INTEGRAL FOLIATED SIMPLICIAL VOLUME AND CIRCLE FOLIATIONS

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Abstract. We show that the integral foliated simplicial volume of a compact oriented smooth manifold with a regular foliation by circles vanishes.

1. Introduction

In his proof of Mostow rigidity in [14], Gromov introduced the concept of simplicial volume for an oriented compact connected topological manifold M. This is a homotopy invariant of M, which measures the complexity of singular fundamental cycles of M with $\mathbb{R}$-coefficients. When M is a smooth manifold, its simplicial volume, despite being only a homotopy invariant, has very deep connections with the geometrical properties of M. For example, if M admits a complete Riemannian metric of negative sectional curvature, its simplicial volume has to be positive (see [14, 31, 18]). Furthermore Gromov showed that the simplicial volume is dominated by the minimal volume of the manifold (see [14]). In this light, positive simplicial volume is an obstruction to the existence of a sequence of collapsing Riemannian metrics, satisfying a uniform sectional curvature bound, on the given manifold (see [25]).

A long-standing conjecture about this interplay between topology and geometry was stated by Gromov [15, $8.A_4(+)$], [17, 3.1 (e) on p. 769]:

Conjecture A. *Let M be an oriented closed (compact without boundary) connected aspherical manifold. If the simplicial volume of M vanishes, then the L^2-Betti numbers of M vanish. In particular, the Euler characteristic of M also vanishes.*

An approach to give a positive answer to Conjecture A is laid out in [16, 28]: consider a suitable integral approximation of the simplicial volume, then apply a Poincaré duality argument to bound the L^2-Betti numbers in terms of this integral approximation. Finally, relate the integral approximation to the simplicial volume on aspherical manifolds. An instance of such an approximation is the integral foliated simplicial volume (see [16, 28]), which we consider here.

In this article we show that in the presence of a circle foliation and a big fundamental group, the integral foliated simplicial volume vanishes:

Theorem B. *Let M be a compact oriented smooth manifold equipped with a regular smooth circle foliation $\mathcal{F}$, such that the inclusion of each leaf of $\mathcal{F}$*

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is π_1-injective and has finite holonomy. Then the (relative) integral foliated simplicial volume of M vanishes:

$$\mathbf{|}\, M, \partial M \,\mathbf{|} = 0.$$

The work of Schmidt in [28] provides the following corollary to Theorem B.

Corollary C. *Let $(M, \mathcal{F})$ be a closed connected oriented aspherical smooth manifold, and $\mathcal{F}$ a regular smooth circle foliation, such that the inclusion of each leaf is π_1-injective and has finite holonomy. Then the L^2-Betti numbers of M vanish. In particular $\chi(M) = 0$.*

The work of Löh in [19] provides another corollary to Theorem B, involving a different invariant, namely the cost. The cost is a randomized version of the minimal number of generators of a group.

Corollary D. *Let $(M, \mathcal{F})$ be a closed connected oriented aspherical smooth manifold, and $\mathcal{F}$ a regular smooth circle foliation, such that the inclusion of each leaf is π_1-injective and has finite holonomy. Then $\mathrm{Cost}(\pi_1(M)) = 1$, that is the manifold M is cheap.*

Additionally, we obtain some information on the stable integral simplicial volume of manifolds with residually finite fundamental group:

Corollary E. *Let M be a compact oriented smooth manifold with residually finite fundamental group, equipped with a regular smooth circle foliation $\mathcal{F}$, such that the inclusion of each leaf of $\mathcal{F}$ is π_1-injective and has finite holonomy. If the foliation restricted to the preimage of the manifold part of the orbifold $M/\mathcal{F}$ is an orientable bundle, then the (relative) stable integral simplicial volume of M vanishes:*

$$\|M, \partial M\|_{\mathbb{Z}}^{\infty} = 0.$$

This, and a slightly stronger statement, will be a corollary of the proof of Theorem B and will be explained at the end of Section 3 (see Remark 3.9). It can be seen as the generalisation of [8, Corollary 1.3] to our setting.

Given a smooth manifold M, we may consider different notions of "symmetry" for M. A general notion is to split M into a family of submanifolds, that retain certain desired geometric or topological properties. In [5], Cheeger and Gromov studied a type of decomposition called an F-structure. This is a direct generalization of a smooth torus action on a smooth manifold. In particular, they show that if a Riemannian manifold M admits a polarized F-structure of positive dimension, then the minimal volume of M vanishes. Since the minimal volume dominates the simplicial volume, the simplicial volume of M also vanishes. Furthermore, in this case the Euler characteristic of M is also zero (see [5, Proposition 1.5] [25, p. 75]).

In the particular case when the splitting of M is given by the orbits of a smooth circle action, the Corollary to the Vanishing Theorem [14, p. 41] implies that the simplicial volume of M vanishes. This fact was independently proven by Yano in [33]. His proof is quite geometric and relies heavily on the stratification of the orbit space by orbit type. This proof has been extended to the setting of the integral foliated simplicial volume by Fauser in [8], when one adds the assumption that the inclusion of every orbit in M is injective at the level of fundamental groups. We point out that this condition is not satisfied when the action has fixed points. A more general notion of symmetry can be found in the context of foliations. They arise naturally as solutions to differential equations (see [21, Chapter 1]). Moreover, smooth group actions, as well as the fibers of a smooth fiber bundle, are some examples of smooth foliations.

We follow the approach of Yano and Fauser [8, 33] to prove Theorem B. The hypothesis requiring the holonomy groups to be finite is necessary to obtain the stratification of the leaf space. Indeed, Sullivan in [30] constructed a 5-dimensional smooth manifold with a foliation by circles, with leaves having infinite holonomy. For this particular example the leaf space does not admit an orbifold structure. Furthermore, the condition that for any leaf its inclusion into the manifold induces an injective map between the fundamental groups is also necessary for our proof. Simple examples of a regular foliation by circles where the inclusion is not π_1-injective are given by the Hopf fibrations $\mathbb{S}^{2n+1} \to \mathbb{C}P^n$. We recall that the integral foliated simplicial volume of simply-connected manifolds, and in particular of $\mathbb{S}^{2n+1}$, is bounded below by 1 (see [28, Proposition 5.29]).

Any smooth circle action gives rise to a circle foliation, but there are many instances of circle foliations not coming from circle actions. A simple example is given by considering non-orientable circle bundles over a non-orientable base. For example, the unit tangent bundle of a closed non-orientable surface is a circle foliation not coming from a circle action. This holds since the unit tangent bundle is not a principal circle bundle: if it were, then the structure group of the tangent bundle would be $SO(2)$, but since the surface is non-orientable the structure group of the tangent bundle cannot be reduced to $SO(2)$. Also observe that the total space of the tangent bundle of any manifold is an orientable manifold. In particular the tangent bundle of the Möbius band $\{(\theta, y) \in [0,1] \times [-1,1]\}/(0,y) \sim (1,-y)$ is diffeomorphic to $\mathbb{S}^1 \times [-1,1] \times \mathbb{R}^2$. Another concrete example is given by the unit tangent bundle of the Klein bottle. We remark that for these examples, there is a finite cover of the manifold with a smooth circle action such that the orbits of the action cover the leaves of the foliation. Other non-homogeneous examples are given by non-orientable Seifert fibrations over a non-orientable surface (see [24, Section 5.2]). We point out that the universal cover of a Seifert fibration is one of $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{R}^3$ (see [27, Proposition 1]), and they all admit a smooth circle action. We remark that, for these cases, the conclusion of the main theorem can also be obtained by the work of Fauser, Friedl, Löh in [9, Theorem 1.7].

To prove Theorem B, we need to check that Yano's and Fauser's techniques from [8, 33] apply in the case of foliations instead of circle actions. After the hollowing procedure, we are left with a certain circle bundle $M_{n-2} \to M_{n-2}/\mathcal{F}_{n-2}$. If it is an orientable bundle, our proof is a straightforward translation of the proof in [8]. But the bundle can be non-orientable over a non-orientable base (thus still having orientable total space): this is a novelty of foliations with respect to circle actions. Then we need an additional argument that exploits the flexibility of the integral foliated simplicial volume (see Section 3, in particular Setup 3.2).

In general, regular foliations by circles are instances of singular circle fiberings over polyhedra, introduced by Edmonds and Fintushel in [6]. They proved that a singular circle fibering on a smooth manifold is given by a group action if and only if the bundle part of the fibering is orientable (see [6, Theorem 3.8]). To the best of our knowledge, it is not known whether an arbitrary smooth manifold with a foliation by circles with finite holonomy admits a finite cover with a group action. If so, our main result would follow by multiplicativity of the integral foliated simplicial volume and the main result in [8].

The condition of π_1-injectivity is a technical one: it is used so that we have only a global representation of $\pi_1(M)$ over a fixed essentially free standard Borel space to consider, instead of a potentially different one for every leaf. This technicality might be overcome by constructing a suitable family of representations and essentially free standard Borel spaces at each step of the hollowing construction (the hollowing construction is presented in Section 2.5).

We organize our work as follows: in Section 2 we present the preliminaries as well as a series of clarifying examples. In Section 3 we prove Theorem B.

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2. Preliminaries

2.1. **Simplicial volume.** We begin by defining the *simplicial volume*, also known as the *Gromov invariant* or *Gromov norm*, of a topological compact connected oriented manifold M, with possibly empty boundary ∂M. Given a singular k-chain $z = \sum_i a_i\sigma_i \in C_k(M, \mathbb{R})$, we define its ℓ_1-*norm* as

$$\|z\|_1 = \sum_i |a_i| \,,$$

where each $\sigma_i \colon \Delta^k \to M$ is a singular simplex of dimension k.

The *simplicial volume* of M is the infimum over the ℓ_1-norms of the (relative) real cycles representing the fundamental class:

$$\|M, \partial M\| = \inf\left\{ \sum_i |a_i| \,\middle|\, [M, \partial M] = \left[\sum_i a_i\sigma_i\right] \in H_n(M, \partial M; \mathbb{R}) \right\}.$$

2.2. **Integral foliated simplicial volume.** Consider a topological compact connected oriented manifold M and its universal cover $\widetilde{M}$. Denote by $\Gamma = \pi_1(M)$ the fundamental group of M. Then Γ acts on $\widetilde{M}$ by deck transformations. This induces a natural action of Γ on $C_k(\widetilde{M}, \mathbb{Z})$. There is a natural identification

$$C_k(M, \mathbb{Z}) \cong \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_k(\widetilde{M}, \mathbb{Z}).$$

A *standard Borel space* is a measurable space that is isomorphic to a Polish space with its Borel σ-algebra $\mathcal{B}$. Recall that a *Polish space* is a separable completely metrizable topological space. Let Z be a *standard Borel probability space*, that is, a standard Borel space endowed with a probability measure μ. Suppose now that Γ acts (on the left) on such a standard Borel probability space $(Z, \mathcal{B}, \mu)$ in a measurable and measure-preserving way. Denote this action by $\alpha\colon \Gamma \to \operatorname{Aut}(Z, \mu)$. Set

$$L^\infty(Z, \mu; \mathbb{Z}) = \{f\colon Z \longrightarrow \mathbb{Z} \mid \exists C \in \mathbb{Z} : |f(z)| \leq C \text{ for } \mu\text{-a. e. } z \in Z\},$$

the space of essentially bounded functions with integer values on Z. We define a right Γ-action on $L^\infty(Z, \mu; \mathbb{Z})$ by setting

$$(f \cdot \gamma)(z) = f(\gamma z),\ \forall f \in L^\infty(Z, \mu; \mathbb{Z}),\ \forall \gamma \in \Gamma,\ \forall z \in Z.$$

There is a natural inclusion for any $k \in \mathbb{N}$:

$$\begin{array}{rrcl} i_\alpha\colon & C_k(M, \mathbb{Z}) \cong \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_k(\widetilde{M}, \mathbb{Z}) & \longrightarrow & L^\infty(Z, \mu; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_k(\widetilde{M}, \mathbb{Z}) \\ & 1 \otimes \sigma & \longmapsto & \mathrm{const}_1 \otimes \sigma, \end{array}$$

where the tensor products are taken over the given actions. We will write $C_*(M; \alpha)$ for the complex $L^\infty(Z, \mu; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}, \mathbb{Z})$. We call its elements *parametrized chains*.

