZ_{n-1}g^{-1}, 1, ...)\mathrm{Rot}_{n-2}^{-1}(1) \\
&= diag(gZ_1g^{-1}, ..., gZ_{n-2}g^{-1}, 1, ...)diag(\mathbb{I}_{n-3}, \mathrm{Rot}(1), 1, ...) \\
&\quad diag(\mathbb{I}_{n-3}, ..., gZ_{n-1}g^{-1}, 1, ...)diag(\mathbb{I}_{n-3}, \mathrm{Rot}(1)^{-1}, 1, ...) \\
&= diag(gZ_1g^{-1}, ..., gZ_{n-2}g^{-1}, 1, ...)diag(\mathbb{I}_{n-3}, \begin{bmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{bmatrix}, 1, ...) \\
&\quad diag(\mathbb{I}_{n-3}, ..., gZ_{n-1}g^{-1}, 1, ...)diag(\mathbb{I}_{n-3}, \begin{bmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{bmatrix}, 1, ...) \\
&= diag(gZ_1g^{-1}, ..., gZ_{n-2}g^{-1}, 1, ...)diag(\mathbb{I}_{n-3}, \begin{bmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{bmatrix}, 1, ...) \\
&\quad diag(\mathbb{I}_{n-3}, \begin{bmatrix} 0 & gZ_{n-1}g^{-1} \\ -\mathbb{I}_n & 0 \end{bmatrix}, 1, ...) \\
&= diag(gZ_1g^{-1}, ..., gZ_{n-2}g^{-1}, 1, ...)diag(\mathbb{I}_{n-3}, \begin{bmatrix} \mathbb{I}_n & 0 \\ 0 & gZ_{n-1}g^{-1} \end{bmatrix}, 1, ...) \\
&= diag(gZ_1g^{-1}, ..., gZ_{n-2}g^{-1}, 1, ...)diag(\mathbb{I}_{n-2}, gZ_{n-1}g^{-1}, 1, ...) \\
&= diag(gZ_1g^{-1}, ..., gZ_{n-1}g^{-1}, 1, ...) = W_\infty([g], Z)
\end{aligned}$$

Therefore $\hat{W}$ and W_∞ are homotopic maps △

Thus the induced maps in cohomology are equal.

$$\hat{W}^* = W_\infty^* : H^*(\mathrm{SU}(\infty); \mathbb{Q}) \to H^*(\mathrm{SU}(n)/\mathbb{T}; \mathbb{Q}) \otimes H^*(\mathbb{T}; \mathbb{Q})$$

And so the following diagram commutes:

$$\begin{array}{ccc}
H^*(\mathrm{SU}(\infty); \mathbb{Q}) & \xrightarrow{W_\infty^*} & H^*(\mathrm{SU}(n)/\mathbb{T}; \mathbb{Q}) \otimes H^*(\mathbb{T}; \mathbb{Q}) \\
& \searrow^{\iota^*} \quad \square \quad \nearrow^{p^*} & \\
& H^*(\mathrm{SU}(n); \mathbb{Q}) &
\end{array}$$

9.1 Loop Spaces and Line Bundles

The Weyl map has been described earlier as the map:

$$\begin{aligned}
W : \mathrm{SU}(n)/\mathbb{T} \times \mathbb{T} &\to \mathrm{SU}(n) \\
([g], Z) &\mapsto gZg^{-1}
\end{aligned}$$

We have also described the maps from n-1 copies of $\mathbb{S}^1$ to $\mathbb{T}$ and from SU(n) to SU(∞) (which can then be included into U), thus we can extend this to a map:

$$\mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1 \hookrightarrow \mathrm{SU}(n)/\mathbb{T} \times \mathbb{T} \to \mathrm{SU}(n) \hookrightarrow \mathrm{SU}(\infty) \hookrightarrow \mathrm{U}$$

where if the inclusion $\mathbb{S}^1 \hookrightarrow \mathbb{T}$ is given by $z \mapsto Z = diag(z,\ \mathbb{I}_{n-2},\ \overline{z})$, then we have a map:

$$\begin{aligned} \hat{P} : \mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1 &\to \mathrm{U} \\ ([g], z) &\mapsto diag(gZg^{-1}, 1, ...) \end{aligned}$$

$\hat{P}$ factors through the following map:

$$\begin{aligned} P : S(\mathrm{SU}(n)/\mathbb{T}) &\to \mathrm{U} \\ [[g], t] &\mapsto diag(gZg^{-1}, 1, ...) \end{aligned}$$

where $Z = diag(exp(2\pi \boldsymbol{i} t),\ \mathbb{I}_{n-2},\ exp(\text{-}2\pi \boldsymbol{i} t))$ S: **Top** $\to$ **Top** is a covariant functor called the suspension that maps objects X to $SX := (X \times [0,\ 1])/{\sim}$ where $(x,\ 0) \sim (x',\ 0)$ and $(x,\ 1) \sim (x',\ 1)$, and maps morphisms f: $X \to Y$ to morphisms Sf: $SX \to SY$ where we define $Sf([x,\ t]) = [f(x),\ t]$. Notice that our map P is not a pointed map of topological spaces.

An adjunction isomorphism then gives us a map:

$$\begin{aligned} Q : \mathrm{SU}(n)/\mathbb{T} \to \quad & \Omega\mathrm{U} \\ [g] \mapsto G : \mathbb{S}^1 &\to \mathrm{U} \\ z &\mapsto diag(gZg^{-1}, 1, ...) \end{aligned}$$

To better understand this loop let us attempt to factor it through the loop space of a space that is homeomorphic to U.

Let Gr be the space of Hermitian projections E: $\mathbb{C}^\infty \to \mathbb{C}^\infty$ of finite positive rank. i.e. $E^* = E = E^2$ and $rank(E) \geq 0$

Let $X_\bullet$ be the simplicial space defined as follows: $X_0 := (pt)$ and

$$X_n := \{(E_1, ..., E_n) \in Gr^n \mid E_i E_j = 0 \text{ for } i \neq j\}$$

with face maps:

$$\begin{aligned} d_{n,0}(E_1, ..., E_n) &= (E_2, ..., E_n) \\ d_{n,i}(E_1, ..., E_n) &= (E_1, ..., E_i + E_{i+1}, ..., E_n) \text{ for } 1 \leq i \leq n-1 \\ d_{n,n}(E_1, ..., E_n) &= (E_1, ..., E_{n-1}) \end{aligned}$$

and degeneracy maps:

$$s_{n,i}(E_1, ..., E_n) = (E_1, ..., E_i, 0, E_{i+1}, ..., E_n) \text{ for } 0 \leq i \leq n$$

It will be easier to describe the geometric realisation of this simplicial space if the elements of the n-simplex are represented by an n-tuple as opposed to an (n+1)-tuple as we have in the past.

For $(x_0, ..., x_n) \in \Delta^n$, let $t_i = \sum_{j=i}^{n} x_j$, then an element of Δ^n can be written as a n-tuple $(t_1, ..., t_n)$ where $1 \geq t_1 \geq ... \geq t_n \geq 0$.

In this context, the coface maps are the maps:

$$\begin{aligned} d^{n,0}(t_1, ..., t_{n-1}) &= (1, t_1, ..., t_{n-1}) \\ d^{n,i}(t_1, ..., t_{n-1}) &= (t_1, ..., t_{i-1}, t_i, t_i, t_{i+1}, ..., t_{n-1}) \\ d^{n,n}(t_1, ..., t_{n-1}) &= (t_1, ..., t_{n-1}, 0) \end{aligned}$$

The codegeneracy maps are the maps:

$$s^{n,i}(t_1, ..., t_{n+1}) = (t_1, ..., t_i, t_{i+2} ..., t_{n+1})$$

There is a homeomorphism shown by Harris [16]

$$\begin{aligned} \phi : |X_\bullet| &\to \mathrm{U} \\ ((E_1, ..., E_n), (t_1, ..., t_n)) &\mapsto exp(2\pi \boldsymbol{i}(t_1 E_1 + ... + t_n E_n)) \end{aligned}$$

Consider $(E, t) \in X_1 \times \Delta^1 \subset |X_\bullet|$, then:

$$\begin{aligned} \phi(E, t) &= exp(2\pi \boldsymbol{i} t E) \\ &= \sum_{j=0}^{\infty} \frac{(2\pi \boldsymbol{i} t)^j}{j!} E^j \\ &= \mathbb{I} + \sum_{j=1}^{\infty} \frac{(2\pi \boldsymbol{i} t)^j}{j!} E \\ &= \mathbb{I} + E \sum_{j=1}^{\infty} \frac{(2\pi \boldsymbol{i} t)^j}{j!} \\ &= \mathbb{I} + E(z - 1) \end{aligned}$$

where $z = exp(2\pi \boldsymbol{i} t)$

We would like to be able to describe a loop in $|X_\bullet|$ that will map to $Q([g]) = (t \mapsto diag(gZg^{-1}, 1, ...)) \in \Omega\mathrm{U}$ by ϕ.

Let E_i be the projection $diag(\mathbf{0}_{i-1}, 1, 0, ...)$ where $\mathbf{0}_n$ is the $n \times n$ matrix with every entry 0. Let us see if the following path does the job:

$$H([g], t) = \begin{cases} ((gE_n g^{-1}, gE_1 g^{-1}), (1 - t, t)), & 0 \leq t \leq 1/2 \\ ((gE_1 g^{-1}, gE_n g^{-1}), (t, 1 - t)), & 1/2 < t \leq 1 \end{cases}$$

To begin, this path is continuous:

$$\begin{aligned} \text{from below } H([g], 1/2) &= ((gE_n g^{-1}, gE_1 g^{-1}), (1/2, 1/2)) \\ &\sim (gE_n g^{-1} + gE_1 g^{-1}, 1/2) \\ &= (gE_1 g^{-1} + gE_n g^{-1}, 1/2) \\ &\sim ((gE_1 g^{-1}, gE_n g^{-1}), (1/2, 1/2)) \\ &= H([g], 1/2) \text{ from above} \end{aligned}$$

and in fact, the path is a loop through the base point of $|X_\bullet|$:

$$\begin{aligned} H([g],0) &= ((gE_ng^{-1}, gE_1g^{-1}),(1,0)) \\ &\sim (gE_ng^{-1},1) \\ &\sim (pt,pt) \in X_0 \times \Delta^0 \\ \text{and } H([g],1) &= ((gE_1g^{-1}, gE_ng^{-1}),(1,0)) \\ &\sim (gE_1g^{-1},1) \\ &\sim (pt,pt) \in X_0 \times \Delta^0 \end{aligned}$$

Now to see how ϕ transforms the loop, notice that $E_1E_n = E_nE_1$ since both are zero by definition, therefore $\forall$ λ, $\mu \in \mathbb{C}$, and $g \in \mathrm{SU}(n)/\mathbb{T}$ we have $exp(\lambda gE_1g^{-1} + \mu gE_ng^{-1}) = exp(\lambda gE_1g^{-1})exp(\mu gE_ng^{-1})$:

$$\begin{aligned} \phi(H([g],t)) &= exp(2\pi \boldsymbol{i}(tgE_1g^{-1} + (1-t)gE_ng^{-1})) \\ &= exp(2\pi \boldsymbol{i}tgE_1g^{-1})exp(2\pi\boldsymbol{i}(1-t)gE_ng^{-1}) \\ &= \phi(gE_1g^{-1},t)\phi(gE_ng^{-1},1-t) \\ &= (\mathbb{I} + gE_1g^{-1}(z-1))((\mathbb{I} + gE_ng^{-1}(\overline{z}-1)) \text{ where } z = exp(2\pi\boldsymbol{i}t) \\ &= \mathbb{I} + gE_1g^{-1}(z-1) + gE_ng^{-1}(\overline{z}-1) + gE_1g^{-1}(z-1)gE_ng^{-1}(\overline{z}-1) \\ &= g\mathbb{I}g^{-1} + g((z-1)E_1)g^{-1} + g((\overline{z}-1)E_n)g^{-1} + (z-1)(\overline{z}-1)gE_1g^{-1}gE_ng^{-1} \\ &= g(\mathbb{I} + (z-1)E_1 + (\overline{z}-1)E_n)g^{-1} + (z-1)(\overline{z}-1)gE_1E_ng^{-1} \\ &= g(\mathbb{I} + diag(z-1,0,...) + diag(\mathbf{0}_{n-1}, \overline{z}-1,0,...))g^{-1} \\ &= g(diag(z,\mathbb{I}_{n-2},\overline{z},1,...))g^{-1} \\ &= g(diag(Z,1,...))g^{-1} \\ &= diag(gZg^{-1},1,...) \end{aligned}$$

Therefore this loop is indeed the correct choice as it maps to the loop we had in ΩU.

$H([g],\ t)$ is a loop wholly contained in the 2-skeleton of $|X_\bullet|$, it will be advantageous for it to be homotopic to a loop wholly contained in the 1-skeleton of $|X_\bullet|$.

$$K([g],t) = \begin{cases} (gE_ng^{-1}, 1-2t), & 0 \le t \le 1/2 \\ (gE_1g^{-1}, 2t-1), & 1/2 < t \le 1 \end{cases}$$

$K([g],\ t)$ is a continuous loop as $K([g],\ 0) \sim K([g],\ 1/2) \sim K([g],\ 1) \sim (pt,\ pt) \in |X_\bullet|$.

There is a homotopy from $H([g],\ t)$ to $K([g],\ t)$.

$$\Psi([g],t,s) = \begin{cases} ((gE_ng^{-1}, gE_1g^{-1}),(1-t-st, t-st)), & 0 \le t \le 1/2 \\ ((gE_1g^{-1}, gE_ng^{-1}),(t-s(1-t), 1-t-s(1-t)), & 1/2 < t \le 1 \end{cases}$$

$\Psi([g],\ t,\ s)$ is continuous:

$$
\begin{aligned}
\text{from below } \Psi([g], 1/2, s) &= ((gE_ng^{-1}, gE_1g^{-1}), ((1-s)/2, (1-s)/2)) \\
&\sim (gE_ng^{-1} + gE_1g^{-1}, (1-s)/2) \\
&= (gE_1g^{-1} + gE_ng^{-1}, (1-s)/2) \\
&\sim ((gE_1g^{-1}, gE_ng^{-1}), ((1-s)/2, (1-s)/2)) \\
&= \Psi([g], 1/2, s) \text{ from above}
\end{aligned}
$$

$\Psi([g],\ t,\ s)$ is a homotopy from $H([g],\ t)$ to $K([g],\ t)$:

$$
\begin{aligned}
\Psi([g], t, 0) &= \begin{cases} ((gE_ng^{-1}, gE_1g^{-1}), (1-t, t)), & 0 \le t \le 1/2 \\ ((gE_1g^{-1}, gE_ng^{-1}), (t, 1-t), & 1/2 < t \le 1 \end{cases} \\
&= H([g], t) \\
\Psi([g], t, 1) &= \begin{cases} ((gE_ng^{-1}, gE_1g^{-1}), (1-t-t, t-t)), & 0 \le t \le 1/2 \\ ((gE_1g^{-1}, gE_ng^{-1}), (t-(1-t), 1-t-(1-t)), & 1/2 < t \le 1 \end{cases} \\
&\sim \begin{cases} (gE_ng^{-1}, 1-2t), & 0 \le t \le 1/2 \\ (gE_1g^{-1}, 2t-1), & 1/2 < t \le 1 \end{cases} \\
&= K([g], t)
\end{aligned}
$$

The 1-skeleton of ΩU is homeomorphic to the space BU. $K([g],\ t)$ therefore factors through $B\text{U} \cong Gr$. We will attempt to determine how this factorisation works.

Let γ: $Gr \to \Omega\text{U}$ be the map sending a projection E to the loop $exp(2\pi \boldsymbol{i} tE)$ for $0 \le t \le 1$.

The loop $\Omega\phi \circ K([g],\text{-})$ is the concatenation of two such loops:

$$\Omega\phi \circ K([g], -) = \gamma(gE_ng^{-1})^{-1} * \gamma(gE_1g^{-1})$$

There are n-1 inclusions $\mathbb{S}^1 \hookrightarrow \mathbb{T}$ given by $z \mapsto diag(\mathbb{I}_{i-1},\ z,\ \mathbb{I}_{n-i-1},\ \overline{z})$. We have thoroughly examined the case when $i = 1$ but an identical method with an identity matrix block shifted along the diagonal allows us to achieve the result that these inclusions composed with the Weyl map and further included into U are adjoint to maps that are homotopic to the concatenation of two loops and for the i^{th} inclusion, this concatenation is given as $\gamma(gE_ng^{-1})^{-1} * \gamma(gE_ig^{-1})$.

We have $B\text{U}(n) = \{E \in Gr \mid rank(E) = n\}$ and since $E \in Gr \implies E$ is an idempotent, we have $rank(E) = tr(E)$.

