A VARIATIONAL APPROACH TO STRAIN-LIMITING VISCOELASTICITY IN ONE SPACE DIMENSION

L. BACHMANN, A. SCHLÖMERKEMPER, AND Y. ŞENGÜL

Abstract. In this work we investigate existence of solutions for a strain-rate type model of viscoelastic material response in the context of strain-limiting theory. We consider a strain-rate type constitutive relation and the equation of motion to derive a partial differential equation where unlike classical equations in nonlinear elasticity the unknown is the stress rather than the deformation. Here, we introduce a variational framework where we prove existence of solutions to this equation for a special case of the nonlinearity by considering the Euler-Lagrange equations of a functional. Finally, we apply the method of time-discretization in order to solve the problem.

1. Introduction

Many materials we come across in our daily lives show viscoelastic response, such as aluminium, polymers and human tissue. One particularly interesting viscoelastic response is the strain-limiting response, which has been experimentally observed in a wide class of materials such as titanium alloys and biological fibres such as collagen (see [15] and references therein). This kind of material response has been modelled successfully using the recently developed implicit constitutive theory. Among other advantages, it leads to a different small strain theory allowing for a nonlinear relationship between the linearized strain and the stress.

The main aim of this work is to investigate such a strain-limiting model for the response of strain-rate type viscoelastic materials. More precisely, we are interested in studying a model where the sum of the linearized strain and the rate of the linearized strain is given as a function of the stress. However, different from the earlier work on strain-limiting response of materials, we adopt a variational approach to get existence of solutions. This method is novel in two ways. Firstly, we define a suitable minimization problem where the unknown is the stress. Moreover, in order to be able to use classical results in the calculus of variations, we use time-discretization so that a static minimization problem is considered at each time step. Secondly, for compactness at each time step, we obtain a priori bounds using the energy which is expressed in terms of the stress, rather than in terms of the deformation or displacement.

Key words and phrases. viscoelasticity, implicit constitutive theory, variational methods, time-discretization.

Typically, the main difficulty for studying physically meaningful mathematical models describing behaviour of materials is the lack of fundamental mathematical theory underlying the models. Such models often result in a variational problem or a system of nonlinear partial differential equations. Hence, an important mathematical task in describing material response is to develop new analytic tools to find solutions of such systems. In this paper we combine variational methods with continuum mechanical tools leading to a new approach to study such models. We hope the framework introduced in this manuscript will be generalized to higher space dimensions as well as adopted for a larger class of problems.

In Cauchy elasticity, the relation between the stress and the strain is given by explicit relations where the stress is described as a function of the strain. Even though these kinds of explicit constitutive relations are quite successful in describing the response of a wide variety of solids, they are not able to capture many important observed features such as the nonlinear relationship between the stress and the strain which can hold even when the strains are very small. As a result, Rajagopal [10, 11, 12, 13] introduced a more general framework to describe material response, namely by means of an implicit relation between the stress and the strain resulting in a nonlinear relationship after linearization. An important advantage of this framework is that the strain can be considered as a function of the stress so that as a result of linearization one obtains a non-linear relationship between the stress and the strain. This means they allow for the gradient of the displacement to stay small so that one could treat the linearized strain, even for arbitrary large values of the stress. Such models have recently attracted a considerable amount of attention due to the fact that various phenomena, including cracks, are successfully described by them, as well as the fact that in the classical linear elasticity theory such nonlinear response cannot be explained, see [2, 14] and references therein for related work.

In this work, we are interested in the analysis of a nonlinear differential equation resulting from a viscoelastic constitutive relation specifying the relation between the linearized strain, the rate of the linearized strain and the stress. In Section 2 we recall the nonlinear partial differential equation to study in the rest of the paper by using the viscoelastic constitutive relation and the equation of motion. Afterwards we summarize the related earlier work. The model is stated in a mathematical setting in Section 3. In Section 4 we introduce a variational framework to solve the initial-boundary-value problem corresponding to the nonlinear partial differential equation we derived. Inspired by work of Friesecke and Dolzmann [9] on time discretization for a quasi-linear evolution equation, we propose an equivalent minimization problem and adopt time-discretization which allows for solving a static problem at each time step. Using compactness arguments for passing to the limit as the time step goes to zero, we prove existence of solutions for the initial and boundary-value problem for a linear form of the constitutive equation.

2. The model and related past work

In this work we want to investigate an evolutionary equation in terms of the stress $T = T(x,t)$. The derivation of this equation follows the approach described in [7]. Let $\Omega \subset \mathbb{R}$ be a bounded open domain. The equation of motion for a homogeneous, viscoelastic medium in one space dimension is given by

$$\rho u_{tt} = T_x \quad \text{on } \Omega \times [0,\infty) \tag{2.1}$$

with suitable boundary conditions. Here, $\rho \in \mathbb{R}$ denotes the mass density, $u : \Omega \times [0,\infty) \to \mathbb{R}$ the deformation/flow map and $T : \Omega \times [0,\infty) \to \mathbb{R}$ the (one-dimensional) stress tensor. Differentiating (2.1) with respect to x under the assumption of suitable regularity, and using the definition of the (one-dimensional) linearized strain ϵ, one obtains

$$\rho \epsilon_{tt} = T_{xx} \quad \text{on } \Omega \times [0,\infty). \tag{2.2}$$