Given a parametrized k-chain $z = \sum_i f_i \otimes \sigma_i \in L^\infty(Z, \mu; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_k(\widetilde{M}, \mathbb{Z})$ we define its *parametrized ℓ_1-norm* as

$$|z|_1 = \sum_i \int_Z |f_i| d\mu,$$

where each $\sigma_i\colon \Delta^k \to M$ is a singular simplex of dimension k. Here we assume that z is in reduced form, that is, all the singular simplices σ_i belong to different Γ-orbits. For more details on the integral foliated simplicial volume, see for example [20].

Note that the action of Γ on $\widetilde{M}$ restricts to the preimage of the boundary of M under the covering map $p\colon \widetilde{M} \to M$. Thus we get an action

$$\Gamma \longrightarrow \operatorname{Homeo}(p^{-1}(\partial M)).$$

This restricted action allows us to define the subcomplex

$$C_*(\partial M; \alpha) = L^\infty(Z, \mu; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(p^{-1}(\partial M), \mathbb{Z}) \subset C_*(M; \alpha),$$

and hence the quotient

$$C_*(M, \partial M; \alpha) = C_*(M; \alpha)/C_*(\partial M; \alpha).$$

This last quotient is naturally isomorphic to the chain complex

$$L^\infty(Z, \mu; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}, p^{-1}(\partial M); \mathbb{Z}).$$

From now on we set:

$$H_*(M, \partial M; \alpha) = H_*(C_*(M, \partial M; \alpha)).$$

The *(relative) integral foliated simplicial volume* of M is the infimum over the parametrized ℓ_1-norms of the (relative) parametrized cycles representing the fundamental class:

$$❙ M, \partial M ❙ = \inf_{\alpha,(Z,\mu)} \{ ❙ M, \partial M ❙^{\alpha} | \alpha \colon \Gamma \longrightarrow \mathrm{Aut}(Z,\mu)\},$$

where $❙ M, \partial M ❙^{\alpha}$ is given by

$$\inf \left\{ \sum_i \int_Z |f_i| d\mu \,\middle|\, [M, \partial M]^{\alpha} = \left[\sum_i f_i \otimes \sigma_i \right] \in H_n(M, \partial M; \alpha) \right\}.$$

Here $[M, \partial M]^{\alpha}$ denotes the image of the fundamental class $[M, \partial M] \in H_n(M, \partial M; \mathbb{Z})$ in $H_n(M, \partial M; \alpha)$ under the map induced by i_α.

Remark 2.1. For every compact connected oriented n-manifold M, we have the following inequality (see [20, Proposition 4.6]):

$$\|M, \partial M\| \leq ❙ M, \partial M ❙ .$$

Note also that the infimum in the definition of $❙ M, \partial M ❙$ is achieved by an *essentially free* action α, meaning that $\mu(\{z \in Z \mid Stab_\alpha(z) \neq \{\mathrm{id}_\Gamma\}\}) = 0$ [20, Corollary 4.14].

From now on, we consider only smooth connected orientable manifolds.

2.3. **Foliations.** We proceed to define a smooth regular foliation on a smooth connected manifold M and state some of its properties. A smooth *regular foliation* of M is a partition $\mathcal{F} = \{L_p \mid p \in M\}$ of M where each *leaf* L_p is an embedded smooth submanifold $L_p \subset M$, called the *leaf through* p, and such that the tangent spaces of the leaves are smooth subbundles of TM (see [21, p. 9 (iii)]).

Remark 2.2. Observe that there might exist foliations where the leaves are not embedded submanifolds [21, Section 1.1], but we will not consider such foliations in the present work.

The following phenomena give rise to foliations: let $p \colon M \to B$ be a smooth submersion. Then the partition induced by the fibers of p produces a foliation. More generally, any involutive subbundle of TM induces a foliation on M. Consider a compact Lie group G acting smoothly on M with finite stabilizer subgroups. The partition

$$\mathcal{F} = \{G(p) = \{gp \mid g \in G\} \mid p \in M\},$$

is a regular foliation (see [1, Theorem 3.65 and Example 5.3][21, p. 16]).

The *codimension* $\mathrm{codim}(M, L)$ of a foliation on a connected manifold M is the codimension of any leaf L. The quotient space $M/\mathcal{F}$ induced by the partition, equipped with the quotient topology, is called the *leaf space of* $\mathcal{F}$. We denote the projection map by $\pi \colon M \to M/\mathcal{F}$, and the image under π of a subset $A \subset M$ by A^*.

2.3.1. *Germs.* Given two smooth manifolds M and N, a point $x \in M$, and two diffeomorphisms $f\colon M \to N$ and $g\colon M \to N$, we say that f and g *define the same germ at* x, and denote it by $f \sim_x g$, if there exists an open neighborhood $U \subset M$ of x such that $f|_U = g|_U$. The *germ of* f *at* x is:

$$[f]_x = \{g\colon M \longrightarrow N \mid f \sim_x g \text{ and } g \text{ is a diffeomorphism}\}.$$

If $N = M$, we denote the group of germs at x of diffeomorphisms f with $f(x) = x$ by $\mathrm{Diff}_x(M)$.

2.3.2. *Holonomy.* Fix a leaf $L \in \mathcal{F}$, and fix two points $p, q \in L$. Consider two *transversal sections* T and S to L at p and q. That is, T and S are submanifolds in M such that $T_pM = T_pL \oplus T_pT$ and $T_qM = T_qL \oplus T_qS$. Consider $\alpha\colon [0,1] \to L$ any path joining p to q.

Proposition 2.3 (Section 2.1 [21])**.** *There exists an open neighborhood $A \subset T$ of p, contained in some foliated chart, and a diffeomorphism $f\colon A \to S$, given by $\alpha\colon [0,1] \to L$ such that $f(p) = q$ and for any $x \in A$, the image $f(x)$ is contained in the same leaf as x (see Figure 2.1).*

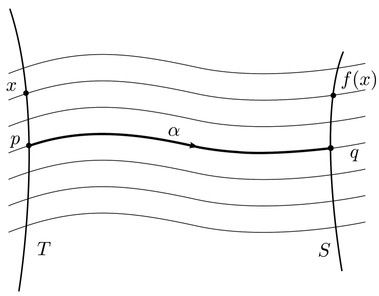


FIGURE 2.1. Local picture of holonomy

For the path α we set

$$\mathrm{hol}(\alpha)^{S,T} = [f]_p.$$

Remark 2.4. Observe that

(1) for a fixed foliated chart, the diffeomorphism f of Proposition 2.3 is unique;
(2) the definition of the holonomy does not depend on the choice of the chart [21, p. 21].

Consequently the holonomy $\mathrm{hol}(\alpha)^{S,T}$ is well-defined.

Proposition 2.5 ([21])**.** *The following hold:*

(i) Let α be a path in L from p to q and β be a path in L from q to w. Let T, S and R be transversal sections to L at p, q and w respectively. Then for the concatenation of paths $\beta\alpha$ we have:

$$\mathrm{hol}(\beta\alpha)^{R,T} = \mathrm{hol}(\beta)^{R,S} \circ \mathrm{hol}(\alpha)^{S,T}.$$

(ii) Consider α and β homotopic paths, relative to the fixed end points, in L from p to q, and fix T and S transversal sections at p, respectively q. Then

$$\mathrm{hol}(\alpha)^{S,T} = \mathrm{hol}(\beta)^{S,T}.$$

(iii) Let α be a path in L from p to q, and consider T and T' two transversal sections at p and S and S' two transversal sections at q. Denote the constant path with image p by c_p. Then

$$\mathrm{hol}(\alpha)^{S',T'} = \mathrm{hol}(c_q)^{S',S} \circ \mathrm{hol}(\alpha)^{S,T} \circ \mathrm{hol}(c_p)^{T,T'}.$$

Considering closed loops in L with base point $p \in L$, we point out that by Proposition 2.5 (i) and (ii), for a transversal section T at p we have a group homomorphism

$$\mathrm{hol}^T\colon \pi_1(L,p) \longrightarrow \mathrm{Diff}_p(T).$$

Moreover taking $k = \mathrm{codim}(M,L)$, since for any transversal section T we can identify $\mathrm{Diff}_p(T) = \mathrm{Diff}_0(\mathbb{R}^k)$, by Proposition 2.5 (iii), we obtain a group homomorphism

$$\mathrm{hol}(L,p)\colon \pi_1(L,p) \longrightarrow \mathrm{Diff}_0(\mathbb{R}^k).$$

We define the *holonomy group of the leaf L at p*, denoted by $\mathrm{Hol}(L,p)$, to be the image of $\pi_1(L,p)$ under the homomorphism $\mathrm{hol}(L,p)$. This group is independent of the base point p up to conjugation in $\mathrm{Diff}_0(\mathbb{R}^k)$. We denote by K the kernel of the short exact sequence:

$$1 \longrightarrow K \longrightarrow \pi_1(L,p) \longrightarrow \mathrm{Hol}(L,p) \longrightarrow 1. \tag{2.1}$$

The holonomy group of a leaf measures roughly "how twisted" the foliation is around the leaf. A foliation induced by a submersion has trivial holonomy for any leaf. Observe that, by considering derivatives of germs at the origin, we obtain a new representation:

$$\mathrm{Dhol}(L,p)\colon \pi_1(L,p) \longrightarrow \mathrm{GL}_k(\mathbb{R}).$$

Remark 2.6. When a smooth Lie group H acts on a smooth manifold M in such a way that the map $H\times M \to M\times M$ given by $(h,p)\mapsto(hp,p)$ is proper, there exists a smooth Riemannian metric g on M such that the action of H is by isometries (see [1, Theorem 3.65]). Such an action is called a *proper action*. Fix $p \in M$ and consider the elements of H as diffeomorphisms of M onto itself. By taking the derivative of $h \in H$ we have a linear map $D_p h\colon T_pM \to T_{hp}M$. For $h \in H$ such that $hp = p$ it becomes $D_p h\colon T_pM \to T_pM$. Since h maps minimizing geodesics to minimizing geodesics, for $v \in T_pM$ small enough we have:

$$h \exp_p(v) = \exp_p(D_p h(v)). \tag{2.2}$$

Thus, h is a linear map with respect to the normal coordinates $\exp_p\colon U \subset T_pM \to V \subset M$. Assume p is fixed by the action of H, *i.e.* $hp = p$ for all $h \in H$. From Equation (2.2) it follows that the linear action of H on T_pM is effective if and only if the action of H is effective. Observe that for any finite group H acting smoothly on a compact manifold M, the action is proper. Thus if $p \in M$ is in the fixed point set of the action of H, the action is linear under the normal coordinates of any H-invariant Riemannian metric. See also [2, Theorem 1] and [23, Lemma 2.1 (1) and (2)].

We now describe a tubular neighborhood of a leaf L in M with finite holonomy. Let $\bar{L} \to L$ be the covering space of L associated with the subgroup K in (2.1). Note that there is an action of $\mathrm{Hol}(L,p)$ on $\bar{L}$. Consider a transversal section T at a point $p \in L$ and the holonomy action of the fundamental group $\pi_1(L,p)$ on T described above. We consider the diagonal action of $H = \mathrm{Hol}(L,p)$ on $\bar{L} \times T$ given by $h \cdot (q,x) = (h \cdot q, h \cdot x)$. Then we have the following theorem:

Theorem 2.7 (Local Reeb stability, Theorem 2.9 in [21])**.** *Let $(M,\mathcal{F})$ be a regular foliation. For a compact leaf L with finite holonomy $H = \mathrm{Hol}(L,p)$, there exists a foliated open neighborhood V of L in M and a diffeomorphism*

$$\bar{L} \times_H T = (\bar{L} \times T)/H \longrightarrow V.$$

Moreover for any point $q \in V$, the whole leaf L_q is contained in V, and, setting $H_q = \{h \in H \mid hq = q\}$, is homeomorphic to $\bar{L}/H_q$.