As before, let $E_i = diag(\mathbf{0}_{i\text{-}1},\ 1,\ 0,\ ...)$, then:

$$
\begin{aligned}
rank(gE_ig^{-1}) &= tr(gE_ig^{-1}) \\
&= tr(gg^{-1}E_i) \\
&= tr(E_i) \\
&= 1
\end{aligned}
$$

Thus the maps:

$$\begin{aligned} \Gamma_i : \mathrm{SU}(n)/\mathbb{T} &\to \Omega\mathrm{U} \\ [g] &\mapsto \gamma(gE_i g^{-1}) \end{aligned}$$

must factor through $B\mathrm{U}(1) \subset Gr$

$$\begin{array}{ccccc} \mathrm{SU}(n)/\mathbb{T} & & \xrightarrow{\quad \Gamma_i \quad} & & \Omega\mathrm{U} \\ & \searrow^{\varepsilon_i} & \square & \nearrow_{\gamma} & \\ & & B\mathrm{U}(1) & & \end{array}$$

where $\varepsilon_i([g]) = gE_i g^{-1}$.

The homotopy class of ε_i: $\mathrm{SU}(n)/\mathbb{T} \to B\mathrm{U}(1)$ corresponds to a line bundle over $\mathrm{SU}(n)/\mathbb{T}$.

$E\mathrm{U}(1) := \{(E, \mathbf{v}) \in B\mathrm{U}(1) \times \mathbb{C}^\infty \mid \mathbf{v} \in Im(E)\}$ is the total space of the universal line bundle over $B\mathrm{U}(1)$. The line bundle corresponding to the homotopy class of ε_i is the pullback of $E\mathrm{U}(1)$ by ε_i:

$$\begin{array}{ccc} \varepsilon_i^*(E\mathrm{U}(1)) & \longrightarrow & E\mathrm{U}(1) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{SU}(n)/\mathbb{T} & \xrightarrow[\varepsilon_i]{} & B\mathrm{U}(1) \end{array}$$

Then $\varepsilon_i^*(E\mathrm{U}(1)) = \{([g], E, \mathbf{v}) \in \mathrm{SU}(n)/\mathbb{T} \times E\mathrm{U}(1) \mid \varepsilon_i([g]) = E\}$ or equivalently, $\varepsilon_i^*(E\mathrm{U}(1)) = \{([g], \mathbf{v}) \in \mathrm{SU}(n)/\mathbb{T} \times \mathbb{C}^\infty \mid \mathbf{v} \in Im(gE_i g^{-1})\}$

There are n natural projections π_i: $\mathbb{T} \to \mathbb{S}^1$ given by $\pi_i(Z) = z_i$ where, as usual $Z = diag(z_1, ..., z_n)$. For each $1 \le i \le n$, let us define the following space: $L_i := \mathrm{SU}(n) \times_{\pi_i} \mathbb{C} := (\mathrm{SU}(n) \times \mathbb{C})/\sim$ where $\sim$ is the following equivalence relation:

$(g, \pi_i(Z)\lambda) \sim (gZ, \lambda)$ for some $Z \in \mathbb{T}$.

The maps l_i: $L_i \to \mathrm{SU}(n)/\mathbb{T}$ given by $l_i([g, \lambda]) = [g]$ are line bundles.

Lemma 58. The line bundle corresponding to the homotopy class of ε_i is l_i.

Proof. It must be shown that there is an isomorphism Ξ: $L_i \to \varepsilon_i^*(E\mathrm{U}(1))$ and that the following diagram commutes:

$$\begin{array}{ccccc} L_i & & \xrightarrow{\quad \Xi \quad} & & \varepsilon_i^*(E\mathrm{U}(1)) \\ & \searrow^{l_i} & \square & \swarrow & \\ & & \mathrm{SU}(n)/\mathbb{T} & & \end{array}$$

To ensure that the diagram commutes, we require that $\Xi([g, \lambda]) = ([g], \mathbf{v})$ for some vector $\mathbf{v} \in Im(gE_i g^{-1})$. Let us test whether Ξ is a well defined isomorphism when $\mathbf{v} = \lambda g\mathbf{e}_i$.

First we must find some $\mathbf{u} \in \mathbb{C}^\infty$ such that $gE_ig^{-1}\mathbf{u} = \lambda g\mathbf{e}_i$ to ensure that $\lambda g\mathbf{e}_i \in Im(gE_ig^{-1})$.

Let $\mathbf{u} = \lambda g\mathbf{e}_i$

$$\begin{aligned} gE_ig^{-1}\lambda g\mathbf{e}_i &= \lambda gE_ig^{-1}g\mathbf{e}_i \\ &= \lambda gE_i\mathbf{e}_i \\ &= \lambda g\mathbf{e}_i \end{aligned}$$

Thus $\lambda g\mathbf{e}_i \in Im(gE_ig^{-1})$.

We must show that for any $Z = diag(z_1, ..., z_n) \in \mathbb{T}$:

$$\begin{aligned} \Xi([g, z_i\lambda]) &= \Xi([gZ, \lambda]). \\ \Xi([g, z_i\lambda]) &= ([g], z_i\lambda g\mathbf{e}_i) \\ \Xi([gZ, \lambda]) &= ([gZ], \lambda gZ\mathbf{e}_i) \\ &= ([g], \lambda gz_i\mathbf{e}_i) \\ &= ([g], z_i\lambda g\mathbf{e}_i) \text{ as required} \end{aligned}$$

Therefore Ξ is well defined.

Since L_i and $\varepsilon_i^*(E\mathrm{U}(1))$ are the total spaces of line bundles, they necessarily have the same rank = 1 and thus, we must only check that Ξ is injective.

Since $\lambda g\mathbf{e}_i = \mathbf{0} \implies \lambda = 0$, Ξ is injective, and thus an isomorphism. △

Lemma 59. The Chern classes of the line bundles l_i: $L_i \to \mathrm{SU}(n)/\mathbb{T}$ are generators of the cohomology ring $H^*(\mathrm{SU}(n)/\mathbb{T}; \mathbb{Z})$

Proof. We have seen that $\mathrm{SU}(n)/\mathbb{T}$ is homeomorphic to $F_n(\mathbb{C}^n)$ via the map:

$$\begin{aligned} &\Psi : \mathrm{SU}(n)/\mathbb{T} \to F_n(\mathbb{C}^n) \\ &\quad [[\mathbf{v}_1|...|\mathbf{v}_n]] \mapsto (Span(\mathbf{v}_1) \subset ... \subset Span(\mathbf{v}_1, ..., \mathbf{v}_{n-1}) \subset \mathbb{C}^n) \end{aligned}$$

Let the maps γ_i: $\Gamma_i \to F_n(\mathbb{C}^n)$ be the tautological line bundles over $F_n(\mathbb{C}^n)$, i.e. $\Gamma_i := \{(\mathrm{V}, \mathbf{v}) \in F_n(\mathbb{C}^n) \times \mathbb{C}^n \mid \mathbf{v} \in \mathrm{V}_i, \mathbf{v} \perp \mathbf{w}\ \forall\ \mathbf{w} \in \mathrm{V}_{i\text{-}1}\}$ and γ_i is the projection to the flag component.

$H^*(F_n(\mathbb{C}^n); \mathbb{Z})$ is generated by the elements $c_1(\Gamma_i)$ for $1 \leq i \leq n$, so the pullback of these elements by Ψ will generate $H^*(\mathrm{SU}(n)/\mathbb{T}; \mathbb{Z})$. Since Chern classes are compatible with pullbacks, we need to show that each map γ_i is pulled back onto a line bundle over $\mathrm{SU}(n)/\mathbb{T}$ with total space homeomorphic to L_i.

$$\begin{array}{ccc} \Psi^*(\Gamma_i) & \longrightarrow & \Gamma_i \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \gamma_i} \\ \mathrm{SU}(n)/\mathbb{T} & \xrightarrow[\Psi]{} & F_n(\mathbb{C}^n) \end{array}$$

Let us examine the space $\Psi^*(\Gamma_i)$:

$$\begin{aligned}\Psi^*(\Gamma_i) &= \{([g], (\mathrm{V}, \mathbf{v})) \in \mathrm{SU}(n)/\mathbb{T} \times \Gamma_i \mid \Psi([g]) = \gamma_i(\mathrm{V}, \mathbf{v})\}\\ &= \{([g], (\mathrm{V}, \mathbf{v})) \in \mathrm{SU}(n)/\mathbb{T} \times \Gamma_i \mid \Psi([g]) = \mathrm{V}\}\\ &= \{([g], (\mathrm{V}, \mathbf{v})) \in \mathrm{SU}(n)/\mathbb{T} \times \Gamma_i \mid Span(g\mathbf{e}_1, ..., g\mathbf{e}_j) = \mathrm{V}_j,\ 1 \le j \le n\}\\ &= \{([g], \mathbf{v}) \in \mathrm{SU}(n)/\mathbb{T} \times \mathbb{C}^n \mid \mathbf{v} \in Span(g\mathbf{e}_1, ..., g\mathbf{e}_i), \mathbf{v} \perp \mathbf{w}\ \forall\ \mathbf{w} \in Span(g\mathbf{e}_1, ..., g\mathbf{e}_{i-1})\}\\ &= \{([g], \mathbf{v}) \in \mathrm{SU}(n)/\mathbb{T} \times \mathbb{C}^n \mid \mathbf{v} \in Span(g\mathbf{e}_i)\}\end{aligned}$$

To show that this space is homeomorphic to L_i, let us attempt to construct the homeomorphism:

$$\begin{aligned}k : L_i &\to \Psi^*(\Gamma_i)\\ [g, \lambda] &\mapsto ([g], \lambda g\mathbf{e}_i)\end{aligned}$$

To show that this map is well defined, let $Z = diag(z_1, ..., z_n)$ be an element of the maximal torus $\mathbb{T} \subseteq \mathrm{SU}(n)$. We require that $k([gZ, \lambda]) = k([g, z_i\lambda])$ which can be seen since:

$$\begin{aligned}k([gZ, \lambda]) &= ([gZ], \lambda z_i g\mathbf{e}_i)\\ &= ([g], z_i\lambda g\mathbf{e}_i)\\ k([g, z_i\lambda]) &= ([g], z_i\lambda g\mathbf{e}_i)\end{aligned}$$

Since L_i and $\Psi^*(\Gamma_i)$ are the total spaces of line bundles over the same space, thanks to work by Milnor and Stasheff [26], if we can show that k is continuous and an isomorphism on each fibre, then we will know that k is a homeomorphism.

Considering $\Psi^*(\Gamma_i)$ as a subspace of $\mathrm{SU}(n)/\mathbb{T} \times \mathbb{C}^n$, if $\pi_{\mathrm{SU}(n)/\mathbb{T}} \circ \iota \circ k$ and $\pi_{\mathbb{C}^n} \circ \iota \circ k$ are both continuous, then k is continuous.

$$\begin{aligned}(\pi_{\mathrm{SU}(n)/\mathbb{T}} \circ \iota \circ k)([g, \lambda]) &= (\pi_{\mathrm{SU}(n)/\mathbb{T}} \circ \iota)([g], \lambda g\mathbf{e}_i)\\ &= [g]\\ (\pi_{\mathbb{C}^n} \circ \iota \circ k)([g, \lambda]) &= (\pi_{\mathbb{C}^n} \circ \iota)([g], \lambda g\mathbf{e}_i)\\ &= \lambda g\mathbf{e}_i\end{aligned}$$

Since we have seen that k is well defined, and we know that quotient maps and multiplication are continuous, both of these maps are necessarily continuous, thus so is k.

To show that k is a fibre-wise isomorphism let us fix a point $[g] \in \mathrm{SU}(n)/\mathbb{T}$ and examine the fibres over $[G]$:

$$\begin{aligned}\{[g, \lambda] \in L_i \mid l_i([g, \lambda]) = [G]\} &= \{[G, \lambda] \in L_i\}\\ &\cong \mathbb{C}\\ \{([g], \mathbf{v}) \in \Psi^*(\Gamma_i) \mid \Psi^*(\gamma_i)([g], \mathbf{v}) = [G]\} &= \{([G], \mathbf{v}) \in \Psi^*(\Gamma_i)\}\\ &\cong Span(G\mathbf{e}_i) \cong \mathbb{C}\end{aligned}$$

Since $k([G, \lambda]) = ([G], \lambda G\mathbf{e}_i) \in \{([G], \mathbf{v}) \in \Psi^*(\Gamma_i)\}$, we can construct a restriction map:

$$k_{[G]} : \{[G, \lambda] \in L_i\} \to \{([G], \mathbf{v}) \in \Psi^*(\Gamma_i)\}$$
$$[G, \lambda] \mapsto ([G], \lambda G\mathbf{e}_i)$$

Since $G\mathbf{e}_i$ is a non-zero vector in $\mathbb{C}^n$, $\lambda G\mathbf{e}_i = \mathbf{0}$ iff $\lambda = 0$, therefore $k_{[G]}$ is injective, and since both the domain and codomain of $k_{[G]}$ have the same rank, it must be an isomorphism.

Since k is continuous and fibre-wise bijective, k is a homeomorphism and thus each L_i is homeomorphic to the space $\Psi^*(\Gamma_i)$, therefore the Chern classes $c_1(L_i)$ for $1 \leq i \leq n$ form a generating set of $H^*(\mathrm{SU}(n)/\mathbb{T}; \mathbb{Z})$. △

9.2 Differences of Line Bundles

Before we can get into how this all fits together, there is one more thing we must check.

Let us consider the space $[X, \Omega\mathrm{U}]$, the set of all homotopy classes of maps from X to $\Omega\mathrm{U}$. We may equip this set with two binary operations.

Let us equip $[X, \Omega\mathrm{U}]$ with the operation of concatenation of loops. For two loops a,b: $X \to \Omega\mathrm{U}$ we have:

$$(a * b)(x)(t) = \begin{cases} a(x)(2t), & 0 \leq t \leq 1/2 \\ b(x)(2t-1), & 1/2 < t \leq 1 \end{cases}$$

This operation has a unit $[1_*] \in [X, \Omega\mathrm{U}]$, that is the class of maps homotopic to the representative 1_*: $X \to \Omega\mathrm{U}$ where $1_*(x)(t) = \mathbb{I} \in \mathrm{U}$.

Each loop a has a concatenative inverse a_*^{-1} such that $a{*}a_*^{-1}$, $a_*^{-1}{*}a$ and 1_* are all homotopic given by $a_*^{-1}(x)(t) = a(x)(1\text{-}t)$

We may also equip $[X, \Omega\mathrm{U}]$ with the operation of multiplication in U. For two loops a,b: $X \to \Omega\mathrm{U}$ we have:

$$(a \cdot b)(x)(t) = a(x)(t) \cdot b(x)(t)$$

This operation also has a unit $[1_.] \in [X, \Omega\mathrm{U}]$, that is the class of maps homotopic to the representative $1_.$: $X \to \Omega\mathrm{U}$ where $1_.(x)(t) = \mathbb{I} \in \mathrm{U}$.

Again each loop a has a multiplicative inverse $a_.^{-1}$ such that $a{\cdot}a_.^{-1}$, $a_.^{-1}{\cdot}a$ and $1_.$ are all homotopic given by $a_.^{-1}(x)(t) = (a(x)(t))^{-1} \in \mathrm{U}$.

If these operations satisfy the following identity, then by the Eckmann-Hilton argument [8], the two operations are the same and more than that, they are commutative and associative.

We must first show that for any loops a,b,c,d we have:

$$(a * b) \cdot (c * d) = (a \cdot c) * (b \cdot d)$$

Which can be seen to hold in this case as:

$$((a*b)\cdot(c*d))(x)(t) = \begin{cases} a(x)(2t)\cdot c(x)(2t), & 0 \le t \le 1/2 \\ b(x)(2t-1)\cdot d(x)(2t-1), & 1/2 < t \le 1 \end{cases}$$

$$\text{and } ((a\cdot c)*(b\cdot d))(x)(t) = \begin{cases} a(x)(2t)\cdot c(x)(2t), & 0 \le t \le 1/2 \\ b(x)(2t-1)\cdot d(x)(2t-1), & 1/2 < t \le 1 \end{cases}$$

Since this identity holds on loops, it will certainly hold on homotopy classes of loops, and thus on $[X, \Omega\mathrm{U}]$ the operations of concatenation and multiplication in U are the same operation which is commutative and associative. Therefore, since we have defined inverses, $[X, \Omega\mathrm{U}]$ is an abelian group under this operation, hereby denoted with standard addition and subtraction notation.

Now we can begin.

We have a group $[\mathrm{SU}(n)/\mathbb{T}, \Omega\mathrm{U}]$. We would like to know how to describe the map we had before Q: $\mathrm{SU}(n)/\mathbb{T} \to \Omega\mathrm{U}$ where $Q([g])(z) = diag(gZg^{-1}, 1, ...)$ where $Z = diag(z, \mathbb{I}_{n-2}, \overline{z}, 1, ...)$ for $z \in \mathbb{S}^1$, in terms of isomorphism classes of principal bundles over $\mathrm{SU}(n)/\mathbb{T}$.

We considered a space

$$|X_\bullet| = \{((E_1, ..., E_n), (t_1, ..., t_n)) \in \coprod_{n=0}^{\infty} Gr^n \times \Delta^n \mid E_iE_j = 0 \text{ for } i \neq j\}$$

and a homeomorphism ϕ: $|X_\bullet| \to \mathrm{U}$ given by $\phi((E_1, ..., E_n), (t_1, ..., t_n)) = exp(2\pi\boldsymbol{i}(E_1t_1 + ... + E_nt_n))$.

This gives us a clear map $[\Omega\phi \circ \text{-}]$: $[\mathrm{SU}(n)/\mathbb{T}, \Omega|X_\bullet|] \to [\mathrm{SU}(n)/\mathbb{T}, \Omega\mathrm{U}]$ which is a group isomorphism since ϕ is a homeomorphism.