The constitutive equation for a viscoealstic material showing strain-limiting behaviour in the strain-rate setting is

$$\epsilon + \nu \epsilon_t = h(T) \quad \text{on } \Omega \times [0,\infty) \tag{2.3}$$

where $\nu > 0$ is a constant, and $h : \mathbb{R} \to \mathbb{R}$ with $h \in C^2(\mathbb{R},\mathbb{R})$ being a (possibly) nonlinear function (see [15]). Here we use the abbreviated notation $h(T)$ for $h(T(\cdot,\cdot))$. Differentiating (2.3) twice with respect to time and substituting (2.2) into the result thereof yields

$$h(T)_{tt} - \frac{\nu}{\rho} T_{xxt} = \frac{1}{\rho} T_{xx} \quad \text{on } \Omega \times [0,\infty). \tag{2.4}$$

We note here that the time derivatives of $h(T)$ are total derivatives; for ϵ and T there is no difference between partial and total derivatives since x is not a function of t. For simplicity we will use the dimensionless quantities introduced in [7] and work with

$$h(T)_{tt} - \nu T_{xxt} = T_{xx} \quad \text{on } \Omega \times [0,\infty). \tag{2.5}$$

In [7] Erbay and Şengül studied travelling wave solutions corresponding to (2.5) for different nonlinear functions h. Then, Erbay, Erkip and Şengül [6] proved local existence of strong solutions to the initial-value problem corresponding to (2.5) when h is strictly increasing. Thereafter it was proved by Şengül in [17] that the local-in time solutions obtained in [6] are actually global solutions. More recently, in [16] Şengül proved that travelling wave solutions can be found analytically and numerically when the nonlinearity term in (2.5) was of the form of an arctangent function.

In the stress-rate type setting using the constitutive equation

$$\epsilon = h(T) - \gamma(T) T_t, \tag{2.6}$$

one can obtain a similar partial differential equation given by

$$h(T)_{tt} - (\gamma(T) T_t)_{tt} = T_{xx}, \tag{2.7}$$

where γ is a (possibly nonlinear) function of the stress, and h is a (possibly) nonlinear function satisfying $h(0) = 0$ (see [8] for a detailed derivation). In [8] Erbay and Şengül considered a special case in one space dimension when $\gamma(T)$ is a constant, and showed that $\gamma > 0$ must necessarily hold in order for (2.6) to be consistent with the first and second law of thermodynamics. Clearly, if $\gamma = 0$, the model reduces to the elastic case. For the stress-rate type model (2.6), another work that exists in the literature is by Duman and Şengül [5], where travelling wave solutions are investigated. The work by Bachmann, De Anna, Schlömerkemper and Şengül [1] is also on stress-rate type viscoelastic models, more precisely on (2.6) with constant γ, where the existence of local-in-time smooth solutions to the linearized model is proved in certain Gevrey classes.

3. The problem

Let $\Omega \subset \mathbb{R}$ be an open, connected and bounded set. For the function $T\colon \Omega\times[0,\infty) \to \mathbb{R}$ we consider the following initial-boundary-value problem

$$T_{xx} + \nu T_{xxt} = h(T)_{tt} \quad \text{in } \Omega \times [0,\infty), \tag{3.1}$$

$$T = \alpha \quad \text{on } \partial\Omega \times [0,\infty), \tag{3.2}$$

$$T_x = \beta \quad \text{on } \partial\Omega \times [0,\infty), \tag{3.3}$$

$$T = T_0 \quad \text{in } \Omega \times \{0\}, \tag{3.4}$$

$$T_t = S_0 \quad \text{in } \Omega \times \{0\}, \tag{3.5}$$

where $\alpha,\ \beta : \Omega \times [0,\infty) \to \mathbb{R}$ and $T_0,\ S_0 : \Omega \times \{0\} \to \mathbb{R}$ are given functions. We assume that

(A0) $\alpha_t\beta = \alpha_t\beta_t = 0$ on $\partial\Omega \times [0,\infty)$.

It is worth noting that, if α is a fixed constant in x and the relation $\beta = \alpha_x$ holds, then assumption (A0) is automatically satisfied. We also make the following assumptions on the function h:

(A1) The primitive function H defined by $h(z) = H'(z)$, $z \in \mathbb{R}$, satisfies $H \in C^3(\mathbb{R},\mathbb{R})$. Moreover, there exists a constant $c_1 > 0$ such that

$$0 \le H(z) \le c_1 \left(|z|^2 + 1\right) \qquad \text{for every } z \in \mathbb{R}.$$

(A2) There exists a constant $c_2 > 0$ and a $0 < q < 1$ such that

$$|h(z)| \le c_2 \left(|z|^q + 1\right) \qquad \text{for every } z \in \mathbb{R}.$$

(A3) (i) There is a constant $c_3 > 0$ such that for all $z \neq 0$, we have

$$\left| h'(z) - \frac{h(z)}{z} \right| \le c_3.$$

(ii) There exists a constant $c_4 > 0$ such that $|h'(z)| \le c_4$ for all $z \in \mathbb{R}$.

