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Stability of the Quermassintegral Inequalities in Hyperbolic Space

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Abstract
For the quermassintegral inequalities of horospherically convex hypersurfaces in the $(n+1)$-dimensional hyperbolic space, where $n \geq 2$, we prove a stability estimate relating the Hausdorff distance to a geodesic sphere by the deficit in the quermassintegral inequality. The exponent of the deficit is explicitly given and does not depend on the dimension. The estimate is valid in the class of domains with upper and lower bound on the inradius and an upper bound on a curvature quotient. This is achieved by some new initial value-independent curvature estimates for locally constrained flows of inverse type.

Keywords Quermassintegral inequalities · Hyperbolic space · Curvature flow

Mathematics Subject Classification 53C21 · 52A39

1 Introduction

The isoperimetric inequality is a fundamental result in geometry that relates the volume of a region in the Euclidean, or also in some non-flat spaces, to the surface area of its boundary. In the Euclidean setting, among all bounded domains $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$,

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there holds

$$\left(\frac{|\Omega|}{\omega_{n+1}}\right)^{\frac{n}{n+1}} \leq \frac{|\partial\Omega|}{(n+1)\omega_{n+1}} \tag{1.1}$$

with equality only when Ω is a geodesic ball. Here ω_{n+1} is the volume of the $(n+1)$-dimensional unit ball and $|\cdot|$ stands for the Hausdorff measure of the appropriate dimension. Equality in this inequality is attained if and only if Ω is a ball. Hence it is natural to investigate the stability question, namely how close is Ω to a geodesic ball, provided the deviation in (1.1) from the equality case is small. For the isoperimetric inequality, this question has been addressed to great extent, e.g. [3, 4, 11] and we are not attempting a more detailed overview here.

The quermassintegral inequalities are a generalization of the isoperimetric inequality. They are a collection of geometric inequalities that interrelate the coefficients in the Steiner formula, which is the Taylor expansion of the volume of outer parallel bodies of a convex body $K \subset \mathbb{R}^{n+1}$,

$$\mathrm{vol}(K+\rho B) = \sum_{m=0}^{n+1} \binom{n+1}{m} W_m(K)\rho^m,$$

see [14, p. 208].

In the Euclidean space, the W_m can be expressed as curvature integrals and the corresponding inequalities are written as follows:

$$\left(\int_{\partial\Omega} E_{m-1}\right)^{\frac{n-m}{n+1-m}} \leq C\int_{\partial\Omega} E_m,$$

where $\Omega \subset \mathbb{R}^{n+1}$ is a convex bounded domain and E_m is the (normalized) elementary degree m symmetric polynomial of principal curvatures of $\partial\Omega$ as an embedding in $\mathbb{R}^{n+1}$. The convexity assumption was relaxed to m-convex and starshaped in [7]. In the convex class, the stability for the inequalities has been thoroughly investigated, for example, in [6, 13], while in the non-convex case, the only available result seems to be that of the second author [12]. The purpose of this paper is the transfer of such investigations into the $(n+1)$-dimensional hyperbolic space, where the quermassintegral inequalities were proved by Wang/Xia for horospherically convex domains [15, Thm. 1.1], by using a suitable curvature flow. They proved that if Ω is a bounded smooth h-convex (i.e. all principal curvatures are greater than 1) domain in $\mathbb{H}^{n+1}$, then there holds

$$W_m(\Omega) \geq f_m \circ f_l^{-1}(W_l(\Omega)), \quad 0 \leq l < m \leq n. \tag{1.2}$$

Equality holds if and only if Ω is a geodesic ball. Here W_m is the m^{th} quermassintegral in $\mathbb{H}^{n+1}$ (see section 2 for the definition), $f_m(r) = W_m(B_r)$, and f_l^{-1} is the inverse function of f_l. Hu/Li/Wei gave an alternative proof by using a different flow [10]. We will review their method later, as we are going to use the same flow for our result.

