A Characterisation of Trading Equilibria in Strategic Market Games*

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Abstract

For a strategic market game (as introduced by Shapley and Shubik), following Dubey and Rogawski (1990), we provide a full explicit characterisation of the set of *trading equilibria* (in which all goods are traded at a positive price), for both the "buy and sell" and the "buy or sell" versions of this model under standard assumptions on the utility functions. We interpret and illustrate our equilibrium-characterising conditions; we also provide simple examples of trading equilibria, including those of *non-interior* strategy profiles (in which at least one trader is using the whole endowment in at least one good or money).

Keywords: strategic market game, trading equilibrium, interior profile, buy and sell, buy or sell.

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1 INTRODUCTION

The strategic market game, as introduced by Shapley and Shubik (Shubik,1973; Shapley, 1976; Shapley and Shubik 1977; Dubey and Shubik, 1977, 1978), perhaps is the most well-analysed model to understand the formation of market-price as a strategic mechanism, arguably, outside the realm of the general equilibrium theory.

In this non-cooperative game, individual traders (players) can influence the market price of any good, through their buy and sell "orders" (strategies), when a specified commodity is used as the monetary medium ("money") for buying and selling all other commodities[1]; the price of any good is simply the ratio of the amount of money and the amount of good at that trading-post.

The strategic market game has the properties of a resource reallocation mechanism and also the properties of a non-cooperative game. Traders select their (buying and selling) strategies to maximise their respective utilities; these strategies form an allocation (redistribution) of the existing resources. The definition of the non-cooperative equilibrium in this context is simply that of Nash equilibrium (in pure strategies) of this strategic form game, in which each trader chooses a (buy and sell) bid profile that indeed maximises the respective utility, given the bids of others. As a consequence, "no-trade", by all players, is an equilibrium for such a market game, because, doing nothing when the others are not bidding is always the best response for any player.

Starting with Dubey and Shubik (1977, 1978), there is a literature on general existence and multiplicity of equilibria (see Peck, Shell and Spear, 1992) for such games. Although, the equilibria of such games are typically inefficient (see Dubey and Rogawski, 1992), this simple model has generated a vast literature over the last four decades; more complicated models and solution concepts out of this basic game have been analysed (see survey papers by Giraud, 2003; Levando, 2012 and Dickson and Tonin, 2021).[2] Moreover, these games have been used as basic set-up in different areas in microeconomics, macroeconomics and in finance.

It is indeed well-known for some time (Dubey and Shubik, 1978; Dubey and Rogawski, 1992), that at any fully "interior" Nash Equilibrium, (some) "gradient" of the utility function of any player must be equal to a scalar multiple of (some) "personal" price vector for that player. A characterisation theorem would be both fundamental and necessary to have for any current and future research in this area. However, to the best of our knowledge, there is no explicit study of equilibrium conditions (resulting from the utility maximisation problem of the traders) and hence no usable characterisation of the set

[1] Other variations of this simple "buy and sell" game are "sell-all", in which the entire endowments for all commodities other than money are brought to the market for sale, and, "buy or sell", in which traders cannot both buy and sell a particular good, that is, where, each agent, for any good, chooses either to just sell some amount of its endowment or to buy back some of this good using some money endowment.

[2] As readers will appreciate, we are not listing all recent developments here.

of (Nash) equilibria even for a simple strategic market game.[3]

We address this issue in this paper, by identifying conditions for trading equilibria and thereby fully characterising the set of trading equilibria for a basic buy and sell game (under some standard assumptions on the utility function of each trader). The main result in this paper states that the equilibrium price of each good is a specific ratio, a constant for all traders. We explicitly find this constant. We interpret this characterising condition as follows: for any interior outcome to be an equilibrium requires that for each traded commodity j, the product of (i) the marginal rate of substitution between good j and money for any agent i and (ii) the ratio of the *value* of the amount of good j bought and sold by all agents other than agent i, must be a constant across agents; moreover, this constant is indeed the (equilibrium) price of good j.

Our key equilibrium condition applies to any "interior" profile in which no trader brings the entire endowment of any good (or money) to the market and thus everyone consumes at least some amount of all the goods and money. We however provide a full characterisation involving "non-interior" profiles where some traders might be inactive on some markets (corner solutions) by carefully studying the complementary slackness conditions of the non-negativity constraints of the corresponding Kuhn-Tucker method. We also derive equilibrium conditions when traders are allowed to either buy or sell but are not allowed to do both (that is, in a "buy or sell" game).

2 MARKET GAMES

We first briefly present our game, the strategic market game, *a la* Shapley and Shubik (Shubik, 1973; Shapley, 1976; Shapley and Shubik, 1977; Dubey and Shubik, 1977, 1978).

A market is denoted by a four-tuple $\Xi = (N, X, E, U)$. $N = \{1, \dots, n\}$ is a finite set of individual traders. $X = (X_1 \times \dots \times X_l) \times X_{l+1 \equiv m} \in \Re_+^{l+1}$ is the commodity space, where $\Re_+$ is the non-negative orthant of the real line $\Re$. The $(l+1)^{\text{th}}$ commodity is the numeraire, "money" that henceforth we denote by m. $E = (e_i = (e_{i1}, \dots, e_{il}, e_{im}) : i \in N)$ is an indexed collection of points in X representing the endowments of the traders. The endowment vector is assumed to be non-negative and non-zero in at least one component for each individual. $U = (U_i : i \in N)$ is an indexed collection of functions from X to $\Re$ representing the utility functions of the traders, that are assumed to be strictly increasing.

Consider a market $\Xi = (N, X, E, U)$. Let us imagine l separate trading posts, one for each of the l commodities. Each individual i supplies q_{ij}, $q_{ij} \geq 0$, to each trading post $j \in \{1, \dots, l\}$. Denote

[3] Perhaps as a result of this lack of an explicit characterisation, we also do not have many explanatory worked-out examples of equilibrium profile (as in Ray, 2001, even though it is not really based on any formal characterisation); of course, within the specific case of "bilateral oligopolies" (Gabszewicz and Michel, 1997), there are numerous studies (such as, Busetto and Codognato, 2006, Dickson, 2013 and Dickson and Hartley, 2013) and examples in the literature (see Dickson and Tonin, 2021 for examples derived from the first principle of Nash equilibrium behaviour in such models).

$q_i := (q_{i1}, \ldots, q_{il})$. Let $Q(j) = \sum_{i \in N} q_{ij}$, for all $j \in \{1, \ldots, l\}$. Each individual trader $i \in N$ makes bids by allocating amounts b_{ij} of his money (that is, the m-th commodity) to trading post j, for each $j \in \{1, \ldots, l\}$. We shall denote his buying strategy by the vector $b_i = (b_{i1}, \ldots, b_{il})$, with the constraints (a) $\sum_{j=1}^{l} b_{ij} \leq e_{im}$ and (b) $b_{ij} \geq 0$. Denote $B(j) := \sum_{i \in N} b_{ij}$.

Given a market $\Xi = (N, X, E, U)$, if we assume that each trader can either buy or sell but not both, then we have a *buy or sell strategic market game*. The difference between a strategic market game and a buy or sell strategic market game is just in terms of admissible strategy profiles; while any admissible strategy profile (q, b) for a buy or sell strategic market game is also a strategy profile for the strategic market game, the converse is clearly not true.

With slight abuse of notation, define $(q, b) := ((q_i, b_i)_{i \in N})$ as an indexed collection of strategies or a strategy profile. The "market-clearing" price $p(q, b)$ is formed as a ratio of total bid to total supply (if positive): $p = \sum_{i=1}^{n} b_i / \sum_{i=1}^{n} q_i$ if $\sum_{i=1}^{n} q_i > 0$; $p = 0$ if $\sum_{i=1}^{n} q_i = 0$.

Definition 1 *Given a market $\Xi = (N, X, E, U)$, a strategy profile (q, b) is a trading profile of the corresponding market game if for each good j, $Q(j)$ and $B(j)$ both are strictly positive.*

As a result, at a trading profile, the price for each good j, $p_j = (B(j)/Q(j)) > 0$.