These assumptions are for instance satisfied for $h(z) = \arctan z$. The main result of this manuscript is on existence of weak solutions of system (3.1)–(3.5), cf. Section 6. The estimates that are necessary for ensuring existence of a minimizer at each time step are obtained for general functions $h(\cdot)$ satisfying (A1), (A2) and (A3). Moreover, we prove the existence result for linear $h(z) = \eta z$ with a constant $\eta > 0$. (We remark that assumption (A2) is not satisfied for the linear case, thus it requires its own proof.) The compactness results and the proof of the main result are then given for this linear case.

4. Variational framework

4.1. **Energy-dissipation law.** To derive the energy-dissipation law of the initial-boundary-value problem (3.1)–(3.5), we assume that (3.1) has a smooth solution T. We multiply (3.1) by T_t and integrate over Ω, which yields

$$\int_\Omega T_{xx} T_t \,\mathrm{d}x + \nu \int_\Omega T_{xxt} T_t \,\mathrm{d}x = \int_\Omega h(T)_{tt} T_t \,\mathrm{d}x \qquad \text{for any } t \in [0, \infty].$$

By assumption (A0), $T_x T_t = \beta \alpha_t = 0$ as well as $T_{xt} T_t = \beta_t \alpha_t = 0$ at $\partial\Omega \times [0, \infty)$. Hence, when we integrate by parts in the first and second terms, the contribution from the boundary vanishes. Moreover, we apply the product rule and Schwarz' lemma to obtain

$$-\int_\Omega \left(\tfrac{1}{2} T_x^2\right)_t \mathrm{d}x - \nu \int_\Omega T_{xt}^2 \,\mathrm{d}x = \int_\Omega h(T)_{tt} T_t \,\mathrm{d}x. \tag{4.1}$$

We integrate this equation in time over the time interval $(0, t)$ for an arbitrary $t > 0$ and obtain

$$\begin{aligned} &-\int_\Omega \tfrac{1}{2} T_x^2(x,t) \,\mathrm{d}x - \nu \int_0^t \int_\Omega T_{x\tau}^2(x,\tau) \,\mathrm{d}x \,\mathrm{d}\tau \\ &\quad = \int_0^t \int_\Omega h(T)_{\tau\tau} T_\tau \,\mathrm{d}x \,\mathrm{d}\tau - \int_\Omega \tfrac{1}{2} T_x^2(x,0) \,\mathrm{d}x \\ &\quad = \int_0^t \int_\Omega h(T)_{\tau\tau} T_\tau \,\mathrm{d}x \,\mathrm{d}\tau - \int_\Omega \tfrac{1}{2} (T_{0\,x})^2(x) \,\mathrm{d}x. \end{aligned} \tag{4.2}$$

Next we assume that there exists a function $f(v, w)$ with $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $f \in C^2$ and

$$\Big(h(T(x,t))\Big)_{tt} = \frac{\partial f}{\partial v}\big((T(x,t), T_t(x,t)\big) - \left(\frac{\partial f}{\partial w}\big(T(x,t), T_t(x,t)\big)\right)_t \tag{4.3}$$

for every $(x,t) \in \Omega \times [0, \infty)$, see Remark 4.1 for examples. By the product rule and since T is assumed smooth here, we then obtain that

$$\begin{aligned} h(T)_{tt} T_t &= \frac{\partial f}{\partial v} T_t - \left(\frac{\partial f}{\partial w}\right)_t T_t = \frac{\partial f}{\partial v} T_t + \frac{\partial f}{\partial w} T_{tt} - \left(\frac{\partial f}{\partial w} T_t\right)_t \\ &= (f(T, T_t))_t - \left(\frac{\partial f}{\partial w} T_t\right)_t. \end{aligned}$$

We plug this into (4.2) to get

$$
\begin{aligned}
&-\int_\Omega \tfrac{1}{2}T_x^2(x,t)\,\mathrm{d}x - \nu\int_0^t\int_\Omega T_{x\tau}^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad = \int_0^t\int_\Omega \left[(f(T,T_\tau))_\tau - \left(\frac{\partial f}{\partial w}T_\tau\right)_\tau\right]\mathrm{d}x\,\mathrm{d}\tau - \int_\Omega \tfrac{1}{2}(T_{0x})^2(x)\,\mathrm{d}x.
\end{aligned}
\tag{4.4}
$$

For any $t \geq 0$ we define the energy functional by

$$
E(T(t),T_t(t)) := \int_\Omega \left[\tfrac{1}{2}T_x^2(x,t) + f(T(x,t),T_t(x,t)) - \left(\frac{\partial f}{\partial w}T_t\right)(x,t)\right]\mathrm{d}x,
\tag{4.5}
$$

where we write $T(t)$ as shorthand for the function $T(\cdot,t)$ depending on the spatial variable x. Similarly, we write $T_t(t)$ and $T_{xt}(t)$ as shorthand notations for corresponding functions. The dissipation functional is defined by

$$
D(T_{xt}(t)) = -\nu\int_0^t\int_\Omega T_{x\tau}^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau.
$$

Then we obtain the energy-dissipation law

$$
E(T(t),T_t(t)) - E(T(0),T_t(0)) = D(T_{xt}(t)),
\tag{4.6}
$$

for any $t \geq 0$. Note that

$$
E(T(0),T_t(0)) = \int_\Omega \left[\tfrac{1}{2}(T_{0x})^2(x) + f(T_0(x),S_0(x)) - \frac{\partial f}{\partial w}(x,0)S_0(x)\right]\mathrm{d}x.
$$

Hence, by $D(T_{xt}(t)) \leq 0$ and by the assumption on the finiteness of the initial energy, we obtain that $E(T(\cdot),T_t(\cdot))$ is bounded in $L^\infty(0,\infty)$.