We have a map H: $\mathrm{SU}(n)/\mathbb{T} \to \Omega|X_\bullet|$ given by

$$H([g])(t) = \begin{cases} ((gE_ng^{-1}, gE_1g^{-1}), (1-t, t)), & 0 \le t \le 1/2 \\ ((gE_1g^{-1}, gE_ng^{-1}), (t, 1-t)), & 1/2 < t \le 1 \end{cases}$$

which we determined satisfied the property $\Omega\phi \circ H = Q$ therefore clearly after taking homotopy classes, $[\Omega\phi \circ \text{-}]([H]) = [Q]$.

H was determined to be homotopy equivalent to the map K: $\mathrm{SU}(n)/\mathbb{T} \to \Omega|X_\bullet|$ given by

$$K([g])(t) = \begin{cases} (gE_ng^{-1}, 1-2t), & 0 \le t \le 1/2 \\ (gE_1g^{-1}, 2t-1), & 1/2 < t \le 1 \end{cases}$$

and it was shown that $\Omega\phi \circ K = {\Gamma_n}^{-1} * \Gamma_1$ where Γ_i: $\mathrm{SU}(n)/\mathbb{T} \to \Omega\mathrm{U}$ is the loop given by $\Gamma_i([g])(t) = exp(2\pi\boldsymbol{i}tgE_ig^{-1})$

Since $[H] = [K]$ necessarily, and $[\Omega\phi \circ \text{-}]$ is a well defined group isomorphism, it must be the case that $[Q] = [{\Gamma_n}^{-1} * \Gamma_1] = [\Gamma_n]^{-1} + [\Gamma_1] = [\Gamma_1] - [\Gamma_n]$.

Each Γ_i was shown to be the composition of a map ε_i: $\mathrm{SU}(n)/\mathbb{T} \to B\mathrm{U}(1)$ given by $\varepsilon_i([g]) = gE_ig^{-1}$ and the map γ: $B\mathrm{U} \to \Omega\mathrm{U}$ given by $\gamma(E)(t) = exp(2\pi\boldsymbol{i}tE)$, permitted since $B\mathrm{U}(1)$ is a subspace of $B\mathrm{U}$, $\Gamma_i = \gamma \circ \varepsilon_i$.

In turn, γ is the composition of two maps, let ω: $B\mathrm{U} \to \Omega|X_\bullet|$ be given by $\omega(E) = (E,\, t)$ for $0 \leq t \leq 1$, then $\gamma = \Omega\phi \circ \omega$. Clearly, ω is injective so since $\Omega\phi$ is an isomorphism, γ is also injective.

We now have a map $[\gamma|_{B\mathrm{U}(1)} \circ$ -]: $[\mathrm{SU}(n)/\mathbb{T},\, B\mathrm{U}(1)] \to [\mathrm{SU}(n)/\mathbb{T},\, \Omega\mathrm{U}]$ that is injective where $[\gamma|_{B\mathrm{U}(1)} \circ$ -$]([\varepsilon_i]) = [\Gamma_i]$.

We have discussed the isomorphism Φ: $[X,\, BG] \to \mathcal{P}_G(X)$ which in the case of $X = \mathrm{SU}(n)/\mathbb{T}$ and $G = \mathrm{U}(1)$ translates into finding an appropriate isomorphism class of line bundles over $\mathrm{SU}(n)/\mathbb{T}$ for any homotopy class of maps from $\mathrm{SU}(n)/\mathbb{T}$ to $B\mathrm{U}(1)$. We discovered that $\Phi([\varepsilon_i]) = [l_i]$ where l_i: $\mathrm{SU}(n) \times_{\pi_i} \mathbb{C} \to \mathrm{SU}(n)/\mathbb{T}$

Let $\Psi = [\gamma|_{B\mathrm{U}(1)} \circ$ -$] \circ \Phi^{-1}$. Ψ: $\mathcal{P}_{\mathrm{U}(1)}(\mathrm{SU}(n)/\mathbb{T}) \to [\mathrm{SU}(n)/\mathbb{T},\, \Omega\mathrm{U}]$ is a group homomorphism and we know that $\Psi([l_i]) = [\Gamma_i]$.

And so $[Q] = [\Gamma_1]$ - $[\Gamma_n] = \Psi([l_1])$ - $\Psi([l_n]) = \Psi([l_1]$ - $[l_n])$.

Similarly, the other inclusions $\mathbb{S}^1 \hookrightarrow \mathbb{T}$ induce maps that are adjoint to maps that are homotopic to the difference of line bundles $[l_i]$ - $[l_n]$ for each $1 \leq i \leq n$-1

9.3 The Class of the Weyl Map in K-theory

The Weyl map is given by:

$$W : \mathrm{SU}(n)/\mathbb{T} \times \mathbb{T} \to \mathrm{SU}(n)$$
$$([g], Z) \mapsto gZg^{-1}$$

and by the inclusions of $\mathrm{SU}(n)$ into $\mathrm{SU}(\infty)$ and $\mathrm{SU}(\infty)$ into U, we have a map:

$$\iota \circ W : \mathrm{SU}(n)/\mathbb{T} \times \mathbb{T} \to \mathrm{U}$$
$$([g], Z) \mapsto diag(gZg^{-1}, 1, ...)$$

We can write elements of $\mathbb{T}$ in the form $diag(z_1, ..., z_{n-1}, \overline{z_1...z_{n-1}})$, let us write $Z_i = diag(\mathbb{I}_{i-1}, z_i, \mathbb{I}_{n-i-1}, \overline{z_i})$, we can construct another map

$$W_\infty : \mathrm{SU}(n)/\mathbb{T} \times \mathbb{T} \to \mathrm{U}$$
$$([g], diag(z_1, ..., z_{n-1}, \overline{z_1...z_{n-1}})) \mapsto diag(gZ_1g^{-1}, ..., gZ_{n-1}g^{-1}, 1, ...)$$

We have seen that W is homotopic to W_∞.

There are n - 1 projections $\pi_{\mathbb{S}^1(i)}$: $\mathbb{T} \to \mathbb{S}^1$ given by $\pi_{\mathbb{S}^1(i)}(diag(z_1, ..., z_{n-1}, \overline{z_1...z_{n-1}})) = z_i$

Let us then construct maps w_i for $1 \leq i \leq n$ - 1:

$$w_i : \mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1 \to \mathrm{U}$$
$$([g], z_i) \mapsto diag(\mathbb{I}_{n(i-1)}, gZ_ig^{-1}, 1, ...)$$

Now we may define W_i: $\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T} \to \mathrm{U}$ as $W_i := w_i \circ (\mathrm{id}_{\mathrm{SU}(n)/\mathbb{T}} \times \pi_{\mathbb{S}^1(i)})$.

Then $W_\infty = W_1 \oplus ... \oplus W_{n\text{-}1}$.

$K^1(X) = [X, \mathrm{U}]$ so in $K^1(\mathrm{SU}(n)/\mathbb{T})$ we have:

$$[W] = [W_\infty] = [W_1 \oplus ... \oplus W_{n-1}] = \sum_{i=1}^{n-1}[W_i]$$

Each map $\mathrm{id}_{\mathrm{SU}(n)/\mathbb{T}} \times \pi_{\mathbb{S}^1(i)}$ induces a map

$$(\mathrm{id}_{\mathrm{SU}(n)/\mathbb{T}} \times \pi_{\mathbb{S}^1(i)})^*\colon\ K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1) \to K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T})$$

and so $(\mathrm{id}_{\mathrm{SU}(n)/\mathbb{T}} \times \pi_{\mathbb{S}^1(i)})^*([w_i]) = [W_i]$. We can use the Künneth formulaula and the naturality of the isomorphisms to obtain a commutative diagram:

$$\begin{array}{ccc}
K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1) & \xrightarrow{(\mathrm{id}_{\mathrm{SU}(n)/\mathbb{T}} \times \pi_{\mathbb{S}^1(i)})^*} & K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T}) \\
\cong\downarrow & \square & \downarrow\cong \\
K^0(\mathrm{SU}(n)/\mathbb{T}) \otimes K^1(\mathbb{S}^1) & \xrightarrow[\oplus \mathrm{id}_{K^1} \otimes\ \pi^*_{\mathbb{S}^1(i)}]{\mathrm{id}_{K^0} \otimes \pi^*_{\mathbb{S}^1(i)}} & K^0(\mathrm{SU}(n)/\mathbb{T}) \otimes K^1(\mathbb{T}) \\
\oplus\ K^1(\mathrm{SU}(n)/\mathbb{T}) \otimes K^0(\mathbb{S}^1) & & \oplus\ K^1(\mathrm{SU}(n)/\mathbb{T}) \otimes K^0(\mathbb{T})
\end{array}$$

and so we must determine the behaviour of the induced maps $\pi^*_{\mathbb{S}^1(i)}\colon K^k(\mathbb{S}^1) \to K^k(\mathbb{T})$.

$\mathbb{T}$ is the maximal torus of $\mathrm{SU}(n)$ so in our case it is really $\mathbb{T}^{n-1}$, the cartesian product of n - 1 copies of $\mathbb{S}^1$. $K^0(\mathbb{S}^1) \cong \mathbb{Z}$ and $K^1(\mathbb{S}^1) \cong \mathbb{Z}$ so let us use the Künneth formulaula to construct $K^0(\mathbb{T})$ and $K^1(\mathbb{T})$.

Lemma 60. In the category of groups, $K^0(\mathbb{T}^k) \cong \mathbb{Z}^{2^{k-1}}$ and $K^1(\mathbb{T}^k) \cong \mathbb{Z}^{2^{k-1}}$

Proof. We will perform an induction argument.

The base case is known to hold; $\mathbb{T}^1 = \mathbb{S}^1$ and so $K^0(\mathbb{T}^1) \cong K^0(\mathbb{S}^1) \cong \mathbb{Z} = \mathbb{Z}^{2^0}$ and $K^1(\mathbb{T}^1) \cong K^1(\mathbb{S}^1) \cong \mathbb{Z} = \mathbb{Z}^{2^0}$ as required.

Let us assume $K^0(\mathbb{T}^k) \cong \mathbb{Z}^{2^{k-1}}$ and $K^1(\mathbb{T}^k) \cong \mathbb{Z}^{2^{k-1}}$, since $\mathbb{T}^k$ is the cartesian product of k copies of $\mathbb{S}^1$, we have $\mathbb{T}^{k+1} \cong \mathbb{T}^k \times \mathbb{S}^1$ and so we may apply the Künneth formulaula:

$$\begin{aligned}
K^0(\mathbb{T}^{k+1}) &\cong K^0(\mathbb{T}^k) \otimes K^0(\mathbb{S}^1) \oplus K^0(\mathbb{T}^k) \otimes K^0(\mathbb{S}^1) \\
&\cong \mathbb{Z}^{2^{k-1}} \otimes \mathbb{Z} \oplus \mathbb{Z}^{2^{k-1}} \otimes \mathbb{Z} \\
&\cong \mathbb{Z}^{2^{k-1}} \oplus \mathbb{Z}^{2^{k-1}} \\
&\cong \mathbb{Z}^{2^k} \\
\text{and } K^1(\mathbb{T}^{k+1}) &\cong K^0(\mathbb{T}^k) \otimes K^1(\mathbb{S}^1) \oplus K^1(\mathbb{T}^k) \otimes K^0(\mathbb{S}^1) \\
&\cong \mathbb{Z}^{2^{k-1}} \otimes \mathbb{Z} \oplus \mathbb{Z}^{2^{k-1}} \otimes \mathbb{Z} \\
&\cong \mathbb{Z}^{2^{k-1}} \oplus \mathbb{Z}^{2^{k-1}} \\
&\cong \mathbb{Z}^{2^k}
\end{aligned}$$

△

The copies of $\mathbb{Z}$ in $K^k(\mathbb{T}^n)$ can be efficiently labelled using the Künneth formula, each one is isomorphic to the tensor product of a copy of either $K^0(\mathbb{S}^1)$ or $K^1(\mathbb{S}^1)$ for each of the component circles and so we may label them by an n-tuple $(i_1, \ldots, i_n)$ where each i_j is 0 if the tensor product includes $K^0(\mathbb{S}^{1(j)})$ or 1 if it includes $K^1(\mathbb{S}^{1(j)})$, the K theory groups of the j^{th} component circle.

Naturally,

$$\mathbb{Z}_{(i_1,\ldots,i_n)} \subset K^k(\mathbb{T}^n) \text{ iff } \sum_{j=1}^{n} i_j \in k + 2\mathbb{Z}$$

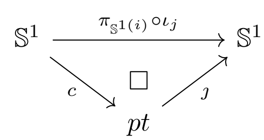
Lemma 61. The restriction maps $\pi_{\mathbb{S}^{1(i)}}\colon\ \mathbb{T} \to \mathbb{S}^1$ induce maps in K-theory $K^0(\pi_{\mathbb{S}^{1(i)}})$ and $K^1(\pi_{\mathbb{S}^{1(i)}})$ such that $Im(K^0(\pi_{\mathbb{S}^{1(i)}})) \cong \mathbb{Z}_{(0,\ldots,0)} \subseteq K^0(\mathbb{T})$ and $Im(K^1(\pi_{\mathbb{S}^{1(i)}})) \cong \mathbb{Z}_{(\mathbf{0}_{i-1},1,\mathbf{0}_{n-i-1})} \subseteq K^1(\mathbb{T})$.

Proof. The restriction maps $\pi_{\mathbb{S}^{1(i)}}\colon\ \mathbb{T} \to \mathbb{S}^1$ each have right inverses in the inclusion maps $\iota_i\colon\ \mathbb{S}^1 \to \mathbb{T}$, so since K^k is a functor we must have $(\pi_{\mathbb{S}^{1(i)}} \circ \iota_i)^* = \mathrm{id}_{K^k(\mathbb{S}^1)}$. For any other inclusion of a component circle into the torus, i.e. for $i \neq j$, we have:

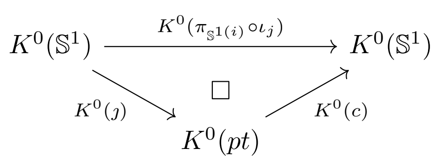
$$\begin{aligned} \pi_{\mathbb{S}^{1(i)}} \circ \iota_j : \mathbb{S}^1 &\to \mathbb{S}^1 \\ z &\mapsto 1 \end{aligned}$$

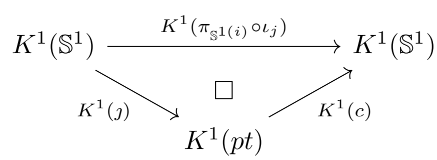
where 1 is the basepoint of the circle.

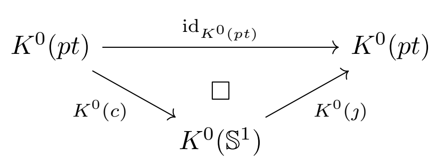
If c: $\mathbb{S}^1 \to pt$ is the map collapsing a circle to a single point and $\jmath$: $pt \to \mathbb{S}^1$ is the incluion of a point to the basepoint of the circle, we see that the following diagram commutes:

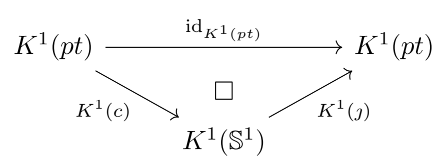


Since additionally, $c \circ \jmath = \mathrm{id}_{pt}$, we must have that the following diagrams all commute:









Since $K^1(pt) \cong 0$ and $K^1(\mathbb{S}^1) \cong \mathbb{Z}$, $K^1(\jmath)$ must be the zero map and $K^1(c)$ must be the inclusion of 0 into $\mathbb{Z}$, therefore $K^1(\pi_{\mathbb{S}^{1(i)}} \circ \iota_j)$ must also be the zero map.

Since $K^0(pt) \cong \mathbb{Z}$ and $K^0(\mathbb{S}^1) \cong \mathbb{Z}$, $K^0(\jmath)$ and $K^0(c)$ must be isomorphisms since the only group homomorphisms p,q: $\mathbb{Z} \to \mathbb{Z}$ such that $q \circ p = \mathrm{id}_{\mathbb{Z}}$ are $p = q = \mathrm{id}_{\mathbb{Z}}$ and $p = q =$ -$\mathrm{id}_{\mathbb{Z}}$. Since in either case $p \circ q = \mathrm{id}_{\mathbb{Z}}$, $K^0(\pi_{\mathbb{S}^1(i)} \circ \iota_j)$ must also be the identity.