At a trading profile, let us denote the vector of i's final allocations by $x_i(q, b)$, computed as follows:
$x_i(q, b) = (x_{i1}(q, b), \ldots, x_{il}(q, b), x_{im}(q, b) \in X$, where,
$x_{ij}(q, b) = e_{ij} - q_{ij} + (b_{ij}/p_j)$ for each $j \in \{1, \ldots, l\}$, and $x_{im}(q, b) = e_{im} - \sum_{j=1}^{l} b_{ij} + \sum_{j=1}^{l} p_j q_{ij}$.

Now agent i's utility maximisation problem (UMP) is to choose (q_i, b_i), given $((q_k, b_k)_{k \in N \setminus \{i\}})$ from all agents other than i, to maximise $U_i(x_i(q, b))$ subject to $q_{ij} \in [0, e_{ij}]$, $b_{ij} \geq 0$ for each $j = 1, \ldots, l$, and $\sum_{j=1}^{l} b_{ij} \leq e_{im}$.

Definition 2 *Given a market $\Xi = (N, X, E, U)$, a trading profile (b, q) is a trading equilibrium of the corresponding market game if for each agent $i \in N$, given $((q_k, b_k)_{k \in N \setminus \{i\}})$, (q_i, b_i) is a solution to agent i's UMP.*

Autarky is the strategy profile (q, b) with $q_{ij} = b_{ij} = 0$, for all $i \in N$ and all $j \in \{1, \ldots, l\}$. In autarky, all traders are inactive in the sense that they are neither buying nor selling any commodities. Clearly it is not a trading profile, however, it is trivially a Nash equilibrium of the market game, as, no agent, given that all others are inactive, has any incentive to deviate and use a strategy with either sell or buy offers being strictly positive.

Note that in any trading profile (and any trading equilibrium), we require the markets of every single good be open with a strictly positive price, which is clearly more demanding than just being in non-autarky.

It is also clear that at a trading equilibrium (q, b), for every trader $i \in N$ and all $j \in \{1, \ldots, l\}$, $\sum_{k \in N \setminus \{i\}} q_k > 0$ holds, otherwise, an individual trader could have captured the whole quantity at the trading post for good j with an infinitesimal amount. Hence, the number of traders with $q_{ij} > 0$ at a trading equilibrium has to be more than one.

We now make a couple of further assumptions on the strictly increasing utility functions, $U_i(x_i(q, b))$ for each trader $i \in N$, for the rest of the paper.

Assumption 1a. $U_i(x_i(q, b))$, for any $i \in N$, is continuously differentiable.

Consider any $i \in N$ and fix any trading decision $(q_j, b_j)_{j \in N \setminus \{i\}} \in \Re_{++}^{2l \times (n-1)}$ for all individuals other than i, where, $\Re_{++}$ is the positive orthant of the real line $\Re$. For each $(q_i, b_i) \in \Re_{++}^{2l}$, define $V_i(q_i, b_i) := U_i(x_i((q_i, b_i), (q_j, b_j)_{j \in N \setminus \{i\}}))$. Therefore, for any given trading decision of others, $(q_j, b_j)_{j \in N \setminus \{i\}} \in \Re_{++}^{2l \times (n-1)}$, the associated function $V_i : \Re_{++}^{2l} \to \Re$ represents the utility that trader i can have for different choices of (q_i, b_i).

The gradient vector of the function $V_i(q_i, b_i)$ is denoted by:

$\nabla V_i(q_i, b_i) = \left(\frac{\partial V_i(q_i,b_i)}{\partial q_{i1}}, \ldots, \frac{\partial V_i(q_i,b_i)}{\partial q_{il}}, \frac{\partial V_i(q_i,b_i)}{\partial b_{i1}}, \ldots, \frac{\partial V_i(q_i,b_i)}{\partial b_{il}} \right)$.

Observe that for any $(q_i, b_i), (q_i', b_i') \in \Re_{++}^{2l}$, $\nabla V_i(q_i, b_i) \cdot [(q_i', b_i') - (q_i, b_i)] = \sum_{k=1}^{l} (q_{ik}' - q_{ik}) \frac{\partial V_i(q_i,b_i)}{\partial q_{ik}} + \sum_{k=1}^{l} (b_{ik}' - b_{ik}) \frac{\partial V_i(q_i,b_i)}{\partial q_{ik}}$.

Definition 3 *A continuously differentiable utility function* $\widetilde{\mathrm{U}}_i((q, b)) \equiv U_i(x_i(q, b))$ *is pseudo-concave in* (q, b) *if for any given* $(q_j, b_j)_{j \in N \setminus \{i\}} \in \Re_{++}^{2l \times (n-1)}$*, the associated real-valued differentiable function* $V_i : \Re^{2l} \to \Re$ *satisfies the following property: for all* $(q_i, b_i), (q_i', b_i') \in \Re_{++}^{2l}$ *such that* $\sum_{k=1}^{l} (q_{ik}' - q_{ik}) \frac{\partial V_i(q_i,b_i)}{\partial q_{ik}} + \sum_{k=1}^{l} (b_{ik}' - b_{ik}) \frac{\partial V_i(q_i,b_i)}{\partial b_{ik}} \leq 0$*, we have* $V_i(q_i', b_i') \leq V_i(q_i, b_i)$*.*

Pseudo-concavity implies a restriction on the function $V_i(.,.)$. As is well-known, note that, if the utility function $U_i(x_i(q, b))$ is twice differentiable in (q_i, b_i), then the property of pseudo-concavity (for $\widetilde{\mathrm{U}}_i(.)$) is equivalent to that of quasi-concavity (see Chapter 2, page 88 in Kall and Mayer, 2006).

Assumption 1b. $\widetilde{\mathrm{U}}_i(.)$, for any $i \in N$, is pseudo-concave.

From here onwards in this paper, we will call $U_i(.)$ pseudo-concave, for notational simplicity.

The assumption of pseudo-concavity of the utility function of any trader i in this game ensures that the Kuhn-Tucker conditions are sufficient for each trader's utility maximisation problem and thus will be used to prove our results in the next section. Specifically, if the objective function is pseudo-concave and all constraints are quasi-convex, then Kuhn-Tucker necessary conditions are also sufficient for a maximum. One may not in general know whether the objective function in a maximisation problem is pseudo-concave or not (see Chapter 9, pages $559 - 560$ in Miller, 2011). In such a case, one has to identify the set of solutions to the optimisation problem by trying out all the solutions generated by the Kuhn-Tucker necessary conditions.

For any strategy profile (q, b), for any $i \in N$, call $Q_{-i}(j) = Q(j) - q_{ij}$ and $B_{-i}(j) = B(j) - b_{ij}$. Abusing notations, we are going to (re-)write (q, b) simply as, $(q, b) := ((q_i, Q_{-i}), (b_i, B_{-i}))$, and thereby identify the following crucial ratio for any $i \in N$ and any $j = 1, \ldots, l$:

$\Delta_{ij}(q, b) := \Delta_{ij}((q_i, Q_{-i}), (b_i, B_{-i})) = [B_{-i}(j) \{\partial U_i(x_i(q, b))/\partial x_{ij}\}] / [Q_{-i}(j) \{\partial U_i(x_i(q, b))/\partial x_{im}\}]$.

Finally, we identify a class of strategy profiles for a market game that we call *interior* for which the final allocations for all traders are *interior* points.

For any strategy profile (q, b) and any $j \in \{1, \ldots, l\}$, let $\mathcal{Q}_j(q, b) = \{i \in N \mid b_{ij} = 0 \; \& \; q_{ij} = e_{ij}\}$ and $\mathcal{B}_j(q, b) = \{i \in N \mid q_{ij} = 0 \; \& \; \sum_{k=1}^{l} b_{ik} = e_{im}\}$.

If $\mathcal{Q}_j(q, b)$ is is non-empty, that is, if $i \in \mathcal{Q}_j(q, b)$, then individual i sells the whole endowment of good j and does not spend any money to buy back that good (and hence does not consume good j in the final allocation). Similarly, if $\mathcal{B}_j(q, b)$ is non-empty, that is, if $i \in \mathcal{B}_j(q, b)$, then individual i spends the entire money endowment and does not sell good j (and thus consumes at least as much as the endowment of good j).