Remark 4.1. In the definition (4.3) for f, the linear case $h(T) = \eta T$, $\eta > 0$, holds for instance for $f(T,T_t) = -\frac{\eta}{2}T_t^2$. Another example for the linear case is $f(T,T_t) = \frac{\eta}{3}T^3T_t - \frac{\eta}{2}T_t^2$, which shows that f is not necessarily unique.

4.2. **Time-discretization.** In this section we introduce a time-discretized system, which turns our dynamical problem into a static minimization problem at each time step. This approach is inspired by work of Friesecke and Dolzmann [9].

Let $k > 0$ be a fixed time-step size. We discretize the time interval $[0,\infty)$ and consider the slightly bigger time interval $\cup_{j=0}^{\infty}((j-1)k, jk]$. We define the set

$$
X(\Omega,\mathbb{R}) := \left\{T \in W^{1,2}(\Omega,\mathbb{R})\colon T-\alpha \in W_0^{1,2}(\Omega,\mathbb{R})\right\}
$$

for a fixed boundary value $\alpha \in W^{1,2}(\Omega,\mathbb{R})$. For initial data $T_0^k \in X(\Omega,\mathbb{R}),\ S_0^k \in L^2(\Omega,\mathbb{R})$ we define the following time-discretization scheme inductively in j

$$\begin{aligned} T^{k,-1} &:= T_0^k - kS_0^k, \\ T^{k,0} &:= T_0^k, \\ T^{k,j} &:= \operatorname*{arg\,min}_{T\in X(\Omega,\mathbb{R})} J^{k,j}(T), \qquad j \in \mathbb{N}, \end{aligned} \tag{4.7}$$

where the functional $J^{k,j} : X(\Omega,\mathbb{R}) \to \mathbb{R}$, $j \in \mathbb{N}$, is defined as

$$\begin{aligned} J^{k,j}(T) := \int_\Omega \frac{1}{2}\left(T_x\right)^2 &+ \frac{\nu}{2k}\left(T_x - (T^{k,j-1})_x\right)^2 \\ &+ \frac{1}{k^2}\left[h'(T)\left(T - T^{k,j-1}\right)^2 + h(T)\left(T - 2T^{k,j-1} + T^{k,j-2}\right)\right. \\ &\left. -2h(T)\left(T - T^{k,j-1}\right) + H(T)\right] \mathrm{d}x. \end{aligned} \tag{4.8}$$

The main contribution of the current work is the introduction of the variational framework for well-posedness of equation (3.1); this heavily depends on the form of the functional defined in (4.8). Certainly, the choice in (4.8) is not unique. However, for this functional we show that the related minimization problem is well-posed at each time step and the corresponding Euler-Lagrange equation (4.13) is also the desired one.

Proposition 4.2. *Let $k > 0$ be the time-step size, let $T^{k,j-1}, T^{k,j-2} \in X(\Omega,\mathbb{R})$, $j \geq 2$, be the minimizers from the time steps $j-1$ and $j-2$. Assume that h satisfies assumptions (A1)–(A3) or that $h(T) = \eta T$ with a constant $\eta > 0$, then $J^{k,j}$ attains a minimum in $X(\Omega,\mathbb{R})$.*

Proof. Let $s\colon \Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be defined as the integrand of the functional $J^{k,j}$ in (4.8), i.e.

$$J^{k,j}(T) = \int_\Omega s\left(x, T(x), T_x(x)\right) \mathrm{d}x.$$

The function $s(x, z_1, \cdot)$ is obviously convex for every $(x, z_1) \in \Omega\times\mathbb{R}$ because of its quadratic structure in the last variable. To show that s is coercive we introduce the following notation for simplicity

$$\begin{aligned} \beta_1 &:= (T^{k,j-1})_x (= \partial_x \beta_2), \\ \beta_2 &:= T^{k,j-1}, \\ \beta_3 &:= 2T^{k,j-1} - T^{k,j-2}. \end{aligned}$$

Let h satisfy assumptions (A1)–(A3). By assumption (A1) we obtain

$$\begin{aligned}
s(x,z_1,z_2) =& \frac{1}{2}|z_2|^2 + \frac{\nu}{2k}(z_2-\beta_1)^2 + \frac{1}{k^2}\Big(h'(z_1)(z_1-\beta_2)^2 \\
&+ h(z_1)(z_1-\beta_3) - 2h(z_1)(z_1-\beta_2) + H(z_1)\Big) \\
=& \frac{1}{2}|z_2|^2 + \frac{\nu}{2k}(z_2-\beta_1)^2 + \frac{1}{k^2}\left(\left(h'(z_1) - \frac{h(z_1)}{z_1}\right) z_1^2\right. \\
&\left. - \left(h'(z_1) - \frac{h(z_1)}{z_1}\right) 2\beta_2 z_1 + h'(z_1)\beta_2^2 - h(z_1)\beta_3 + H(z_1)\right) \qquad (4.9)\\
\geq& \frac{1}{2}|z_2|^2 + \frac{1}{k^2}\left(h'(z_1) - \frac{h(z_1)}{z_1}\right) z_1^2 \\
&- \frac{1}{k^2}\left|h'(z_1) - \frac{h(z_1)}{z_1}\right| 2|\beta_2||z_1| - |h'(z_1)|\frac{\beta_2^2}{k^2} - |h(z_1)|\frac{|\beta_3|}{k^2}.
\end{aligned}$$

