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Computational aspects of orbifold equivalence

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Abstract
In this paper we study the computational feasibility of an algorithm to prove orbifold equivalence between potentials describing Landau–Ginzburg models. Through a comparison with state-of-the-art results of Gröbner basis computations in cryptology, we infer that the algorithm produces systems of equations that are beyond the limits of current technical capabilities. As such the algorithm needs to be augmented by 'inspired guesswork', and we provide examples of applying this approach.

Mathematics Subject Classification 81-08 · 14B05 · 13C14

1 Introduction

Initially a model to describe superconductivity, Landau–Ginzburg models were promoted in the late '80 s to 2-dimensional (2, 2)-supersymmetric sigma models completely characterized by a polynomial f called *potential* (Vafa and Warner 1989). Landau–Ginzburg models gained importance in string theory and algebraic geometry as they form a family of quantum field theories which are related under homological mirror symmetry (Fan et al. 2013; Witten 1993). Furthermore, they are connected to cohomological field theories via (Polishchuk and Vaintrob 2016). This makes it natural to ask whether we can define some notion of "equivalence" between different potentials. The notion of orbifold equivalence was inspired by the study of (defects in) topological quantum field theories (see Carqueville and Runkel 2016; Davydov et al. 2011; Fröhlich et al. 2010) and it was first defined in the context of the study of equivariant and orbifold completions of the bicategory of Landau–Ginzburg models. Several examples have been explored in detail in the recent years (Carqueville

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et al. 2016; Newton and Ros Camacho 2016; Recknagel and Weinreb 2018), and its connection to other topics like the McKay correspondence (Ionov 2023).

A further reason to study orbifold equivalences is that they may be used to generate examples of the so-called Landau–Ginzburg/conformal field theory (LG/CFT) correspondence (see e.g. Ros Camacho (2020) for a review). This physics result states that the infrared fixed point of a Landau–Ginzburg model with potential f is a 2-dimensional rational conformal field theory (CFT) with central charge c_f. At the defects level, this predicts some relation between two seemingly different mathematical entities: matrix factorizations (which describe defects for Landau–Ginzburg models, Brunner and Roggenkamp 2007) and representations of the vertex operator algebra of the CFT (describing defects for the rational CFT). We lack a precise mathematical statement for this result, yet there are several promising examples available of this correspondence. In the particular case of simple singularities, it was proven in Carqueville et al. (2016) that via orbifold equivalence one finds exactly the predicted equivalences for the $N = 2$ supersymmetric minimal models. Furthermore, there are physics results suggesting that this might not be the only case, involving Landau–Ginzburg models with potentials describing singularities of modality greater than 0 (Cecotti and Del Zotto 2011; Martinec 1989). Hence, finding further orbifold equivalences is potentially a source of further examples of equivalences within the LG/CFT correspondence. This would strongly enhance our mathematical understanding of this intriguing physics result.

The present paper is concerned with finding orbifold equivalences using computer search. The current state of the art is the algorithm presented in Recknagel and Weinreb (2018). As recorded in Proposition 3.2, this algorithm terminates if and only if two potentials are orbifold equivalent. In pertinent examples, we quantify the size of these computations, and compare these sizes to current bests in solving these systems: the Fukuoka MQ challenge (Yasuda et al. 2015). As such, we show that experimental infeasability was not an accident that can be solved by choosing a different implementation (as was speculated in Recknagel and Weinreb 2018) but that these computations lie well beyond what current technology enables.

2 Orbifold equivalence

In this section we introduce the necessary background for defining orbifold equivalence. For the reader more familiar with higher categories, we refer to Appendix 1 for a complete description of orbifold equivalence in the context of the bicategory of Landau–Ginzburg models.

Potentials

Definition 2.1 Let $\Bbbk$ be an algebraically closed field of characteristic zero. We will consider the category $\mathcal{R}$ of polynomial rings in a finite number of variables over $\Bbbk$, each variable endowed with a fixed grading in $\mathbb{Q}_{>0}$.

Given $R \in \mathcal{R}$, we write

$$R = \bigoplus_{q \in \mathbb{Q}_{\geq 0}} R_q$$

for the equal-grading direct summands of R, and we call their elements *quasi-homogeneous*. Note that $R_0 = \mathbb{k}$.

Definition 2.2 For $R = \mathbb{k}[x_1, \ldots, x_n] \in \mathcal{R}$ and $f \in R$, the *Jacobian ideal* I_f of f is the ideal generated by the partial derivatives of f:

$$I_f = (\partial_{x_1} f, \cdots, \partial_{x_n} f)$$

The *Jacobian* of f is $\mathrm{Jac}\, f = R/I_f$. We call (R, f) a *potential* if f is quasi-homogeneous and if $\mathrm{Jac}\, f$ is a non-zero finite-dimensional $\mathbb{k}$-vector space. We often write f to represent the pair (R, f), and we may similarly write 'let $f \in R$ be a potential'. We write $\mathcal{P}_{\mathbb{k}}$ for the set of potentials.