Definition 4 *A strategy profile (q, b) is said to be an interior profile if $\mathcal{Q}_j(q, b) \cup \mathcal{B}_j(q, b) = \emptyset$ for all $j \in \{1, \ldots, l\}$.*

For any non-interior strategy profile (q, b), there exists $j \in \{1, \ldots, l\}$ such that $\mathcal{Q}_j(q, b) \cup \mathcal{B}_j(q, b) \neq \emptyset$; that is, in a *non-interior* strategy profile, at least one trader is using the whole endowment in at least one good or money.

3 RESULTS

We first state our main result.

Theorem 1 *Consider a market game with a market $\Xi = (N, X, E, U)$, under Assumptions* 1a *and* 1b.

1. *An interior trading profile (q, b) is a trading equilibrium for this game if and only if $p_j^2 = \Delta_{ij}(q, b)$, for all $i \in N$ and all $j \in \{1, \ldots, l\}$.*
2. *A non-interior trading profile (q, b) is a trading equilibrium for this game if and only if we have the following for all $i \in N$:*
   - *(a) $p_j^2 \leq \Delta_{ij}(q, b)$, for any j for which $i \in \mathcal{B}_j(q, b)$,*
   - *(b) $p_j^2 \geq \Delta_{ij}(q, b)$, for any j for which $i \in \mathcal{Q}_j(q, b)$ and*
   - *(c) $p_j^2 = \Delta_{ij}(q, b)$, for any j for which $i \notin [\mathcal{Q}_j(q, b) \cup \mathcal{B}_j(q, b)]$.*

We prove Theorem 1 in several steps. We state a couple of lemmata about bids (b) and offers (q) at a trading equilibrium. The first lemma is related to the offers (q) at equilibrium.

Lemma 1 *At a trading equilibrium, for any $i \in N$ and any $j \in \{1, \ldots, l\}$,*

(a) if $q_{ij} \in (0, e_{ij})$, then $p_j^2 = \Delta_{ij}(q, b)$,

(b) if $q_{ij} = 0$, then $p_j^2 \leq \Delta_{ij}(q, b)$,

(c) if $q_{ij} = e_{ij} > 0$, then $p_j^2 \geq \Delta_{ij}(q, b)$.

Similarly, we provide conditions related to the bids (b) at equilibrium in our next lemma.

Lemma 2 *At a trading equilibrium, for any $i \in N$ and any $j \in \{1, \ldots, l\}$,*

(a) if $\sum_{k=1}^{l} b_{ik} < e_{im}$, then $p_j^2 \geq \Delta_{ij}(q, b)$,

(b) if $b_{ij} > 0$, then $p_j^2 \leq \Delta_{ij}(q, b)$.

The proofs of Lemma 1 and Lemma 2 heavily use the Kuhn-Tucker conditions for the optimisation problem for each individual trader. These proofs have been postponed to the Appendix of this paper. Lemma 1 and Lemma 2 together lead to the following proposition.

Proposition 1 *A trading strategy profile (q, b) is a trading equilibrium if and only if the following conditions hold for any $i \in N$ and any $j \in \{1, \ldots, l\}$:*

(K1) If $q_{ij} \in (0, e_{ij})$, then $p_j^2 = \Delta_{ij}(q, b)$.

(K2) If $q_{ij} = 0$ and $\sum_{k=1}^{l} b_{ik} < e_{im}$, then $p_j^2 = \Delta_{ij}(q, b)$.

(K3) If $q_{ij} = 0$ and $\sum_{k=1}^{l} b_{ik} = e_{im}$, then $p_j^2 \leq \Delta_{ij}(q, b)$.

(K4) If $q_{ij} = e_{ij} > 0$ and $b_{ij} > 0$, then $p_j^2 = \Delta_{ij}(q, b)$.

(K5) If $q_{ij} = e_{ij} > 0$ and $b_{ij} = 0$, then $p_j^2 \geq \Delta_{ij}(q, b)$.

Proposition 1 asserts that given our Assumptions $1a$ and $1b$ and given that the constraints in the individual maximisation problem are linear (hence, quasi-convex), the Kuhn-Tucker conditions are both necessary and sufficient to characterise equilibrium outcomes.

Using Proposition 1, the proof of Theorem 1 is now immediate. Specifically, from $(K1)$, $(K2)$ and $(K4)$, we get the first condition and (c) of the second condition in Theorem 1; from $(K3)$, we get (a) of the second condition and from $(K4)$, we get (b) of the second condition in Theorem 1.[4]

[4] A remark mentioned in Dubey and Rogawski (1990) can be used to provide another analytical proof of the main message of our result (for interior equilibrium profiles). From Lemma 4 in Section 2.1 in Dubey and Shubik (1978) and from the second display on page 301 in Dubey and Rogawski (1990), one can deduce the following: given a collection of strategies $((q_k, b_k)_{k \in N \setminus \{i\}})$ of all agents other than i, the set of vectors of i's final allocations $x_i(q, b)$, is a smooth manifold of co-dimension 1 and its comprehensive hull is convex; moreover, using the normal at any (equilibrium) point and the tangent plane at that point which locally approximates the manifold, one can conclude that at an interior Nash equilibrium, (some) "gradient" of the utility function for any player must be equal to a suitably defined constant.

We now interpret and illustrate our key equilibrium condition, $\Delta_{ij}(q,b) = p_j^2$, obtained for a trading equilibrium with interior strategy profile (the first condition in Theorem 1).

A simplification of this condition is:

$$\frac{\partial U_i(x(q,b))}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial q_{ij}} = -\frac{\partial U_i(x(q,b))}{\partial x_{im}} \frac{\partial x_{im}}{\partial q_{ij}}. \tag{1}$$

Condition (1) states that, in equilibrium, the marginal rise in utility due to a rise in the consumption of x_{ij} caused by an incremental fall in q_{ij} must be equal to the absolute value of the marginal fall in utility due to a fall in the consumption of x_{im} caused by this incremental fall in q_{ij}. Using $p_j \frac{\partial x_{ij}}{\partial b_{ij}} + \frac{\partial x_{ij}}{\partial q_{ij}} = 0$ and $p_j \frac{\partial x_{im}}{\partial b_{ij}} + \frac{\partial x_{im}}{\partial q_{ij}} = 0$, one can rewrite condition (1) as follows:

$$\frac{\partial U_i(x(q,b))}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial b_{ij}} = -\frac{\partial U_i(x(q,b))}{\partial x_{im}} \frac{\partial x_{im}}{\partial b_{ij}}. \tag{2}$$

Like condition (1), condition (2) has a similar interpretation in terms of b_{ij}. The general requirements that $p_j \frac{\partial x_{ij}}{\partial b_{ij}} + \frac{\partial x_{ij}}{\partial q_{ij}} = 0$ and $p_j \frac{\partial x_{im}}{\partial b_{ij}} + \frac{\partial x_{im}}{\partial q_{ij}} = 0$ ensures that we have only one equilibrium condition for interior strategy profiles, that is, $\Delta_{ij} = p_j^2$.

Consider now (a) of the second condition in Theorem 1, which states that, if, in equilibrium, we have $q_{ij} = 0$ and $\sum_{k=1}^{l} b_{ik} = e_{im}$, then either $p_j^2 = \Delta_{ij}(q,b)$ or $p_j^2 < \Delta_{ij}(q,b)$. In case, in equilibrium, we end-up with $p_j^2 < \Delta_{ij}(q,b)$, then it is not possible to equate p_j^2 and $\Delta_{ij}(q,b)$, neither by decreasing q_{ij} (since, $q_{ij} = 0$), nor by increasing b_{ij} (since, $\sum_{k=1}^{l} b_{ik} = e_{im}$).

Similarly, consider (b) of the second condition in Theorem 1, which states that, if, in equilibrium, we have $q_{ij} = e_{ij}$ and $b_{ij} = 0$, then either $p_j^2 = \Delta_{ij}(q,b)$ or $p_j^2 > \Delta_{ij}(q,b)$. Here as well, in case, in equilibrium, we end-up with $p_j^2 > \Delta_{ij}(q,b)$, then it is not possible to equate p_j^2 and $\Delta_{ij}(q,b)$, neither by increasing q_{ij} (since, $q_{ij} = e_{ij}$), nor by decreasing b_{ij} (since, $b_{ij} = 0$).