In this paper, we study the stability of these inequalities in the hyperbolic space. In particular, we prove the following result, which controls the Hausdorff distance of an h-convex hypersurface in $\mathbb{H}^{n+1}$ to a geodesic sphere by the deviation of the inequality (1.2) from the equality case:

Theorem 1.1 *Let $n \geq 2$, $\Omega \subset \mathbb{H}^{n+1}$ be an h-convex domain, and $1 \leq m \leq n-1$. Then there exists a constant $C = C\left(n, \rho_-(\Omega), \max_{\partial\Omega} E_m/E_{m-1}\right)$ and a geodesic sphere $S_{\mathbb{H}}$ such that*

$$\operatorname{dist}(\partial\Omega, S_{\mathbb{H}}) \leq C\left(W_{m+1}(\Omega) - f_{m+1} \circ f_m^{-1}\left(W_m(\Omega)\right)\right)^{\frac{1}{m+2}}. \tag{1.3}$$

Here $\rho_-(\Omega)$ is the inradius of the domain Ω. The dependence of C on $\rho_-(\Omega)$ means that we neither control C when $\rho_-(\Omega)$ tends to zero, nor when it tends to infinity.

Remark 1.2 (i) Note that the curvature dependence of C does allow for curvature blowup in a certain sense. Namely, the quantity E_m/E_{m-1} may remain bounded, even if $|A|^2$ becomes unbounded, as can be seen from the example $n-1 = m = 2$, for which

$$\frac{E_2}{E_1} = c_n \frac{\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3}{\kappa_1 + \kappa_2 + \kappa_3}$$

remains bounded, unless merely κ_2 goes to infinity.

(ii) Also note that we do not assume $\partial\Omega$ to be nearly spherical, as it is done, for example, in the recent paper [16], where the authors a priori assume $W^{2,\infty}$ closeness to a sphere and obtain stability of the Fraenkel asymmetry.

In particular, from the previous theorem, we get an estimate in terms of W_2 and W_1 with exponent $1/3$, if we choose $m = 1$ and impose a bound on the mean curvature $H = nE_1$. It turns out that under the same assumption, we can extend this to arbitrary m with the same exponent.

Theorem 1.3 *Let $n \geq 2$, $\Omega \subset \mathbb{H}^{n+1}$ be an h-convex domain, and $1 \leq m \leq n-1$. Then there exists a constant $C = C\left(n, \rho_-(\Omega), \max_{\partial\Omega} H\right)$ and a geodesic sphere $S_{\mathbb{H}}$ such that*

$$\operatorname{dist}(\partial\Omega, S_{\mathbb{H}}) \leq C\left(W_{m+1}(\Omega) - f_{m+1} \circ f_m^{-1}\left(W_m(\Omega)\right)\right)^{\frac{1}{3}}.$$

The idea of the proof combines two major inputs drawn from different directions. The first one, which is also deeply involved in the actual proof of the quermassintegral inequalities (1.2), is the use of a suitable curvature flow to be defined later, which preserves $W_m(\Omega)$ and decreases $W_{m+1}(\Omega)$. The flow exists for all times and converges to a geodesic sphere. This proves the inequality. To characterize the equality case, it is observed that $W_{m+1}(\Omega)$ is only strictly decreasing, when the traceless second fundamental form is not zero. For the proof of (1.2), this was sufficient, but for the proof of (1.3), we will make this quantitative and obtain an estimate on the traceless second fundamental form. The second input is an estimate relating the Hausdorff distance to a geodesic sphere with the traceless second fundamental form. Such an estimate, in the form in which we need it, is due to De-Rosa/Gioffré [2]. The combination of these two ingredients will complete the proof.

After reviewing preliminaries in Sect. 2, we prove new a priori estimates for the locally constrained flow of h-convex hypersurfaces in Sect. 3, which are of independent interest. In Sect. 4, we complete the proof.

2 Preliminaries

To study the curvature flow which is used to prove the quermassintegral inequality and their stability, it is useful to view the pointed hyperbolic space $\mathbb{H}^{n+1}$ as the warped product manifold, coming from polar coordinates around a given origin o,

$$\mathbb{H}^{n+1}\backslash\{o\} = (0, \infty) \times \mathbb{S}^n,$$

equipped with the metric

$$\bar{g} = dr^2 + \lambda^2(r) g_{\mathbb{S}^n},$$

where $\lambda(r) = \sinh(r)$ and $g_{\mathbb{S}^n}$ is the standard round metric on the n-dimensional unit sphere. We will also occasionally write $\langle \cdot, \cdot \rangle$ for $\bar{g}$. In this paper, $\mathrm{d}_{\mathbb{H}^{n+1}}$ will always denote the geodesic distance of two points in hyperbolic space, while

$$\mathrm{dist}(K, L) = \inf\{\delta > 0\colon K \subset B_\delta(L) \ \wedge \ L \subset B_\delta(K)\}$$

denotes the Hausdorff distance of two compact sets.