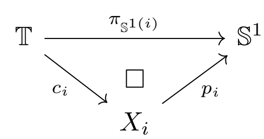
Let us denote by X_i the following space:

$$X_i := \prod_{k=1}^{i-1} pt \times \mathbb{S}^1 \times \prod_{k=1}^{n-i-1} pt$$

Let us denote by $\jmath_i$: $X_i \to \mathbb{T}$ the natural inclusions of each X_i into $\mathbb{T}$. Each $\jmath_i$ has a clear left inverse we will denote c_i: $\mathbb{T} \to X_i$ that collapses all but the i^{th} component circle of $\mathbb{T}$ to a point. Observe how these maps may be written:

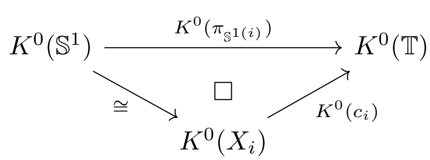
$$\jmath_i = \prod_{k=1}^{i-1} \jmath \times \mathrm{id} \times \prod_{k=1}^{n-i-1} \jmath$$
$$c_i = \prod_{k=1}^{i-1} c \times \mathrm{id} \times \prod_{k=1}^{n-i-1} c$$

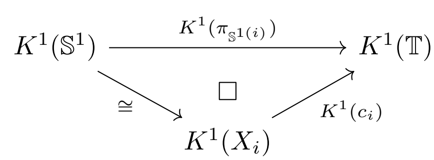
Clearly, each X_i is homeopmorphic to $\mathbb{S}^1$, let us denote the homeomorphism that fits into the following commutative diagram p_i: $X_i \to \mathbb{S}^1$:



Necessarily, $K^0(p_i)$ and $K^1(p_i)$ will be isomorphisms.

Now to the question we actually asked, we would like to understand the behaviour of $K^0(\pi_{\mathbb{S}^1(i)})$ and $K^1(\pi_{\mathbb{S}^1(i)})$, and so since the following diagrams commute:





all we must do is investigate the bahviour of the maps $K^0(c_i)$ and $K^1(c_i)$ which, by the naturality of the Künneth formulaula isomorphisms, decompose once more into the direct sum of the tensor product of a combination of only four maps.

Helpfully, since $K^1(c)$ is the inclusion of 0 into $\mathbb{Z}$, $K^0(c)$ is an isomorphism and both of the other maps are identities, our maps $K^0(c_i)$ and $K^1(c_i)$ factor exclusively through isomorphisms and inclusions.

$$\begin{array}{ccc}
K^0(X_i) & \xrightarrow{\quad K^0(c_i) \quad} & K^0(\mathbb{T}) \\
{\scriptstyle\cong}\big\downarrow & \square & \big\uparrow \\
\bigotimes_{k=1}^{i-1} K^0(pt) \otimes K^0(\mathbb{S}^1) \bigotimes_{k=1}^{n-i-1} K^0(pt) & \xrightarrow[\cong]{} & \mathbb{Z}_{(0,\dots,0)}
\end{array}$$

$$\begin{array}{ccc}
K^1(X_i) & \xrightarrow{\quad K^1(c_i) \quad} & K^1(\mathbb{T}) \\
{\scriptstyle\cong}\big\downarrow & \square & \big\uparrow \\
\bigotimes_{k=1}^{i-1} K^0(pt) \otimes K^1(\mathbb{S}^1) \bigotimes_{k=1}^{n-i-1} K^0(pt) & \xrightarrow[\cong]{} & \mathbb{Z}_{(\mathbf{0}_{i-1},1,\mathbf{0}_{n-i-1})}
\end{array}$$

Therefore the map $K^0(\pi_{\mathbb{S}^{1(i)}})$ is an isomorphism onto $\mathbb{Z}_{(0,\dots,0)} \subseteq K^0(\mathbb{T})$ and $K^1(\pi_{\mathbb{S}^{1(i)}})$ is an isomorphism onto $\mathbb{Z}_{(\mathbf{0}_{i-1},1,\mathbf{0}_{n-i-1})} \subseteq K^1(\mathbb{T})$. △

Lemma 62. The class of the Weyl map $[W] \in K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T})$ has non trivial components only in the subgroup:

$$\tilde{K}^0(\mathrm{SU}(n)/\mathbb{T}) \otimes \bigoplus_{i=1}^{n-1} \mathbb{Z}_{(\mathbf{0}_{i-1},1,\mathbf{0}_{n-i-1})} \subseteq K^1(\mathrm{SU}(n)/\mathbb{T})$$

Proof. We have seen that the class of the Weyl map can be decomposed in the following manner:

$$[W] = \sum_{i=1}^{n-1} [W_i] = \sum_{i=1}^{n-1} (\mathrm{id}_{\mathrm{SU}(n)/\mathbb{T}} \times \pi_{\mathbb{S}^{1(i)}})^*([w_i])$$

and thanks to the previous lemma, we know that the maps $(\mathrm{id}_{\mathrm{SU}(n)/\mathbb{T}} \times \pi_{\mathbb{S}^{1(i)}})^*$: $K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1) \to K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T})$ must first factor through a specific space:

$$\begin{array}{ccc}
K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1) & \xrightarrow{(\mathrm{id}_{\mathrm{SU}(n)/\mathbb{T}} \times \pi_{\mathbb{S}^{1(i)}})^*} & K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T}) \\
{\scriptstyle\cong}\big\downarrow & \square & \big\uparrow \\
\begin{array}{c} K^0(\mathrm{SU}(n)/\mathbb{T}) \otimes K^1(\mathbb{S}^1) \\ \oplus\, K^1(\mathrm{SU}(n)/\mathbb{T}) \otimes K^0(\mathbb{S}^1) \end{array} & \xrightarrow[\cong]{} & \begin{array}{c} K^0(\mathrm{SU}(n)/\mathbb{T}) \otimes \mathbb{Z}_{(\mathbf{0}_{i-1},1,\mathbf{0}_{n-i-1})} \\ \oplus\, K^1(\mathrm{SU}(n)/\mathbb{T}) \otimes \mathbb{Z}_{(\mathbf{0}_{i-1},0,\mathbf{0}_{n-i-1})} \end{array}
\end{array}$$

Therefore we must have:

$$\begin{aligned}
[W] \in K^0(\mathrm{SU}(n)/\mathbb{T}) \otimes \bigoplus_{i=1}^{n-1} \mathbb{Z}_{(\mathbf{0}_{i-1},1,\mathbf{0}_{n-i-1})} \\
\oplus\, K^1(\mathrm{SU}(n)/\mathbb{T}) \otimes \mathbb{Z}_{(0,\dots,0)} \subseteq K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T})
\end{aligned}$$

We must determine the nature of the classes $[w_i] \in K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1)$.

Consider the space $A := (\{[1]\} \times \mathbb{S}^1) \cup (\mathrm{SU}(n)/\mathbb{T} \times \{1\}) \subset \mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1$. The inclusion $\iota\colon\ A \to \mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1$ is a cofibration, therefore the pair $(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1,\ A)$ has the homotopy extension property with respect to all spaces, particularly, with respect to the space U.

We need to determine the behaviour of w_i on elements of A which are either of the form $([g],\ 1)$ or $([\mathbb{I}_n],\ z)$ for $[g] \in \mathrm{SU}(n)/\mathbb{T}$, $z \in \mathbb{S}^1$.

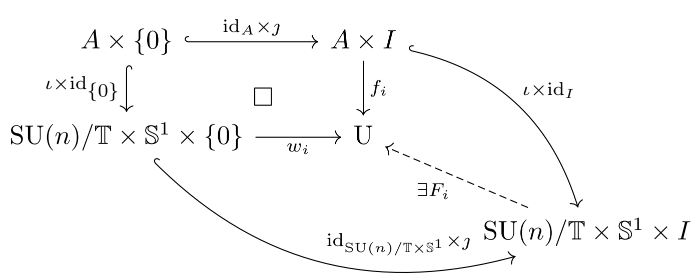
$$\begin{aligned} w_i([g], 1) &= diag(\mathbb{I}_{n(i-1)}, g\mathbb{I}_n g^{-1}, 1, ...) \\ &= \mathbb{I} \\ w_i([\mathbb{I}_n], z) &= diag(\mathbb{I}_{n(i-1)}, \mathbb{I}_n diag(\mathbb{I}_{i-1}, z, \mathbb{I}_{n-i-1}, \overline{z})\mathbb{I}_n, 1, ...) \\ &= diag(\mathbb{I}_{(n+1)(i-1)}, z, \mathbb{I}_{n-i-1}, \overline{z}, 1, ...) \end{aligned}$$

Let us define a homotopy $f_i\colon\ A \times I \to \mathrm{U}$ in the following manner:

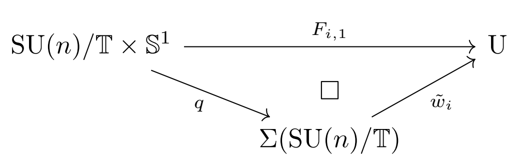
$$f_i([g], z, t) = \begin{cases} \mathbb{I}, & ([g], z) \in \mathrm{SU}(n)/\mathbb{T} \times \{1\} \subset A \\ diag(\mathbb{I}_{(n+1)(i-1)}, exp(2\pi \boldsymbol{i}(1-t)s), & \\ \quad \mathbb{I}_{n-i-1}, exp(-2\pi \boldsymbol{i}(1-t)s), 1, ...), & ([g], z) \in \{[\mathbb{I}_n]\} \times \mathbb{S}^1 \subset A \end{cases}$$

where $z = exp(2\pi \boldsymbol{i} s)$ for some $s \in I$.

By design we have that $f_i([g],\ z,\ 0) = w_i([g],\ z)$ for $([g],\ z) \in A$, so the conditions of the homotopy extension property hold and we can construct a homotopy F_i to fit into the commutative diagram:



Additionally, we have that $f_i([g],\ z,\ 1) = \mathbb{I}$, so $F_{i,1}([g],\ z) := F_i([g],\ z,\ 1) = \mathbb{I}$ for $([g],\ z) \in A$, therefore $F_{i,1}$ necessarily factors through $\Sigma(\mathrm{SU}(n)/\mathbb{T})$. Let $\tilde{w}_i$ be the map that makes the following diagram commute:



In K-theory therefore, we have $[w_i] = [F_{i,1}]$ since w_i and $F_{i,1}$ are homotopic, and thus $[w_i] = q^*[\tilde{w}_i]$.

So in order to better understand our class $[W]$, now we must determine the behaviour of $q^*\colon\ K^1(\Sigma(\mathrm{SU}(n)/\mathbb{T})) \to K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1)$.

Since $\Sigma(\mathrm{SU}(n)/\mathbb{T}) = (\mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1)/A$, let us consider the long exact sequence induced by the pair $(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1,\, A)$ where ι_X is the map induced by the inclusion of $(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1,\, \emptyset)$ into $(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1,\, A)$ and ι_A is the map induced by the inclusion of A into $\mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1$:

$$\begin{array}{ccccc}
K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1, A) & \xrightarrow{\iota_X} & K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1) & \xrightarrow{\iota_A} & K^1(A) \\
\cong\uparrow & \square & \uparrow q* & & \\
K^1(\Sigma(\mathrm{SU}(n)/\mathbb{T}), pt) & \xrightarrow[\cong]{} & K^1(\Sigma(\mathrm{SU}(n)/\mathbb{T})) & &
\end{array}$$

Form this diagram we can see that $Im(q^*) = Im(\iota_X) = Ker(\iota_A)$. Notice additionally, that while $(\{[1]\} \times \mathbb{S}^1) \cup (\mathrm{SU}(n)/\mathbb{T} \times \{1\}) = A$, we also have $(\{[1]\} \times \mathbb{S}^1) \cap (\mathrm{SU}(n)/\mathbb{T} \times \{1\}) = \{pt\}$ and so let us use a Mayer-Vietoris sequence:

$$\begin{array}{ccccc}
K^1(A) & \xrightarrow{(\jmath_1^*, \jmath_2^*)} & K^1(\{[1]\} \times \mathbb{S}^1) \oplus K^1(\mathrm{SU}(n)/\mathbb{T} \times \{1\}) & \xrightarrow{\iota_1^* - \iota_2^*} & K^1(pt) \\
\uparrow & & & & \downarrow \\
K^0(pt) & \xleftarrow[\iota_1^* - \iota_2^*]{} & K^0(\{[1]\} \times \mathbb{S}^1) \oplus K^0(\mathrm{SU}(n)/\mathbb{T} \times \{1\}) & \xleftarrow[(\jmath_1^*, \jmath_2^*)]{} & K^0(A)
\end{array}$$

where ι_1, ι_2 are the inclusions of a point into $\mathbb{S}^1$ and $\mathrm{SU}(n)/\mathbb{T}$ respectively, and $\jmath_1$, $\jmath_2$ are the inclusions of $\{[1]\} \times \mathbb{S}^1$ and $\mathrm{SU}(n)/\mathbb{T} \times \{1\}$ into A respectively.

Since we know that $K^1(pt) \cong 0$, $K^0(pt) \cong \mathbb{Z}$, and $K^1(\mathbb{S}^1) \cong \mathbb{Z}$, we can rewrite our sequence:

$$\begin{array}{ccccc}
K^1(A) & \overset{(\jmath_1^*, \jmath_2^*)}{\twoheadrightarrow} & K^1(\{[1]\} \times \mathbb{S}^1) \oplus K^1(\mathrm{SU}(n)/\mathbb{T} \times \{1\}) & \longrightarrow & 0 \\
\uparrow & & & & \downarrow \\
\mathbb{Z} & \xleftarrow[\iota_1^* - \iota_2^*]{} & \mathbb{Z} \oplus K^0(\mathrm{SU}(n)/\mathbb{T} \times \{1\}) & \underset{(\jmath_1^*, \jmath_2^*)}{\leftarrowtail} & K^0(A)
\end{array}$$

Additionally, since ι_1 is the inclusion of a point into the circle, $K^0(\iota_1)$ is an isomorphism, and so ι_1^*-ι_2^*: $\mathbb{Z} \oplus K^0(\mathrm{SU}(n)/\mathbb{T} \times \{1\}) \to \mathbb{Z}$ is a surjection as $(\iota_1^*\text{-}\iota_2^*)(z,\, 0) = \iota_1^*(z)$, so $Im(\iota_1^*\text{-}\iota_2^*) = Im(\iota_1^*) = \mathbb{Z}$, and therefore the connecting map into $K^1(A)$ is the zero map. Thus by exactness:

$$(\jmath_1^*, \jmath_2^*) : K^1(A) \to K^1(\{[1]\} \times \mathbb{S}^1) \oplus K^1(\mathrm{SU}(n)/\mathbb{T} \times \{1\})$$

is an isomorphism.

Now that we have products again, we can use the Künneth formulaula to show that $K^1(A)$ is isomorphic to $K^0(pt) \otimes K^1(\mathbb{S}^1) \oplus K^1(\mathrm{SU}(n)/\mathbb{T}) \otimes K^0(pt)$. Thus in order to find the kernel of ι_A, all we must do is determine the kernel of the map naturally induced by the Künneth formulaula:

$$\begin{array}{ccc}
K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1) & \xrightarrow{\cong} & K^0(\mathrm{SU}(n)/\mathbb{T}) \otimes K^1(\mathbb{S}^1) \oplus K^1(\mathrm{SU}(n)/\mathbb{T}) \otimes K^0(\mathbb{S}^1) \\
\iota_A \downarrow & \square & \downarrow (\iota_2^* \otimes \mathrm{id}_{\mathbb{S}^1}^*) \oplus (\mathrm{id}_{\mathrm{SU}(n)/\mathbb{T}}^* \otimes \iota_1^*) \\
K^1(A) & \xrightarrow[\cong]{} & K^0(pt) \otimes K^1(\mathbb{S}^1) \oplus K^1(\mathrm{SU}(n)/\mathbb{T}) \otimes K^0(pt)
\end{array}$$

The two identities and ι_1^* are all isomorphisms so all of their kernels are 0, let us see how that affects the kernel:

$$\begin{aligned}
Ker((\iota_2^* \otimes \mathrm{id}_{\mathbb{S}^1}^*) \oplus (\mathrm{id}_{\mathrm{SU}(n)/\mathbb{T}}^* \otimes \iota_1^*)) &= Ker(\iota_2^* \otimes \mathrm{id}_{\mathbb{S}^1}^*) \oplus Ker(\mathrm{id}_{\mathrm{SU}(n)/\mathbb{T}}^* \otimes \iota_1^*) \\
&= (Ker(\iota_2^*) \otimes K^1(\mathbb{S}^1) \cup K^0(\mathrm{SU}(n)/\mathbb{T}) \otimes 0) \\
&\quad \oplus (0 \otimes K^0(\mathbb{S}^1) \cup K^1(\mathrm{SU}(n)/\mathbb{T}) \otimes 0) \\
&= Ker(\iota_2^*) \otimes K^1(\mathbb{S}^1) \cup 0 \oplus 0 \cup 0 \\
&= Ker(\iota_2^*) \otimes K^1(\mathbb{S}^1) \\
&= \tilde{K}^0(\mathrm{SU}(n)/\mathbb{T}) \otimes K^1(\mathbb{S}^1)
\end{aligned}$$

since ι_2 is the inclusion of a point into $\mathrm{SU}(n)/\mathbb{T}$.