The equilibrium condition for any interior strategy profile is now quite transparent. It requires that for each traded commodity j, the marginal rate of substitution (MRS) between good j and money for any agent i ($MRS_{x_{ij},x_{im}}$) times the ratio of the *value* of the amount of good j bought and sold by all agents other than i must be a constant across agents. Moreover, this constant is the equilibrium price of good j. Formally, for each commodity $j \in \{1, \ldots, l\}$,

$$p_j^* = \frac{\left(\frac{B_{-i}(j)}{p_j^*}\right) \frac{\partial U_i}{\partial x_{ij}}}{Q_{-i}(j) \frac{\partial U_i}{\partial x_{im}}} = \left(\frac{B_{-i}(j)}{p_j^* Q_{-i}(j)}\right) MRS_{x_{ij},x_{im}}(x^*(b^*,q^*)) \;\forall\; i \in N.$$

We could easily rephrase our main theorem for any buy or sell market game as well, in which for all $i \in N$ and all $j \in \{1, \ldots, l\}$, $q_{ij} \geq 0$, $b_{ij} \geq 0$ and $q_{ij}.b_{ij} = 0$; that is, for all $i \in N$ and all $j \in \{1, \ldots, l\}$, either $q_{ij} = 0$ (and $b_{ij} > 0$: individual i is only buying good j) or $b_{ij} = 0$ (and $q_{ij} > 0$: individual i is only selling good j). Subject to the above restriction, we can define $\mathcal{B}$-profile, $\mathcal{Q}$-profile and the interior strategy profile for any buy or sell market game exactly the same way as we did for a strategic market game earlier. To identify the conditions for a trading equilibrium for such a buy or sell model, we have the following result.

Theorem 2 *Consider a buy or sell strategic market game with a market $\Xi = (N, X, E, U)$, under Assumptions* 1a *and* 1b*.*

1. *An interior trading strategy profile (q, b) for this game is a trading equilibrium if and only if $p_j^2 = \Delta_{ij}(q, b)$, for all $i \in N$ and all $j \in \{1, \ldots, l\}$.*
2. *A non-interior trading strategy profile (q, b) for this game is a trading equilibrium if and only if we have the following for all $i \in N$:*
   - *(a) $p_j^2 \leq \Delta_{ij}(q, b)$, for any j for which $i \in \mathcal{B}_j(q, b)$ and*
   - *(b) $p_j^2 \geq \Delta_{ij}(q, b)$, for any j for which $i \in \mathcal{Q}_j(q, b)$.*

The proof of Theorem 2 is very similar to that Theorem 1 (using Proposition 1) above, and thus has been postponed to the Appendix.

4 EXAMPLES

We illustrate our main result using several examples. We start with Cobb-Douglas utility functions.

Consider a specific market, denoted by Ξ_2^{C-D}, with just two individuals, 1 and 2, and two goods, x and y, with the following Cobb-Douglas preferences and endowments: for individual 1, $U_1(x, y) = A_1 x^{\alpha_1} y^{1-\alpha_1}$, $A_1 > 0$, $\alpha_1 \in (0, 1)$, $\omega^1 = (\omega_x^1, \omega_y^1) >> 0$ and for individual 2, $U_2(x, y) = A_2 x^{\alpha_2} y^{1-\alpha_2}$, $A_2 > 0$, $\alpha_2 \in (0, 1)$, $\omega^2 = (\omega_x^2, \omega_y^2) >> 0$. We have the following characterisation for any interior equilibrium for the corresponding market game.

Proposition 2 *Any trading profile (q, b) with $b_1 - q_1 = q_2 - b_2 = k > 0$ (implying $b_1 > k$ and $q_2 > k$) for a market game with the market Ξ_2^{C-D} constitutes an interior equilibrium with the equilibrium price $p = (b_1 + b_2)/(q_1 + q_2) = 1$ if and only if:*

$$\alpha_1 b_2 \omega_y^1 = (1 - \alpha_1) q_2 \omega_x^1 + k[\alpha_1 b_2 + (1 - \alpha_1) q_2],\ \omega_x^1 \geq q_1,\ \omega_y^1 \geq b_1 \geq k. \quad (3)$$

$$(1 - \alpha_2) q_1 \omega_x^2 = \alpha_2 b_1 \omega_y^2 + k[\alpha_2 b_1 + (1 - \alpha_2) q_1],\ \omega_x^2 \geq q_2 \geq k,\ \omega_y^2 \geq b_2. \quad (4)$$

The proof of Proposition 2 is in the Appendix. The following numerical example provides a simple illustration of the conditions in Proposition 2.

Example 1 *Consider a market game with two individuals* 1 *and* 2 *with the following preferences and endowments: for individual* 1, $U_1(x,y) = A_1 x^{\frac{1}{3}} y^{\frac{2}{3}}$, $A_1 > 0$, $\omega_x^1 = 2t_1$ *and* $\omega_y^1 = 5t_1 + 7$ *with* $t_1 \geq 1$ *and for individual* 2, $U_2(x,y) = A_2 x^{\frac{3}{4}} y^{\frac{1}{4}}$, $A_2 > 0$, $\omega_x^2 = 48t_2 + 14$ *and* $\omega_y^2 = 8t_2$ *with* $t_2 \geq 1$. *Then the trade vector* $(q_1 = 2, b_1 = 4; q_2 = 10, b_2 = 8)$ *is an equilibrium strategy profile with equilibrium price* $p^* = 1$.

One may check the conditions in Proposition 2 to show that the strategy profile $(q_1 = 2, b_1 = 4; q_2 = 10, b_2 = 8)$ with $k = 2$ and $p = 1$ is indeed an equilibrium. Note that this profile is in equilibrium for a range of games with different endowments, determined by the choice of parameters, $t_1 \geq 1$ and $t_2 \geq 1$; in particular, for $t_1 = t_2 = 1$, we have the endowment vectors $(2, 12)$ and $(62, 8)$.

Our next example is from Dickson and Tonin (2021) and it uses a "bilateral oligopoly" (Gabszewicz and Michel, 1997) in which all agents have positive endowments in only one of the two goods.

Example 2 *Consider a market game with two goods and four individuals of who individuals* 1 *and* 2 *are identical with utility functions and endowments, respectively,* $\ln(1+x)+y$ *and* $(3,0)$, *while individuals* 3 *and* 4 *are identical with preferences and endowments given by* $3x - \frac{1}{2}x^2 + y$ *and* $(0,5)$. *The trade vector* $(q_1 = q_2 = \frac{7-\sqrt{17}}{2}; b_3 = b_4 = \sqrt{17} - 3)$ *is an equilibrium strategy profile with price* $p^* = \frac{\sqrt{17}-1}{4}$.

The fact that the strategy profile in Example 2 is an equilibrium has been proved directly from the first principles of maximisation in Dickson and Tonin, 2021 (Example 3 in their paper). We can check these values with our equilibrium condition easily. To see this, let's take the condition just for individual 1 (or 2) in this example. The allocation for individual 1 at equilibrium is given by $x = \frac{\sqrt{17}-1}{2}$ and $y = \sqrt{17} - 3$. At these values, our ratio $\Delta_1(q,b)$ becomes:

$\Delta_1(q,b) = \frac{4(\sqrt{17}-3)(\sqrt{17}-1)}{8(7-\sqrt{17})} = \frac{(\sqrt{17}-1)(\sqrt{17}-3)(\sqrt{17}+3)}{2(7-\sqrt{17})(\sqrt{17}+3)} = \frac{\sqrt{17}-1}{\sqrt{17}+1} = \frac{(\sqrt{17}-1)^2}{16}$,

which is equal to $(\frac{\sqrt{17}-1}{4})^2 = p^2$, confirming our condition obtained in the first part of Theorem 1.

Following examples illustrate non-interior profiles as equilibria, using the second part of our Theorem 1.