By Young's inequality, $|\beta_2||z_1| \leq \frac{1}{2}(|\beta_2|^2 + |z_1|^2)$. Importantly, the terms involving $|z_1|^2$ cancel. We thus obtain

$$s(x,z_1,z_2) \geq \frac{1}{2}|z_2|^2 - \frac{1}{k^2}\left|h'(z_1) - \frac{h(z_1)}{z_1}\right| |\beta_2|^2 - |h'(z_1)|\frac{\beta_2^2}{k^2} - |h(z_1)|\frac{|\beta_3|}{k^2}.$$

Then, by assumptions (A3)(i) and (A3)(ii) and another application of Young's inequality, we get

$$\begin{aligned}
s(x,z_1,z_2) \geq& \frac{1}{2}|z_2|^2 - \frac{c_3}{k^2}|\beta_2|^2 - \frac{c_4\beta_2^2}{k^2} - \frac{|h(z_1)||\beta_3|}{k^2} \\
\geq& \frac{1}{2}|z_2|^2 - \frac{c_3}{k^2}|\beta_2|^2 - \frac{c_4\beta_2^2}{k^2} - \frac{|h(z_1)|^2}{2k^2} - \frac{|\beta_3|^2}{2k^2}.
\end{aligned}$$

Finally, by (A2) we obtain

$$s(x,z_1,z_2) \geq \frac{1}{2}|z_2|^2 + \alpha_1|z_1|^q + \alpha_2(x)$$

for some $\alpha_1 \in \mathbb{R}$ and $\alpha_2 \in L^1(\Omega)$, which implies that the functional $J^{k,j}$ is weakly lower semicontinuous on $W^{1,2}(\Omega,\mathbb{R})$ and allows an application of a standard existence theorem in the direct method of the calculus of variation, see [4, Theorem 3.30].

Next we prove the lower bound in the case of $h(T) = \eta T$ for some $\eta > 0$. Starting with (4.9), we have

$$\begin{aligned}
s(x,z_1,z_2) =& \frac{1}{2}|z_2|^2 + \frac{\nu}{2k}(z_2-\beta_1)^2 + \frac{\eta}{k^2}\left(\beta_2^2 - z_1\beta_3 + \frac{1}{2}z_1^2\right) \\
\geq& \frac{1}{2}|z_2|^2 - \frac{\eta}{2k^2}|z_1||\beta_3| + \frac{\eta}{2k^2}\frac{1}{2}z_1^2.
\end{aligned}$$

An application of Young's inequality to the second term and a cancellation of the terms involving z_1^2 yields

$$s(x, z_1, z_2) \geq \frac{1}{2}|z_2|^2 - \frac{\eta}{2k^2}|\beta_3|^2$$

and hence the coercivity required for an application of the above cited existence theorem. For the upper bound we start again from (4.9) and apply assumption (A3) to obtain for some constant $c > 0$ (which may change from line to line)

$$\begin{aligned} s(x, z_1, z_2) \leq & c\big(|z_2|^2 + |\beta_1||z_2| + \beta_1^2 + z_1^2 + |\beta_2||z_1| + \beta_2^2 \\ & + |h(z_1)||\beta_3| + H(z_1)\big). \end{aligned}$$

Finally, another application of Young's inequality implies

$$s(x, z_1, z_2) \leq c\big(|z_1|^2 + |z_2|^2 + |\beta_1|^2 + |\beta_2|^2 + |\beta_3|^2 + |h(z_1)|^2 + H(z_1)\big).$$

We remark that $|h(z)|^2 \leq c(|z|^{2q} + |z|^q + 1) \leq c(|z|^2 + 1)$ for any $z \in \mathbb{R}$ by (A2). Thus we obtain, applying also (A1), that

$$s(x, z_1, z_2) \leq c\big(|z_1|^2 + |z_2|^2 + |\beta_1|^2 + |\beta_2|^2 + |\beta_3|^2 + 1\big).$$

Therefore it holds that

$$\inf_{T \in X(\Omega, \mathbb{R})} \left\{ \int_\Omega s(x, T(x), T_x(x))\, \mathrm{d}x \right\} < \infty.$$

By the direct method of the calculus of variations, the functional attains a minimum in $X(\Omega, \mathbb{R})$, see e.g. [4, Thm 3.30]). □

Next we consider the first variation of the functional in (4.8). Let $\lambda \in (-\lambda^*, \lambda^*)$ for some suitably small $\lambda^* \in \mathbb{R}$ and $\phi \in C_0^\infty(\Omega, \mathbb{R})$. The first variation is then given by