Therefore we must have that $Ker(\iota_A) = \tilde{K}^0(\mathrm{SU}(n)/\mathbb{T}) \otimes K^1(\mathbb{S}^1)$ as a subset of $K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{S}^1)$, and since $Ker(\iota_A) = Im(\iota_X) = Im(q^*)$ and $[w_i] \in Im(q^*)$, we must have $[w_i] \in \tilde{K}^0(\mathrm{SU}(n)/\mathbb{T}) \otimes K^1(\mathbb{S}^1)$ which in turn ensures that:

$$\begin{aligned}
[W] \in\ & \tilde{K}^0(\mathrm{SU}(n)/\mathbb{T}) \otimes \bigoplus_{i=1}^{n-1} \mathbb{Z}_{(\mathbf{0}_{i-1},1,\mathbf{0}_{n-i-1})} \\
&\subseteq K^0(\mathrm{SU}(n)/\mathbb{T}) \otimes K^1(\mathbb{T}) \\
&\subset K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T})
\end{aligned}$$

$\triangle$

Theorem 63. The class of the Weyl map in $K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T})$ is given by:

$$[W] = \bigoplus_{i=1}^{n-1} \Phi([L_i] - [L_n]) \otimes z_i$$

where Φ is an isomorphism, and for each i, z_i is a generator of $K^1(\mathbb{S}^{1(i)})$, and L_i is the i^{th} tautological line bundle over $\mathrm{SU}(n)/\mathbb{T}$.

Proof. Earlier we have seen that, in unreduced K-theory, the class of the map

$$\begin{aligned}
P_1 : S(\mathrm{SU}(n)/\mathbb{T}) &\to \mathrm{U} \\
[[g], t] &\mapsto diag(g.Z_1.g^{-1}, 1, ...)
\end{aligned}$$

where $Z_1 = diag(exp(2\pi \boldsymbol{i} t),\ \mathbb{I}_{n-2},\ exp(\text{-}2\pi \boldsymbol{i} t))$, is equal to the difference of the classes of two line bundles $[L_1]$ - $[L_n]$ (after a fair few isomorphisms), and without much further effort, we can see that the class of the map:

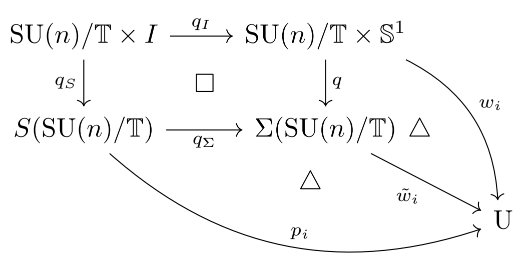
$$\begin{aligned}
P_i : S(\mathrm{SU}(n)/\mathbb{T}) &\to \mathrm{U} \\
[[g], t] &\mapsto diag(g.Z_i.g^{-1}, 1, ...)
\end{aligned}$$

where $Z_i = diag(\mathbb{I}_{i-1},\ exp(2\pi \boldsymbol{i} t),\ \mathbb{I}_{n-i-1},\ exp(\text{-}2\pi \boldsymbol{i} t))$, is in turn equal to the difference of the classes of two line bundles $[L_i]$ - $[L_n]$.

We could easily produce a homotopy, thus ensuring they represent the same class in $K^1(S(\mathrm{SU}(n)/\mathbb{T})$, between each map P_i and a counterpart map:

$$p_i : S(\mathrm{SU}(n)/\mathbb{T}) \to \mathrm{U}$$
$$[[g], t] \mapsto diag(\mathbb{I}_{n(i-1)}, P_i([g], t))$$

In the following diagram we have quotient maps q_I, q_S and q_Σ that together with our original quotient map q, form the innermost square which commutes as $q \circ q_I = q_\Sigma \circ q_S$. Additionally, since the maps w_i have been seen to factor through $S(\mathrm{SU}(n)/\mathbb{T})$, the outermost square also commutes as $w_i \circ q_I = p_i \circ q_S$. Additionally, additionally, as we have already described, there is a homotopy between w_i and $\tilde{w}_i \circ q$, thus by composition with q_I there is a homotopy between $w_i \circ q_I$ and $\tilde{w}_i \circ q \circ q_I$, thus by using the commutative squares there is a homotopy between $p_i \circ q_S$ and $\tilde{w}_i \circ q_\Sigma \circ q_S$, and thus there is a homotopy between p_i and $\tilde{w}_i \circ q_\Sigma$:



Since $\mathrm{SU}(n)/\mathbb{T}$ is a based CW-complex, q_Σ is a homotopy equivalence and thus induces isomorphisms ${q_\Sigma}^*$: $K^k(\Sigma(\mathrm{SU}(n)/\mathbb{T})) \to K^k(S(\mathrm{SU}(n)/\mathbb{T}))$.

The homotopy between p_i and $\tilde{w}_i \circ q_\Sigma$ ensures that $[p_i] = {q_\Sigma}^*[\tilde{w}_i] \in K^1(S(\mathrm{SU}(n)/\mathbb{T}))$, which, since ${q_\Sigma}^*$ is an isomorphism, means that $[\tilde{w}_i] = {q_\Sigma}^{*\text{-}1}[p_i] \in K^1(\Sigma(\mathrm{SU}(n)/\mathbb{T}))$. Therefore, since $[w_i] = q^*[\tilde{w}_i]$ as we have seen, we must have $[w_i] = q^*({q_\Sigma}^{*\text{-}1}[p_i])$.

We know that $[p_i] = \Xi([L_i]$ - $[L_n])$ therefore:

$$[W] = \bigoplus_{i=1}^{n-1} q^*(q_\Sigma^{*-1}(\Xi([L_i] - [L_n]))) \otimes z_i$$

and since we only require information about the restriction of q^* to the map $q^*|$: $\{[w_i] \mid 1 \leq i \leq n\text{-}1\} \to \{[\tilde{w}_i] \mid 1 \leq i \leq n\text{-}1\}$ which is an isomorphism of sets, we achieve the result that

$$[W] = \bigoplus_{i=1}^{n-1} \Phi([L_i] - [L_n]) \otimes z_i$$

and thus our theorem holds. $\triangle$

10 Finally Twisting K-Theory

10.1 The Tensor Chern Character

For each fixed integer d we have described a strict symmetric monoidal category $\mathscr{C}_\otimes$ with objects $\mathbb{N}$ and morphism sets

$$\mathrm{Hom}_{\mathscr{C}_\otimes}(n,m) = \begin{cases} \mathrm{U}((\mathbb{C}^d)^{\otimes n}), & n = m \\ \emptyset, & else \end{cases}$$

Since $\mathscr{C}_\otimes$ is a strict symmetric monoidal category, we were able to construct the reduced cohomology theory $h^*_\otimes$: **Top*** $\to$ **AbGrp** defined as

$$h^k_\otimes(X) = \begin{cases} [X, \Omega^{(-k+1)}B(B\mathrm{U}[\frac{1}{d}] \times \mathbb{Z})], & k \leq 0 \\ [X, B^{(k)}(B\mathrm{U}[\frac{1}{d}] \times \mathbb{Z})], & k > 0 \end{cases}$$

Theorem 64. There is a natural transformation *logch*: $h^*_\otimes \to H^*_{per}$ that will be called the **tensor Chern character** at it is analogous to the Chern character ch: $K^* \to H^*_{per}$ that we are already familiar with.

Proof. It can be seen that $h^0_\otimes(X) \subseteq \mathrm{GL}(1,\ K^0(X) \otimes \mathbb{Z}[\frac{1}{d}])$.

$$\begin{aligned} h^0_\otimes(X) &= [X, \Omega B(B\mathrm{U}[\frac{1}{d}] \times \mathbb{Z})] \\ &= [X, B\mathrm{U}[\frac{1}{d}] \times \mathbb{Z}] \\ &\subseteq [X, B\mathrm{U}[\frac{1}{d}] \times \mathbb{Z}[\frac{1}{d}]] \\ &... \\ &= \mathrm{GL}(1, K^0(X) \otimes \mathbb{Z}[\frac{1}{d}]) \end{aligned}$$

We will be able to make good use of this in order to construct *logch*.

ch: $K^0(X) \to H^0_{per}(X;\ \mathbb{Q})$ is the Chern character we are already familiar with. We may tensor with the identity on $\mathbb{Z}[\frac{1}{d}]$ to achieve $ch \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{d}]}$: $K^0(X) \otimes \mathbb{Z}[\frac{1}{d}] \to H^0_{per}(X;\ \mathbb{Q}) \otimes \mathbb{Z}[\frac{1}{d}]$ which can then be composed with the isomorphism Ψ: $H^0_{per}(X;\ \mathbb{Q}) \otimes \mathbb{Z}[\frac{1}{d}] \to H^0_{per}(X;\ \mathbb{Q})$ given by:

$$\Psi(\sum_{i=0}^{n} \phi_i \otimes z_i) = \sum_{i=0}^{n} z_i \phi_i$$

possible because $\mathbb{Z}[\frac{1}{d}]$ is a subset of $\mathbb{Q}$.

Since ch, $\mathrm{id}_{\mathbb{Z}[\frac{1}{d}]}$, and Ψ are all ring homomorphisms, restricting to invertible elements in the source space also restricts to invertible elements in the target space and so we have:

$$\mathrm{GL}(1, \Psi \circ (ch \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{d}]})) : \mathrm{GL}(1, K^0(X) \otimes \mathbb{Z}[\frac{1}{d}]) \to \mathrm{GL}(1, H^0_{per}(X; \mathbb{Q}))$$

and with a further restriction:

$$\mathrm{GL}(1, \Psi \circ (ch \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{d}]}))|_{h^0_\otimes(X)} : h^0_\otimes(X) \to \mathrm{GL}(1, H^0_{per}(X;\mathbb{Q}))$$

Now to discuss the map from GL(1, $H^0_{per}(X;\mathbb{Q})$) to $H^0_{per}(X;\mathbb{Q})$.

For X a CW-complex $H^i(X;\mathbb{Q}) \cong 0$ for $i > N$ where N is the dimension of X and $i < 0$ and so $H^0_{per}(X;\mathbb{Q})$ is a graded commutative ring where for all i,$j \in 2\mathbb{N}$, $H^i(X;\mathbb{Q}) \smile H^j(X;\mathbb{Q}) \subseteq H^{i+j}(X;\mathbb{Q})$ and $H^n(X;\mathbb{Q}) = 0$ for $n > M$ where $M \in 2\mathbb{N}$ is such that N - 1 $\leq M \leq N$ where N is the dimension of X.

Elements of $H^0_{per}(X;\mathbb{Q})$ will be written as $\phi_0 + \phi_2 + ... + \phi_M$ where each ϕ_i is a class in $H^i(X;\mathbb{Q})$.

We must determine what elements of $H^0_{per}(X;\mathbb{Q})$ are elements of GL(1, $H^0_{per}(X;\mathbb{Q})$).

Lemma 65. If R is a graded commutative ring

$$R = \bigoplus_{n\in\mathbb{N}} R_n$$

with $R_i.R_j \subseteq R_{i+j}$ and $R_n = 0$ for $n > N$ for some $N \in \mathbb{N}$,

then GL(1, R) $= \{r_0 + ... + r_N \in R \mid r_0 \in$ GL(1, R_0)$\}$

Proof. Let $r = r_0 + ... + r_N$ be an invertible element of R. Then $\exists\ s = s_0 + ... + s_N \in R$ such that $r.s = 1$.

$$\begin{aligned} r.s &= 1 \\ (r_0 + ... + r_N).(s_0 + ... + s_N) &= 1 \\ r_0.s_0 + r_0.(s - s_0) + (r - r_0).s_0 + (r - r_0).(s - s_0) &= 1 \\ \text{so } r_0.s_0 &= 1 \\ \text{and } r_0.(s - s_0) + (r - r_0).s_0 + (r - r_0).(s - s_0) &= 0 \end{aligned}$$

Therefore, if $r = r_0 + ... + r_N$ is an invertible element of R, we require that r_0 be an invertible element of R_0.

Now, let $r = r_0 + ... + r_N$ be an element of R with r_0 an invertible element of R_0. We would like to show that r is an invertible element of R.

Consider an element of R of the form 1 - x where $x \in R\backslash R_0$. It is easy to show that 1 - x is an invertible element of R.

$$\begin{aligned} (1-x).(1+x) &= 1 - x^2 \\ (1-x).(1+x).(1+x^2) &= 1 - x^4 \\ &... \\ (1-x).\prod_{i=0}^{k-1}(1+x^{2^i}) &= 1 - x^{2^k} = 1 \end{aligned}$$

since there will always be some k such that $2^k > N$, $x^n \in R\backslash(\bigoplus_{i=0}^{N} R_i)$ for $n > N$, and $R_n = 0$ for $n > N$. Therefore 1 - x is an invertible element of R.

$r = r_0 + ... + r_N$ is an element of R where r_0 is an invertible element of R_0.

$$\begin{aligned} r.r_0^{-1} = (r_0 + r - r_0).r_0^{-1} &= 1 + (r - r_0).r_0^{-1} \\ &= 1 + (r.r_0^{-1} - 1) \\ &= 1 - (1 - r.r_0^{-1}) \end{aligned}$$

$$r.r_0^{-1}.\prod_{i=0}^{k-1}(1 + (1 - r.r_0^{-1})^{2^i}) = 1 - (1 - r.r_0^{-1})^{2^k} = 1$$

Therefore if r_0 is an invertible element of R_0, r is an invertible element of R and thus GL(1, R) $= \{r_0 + ... + r_N \mid r_0 \in$ GL(1, R_0)$\}$. △

From this lemma, we can see that the elements of GL(1, $H^0_{per}(X; \mathbb{Q})$) are those elements $\phi_0 + \phi_2 + ... + \phi_M \in H^0_{per}(X; \mathbb{Q})$ such that ϕ_0 is an invertible element of $H^0(X; \mathbb{Q})$.

For X a path connecteded CW-complex, $H^0(X; \mathbb{Q}) \cong \mathbb{Q}$ so GL(1, $H^0_{per}(X; \mathbb{Q})$) are those elements $\phi_0 + \phi_2 + ... + \phi_M \in H^0_{per}(X; \mathbb{Q})$ such that $\phi_0 \neq 0$.

Finally we need to describe a map Ӝ: GL(1, $H^0_{per}(X; \mathbb{Q})$) $\to H^0_{per}(X; \mathbb{Q})$ such that Ӝ$(\phi \smile \psi)$ = Ӝ(ϕ) + Ӝ(ψ)

The initial thought is that Ӝ should be the logarithm, we will use the Taylor expansion:

$$\begin{aligned} log(1 + x) &= x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + ... \\ &= \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k!} \end{aligned}$$

however, let us apply the logarithm to an element of GL(1, $H^0_{per}(X; \mathbb{Q})$) of the form $\phi_0 + \phi$:

$$\begin{aligned} log(\phi_0 + \phi) &= log(\phi_0 \smile (1 + \phi \smile \phi_0^{-1})) \\ &= log(\phi_0) + log(1 + \phi \smile \phi_0^{-1}) \\ &= log(\phi_0) + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{(\phi \smile \phi_0)^k}{k!} \end{aligned}$$

Since $log(\phi_0 + \phi)$ - log(ϕ_0) is a polynomial in elements of $H^0_{per}(X; \mathbb{Q})$, it too is an element of $H^0_{per}(X; \mathbb{Q})$. However, there is no guarantee that $log(\phi_0) \in H^0_{per}(X; \mathbb{Q})$ so it cannot be that Ӝ = *log* as simply as that. Luckily, in order to define *logch* we don't require that Ӝ be defined on all of GL(1, $H^0_{per}(X; \mathbb{Q})$), just on the subset Im(GL(1, $\Psi \circ (ch \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{d}]})))|_{h^0_{\otimes}(\mathrm{X})}$) and it is only the component in degree 0 that needs special attention.

Since $h^0_\otimes(X) \subset$ GL(1, $K^0(X) \otimes \mathbb{Z}[\frac{1}{d}]$), Im(GL(1, $\Psi \circ (ch \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{d}]})))|_{h^0_\otimes(\mathrm{X})}$) $\subset Im$(GL(1, $\Psi \circ (ch \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{d}]})$)). We will define a map using this slightly larger space for ease of use.

We need to find a map

$$Ш : Im(\mathrm{GL}(1, \Psi \circ (ch \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{d}]}))) \cap H^0(X;\mathbb{Q}) \to H^0_{per}(X;\mathbb{Q})$$

such that Ш($\phi \smile \psi$) = Ш(ϕ) + Ш(ψ).

Then Ж($\phi_0 + \phi$):= Ш(ϕ_0) + $log(\phi_0 + \phi)$ - $log(\phi_0)$ will behave as we require.

First we must determine the invertible elements of $K^0(X) \otimes \mathbb{Z}[\frac{1}{d}]$.

For X path connected

GL(1, $K^0(X) \otimes \mathbb{Z}[\frac{1}{d}]$) $\cong$ {[E] - [F] $\in K^0(X)$ | dim([E] - [F]) $\in$ GL(1, $\mathbb{Z}[\frac{1}{d}]$)}

$$\mathrm{GL}(1, \mathbb{Z}[\frac{1}{d}]) \cong \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{p|d} \mathbb{Z}$$

where p is prime.