Example 3 *Consider a market game with two goods and three individuals with utility functions and endowments, respectively, as below:*

Individual 1: utility function $U_1(x_1, y_1) = \sqrt{x_1 y_1}$ *and endowment* $(e_{1x} = a, e_{1y} = a)$, *where* $a > 1$;

Individual 2: utility function $U_2(x_2, y_2) = 2\sqrt{x_2} + y_2$ *and endowment* $\left(e_{2x} = \frac{1}{3}, e_{2y} = \frac{1}{4}\right)$;

Individual 3: utility function $U_3(x_3, y_3) = x_3 + 2y_3$ *and endowment* $\left(e_{3x} = \frac{1}{4}, e_{3y} > 0\right)$, *where* $e_{3y} > 0$;

The trade vector $((q_1^* = b_1^* = 1); \left(q_2^* = 0, b_2^* = \frac{1}{4}\right); \left(q_3^* = \frac{1}{4}, b_3^* = 0\right))$ *is an equilibrium strategy profile with price* $p^* = 1$. *The corresponding allocation vector is given by:*

$((x_1^* = y_1^* = a); \left(x_2^* = \frac{7}{12}, y_2^* = 0\right); \left(x_3^* = 0, y_3^* = e_{3y} + \frac{1}{4}\right))$.

The fact that the strategy profile in Example 3 is an equilibrium can of course be proved directly from first principles; however, we can make use of our conditions to check for the (non-interior) equilibrium profiles presented in this paper as below:

$$\Delta_1(q^*, b^*) = \left(\frac{b_2^* + b_3^*}{q_3^* + q_3^*}\right)\left(\frac{\frac{\partial U_1}{\partial x_1}}{\frac{\partial U_1}{\partial y_1}}\right) = \left(\frac{(1/4)+0}{0+(1/4)}\right)\left(\frac{y_1^*}{x_1^*}\right) = \left(\frac{(1/4)}{(1/4)}\right)\left(\frac{a}{a}\right) = 1 = p^*.$$

$$\Delta_2(q^*, b^*) = \left(\frac{b_1^* + b_3^*}{q_1^* + q_3^*}\right)\left(\frac{\frac{\partial U_2}{\partial x_2}}{\frac{\partial U_2}{\partial y_2}}\right) = \left(\frac{1+0}{1+(1/4)}\right)\left(\frac{1}{\sqrt{x_2^*}}\right) = \left(\frac{4}{5}\right)\left(\frac{\sqrt{12}}{\sqrt{7}}\right) > 1 = p^*.$$

$$\Delta_3(q^*, b^*) = \left(\frac{b_1^* + b_2^*}{q_1^* + q_2^*}\right)\left(\frac{\frac{\partial U_3}{\partial x_3}}{\frac{\partial U_3}{\partial y_3}}\right) = \left(\frac{1+(1/4)}{1+0}\right)\left(\frac{1}{2}\right) = \left(\frac{5}{8}\right) < 1 = p^*.$$

The above three (in-)equalities confirm that the (non-interior) profile here is indeed an equilibrium.

A careful look at the construction reveals an interesting structure of this profile. At equilibrium, both individuals 2 and 3 are strategically choosing to only buy or sell; individual 2 is only buying the commodity (good 1) while individual 3 is only selling it. Moreover, these two individuals (2 and 3) are bringing their entire endowments of money and commodity, respectively ($\frac{1}{4}$ each), to the trading post. As a consequence, individual 1 faces the total opponents' bid as an *interior* point $(\frac{1}{4}, \frac{1}{4})$ and thereby chooses the best response that also turns out to be an *interior* point $(1, 1)$ at the equilibrium price of 1. As if, individual 1 is serving the role of a middleman, facilitating the trade or the exchange between individuals 2 and 3 (by buying the commodity from individual 3 and selling it to individual 2) at the unit price.

It is easy to conjecture that the equilibrium price for such an equilibrium phenomenon need not necessarily have $p^* = 1$; one may of course obtain the above middleman structure at an equilibrium for a market price other than unity, as shown in the following example of a market game with two goods and three individuals.

Example 4 *Consider and fix strictly positive numbers a_1, a_2, a_3 such that $a_1 > 2a_2$ and $a_2 > 2a_3$. Now, take a market game with utility functions and endowments, respectively, as below:*

Individual 1: utility function $U_1(x_1, y_1) = \sqrt{x_1 y_1}$ and endowment $(e_{1x} = a_1, e_{1y} = 2a_1)$;

Individual 2: utility function $U_2(x_2, y_2) = 4x_2 + y_2$ and endowment $(e_{2x} > 0, e_{2y} = 2a_3)$;

Individual 3: utility function $U_3(x_3, y_3) = x_3 + 3y_3$ and endowment $(e_{3x} = a_3, e_{3y} > 0)$;

The trade vector $((q_1^* = a_2, b_1^* = 2a_2); (q_2^* = 0, b_2^* = 2a_3); (q_3^* = a_3, b_3^* = 0))$ is an equilibrium strategy profile with price $p^* = 2$.

The corresponding allocations are given by:

$(x_1^* = a_1, y_1^* = 2a_1)$, $(x_2^* = e_{2x} + a_3, y_2^* = 0)$, $(x_3^* = 0, y_3^* = 2a_3 + e_{3y})$.

One may check our conditions to confirm that the (non-interior) equilibrium profiles as below:

$$\Delta_1(q^*, b^*) = \left(\frac{b_2^* + b_3^*}{q_3^* + q_3^*}\right)\left(\frac{\frac{\partial U_1}{\partial x_1}}{\frac{\partial U_1}{\partial y_1}}\right) = \left(\frac{2a_3 + 0}{0 + a_3}\right)\left(\frac{y_1^*}{x_1^*}\right) = \left(\frac{2a_3}{a_3}\right)\left(\frac{2a_1}{a_1}\right) = 4 = (2)^2 = (p^*)^2.$$

$$\Delta_2(q^*, b^*) = \left(\frac{b_1^* + b_3^*}{q_1^* + q_3^*}\right)\left(\frac{\frac{\partial U_2}{\partial x_2}}{\frac{\partial U_2}{\partial y_2}}\right) = \left(\frac{2a_2 + 0}{a_2 + a_3}\right)\left(\frac{4}{1}\right) = 4\left(\frac{2a_2}{a_2 + a_3}\right) > 4 = (p^*)^2.$$

$$\Delta_3(q^*, b^*) = \left(\frac{b_1^* + b_2^*}{q_1^* + q_2^*}\right)\left(\frac{\frac{\partial U_3}{\partial x_3}}{\frac{\partial U_3}{\partial y_3}}\right) = \left(\frac{2a_2 + 2a_3}{a_2 + 0}\right)\left(\frac{1}{3}\right) = \left(\frac{2a_2 + 2a_3}{2a_2 + a_2}\right) < 1 < (p^*)^2.$$

The above three (in-)equalities confirm that the (non-interior) profile here is indeed an equilibrium.

5 REMARKS

We, in this paper, identify the conditions for trading equilibria and thereby fully characterise the set of trading equilibria for a simple form of strategic market games. Our main result (Theorem 1) precisely provides an explicit study of these equilibrium conditions. Note that, under the assumptions imposed, the conditions in Theorem 1 are necessary and sufficient for identifying equilibrium profiles, if an equilibrium exists; however, our assumptions do not imply the *existence* of a trading equilibrium. We are not addressing this issue here and we doubt whether anything can be said about this from existing results in the literature along with our assumptions.

One may view our main result as the non-competitive version of the equality between the price ratio and marginal rate of substitution in competitive economies that resembles with the notion of "marginal price" by Okuno, Postlewaite and Roberts (1980). One may also use our approach for wider scope. The characterisation presented in this paper depends crucially on the fact that the price of any commodity can be computed from strategy profiles. Our particular result presented in this paper may not apply to other market models, such as window models, in which the price of a commodity do not necessarily exist in the same way we have here; however, we believe, our technique can be appropriately extended to these models as well.

Dubey and Rogawski (1990) provides comments and remarks that are qualitatively equivalent to our main result, in particular, the statement of the first condition (for interior profiles). We also acknowledge that the remark mentioned in Dubey and Rogawski (1990) can provide a mathematically correct proof of the main theme of our result, by directly using certain known facts from the literature. However, our contribution in this paper is to make such an intuitive result explicit. Our Theorem 1 precisely characterises this constant using an explicit study of equilibrium conditions (resulting from the utility maximisation problem of the traders).

There are clear benefits of having such an equilibrium characterisation. One can derive the equilibrium conditions for specific forms of market games, as we have demonstrated here, using Cobb-Douglas utility functions; one can also construct specific numerical examples of equilibrium profiles even with non-interior strategy profiles.