In the case where $\mathcal{F}$ is induced by a group action, Theorem 2.7 is known as the Slice Theorem (see [1, Theorem 3.57]). We say that a regular smooth foliation $\mathcal{F}$ has *finite holonomy* if every leaf has finite holonomy. For more information on foliations, see for example [21, 4].

2.3.3. *Orbifolds.* For the sake of completeness, we recall the notion of an orbifold. Consider a topological space X, and fix $n \geq 0$. An *n-dimensional orbifold chart* on X is given by an open subset $\widetilde{U} \subset \mathbb{R}^n$, a finite group G of smooth automorphisms of $\widetilde{U}$, and a G-invariant map $\phi\colon \widetilde{U} \to X$, which induces a homeomorphism of $\widetilde{U}/G$ onto some open subset $U \subset X$. By G-invariant we mean that $\phi(g \cdot u) = \phi(u)$ for any $g \in G$ and $u \in \widetilde{U}$. An *embedding* $\lambda\colon (\widetilde{U}, G, \phi) \hookrightarrow (\widetilde{V}, H, \psi)$ between two orbifold charts is a smooth embedding $\lambda\colon \widetilde{U} \to \widetilde{V}$ such that $\psi\lambda = \phi$. Given two charts $(\widetilde{U}, G, \phi)$ and $(\widetilde{V}, H, \psi)$ with $\phi(\widetilde{U}) = U$ and $\psi(\widetilde{V}) = V$, we say that they are compatible if for any $x \in U \cap V$ there exists $W \subset U \cap V$, an open neighborhood of x, and a chart $(\widetilde{W}, K, \mu)$ with $\mu(\widetilde{W}) = W$, such that there are embeddings $(\widetilde{W}, K, \mu) \hookrightarrow (\widetilde{U}, G, \phi)$ and $(\widetilde{W}, K, \mu) \hookrightarrow (\widetilde{V}, H, \psi)$. The space X equipped with an atlas of orbifold charts is called an *orbifold*, and we use the notation $\mathcal{O}_X$ to distinguish it from the original topological space X.

Let $(M,\mathcal{F})$ be a regular foliation with all leaves compact with finite holonomy. From Theorem 2.7, an open neighborhood of $p^* \in M/\mathcal{F}$ is given by $T/\mathrm{Hol}(L_p, p)$. Thus the transversal sections to the leaves, the holonomy, and the projection map π induce an orbifold atlas on $M/\mathcal{F}$.

Theorem 2.8 (Theorem 2.15 in [21])**.** *Let $(M,\mathcal{F})$ be a foliation of codimension k such that any leaf of $\mathcal{F}$ is compact with finite holonomy group. Then the space of leaves $M/\mathcal{F}$ has a canonical orbifold structure of dimension k.*

Consider an n-dimensional orbifold X, and for $x \in X$, let $(\widetilde{U}, G, \phi)$ be a chart around x. Take $y \in \widetilde{U}$ such that $\phi(y) = x$. The *local group* or *isotropy* at x is the conjugacy class (see [21, Section 2.4]) in $\mathrm{Diff}_0(\mathbb{R}^n)$ of

$$G_x = \{g \in G \mid gy = y\}.$$

Consider $(M,\mathcal{F})$ a regular foliation of codimension k, with compact leaves of finite holonomy. Given $p^* \in M/\mathcal{F}$, the isotropy G_p is the conjugacy class of $\mathrm{Hol}(L_p, p)$ in $\mathrm{Diff}_0(\mathbb{R}^k)$ (see [21, Theorem 2.15]).

2.4. **Triangulation of the leaf space.** Let M be a compact manifold. We consider a regular foliation $(M,\mathcal{F})$ by circles, *i.e.* any leaf L of $\mathcal{F}$ is homeomorphic to a circle. We observe that in this case the holonomy of any leaf L is isomorphic to either the trivial group, $\mathbb{Z}/k\mathbb{Z}$, for $k \in \mathbb{N}_{\geq 2}$, or $\mathbb{Z}$, since $\mathbb{Z}$, $k\mathbb{Z}$ and the trivial group are the only subgroups of $\mathbb{Z} = \pi_1(\mathbb{S}^1)$. We will assume from now on that $\mathrm{Hol}(L,p)$ is finite, *i.e.* of the form $\mathbb{Z}/k\mathbb{Z}$ or the trivial group, for any leaf. In this case, by Theorem 2.8, the leaf space $M/\mathcal{F}$ is a compact orbifold. For a general regular foliation $(M,\mathcal{F})$ with compact leaves and finite holonomy we will now define a decomposition into strata, and recall that we can triangulate the leaf space $M/\mathcal{F}$ with respect to this decomposition. The triangulation of the leaf space is such that the interior of each simplex lies in a smooth submanifold contained in $M/\mathcal{F}$. Furthermore the projection map $\pi\colon M \to M/\mathcal{F}$ is a smooth submersion between smooth manifolds over the interior of each simplex.

2.4.1. *The stratification.* We recall that a *stratification* of a topological space X is a partition of X into subsets $\{\Sigma_\alpha\}_{\alpha\in\Lambda}$ such that:

(i) The partition is locally finite, *i.e.* each compact subset of X only intersects a finite number of strata.
(ii) If $\Sigma_\beta \cap \mathrm{cl}(\Sigma_\alpha) \neq \emptyset$, then $\Sigma_\beta \subset \mathrm{cl}(\Sigma_\alpha)$.
(iii) If X is a smooth manifold, the strata Σ_α are embedded smooth submanifolds.

Let $H < \mathrm{Diff}_0(\mathbb{R}^n)$ be a finite subgroup. We denote by (H) its conjugacy class in $\mathrm{Diff}_0(\mathbb{R}^n)$. For a regular foliation $(M,\mathcal{F})$ with compact leaves and finite holonomy, we consider the set

$$\Sigma_{(H)} = \{p^* \in M/\mathcal{F} \mid (\mathrm{Hol}(L_p,p)) = (H)\} \subset M/\mathcal{F}.$$

From the description of the tubular neighborhood of a leaf L_p given by Theorem 2.7, we can see that $\{\Sigma_{(H)} \mid H = \mathrm{Hol}(L_p,p)\}$ gives a stratification of $M/\mathcal{F}$. We can describe $\Sigma_{(H)}$ locally as follows: for $p^* \in \Sigma_{(H)}$ we consider $\bar{L}_p \times_H T$, the tubular neighborhood of L_p described in Theorem 2.7. The points in $\Sigma_{(H)}$ correspond to the projection of the fixed points of the action of $\pi_1(L_p)$ on T. Thus locally, the stratification of $M/\mathcal{F}$ is induced from the stratification of the orbit space T/H (see [26, Section 4.3], [29, Sections 4.2 and 4.3]). From the fact that the transverse spaces T to the leaves give an atlas for the orbifold $M/\mathcal{F}$, we see that indeed the subsets $\Sigma_{(H)}$ of $M/\mathcal{F}$ induce a stratification on the orbifold $M/\mathcal{F}$ (see [22, Section 1.2]).

Remark 2.9. In general for an orbifold X, the isotropy groups yield a stratification of X, by setting $\Sigma_{(H)} = \{x \in X \mid (H) = (G_x)\}$.

For a foliation by circles $(M,\mathcal{F})$ with finite holonomy over a compact manifold M we describe the holonomy stratification. The stratification of $M/\mathcal{F}$ is given by $\{\Sigma_{(\mathbb{Z}/k\mathbb{Z})} \mid k \in \mathbb{N}\}$. Since M is compact, there are finitely many conjugacy types of holonomy groups.

Proposition 2.10. *Let $(M,\mathcal{F})$ be a regular foliation with compact leaves and finite holonomy. Consider a connected component Σ of a holonomy stratum. Then for $S = \pi^{-1}(\Sigma)$, the projection $\pi\colon S \to \Sigma$ is a fiber bundle.*

Proof. Fix $p \in S$, consider T a fixed transversal to L_p at p, and assume that H is the holonomy group of L_p at p. Observe that, by construction, for any $x \in S$ we have $L_x \subset S$. Thus the foliation $\mathcal{F}$ induces a regular foliation on S, which we also denote by $\mathcal{F}$. Recall that $q \in S \cap T$ is fixed by H. Then by Theorem 2.7, we have that $L_q = \bar{L}_p/H$ is a covering space of L_p of degree 1. Since q was arbitrary, we conclude that all the leaves of $(S, \mathcal{F})$ induce a degree 1 covering of L_p.

We now consider $(S, \mathcal{F})$, and denote by G the holonomy group of L_p at p for the restricted foliation $\mathcal{F}$ to S. Since we know that all leaves in $(S, \mathcal{F})$ are covering spaces of L_p corresponding to the trivial group, and are the fixed points of the faithful action of G on $S \cap T$, we conclude that G is the trivial group. Then, by Theorem 2.7, we obtain that locally the foliation $(S, \mathcal{F})$ is a product foliation. □

2.4.2. *Triangulation.* Given an orbifold X, there exists a triangulation $\mathcal{T}$ of X, such that the closures of the strata $\Sigma_{(H)}$ are contained in subcomplexes of $\mathcal{T}$ (see [13, 32]). By taking a subdivison of $\mathcal{T}$ we can assume that for a simplex σ of $\mathcal{T}$, the isotropy groups of the points in the interior of σ are the same, and are subgroups of the isotropy groups of the points in the boundary of σ. Furthermore we can assume that there is a face $\sigma' \subset \sigma$, such that the isotropy is constant on $\sigma \setminus \sigma'$, and possibly larger on σ'. Thus, any simplex σ of $\mathcal{T}$ has a vertex $v \in \sigma$ with maximal isotropy. This means that for any point $x \in \sigma$, a conjugate of G_x is contained in G_v.

2.5. **Hollowings.** In this section, we introduce our main geometric tool, *hollowings*. The right category in which to consider this construction is that of *manifolds with corners*, so we first define these for the sake of completeness. We follow the presentations in [7, Appendix B] and [33], and refer the reader interested in more details on manifolds with corners and their submanifolds to [7, Appendix B].

Let $n \in \mathbb{N}$ and let $k \in \{0, ..., n\}$. A *sector of dimension n with index k* is a set of the form

$$A = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_{i_1} \geq 0, ..., x_{i_k} \geq 0\},$$

with pairwise distinct $i_1, ..., i_k \in \{1, ..., n\}$. Note that $\mathbb{R}^n$ is a sector of dimension n with index 0. The *index* of a point $x \in A$ is the number of its coordinates that are equal to 0 among $x_{i_1}, ..., x_{i_k}$. The index of $0 \in \mathbb{R}^n$ is k, the index of A.

Let $m \in \mathbb{N}$ be arbitrary and $A \subset \mathbb{R}^n$ a sector. Let $U \subset A$ be open, and $V \subset \mathbb{R}^m$ an arbitrary subset. A map $\phi\colon U \to V$ is *smooth* if for every $x \in U$, there exists an open neighborhood $U_x \subset \mathbb{R}^n$ of x and a smooth map $\phi_x\colon U_x \to V$ such that ϕ_x and ϕ coincide on $U \cap U_x$.

Let M be a topological space. An *atlas with corners for M of dimension n* is a family $(U_i, V_i, \phi_i)_{i \in I}$ where $(V_i)_{i \in I}$ is an open cover of M, each U_i is an open subset in a sector A_i of dimension n and each $\phi_i\colon U_i \to V_i$ is a homeomorphism (called *chart*) such that all transition maps are smooth, *i.e.* for all $i, j \in I$, the map

$$(\phi_j^{-1} \circ \phi_i)|_{U_i \cap \phi_i^{-1}(V_j)}$$

is smooth (in the above sense).