The isomorphism is given as follows for $d = p_1^{a_1}...p_n^{a_n}$:

$$\Phi : \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{p|d} \mathbb{Z} \to \mathrm{GL}(1, \mathbb{Z}[\frac{1}{d}])$$
$$(u, b_1, ..., b_n) \mapsto u \cdot p_1^{b_1}...p_n^{b_n}$$

$$\mathrm{GL}(1, \Psi \circ (ch \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{d}]}))(\sum_{i=0}^{N}([E_i] - [F_i]) \otimes z_i) = \sum_{i=0}^{N} z_i \cdot ch([E_i] - [F_i])$$

$$ch([E] - [F]) = dim([E] - [F]) + \sum_{k=1}^{\infty} \frac{s_k(c([E] - [F]))}{k!}$$

Im(GL(1, $\Psi \circ (ch \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{d}]})$)) $\cap H^0(X; \mathbb{Q}) \cong$ GL(1, $\mathbb{Z}[\frac{1}{d}]$)

And so we need to define a map w: GL(1, $\mathbb{Z}[\frac{1}{d}]$) $\to \mathbb{Q}$ with $w(xy) = w(x) + w(y)$ in order to define Ш.

$$\begin{array}{ccc} Im(\mathrm{GL}(1, \Psi \circ (ch \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{d}]}))) \cap H^0(X;\mathbb{Q}) & \xrightarrow{Ш} & H^0_{per}(X;\mathbb{Q}) \\ \cong\downarrow & \square & \downarrow\cong \\ \mathrm{GL}(1, \mathbb{Z}[\frac{1}{d}]) & \xrightarrow[w]{} & \mathbb{Q} \end{array}$$

For $d = p^k$ for some prime p and some $k \in \mathbb{N}$ this is easy. GL(1, $\mathbb{Z}[\frac{1}{d}]$) = $\{u \oplus p^n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$ so we can construct the map w_d as follows:

$$w_d : \mathrm{GL}(1, \mathbb{Z}[\frac{1}{d}]) \to \mathbb{Q}$$
$$u \oplus p^n \mapsto \frac{n}{k}$$

and so we have

$$\begin{aligned} w_d((u \oplus p^n) \cdot (v \oplus p^m)) &= w_d((uv \oplus p^{n+m})) \\ &= \frac{n+m}{k} \\ &= \frac{n}{k} + \frac{m}{k} \\ &= w_d(u \oplus p^n) + w_d(v \oplus p^m) \end{aligned}$$

as required

Then the tensor Chern character can be given as:

$$logch = \text{ЖK} \circ \mathrm{GL}(1, \Psi \circ (ch \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{d}]}))|_{h^0_\otimes(X)} : h^0_\otimes(X) \to H^0_{per}(X; \mathbb{Q})$$

△

Since $\breve{H}^*_{per}$ is a graded vector space over $\mathbb{Q}$, any homomorphism of graded rings $\breve{\eta}$: $\breve{k}^* \to \breve{H}^*_{per}$ will extend to a natural transformation η: $k^* \to H^*_{per}$

We need to determine the nature of $\breve{h}^*_\otimes$. And so we must evaluate $h^n_\otimes(\mathbb{S}^0)$

$$h^n_\otimes(\mathbb{S}^0) = \begin{cases} [\mathbb{S}^0, \Omega^{(-n+1)} B(B\mathrm{U}[\frac{1}{d}] \times \mathbb{Z})], & n \leq 0 \\ [\mathbb{S}^0, B^{(n)}(B\mathrm{U}[\frac{1}{d}] \times \mathbb{Z})], & n > 0 \end{cases}$$

$$h^n_\otimes(\mathbb{S}^0) = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}[\frac{1}{d}], & n \in 2\mathbb{Z} \cap (\mathbb{Z} \backslash \mathbb{N}) \\ 0, & else \end{cases}$$

10.2 The Class of the Weyl Map in $h^*_\otimes$

As we have seen, we must have a strong symmetric monoidal functor between strict symmetric monoidal categories in order to produce a natural transformation of cohomology theories. Our focus is on exponential functors which are only required to be monoidal functors, however there is a family of exponential functors that are strong symmetric monoidal functors.

10.2.1 The Determinant and Powers Thereof

The exponential functors det^m: $\mathscr{C}_\oplus \to \mathscr{C}_\otimes$ are strong symmetric monoidal functors as $det^m(AB) = det^m(A)det^m(B) = det^m(B)det^m(A) = det^m(BA)$ since multiplication in the underlying field must be commutative. Therefore, each of these exponential functors induces a natural transformations of cohomology theories τ: $h^*_\oplus \to h^*_\otimes$.

Since $det^m(\mathbb{C}) = \mathbb{C}$, we know that these instances of $\mathscr{C}_\otimes$ are when $d = 1$, and we have already discussed the cohomology theory generated by this strict symmetric monoidal category:

$$h^k_\otimes(X) = \tilde{H}^k(X; \mathbb{Z}) \times \tilde{H}^{k+2}(X; \mathbb{Z})$$

Now let us concern ourselves with the Weyl map again. We have seen that the map $W\colon \mathrm{SU}(n)/\mathbb{T} \times \mathbb{T} \to \mathrm{SU}(n)$ includes into U and therefore defines a class $[W] \in K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T})$. We have also seen that our cohomology theory constructed from $\mathscr{C}_\oplus$ is connective K-theory, so $h^1_\oplus(X) \cong K^1(X)$ and clearly $[W]$ is also a class in this cohomology theory.

Theorem 66. The natural transformation of cohomology theories generated by the symmetric exponential functor det^m ensures we have a class $\tau^1([W]) \in h^1_\otimes(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T})$. This class is given by:

$$\tau^1([W]) = \sum_{i=1}^{n-1} m.(c_1([L_1]) - c_1([L_n])) \otimes z_i$$

Proof. $h^1_\otimes(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T}) \cong H^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T}; \mathbb{Z}) \times H^3(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T}; \mathbb{Z})$ but we can show that the class $\tau^1([W])$ lies in the subset $\{0\} \times H^3(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T}; \mathbb{Z})$.

The functor $det^m\colon \mathscr{C}_\oplus \to \mathscr{C}_\otimes$ induces a continuous map between topological spaces $|N(det)|\colon |N(\mathscr{C}_\oplus)| \to |N(\mathscr{C}_\otimes)|$ which in turn induces a continuous map on the classifying spaces $B|N(det)|\colon B|N(\mathscr{C}_\oplus)| \to B|N(\mathscr{C}_\otimes)|$.

We have seen that $B|N(\mathscr{C}_\oplus)| \simeq \mathrm{U}$, $\mathscr{C}_\otimes \cong \mathscr{N} \times \mathscr{U}(1)$, products commute with the classifying space construction up to homotopy, and it is not too tricky to see that $|N(\mathscr{N})| \simeq \mathbb{N}$ since for any set X, if $\mathscr{X}$ is the category with X as a collection of objects and only the required identity morphisms from each object to itself, then $|N(\mathscr{X})| \simeq X$.

Additionally, since $B(\mathbb{N}) \simeq \mathrm{U}(1)$, we would like to show that the following diagrams commute up to homotopy:

$$\begin{array}{ccc} B|N(\mathscr{C}_\oplus)| & \xrightarrow{B|N(det^m)|} & B|N(\mathscr{C}_\otimes)| \\ \Xi\downarrow & \triangle & \downarrow\simeq \\ B|N(\mathscr{N})| & \xleftarrow[\pi_{B|N(\mathscr{N})|}]{} & B|N(\mathscr{N})| \times B|N(\mathscr{U}(1))| \end{array} \qquad \begin{array}{ccc} B|N(\mathscr{C}_\oplus)| & \xrightarrow{\Xi} & B|N(\mathscr{N})| \\ \simeq\downarrow & \triangle & \downarrow\simeq \\ \mathrm{U} & \xrightarrow[det]{} & \mathrm{U}(1) \end{array}$$

where $\Xi\colon B|N(\mathscr{C}_\oplus)| \to B|N(\mathscr{N})|$ is the continuous map induced by the functor $\xi\colon \mathscr{C}_\oplus \to \mathscr{N}$ that sends each object to itself (since both are labeled by $\mathbb{N}$), and each morphism to the appropriate identity morphism.

It is not too tricky to show that the first diagram commutes up to homotopy as every map is induced by a functor and clearly the following diagram commutes in **Cat**:

$$\begin{array}{ccc} \mathscr{C}_\oplus & \xrightarrow{det^m} & \mathscr{C}_\otimes \\ \xi\downarrow & \square & \downarrow\cong \\ \mathscr{N} & \xleftarrow[\pi_{\mathscr{N}}]{} & \mathscr{N} \times \mathscr{U}(1) \end{array}$$

After applying a series of functors we achieve the diagram we want but it does not commute on the nose since $B|N(\mathscr{N} \times \mathscr{U}(1))| \simeq B|N(\mathscr{N})| \times B|N(\mathscr{U}(1))|$ is only a homotopy equivalence.

The second diagram is a question of showing that two maps are representatives of the same class in $[B|N(\mathscr{C}_\oplus)|,\ \mathrm{U}(1)] \cong H^1(\mathrm{U};\ \mathbb{Z})$ since $B|N(\mathscr{C}_\oplus)| \simeq \mathrm{U}$, and $\mathrm{U}(1) \simeq K(\mathbb{Z},\ 1)$.

The universal coefficient theorem states that:

$$H^1(\mathrm{U};\mathbb{Z}) \cong [H_1(\mathrm{U}),\mathbb{Z}] \oplus Ext(H_0(\mathrm{U}),\mathbb{Z})$$

and since $H_0(X)$ is a free abelian group for any topological space and $Ext(A,\ B) = 0$ if A is a free group, we achieve $H^1(\mathrm{U};\ \mathbb{Z}) \cong [H_1(\mathrm{U}),\ \mathbb{Z}]$. furthermore, since $H_1(X) \cong \pi_1(X)_{\mathrm{ab}}$, $[G,\ A] \cong [G_{\mathrm{ab}},\ A]$ for any group G and abelian group A, and $\mathbb{Z}$ is abelian, $[H_1(\mathrm{U}),\ \mathbb{Z}] = [\pi_1(\mathrm{U}),\ \mathbb{Z}]$. Since also, $\pi_1(\mathrm{U}(1)) \cong \mathbb{Z}$, then $[\pi_1(\mathrm{U}),\ \mathbb{Z}] = [\pi_1(\mathrm{U}),\ \pi_1(\mathrm{U}(1))]$.

The functor $\xi\colon\ \mathscr{C}_\oplus \to \mathscr{N}$, as well as inducing the map $\Xi\colon\ B|N(\mathscr{C}_\oplus)| \to B|N(\mathscr{N})|$, further induces a map $\Omega\Xi\colon\ \Omega B|N(\mathscr{C}_\oplus)| \to \Omega B|N(\mathscr{N})|$ which makes the following diagram commute up to homotopy:

$$\begin{array}{ccc} \Omega B|N(\mathscr{C}_\oplus)| & \xrightarrow{\Omega\Xi} & \Omega B|N\mathscr{N})| \\ \simeq\big\downarrow & \square & \big\downarrow\simeq \\ B\mathrm{U} \times \mathbb{Z} & \xrightarrow[\pi_{\mathbb{Z}}]{} & \mathbb{Z} \end{array}$$

Therefore, since $B\mathrm{U}$ is connected, the induced map $\pi_0(\Omega\Xi)\colon\ \pi_0(\Omega B|N(\mathscr{C}_\oplus)|) \to \pi_0(\Omega B|N(\mathscr{N})|)$ makes the following diagram commute:

$$\begin{array}{ccc} \pi_0(\Omega B|N(\mathscr{C}_\oplus)|) & \xrightarrow{\pi_0(\Omega\Xi)} & \pi_0(\Omega B|N\mathscr{N})|) \\ \cong\big\downarrow & \square & \big\downarrow\cong \\ \mathbb{Z} & \xrightarrow[\mathrm{id}_{\mathbb{Z}}]{} & \mathbb{Z} \end{array}$$

Therefore since $\pi_0(\Omega X) \cong \pi_1(X)$ for any pointed topological space X by the suspension loop adjunction, $[\pi_1(\Xi)] = [\mathrm{id}_{\mathbb{Z}}] \in [\pi_1(\mathrm{U}),\ \pi_1(\mathrm{U}(1))]$

det: $\mathrm{U} \to \mathrm{U}(1)$ also induces a map $\pi_1(det)\colon\ \pi_1(\mathrm{U}) \to \pi_1(\mathrm{U}(1))$. The following map is a generator of $\pi_1(\mathrm{U})$:

$$\begin{aligned} g : \mathbb{S}^1 &\to \mathrm{U} \\ z &\mapsto diag(z, 1) \end{aligned}$$

and since clearly $det \circ g = \mathrm{id}_{\mathbb{S}^1}$, and $\mathrm{id}_{\mathbb{S}^1}$ is a generator of $\pi_1(\mathrm{U}(1))$, $\pi_1(det)$ is an isomorphism and $[\pi_1(det)] = [\mathrm{id}_{\mathbb{Z}}]$.

Therefore the two maps are in the same class, and thus the diagram commutes up to homotopy.

All this to say that, since the Weyl map factors through $\mathrm{SU}(\infty)$, and $det^m(A) = 1$ for any $A \in \mathrm{SU}(\infty)$, $\pi_{H^1}[B|N(det^m)| \circ W] = [0] \in H^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T};\ \mathbb{Z})$.

We therefore know that the image of the Weyl map lies totally inside the supspace $\{0\} \times H^3(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T};\ \mathbb{Z}) \subseteq h^1_\otimes$

The strong symmetric exponential functor det^m induces a natural transformation of cohomology theories τ. Thus for any topological space X, we have the following commutative diagram in the category of graded rings:

$$\begin{array}{ccc} h^*_\oplus(X \times \mathbb{T}^n) & \xrightarrow{\tau_{X\times\mathbb{T}^n}} & h^*_\otimes(X \times \mathbb{T}^n) \\ \cong\downarrow & \square & \downarrow\cong \\ h^*_\oplus(X) \otimes \Lambda^*_\mathbb{Z}[z_1, ..., z_n] & \xrightarrow[\tau_X\otimes \mathrm{id}]{} & h^*_\otimes(X) \otimes \Lambda^*_\mathbb{Z}[z_1, ..., z_n] \end{array}$$

Since the class of the Weyl map lies in $h^1_\oplus(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T})$, we only need to concern ourselves with the following commutative diagram in the category of abelian groups:

$$\begin{array}{ccc} h^1_\oplus(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T}) & \xrightarrow{\tau^1_{\mathrm{SU}(n)/\mathbb{T}\times\mathbb{T}}} & h^1_\otimes(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T}) \\ \cong\downarrow & \square & \downarrow\cong \\ \bigoplus_{j=0}^{n-1} h^{1-j}_\oplus(\mathrm{SU}(n)/\mathbb{T}) \otimes \Lambda^j_\mathbb{Z}[z_1, ..., z_{n-1}] & \xrightarrow[(\tau_{\mathrm{SU}(n)/\mathbb{T}}\otimes \mathrm{id})]{} & \bigoplus_{j=0}^{n-1} h^{1-j}_\otimes(\mathrm{SU}(n)/\mathbb{T}) \otimes \Lambda^j_\mathbb{Z}[z_1, ..., z_{n-1}] \end{array}$$

At this point we can further reduce the scope of our diagram. We have seen that $h^n_\oplus(X) \cong K^n(X)$ for any topological space X and $n \leq 1$, and also we have seen that $K^0(\mathbb{T}^n) \cong \Lambda^{even}_\mathbb{Z}[z_1, ..., z_n]$, and $K^1(\mathbb{T}^n) \cong \Lambda^{odd}_\mathbb{Z}[z_1, ..., z_n]$. These facts together show us that the left vertical map in our diagram must be the Künneth formula isomorphism.

We described the class of the Weyl map as an element of a subset:

$$[W] \in \bigoplus_{i=1}^{n-1} \tilde{K}^0(\mathrm{SU}(n)/\mathbb{T}) \otimes \mathbb{Z}_{\mathbf{0}_{i-1},1,\mathbf{0}_{n-i-1}} \subseteq K^0(\mathrm{SU}(n)/\mathbb{T}) \otimes K^1(\mathbb{T})$$

and here we find a natural identification, $\mathbb{Z}_{\mathbf{0}_{i-1},1,\mathbf{0}_{n-i-1}} \cong \Lambda^1_\mathbb{Z}[z_i]$. Our diagram naturally simplifies to:

$$\begin{array}{ccc} K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T}) & \xrightarrow{\tilde{\tau}^1_{\mathrm{SU}(n)/\mathbb{T}\times\mathbb{T}}} & h^1_\otimes(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T}) \\ \iota\uparrow & \square & \uparrow\iota \\ K^0(\mathrm{SU}(n)/\mathbb{T}) \otimes \Lambda^1_\mathbb{Z}[z_1, ..., z_{n-1}] & \xrightarrow[\tilde{\tau}^0_{\mathrm{SU}(n)/\mathbb{T}}\otimes \mathrm{id}]{} & h^0_\otimes(\mathrm{SU}(n)/\mathbb{T}) \otimes \Lambda^1_\mathbb{Z}[z_1, ..., z_{n-1}] \end{array}$$

where $\tilde{\tau}^i_X$ is the precomposition of τ^i_X with the isomorphism $K^i(X) \to h^i_\oplus(X)$ for $i \leq 1$.