Remark 2.3 The polynomial f is quasi-homogeneous of degree $d \in \mathbb{Q}_{>0}$ if and only if it satisfies:

$$\frac{|x_1|}{d} x_1 \partial_{x_1} f + \cdots + \frac{|x_n|}{d} x_n \partial_{x_n} f = f$$

where $|x_i|$ denotes the degree associated to the variable x_i. In particular, this implies that $f \in I_f$. We have an interesting converse in the case of power series (Saito 1971): there is a coordinate transformation making f quasi-homogeneous if and only if $f \in I_f$.

For future use, we record the following result.

Lemma 2.4 *If f is a potential, then there exists an $N \in \mathbb{N}$ such that $(x_1, \ldots, x_q)^N \subseteq I_f$.*

Proof This only uses the facts that I_f is quasi-homogeneous (i.e. for every $g \in I_f$ with quasi-homogeneous decomposition $g = \sum_\ell g_\ell$, we have $g_\ell \in I_f$ for all ℓ) and that R/I_f is finite dimensional over $\mathbb{k}$.

Pick a variable x_i. We will first prove that $x_i^{M_i} \in I_f$ for some M_i. For this, pick a lexicographical monomial order such that x_i is smaller than all other variables. Under this order, x_i^M ($M \geq 1$) can only be a leading monomial of a polynomial g if g is a function of only x_i and no other variables. Let G be a Gröbner basis of I_f with respect to this monomial order. Because I_f is quasi-homogeneous, we may choose G such that every $g \in G$ is quasi-homogeneous as well.

Because R/I_f is finite-dimensional, for large M, x_i^M must be reducible by G. That means G contains a divisor of x_i^M as a leading monomial, and we write M_i so that $x_i^{M_i}$ is a leading monomial of some $g \in G$. But with the chosen monomial order g

is a function of only x_i, and with g being quasi-homogeneous, we find $g = cx_i^{M_i}$ for some $c \in \Bbbk^\times$. Then $x_i^{M_i} \in I_f$.

To see that $(x_1, \ldots, x_q)^N \subseteq I_f$, we need to show that monomials of total degree N are in I_f for large enough N. But for

$$N > q \sum_i M_i$$

at least one variable x_i has, in such a monomial, an exponent greater than M_i, and so the monomial is a multiple of $x_i^{M_i} \in I_f$. It is therefore an element of I_f. □

Graded modules

Convention 2.5 *While R has a grading with values in $\mathbb{Q}_{\geq 0}$,* graded R-modules *have a $\mathbb{Q}$-grading.*

Definition 2.6 For $n \in \mathbb{Q}$ we define the graded R-module $R(n)$ as follows. As a non-graded R-module, it is isomorphic to R, and its grading is given by

$$R(n)_m = \begin{cases} R_{n+m} & \text{if } n + m \geq 0 \\ \{0\} & \text{if } n + m < 0. \end{cases}$$

A choice of grading on two R-modules induces a unique grading on the space of maps between such modules. Let us make this explicit for maps from $R(n)$ to $R(m)$. As non-graded modules we have

$$\mathrm{Hom}_R(R(n), R(m)) \cong \mathrm{Hom}_R(R, R) \cong R.$$

Comparing the quasi-homogeneous components of the left hand side and the right hand side, one readily obtains the following explicit form:

$$\mathrm{Hom}_R(R(n), R(m))_\ell \cong R_{m-n+\ell}.$$

Convention 2.7 *We use the term* quasi-homogeneous map *for maps of any degree, whereas* morphism *is reserved for quasi-homogeneous maps of degree zero.*

In particular, this convention implies that even though there is an invertible quasi-homogeneous map between $R(n)$ and $R(m)$ for any n, m, they are iso*morphic* if and only if $n = m$.

Definition 2.8 A *finitely generated, free, graded R-module X* is a graded R-module X that has a decomposition

$$X \cong R(n_1) \oplus \cdots \oplus R(n_\ell)$$

for some $n_1 \geq \cdots \geq n_\ell \in \mathbb{Q}$.

The choice of such a decomposition is equivalent to the choice of an R-basis consisting of quasi-homogeneous elements.

Multi-variate residues

We will make use of the multi-variate residue symbol as described by Lipman (1987). It is completely characterized by three simple facts that we describe in this section. With a view towards our computational objective, we will prove that this characterization is effective, i.e. it gives an algorithm for computing it.

These three facts are as follows:

(F1)

$$\operatorname{res}\left(\frac{g\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_q}{f_1, \cdots, f_q}\right) = 0 \text{ if } g \in (f_1, \ldots, f_q)$$

(F2)

$$\operatorname{res}\left(\frac{g\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_q}{x_1^{d_1}, \cdots, x_q^{d_q}}\right) = \left(\text{the } x_1^{d_1-1} \cdots x_q^{d_q-1}\text{-coefficient of } g\right)$$

for all $d_1, \ldots, d_q \in \mathbb{N}$.

(F3) The transformation rule:

$$\operatorname{res}\left(\frac{g\det(M)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_q}{M(f_1, \cdots, f_q)}\right) = \operatorname{res}\left(\frac{g\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_q}{f_1, \cdots, f_q}\right)$$

for any R-linear transformation $M\colon R^q \to R^q$.