We say that a Hausdorff second countable topological space M is an *n-dimensional manifold with corners*, if it admits an atlas with corners as above. For such an M, define the subsets $\partial M =<^{(1)} M \supset ... \supset<^{(n)} M$ as the image under the chart homeomorphisms of all subsets of points of index at least $1 \leq ... \leq n$. This does not depend on the choice of the charts.

In order to define submanifolds with corners, we need the notion of *subsector*. Let $A = \{x \in \mathbb{R}^n \mid x_{i_1} \geq 0, ..., x_{i_k} \geq 0\}$ be a sector of dimension n with index k. Let $p, \ell, j \in \mathbb{N}$ with $p \geq \ell$. A *subsector of A (of codimension p, coindex ℓ in A and complementary index j)* is a subset $A' \subset A$ determined by sets $L \subset \{i_1, ..., i_k\}$ with ℓ elements, $P \subset \{1, ..., n\} \setminus \{i_1, ..., i_k\}$ with $p - \ell$ elements and $J \subset \{1, ..., n\} \setminus (\{i_1, ..., i_k\} \cup P)$ with j elements such that

$$A' = \{(x_1, ..., x_n) \in A \mid x_i = 0 \, \forall i \in L,\, x_i = 0 \, \forall i \in P,\, x_i \geq 0 \, \forall i \in J\}.$$

Let M be a manifold with corners. A closed subspace $N \subset M$ is called a *submanifold (with corners)*, if for all $x \in N$ there exists a chart $\phi\colon U \to V$ of M, where U is contained in a sector A, and a subsector $A' \subset A$ such that $\phi(0) = x$ and $\phi^{-1}(V \cap N) = U \cap A'$.

With these definitions, we are able to define *hollowings*. Let M be a manifold with corners and N a submanifold with corners of M, such that N is transverse to each $\left(<^{(k)} M\right) \setminus \left(<^{(k+1)} M\right)$ and $<^{(k)} N = N \cap \left(<^{(k)} \ M\right)$. The tubular neighborhood $\nu(N)$ of N in M has the structure of a disk bundle over N. Let $\psi\colon \nu_S(N) \times [0,1] \to \nu(N)$ be the parametrization of $\nu(N)$ in polar coordinates, that is $\nu_S(N)$ is the total space of the associated sphere bundle, $\psi|_{\nu_S(N)\times\{1\}} = \mathrm{id}_{\nu_S(N)}$ and $\psi|_{\nu_S(N)\times\{0\}}$ is the projection of the bundle $\nu(N) \to N$. Take

$$M' = \mathrm{cl}\,(M \setminus \nu(N)) \bigcup_{\psi|_{\nu_S(N)\times\{1\}}} \nu_S(N) \times [0,1].$$

There is a natural map $p\colon M' \to M$ defined by $p|_{M\setminus\nu(N)} = \mathrm{id}_{M\setminus\nu(N)}$ and $p|_{\nu_S(N)\times[0,1]} = \psi$. The space M' has a canonical structure as a manifold with corners, making p a differentiable map between manifolds with corners.

The *hollowing* of M at N is the map $p\colon M' \to M$. In the case where $N = \emptyset$, we define $M' = M$ and $p = \mathrm{id}_M$. The submanifolds $N \subset M$ and $p^{-1}(N) \subset M'$ are called the *trace* and the *hollow wall* of p, respectively. For a submanifold $L \subset M$, denote by $\bar{p}(L)$ the submanifold $\mathrm{cl}(p^{-1}(L\setminus(L\cap N))) \subset M'$.

Let $(M, \mathcal{F})$ be a compact oriented n-dimensional manifold, with possibly empty boundary ∂M, equipped with a regular foliation of dimension q, such that each leaf has finite holonomy. Consider a triangulation of the leaf space $M/\mathcal{F}$ as described in Section 2.4.2, and denote by $(M/\mathcal{F})^{(k)}$ the k-th skeleton of the triangulation. We define a sequence of hollowings

$$M_{n-q-1} \xrightarrow[p_{n-q-2}]{} M_{n-q-2} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow[p_0]{} M_0 = M$$

in the following way. For $k = 0, ..., n-q-2$, we define inductively the map $p_k\colon M_{k+1} \to M_k$ to be the hollowing at

$$X_k = \bar{p}_{k-1}(\ldots \bar{p}_0(\pi^{-1}((M/\mathcal{F})^{(k)}))\ldots),$$

for $k \geq 1$, and $X_0 = \pi^{-1}((M/\mathcal{F})^{(0)})$. The following proposition shows that at each step of the hollowing we obtain a foliated manifold.

Proposition 2.11. *Let $(M, \mathcal{F})$ be a compact smooth manifold with a regular foliation with finite holonomy. For each hollowing $p_i \colon M_{i+1} \to M_i$, there are regular foliations $\mathcal{F}_i$ on M_i, and $\mathcal{F}_{i+1}$ on M_{i+1}, such that p_i maps $\mathcal{F}_{i+1}$ to $\mathcal{F}_i$.*

Proof. We will prove this by induction on i. Assume that $(M_i, \mathcal{F}_i)$ is a regular foliation with finite holonomy. Consider U a tubular neighborhood of X_i in M_i, and fix $q \in U$. Then q belongs to a tubular neighborhood of some leaf $L_p \subset M_i$ of $\mathcal{F}_i$ with $p \in X_i$. Denote by H the holonomy group of L_p, and T a transverse subspace to L_p at p. Since a tubular neighborhood of L_p is foliated diffeomorphic to $\bar{L}_p \times_H T$, the leaf L_q is contained in this tubular neighborhood, and thus in the tubular neighborhood U of X_i. This implies that the tubular neighborhood of X_i in M_i is foliated. Since we are hollowing M_i at X_i, we see that the foliation $\mathcal{F}_i$ restricted to U minus the zero-section X_i induces a foliation on $\nu_S(X_i) \times [0, 1]$, with finite holonomy. This proves the proposition. ☐

Remark 2.12. In general, let $(M, \mathcal{F})$ be a compact smooth manifold with a regular foliation with finite holonomy. Consider a fixed stratum $\Sigma_{(H)}$ of $M/\mathcal{F}$. For $p^* \in \Sigma_{(H)}$ we have, from Theorem 2.7, that a neighborhood is given by T/H. Furthermore, we may assume that T is homeomorphic to an $(n - q)$-disk, where q is the dimension of $\mathcal{F}$, and that H acts linearly on it (see Remark 2.6). This implies that the fixed point set of the action of H on T is a linear subspace. Thus locally, the stratum is a submanifold in T/H. Since, by construction, the simplices of the triangulation are contained in strata, from the description of the hollowing and the local description of the foliation, we see that the map $p_i^* \colon M_{i+1}/\mathcal{F}_{i+1} \to M_i/\mathcal{F}_i$ is a hollowing on the i-th skeleton.

Remark 2.13. For the particular case that the regular foliation $\mathcal{F}$ is given by circles, the foliation $\mathcal{F}_i$ is also given by circles.

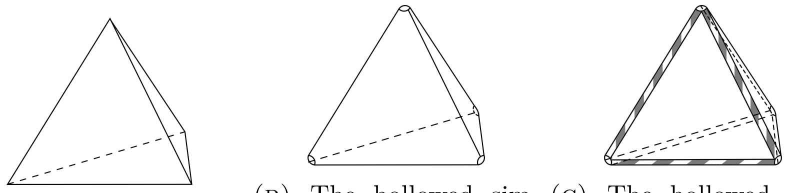
2.5.1. *Examples.* We present some examples of hollowings.

Example 2.14.

(1) For the standard n-dimensional simplex Δ^n, we denote by $(\Delta^n)^{(k)}$ the k-th skeleton of Δ^n. We can define a sequence of hollowings inductively. Setting $\Delta^n_0 = \Delta^n$, we define $q_k \colon \Delta^n_{k+1} \to \Delta^n_k$ the hollowing at $\bar{q}_{k-1} \cdots \bar{q}_0((\Delta^n)^{(k)})$ (see Figure 2.2).

(2) The following is a hollowing for a singular foliation, but we find it helpful as an example. Let $\mathbb{S}^1$ act on $\mathbb{S}^2$ by rotations along the north-south axis. This gives a foliation $\mathcal{F}$ of $\mathbb{S}^2$ by the orbit circles, except for the two fixed points N, S. The leaf space $(\mathbb{S}^2)^*$ is a segment with orbifold structure given by the open segment and its two boundary points corresponding to the fixed points N, S.

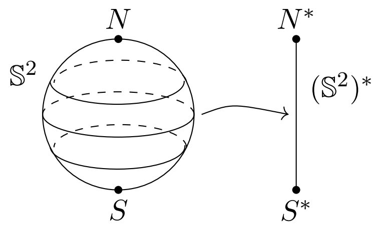
A hollowing p at the submanifold $\{N, S\}$ is described as follows: in $(\mathbb{S}^2)'$ the tubular neighborhoods $\nu(N), \nu(S) \subset \mathbb{S}^2$ of N and S (two 2-disks) are replaced by two annuli $[0, 1] \times (\mathbb{S}^1 \cup \mathbb{S}^1)$, by gluing $\{1\} \times$



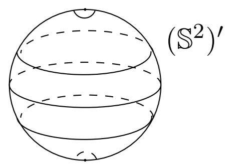
(A) The simplex Δ_0^3 (B) The hollowed simplex Δ_1^3 (C) The hollowed simplex Δ_2^3

FIGURE 2.2. Hollowings of the simplex

$(\mathbb{S}^1 \cup \mathbb{S}^1)$ via the identity to $\mathbb{S}^2 \setminus (\nu(N) \cup \nu(S))$ along the boundaries $\mathbb{S}^1 \cup \mathbb{S}^1$ (see Figure 2.3). In this case the hollowed manifold $(\mathbb{S}^2)'$ is a closed annulus, that is $\mathbb{S}^2$ with two disjoint open disks removed.



(A) Foliation and leaf space



(B) Hollowing (small open disks are missing at the two poles)

FIGURE 2.3. Action of $\mathbb{S}^1$ by rotations on $\mathbb{S}^2$.

(3) Let $\mathbb{S}^3 = \{(z_0, z_1) \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\}$. Fix $p, q \in \mathbb{Z}$ with $(p, q) = 1$ and let $\mathbb{S}^1$ act on $\mathbb{S}^3$ via

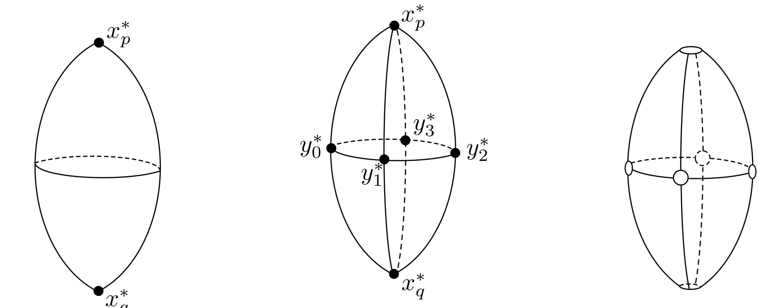
$$e^{2i\pi\theta} \cdot (z_0, z_1) = (e^{2i\pi\theta p} z_0, e^{2i\pi\theta q} z_1).$$

This action is called a *weighted Hopf fibration*; when $p, q = 1$ we recover the classical Hopf fibration. The action has no fixed points, but the circles

$$\{(z_0, 0) \mid |z_0| = 1\},\ \{(0, z_1) \mid |z_1| = 1\} \subset \mathbb{S}^3$$

have non-trivial isotropy isomorphic to $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$, respectively. All other points have trivial isotropy. The orbit of each point is a circle. The orbit space is $\mathbb{S}^2$, with orbifold structure a cylinder with upper and lower boundary circles collapsed to points x_p^* and x_q^*, respectively, corresponding to the two orbits with isotropy $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$, respectively. Take a triangulation of this $\mathbb{S}^2$ such that x_p^*, x_q^* are among the vertices: for example add four vertices $y_0^*, ..., y_3^*$ on the equator and join each of them by edges to x_p^* and

x_q^* (see Figure 2.4). In this fashion, we obtain a triangulation of $\mathbb{S}^2$ with 6 vertices, 12 edges and 8 triangles. The hollowings consist here of a single map $p_0 \colon M_1 \to \mathbb{S}^3$, that is the hollowing at $\pi^{-1}((\mathbb{S}^3/\mathcal{F})^{(0)})$: we take tubular neighborhoods $\nu_p, \nu_q, \nu_{y_0}, \ldots, \nu_{y_3}$ of the 6 circles $\pi^{-1}(x_p^*), \pi^{-1}(x_q^*), \pi^{-1}(y_0^*), ..., \pi^{-1}(y_3^*)$ out of $\mathbb{S}^3$ and glue back the products $[0,1] \times \partial\nu_p, [0,1] \times \partial\nu_q, [0,1] \times \partial\nu_{y_0}, \ldots, [0,1] \times \partial\nu_{y_3}$ using the identity map along the associated sphere bundles $\partial\nu_p, \partial\nu_q, \partial\nu_{y_0}, \ldots, \partial\nu_{y_3}$.