The vector field $\lambda\partial_r$ on $\mathbb{H}^{n+1}$ is a conformal Killing field, i.e.

$$\overline{\nabla}(\lambda\partial_r) = \lambda' \bar{g},$$

where $\overline{\nabla}$ is the Levi-Civita connection of $\bar{g}$.

Let M be a smooth closed hypersurface in $\mathbb{H}^{n+1}$ with outward unit normal ν, then we define the support function of the hypersurface by

$$u = \langle \lambda(r)\partial_r, \nu \rangle.$$

Writing (g_{ij}) for the metric induced on M with inverse (g^{ij}) and Levi-Civita connection ∇, h_{ij} the second fundamental form and $A = (h^i_j) = (g^{ik}h_{kj})$ the Weingarten operator, we have the following equation, which follows from the conformal Killing property and the Weingarten equation:

$$\nabla_i u = \langle \lambda\partial_r, e_k \rangle h^k_i, \tag{2.1}$$

where $e_1, \dots, e_n$ is a basis of the tangent space of M.

Using the change of variables,

$$r = \log(2+\rho) - \log(2-\rho), \qquad \rho \in (-2, 2),$$

we obtain

$$\bar{g} = e^{2\phi}\left(d\rho^2 + \rho^2 g_{\mathbb{S}^n}\right) \equiv e^{2\phi}\tilde{g}, \tag{2.2}$$

where

$$e^{2\phi} = \frac{16}{(4-\rho^2)^2}.$$

As a result, the hyperbolic space can now be viewed as a conformally flat space. We will need a simple lemma about the surface area of a submanifold of $\mathbb{H}^{n+1}$, when viewed as a Euclidean submanifold.

Lemma 2.1 *Let (M, g) be the embedding of a compact smooth manifold M into $\mathbb{H}^{n+1}$ with*

$$\max_M r \leq \Lambda_0.$$

Then the Euclidean conformal image $\tilde{M}$ in $B_2(0)$ as in (2.2) satisfies

$$\frac{1}{C}|\tilde{M}| \leq |M| \leq C|\tilde{M}|,$$

with $C = C(\Lambda_0)$.

Proof We have with some local parametrization $X : U \to M$,

$$|M| = \int_U \sqrt{\det g_{ij}} = \int_U e^{n\phi}\sqrt{\det \tilde{g}_{ij}} = \int_{\tilde{M}} e^{n\phi}$$

□

The notion of *convexity by horospheres* or short *h-convexity* is crucial for our result:

Definition 2.2 A smooth bounded domain $\Omega \subseteq \mathbb{H}^{n+1}$ is said to be h-convex, if the principal curvatures of the boundary $\partial\Omega$ satisfy $\kappa_i \geq 1$ for all $i = 1, \cdots, n$. Then we also call $\partial\Omega$ h-convex.

Such h-convex domains already enjoy a quite rigid geometry, and several of their geometric quantities are already controlled by the inradius: Let $\rho_-(\Omega)$ be the inradius of Ω, i.e. the largest number, such that a ball of radius equal to that number fits into Ω. Let o be the centre of that ball. In [1, Thm. 1], it is shown that

$$\max_{\partial\Omega} r = \max_{x\in\partial\Omega} \mathrm{d}_{\mathbb{H}^{n+1}}(o, x) \leq \rho_-(\Omega) + \log 2. \tag{2.3}$$

Furthermore, one can extract an estimate on the support function. Due to (2.1), where u attains a minimum, ∇r must be zero, since A is invertible. However, $\min_{\partial\Omega} r = \rho_-(\Omega)$ and hence

$$\min_{\partial\Omega} u = \min_{\partial\Omega} \lambda(r) = \lambda(\rho_-(\Omega)).$$

The h-convexity of a hypersurface of $\mathbb{H}^{n+1}$ translates into convexity of the conformal image:

Lemma 2.3 *Let (M, g) be an h-convex hypersurface of $\mathbb{H}^{n+1}$. Then its conformal Euclidean image $\tilde{M}$ in $B_2(0)$ as in (2.2) is convex.*