Our characterisation turns out to be extremely handy to check whether a profile (or an outcome) is an equilibrium or not, using our simple condition. It can immediately help one to identify testable restrictions (see the survey by Carvajal, Ray and Snyder, 2004), if any, for equilibria in such games, using similar analysis by Carvajal, Deb, Fenske and Quah (2013) who provide tests for equilibria in the Cournot model. In a parallel paper (Mitra, Ray and Roy, 2024), we address this particular issue, complementing the results on market games presented in Forges and Minelli (2009).

6 APPENDIX (PROOFS)

We collect the proofs of our results in this section.

Proof of Lemma 1. Given a market game with the market $\Xi = (N, X, E, U)$, if (q, b) is a trading equilibrium, then for each $i \in N$, (q_i, b_i) (given $(q_j, b_j)_{j \in N \setminus \{i\}}$) maximises $U_i(x_i(q, b))$ subject to $q_{ij} \in [0, e_{ij}]$, $b_{ij} \geq 0$ for $j = 1, \ldots, l$, and $\sum_{k=1}^{l} b_{ik} \leq e_{im}$.

In general, we have $\frac{\partial x_{ij}}{\partial q_{ij}} = -(1 - \frac{b_{ij}}{B(j)}) \leq 0$, $p_j \frac{\partial x_{ij}}{\partial b_{ij}} + \frac{\partial x_{ij}}{\partial q_{ij}} = 0$, $\frac{\partial x_{im}}{\partial b_{ij}} = -(1 - \frac{q_{ij}}{Q(j)}) \leq 0$ and $p_j \frac{\partial x_{im}}{\partial b_{ij}} + \frac{\partial x_{im}}{\partial q_{ij}} = 0$.

Define $\lambda_i = (\lambda_{i1}, \ldots, \lambda_{il}) \in \Re^l_+$, $\gamma_i = (\gamma_{i1}, \ldots, \gamma_{il}) \in \Re^l_+$, $\beta_i = (\beta_{i1}, \ldots, \beta_{il}) \in \Re^l_+$ and $\delta_i \in \Re_+$. The Lagrangian function for the optimisation problem of traders $i \in N$ is the following:

$$L(q, b, \lambda_i, \gamma_i, \beta_i, \delta_i) = U_i(x_i(q, b)) + \sum_{k=1}^{l} \lambda_{ik} q_{ik} + \sum_{k=1}^{l} \gamma_{ik} (e_{ik} - q_{ik}) + \sum_{k=1}^{l} \beta_{ik} b_{ik} + \delta_i \left(e_{im} - \sum_{k=1}^{l} b_{ik} \right). \quad (5)$$

The Lagrangian function $L(q, b, \lambda_i, \gamma_i, \beta_i, \delta_i)$ (defined above) is not complete in the sense that it does not incorporate the fact that in equilibrium, the price $p_j := B(j)/Q(j)$, for each j, is positive. This means that along with the Lagrangian function $L(q, b, \lambda_i, \gamma_i, \beta_i, \delta_i)$, we must also add, for each j, a term which is the product of $B(j)$ and an associated Lagrangian multiplier $\kappa_j \geq 0$ and another term consisting of the product of $Q(j)$ and another Lagrangian multiplier $\nu_j \geq 0$. However, we do not need to consider these added multipliers due to the following reason: positivity of price means that $B(i) > 0$ and $Q(j) > 0$ and hence, from complementary slackness conditions it follows that $\kappa_j = \mu_j = 0$ for each j. Hence, after incorporating the fact that, in equilibrium, $p_j > 0$ for all j, the Lagrangian function indeed reduces to $L(q, b, \lambda_i, \gamma_i, \beta_i, \delta_i)$.

From (5) The Kuhn-Tucker conditions are the following:

$$\frac{\partial L}{\partial q_{ij}} = \frac{\partial U_i}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial q_{ij}} + \frac{\partial U_i}{\partial x_{im}} \frac{\partial x_{im}}{\partial q_{ij}} + \lambda_{ij} - \gamma_{ij} \leq 0 \text{ and } q_{ij} \frac{\partial L}{\partial q_{ij}} = 0, \text{ for each good } j, \quad (6)$$

$$\lambda_{ij} \geq 0, q_{ij} \geq 0 \text{ and } \lambda_{ij} q_{ij} = 0, \text{ for each multiplier } \lambda_{ij}, \text{ given } i, \quad (7)$$

$$\gamma_{ij} \geq 0, e_{ij} \geq q_{ij} \text{ and } \gamma_{ij}(e_{ij} - q_{ij}) = 0, \text{ for each multiplier } \gamma_{ij}, \text{ given } i, \quad (8)$$

$$\frac{\partial L}{\partial b_{ij}} = \frac{\partial U_i}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial b_{ij}} + \frac{\partial U_i}{\partial x_{im}} \frac{\partial x_{im}}{\partial b_{ij}} + \beta_{ij} - \delta_i \leq 0 \text{ and } b_{ij} \frac{\partial L}{\partial b_{ij}} = 0, \text{ for each good } j, \quad (9)$$

$$\beta_{ij} \geq 0, b_{ij} \geq 0 \text{ and } \beta_{ij} b_{ij} = 0, \text{ for each multiplier } \beta_{ij} \text{ given } i, \text{ and} \quad (10)$$

$$\delta_i \geq 0, \ e_{im} \geq \sum_{k=1}^{l} b_{ik} \text{ and } \delta_i \left(e_{im} - \sum_{k=1}^{l} b_{ik} \right) = 0 \text{ for the multiplier } \delta_i. \quad (11)$$

Since our Lagrangian function (5) has the terms $\sum_{k=1}^{l} \lambda_{ik} q_{ik}$, with $q_{ik} \geq 0$, the first part of condition (6) becomes:

$\frac{\partial L}{\partial q_{ij}} = \frac{\partial U_i}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial q_{ij}} + \frac{\partial U_i}{\partial x_{im}} \frac{\partial x_{im}}{\partial q_{ij}} + \lambda_{ij} - \gamma_{ij} = 0$, for each good j.

Hence,

$$-\left(\frac{B_{-i}(j)}{B(j)}\right)\frac{\partial U_i}{\partial x_{ij}} + p_j\left(\frac{Q_{-i}(j)}{Q(j)}\right)\frac{\partial U_i}{\partial x_{im}} + \lambda_{ij} - \gamma_{ij} = 0, \tag{12}$$

Similarly, and from the first part of condition (9) it follows that for each good j,

$$\frac{1}{p_j}\left(\frac{B_{-i}(j)}{B(j)}\right)\frac{\partial U_i}{\partial x_{ij}} - \left(\frac{Q_{-i}(j)}{Q(j)}\right)\frac{\partial U_i}{\partial x_{im}} + \beta_{ij} - \delta_i = 0. \tag{13}$$

From (12) it follows that

$$\left(\frac{Q_{-i}(j)}{p_jQ(j)}\right)\frac{\partial U_i}{\partial x_{im}}\left(p_j^2 - \Delta_{ij}(q,b)\right) = -\lambda_{ij} + \gamma_{ij}. \tag{14}$$

If $q_{ij} \in [0, e_{ij})$, then given $\gamma_{ij}(e_{ij} - q_{ij}) = 0$ and $e_{ij} > q_{ij}$ we get $\gamma_{ij} = 0$. Hence, using $\gamma_{ij} = 0$, $[Q_{-i}(j)]/[p_jQ(j)] > 0$ and $\frac{\partial U_i}{\partial x_{im}} > 0$, from (14) we get the following:

(R1) If $q_{ij} \in [0, e_{ij})$, then $p_j^2 \leq \Delta_{ij}(q, b)$.

Pre-multiplying (12) by q_{ij} and using $\lambda_{ij}q_{ij} = 0$ and $q_{ij}\left[\partial L/\partial q_{ij}\right] = 0$, we get,

$$q_{ij}\left[-\left(\frac{B_{-i}(j)}{B(j)}\right)\frac{\partial U_i}{\partial x_{ij}} + p_j\left(\frac{Q_{-i}(j)}{Q(j)}\right)\frac{\partial U_i}{\partial x_{im}} - \gamma_{ij}\right] = 0. \tag{15}$$

If $q_{ij} > 0$, then we have

$$\left(\frac{Q_{-i}(j)}{p_jQ(j)}\right)\frac{\partial U_i}{\partial x_{im}}\left(p_j^2 - \Delta_{ij}(q,b)\right) = \gamma_{ij}. \tag{16}$$

Since $\gamma_{ij} \geq 0$, $Q_{-i}(j)/Q(j) > 0$ and $\frac{\partial U_i}{\partial x_{im}} > 0$, (16) implies $p_j^2 \geq \Delta_{ij}(q, b)$. Hence, we have,

(R2) If $q_{ij} > 0$, then $p_j^2 \geq \Delta_{ij}(q, b)$.