(A) Orbit space (B) Triangulated orbit space (C) Hollowing

FIGURE 2.4. Hollowing for the orbit space of the weighted Hopf fibration

2.5.2. *Decomposition of the manifold.* Let $(M, \mathcal{F})$ be a regular foliation by circles with finite holonomy. In this case the leaf space has dimension $n-1$. We will show that when M is orientable, the $(n-3)$-skeleton of the triangulation of $M/\mathcal{F}$ given in Subsection 2.4.2 contains all the strata of non-trivial holonomy.

Lemma 2.15. *Let M be a compact oriented smooth manifold equipped with a regular smooth circle foliation $\mathcal{F}$. Then the strata with non-trivial holonomy are contained in the $(n-3)$-skeleton of $M/\mathcal{F}$.*

Proof. Suppose there is a connected component Σ of a stratum with non-trivial holonomy that lies in $(M/\mathcal{F})^{(n-2)}$ and not in $(M/\mathcal{F})^{(n-3)}$. Fix $p \in M$ such that $p^* \in \Sigma$, and denote by H the holonomy of the leaf L_p through p at p. We are going to study the action of H on a small neighborhood of p in M.

For $S = \pi^{-1}(\Sigma)$ we have a circle bundle $\pi \colon S \to \Sigma$ by Proposition 2.10. Thus over a closed neighborhood $U \subset \Sigma$ around p^* we have that $\pi^{-1}(U)$ is diffeomorphic to $\mathbb{S}^1 \times U$, and S has codimension 1 in M. Moreover since Σ is an $(n-2)$-manifold, we can assume that U is homeomorphic to $\mathbb{D}^{n-2}$.

We now consider $\operatorname{proj} \colon \operatorname{Tub}(S) \to S$ a small tubular neighborhood of S in M, and set $\operatorname{Tub}(\pi^{-1}(U)) = (\operatorname{proj})^{-1}(\pi^{-1}(U))$. Then for a small closed set $\bar{V} \subset \pi^{-1}(U)$ we have that $V = \operatorname{proj}^{-1}(\bar{V})$ is homeomorphic to $[-1,1] \times \bar{V}$ (see Figure 2.5). Since $\pi^{-1}(U) \cong \mathbb{S}^1 \times \mathbb{D}^{n-2}$, we can assume that V is homeomorphic to $[-1,1] \times [-1,1] \times \mathbb{D}^{n-2}$, but with the first factor $[-1,1]$ corresponding to the intersection of V with the circle leaves of the foliation.

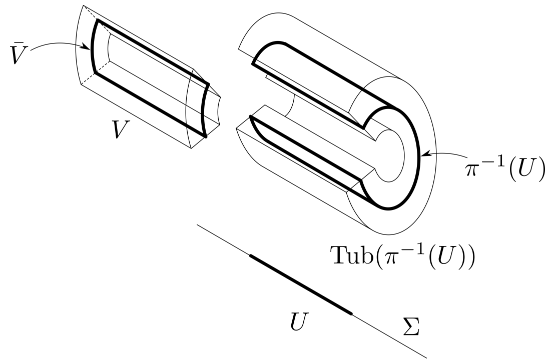


FIGURE 2.5. Neighborhood V

A small neighborhood A of p^* in $M/\mathcal{F}$ is homeomorphic to $([-1,1] \times \mathbb{D}^{n-2})/H$, since $[-1,1] \times \mathbb{D}^{n-2}$ corresponds to the intersection of V with a transversal section T to the leaf L_p at p. Observing that $\mathbb{D}^{n-2}$ corresponds to the intersection of the transversal section T with S, and this intersection is fixed by H, we conclude that H acts trivially on $\mathbb{D}^{n-2}$. Consequently H acts faithfully linearly on $\mathbb{R}^1$ (see Remark 2.6), the tangent space to the factor $[-1,1]$. Thus H is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Now consider a closed tubular neighborhood $\mathrm{Tub}(L_p)$. By the Local Reeb stability Theorem 2.7, this neighborhood is foliated diffeomorphic to $\mathbb{S}^1 \times_H ([-1,1] \times \mathbb{D}^{n-2})$. As we said above, the action of $\mathbb{Z}/2\mathbb{Z}$ on the factor $\mathbb{D}^{n-2}$ is trivial, so $\mathrm{Tub}(L_p)$ is foliated diffeomorphic to $(\mathbb{S}^1 \times_{\mathbb{Z}/2\mathbb{Z}} [-1,1]) \times \mathbb{D}^{n-2}$. Observe that $\mathbb{S}^1 \times_{\mathbb{Z}/2\mathbb{Z}} [-1,1]$ is diffeomorphic to the Möbius band. Since M is orientable, then $\mathrm{Tub}(L_p)$ is orientable too. This is a contradiction. $\square$

The previous lemma implies, as in Yano [33], that the hollowing constructed in Section 2.5 is as follows:

$$M_{n-2} \xrightarrow[p_{n-3}]{} M_{n-3} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow[p_0]{} M_0 = M.$$

As in [8, 7] we extend this sequence to account for the case when M has non-empty boundary. We set $M_{-1} = M$, and $p_{-1} = \mathrm{id}_M$. We set $X_{-1} = \partial M \subset M_{-1}$, and define $N_j = p_j^{-1}(X_j) \subset M_{j+1}$. For $j < i$ we set

$$p_{i,j} = p_j \circ p_{j+1} \circ \cdots \circ p_{i-1} \colon M_i \longrightarrow M_j.$$

Define $\widetilde{N}_j = p_{n-2,j+1}^{-1}(N_j) = p_{n-2,j}^{-1}(X_j) \subset M_{n-2}$, and for $\{j_1, \ldots, j_k\}$ a set of distinct indices with $-1 \leq j_i \leq n-3$ we define

$$\widetilde{N}_{j_1,\ldots,j_k} = \widetilde{N}_{j_1} \cap \ldots \cap \widetilde{N}_{j_k} \subset M_{n-2}$$

and

$$X_{j_1,\ldots,j_k} = p_{n-2,j_1}(\widetilde{N}_{j_1,\ldots,j_k}) \subset X_{j_1}.$$

Remark 2.16. Note that $\widetilde{N}_{j_1,\ldots,j_k}$ does not depend on the order of the indices $j_1, \ldots, j_k$, while $X_{j_1,\ldots,j_k}$ depends on which index appears first. There is no assumption of any kind on the order relations between the j_i's.

We recover for the foliated context a series of lemmas from [33] and [8]. For example we have the following:

Lemma 2.17 (Lemma 4 in [33]). *Let $(M, \mathcal{F})$ be a regular foliation by circles with finite holonomy. For $j_1, \dots, j_k \geq 0$, each connected component of $X_{j_1,\dots,j_k}/\mathcal{F}_{j_1}$ is contractible and can be identified with $\Delta^{j_1-k+1}_{j_1-k+1}$.*

Proof. We show first that this holds for X_{j_1}. The set X_{j_1} is the pullback via $p_{j_1-1,0} \circ \pi \colon M_{j_1} \to M/\mathcal{F}$ of the j_1-skeleton. For every connected component X'_{j_1} of X_{j_1}, we have that $X'_{j_1}/\mathcal{F}_{j_1}$ is equal to Δ^{ℓ}_{ℓ}, where $\ell = j_1$ (see Example 2.14 (1)). Under this identification, each connected component of $X_{j_1,\dots,j_k}/\mathcal{F}_{j_1}$ is diffeomorphic to $\Delta^{\ell-k+1}_{\ell-k+1}$. This space is contractible. □

From the construction of the triangulation (Section 2.4.2) we obtain:

Lemma 2.18 (Lemma 2.1 in [8]). *Let $(M, \mathcal{F})$ be a regular foliation by circles with finite holonomy. For $j_1, \dots, j_k \geq 0$, the space $X_{j_1,\dots,j_k}$ is foliated diffeomorphic to $(X_{j_1,\dots,j_k}/\mathcal{F}_{j_1}) \times \mathbb{S}^1$.*

Proof. From the construction of the triangulation of $M/\mathcal{F}$, over each face of an $(n-1)$-simplex we have leaves of $\mathcal{F}$ with same holonomy. By construction of $X_{j_1,\dots,j_k}$, this fact also holds for $\mathcal{F}_{j_1}$. From Lemma 2.17 it follows that over a connected component of $X_{j_1,\dots,j_k}$ we have constant holonomy. Thus by Proposition 2.10 $X_{j_1,\dots,j_k}$ is the union of total spaces of circle bundles over the connected components of $X_{j_1,\dots,j_k}/\mathcal{F}_{j_1}$, which are contractible. Therefore it is trivial. □

A main difference with the work of [8, 33] is that the circle bundle $\mathbb{S}^1 \to M_{n-2} \to (M_{n-2}/\mathcal{F}_{n-2})$ is not trivial in general. Nonetheless, we have the following lemma:

Lemma 2.19. *Let $(M, \mathcal{F})$ be a regular foliation by circles with finite holonomy on a connected manifold. For the last hollowing M_{n-2}, the leaf space $(M_{n-2}/\mathcal{F}_{n-2})$ has the homotopy type of a compact connected 1-complex and M_{n-2} is aspherical.*

Proof. Let $\Sigma = \cup_{H \neq \{\mathrm{id}\}} \Sigma_{(H)} \subset M/\mathcal{F}$ be the union of all strata corresponding to non-trivial holonomy groups. By Lemma 2.15, $\Sigma/\mathcal{F}$ is contained in simplices of dimension at most $n-3$ in $M/\mathcal{F}$.

From this it follows that $\bar{p}_{n-2,0}(\pi^{-1}(\Sigma)) = \emptyset$, since at this point we have removed all the preimages of the simplices of dimension less than $n-2$. Thus over M_{n-2} the foliation does not have holonomy, and induces a circle bundle. Observe that $M_{n-2}/\mathcal{F}_{n-2} \cong M/\mathcal{F} \setminus (M/\mathcal{F})^{(n-3)}$ has the homotopy type of a compact 1-complex, that is a finite graph. Since M is connected, this graph is connected. The fundamental group of a finite graph is a finitely generated free group. From the long exact sequence of homotopy groups of the fibration $\mathbb{S}^1 \to M_{n-2} \to M_{n-2}/\mathcal{F}_{n-2}$ we see that M_{n-2} is aspherical. □

Furthermore, by the construction of the hollowings and the triangulation, the following proposition holds:

Proposition 2.20 (See Proposition 2.2 in [8]). *For all pairwise distinct $j_1, \dots, j_k \in \{0, \dots, n-3\}$ we have that $X_{j_1,\dots,j_k,-1}$ is the union of the connected components $Y \subset X_{j_1,\dots,j_k}$ that satisfy*

$$Y \subset p_{n-2,j_1}(\widetilde{N}_{-1}).$$

Proof. Let $j \in \{0, \dots, n-3\}$. We first show the statement for $X_{j,-1} \subset X_j$. We work in the space $X_j/\mathcal{F}_j$ and show the statement there.