We have also seen that, in the case of powers of the determinant, $h^n_\otimes(X) \cong H^n(X; \mathbb{Z}) \times H^{n+2}(X; \mathbb{Z})$, and that the class of the Weyl map is wholly contained in a single copy of $H^3(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T}; \mathbb{Z}) \subseteq h^1_\otimes(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T})$ so we only need to concern ourselves with the elements of degree 3 on the right hand side of our diagram to achieve:

$$\begin{array}{ccc} K^1(\mathrm{SU}(n)/\mathbb{T}\times\mathbb{T}) & \xrightarrow{\pi_{H^3}\circ\tilde{\tau}^1_{\mathrm{SU}(n)/\mathbb{T}\times\mathbb{T}}} & H^3(\mathrm{SU}(n)/\mathbb{T}\times\mathbb{T};\mathbb{Z}) \\ \iota\uparrow & \square & \uparrow\iota \\ K^0(\mathrm{SU}(n)/\mathbb{T})\otimes\Lambda^1_{\mathbb{Z}}[z_1,...,z_{n-1}] & \xrightarrow[\pi_{H^2}\circ\tilde{\tau}_{\mathrm{SU}(n)/\mathbb{T}}\otimes\mathrm{id}]{} & H^2(\mathrm{SU}(n)/\mathbb{T};\mathbb{Z})\otimes\Lambda^1_{\mathbb{Z}}[z_1,...,z_{n-1}] \end{array}$$

and from this we can see that in order to determine where the class of the Weyl map ends up in $h^1_{\otimes}(\mathrm{SU}(n)/\mathbb{T}\times\mathbb{T})$, the only thing we need to do is describe the maps $K^0(\mathrm{SU}(n)/\mathbb{T})\to H^2(\mathrm{SU}(n)/\mathbb{T};\,\mathbb{Z})$ induced by powers of the determinant.

Recall that:

$$[W]=\bigoplus_{i=1}^{n-1}([L_i]-[L_n])\otimes z_i\in\tilde{K}^0(\mathrm{SU}(n)/\mathbb{T})\otimes\Lambda^1_{\mathbb{Z}}[z_1,...,z_{n-1}]\subseteq K^1(\mathrm{SU}(n)/\mathbb{T}\times\mathbb{T})$$

where each L_i is the total space of a tautological line bundle over $\mathrm{SU}(n)/\mathbb{T}$.

The exponential functor $det^m\colon\mathscr{C}_{\oplus}\to\mathscr{C}_{\otimes}$ is built using maps:

$$\begin{aligned} det^m:\mathrm{U}(n)&\to\mathrm{U}(1)\\ A&\mapsto(det(A))^m \end{aligned}$$

Since $\mathrm{U}=colim(\mathrm{U}(0)\hookrightarrow\mathrm{U}(1)\hookrightarrow...\hookrightarrow\mathrm{U}(n)\hookrightarrow...)$, and each of the following diagrams commute:

$$\begin{array}{ccc} \mathrm{U}(n) & \xrightarrow{det^m} & \mathrm{U}(1) \\ \iota\downarrow & \square & \downarrow= \\ \mathrm{U}(n+1) & \xrightarrow[det^m]{} & \mathrm{U}(1) \end{array}$$

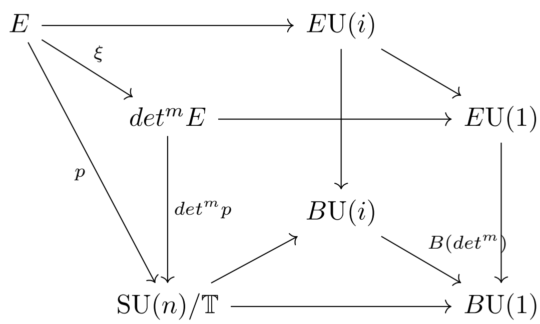
where $\iota(A)=diag(A,\,1)$, since $det(A)=det(diag(A,\,1))$. We can easily define a group homomorphism $det^m\colon\,\mathrm{U}\to\mathrm{U}(1)$ since every element of U has a representative of the form $diag(A,\,1,\,...)$ for $A\in\mathrm{U}(N)$ for some sufficiently large N. Since this map is a group homomorphism, we can construct a map on the classifying spaces $B(det^m)\colon\,B\mathrm{U}\to B\mathrm{U}(1)$.

Post composition with projection onto the $B\mathrm{U}$ component and this map together with the isomorphism $c_1\colon\,[X,\,B\mathrm{U}(1)]\to H^2(X;\,\mathbb{Z})$ that sends an isomorphism class of complex line bundles to it's first Chern class is exactly the map we were after:

$$K^0(\mathrm{SU}(n)/\mathbb{T})\xrightarrow{\cong}[\mathrm{SU}(n)/\mathbb{T},B\mathrm{U}\times\mathbb{Z}]\xrightarrow{B(det^m)^*}[\mathrm{SU}(n)/\mathbb{T},B\mathrm{U}(1)]\xrightarrow{c_1}H^2(\mathrm{SU}(n)/\mathbb{T};\mathbb{Z})$$

Since the colimit and classifying space construction commute, we can investigate the nature of the map $B(det^m)^*\colon\,[\mathrm{SU}(n)/\mathbb{T},\,B\mathrm{U}\times\mathbb{Z}]\to[\mathrm{SU}(n)/\mathbb{T},\,B\mathrm{U}(1)]$ by investigating how each of the maps $B(det^m)^*\colon\,[\mathrm{SU}(n)/\mathbb{T},\,B\mathrm{U}(\mathrm{i})]\to[\mathrm{SU}(n)/\mathbb{T},\,B\mathrm{U}(1)]$ behave.

Since the set of homotopy classes of maps from a space X into the classifying space $B\mathrm{U}(i)$ is isomorphic to the set of isomorphism classes of i-dimensional complex vector bundles, let $p\colon E \to \mathrm{SU}(n)/\mathbb{T}$ be an i-dimensional vector bundle, then the vector bundle $det^m p\colon det^m E \to \mathrm{SU}(n)/\mathbb{T}$ is the line bundle the makes the following diagram commute:



The map $\xi\colon E \to det^m E$ sends each fibre of $p_x \subseteq E$ to the fibre $(det^m p)_x \subseteq det^m E$ for each $x \in \mathrm{SU}(n)/\mathbb{T}$ and the restriction map is given by:

$$\begin{aligned} \xi|_{p_x} : p_x &\to (det^m p)_x \\ A &\mapsto (det(A))^m \end{aligned}$$

Again thanks to the colimit we are able to use these maps to define a group homomorphism that sends any isomorphism class of vector bundles to the isomorpism class of it's determinant bundle.

It can be shown fibrewise that if $p\colon E \to \mathrm{SU}(n)/\mathbb{T}$ and p'$\colon E$' $\to \mathrm{SU}(n)/\mathbb{T}$ are representatives of two isomorphism classes of vector bundles over $\mathrm{SU}(n)/\mathbb{T}$ then $det^m(E) \cong det(E)^{\otimes m}$, and $det^m(E \oplus E') \cong det^m(E) \otimes det^m(E')$ and so $c_1(det^m(E \oplus E')) = c_1(det^m(E)) + c_1(det^m(E')) = m(c_1(det(E))$ - $c_1(det(E')))$ once we take Chern classes. Therefore we have our map:

$$\begin{aligned} (det^m)^* : K^0(\mathrm{SU}(n)/\mathbb{T}) &\to H^2(\mathrm{SU}(n)/\mathbb{T}; \mathbb{Z}) \\ [E] - [F] &\mapsto m(c_1(det(E)) - c_1(det(F))) \end{aligned}$$

Additionally, if a vector bundle $p\colon E \to \mathrm{SU}(n)/\mathbb{T}$ can be written as a direct sum of line bundles, i.e. $E \cong L_1 \oplus ... \oplus L_k$ and $p \cong p_{L_1} \oplus ... \oplus p_{L_k}$ where each $p_{L_i}\colon L_i \to \mathrm{SU}(n)/\mathbb{T}$ is a complex line bundle, then:

$$\begin{aligned} c_1(det^m(E)) &= c_1(det^m(L_1 \oplus ... \oplus L_k)) \\ &= c_1(det(L_1 \oplus ... \oplus L_k)^{\otimes m}) \\ &= m.c_1(det(L_1 \oplus ... \oplus L_k)) \\ &= m.c_1(det(L_1) \otimes ... \otimes det(L_k)) \\ &= m.c_1(L_1 \otimes ... \otimes L_k) \\ &= m.(c_1(L_1) + ... + c_1(L_k)) \\ &= m.c_1(E) \end{aligned}$$

So since the class of the Weyl map only has information concerning classes of line bundles which are clearly isomorphic to a direct sum of line bundles, we finally achieve the result that the map:

$$\begin{aligned} W : \mathrm{SU}(n)/\mathbb{T} \times \mathbb{T} &\to \mathrm{SU}(n) \\ ([g], Z) &\mapsto gZg^{-1} \end{aligned}$$

when passed through a series of maps, including one induced by the expoential functor det^m: $\mathscr{C}_\oplus \to \mathscr{C}_\otimes$:

$$\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T} \xrightarrow{W} \mathrm{SU}(n) \hookrightarrow \mathrm{SU}(\infty) \hookrightarrow \mathrm{U} \xrightarrow{(det^m)^*} B B\mathrm{U}_\otimes[\tfrac{1}{1}]$$

gives us a class $\tau([W]) \in h^1_\otimes(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T})$ and:

$$\tau([W]) = \sum_{i=1}^{n-1} m(c_1(L_i) - c_1(L_n)) \otimes z_i$$

where each L_i is the total space of the i^{th} tautological line bundle over $\mathrm{SU}(n)/\mathbb{T}$, and each z_i is a generator of $H^1(\mathbb{S}^1; \mathbb{Z})$, one for each natural inclusion of a circle into the torus $\mathbb{T}$. △

10.2.2 Non-Symmetric Exponential Functors

Not all exponential functors are created equally. It is not a necessary condition that they must preserve the symmetry between the categories

Theorem 67. Let F: $\mathscr{C}_\oplus \to \mathscr{C}_\otimes$ be an exponential functor that is not necessarily a strong symmetric monoidal functor. If τ^1: $h^1_\oplus \to h^1_\otimes$ is the natural transformation induced by F between the first degree cohomology groups and W is the Weyl map, then the class $\tau^1([W]) \in h^1_\otimes(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T})$ is given by:

$$\tau^1([W]) = \bigodot_{i=1}^{n-1}([F(L_i) \otimes \bigotimes_{j=1}^{n-1} F(L_j)] \otimes \frac{1}{d^n}) \otimes z_i$$

where $\odot$ is the binary operation of $h^0_\otimes(\mathrm{SU}(n)/\mathbb{T}) \otimes \Lambda^1_\mathbb{Z}[z_1, ..., z_n]$ given by:

$$(a \otimes x) \odot (b \otimes y) = ab \otimes x + y$$

Proof. Since F is not necessarily strong symmetric, we do not necessarily obtain a natural transformation of cohomology theories induced by F, however we do still clearly achieve a homomorphism of topological monoids $|N(F)|$: $|N(\mathscr{C}_\oplus)| \to |N(\mathscr{C}_\otimes)|$ simply by performing the nerve and geometric realisation functors. The monoid operation in each of these monoids is induced by the monoidal product in each of their respective categories.

Since we have a monoid homomorphism, we can perform the classifying space construction to achieve a continuous map of pointed topological spaces

$B|N(F)|\colon\ B|N(\mathscr{C}_\oplus)| \to B|N(\mathscr{C}_\otimes)|$ and it is post composition with this map that induces a natural transformation $\tau^1\colon\ h^1_\oplus \to h^1_\otimes$.

Clearly, we can apply the loop functor to this map to achieve $\Omega B|N(F)|\colon$ $\Omega B|N(\mathscr{C}_\oplus)| \to \Omega B|N(\mathscr{C}_\otimes)|$ and similarly, post composition with this map induces a natural transformation $\tau^0\ h^0_\oplus \to h^0_\otimes$

Critically, these two natural transformations are identically equal to those we can construct if we make an attempt at a full blown natural transformation of cohomology theories using Γ-categories.

Even though we don't have a natural transformation of cohomology theories, for any cohomology theory we do still have the isomorphism:

$$h^*(X \times \mathbb{T}^n) \cong h^*(X) \otimes \Lambda^*_{\mathbb{Z}}[z_1, ..., z_n]$$

and as we have already seen, for the cohomology theory $h^*_\oplus$ this isomorphism of graded rings restricted to the isomorphism of abelian groups in the first degree:

$$h^1_\oplus(X \times \mathbb{T}^n) \cong \bigoplus_i^n h^{1-i}_\oplus(X) \otimes \Lambda^i_{\mathbb{Z}}[z_1, ..., z_n]$$

is identically the map given by the Künneth formula isomorphism:

$$\begin{array}{ccc} h^1_\oplus(X \times \mathbb{T}^n) & \xrightarrow{\qquad\cong\qquad} & K^1(X \times \mathbb{T}^n) \\ \cong\downarrow & \square & \downarrow\cong \\ \bigoplus_{i=0}^n h^{1-i}_\oplus(X) \otimes \Lambda^i_{\mathbb{Z}}[z_1, ..., z_n] & \xrightarrow[\cong]{} & K^1(X) \otimes K^0(\mathbb{T}^n) \oplus K^0(X) \otimes K^1(\mathbb{T}^n) \end{array}$$

We have also seen that the class of the Weyl map lies entirely within a subset of $K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T})$

$$[W] \in \bigoplus_{i=0}^{n-1} \tilde{K}^0(\mathrm{SU}(n)/\mathbb{T}) \otimes \mathbb{Z}_{\mathbf{0}_{i-1},1,\mathbf{0}_{n-i-1}} \subseteq K^1(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T})$$

and we have also already seen that this subset maps by the bottom isomorphism in the diagram to $h^0_\oplus(\mathrm{SU}(n)/\mathbb{T}) \otimes \Lambda^1_{\mathbb{Z}}[z_1,\ ...,\ z_{n\text{-}1}]$.

Therefore, since we do have the required natural transformation in the 0^{th} and 1^{st} degrees, the following diagram commutes:

$$\begin{array}{ccc} h^1_\oplus(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T}) & \xrightarrow{\tau^1_{\mathrm{SU}(n)/\mathbb{T}\times\mathbb{T}}} & h^1_\otimes(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T}) \\ \uparrow & \square & \uparrow \\ h^0_\oplus(\mathrm{SU}(n)/\mathbb{T}) \otimes \Lambda^1_{\mathbb{Z}}[z_1, ..., z_{n-1}] & \xrightarrow[\tau^0_{\mathrm{SU}(n)/\mathbb{T}} \otimes \mathrm{id}]{} & h^0_\otimes(\mathrm{SU}(n)/\mathbb{T}) \otimes \Lambda^1_{\mathbb{Z}}[z_1, ..., z_{n-1}] \end{array}$$

and to understand the class $\tau^1_{\mathrm{SU}(n)/\mathbb{T}\times\mathbb{T}}([W])$ we just need to investigate the map $\tau^0_{\mathrm{SU}(n)/\mathbb{T}}$.

If $F\colon \mathscr{C}_\oplus \to \mathscr{C}_\otimes$ is an expoential functor such that $F(\mathbb{C}) \cong \mathbb{C}^d$, then we have seen that $h^0_\otimes(X) \subseteq \mathrm{GL}(1,\ K^0(X) \otimes \mathbb{Z}[\frac{1}{d}])$.

The invertible elements of $K^0(X)$ under the tensor product are the formal differences of vector bundles $[E_1]$ - $[E_2]$ such that $|dim(E_1)$ - $dim(E_2)| = 1$ and the tensor multiplicative unit of $K^0(X)$ is $[X \times \mathbb{C}] = [X \times \mathbb{C}]$ - $[X]$ (the - $[X]$ can be alighted since $[X]$ is the additive identity of the monoid that we group complete to achieve $K^0(X)$).

Clearly, $[X \times \mathbb{C}] \otimes 1$ is the multiplicative unit of $K^0(X) \otimes \mathbb{Z}[\frac{1}{d}]$ and since the multiplication in this ring is given by extending the relation on simples $([E_1] \otimes p) \cdot ([E_2] \otimes q) = ([E_1 \otimes E_2] \otimes pq)$, any element $[X \times \mathbb{C}] \otimes p$ where $p \in \mathrm{GL}(1,\ \mathbb{Z}[\frac{1}{d}])$ is invertible.

$X \times \mathbb{C}^n \cong (X \times \mathbb{C})^{\oplus n}$, and by the relation $[E^{\oplus n}] \otimes p = [E] \otimes np$ for any $n \in \mathbb{Z}$, we therefore know that

$$[X \times \mathbb{C}^d] \otimes 1 = [X \times \mathbb{C}] \otimes d \in \mathrm{GL}(1,\ K^0(X) \otimes \mathbb{Z}[\tfrac{1}{d}])$$

The Weyl map deals with the classes $[L_i]$ of the tautological line bundles over $\mathrm{SU}(n)/\mathbb{T}$. We have seen that:

$$\bigoplus_{i=1}^{n} L_i \cong \mathrm{SU}(n)/\mathbb{T} \times \mathbb{C}^n$$

Since F is an exponential functor, in terms of vector bundles:

$$\begin{aligned}\bigotimes_{i=1}^{n} F(L_i) &\cong F(\bigoplus_{i=1}^{n} L_i)\\ &\cong F(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{C}^n)\\ &\cong F((\mathrm{SU}(n)/\mathbb{T} \times \mathbb{C})^n)\\ &\cong F(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{C})^{\otimes n}\\ &\cong (\mathrm{SU}(n)/\mathbb{T} \times \mathbb{C}^d)^{\otimes n}\end{aligned}$$

Therefore, in the ring $K^0(X) \otimes \mathbb{Z}[\frac{1}{d}]$ we have:

$$\begin{aligned}\prod_{i=1}^{n}([F(L_i)] \otimes 1) &= [\bigotimes_{i=1}^{n} F(L_i)] \otimes 1\\ &= [(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{C})^{\otimes n}] \otimes 1\\ &= ([\mathrm{SU}(n)/\mathbb{T} \times \mathbb{C}^d] \otimes 1)^n\\ &\in \mathrm{GL}(1, K^0(\mathrm{SU}(n)/\mathbb{T} \otimes \mathbb{Z}[\frac{1}{d}])\end{aligned}$$

as it is the product of n copies of the invertible element $[\mathrm{SU}(n)/\mathbb{T} \times \mathbb{C}^d] \otimes 1$.