Proof We have

$$e^{\phi} h^i_j = \tilde{h}^i_j + d\phi(\tilde{\nu})\delta^i_j,$$

see [5, Equ. (1.1.51)]. There holds

$$\phi = \log e^{\phi} = \log 4 - \log(4 - \rho^2)$$

and hence

$$d\phi = \frac{2\rho}{4-\rho^2} d\rho,$$

which implies

$$\tilde{h}^i_j \geq \frac{4}{4-\rho^2} h^i_j - \frac{2\rho}{4-\rho^2}\delta^i_j \geq \frac{4-2\rho}{4-\rho^2}\delta^i_j = \frac{2}{2+\rho}\delta^i_j.$$

Hence the second fundamental form is positive definite. □

Now we define the hyperbolic quermassintegrals. For any smooth body Ω in the hyperbolic space $\mathbb{H}^{n+1}$ with boundary $M = \partial\Omega$, the k^{th} quermassintegral W_k is defined inductively as follows:

$$W_{k+1}(\Omega) = \frac{1}{n+1}\int_M E_k(\kappa)d\mu - \frac{k}{n+2-k} W_{k-1}(\Omega), \quad k = 1, \ldots, n-1,$$

where

$$W_0(\Omega) = |\Omega|, \quad W_1(\Omega) = \frac{1}{n+1}|M|.$$

Here E_k is the normalized elementary symmetric polynomial in n-variables $\kappa = (\kappa_1, \ldots, \kappa_n)$,

$$E_k(\kappa) = \frac{1}{\binom{n}{k}} \sum_{1\leq i_1 \leq \cdots \leq i_k \leq n} \kappa_{i_1} \cdots \kappa_{i_k}.$$

In this paper, we use the curvature functions

$$F(\kappa_i) = \frac{E_m}{E_{m-1}}, \quad 1 \leq m \leq n-1.$$

For us, only the properties on the positive cone $\Gamma_+ \subset \mathbb{R}^n$ matter, where these functions are monotone, i.e.

$$\frac{\partial F}{\partial \kappa_i} > 0$$

and concave. We may also understand these functions as being defined on the Weingarten operator, or on the second fundamental form and the metric,

$$F = F(\kappa) = F(h^i_j) = F(g, h).$$

Then we write

$$F^{ij} = \frac{\partial F}{\partial h_{ij}}$$

and there holds

$$F^i_j = \frac{\partial F}{\partial h^j_i} = g_{kj} F^{ik}.$$

We refer to [5, Ch. 2] for a thorough treatment.

3 New a Priori Estimates for the Locally Constrained Flow

Wang/Xia [15] proved the quermassintegral inequalities (1.2) in the hyperbolic space by using the following flow: Let $M_0 = \partial\Omega$ be a smooth, h-convex hypersurface in $\mathbb{H}^{n+1}$ with

$$X_0 \colon \mathbb{S}^n \to M_0 \hookrightarrow \mathbb{H}^{n+1}.$$

Then the flow is defined as

$$\begin{aligned} X \colon \mathbb{S}^n \times [0,\infty) \quad &\to \quad \mathbb{H}^{n+1} \\ \frac{\partial}{\partial t} X(\xi, t) \quad &= \quad \left(c(t) - \left(\frac{E_k}{E_l}\right)^{\frac{1}{k-l}} (x,t) \right) \nu(\xi, t) \\ X(\cdot, 0) = X_0, \end{aligned}$$

where ν is the outward normal to the hypersurface, and $c(t)$ is chosen such that the l^{th} quermassintegral is preserved under this flow.

The same inequality (1.2) was proved by Hu/Li/Wei [10] where they used a different flow:

$$\begin{aligned}\frac{\partial}{\partial t}X(\xi,t) &= \left(\frac{\lambda'(r)}{F} - u\right)\nu(\xi,t)\\ X(\cdot,0) &= X_0,\end{aligned} \tag{3.1}$$

with the notation from Sect. 2. This flow preserves the m^{th} quermassintegral $W_m(\Omega_t)$ and decreases $W_{m+1}(\Omega_t)$ monotonically.