Let $Y \subset X_j/\mathcal{F}_j$ be a connected component. We will show that

$$Y \subset X_{j,-1}/\mathcal{F}_j \Longleftrightarrow Y \subset p_{n-2,j}(\widetilde{N}_{-1})/\mathcal{F}_j.$$

The left-to-right implication is true by definition. For the right-to-left implication, as in the proof of Lemma 2.17, we observe that Y is homeomorphic to Δ^j_j, where Δ^j_j is obtained from the standard simplex Δ^j by hollowing inductively along the ℓ-skeleton for all $\ell \in \{0, \dots, j-1\}$ (see Example 2.14 1). From this it follows that we are in one of the following cases:

(1) $Y \subset p_{n-2,j}(\widetilde{N}_{-1})/\mathcal{F}_j$, or
(2) $Y \cap p_{n-2,j}(\widetilde{N}_{-1})/\mathcal{F}_j = \emptyset$.

In the first case, we have

$$\begin{array}{rcr} Y & \subset & X_j/\mathcal{F}_j \cap p_{n-2,j}(\widetilde{N}_{-1})/\mathcal{F}_j \\ & = & p_{n-2,j}(\widetilde{N}_j)/\mathcal{F}_j \cap p_{n-2,j}(\widetilde{N}_{-1})/\mathcal{F}_j \\ & \subset & p_{n-2,j}(\widetilde{N}_{j,-1})/\mathcal{F}_j \\ & = & X_{j,-1}/\mathcal{F}_j, \end{array}$$

where the last inclusion follows from

$$p_{n-2,j}(\widetilde{N}_j \setminus \widetilde{N}_{-1})/\mathcal{F}_j \cap p_{n-2,j}(\widetilde{N}_{-1} \setminus \widetilde{N}_j)/\mathcal{F}_j = \emptyset,$$

which holds by construction of the hollowings. In the second case, we have $Y \cap X_{j,-1}/\mathcal{F}_j = \emptyset$.

For $X_{j_1,\dots,j_k,-1}$ with $k \geq 1$, it suffices to observe that

$$X_{j_1,\dots,j_k,-1}/\mathcal{F}_j = X_{j_1,\dots,j_k}/\mathcal{F}_j \cap X_{j_1,-1}/\mathcal{F}_j.$$

See also [7, Lemma 4.2.8] for more details. □

Remark 2.21. We point out that we can take a refinement of the triangulation on $\partial(M_{n-2}/\mathcal{F}_{n-2})$ so that it is compatible with the decompositions

$$\partial(M_{n-2}/\mathcal{F}_{n-2}) = \bigcup_{i=-1}^{n-3} \widetilde{N}_i/\mathcal{F}_{n-2}$$

and

$$\partial\left(\widetilde{N}_{i_1,\dots,i_k}/\mathcal{F}_{n-2}\right) = \bigcup_{i=-1, i\neq i_1,\dots,i_k}^{n-3} \widetilde{N}_{i_1,\dots,i_k,i}/\mathcal{F}_{n-2}.$$

That is, each $\widetilde{N}_{i_1,\dots,i_k,i}/\mathcal{F}_{n-2}$ is a subcomplex of $\partial(M_{n-2}/\mathcal{F}_{n-2})$.

3. Proof of Theorem B

Now we establish the necessary preliminary results for the proof of our main theorem, which is carried out at the end of the section.

We will then use the triangulation of $M/\mathcal{F}$ to construct a series of triangulations on the holonomy strata that have zero foliated simplicial volume.

Proposition 3.1 (See Proposition 4.1 in [8])**.** *Assume the inclusions of the leaves of the foliation $\mathcal{F}$ are π_1-injective. Take $k \in \{1, \dots, n-2\}$ and let $j_1, \dots, j_k \in \{0, \dots, n-3\}$ be pairwise distinct. Then, for any choice of* *basepoints, the inclusions* $X_{j_1,\dots,j_k} \subset M_{j_1}$ *and* $X_{j_1,\dots,j_k,-1} \subset M_{j_1}$ *are* π_1*-injective.*

Proof. By Proposition 2.20, it suffices to show that the inclusion $X_{j_1,\dots,j_k} \subset M_{j_1}$ is π_1-injective. By Lemma 2.18, we have $X_{j_1,\dots,j_k} \cong (X_{j_1,\dots,j_k}/\mathcal{F}_{j_1}) \times \mathbb{S}^1$. By Lemma 2.17, each connected component of $X_{j_1,\dots,j_k}/\mathcal{F}_{j_1}$ is contractible. Now, by Proposition 2.11, the composition of maps

$$X_{j_1,\dots,j_k} \subset M_{j_1} \xrightarrow{p_{j_1,0}} M_0 = M$$

is the inclusion of leaves into M, and thus π_1-injective by hypothesis. Thus the inclusion $X_{j_1,\dots,j_k} \subset M_{j_1}$ is also π_1-injective. □

We now construct a series of representations of the fundamental groups of the hollowings M_j as follows:

Setup 3.2. Fix $x_{n-2} \in M_{n-2}$, and set $x_i = p_{n-2,i}(x_{n-2}) \in M_i$. We write $\Gamma = \pi_1(M, x_0)$ and consider a fixed essentially free standard Γ-space (Z, μ), with the representation $\alpha_0 = \alpha \colon \Gamma \to \operatorname{Aut}(Z, \mu)$. From this representation, using the hollowing maps $p_{i,0} \colon M_i \to M$, we can define for $\Gamma_i = \pi_1(M_i, x_i)$ a representation $\alpha_i \colon \Gamma_i \to \operatorname{Aut}(Z, \mu)$ by setting

$$\alpha_i = \alpha \circ \pi_1(p_{i,0}).$$

Recall that we have a circle bundle $M_{n-2} \to M_{n-2}/\mathcal{F}_{n-2}$ (see proof of Lemma 2.19), which may be orientable or not. If the bundle structure of M_{n-2} is orientable, then by the classification of oriented $\mathbb{S}^1$-bundles, it is trivial: by Lemma 2.19 $M_{n-2}/\mathcal{F}_{n-2}$ has the homotopy type of a graph, and hence $H^2(M_{n-2}/\mathcal{F}_{n-2}, \mathbb{Z}) = 0$.

If the bundle structure of M_{n-2} is non-orientable, since M_{n-2} is orientable, then the base $M_{n-2}/\mathcal{F}_{n-2}$ is non-orientable. For this case we need to change the Borel space we consider as follows: we take B the oriented double cover of $M_{n-2}/\mathcal{F}_{n-2}$. By pulling back the circle bundle we obtain an orientable double cover W of M_{n-2}. This double cover is homotopy equivalent to an oriented circle bundle over B. Thus we get the following commutative diagram:

$$\begin{tikzcd} W \arrow[r, "\tilde{p}", "2"'] \arrow[d] & M_{n-2} \arrow[d] \\ B \arrow[r, "p", "2"'] & M_{n-2}/\mathcal{F}_{n-2}. \end{tikzcd}$$

Observe that the fundamental group H of W is a subgroup of index 2 of Γ_{n-2}, and thus we obtain a representation β of H on (Z, μ) by restricting α_{n-2}. In the subsequent proofs, we will find parametrized relative fundamental cycles of W, and from them we will obtain appropriate parametrized fundamental cycles of M_{n-2}.

To do so, as in [20, Setup 4.24 and Definition 4.25], set $\gamma_0 = e$ and γ_1 be a fixed representative of the non-identity class in Γ_{n-2}/H. We denote the elements in

$$\Gamma_{n-2} \times_H Z = \Gamma_{n-2} \times Z / \{(\gamma h, z) \sim (\gamma, h \cdot z), \gamma \in \Gamma_{n-2}, h \in H, z \in Z\}$$

by $[\gamma, z]$, and Γ_{n-2} acts on $\Gamma_{n-2} \times_H Z$ by $\gamma'[\gamma, z] = [\gamma'\gamma, z]$. The measure on $\Gamma_{n-2} \times_H Z$ is given as follows: we take the counting measure μ' on Γ_{n-2}/H and then pull back the measure $(1/2)\mu' \otimes \mu$ on $\Gamma_{n-2}/H \times Z$ via the bijection

$$\begin{aligned} \Gamma_{n-2} \times_H Z &\longrightarrow \Gamma_{n-2}/H \times Z \\ [\gamma, z] &\longmapsto (\gamma H, z). \end{aligned}$$

As in [20, Proof of Proposition 4.26], we have the following well-defined $\mathbb{Z}\Gamma_{n-2}$-isomorphism:

$$\psi \colon L^\infty(Z; \mathbb{Z}) \otimes_{\mathbb{Z}H} \mathbb{Z}\Gamma_{n-2} \longrightarrow L^\infty(\Gamma_{n-2} \times_H Z; \mathbb{Z})$$

$$f \otimes \gamma_j \longmapsto \left([\gamma_k, z] \mapsto \begin{cases} f(z), & \text{if } k = j \\ 0, & \text{if } k \neq j \end{cases} \right).$$

The induced map

$$\Psi \colon C_*(W; \beta) \longrightarrow C_*(M_{n-2}; \Gamma_{n-2} \times_H Z)$$

$$f \otimes \sigma \longmapsto \psi(f \otimes e) \otimes \sigma$$

sends parametrized fundamental cycles of W to parametrized fundamental cycles of M_{n-2} [20, Proof of Proposition 4.26]. The parametrized norms behave as follows:

$$\left| \Psi(c) \right|^{\Gamma_{n-2} \times_H Z} \leq \frac{1}{2} \left| c \right|^{Z} \tag{3.1}$$

for all $c \in C_*(W; \beta)$.

Using the hollowings $p_i \colon M_{i+1} \to M_i$, we define $H_i = \pi_1(p_{n-2,i} \circ \tilde{p})(H) < \Gamma_i$. Note that H_i has finite index in Γ_i for every $i \in \{0, \ldots, n-2\}$. Indeed, the maps $\pi_1(p_{n-2,i})$ induced by the hollowings are surjective: the hollowing maps p_i are quotient maps by construction, and their fibers are either a point, or a sphere of dimension at least 1, hence connected. Then [3, Theorem 1.1] applies. Moreover, the index of H in Γ_{n-2} is 2 by definition, so that $[\Gamma_i : H_i] \leq 2$.

We note that by construction of the representations α_i, the restriction of α_i to H_i is an essentially free action on Z. The spaces we consider are $\Gamma_i \times_{H_i} Z$, which have an essentially free action of Γ_i as above, denoted by β_i.

Depending on whether the circle bundle $M_{n-2} \to M_{n-2}/\mathcal{F}_{n-2}$ is orientable or not, for $i \in \{0, \ldots, n-2\}$, we consider V_i equal to Z, respectively $\Gamma_i \times_{H_i} Z$, with a representation ξ_i of Γ_i given by α_i, respectively β_i.