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}\lambda} J^{k,j}\ (T + \lambda\phi)|_{\lambda=0} \\ & = \int_\Omega T_x \phi_x + \frac{\nu}{k}\left(T_x - (T^{k,j-1})_x\right)\phi_x + \frac{1}{k^2}\Big[h''(T)\left(T - T^{k,j-1}\right)^2 \phi \\ & \quad + 2h'(T)\left(T - T^{k,j-1}\right)\phi + h'(T)\Big(T - 2T^{k,j-1} + T^{k,j-2}\Big)\phi \\ & \quad - 2h'(T)\left(T - T^{k,j-1}\right)\phi + h(T)\phi - 2h(T)\phi + h(T)\phi\Big]\,\mathrm{d}x \\ & = \int_\Omega \Bigg\{ - T_{xx} - \frac{\nu}{k}\left(T_{xx} - (T^{k,j-1})_{xx}\right) + \frac{1}{k^2}\Big[h''(T)\left(T - T^{k,j-1}\right)^2 \\ & \quad + h'(T)\Big(T - 2T^{k,j-1} + T^{k,j-2}\Big)\Big] \Bigg\}\phi\,\mathrm{d}x, \end{aligned}$$

where we used integration by parts for the last equality. Hence, a minimizer $T^{k,j} \in X(\Omega, \mathbb{R})$ of $J^{k,j}$, $j \geq 1$, satisfies the weak, discretized Euler-Lagrange equation

$$\int_\Omega \Bigg\{ -(T^{k,j})_{xx} - \frac{\nu}{k}\Big((T^{k,j})_{xx} - (T^{k,j-1})_{xx}\Big) + \frac{1}{k^2}\Big[h''(T^{k,j})\Big(T^{k,j} - T^{k,j-1}\Big)^2 + h'(T^{k,j})\Big(T^{k,j} - 2T^{k,j-1} + T^{k,j-2}\Big)\Big]\Bigg\}\phi\,\mathrm{d}x = 0 \tag{4.10}$$

for every $\phi \in C_0^\infty(\Omega, \mathbb{R})$, which is exactly the weak, time-discretized version of (3.1).

In the following we introduce a time-dependent function $\tilde{T}^k : \Omega\times[0,\infty) \to \mathbb{R}$ by piecewise constant interpolation of the $T^{k,j}$, which are the solutions of the discretized Euler-Lagrange equation. For a time-stepsize $k > 0$ and for $j \in \mathbb{N}\cup\{0\}$ set $I^{k,j} := ((j-1)k, jk]$. Moreover, we construct a time-dependent function $T^k\colon \Omega \times [0,\infty) \to \mathbb{R}$ by linear interpolation of the minimizers $T^{k,j}$ at time kj and $T^{k,j-1}$ at time $k(j-1)$, which, in contrast to the constant interpolation, is differentiable with respect to time. Then the interpolations are defined as follows for any $(x,t) \in \Omega \times [0,\infty)$:

$$\tilde{T}^k(x,t) := T^{k,j}(x), \quad \text{if } t \in I^{k,j},$$
$$T^k(x,t) := \left(j - \frac{t}{k}\right) T^{k,j-1}(x) + \left(\frac{t}{k} - (j-1)\right) T^{k,j}(x), \quad \text{if } t \in I^{k,j}.$$

Correspondingly, the first time derivatives are defined by

$$\tilde{S}^k(x,t) := S^{k,j}(x) := \frac{1}{k}\Big(T^{k,j}(x) - T^{k,j-1}(x)\Big), \quad \text{if } t \in I^{k,j},$$
$$S^k(x,t) := \left(j - \frac{t}{k}\right) S^{k,j-1}(x) + \left(\frac{t}{k} - (j-1)\right) S^{k,j}(x), \quad \text{if } t \in I^{k,j}. \tag{4.11}$$

Finally, for the second-order time derivatives we set

$$\tilde{R}^k(x,t) := R^{k,j}(x) := \frac{1}{k}\Big(S^{k,j}(x) - S^{k,j-1}(x)\Big), \quad \text{if } t \in I^{k,j},$$
$$R^k(x,t) := \left(j - \frac{t}{k}\right) R^{k,j-1}(x) + \left(\frac{t}{k} - (j-1)\right) R^{k,j}(x), \quad \text{if } t \in I^{k,j}.$$

Note that

$$\Big(T^k\Big)_t = \tilde{S}^k, \qquad \Big(S^k\Big)_t = \tilde{R}^k. \tag{4.12}$$

Hence, the weak, time-discretized Euler-Lagrange equation in (4.10) can be rewritten, for every $\phi \in C_0^\infty(\Omega, \mathbb{R})$, as

$$\int_\Omega \Big[-(\tilde{T}^k)_{xx} - \nu(\tilde{S}^k)_{xx} + \Big(h''(\tilde{T}^k)(\tilde{S}^k)^2 + h'(\tilde{T}^k)\tilde{R}^k\Big)\Big]\,\phi\,\mathrm{d}x = 0. \tag{4.13}$$

5. Compactness

For the case $h(T) = \eta T$, $\eta > 0$, we prove the compactness result needed for the proof of the main result of this paper.