We will quantify the proofs from [9] and [10] and employ the flow (3.1) to extract information on the size of the traceless second fundamental form. To exploit this further, we will use the result from De Rosa/Gioffrè's paper [2]. The closeness of the hypersurface to a geodesic sphere can be controlled by the L^p norm of the traceless second fundamental form $\mathring{A}$, whenever $\mathring{A}$ is small. Their result is only for the Euclidean space; however, we note that up to a term coming from the conformal factor, the traceless second fundamental form is conformally invariant, and hence, the umbilicity in the Euclidean and the hyperbolic space is comparable. We will point out the necessary details whenever appropriate. We will also need some refined curvature estimates, which do not depend on their initial values. Therefore, we require some evolution equations and additional a priori estimates, which we develop in the sequel.

It is known that the flow (3.1) has arbitrary spheres as barriers, i.e. for all $(t,\xi) \in [0,\infty) \times \mathbb{S}^n$ there holds due to (2.3),

$$\rho_-(\Omega) = \min_{\partial\Omega} r \le r(\xi,t) \le \max_{\partial\Omega} r \le \rho_-(\Omega) + \log 2. \tag{3.2}$$

Since the flow preserves the h-convexity, we also obtain a uniform C^1-bound via

$$\lambda(\rho_-(\Omega)) \le u(\xi,t) \le \lambda(r(\xi,t)) \le \lambda(\rho_-(\Omega) + \log 2) \le e^{\rho_-(\Omega)}.$$

Let us define the operator

$$\mathcal{L} = \partial_t - \frac{\lambda'}{F^2}F^{ij}\nabla^2_{ij} - \langle \lambda\partial_r, \nabla^{(\cdot)}\rangle.$$

Lemma 3.1 *Along the flow (3.1), the induced metric $g = (g_{ij})$ and second fundamental form (h_{ij}) satisfy the following equations, see [10, Lemma 3.1]*

$$\begin{aligned}\partial_t g_{ij} &= 2\left(\frac{\lambda'(r)}{F} - u\right)h_{ij};\\ \mathcal{L}h_{ij} &= \frac{\lambda'}{F^2}F^{kl,pq}\nabla_i h_{kl}\nabla_j h_{pq} - \left(\frac{\lambda'}{F} + u\right)g_{ij} - 2u(h^2)_{ij}\\ &\quad + \frac{1}{F}\left(\nabla_j F\nabla_i(\frac{\lambda'}{F}) + \nabla_i F\nabla_j(\frac{\lambda'}{F})\right)\\ &\quad + \left(\frac{u}{F} + \lambda' + \frac{\lambda'}{F^2}F^{kl}(h_{rk}h^r_l + g_{kl})\right)h_{ij}.\end{aligned}$$

Lemma 3.2 *The curvature function F satisfies*

$$\mathcal{L}F = (1 - F^{ij}g_{ij})u + \tfrac{\lambda'}{F}(F^2 - F^{ij}(h^2)_{ij}) + \tfrac{2}{F}F^{ij}\nabla_i F\nabla_j\left(\tfrac{\lambda'}{F}\right).$$

Proof We use $F = F(h_{ij}, g_{ij})$, Lemma 3.1 and [5, Equ. (2.1.150)] to compute

$$\begin{aligned}
\mathcal{L}F &= F^{ij}\partial_t h_{ij} + \frac{\partial F}{\partial g_{ij}}\partial_t g_{ij} - \frac{\lambda'}{F^2}F^{ij}\nabla_{ij}F - \langle \lambda\partial_r, \nabla F\rangle \\
&= F^{ij}\partial_t h_{ij} - 2F^{ik}h^j_k h_{ij}\left(\frac{\lambda'}{F} - u\right) - \frac{\lambda'}{F^2}F^{ij}F^{kl}\nabla_{kl}h_{ij} \\
&\quad - \frac{\lambda'}{F^2}F^{ij}F^{kl,rs}\nabla_i h_{kl}\nabla_j h_{rs} - \langle \lambda\partial_r, \nabla F\rangle \\
&= F^{ij}\mathcal{L}h_{ij} - 2F^{ik}h^j_k h_{ij}\left(\frac{\lambda'}{F} - u\right) - \frac{\lambda'}{F^2}F^{ij}F^{kl,rs}\nabla_i h_{kl}\nabla_j h_{rs} \\
&= -F^{ij}\left(\frac{\lambda'}{F} + u\right)g_{ij} + \frac{2}{F}F^{ij}\nabla_i F\nabla_j\left(\frac{\lambda'}{F}\right) \\
&\quad + \left(u + \lambda' F + \frac{\lambda'}{F}F^{kl}(h_{rk}h^r_l + g_{kl})\right) - 2\frac{\lambda'}{F}F^{ik}h^j_k h_{ij} \\
&= (1 - F^{ij}g_{ij})u + \frac{\lambda'}{F}(F^2 - F^{ij}(h^2)_{ij}) + \frac{2}{F}F^{ij}\nabla_i F\nabla_j\left(\frac{\lambda'}{F}\right).
\end{aligned}$$