Let $\widetilde{p}_i \colon \widetilde{M}_{i+1} \to \widetilde{M}_i$ be a fixed lift to the universal covers of the map $p_i \colon M_{i+1} \to M_i$. We define a chain map

$$P_i \colon L^\infty(\xi_{i+1}; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma_{i+1}} C_*(\widetilde{M}_{i+1}, \mathbb{Z}) \longrightarrow L^\infty(\xi_i; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma_i} C_*(\widetilde{M}_i, \mathbb{Z})$$

$$f \otimes \sigma \longmapsto f \otimes (\widetilde{p}_i \circ \sigma).$$

In this way we obtain the following sequence:

$$L^\infty(\xi_{n-2}; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma_{n-2}} C_*(\widetilde{M}_{n-2}, \mathbb{Z}) \xrightarrow{P_{n-3}} \cdots \xrightarrow{P_0} L^\infty(\xi_0; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma_0} C_*(\widetilde{M}, \mathbb{Z}).$$

For $i < j$ we can also consider the maps

$$P_{j,i} \colon L^\infty(\xi_j; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma_j} C_*(\widetilde{M}_j, \mathbb{Z}) \longrightarrow L^\infty(\xi_i; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma_i} C_*(\widetilde{M}_i, \mathbb{Z}),$$

defined as $P_{j,i} = P_i \circ \cdots \circ P_{j-1}$. These maps will be used later in the proof of Theorem B.

For $X_{i_1,\ldots,i_k} \subset M_{i_1}$ we set $\Lambda_{i_1,\ldots,i_k} = \pi_1(X_{i_1,\ldots,i_k})$. As in the observation made in [8, Setup 4.2], these groups are independent of the base points chosen:

Lemma 3.3. *Let x, y be two points in a connected component of $X_{i_1,\ldots,i_k}$. Then $\pi_1(X_{i_1,\ldots,i_k}, x) \cong \pi_1(X_{i_1,\ldots,i_k}, y)$. The fundamental group of any such connected component is $\mathbb{Z}$.*

Proof. By Proposition 3.1 the inclusions $X_{i_1,\ldots,i_k} \hookrightarrow M_{i_1}$ are π_1-injective for all suitable sets of indices $i_1, \ldots, i_k$, and by Lemma 2.18 $X_{i_1,\ldots,i_k} \cong (X_{i_1,\ldots,i_k}/\mathcal{F}_{i_1}) \times \mathbb{S}^1$. Moreover, by Lemma 2.17 the connected components of $X_{i_1,\ldots,i_k}/\mathcal{F}_{i_1}$ are contractible. So for every choice of base point $x \in X_{i_1,\ldots,i_k}$, the corresponding fundamental group $\pi_1(X_{i_1,\ldots,i_k}, x)$ is $\mathbb{Z}$.

Set $x^* = \pi \circ p_{i_1,0}(x), y^* = \pi \circ p_{i_1,0}(y)$ the images of x, y in $\pi \circ p_{i_1,0}(M_{i_1})$. Choose an embedded path $\overline{\gamma}$ from x^* to y^* in $\pi \circ p_{i_1,0}(X_{i_1,\ldots,i_k})$ so that its interior avoids the $(n-3)$-skeleton $(M/\mathcal{F})^{(n-3)}$. Its preimage γ in $X_{i_1,\ldots,i_k}$ is then a circle bundle over the embedded interval $\overline{\gamma}$, thus an annulus. Therefore the two fibers above the two ends of the path $\overline{x}$ and $\overline{y}$ are homotopic. These are exactly the two generating circles of $\pi_1(X_{i_1,\ldots,i_k}, x)$ and $\pi_1(X_{i_1,\ldots,i_k}, y)$. ☐

By Proposition 3.1, we have $\Lambda_{i_1,\ldots,i_k} < \Gamma_{i_1}$. We denote by $\xi'_{i_1,\ldots,i_k}$ the restriction of the representation ξ_{i_1} to $\Lambda_{i_1,\ldots,i_k}$. For the universal cover $q_{i_1} \colon \widetilde{M}_{i_1} \to M_{i_1}$, observe that $q_{i_1}^{-1}(X_{i_1,\ldots,i_k})$ is Γ_{i_1}-invariant. Hence we can consider the subcomplex

$$L^\infty(\xi_{i_1}; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma_{i_1}} C_*(q_{i_1}^{-1}(X_{i_1,\cdots,i_k}), \mathbb{Z}).$$

Proposition 3.4. *For the subcomplex $L^\infty(\xi_{i_1}; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma_{i_1}} C_*(q_{i_1}^{-1}(X_{i_1,\cdots,i_k}), \mathbb{Z})$ and the restriction $\xi'_{i_1,\cdots,i_k}$ we have an isomorphism from*

$$L^\infty(\xi_{i_1}; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma_{i_1}} C_*(q_{i_1}^{-1}(X_{i_1,\cdots,i_k}), \mathbb{Z})$$

onto

$$L^\infty(\xi'_{i_1,\cdots,i_k}; \mathbb{Z}) \otimes_{\mathbb{Z}\Lambda_{i_1,\cdots,i_k}} C_*(X_{i_1,\cdots,i_k}, \mathbb{Z}).$$

Proof. See [8, p. 12]. ☐

For a fixed $\varepsilon > 0$, we will show the existence of an essentially free Γ-space, and a representation of Γ, such that there is a relative parametrized fundamental cycle of M with ℓ^1-norm bounded above by ε. We begin by finding such a cycle for M_{n-2}.

Proposition 3.5. *Let $(M, \mathcal{F})$ be an oriented compact connected smooth n-manifold with a regular foliation by circles with finite holonomy. Assume that the inclusion of each leaf into M is π_1-injective. Set $\Gamma = \pi_1(M, x_0)$ and choose $\varepsilon > 0$. There exists a relative fundamental cycle*

$$z \in C_n(M_{n-2}; \xi_{n-2})$$

that has ℓ^1-norm less than ε.

Proof. We consider two cases: when $M_{n-2}/\mathcal{F}_{n-2}$ is orientable and when it is not. In the first case, since M_{n-2} is orientable, then the circle bundle $M_{n-2} \to M_{n-2}/\mathcal{F}_{n-2}$ is also orientable. Thus, as stated in the proof of Lemma 2.19 and in the Setup 3.2, we have $M_{n-2} \cong (M_{n-2}/\mathcal{F}_{n-2}) \times \mathbb{S}^1$. The subgroup Λ of $\pi_1(M_{n-2}/\mathcal{F}_{n-2}) \times \pi_1(\mathbb{S}^1)$ generated by the circle factor corresponds under this homeomorphism to the subgroup of Γ_{n-2} generated by a leaf. Observe that, by Proposition 2.11, each leaf in M_{n-2} is mapped by $p_{n-3,0}$ to a leaf of $(M, \mathcal{F})$. Since the inclusion of any leaf is π_1-injective, then (Z, μ) is an essentially free standard Λ-space with respect to α', the restriction of α_{n-2} to Λ. From the proof of [11, Lemma 10.8], given any relative fundamental cycle $\bar{z}$ of $M_{n-2}/\mathcal{F}_{n-2}$, there exists a cycle $c_{\mathbb{S}^1} \in C_1(\mathbb{S}^1; \alpha')$ such that the relative fundamental cycle

$$z = \bar{z} \times c_{\mathbb{S}^1} \in C_n(M_{n-2}; \alpha_{n-2})$$

has ℓ^1-norm less than ε, as desired.

For the second case, we consider the oriented double cover $\tilde{p} \colon W \to M_{n-2}$ of M_{n-2}, which is the total space of a trivial circle bundle over the orientable double cover B of $M_{n-2}/\mathcal{F}_{n-2}$. Recall that Z is an essentially free H-space, via the representation β. As in the first case, since B has the homotopy type of a 1-complex, the fundamental group of the fiber $\mathbb{S}^1$ injects into H. Denote by β' the restriction of β to this subgroup. Again by the proof of Lemma 10.8 in [11], for any relative fundamental cycle $\bar{u}$ of B, we can find a parametrized cycle $c_{\mathbb{S}^1} \in C_1(\mathbb{S}^1; \beta')$ such that $\bar{u} \times c_{\mathbb{S}^1}$ has ℓ^1-norm less than 2ε.

Recall that the parametrized norms behave as follows

$$\left| \Psi(c) \right|^{\Gamma_{n-2} \times_H Z} \leq \frac{1}{2} \left| c \right|^{Z}$$

for all $c \in C_*(W; \beta)$. Hence, taking $c = \bar{u} \times c_{\mathbb{S}^1} \in C_n(W; \beta)$, we obtain a $\Gamma_{n-2} \times_H Z$-parametrized relative fundamental cycle $z = \Psi(c)$ for M_{n-2} with

$$\left| z \right|^{\Gamma_{n-2} \times_H Z} < \varepsilon. \qquad \square$$

We will now show the existence of a fundamental cycle of $(M, \partial M)$ with arbitrarily small ℓ^1-norm. Let $\bar{z}$ be any relative fundamental cycle of the manifold $M_{n-2}/\mathcal{F}_{n-2}$. We first consider the case when the circle bundle $M_{n-2} \to M_{n-2}/\mathcal{F}_{n-2}$ is orientable. We fix $\varepsilon > 0$ and consider $z = \bar{z} \times c_{\mathbb{S}^1} \in C_n(M_{n-2}; \alpha_{n-2})$, the cycle obtained from Proposition 3.5. For $n-3 \geq i \geq -1$, we define a cycle $\bar{z}_i \in C_{n-2}(\widetilde{N}_i/\mathcal{F}_{n-2}; \mathbb{Z})$ as the sum of all the simplices in $\partial \bar{z}$ that belong to the subcomplex $\widetilde{N}_i/\mathcal{F}_{n-2} \subset \partial M_{n-2}/\mathcal{F}_{n-2}$. We define

$$z_i = \bar{z}_i \times c_{\mathbb{S}^1} \in C_{n-1}(\widetilde{N}_i; \alpha_{n-2}).$$

In an analogous fashion, for a subset of pairwise distinct indices $i_1, \dots, i_k$ with $n-3 \geq i_j \geq -1$, we inductively define cycles

$$\bar{z}_{i_1,\dots,i_k} \in C_{n-1-k}(\widetilde{N}_{i_1,\dots,i_k}/\mathcal{F}_{n-2}; \mathbb{Z}),$$

as the sum of all the simplices of $\partial \bar{z}_{i_1,\dots,i_{k-1}}$ contained in $\widetilde{N}_{i_1,\dots,i_k}/\mathcal{F}_{n-2}$. For non-pairwise distinct indices we set $\bar{z}_{i_1,\dots,i_k} = 0$. We define

$$z_{i_1,\dots,i_k} = \bar{z}_{i_1,\dots,i_k} \times c_{\mathbb{S}^1} \in C_{n-k}(\widetilde{N}_{i_1,\dots,i_k}; \alpha_{n-2}).$$

When the circle bundle $M_{n-2} \to M_{n-2}/\mathcal{F}_{n-2}$ is not orientable, we use its oriented double cover. Let $W \to B$ be the oriented double cover of $M_{n-2} \to M_{n-2}/\mathcal{F}_{n-2}$, as in Setup 3.2:

$$\begin{tikzcd} W \arrow[r, "\tilde{p}", "2"'] \arrow[d] & M_{n-2} \arrow[d] \\ B \arrow[r, "p", "2"'] & M_{n-2}/\mathcal{F}_{n-2}. \end{tikzcd}$$

We have corresponding preimages

$$\begin{tikzcd} \widetilde{W} \arrow[r] \arrow[d, "q"'] & q_{n-2}^{-1}(\widetilde{N}_j) \subset \widetilde{M}_{n-2} \arrow[d, "q_{n-2}"] \\ \tilde{p}^{-1}(\widetilde{N}_j) \subset W \arrow[r, "\tilde{p}"] \arrow[d] & \widetilde{N}_j \subset M_{n-2} \arrow[d] \\ p^{-1}(\widetilde{N}_j/\mathcal{F}_{n-2}) \subset B \arrow[r, "p"] & \widetilde{N}_j/\mathcal{F}_{n-2} \subset M_{n-2}/\mathcal{F}_{n-2}. \end{tikzcd}$$

We recall that $W \to B$ is a trivial circle bundle. Proposition 3.5 thus gives us a parametrized relative fundamental cycle of the form $u = \bar{u} \times c_{\mathbb{S}^1} \in C_n(W;\beta)$, where $\bar{u}$ is any relative fundamental cycle of B.