Combining $(R1)$ and $(R2)$, we get Lemma 1. ■

Proof of Lemma 2. We prove Lemma 2 following the proof of Lemma 1.

From (13) it follows that

$$\left(\frac{Q_{-i}(j)}{p_j^2Q(j)}\right)\frac{\partial U_i}{\partial x_{im}}\left(\Delta_{ij}(q,b) - p_j^2\right) = -\beta_{ij} + \delta_i. \tag{17}$$

If $\sum_{k=1}^{l} b_{ik} < e_{im}$, then given $\delta_i(e_{im} - \sum_{k=1}^{l} b_{ik}) = 0$ and $\sum_{k=1}^{l} b_{ik} < e_{im}$ we get $\delta_i = 0$. Hence, using $\delta_i = 0$, $[Q_{-i}(j)]/[p_j^2Q(j)] > 0$ and $\frac{\partial U_i}{\partial x_{im}} > 0$, from (17) we get Lemma 2(a).

Pre-multiplying (13) by b_{ij} and using $\beta_{ij}b_{ij} = 0$ and using $b_{ij}\left[\partial L/\partial b_{ij}\right] = 0$, we get,

$$b_{ij}\left[\left(\frac{1}{p_j}\right)\left(\frac{B_{-i}(j)}{B(j)}\right)\frac{\partial U_i}{\partial x_{ij}} - \left(\frac{Q_{-i}(j)}{Q(j)}\right)\frac{\partial U_i}{\partial x_{im}} - \delta_i\right] = 0. \tag{18}$$

If $b_{ij} > 0$, we have,

$$\left(\frac{1}{p_j}\right)\left(\frac{B_{-i}(j)}{B(j)}\right)\frac{\partial U_i}{\partial x_{ij}} - \left(\frac{Q_{-i}(j)}{Q(j)}\right)\frac{\partial U_i}{\partial x_{im}} - \delta_i = 0. \tag{19}$$

From (19) it follows that if $b_{ij} > 0$, then

$$\Delta_{ij}(q,b) = p_j^2 + \frac{p_j^2 \delta_i}{\left(\frac{Q_{-i}(j)}{Q(j)}\right)\left(\frac{\partial U_i}{\partial x_{im}}\right)}. \tag{20}$$

Since $\delta_i \geq 0$, $Q_{-i}(j)/Q(j) > 0$ and $\frac{\partial U_i}{\partial x_{im}} > 0$, from (20) we get $p_j^2 \leq \Delta_{ij}(q,b)$. Hence, we have established Lemma 2(b). ■

Proof of Proposition 1. The necessary conditions stated in Proposition 1, ($K1$)-($K5$), follow immediately from Lemma 1 and Lemma 2. In particular, from Lemma 1(a), we get ($K1$); from Lemma 1(b) and Lemma 2(a), we get condition ($K2$). Given condition ($K2$), from Lemma 2(b), we get ($K3$). From Lemma 1(c) and Lemma 2(b), we get condition ($K4$). Given condition ($K4$), from Lemma 1(c), we get ($K5$).

The complete proof of Proposition 1 would thus require confirmation of the sufficiency part (that ($K1$)-($K5$) are sufficient for maximisation), using pseudo-concavity of the utility functions. To complete the proof of this proposition, we need to check whether pseudo-concavity is sufficient for the conditions ($K1$)-($K5$).

One can verify that for any $i \in N$, any trading decision $(q_j, b_j)_{j \in N \setminus \{i\}} \in \Re_{++}^{2l \times |N|}$ and any commodity $j \in \{1 \ldots, l\}$, we have the following restrictions on the partial derivatives of the associated function $V_i : \Re_{++}^{2l} \to \Re$.

(i) $\frac{\partial V_i(q_i,b_i)}{\partial b_{ij}} = \left(\frac{B_{-i}(j)}{p_j B(j)}\right) \frac{\partial U_i}{\partial x_{ij}} - \left(\frac{Q_{-i}(j)}{Q(j)}\right) \frac{\partial U_i}{\partial x_{im}} = \left(\frac{Q_{-i}(j)}{p_j^2 Q(j)}\right) \frac{\partial U_i}{\partial x_{im}} [\Delta_{ij} - p_j^2]$ and

(ii) $\frac{\partial V_i(q_i,b_i)}{\partial q_{ij}} = -\left(\frac{B_{-i}(j)}{B(j)}\right) \frac{\partial U_i}{\partial x_{ij}} + \left(\frac{p_j Q_{-i}(j)}{Q(j)}\right) \frac{\partial U_i}{\partial x_{im}} = \left(\frac{Q_{-i}(j)}{p_j Q(j)}\right) \frac{\partial U_i}{\partial x_{im}} [p_j^2 - \Delta_{ij}]$.

We also have the restriction (iii) below (that follows immediately from (i) and (ii)):

(iii) $p_j \frac{\partial V_i(q_i,b_i)}{\partial b_{ij}} + \frac{\partial V_i(q_i,b_i)}{\partial q_{ij}} = 0$.

For any $k \in \{1, \ldots, l\}$ and any pair of trading decisions $((q_i', b_i'), (q_i, b_i)) \in \Re_{++}^{2l}$ for trader i, define $A(k) := [(b_{ik}' - b_{ik}) - p_k(q_{ik}' - q_{ik})]\,[\Delta_{ik} - p_k^2]$.

From (i) and (iii) and by using $\frac{\partial U_i}{\partial x_{im}} > 0$, we get the following reduced condition:

$$\text{If } \sum_{k=1}^{l} A(k) \left(\frac{Q_{-i}(k)}{p_k^2 Q(k)}\right) \leq 0, \text{ then } V_i(q_i', b_i') \leq V_i(q_i, b_i). \tag{21}$$

Now suppose (q_i, b_i) satisfies ($K1$)-($K5$), for all $k \in \{1, \ldots, n\}$.

Consider any $(q_i', b_i')(\neq (q_i, b_i))$ and then consider the sum $\sum_{k=1}^{l} A(k)[Q_{-i}(k)/\{p_k^2 Q(k)\}]$.

Note that if $p_k^2 = \Delta_{ik}$, then $A(k) = 0$.

Now, if for trader i, (q_i, b_i) satisfies the conditions $(K1)$-$(K5)$, then for any $(q'_i, b'_i)(\neq (q_i, b_i))$, we have $\sum_{k=1}^{l} A(k)\left(\frac{Q_{-i}(k)}{p_k^2 Q(k)}\right) \leq 0$, because, $A(k) = [(b'_{ik} - b_{ik}) - p_k(q'_{ik} - q_{ik})]\,[\Delta_{ik} - p_k^2] \leq 0$ and $\left(\frac{Q_{-i}(k)}{p_k^2 Q(k)}\right) > 0$, for all k.

Therefore, we have $V_i(q'_i, b'_i) \leq V_i(q_i, b_i)$ from (21) which implies pseudo-concavity. Hence, the conditions $(K1)$-$(K5)$ are also sufficient for pseudo-concavity. ■

Proof of Theorem 2. If (q, b) is a trading equilibrium for a buy or sell strategic market game, then for each $i \in N$, (q_i, b_i) (given $((q_j, b_j)_{j \in N \setminus \{i\}})$ maximises $U_i(x_i(q, b))$ subject to $q_{ij} \in [0, e_{ij}]$, $b_{ij} \geq 0$ for $j = 1, \ldots, l$, $\sum_{k=1}^{l} b_{ik} \leq e_{im}$ and $b_{ij} q_{ij} = 0$.

Define $\lambda_i = (\lambda_{i1}, \ldots, \lambda_{il}) \in \Re_+^l$, $\gamma_i = (\gamma_{i1}, \ldots, \gamma_{il}) \in \Re_+^l$, $\beta_i = (\beta_{i1}, \ldots, \beta_{il}) \in \Re_+^l$, $\delta_i \in \Re_+$ and $\kappa_i \in \Re_+$.