Therefore, since if $a.b$ is invertible in a ring, then both a and b are also invertible, for each i, $[F(L_i)] \otimes 1 \in \mathrm{GL}(1,\ K^0(\mathrm{SU}(n)/\mathbb{T}) \otimes \mathbb{Z}[\frac{1}{d}])$

$([F(L_\text{i})] \otimes 1)^{-1}$ is the element such that $([F(L_\text{i})] \otimes 1) \cdot ([F(L_\text{i})] \otimes 1)^{-1} = [\text{SU}(n)/\mathbb{T} \times \mathbb{C}] \otimes 1$.

$$\begin{aligned}
([F(L_i)] \otimes 1) \cdot (\prod_{j \in [n] \setminus i} ([F(L_j)] \otimes 1)) &= ([\text{SU}(n)/\mathbb{T} \times \mathbb{C}^d] \otimes 1)^n \\
&= ([\text{SU}(n)/\mathbb{T} \times \mathbb{C}] \otimes d)^n \\
&= [\text{SU}(n)/\mathbb{T} \times \mathbb{C}] \otimes d^n \\
\implies ([F(L_i)] \otimes 1)^{-1} &= (\prod_{j \in [n] \setminus i} ([F(L_j)] \otimes 1)) \cdot ([\text{SU}(n)/\mathbb{T} \times \mathbb{C}] \otimes \frac{1}{d^n}) \\
&= (\prod_{j \in [n] \setminus i} [F(L_j)] \otimes 1) \cdot ([\text{SU}(n)/\mathbb{T} \times \mathbb{C}] \otimes \frac{1}{d^n}) \\
&= [\bigotimes_{j \in [n] \setminus i} F(L_j)] \otimes 1) \cdot ([\text{SU}(n)/\mathbb{T} \times \mathbb{C}] \otimes \frac{1}{d^n}) \\
&= [\bigotimes_{j \in [n] \setminus i} F(L_j)] \otimes \frac{1}{d^n}
\end{aligned}$$

The map $\tau^0_{\text{SU}(n)/\mathbb{T}}$ sends formal differences of vector bundles to a subgroup of the invertible elements of $K^0(X) \otimes \mathbb{Z}[\frac{1}{d}]$ but we only need to concern ourselves with the image of the formal differences $[L_i]$ - $[L_n]$.

Since we have already checked the invertibility, and by the nature of an exponential functor, the addition in $h^0_{\oplus}(\text{SU}(n)/\mathbb{T})$ becomes the multiplication in $h^0_{\otimes}(\text{SU}(n)/\mathbb{T})$, we have:

$$\begin{aligned}
\tau^0_{\text{SU}(n)/\mathbb{T}}([L_i] - [L_n]) &= ([F(L_i)] \otimes 1) \cdot ([F(L_n)] \otimes 1)^{-1} \\
&= ([F(L_i)] \otimes 1) \cdot ([\bigotimes_{j=1}^{n-1} F(L_j)] \otimes \frac{1}{d^n}) \\
&= [F(L_i) \otimes \bigotimes_{j=1}^{n-1} F(L_j)] \otimes \frac{1}{d^n}
\end{aligned}$$

We now fully understand every map in our diagram and we can see that:

$$\tau^1_{\text{SU}(n)/\mathbb{T} \times \mathbb{T}}([W]) = \bigodot_{i=1}^{n-1} ([F(L_i) \otimes \bigotimes_{j=1}^{n-1} F(L_j)] \otimes \frac{1}{d^n}) \otimes z_i$$

where $\odot$ is the binary operation that defines the group $h^0_{\otimes}(\text{SU}(n)/\mathbb{T}) \otimes \Lambda^1_{\mathbb{Z}}[z_1, ..., z_{n\text{-}1}]$ where:

$$(a \otimes x) \odot (b \otimes y) = ab \otimes x + y$$

△

10.2.3 The Tensor Chern Character in Action

Finally we can use the tensor Chern character, the natural transformation of cohomology theories *logch*: $h^*_\otimes \to H^*_{per}$(-; $\mathbb{Q}$) we previously constructed where:

$$\begin{gathered} logch^0_X : h^0_\otimes(X) \to H^0_{per}(X;\mathbb{Q}) \\ ([E_1] - [E_2]) \otimes p \mapsto log((ch(E_1) - ch(E_2))p) \end{gathered}$$

where $p \in \mathrm{GL}(1,\ \mathbb{Z}[\frac{1}{d}])$.

Since the tensor Chern character is a natural transformation of cohomology theories, the following diagram commutes in the category of graded rings

$$\begin{array}{ccc} h^*_\otimes(X \times \mathbb{T}^n) & \xrightarrow{logch^*_{X\times\mathbb{T}^n}} & H^*_{per}(X \times \mathbb{T}^n;\mathbb{Q}) \\ \cong\downarrow & \square & \downarrow\cong \\ h^*_\otimes(X) \otimes \Lambda^*_\mathbb{Z}[z_1, ..., z_n] & \xrightarrow[logch^*_X \otimes \mathrm{id}]{} & H^*_{per}(X;\mathbb{Q}) \otimes \Lambda^*_\mathbb{Z}[z_1, ..., z_n] \end{array}$$

Since the class of the Weyl map is contained in just the first degree, and more specifically, just the subgroup $h^0_\otimes(\mathrm{SU}(n)/\mathbb{T}) \otimes \Lambda^1_\mathbb{Z}[z_1,\ ...,\ z_{n\text{-}1}]$, for our purposes we can restrict this diagram to just this degree and use inclusions to restrict again to the subgroup and achieve the following diagram in the category of abelian groups:

$$\begin{array}{ccc} h^1_\otimes(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T}) & \xrightarrow{logch^1_{\mathrm{SU}(n)/\mathbb{T}\times\mathbb{T}}} & H^1_{per}(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T};\mathbb{Q}) \\ \iota\uparrow & \square & \uparrow\iota \\ h^0_\otimes(\mathrm{SU}(n)/\mathbb{T}) \otimes \Lambda^1_\mathbb{Z}[z_1, ..., z_{n-1}] & \xrightarrow[logch^0_{\mathrm{SU}(n)/\mathbb{T}} \otimes \mathrm{id}]{} & H^0_{per}(\mathrm{SU}(n)/\mathbb{T};\mathbb{Q}) \otimes \Lambda^1_\mathbb{Z}[z_1, ..., z_{n-1}] \end{array}$$

And since we already understand the behaviour of the tensor Chern character in the 0^{th} degree, we can easily see that if $\tau_1([W])$ is the class of the Weyl map in $h^1_\otimes(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{T})$ after application of a natural transformation induced by an exponential functor F: $\mathscr{C}_\oplus \to \mathscr{C}_\otimes$, then:

$$\begin{aligned} logch^1(\tau^1([W]) &= (logch^0 \otimes \mathrm{id})(\bigodot_{i=1}^{n-1}([F(L_i) \otimes \bigotimes_{j=1}^{n-1} F(L_j)] \otimes \frac{1}{d^n}) \otimes z_i) \\ &= \sum_{i=1}^{n-1} logch^0([F(L_i) \otimes \bigotimes_{j=1}^{n-1} F(L_j)] \otimes \frac{1}{d^n})) \otimes z_i \\ &= \sum_{i=1}^{n-1} log(ch(F(L_i) \otimes \bigotimes_{j=1}^{n-1} F(L_j))\frac{1}{d^n}) \otimes z_i \\ &= \sum_{i=1}^{n-1} log(ch(F(L_i))(\prod_{j=1}^{n-1} ch(F(L_j)))\frac{1}{d^n}) \otimes z_i \end{aligned}$$

Theorem 68. The result obtained using the tensor Chern character with the exponential functors det^m is identical to the result previously obtained by understanding the cohomology theory $h^*_\otimes$ for $d = 1$.

Proof. Since $d^n = dim(F(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{C}^n))$, $ch(F(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{C}^n)) = d^n$ and so:

$$\begin{aligned}
logch^1(\tau^1([W]) &= \sum_{i=1}^{n-1} log(\frac{ch(F(L_i))(\prod_{j=1}^{n-1} ch(F(L_j)))}{ch(F(\mathrm{SU}(n)/\mathbb{T} \times \mathbb{C}^n))}) \otimes z_i \\
&= \sum_{i=1}^{n-1} log(\frac{ch(F(L_i))(\prod_{j=1}^{n-1} ch(F(L_j)))}{ch(F(\bigoplus_{k=1}^{n} L_k))}) \otimes z_i \\
&= \sum_{i=1}^{n-1} log(\frac{ch(F(L_i))(\prod_{j=1}^{n-1} ch(F(L_j)))}{ch(\bigotimes_{k=1}^{n} F(L_k))}) \otimes z_i \\
&= \sum_{i=1}^{n-1} log(\frac{ch(F(L_i))(\prod_{j=1}^{n-1} ch(F(L_j)))}{\prod_{k=1}^{n} ch(F(L_k))}) \otimes z_i \\
&= \sum_{i=1}^{n-1} log(\frac{ch(F(L_i))}{ch(F(L_n))}) \otimes z_i \\
&= \sum_{i=1}^{n-1} (log(ch(F(L_i))) - log(ch(F(L_n)))) \otimes z_i
\end{aligned}$$

The construction of a class in $h^1_\otimes(\mathrm{SU}(n)/\mathbb{T})$ as we have done for a general exponential functor together with an application of the tensor Chern character can be shown to yield the same result as those we achieve using the natural transformations of cohomology theories constructed from the exponential functors $det^m\colon \mathscr{C}_\oplus \to \mathscr{C}_\otimes$. If τ is the natural transformation of cohomology theories induced by det^m then:

$$\begin{aligned}
logch^1(\tau^1([W])) &= \sum_{i=1}^{n-1}(log(ch(det^m(L_i))) - log(ch(det^m(L_n)))) \otimes z_i \\
&= \sum_{i=1}^{n-1}(log(ch(det(L_i)^{\otimes m})) - log(ch(det(L_n)^{\otimes m}))) \otimes z_i \\
&= \sum_{i=1}^{n-1}(log(ch((L_i)^{\otimes m})) - log(ch((L_n)^{\otimes m}))) \otimes z_i \\
&= \sum_{i=1}^{n-1}(log(ch(L_i)^m) - log(ch(L_n)^m)) \otimes z_i \\
&= \sum_{i=1}^{n-1}(m.log(ch(L_i)) - m.log(ch(L_n))) \otimes z_i \\
&= \sum_{i=1}^{n-1} m(log(exp(c_1(L_i))) - log(exp(c_1(L_n)))) \otimes z_i \\
&= \sum_{i=1}^{n-1} m(c_1(L_i) - c_1(L_n)) \otimes z_i
\end{aligned}$$

Which agrees exactly with our alternative method of applying the functor det^m to the line bundles directly. △

References

[1] Adams, J. F. *Stable homotopy and generalised homology.*
University of Chicago Press (1974)

[2] Atiyah, M., Segal, G.
Twisted K-theory
Ukr. Mat. Visn., 1(3):287–330, (2004)

[3] Baez, J. C.
Some Definitions Everyone Should Know
University of California, Riverside (2004)

[4] Becker, K., Murray, M., Stevenson, D.
The Weyl map and bundle gerbes
J. Geom. Phys., 149:103572, 19, (2020)

[5] Borel, A.
Sur La Cohomologie des Espaces Fibres Principaux et des Espaces Homogenes de Groupes de Lie Compacts
Annals of Mathematics, Second Series, Vol. 57, No. 1 (Jan., 1953), pp. 115-207

[6] Bradley, T., Bryson, T., Terilla, J.
Topology A Categorical Approach
The MIT Press (2020)

[7] Dadarlat, M., Pennig, U.
Unit spectra of K-theory from strongly self- absorbing C-algebras*
Algebr. Geom. Topol., 15(1):137–168, 2015.
A Dixmier-Douady theory for strongly self- absorbing C-algebras*
J. Reine Angew. Math., 718:153–181, 2016.

[8] Eckmann, B., Hilton, P. J.
Group-like structures in general categories. I. Multiplications and comultiplications
Mathematische Annalen, 145 (3): 227–255, doi:10.1007/bf01451367 (1962)

[9] Eilenberg, S., MacLane, S.
Relations between homology and homotopy groups of spaces
Annals of Mathematics (Second Series) 46 (3): 480–509, doi:10.2307/1969165 (1945)

Relations between homology and homotopy groups of spaces

Annals of Mathematics. (Second Series). 51 (3): 514–533. doi:10.2307/1969365 (1950)

On the groups $H(\Pi ,n)$*. III. Operations and obstructions*

Annals of Mathematics. 60 (3): 513–557. doi:10.2307/1969849 (1954)

[10] Eilenberg, S., Steenrod, N.

Foundations of Algebraic Topology

Princeton University Press (1952)

[11] Eilenberg, S., Zilber, J., A.

Semi-simplicial complexes and singular homology

Annals of Mathematics 51:3 (1950), 499–513

[12] Evans, D., Pennig, U.

Equivariant higher twisted K-theory of SU(n) > for exponential functor twists

J. Topol., 15(2):896–949, (2022)

Equivariant higher Dixmier-Douady theory for circle actions on UHF-algebras

Adv. Math., 410(part B):Paper No. 108745, 40, (2022)

[13] Freed, D. S.

Γ*-spaces and Deeloopings*

University of Texas (2012)

[14] Freed, D. S., Hopkins, M. J., Teleman, C.

Loop groups and twisted K-theory I.

J. Topol. 4 (2011), no.4, 737–798.

Loop groups and twisted K-theory II.

J. Amer. Math. Soc.26 (2013), no.3, 595–644.

Loop groups and twisted K-theory III.

Ann. of Math. (2)174 (2011), no.2, 947–1007.

[15] Grothendieck, A.

Techniques de construction et théorèmes d'existence en géométrie algébrique III : préschémas quotients

Séminaire Bourbaki, no. 6 (1961), Exposé no. 212

[16] Harris, B.

Bott Periodicity via Simplicial Spaces

Journal of Algebra, vol. 62, no. 2, Feb. 1980, pp. 450–54. DOI.org (Crossref), doi:10.1016/0021-8693(80)90194-5.

[17] Hatcher, A.

Algebraic Topology

Cambridge University Press (2001)

[18] Hilton, P.

General Cohomology Theory and K-Theory;

Cambridge University Press (1971)

[19] Joyal, A., Street, R.

Braided Monoidal Categories

Macquarie Mathematics Reports (1986)

[20] Kelly, G. M.

On MacLane's conditions for coherence of natural associativities, commutativities, etc.

Journal of Algebra, vol 1, no. 4, Dec. 1964, pp. 397-402

[21] Kelly, G., M.

Basic Concepts of Enriched Category Theory

Cambridge University Press, Lecture Notes in Mathematics 64, (1982)

[22] MacLane, S.

Categories for the Working Mathematician

Graduate Texts in Mathematics vol 5. Springer (1971, 1997) doi:10.1007/978-1-4757-4721-8

[23] Macdonald, I. G.

Symmetric Functions and Hall Polynomials (2^{nd} edition)

Oxford: Clarendon Press (1995)

[24] May, P.

A note on the splitting principle

Topology and its Applications Volume 153, Issue 4, 1 November 2005, Pages 605-609

[25] McDuff,D., Segal, G.

Homology fibrations and the "group-completion" theorem

Invent Math 31, 279–284 (1976)

[26] Milnor, J., Stasheff, J.
Characteristic Classes
Princeton University Press (1974)

[27] Mitchell, S.
Notes on principal bundles and classifying spaces
MIT (Aug 2001)

[28] Park, E.
Complex Topological K-Theory
Cambridge University Press (2008)

[29] Pennig, U.
Exponential Functors, R-matrices, and Twists
Algebr. Geom. Topol. 20 (2020) 1279-1324

[30] Reeder, M.
On the cohomology of compact Lie groups
Enseign. Math., II. Sér. 41, No. 3-4, 181-200 (1995)

[31] Segal, G.
Classifying spaces and spectral sequences.
Inst. Hautes Études Sci. Publ. Math. No. 34 (1968) 105–112

[32] Segal, G.
Categories and Cohomology Theories
Topology, vol. 13, no. 3, 293-312 (1974)

[33] Steenrod, N.
The Topology of Fibre Bundles
Princeton University Press (1951)

[34] Teleman, C.
K-theory and the moduli space of bundles on a surface and deformations of the Verlinde algebra. In Topology, geometry and quantum field theory
volume 308 of London Math. Soc. Lecture Note Ser., pages 358–378. Cambridge Univ. Press, Cambridge, (2004)