For $n-3 \geq i \geq -1$, we define a cycle $\bar{u}_i \in C_{n-2}(p^{-1}(\widetilde{N}_i/\mathcal{F}_{n-2}), \mathbb{Z})$ as the sum of all the simplices in $\partial\bar{u}$ that belong to the subcomplex $p^{-1}(\widetilde{N}_i/\mathcal{F}_{n-2}) \subset \partial B$. We define

$$u_i = \bar{u}_i \times c_{\mathbb{S}^1} \in C_{n-1}(\tilde{p}^{-1}(\widetilde{N}_i);\beta).$$

In an analogous fashion, for a subset of pairwise distinct indices $i_1, \dots, i_k$ with $n-3 \geq i_j \geq -1$, we inductively define cycles

$$\bar{u}_{i_1,\dots,i_k} \in C_{n-1-k}(p^{-1}(\widetilde{N}_{i_1,\dots,i_k}/\mathcal{F}_{n-2}), \mathbb{Z}),$$

as the sum of all the simplices of $\partial\bar{u}_{i_1,\dots,i_{k-1}}$ contained in $p^{-1}(\widetilde{N}_{i_1,\dots,i_k}/\mathcal{F}_{n-2})$. For non-pairwise distinct indices we set $\bar{u}_{i_1,\dots,i_k} = 0$. We define

$$u_{i_1,\dots,i_k} = \bar{u}_{i_1,\dots,i_k} \times c_{\mathbb{S}^1} \in C_{n-k}(\tilde{p}^{-1}(\widetilde{N}_{i_1,\dots,i_k});\beta).$$

Using the map Ψ introduced in Setup 3.2, we write

$$\begin{array}{lcl} z & = & \Psi(u) \in C_n(M_{n-2};\beta_{n-2}), \\ z_{i_1,\dots,i_k} & = & \Psi(u_{i_1,\dots,i_k}) \in C_{n-k}(\widetilde{N}_{i_1,\dots,i_k};\beta_{n-2}). \end{array}$$

With this notation, we have the following three lemmas.

Lemma 3.6 (See [8], Lemma 6.1)**.** *We have*

$$\partial z = \sum_{i=-1}^{n-3} z_i \quad \text{and} \quad \partial z_{i_1,\dots,i_k} = \sum_{i=-1}^{n-3} z_{i_1,\dots,i_k,i}$$

for all $k \in \{1, \dots, n-1\}$ *and pairwise distinct* $i_1, \dots, i_k \in \{-1, \dots, n-3\}$.

Proof. In the orientable case, by definition of z and $z_{i_1,\dots,i_k}$, it is enough to show the analogous statements for $\bar{z}$ and $\bar{z}_{i_1,\dots,i_k}$.

Recall from Remark 2.21 that the boundary $\partial(M_{n-2}/\mathcal{F}_{n-2})$ is a union of subcomplexes $\cup_{i=-1}^{n-3}\widetilde{N}_i^*$ of the simplicial structure on $\partial(M_{n-2}/\mathcal{F}_{n-2})$. It follows that

$$\partial\bar{z} = \partial\bar{z}|_{\partial(M_{n-2}/\mathcal{F}_{n-2})} = \sum_{i=-1}^{n-3}\partial\bar{z}|_{\widetilde{N}_i^*} = \sum_{i=-1}^{n-3}\bar{z}_i.$$

Moreover, we compute that for all $k\in\{1,\dots,n-1\}$ and all pairwise distinct indices $i_1,\dots,i_k\in\{-1,\dots,n-3\}$

$$\begin{aligned}\partial\bar{z}_{i_1,\dots,i_k} &= \partial\bar{z}_{i_1,\dots,i_k}|_{\partial(\widetilde{N}_{i_1,\dots,i_k}/\mathcal{F}_{n-2})}\\ &= \sum_{i\neq i_1,\dots,i_k}\partial\bar{z}_{i_1,\dots,i_k}|_{\widetilde{N}_{i_1,\dots,i_k,i}/\mathcal{F}_{n-2}}\\ &= \sum_{i=-1}^{n-3}\bar{z}_{i_1,\dots,i_k,i}.\end{aligned}$$

In the non-orientable case, we compute:

$$\partial z = \partial\Psi(u) = \Psi(\partial(u)) = \Psi(\partial(\bar{u})\times c_{\mathbb{S}^1}).$$

Then we remark

$$\partial\bar{u} = \partial\bar{u}|_{\partial(p^{-1}(M_{n-2}/\mathcal{F}_{n-2}))} = \sum_{i=-1}^{n-3}\partial\bar{u}|_{p^{-1}(\widetilde{N}_i^*)} = \sum_{i=-1}^{n-3}\bar{u}_i.$$

We insert it in the previous computation and obtain the conclusion.

An analogous reasoning shows also the formula for $\partial z_{i_1,\dots,i_k}$. □

Lemma 3.7 (See [8], Lemma 6.2). *Let $k\in\{1,\dots,n-1\}$ and let $\tau\in\mathrm{Sym}(k)$ be a permutation of $\{1,\dots,k\}$. Then we have*

$$z_{i_1,\dots,i_k} = \mathrm{sign}(\tau)z_{i_{\tau(1)},\dots,i_{\tau(k)}}.$$

Proof. We may assume that τ is a transposition (i_j,i_{j+1}). By definition of $\widetilde{N}_{i_1,\dots,i_k}$ and $z_{i_1,\dots,i_k}$, we may even assume that $\tau=(i_{k-1},i_k)$. Thus we have to show that

$$z_{i_1,\dots,i_{k-1},i_k} = -z_{i_1,\dots,i_{k-2},i_k,i_{k-1}}.$$

By Lemma 3.6, we have

$$0 = \partial\partial z_{i_1,\dots,i_{k-2}} = \partial\left(\sum_{i=-1}^{n-3} z_{i_1,\dots,i_{k-2},i}\right) = \sum_{j=-1}^{n-3}\sum_{i=-1}^{n-3} z_{i_1,\dots,i_{k-2},i,j}.$$

In the orientable case, since $\partial(\widetilde{N}_{i_1,\dots,i_{k-2}}/\mathcal{F}_{n-2})$ is a subcomplex of the simplicial structure on $M_{n-2}/\mathcal{F}_{n-2}$, and from the definition of $z_{i_1,\dots,i_{k-2}}$, it follows that cancellations may occur only between terms with the same set of indices. Hence the only possibility is

$$z_{i_1,\dots,i_{k-1},i_k} = -z_{i_1,\dots,i_{k-2},i_k,i_{k-1}}.$$

In the non-orientable case, the exact same relations hold for the $u_{i_1,\dots,i_k}$. Then apply the chain map Ψ to finish the proof. □

Lemma 3.8 (See [8], Lemma 6.3). *There exist chains*

$$w_{i_1,\dots,i_k} \in C_{n-k+1}(X_{i_1,\dots,i_k}; \xi'_{i_1,\dots,i_k}),$$

$$w_{i_1,\dots,i_k,-1} \in C_{n-k}(X_{i_1,\dots,i_k,-1}; \xi'_{i_1,\dots,i_k,-1}),$$

with $k \in \{1,\dots,n-2\}$ and $i_1, \dots, i_k \in \{0,\dots,n-3\}$, and a constant $C \in \mathbb{R}_+$ depending only on k, such that

(i) the chains $w_{i_1,\dots,i_k}$ and $w_{i_1,\dots,i_k,-1}$ are alternating with respect to permutations of the indices $\{i_1,\dots,i_k\}$;
(ii) the following relations hold:

$$\bullet\ \partial w_{i_1,\dots,i_k} = P_{n-2,i_1}(z_{i_1,\dots,i_k}) - \sum_{i=-1}^{n-3} w_{i_1,\dots,i_k,i},$$

$$\bullet\ \partial w_{i_1,\dots,i_{n-2},-1} = P_{n-2,i_1}(z_{i_1,\dots,i_{n-2},-1});$$

(iii) $|w_{i_1,\dots,i_k}|_1 \le C|z|_1$. The index i_k is allowed to take the value -1.

Proof. Recall that $X_{i_1,\dots,i_k} \cong (X_{i_1,\dots,i_k}/\mathcal{F}_{i_1}) \times \mathbb{S}^1$, and that for both the cases of orientability of the circle bundle $M_{n-2} \to M_{n-2}/\mathcal{F}_{n-2}$ we have a series of essentially free representations of the fundamental groups of the hollowings M_i. To prove this Lemma, we apply Lemmas 3.6, 3.7 above to the proof of [8, Lemma 6.3] for the essentially free representations α_i. □

Now we are ready to prove the main theorem of the present article.

Proof of Theorem B. Given $\varepsilon > 0$, from Proposition 3.5, there exists a (relative) fundamental cycle $z \in C_n(M_{n-2}; \xi_{n-2})$ with ℓ^1-norm less than ε. From Lemma 3.8, we have associated chains $w_i \in C_n(X_i; \xi'_i)$ with ℓ^1-norm less than $C|z|_1$, where C is the constant of Lemma 3.8. We claim that the chain

$$z' = P_{n-2,0}(z) - \sum_{i=0}^{n-3} P_{i,0}(w_i) \in C_n(M; \xi)$$

is a ξ-parametrized relative fundamental cycle of M. From [8, Proposition 3.9], it is sufficient to show that z' is a U-local ξ-parametrised relative fundamental cycle of M, for an arbitrary open subset $U \subset M \setminus p_{n-2,0}(\partial M_{n-2})$ diffeomorphic to an n-disk. Since the boundary of M_{n-2} contains all the preimages of the hollowings, we have $P_{n-2,0}(z) = z'$ on any small ball in $M \setminus p_{n-2,0}(\partial M_{n-2})$. The rest of the proof of the claim follows from the proof of [8, Theorem 1.1].

We recall that for $f \otimes \sigma \in C_n(M_{n-2}; \xi_{n-2})$ we have

$$P_{n-2,0}(f \otimes \sigma) = f \otimes \widetilde{p}_0 \circ \cdots \circ \widetilde{p}_{n-3}(\sigma) \in C_n(M; \xi).$$

Thus we have that

$$|z'|_1 \le |z|_1 + \sum_{i=0}^{n-3} |w_i|_1 < \varepsilon,$$

when we choose $z \in C_n(M_{n-2}; \xi_{n-2})$ such that $|z|_1 < \frac{\varepsilon}{(n-2)C+1}$. □

Remark 3.9. In the case when the circle bundle $M_{n-2} \to M_{n-2}/\mathcal{F}_{n-2}$ is orientable, the parametrized norm vanishes for arbitrary essentially free Γ-spaces. In particular, when Γ is residually finite, its profinite completion is such an essentially free Γ-space. By [12, Theorem 2.6] and [10, Proposition 2.12], this shows that the stable integral simplicial volume of M also vanishes.

If the bundle $M_{n-2} \to M_{n-2}/\mathcal{F}_{n-2}$ is not orientable, our proof shows that starting from any measured Γ-space Z, we find arbitrarily small fundamental cycles parametrized by the space $\Gamma \times_{H_0} Z$ with the action β_0, where $H_0 = \pi_1(p_{n-2,0} \circ \tilde{p})(H)$. This however does not allow us a priori to conclude about the action on the profinite completion of Γ, except if $H_0 = \Gamma$. In this case, the space $\Gamma \times_{H_0} Z$ can be identified with Z; the action β_0 on $\Gamma \times_{H_0} Z$ corresponds under this identification to the original action $\gamma \cdot z = \alpha(\gamma)(z)$ of Γ on Z. Thus, setting Z to be the profinite completion of Γ implies again, by [12, Theorem 2.6] and [10, Proposition 2.12], that the stable integral simplicial volume of M vanishes.

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