□

Corollary 3.3 *Along the flow (3.1), the curvature function satisfies the estimate*

$$1 \le F \le \max_{t=0} F.$$

Proof The lower bound follows immediately from the h-convexity and the monotonicity of F. For the upper bound, we use the estimates from [10, Cor. 2.3], which give

$$F^2 \le F^{ij}h_{ik}h^k_j \le (n + 1 - m)F^2, \quad 1 \le F^{ij}g_{ij} \le m.$$

We conclude that at maximal points of F, we have $\mathcal{L}F \le 0$ and the result follows from the maximum principle.

Lemma 3.4 *Along the flow (3.1), the mean curvature $H = g^{ij}h_{ij}$ evolves as follows.*

$$\begin{aligned}
\mathcal{L}H &= \frac{\lambda'}{F^2}F^{kl,pq}\nabla_i h_{kl}\nabla^i h_{pq} - n\left(\frac{\lambda'}{F} + u\right) - \frac{2\lambda'}{F^3}|\nabla F|^2 + \frac{2}{F^2}\nabla_i\lambda'\nabla^i F \\
&\quad + \left(\frac{u}{F} + \lambda' + \frac{\lambda'}{F^2}F^{kl}(h_{rk}h^r_l + g_{kl})\right)H - 2\frac{\lambda'}{F}|A|^2.
\end{aligned}$$

Proof Using the evolution of g_{ij}, we can easily find the evolution of g^{ij},

$$\frac{\partial}{\partial t} g^{ij} = -2 g^{jk} g^{il} \left(\frac{\lambda'}{F} - u \right) h_{kl}.$$

Hence

$$\begin{aligned}
\mathcal{L}H &= g^{ij} \mathcal{L} h_{ij} - 2 \left(\frac{\lambda'}{F} - u \right) |A|^2 \\
&= \frac{\lambda'}{F^2} F^{kl,pq} \nabla_i h_{kl} \nabla^i h_{pq} - n \left(\frac{\lambda'}{F} + u \right) \\
&\quad + \frac{1}{F} \left(\nabla^i F \nabla_i (\frac{\lambda'}{F}) + \nabla_i F \nabla^i (\frac{\lambda'}{F}) \right) \\
&\quad + \left(\frac{u}{F} + \lambda' + \frac{\lambda'}{F^2} F^{kl} (h_{rk} h^r_l + g_{kl}) \right) H - 2 \frac{\lambda'}{F} |A|^2 \\
&= \frac{\lambda'}{F^2} F^{kl,pq} \nabla_i h_{kl} \nabla^i h_{pq} - n \left(\frac{\lambda'}{F} + u \right) - \frac{2\lambda'}{F^3} |\nabla F|^2 + \frac{2}{F^2} \nabla_i \lambda' \nabla^i F \\
&\quad + \left(\frac{u}{F} + \lambda' + \frac{\lambda'}{F^2} F^{kl} (h_{rk} h^r_l + g_{kl}) \right) H - 2 \frac{\lambda'}{F} |A|^2.
\end{aligned}$$

