THE SPECTRAL FORM OF THE FUNCTIONAL MODEL FOR MAXIMALLY DISSIPATIVE OPERATORS: A LAGRANGE IDENTITY APPROACH

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Abstract. This paper is a contribution to the theory of functional models. In particular, it develops the so-called spectral form of the functional model where the selfadjoint dilation of the operator is represented as the operator of multiplication by an independent variable in some auxiliary vector-valued function space. By using a Lagrange identity, in our version the connection between this auxiliary space and the original Hilbert space will be explicit. A simple example is provided.

Dedicated to the memory of Professor B.S.Pavlov (1936-2016), outstanding mathematician and personality, who made great contributions to the theory of functional models.

1. Introduction

The spectral and scattering properties of non-selfadjoint problems have become a subject of much mathematical and physical interest in recent years. Mathematically these problems pose a challenge, as apart from exceptional cases, the well-developed methods used to examine the spectrum of selfadjoint problems are not applicable. One of the tools to attack non-selfadjoint problems is functional models.

Functional models were introduced by Sz.-Nagy and Foias (see [34, 23] and references therein) to analyse the structure of contractions and relations between an operator, its spectrum and its characteristic function, and independently by de Branges [9]. These works built on the earlier papers [17, 18] of Livšic for the triangular model. The ideas of Sz.-Nagy-Foias inspired great interest in the Soviet school. In particular, Pavlov [27] introduced a very useful symmetric version of the Sz.-Nagy-Foias model; Nikolski and Vasyunin [26] formulated a coordinate-free model; and Tikhonov [35] re-developed the Sz.-Nagy-Foias model in the Nikolski-Vasyunin framework. As pointed out in [25], the coordinate-free model has the advantage of leaving the choice of spectral representation of the dilation to the user in the context of particular applications.

Pavlov was always clear that his symmetric model should be used to solve real physical problems, and indeed his work on quantum switches [30] and Naboko and Romanov's work on time asymptotics for the Boltzmann operator [21] have relied heavily on it. Pavlov himself developed a simplified version of his functional model for Schrödinger operators [29]. Vasyunin [36] and Naboko [20] also introduced simplifications of Pavlov's original model; in particular, Naboko's model for additive perturbations is directly based on Pavlov's model in [29].

A drawback of many functional models is that their constructions require objects which may be difficult to describe explicitly, such as operator square roots, making it hard to apply the results to specific examples. In this context, Naboko's approach had at least two significant advantages: firstly, it gave explicit formulae for all expressions arising in the model, in terms of objects which arise naturally in the description of the original operator (e.g. the imaginary part of the potential of a Schrödinger operator); secondly, unlike approaches based on Cayley transformation to contractions, it also allowed the study of non-dissipative operators. Ryzhov's functional model for the case when the perturbation is only in the boundary conditions [31] enjoys similar advantages, and inspired the work of Cherednichenko, Kiselev and Silva [8] on transmission problems for PDEs.

Our aim in recent work has been to develop a functional model for the case when the non-selfadjointness arises both in additive terms and in the boundary conditions. In a first paper, [6], we considered a general maximally dissipative operator and developed the so-called 'translation form' of the functional model. We presented a construction of the selfadjoint dilation based on the Lagrange identity in the spirit of operator colligations [3, 4]. The flexibility of the choice of the Γ-operators in the Lagrange identity means that these can be chosen so that expressions arising in the dilation are given explicitly in terms of physical parameters (coefficients, boundary conditions and Titchmarsh-Weyl M-function) of the maximally dissipative operator. The presentation of such explicit expressions for the spectral form of the functional model is arguably the main contribution of the present paper.

In the spectral form of the functional model, the dilation is very simple, being the operator of multiplication by an independent variable in some auxiliary vector-valued function space; in our version the connection between this auxiliary space and the original Hilbert space will be explicit (Theorem 4.8). Using the operator colligation setting for our problem, we also obtain an explicit expression for the completely non-selfadjoint part of the operator (Theorem 3.3) and an operator analytic proof of the famous result by Sz.-Nagy-Foias on the pure absolute continuity of the spectrum of the minimal selfadjoint dilation (Theorem 3.4). In the final section of the paper, we consider an example of a limit circle Sturm-Liouville operator.