Given the Lagrangian function $L(b, q, \lambda_i, \gamma_i, \beta_i, \delta_i)$ from (5) for the strategic market game, the Lagrangian function $\bar{L}(.)$ for the optimisation problem of traders $i \in N$ in the buy or sell strategic market game is the following:

$$\bar{L}(q, b, \lambda_i, \gamma_i, \beta_i, \delta_i, \kappa_i) = L(q, b, \lambda_i, \gamma_i, \beta_i, \delta_i) + \kappa_i b_{ij} q_{ij}. \tag{22}$$

The newly added constraint for the Lagrangian function $\bar{L}(q, b, \lambda_i, \gamma_i, \beta_i, \delta_i, \kappa_i)$ is that $\kappa_i \geq 0$, $b_{ij} \geq 0$, $q_{ij} \geq 0$ and $\kappa_i b_{ij} q_{ij} = 0$ for the new multiplier κ_i. The proof of this theorem becomes easy from the following observations.

1. $\frac{\partial \bar{L}}{\partial q_{ij}} = \frac{\partial L}{\partial q_{ij}} + \kappa_i b_{ij}$.
2. $q_{ij} \frac{\partial \bar{L}}{\partial q_{ij}} = q_{ij} \frac{\partial L}{\partial q_{ij}} + \kappa_i b_{ij} q_{ij} = q_{ij} \frac{\partial L}{\partial q_{ij}} = 0$ (since $\kappa_i b_{ij} q_{ij} = 0$).
3. $\frac{\partial \bar{L}}{\partial b_{ij}} = \frac{\partial L}{\partial b_{ij}} + \kappa_i q_{ij}$.
4. $b_{ij} \frac{\partial \bar{L}}{\partial b_{ij}} = b_{ij} \frac{\partial L}{\partial b_{ij}} + \kappa_i b_{ij} q_{ij} = b_{ij} \frac{\partial L}{\partial b_{ij}} = 0$.

Hence, by using arguments similar to the ones used earlier, one can also prove lemmata corresponding to Lemma 1 and Lemma 2 for this buy or sell strategic market game. Specifically, one can show the following:

(a) If $q_{ij} \in (0, e_{ij})$, then $p_j^2 = \Delta_{ij}(b, q)$.

(b) If $q_{ij} = 0$, then $p_j^2 \leq \Delta_{ij}(b, q)$.

(c) If $q_{ij} = e_{ij}$, then $p_j^2 \geq \Delta_{ij}(b, q)$.

(d) If $\sum_{k=1}^{l} b_{ik} < e_{im}$, then $p_j^2 \geq \Delta_{ij}(b, q)$.

From (a)-(d), it follows that for any $i \in N$ and any $j = 1, \ldots, l$, the following conditions hold:

(k1) If $q_{ij} \in (0, e_{ij})$ and $b_{ij} = 0$, then $p_j^2 = \Delta_{ij}(b, q)$.

(k2) If $q_{ij} = 0$ and $\sum_{k=1}^{l} b_{ik} < e_{im}$, then $p_j^2 = \Delta_{ij}(b, q)$.

(k3) If $q_{ij} = 0$ and $\sum_{k=1}^{l} b_{ik} = e_{im}$, then $p_j^2 \leq \Delta_{ij}(b, q)$.

(k4) If $q_{ij} = e_{ij}$ and $b_{ij} = 0$, then $p_j^2 \geq \Delta_{ij}(b, q)$.

Specifically, from (a) we get condition $(k1)$ using the buy or sell restriction. From (b) and (d) we get condition $(k2)$. Given condition $(k2)$, from (b) we also get condition $(k3)$. From (c), we get condition $(k4)$ using the buy or sell restriction. Finally, from $(k1)$ and $(k2)$, we get $(kt1)$ and $(kt2)(c)$ in the statement of Theorem 2; we get $(kt2)(a)$ from $(k3)$ and, from $(k4)$, we get $(kt2)(b)$.

Given our assumption that the utility function $U_i(x_i(q, b))$ of each trader $i \in N$ is continuously differentiable and pseudo-concave and given that the constraints are quasi-convex, the Kuhn-Tucker conditions are both necessary and sufficient to characterise the equilibrium outcomes. The arguments for checking why pseudo-concavity is sufficient for the Kuhn-Tucker conditions $(k1)$-$(k4)$ above is similar to the arguments used in the proof of Theorem 1 (and Proposition 1) and hence is omitted. ■

Proof of Proposition 2. In a market game with the market Ξ_2^{C-D}, the equilibrium consumptions of the two goods x and y for the two individuals are:

- $x_1 = \omega_x^1 + b_1/p - q_1$ so that $x_1 = \omega_x^1 + b_1 - q_1 = \omega_x^1 + k$,
- $y_1 = \omega_y^1 - b_1 + pq_1$ so that $y_1 = \omega_y^1 - b_1 + q_1 = \omega_y^1 - k$,
- $x_2 = \omega_x^2 + b_2/p - q_2$ so that $x_2 = \omega_x^2 + b_2 - q_2 = \omega_x^2 - k$, and
- $y_2 = \omega_y^2 - b_2 + pq_2$ so that $y_2 = \omega_y^2 - b_2 + q_2 = \omega_y^2 + k$.

Given the Cobb-Douglas utility function, for any $i \in \{1, 2\}$,
$MRS_{x,y}^i = \frac{\partial U_i(x_i, y_i)}{\partial x_i} / \frac{\partial U_i(x_i, y_i)}{\partial y_i} = (\alpha_i/(1-\alpha_i))(y_i/x_i)$.
Equilibrium condition of the market game is:
$\left(\frac{b_2}{q_2}\right) MRS_{x,y}^1 = \left(\frac{b_1}{q_1}\right) MRS_{x,y}^2 = p^2 = 1$.
From the equilibrium condition, we have the following implications.

For individual 1, we have,
$\left(\frac{b_2}{q_2}\right)\left(\frac{\alpha_1}{1-\alpha_1}\right)\left(\frac{\omega_y^1 - k}{\omega_x^1 + k}\right) = 1 \Rightarrow \alpha_1 b_2(\omega_y^1 - k) = (1-\alpha_1) q_2 (\omega_x^1 + k)$
$\Rightarrow \alpha_1 b_2(\omega_y^1 - k) = (1-\alpha_1)(b_2 + k)(\omega_x^1 + k)$
$\Rightarrow \alpha_1 b_2 \omega_y^1 = (1-\alpha_1)(b_2 + k)\omega_x^1 + k[(1-\alpha_1)(b_2 + k) + \alpha_1 b_2]$
$\Rightarrow \alpha_1 b_2 \omega_y^1 = (1-\alpha_1)(b_2 + k)\omega_x^1 + k[b_2 + (1-\alpha_1)k]$
$\Rightarrow \alpha_1 b_2 \omega_y^1 = (1-\alpha_1) q_2 \omega_x^1 + k[\alpha_1 b_2 + (1-\alpha_1) q_2]$.

Therefore, for individual 1, we have (3) as a restriction for an interior equilibrium.

Similarly, for individual 2, we have

$\left(\frac{b_1}{q_1}\right)\left(\frac{\alpha_2}{1-\alpha_2}\right)\left(\frac{\omega_y^2+k}{\omega_x^2-k}\right) = 1 \Rightarrow \alpha_2 b_1(\omega_y^2 + k) = (1-\alpha_2)q_1(\omega_x^2 - k)$

$\Rightarrow \alpha_2 b_1(\omega_y^2 + k) = (1-\alpha_2)(b_1 - k)(\omega_x^2 - k)$

$\Rightarrow (1-\alpha_2)(b_1 - k)\omega_x^2 = \alpha_2 b_1 \omega_y^2 + k[(1-\alpha_2)(b_1 - k) + \alpha_2 b_1]$

$\Rightarrow (1-\alpha_2)(b_1 - k)\omega_x^2 = \alpha_2 b_1 \omega_y^2 + +k[b_1 - (1-\alpha_2)k]$

$\Rightarrow (1-\alpha_2)q_1\omega_x^2 = \alpha_2 b_1 \omega_y^2 + +k[\alpha_2 b_1 + (1-\alpha_2)q_1]$.

Thus, for individual 2, (4) is a restriction for an interior equilibrium. ■

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