Proposition 5.1. *Assume that $h(T) = \eta T$ with a constant $\eta > 0$ in (3.1). Also, assume that the initial energy is bounded, that is,*

$$\sup_{k>0} E(T_0^k, S_0^k) < \infty.$$

Then the following quantities defined in (4.7) weakly converge as the time step $k \to 0$, up to suitably chosen subsequences,

$$\begin{aligned} T^k\big|_{t=0} &= T_0^k \rightharpoonup T_0 \quad \text{in } W^{1,2}(\Omega, \mathbb{R}), \\ S^k\big|_{t=0} &= S_0^k \rightharpoonup S_0 \quad \text{in } L^2(\Omega, \mathbb{R}). \end{aligned} \tag{5.1}$$

Moreover, for every $t > 0$, after choosing suitable subsequences we obtain the following convergences as $k \to 0$;

$$\tilde{T}^k \overset{*}{\rightharpoonup} \tilde{T} \text{ in } L^\infty\left(0, t; W^{1,2}(\Omega, \mathbb{R})\right) \cap W^{1,\infty}\left(0, t; L^2(\Omega, \mathbb{R})\right), \tag{5.2}$$

$$T^k \overset{*}{\rightharpoonup} T \text{ in } L^\infty\left(0, t; W^{1,2}(\Omega, \mathbb{R})\right) \cap W^{1,\infty}\left(0, t; L^2(\Omega, \mathbb{R})\right), \tag{5.3}$$

$$\tilde{S}^k \overset{*}{\rightharpoonup} \tilde{S} \text{ in } L^\infty\left(0, t; L^2(\Omega, \mathbb{R})\right) \cap W^{1,\infty}\left(0, t; L^2(\Omega, \mathbb{R})\right), \tag{5.4}$$

$$S^k \overset{*}{\rightharpoonup} S \text{ in } L^\infty\left(0, t; L^2(\Omega, \mathbb{R})\right) \cap L^2\left(0, t; W^{1,2}(\Omega, \mathbb{R})\right), \tag{5.5}$$

$$\tilde{R}^k \overset{*}{\rightharpoonup} \tilde{R} \text{ in } L^2\left(0, t; W^{-1,2}(\Omega, \mathbb{R})\right). \tag{5.6}$$

Proof. Firstly, we note that the choice of h given in the statement of the theorem allows one to have $f(T, T_t) = -\frac{\eta}{2}T_t^2$, in which case the energy becomes

$$E(T(t), T_t(t)) := \int_\Omega \left(\tfrac{1}{2}T_x^2(x,t) + \tfrac{\eta}{2}T_t^2(x,t)\right) \mathrm{d}x.$$

By this, which ensures having the term $|S_0^k|^2$ in the initial energy in the discrete setting, and the assumption on finiteness of the initial energy we obtain the assertions in (5.1). To prove (5.2)–(5.6), we use (4.6) for the discretized quantities T^k and S^k given by

$$\int_\Omega \left(\tfrac{1}{2}(T_x^k)^2(x,t) + \tfrac{\eta}{2}(S^k)^2(x,t)\right) \mathrm{d}x + \nu \int_0^t \int_\Omega \left(S_x^k\right)^2 \mathrm{d}x\,\mathrm{d}t = E(T_0^k, S_0^k) < \infty. \tag{5.7}$$

By (5.7), we obtain (5.2), (5.3), (5.4) and (5.5). To prove (5.6), note that $h(z) = z$ implies $h''(\tilde{T}^k)(\tilde{S}^k)^2 + h'(\tilde{T}^k)\tilde{R}^k = \tilde{R}^k$. Therefore, from (4.13), we have

$$\|\tilde{R}^k(t)\|_{W^{-1,2}} \le \|\tilde{T}_x^k(t)\|_{L^2} + \nu\|\tilde{S}_x^k(t)\|_{L^2} \quad \forall t > 0,$$

where for $q \ge 1$, the norm $\|\cdot\|_{W^{-1,q}}$ and $\|\cdot\|_{(L^q)'}$ are defined as

$$\|f\|_{W^{-1,q}} = \sup_{\phi \in W_0^{1,q}\setminus\{0\}} \frac{\left|\int_\Omega f\phi\,\mathrm{d}x\right|}{\|\phi\|_{W^{1,q}}} \text{ and } \|f\|_{(L^q)'} = \sup_{\|\phi\|_{L^q}} \left|\int_\Omega f\phi\,\mathrm{d}x\right|.$$

Now, the first term is bounded in $L^\infty(0,t;L^2(\Omega,\mathbb{R}))$ and the last term in $L^2(0,t;L^2(\Omega,\mathbb{R}))$ so both terms are bounded in $L^\infty(0,t;L^2(\Omega,\mathbb{R}))$. □

We are now in a position to state the below result, which yields further information on the limit functions.