Remark 2.9 Note that (F3) preserves the applicability of (F1): if $g \in (f_1, \ldots, f_q)$, then also $g \det(M) \in (Mf_1, \ldots, Mf_q)$. Namely, write $f = (f_1, \ldots, f_q)$ and suppose $g = \beta f$ for some R-linear $\beta\colon R^q \to R$. Writing $M^\dagger$ for the adjoint of M, we have $M^\dagger M = \det(M)\,\mathrm{Id}$, and so we can write $g \det(M) = (\beta M^\dagger)(Mf)$, which expresses $g \det(M)$ in the generators of $Mf = (Mf_1, \ldots, Mf_q)$.

These facts suffice to compute any residue symbol:

Lemma 2.10 *Let $R \in \mathcal{R}$ and let $f_1, \ldots, f_q \in I$ be generators for an ideal $I \subseteq R$ such that $(x_1, \ldots, x_n)^N \subseteq I$ for some $N \in \mathbb{N}$. Then there exists a $q \times q$ matrix M with coefficients in R such that for every i, $\sum_j M_{ij} f_j = x_i^{d_i}$ for some $d_i \in \mathbb{N}$. Moreover, this matrix can be computed explicitly.*

Proof The assumption guarantees that for every i, some power $x_i^{d_i}$ is an element of I, and this power d_i can be found algorithmically by a Gröbner basis computation as outlined in the proof of Lemma 2.4. This computation yields the coefficients M_{ij} for all j. Repeating the computation for all i yields the matrix M. □

Proposition 2.11 *For given $g \in R$ and $I = (f_1, \dots, f_q)$ such that R/I is finite-dimensional, the residue symbol*

$$\operatorname{res}\left(\frac{g\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_q}{f_1, \cdots, f_q}\right)$$

can be computed algorithmically.

Proof Write $I = (f_1, \dots, f_q)$. We first compute a Gröbner basis G of I. Then, we check whether $g \in I$. If it is, the residue is 0 and we have finished the computation.

If $g \notin I$, then we compute the matrix M such that $M \cdot (f_1, \dots, f_q)$ consists of a vector of monomials (Lemma 2.10). We can then use (F3) to replace g by $g \det(M)$, and (F2) to compute the residue as the appropriate coefficient of $g \det(M)$. □

Matrix factorizations

Definition 2.12 A finitely generated, free, graded R-module X is *supergraded* if it has a decomposition

$$X = X_+ \oplus X_-$$

into an *even* and *odd* part, respectively, both of which are f.g., free, graded R-modules themselves.

Convention 2.13 *There is some risk of confusion from using two gradings: the $\mathbb{Q}$-grading on R-modules and maps between them is not to be confused with the supergrading on $X_+ \oplus X_-$. These are our conventions:*

- *We use 'grade', 'grading', and 'quasi-homogeneous' exclusively to refer to the $\mathbb{Q}$-grading. We use 'even' and 'odd' exclusively to refer to the supergrading. We use 'even/odd' for super-homogeneity.*
- *Just like in the case of the $\mathbb{Q}$-grading (see Convention 2.7), maps may be even or odd, but morphisms are assumed even.*
- *We use the Koszul sign rule for tensor products of supergraded modules. In order to highlight its effect on the trace operator, we write* str *or* supertrace *to emphasize this. Explicitly, it is given by*

$$\operatorname{str} e_i \otimes e^j = (-1)^{\operatorname{sign}(e_i)\operatorname{sign}(e^j)}\delta_i^j$$

  for a basis $\{e_i\}_i$ with dual basis $\{e^i\}_i$.

Definition 2.14 Let $f \in R$ be a potential. A *matrix factorization of f* is a finitely generated, graded, supergraded R-module X together with an odd, homogeneous map d_X such that $d_X^2 = f \cdot \mathrm{Id}_X$.

Notation 2.15 *We will write X to represent the pair (X, d_X) from this definition.*

Orbifold equivalence

Definition 2.16 Let two potentials $f \in R$ and $g \in S$ be given. Write $T = R \otimes_{\Bbbk} S$. Then a *matrix factorization of* $f - g$ is a matrix factorization Q over T of the potential

$$f \otimes 1 - 1 \otimes g \in T$$

Note that the existence of Q implies that f and g have the same grading, since d_Q and therefore d_Q^2 are quasi-homogeneous maps by assumption, and therefore so is $(f - g) \cdot \mathrm{Id}_Q$.

Definition 2.17 Let $f \in \Bbbk[x_1, \ldots, x_m]$, $g \in \Bbbk[y_1, \ldots, y_n]$, and Q a matrix factorization of $f - g$. Its *quantum dimension with respect to* f is

$$\mathrm{qdim}_f\, Q = \mathrm{res}\left(\frac{\mathrm{str}\, \partial_{x_1} Q \cdots \partial_{x_m} Q \partial_{y_1} Q \cdots \partial_{y_n} Q \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m}{\partial_{x_1} f, \cdots, \partial_{x_m} f}\right)$$

The *left* and *right quantum dimensions* are, respectively, the quantum dimensions w.r.t. f and w.r.t. g.