□

Corollary 3.5 *Along the flow (3.1) and up to time $t = 1$, the curvature function satisfies the estimate*

$$n \leq H \leq \frac{C(n, \rho_-(\Omega), \max_{M_0} F)}{t}.$$

Proof We proceed similarly to the proof of Corollary 3.3. At maximal points of H, we have, using $|A|^2 \geq H^2/n$ and the concavity of F,

$$\begin{aligned}
\mathcal{L}H &\leq -n \left(\frac{\lambda'}{F} + u \right) - \frac{2\lambda'}{F^3} |\nabla F|^2 + \frac{2}{F^2} \nabla_i \lambda' \nabla^i F \\
&\quad + \left(\frac{u}{F} + \lambda' + \frac{\lambda'}{F^2} F^{kl} (h_{rk} h^r_l + g_{kl}) \right) H - \frac{2}{n} \frac{\lambda'}{F} H^2 \\
&\leq C - \frac{1}{nF} H^2 \\
&\leq C - \frac{1}{C} H^2,
\end{aligned}$$

where in the last step, we used Corollary 3.3. We have also used Cauchy-Schwarz to absorb ∇F and first-order terms in H. The result again follows from a simple ODE comparison argument. □

4 Proof of Theorems 1.1 and 1.3

In this section, we prove Theorem 1.1. In the following proof, we take $C = C(n, \rho_-(\Omega), \max_{\partial\Omega} F)$ to be a generic constant depending on the quantities mentioned.

Proof Let $\epsilon > 0$ be such that

$$W_{m+1}(\Omega) = f_{m+1} \circ f_m^{-1}(W_m(\Omega)) + \epsilon.$$

Let $\rho_-(\Omega)$ be the inradius of Ω and pick the origin o as the centre of the corresponding inball. Under the flow (3.1) with initial surface $\partial\Omega$, $W_{m+1}(\Omega_t)$ evolves as (see [15, Prop. 3.1] for details)

$$\frac{\partial}{\partial t} W_{m+1}(\Omega_t) = \frac{n-m}{n+1} \int_{M_t} \left(\lambda'(r) \frac{E_{m-1}}{E_m} - u \right) E_{m+1},$$

where $M_t = \partial\Omega_t$. We compute

$$\begin{aligned}
\int_0^\infty \int_{M_t} \lambda' \left(\frac{E_{m+1}E_{m-1}}{E_m} - E_m \right) &= \int_0^\infty \int_{M_t} \left(\frac{\lambda' E_{m-1}}{E_m} - u \right) E_{m+1} \\
&= \frac{n+1}{n-m} \int_0^\infty \frac{\partial}{\partial t} W_{m+1}(\Omega_t)\, dt \\
&= \frac{n+1}{n-m} (W_{m+1}(B) - W_{m+1}(\Omega)) \\
&= -\frac{n+1}{n-m} \epsilon.
\end{aligned} \tag{4.1}$$

In the first line of this calculation, we have used the Minkowski formula proved, for example, in Guan/Li [8]

$$\int_{M_t} \lambda'(r) E_m = \int_{M_t} u E_{m+1}.$$

We have also used that Ω converges to a round ball at infinite time, $\Omega_\infty = B$ where (1.2) holds with equality, and W_m is preserved under the flow, $W_m(B) = W_m(\Omega)$. Along the flow, we have

$$-\lambda(r) \leq \frac{\lambda'(r)}{F} - \frac{\lambda(r)}{v} \leq \lambda'(r)$$

and hence, using $\lambda \leq \lambda'$,

$$\left| \frac{\lambda'(r)}{F} - u \right| \leq \lambda'(\max_{\partial\Omega} r) \leq \cosh(\rho_-(\Omega) + \log 2) \leq 2\cosh(\rho_-(\Omega)), \tag{4.2}$$

where we used (3.2) and [1, Thm. 1].

Using the above bound, we want to estimate the Hausdorff distance between M_t and $M_0 = \partial\Omega$. Let $X(\xi, 0)$ and $X(\xi, t)$ be two points in M_0 and M_t, respectively. Let $\gamma : [0, t] \to \mathbb{H}^{n+1}$ be a curve defined as

$$\gamma(\tau) = X(\xi, \tau).$$

Then we have due to (4.2),

$$\mathrm{d}_{\mathbb{H}^{n+1}}(X(\xi, 0), X(\xi, t)) \leq \max_{[0,t]} |\partial_\tau \gamma| t \leq 2\cosh(\rho_-(\Omega))t.$$

From this, we get

$$\mathrm{dist}(M_t, \partial\Omega) \leq Ct, \quad \forall t \geq 0.$$

From (4.1) and $\lambda' \geq 1$, we get

$$\int_0^\infty \int_{M_t} \left(E_m - \frac{E_{m+1}E_{m-1}}{E_m} \right) \leq \frac{n+1}{n-m}\epsilon.$$