Theorem 5.2. *Assume that $h(T) = \eta T$, $\eta > 0$. Then, the limits obtained in Proposition 5.1 satisfy $\tilde{T} = T$, $\tilde{S} = S = T_t$, $\tilde{R} = R = S_t$. In particular,*

$$T \in L^\infty(W^{1,2}) \cap W^{1,\infty}(L^2) \cap W^{2,\infty}(L^2)$$

and the equality

$$\int_0^t \int_\Omega \Big\{(T_x + \nu S_x)\phi_x + \eta S\phi_t\Big\}\,\mathrm{d}x\,\mathrm{d}\tau + \int_\Omega \eta\, S\phi|_{\tau=t} - \int_\Omega \eta\, S\phi|_{\tau=0} = 0$$

holds for every $t > 0$, $\phi \in C_0^\infty(\Omega \times [0,\infty),\mathbb{R})$, and we have $T|_{t=0} = T_0$ and $S|_{t=0} = S_0$.

Proof. By [9, Lemma 2.4] we know that the weak limits of piecewise constant and piecewise linear interpolations of a function given at discrete time steps coincide. Hence the assertion follows with the help of Proposition 5.1. □

In order to show that T is a weak solution of (3.1)–(3.5), it only remains to pass to the limit. Before embarking upon this, however, we state the following Aubin-Lions type lemma (see e.g. [3]) which will be referred to later.

Lemma 5.3. *Let X_0, X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$ where the first embedding being compact and the second one continuous. For $1 \leq p, q \leq \infty$, let*

$$W = \{z \in L^p([0,t];X_0) : z_t \in L^q([0,t];X_1)\}.$$

If $p < \infty$, then the embedding of W into $L^p([0,t];X)$ is compact. If $p = \infty$, and $q > 1$, then the embedding of W into $C([0,t];X)$ is compact.

Proposition 5.4. *Let T_0^k and S_0^k be as in Proposition 5.1, and let k be the index of the subsequence for which the convergences* (5.1) *and* (5.2)*–*(5.6) *hold. Then, T^k, S^k and R^k converge strongly in $L^2(0,t;L^2(\Omega,\mathbb{R}))$.*

Proof. We first note that by (5.3) and (5.5), we know that T^k converges to T in the weak topology of $L^\infty(W^{1,2}) \cap W^{1,\infty}(L^2)$. By Lemma 5.3, we can conclude that this convergence is strong in $L^2(L^2)$. Applying the same result for S^k using (5.5) and (5.6) with Theorem 5.2, we can conclude that S^k converges strongly to S in $L^2(L^2)$. □

6. The main result

We now state and prove the main result of this work.

Theorem 6.1. *Let $\Omega \subset \mathbb{R}$ be an open, connected and bounded set and let $h(T) = \eta T$, $\eta > 0$. Let $\alpha \in W^{1,2}(\Omega, \mathbb{R})$, (A0) be satisfied and the initial data belong to*

$$T_0 \in X(\Omega, \mathbb{R}) := \Big\{ T \in W^{1,2}(\Omega, \mathbb{R}) \colon T - \alpha \in W_0^{1,2}(\Omega, \mathbb{R}) \Big\}.$$

There exists a

$$T \in L^\infty \left(0, \infty; W^{1,2}\right) \cap W^{1,\infty} \left(0, \infty; L^2\right) \cap W^{2,\infty} \left(0, \infty; L^2\right) \cap W^{2,2} \left(0, \infty; W^{-1,2}\right)$$

which is a global weak solution to (3.1)–(3.5)*, that is,*

$$\int_0^\infty \int_\Omega \Big(T_x \phi_x + \nu T_{xx} \phi_t + (h(T))_{tt} \phi \Big) \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{6.1}$$

with $T|_{t=0} = T_0$ and $S|_{t=0} = S_0$, where $\phi \in C_0^\infty(\Omega \times [0, \infty), \mathbb{R})$. Moreover, the energy-dissipation equality given by

$$E(T(t), T_t(t)) - E(T(0), T_t(0)) = -\nu \int_0^t \int_\Omega T_{xt}^2(x, t) \, \mathrm{d}x \, \mathrm{d}t,$$

for every $t > 0$, is satisfied.

Proof. The assertion follows from Proposition 4.2, Proposition 5.1 and Theorem 5.2. □

7. Conclusion

In this paper, the main aim is to place the already-developed theory for strain-rate type viscoelasticity with limiting strain into the variational framework of minimization problems. Even though some of the results presented here are proved for general nonlinearities $h(T)$ satisfying (A1)–(A3) as well as for the linear case, the convergences, and hence the existence of solutions are proved only for $h(T) = \eta T$, $\eta > 0$, which leaves the nonlinear cases open for further work. Moreover, as a result of being able to work in this variational setting, there are more generalizations to be made for the elastic part of the energy, which we believe will be the main contributions of some forthcoming papers.

8. Acknowledgments

This research was facilitated by the Heilbronn Small Grant awarded by Heilbronn Institute for Mathematical Research (HIMR) to YŞ between 2022-2023 during which AS and LB were able to visit Cardiff University.

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(L. Bachmann) University of Würzburg, Institute of Mathematics, Emil-Fischer-Str. 40, 97074 Würzburg, Germany
Email address: `luisa.bachmann@uni-wuerzburg.de`

(A. Schlömerkemper) University of Würzburg, Institute of Mathematics, Emil-Fischer-Str. 40, 97074 Würzburg, Germany
Email address: `anja.schloemerkemper@uni-wuerzburg.de`

(Y. Şengül) School of Mathematics, Cardiff University, Cardiff, CF24 4AG, UK
Email address: `sengultezely@cardiff.ac.uk`