Remark 2.18 Since at present we are only interested in the (non)zero-ness of quantum dimensions, we omit the signs (Carqueville and Murfet 2016; Carqueville et al. 2016).

Definition 2.19 The potentials f and g are *orbifold equivalent* if there is a matrix factorization of $f - g$ with nonzero left and right quantum dimensions.

It is not quite trivial to see that this is an equivalence relation; in fact, even reflexivity already requires a rather complicated matrix factorization Q. Similarly, transitivity is ‘almost’ easy to obtain, namely through a suitably defined tensor product of bimodules, but this results in a module that is not finitely generated. The hard part is obtaining the desired finitely generated one from this starting point.

Here, we will content ourselves with citing the result, contained at Sect. 2.1 of Carqueville et al. (2016):

Theorem 2.20 *Orbifold equivalence is an equivalence relation on the set of potentials* $\mathcal{P}_{\Bbbk}$. □

3 Search algorithm

Our task is as follows: given potentials $f \in R$ and $g \in S$, find out whether they are orbifold equivalent. We will present an algorithm that finishes in finite time if they are. It is not a decision procedure, however: the algorithm does not terminate if they are not. This section offers an exposition of parts of Recknagel and Weinreb (2018), tailored towards our use in Sect. 4.

Let’s first describe an easy instance of the algorithm.

Example 3.1 Assume the following potentials to be quasi-homogeneous of degree 2. Out of reflexivity of equivalence relations, it is clear that x^3 is orbifold equivalent to y^3, but let us analyze this case as an illustration. One way of finding an orbifold equivalence is splitting the total grading 2 into $\frac{4}{3} + \frac{2}{3}$ and then writing the most general rank 2 odd matrix with entries of those gradings respectively:

$$d_Q = \begin{pmatrix} 0 & c_1 x + c_2 y \\ c_3 x^2 + c_4 xy + c_5 y^2 & 0 \end{pmatrix}$$

with indeterminates $c_1, \ldots, c_5 \in \Bbbk$. Then the equation

$$d_Q^2 = (x^3 - y^3) \cdot \mathrm{Id}_Q$$

is equivalent to a set of equations in the variables $c_1, \ldots, c_5$. In detail, we find 4 distinct quadratic equations—one for each degree-3 monomial—in 5 variables.

We add to these equations the requirement that the quantum dimensions do not vanish. Thanks to Proposition 2.11, we can compute e.g. the left quantum dimension. It is a polynomial q_ℓ in $c_1, \ldots, c_5$, namely

$$q_\ell = -\frac{2}{3} c_2 c_3 + \frac{1}{3} c_1 c_4$$

Following (Recknagel and Weinreb 2018), we encode the non-vanishing by adding a helper variable c_ℓ and adding

$$c_\ell q_\ell - 1 = 0$$

to our equations. This has at least one solution for $c_\ell, c_1, \ldots, c_5$ if and only if the original system has at least one solution for which q_l does not vanish.

Adding two such equations, for left and right quantum dimension respectively, we find 6 equations in 7 variables, and if they admit a solution in $\Bbbk^7$, we found a matrix factorization proving orbifold equivalence of x^3 and y^3.

The existence of such a solution can be established or refuted, thanks to the weak Nullstellensatz, by checking whether the ideal generated by these equations is not equal to the trivial ideal (1). Algorithmically, this can be decided by computing a Gröbner basis.

It is straightforward to generalize this example to a search procedure. For this, we note the following:

- There are only countably many ranks $2m \in 2\mathbb{N}$ for Q;
- For every m, we can enumerate the possible gradings $(n_1, \ldots, n_{2m})$ of the free summands in

$$Q = R(n_1) \oplus \cdots \oplus R(n_{2m})$$

Through a standard diagonal procedure, we can enumerate the union of all modules appearing in this way. The gradings $n_1, \ldots, n_{2m}$ fix the grading of the entries in d_Q through $|(d_Q)_{ij}| = n_j - n_i + |d_Q|$. Then a 'most general' version of $(d_Q)_{ij}$ for these gradings is given by a polynomial with

$$\dim_{\Bbbk} T_{n_j - n_i + |d_Q|}$$

free variables at the (i, j) entry—one free variable for every quasi-homogeneous monomial of grading $n_j - n_i + |d_Q|$ in T.

Having found this most general form, we compute the coefficient equations from the matrix equation

$$d_Q^2 = (f - g) \cdot \mathrm{Id}_Q$$

Suppose they are given by

$$\{s_i(c_1, \ldots, c_N) = 0\}_{i \in S}$$

for some finite index set S. We augment this set with the two equations

$$c_\ell q_\ell(c_1, \ldots, c_N) - 1 = 0$$
$$c_r q_r(c_1, \ldots, c_N) - 1 = 0$$

Just like in the example, the weak Nullstellensatz implies that determining whether these allow a simultaneous solution in $\Bbbk^{N+2}$ is a finite computation.