Then using [12, Lemma 4.2] and Corollary 3.5, we get for $\delta > 0$,

$$\int_\delta^{2\delta} \int_{M_t} |\mathring{A}|^2 \leq C \max_{[\delta,2\delta]} E_{m-1} \int_\delta^{2\delta} \int_{M_t} \frac{E^2_{m+1,n1}|\mathring{A}|^2}{E_m} \leq \frac{C}{\delta^{m-1}}\epsilon, \tag{4.3}$$

where we also used $E^2_{m+1,ij} = \frac{\partial^2 E_{m+1}}{\partial\kappa_i \kappa_j} \geq 1$. Hence there exists $t_\delta \in [\delta, 2\delta]$, such that

$$\|\mathring{A}\|_{L^2(M_{t_\delta})} \leq C\delta^{-\frac{m}{2}}\sqrt{\epsilon}.$$

Now put

$$\delta = \epsilon^{\frac{1}{m+2}}$$

to obtain

$$\mathrm{dist}(M_{t_\delta}, \partial\Omega) + \|\mathring{A}\|_{L^2(M_{t_\delta})} \leq C\epsilon^{\frac{1}{m+2}}. \tag{4.4}$$

In order to apply [2, Thm. 1.2], we view M_{t_δ} as a Riemannian submanifold of the Euclidean ball of radius 2, which is conformal to $\mathbb{H}^{n+1}$ as in (2.2). Due to Lemma 2.3 and furnishing the Euclidean geometric tensors by a tilde, we see that $\tilde{M}_{t_\delta}$ is convex. Now we have to normalize $\tilde{M}_{t_\delta}$,

$$\hat{M}_{t_\delta} = \left(\frac{|\mathbb{S}^n|}{|\tilde{M}_{t_\delta}|} \right)^{\frac{1}{n}} \tilde{M}_{t_\delta} \equiv \gamma \tilde{M}_{t_\delta}.$$

Note that $|M_{t_\delta}|$ is controlled from above and below in terms of $\rho_-(\Omega)$, due to the convergence of the surface area-preserving curvature flow

$$\frac{\partial}{\partial t}X = \left(\frac{\lambda'}{E_1} - u\right)\nu,$$

which converges to a geodesic sphere with radius between $\rho_-(\Omega)$ and $\rho_-(\Omega) + \log 2$. Due to Lemma 2.1, we have $\gamma = \gamma(n, \rho_-(\Omega))$. [2, Thm. 1.2] gives, provided that $\epsilon \leq \epsilon_0(n, \rho_-(\Omega), \max_{\partial\Omega} F)$ with ϵ_0 sufficiently small, a parametrization

$$\psi : \mathbb{S}^n \to \hat{M}_{t_\delta} \subset B_2(0) \subset \mathbb{R}^{n+1}$$

and a point $\mathcal{O} \in \mathbb{R}^{n+1}$, such that ψ satisfies the estimate

$$\|\psi - \mathrm{id} - \mathcal{O}\|_{W^{2,2}(\mathbb{S}^n)} \leq C\|\mathring{\hat{A}}\|_{L^2(\hat{M}_{t_\delta})} \leq C\|\mathring{A}\|_{L^2(M_{t_\delta})} \leq C\epsilon^{\frac{1}{m+2}}.$$

This implies that $\hat{M}_{t_\delta}$ is Hausdorff-close to the Euclidean unit sphere, that $\tilde{M}_{t_\delta}$ is close to a Euclidean sphere of radius γ^{-1} and that in turn M_{t_δ} is close to a hyperbolic sphere, with exactly the same error estimate,

$$\mathrm{dist}(M_{t_\delta}, S_{\mathbb{H}}) \leq C\epsilon^{\frac{1}{m+2}}.$$

Employing (4.4) finishes the proof for $\epsilon \leq \epsilon_0$. However, if $\epsilon > \epsilon_0$, the estimate is trivial due to

$$\max_{\partial\Omega} r \leq \rho_-(\Omega) + \log 2.$$

To prove Theorem 1.3, we reconvene at (4.3) and do not estimate $\max E_m$ using Corollary 3.5, but the constant itself is now allowed to depend on H. Hence the factor δ^{-m+1} is simply not present and in the subsequent computations, we can pretend m would be one. The proof can then literally be completed as above. □

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