We can summarize the discussion above in the following result:

Proposition 3.2 *There is an algorithm that, given two potentials $f \in \Bbbk[x_1, \ldots, x_q]$ and $g \in \Bbbk[y_1, \ldots, y_n]$, terminates if and only if f and g are orbifold equivalent.* □

4 Computational feasibility

The algorithm described above consists of a discrete part and a continuous part: The discrete part is concerned with enumerating possible ranks and gradings, and the continuous part is concerned with solving geometric equations.

Compared to the way it is described above, it is possible to significantly optimize the enumeration of possible gradings by taking into account the possible factorizations of the monomials appearing in f and g. In fact, it is necessary to do so to avoid a combinatorial explosion. Details for such a significant optimization are provided in Recknagel and Weinreb (2018).

In this section we look at the feasibility of the continuous part. It is well known that Gröbner basis computations have a tendency to blow up; in fact, doubly-exponential runtime has been proved for pathological cases (Mayr and Meyer 1982). For this reason, algebraic problems such as the present one have attracted the interest of the cryptology community as a potentially quantum-computer resistant replacement for digital signatures now commonly implemented through a discrete logarithm problem (Shor 1999; Makarim et al. 2017).

To quantify computational difficulty and feasibility, this community maintains lists of open problems for the public at large to submit solutions. One of these challenges is the *Fukuoka MQ Challenge* (Yasuda et al. 2015). One of their published lists consists of $2N$ quadratic equations in N variables—much like the ones we encountered in the previous section—for ever increasing N.

In the remainder of this section, we will compare the difficulty of the Gröbner basis computation corresponding to known matrix factorizations to the top contenders in the MQ Challenge as of July 2023. This should give an indication of the workability of this algorithm in practice.

Remark 4.1 In contrast to our present work, cryptology focuses on finite fields and the MQ Challenge is no exception. We believe that a comparison for feasibility still makes sense, as finite fields often have very efficient computer implementations. If anything, a problem stated over a field of characteristic zero will be *less* feasible. If this belief holds true, the MQ Challenge offers a *lower bound* for the difficulty of the problem we are trying to tackle.

Another difference is that the MQ Challenge concerns itself with dense polynomials; i.e. with polynomials where almost all monomials of degree at most two have a nonzero coefficient. The polynomials that appear for us are less dense than that. In particular, no linear terms appear. We still believe that denseness is a reasonable comparison.

To explain Table 1, let us go over one of its entries in detail. The three-variable potentials describing the singularities Q_{10} and E_{14} are known to be orbifold equivalent (Newton and Ros Camacho 2016). Explicitly, they are given by $f_{E_{14}} = x^4 + y^3 + xz^2$ and $f_{Q_{10}} = u^4 w + v^3 + w^2$ respectively.

The matrix factorization testifying that is given by

$$
\begin{aligned}
Q = T \oplus T(\frac{1}{4}) \oplus T(\frac{1}{3}) \oplus T(\frac{7}{12}) \\
\oplus T \oplus T(\frac{1}{4}) \oplus T(\frac{1}{3}) \oplus T(\frac{7}{12})
\end{aligned}
$$

as a $\mathbb{Q}$-graded module over $T = \Bbbk[x, y, z, u, v, w]$. That implies that d_Q's entries have gradings given by the following matrix:

$$
\frac{1}{12}\begin{pmatrix}
 & & & & 12 & 15 & 16 & 19 \\
 & & & & 9 & 12 & 13 & 16 \\
 & & & & 8 & 11 & 12 & 15 \\
 & & & & 5 & 8 & 9 & 12 \\
12 & 15 & 16 & 19 & & & & \\
9 & 12 & 13 & 16 & & & & \\
8 & 11 & 12 & 15 & & & & \\
5 & 8 & 9 & 12 & & & &
\end{pmatrix}
$$

Table 1 Gröbner basis challenge size for several known orbifold equivalences

Equivalence	Indeterminates	Equations
$f_{Q_{10}} \sim f_{E_{14}}$	108	237
$f_{Q_{18}} \sim f_{E_{30}}$	140	341
$f_{Q_{12}} \sim f_{E_{18}}$	116	263

Following the procedure from the last section, this results in the variables $c_1, \dots, c_{106}$ to describe the most general version of d_Q with these gradings.

When taken coefficient-by-coefficient (both of the matrix and of the polynomial entries), the equation

$$d_Q^2 = (f_{E_{14}} - f_{Q_{10}}) \cdot \mathrm{Id}_Q$$

gives 470 equations in $c_1, \dots, c_{106}$. Adding the quantum dimension helper variables and constraints, we are faced with a system of 472 equations in 108 variables.

A significant optimization can be made. Since d_Q is odd, it is of the form

$$d_Q = \begin{pmatrix} 0 & d_Q^\sharp \\ d_Q^\flat & 0 \end{pmatrix}$$

and $d_Q^2 = (f - g)\,\mathrm{Id}_Q$ reduces to the two equations

$$\begin{aligned} d_Q^\sharp d_Q^\flat &= (f_{E_{14}} - f_{Q_{10}}) \cdot \mathrm{Id}_{Q_+} \\ d_Q^\flat d_Q^\sharp &= (f_{E_{14}} - f_{Q_{10}}) \cdot \mathrm{Id}_{Q_-} \end{aligned}$$

However, these two equations are equivalent to one another. We may therefore consider only the constraints arising from either one of them, and this cuts the number of independent constraints on $c_1, \dots, c_{106}$ roughly in half. In the specific case above, we are left with 237 equations in 108 variables.

For comparison, the current top contender in the MQ Challenge solved a system of 160 equations in 80 variables over the field of 2 elements. This strongly suggests that the described algorithm would not have been able to find this orbifold equivalence within reasonable time.

Table 1 lists similar outcomes for different equivalences. One is the example treated in the next Section $f_{Q_{18}} \sim f_{E_{30}}$, while the second involves an equivalence already known from existing ones, $f_{Q_{12}} \sim f_{E_{18}}$.

5 'Inspired guessing'

Given this rather sobering view on computer explorations, it is useful to combine them with some 'inspired guessing': this can reduce the number of equations and indeterminates and in this way make the computer approach feasible.

A way to detect natural candidates for orbifold equivalence is via the following result:

Lemma 5.1 *Let $f \in \Bbbk[x_1, \ldots, x_n]$ be a potential with a $\mathbb{Q}$-grading assigned to each variable which we will denote as $|x_i|$. Define the central charge associated to f to be:*

$$c_f = \sum_{i=1}^{n} (1 - |x_i|) .$$

If two potentials f, g are orbifold equivalent potentials, then they have the same central charge.[1]

For a proof see (Carqueville and Runkel 2016, Proposition 6.4). This is a necessary yet not sufficient condition, but it is still a useful source of potential candidates for orbifold equivalences. Here we focus on some instance related to the so-called bimodal singularities (see e.g. Ebeling and Ploog (2013)), not necessarily new per se (since it can be derived from already known equivalences[2]) but not previously contained anywhere in the literature: $f_{Q_{18}} \sim f_{E_{30}}$.

In the following we will describe in detail the procedure for this case, described by the potentials

$$f_{Q_{18}} = x^8 + y^3 + xz^2$$
$$f_{E_{30}} = u^8 w + v^3 + w^2$$

(both with central charge $c_{Q_{18}} = \frac{17}{15} = c_{E_{30}}$).

- First we split the total grading 2 into 3 different pairs of two adequate summands (consistent with the degree assigned to each of the variables). Note that in this case, we have:

$$|x| = \frac{1}{4}, \qquad |u| = \frac{1}{8}$$
$$|y| = \frac{2}{3}, \qquad |v| = \frac{2}{3}$$
$$|z| = \frac{7}{8}, \qquad |w| = 1$$

  Inspired by the charge of the entries at Kajiura et al. (2009) for Q_{12}, we choose to split 2 in the following way: $2 = 1 + 1 = \frac{4}{3} + \frac{2}{3} = \frac{9}{8} + \frac{7}{8}$.

[1] One can relate this to the so-called strange duality of singularities as described by Arnol'd (1975); Arnol'd et al. (1985) (see e.g. Newton and Ros Camacho 2016 for a detailed discussion in the case of unimodal singularities).

[2] Indeed, under a suitable change of variables $f_{Q_{18}}$ can be decomposed as a sum of f_{D_9} and f_{A_2}, and $f_{E_{30}}$ as the sum of $f_{A_{15}}$ and f_{A_2}. The equivalence between f_{D_9} and $f_{A_{15}}$ was proven in Carqueville et al. (2016).

- Then we distribute these entries in a $2^3 = 8$ odd matrix (again inspired by the V_0 indecomposable for $f_{Q_{12}}$ of Kajiura et al. (2009)) as in:

$$\frac{1}{24}\begin{pmatrix} & & & & 21 & 32 & 24 & 29 \\ & & & & 16 & 27 & 19 & 24 \\ & & & & 24 & 35 & 27 & 32 \\ & & & & 13 & 24 & 16 & 21 \\ 27 & 32 & 24 & 35 & & & & \\ 16 & 21 & 13 & 24 & & & & \\ 24 & 29 & 21 & 32 & & & & \\ 19 & 24 & 16 & 27 & & & & \end{pmatrix} \tag{1}$$

- Here, notice that:
  - The most general polynomial we can generate of charge $\frac{2}{3}$ is $c_1u + c_2v$, and of charge $\frac{4}{3}$ is $c_1u^2 + c_2uv + c_3v^2$ ($c_i \in \mathbb{C}$).
  - With this grading, we cannot generate monomials of degree $\frac{13}{24}$ and $\frac{29}{24}$, and these entries will be straightforward zero. For the entries with degree $\frac{19}{24}$ and $\frac{35}{24}$, we set them by hand to be zero as part of the 'inspired guess'.
  - Monomials potentially generating x^8, w^2 and xz^2 could be x^4 and w (both of charge 1) and z and xz (each of charge $\frac{7}{8}$ and $\frac{9}{7}$) respectively.
  - Let us specify the non-zero blocks of the twisted differential as in Eq. 1. We insert these entries in the matrix and adjust ± 1 coefficients so the determinant of the $d_Q^\sharp$ is $v^3 + w^2 - x^8 - y^3 - xz^2 (= f_{E_{30}} - f_{Q_{18}} - u^8 w)$ squared:

$$d_Q^1 = \begin{pmatrix} z & v^2 + yv + y^2 & x^4 + w & 0 \\ y - v & -xz & 0 & x^4 + w \\ x^4 - w & 0 & -xz & -(v^2 + yv + y^2) \\ 0 & x^4 - w & v - y & z \end{pmatrix}$$

  (again with $d_Q^0 = \sqrt{Det[d_Q^1]}(d_Q^1)^{-1}$).

- At this point, we write for each entry in the matrix all possible remaining monomials making them the most general instance of a polynomial of each charge we can have. We get 84 variables.
- Then we impose $d_Q^2 = \left(f_{E_{30}} - f_{Q_{18}}\right) \cdot \mathrm{Id}_Q$, and we reduce the amount of variables and equations to be satisfied solving by hand as many linear equations as possible (77 in total). We are then left with a system of 5 equations in 7 variables.
- And last we compute its left and right quantum dimensions. Imposing them to be non-zero we obtain two more inequalities to be satisfied.

Remark 5.2 The reader may notice that this method of reducing the amount of equations and variables in steps is similar to what was called "progressive perturbation" in Newton and Ros Camacho (2016), where the shape of our starting ansatz is again suggested by the indecomposables of the triangulated categories of matrix factorizations in Kajiura et al. (2009).

In this way we construct a matrix factorization with non-zero quantum dimensions. As described in Sect. 3, a Gröbner basis computation now determines whether this system admits a solution. The size of the system is now sufficiently small to complete this in reasonable time, so we have proven that these two potentials are orbifold equivalent.

Remark 5.3 Observe that the potentials involved in this equivalence have the following nice property. Let us write $f_{Q_{18}}$ as $f_{Q_{18}} = \sum_{i=1}^{3} \prod_{j=1}^{3} x_j^{A_{ij}}$ with $A_{ij} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ the matrix of coefficients. It turns out that $f_{E_{30}} = \sum_{i=1}^{3} \prod_{j=1}^{3} x_j^{A_{ji}}$. This is what is called the 'Berglund-Hübsch transposed potential', a well-known way to generate mirror symmetric Landau–Ginzburg potentials and we refer to the literature for further details on this (Berglund and Hübsch 1993; Krawitz 2010; Ebeling 2016).

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A. Categorical origins of orbifold equivalence

The concept of orbifold equivalence was first introduced in the context of the study of bicategories, and in particular that of Landau–Ginzburg models. Here, we aim to review the categorical origins of the definition of orbifold equivalence (Carqueville and Runkel 2016).

First, consider the following categories of matrix factorizations:

mf (S, f): given a potential $f \in S$, objects are matrix factorizations of f as in Definition 2.14, and given two objects (X, d_X), (Y, d_Y) morphisms are S-linear maps $\varphi \colon X \to Y$. This category is differential supergraded, and for such a φ there is a differential in the morphism space given by: $\delta\varphi = d_Y \circ \varphi - (-1)^{|\varphi|} \varphi \circ d_X$, where $|\varphi|$ is the degree of φ.

We say that two morphisms $\varphi, \psi : M \to N$ are *homotopy equivalent* if there exist a morphism θ of degree one such that $\varphi - \psi = d_N \circ \theta + \theta \circ d_M$. Homotopy equivalence is an equivalence relation.

hmf (S, f): objects are those of mf (S, f), and morphisms are those of mf (S, f) that are even and compatible with the twisted differential (i.e. satisfying that $d_Y \circ \varphi = \varphi \circ d_X$) modulo homotopy.

hmf $(S, f)^\omega$: idempotent completion of the category hmf (S, f). That means, we take objects isomorphic to direct summands of objects of hmf (S, f).

Next, let us define a tensor product of matrix factorizations. Let $f_1 \in S_1$, $f_2 \in S_2$, $f_3 \in S_3$ be three potentials, X be a matrix factorization of $f_1 - f_2$ and Y be a matrix factorization of $f_2 - f_3$. The tensor product matrix factorization $X \otimes_{S_2} Y$ is the matrix factorization of $f_1 - f_3$ with base module over $S_1 \otimes_{\Bbbk} S_3$ and twisted differential $d_{X \otimes Y} = d_X \otimes \mathrm{Id}_Y + \mathrm{Id}_X \otimes d_Y$.

Remark A.1 Notice here that for $S_2 \neq \Bbbk$, $X \otimes_{S_2} Y$ is of infinite rank over $S_1 \otimes_{\Bbbk} S_3$. Yet the resulting matrix factorization is actually isomorphic to one of finite rank (Khovanov and Rozansky 2008).

For the case $S_1 = S_2 = S$, note that under this tensor product,

Proposition A.2 (Carqueville and Murfet (2016); Carqueville and Runkel (2010)) hmf $\left(S^{\otimes 2}, f \otimes 1 - 1 \otimes f\right)^\omega$ *is a tensor category.*

In fact one can prove more general cases than just this one (Carqueville and Murfet 2016) and even compute dual matrix factorizations as well (Carqueville and Murfet 2016; Carqueville and Runkel 2012), for which we refer to the literature. Hence a legitimate question is if this category is in addition pivotal. For future convenience, in order to answer this question let us go one step higher and define the following bicategory that we will denote as $\mathcal{LG}_{\Bbbk}$:

- Objects are potentials as in Definition 2.2,
- For any two objects (S_1, f_1), (S_2, f_2), the morphism category is hmf $(S_1 \otimes_{\Bbbk} S_2, f_1 - f_2)^\omega$.

This is indeed a bicategory (Carqueville and Runkel 2010). Furthermore,

Theorem A.3 $\mathcal{LG}_{\Bbbk}$ *is a graded pivotal bicategory.*

Graded pivotality means that the bicategory is pivotal up to shifts, and one needs a detailed discussion of how these and adjunction maps are compatible. For details we refer to the original source (Carqueville and Murfet 2016). But, notice here that:

Remark A.4 The subbicategory $\mathcal{LG}'_{\Bbbk}$ whose objects are potentials with an even number of variables is pivotal.

Moreover, we have explicit formulas for the adjunctions and more precisely of the evaluation and coevaluation maps. These were constructed in the one-variable case in Carqueville and Runkel (2012) and then for more general cases in Carqueville and Murfet (2016). One may combine these for example to get the explicit expressions of the so-called *left and right quantum dimensions* as stated in Definition 2.17.

Using the theory of equivariant and orbifold completion of bicategories (Carqueville and Runkel 2010), one finds the following result for $\mathcal{LG}'_{\Bbbk}$:

Theorem A.5 *Let $f_1 = f_1(x_1, \ldots, x_m)$, $f_2 = f_2(y_1, \ldots, y_n)$ be two potentials and $(M, d_M) \in \mathrm{hmf}(f_2 - f_1)$ with invertible quantum dimension. Then*

$$(M, d_M) : \left((S_1, f_1), M^\dagger \otimes M\right) \rightleftarrows \left((S_2, f_2), I_{f_2 \otimes 1 - 1 \otimes f_2}\right) : (M, d_M)^\dagger$$

(where $\left(M, d^M\right)^\dagger$ is the right adjoint of $\left(M, d^M\right)$) is an adjoint equivalence in $\mathcal{LG}'_{\Bbbk}$, and $M^\dagger \otimes M$ is a symmetric separable Frobenius algebra object in $\mathrm{hmf}(S_1 \otimes_{\Bbbk} S_1, f_1 \otimes 1 - 1 \otimes f_1)^\omega$.

Let's reformulate this theorem as an equivalence relation:

Definition A.6 Let $f_1 = f_1(x_1, \ldots, x_m)$, $f_2 = f_2(y_1, \ldots, y_n)$ be two potentials and $\left(M, d^M\right) \in \mathrm{Ob}(\mathrm{hmf}(f_2 - f_1))^\omega$. Assign to (M, d_M) two elements in $\Bbbk$, the left and right quantum dimensions $\mathrm{qdim}_l(M)$ $\mathrm{qdim}_r(M)$ as in Definition 2.17. If there exists such an (M, d_M), then we say that f_1 and f_2 are *orbifold equivalent*.

Remark A.7 Notice that this definition is equivalent to Definition 2.19.

Proposition A.8 (Carqueville et al. (2016)) *Denote as $\mathcal{P}_{\Bbbk}$ the set of potentials with any number of variables with coefficients in the field $\Bbbk$. Orbifold equivalence is an equivalence relation in $\mathcal{P}_{\Bbbk}$.*

Notice here that:

- Following the notation in Definition A.6, if f_1 and f_2 are orbifold equivalent then clearly $m = n$ mod 2.
- We are considering implicitly a $\mathbb{Q}$-graded setting, and so the quantum dimensions take values in $\Bbbk$. This can be seen from counting degrees in the formulas of Definition 2.18.
- Quantum dimensions are independent of the $\mathbb{Q}$-grading of a matrix factorization.

Given two potentials f_1, f_2 and a matrix factorization X of $f_1 - f_2$ proving that f_1 and f_2 are orbifold quivalent, one finds as a corollary of Theorem A.5 that the following equivalence of categories holds:

Proposition A.9

$$\mathrm{hmf}(S, f_2)^\omega \simeq \mathrm{mod}\left(X^\dagger \otimes X\right)$$

In the Introduction it was mentioned that orbifold equivalence could be used as a source of equivalences of categories in the context of the Landau–Ginzburg/conformal field theory correspondence, and Proposition A.9 is the key to do it. In the case of simple singularities (Carqueville et al. 2016), we found equivalences of categories of matrix factorizations of these potentials and the expected from CFT categories of modules over separable symmetric Frobenius algebra objects (Ostrik 2003). For more details we refer to Davydov et al. (2018); Ros Camacho (2020). For the remaining existing orbifold equivalences we hope to find similar equivalences and their respective CFT counterpart (hopefully not so distant) in the future (Ros Camacho 2023).

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