Throughout the paper we use the following notation: For a complex number $z \in \mathbb{C}$, let $\Im z$ denote its imaginary part and $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$, $\mathbb{C}_- = \{z \in \mathbb{C} : \Im z < 0\}$. The positive half-line will be denoted by $\mathbb{R}_+$. For an operator A in a Hilbert space H, we denote its range by $\operatorname{Ran} A$, its kernel by $\ker(A)$, its adjoint by A^* and its spectrum and resolvent set by $\sigma(A)$ and $\rho(A)$, respectively. The inner product on H will be linear in the first component. The set of bounded linear operators in H is denoted $B(H)$. The Lebesgue space of square-integrable functions on the half-line is $L_2(\mathbb{R}_+)$, while $H^s(\mathbb{R}_+)$ denotes the usual Sobolev space of order s; $H^s_0(\mathbb{R}_+)$ denotes the closure in H^s-norm of the smooth, compactly supported functions on the half-line.

2. Preliminaries

This section reviews some classical results on dissipative operators - for more on the subject, we refer the reader to [13, 16, 34] - and results from our previous paper [6], where most of the proofs can be found. We start with some basic definitions.

Definition 2.1. *A densely defined linear operator A with domain $D(A)$ in a Hilbert space H is called dissipative if $\Im \langle Ah, h\rangle \geq 0$ for all $h \in D(A)$. A is called anti-dissipative if $(-A)$ is dissipative. Dissipative operators which have no non-trivial dissipative extensions are called maximally dissipative operators (MDO).*

The Cayley transform, an operator version of the Möbius transform, defined by

$$T = I - 2i(A+i)^{-1} = (A - iI)(A + iI)^{-1} \tag{2.1}$$

is a bijective map between the class of MDOs and contractions that do not have 1 as an eigenvalue. Thus many results for MDOs can be obtained from studying contractions, and vice versa.

We next introduce reducing subspaces and the concept of complete non-selfadjointness.

Definition 2.2. *Let A be an operator on a Hilbert space H, $H_1 \subseteq H$ a subspace and P_{H_1} the orthogonal projection of H onto H_1. The subspace H_1 is invariant with respect to A if $P_{H_1}D(A) \subseteq D(A)$ and $AP_{H_1}h \in H_1$ for all $h \in D(A)$. It is a reducing subspace for A if both H_1 and $H \ominus H_1$ are invariant with respect to A.*

Definition 2.3. *Let A be an MDO. A is completely non-selfadjoint (cns) if there exists no reducing subspace $H_1 \subseteq H$ such that $A|_{H_1}$ is selfadjoint.*

The Langer decomposition [6, 15, 20] gives an explicit formula for the completely non-selfadjoint part of the operator. In the case of relatively bounded imaginary part the formula is simple. For more general situations it involves operators which are regularisations of the (possibly non-existing) imaginary part of the operator. In our setting, we will determine a more explicit formula for the completely non-selfadjoint part of an MDO in Theorem 3.3.

Proposition 2.4 (Sz.-Nagy)**.** *For any MDO A on a Hilbert space H there exists a selfadjoint operator $\mathcal{L}$ on a Hilbert space $\mathcal{H} \supseteq H$ such that*

$$e^{itA} = P_H e^{it\mathcal{L}}|_H,\ t \geq 0 \quad \text{or equivalently} \quad (A-\lambda)^{-1} = P_H(\mathcal{L}-\lambda)^{-1}|_H, \quad \lambda \in \mathbb{C}_-.$$

Moreover, $(A^* - \lambda)^{-1} = P_H(\mathcal{L}-\lambda)^{-1}|_H$ for $\lambda \in \mathbb{C}_+$. The operator $\mathcal{L}$ is called a selfadjoint dilation of A.

The selfadjoint dilation is a very useful tool in studying an MDO A. By decomposing A into its selfadjoint and completely non-selfadjoint parts, it is sufficient to construct a selfadjoint dilation of the completely non-selfadjoint part to obtain a selfadjoint dilation for A. The next lemma, whose proof illustrates the use of the selfadjoint dilation, will be needed later on.

Lemma 2.5. *Let A be a maximally dissipative operator in Hilbert space. Then for any $k \in \mathbb{R}$*

$$i\tau(A + k + i\tau)^{-1} \overset{s}{\to} I$$

in the strong operator topology as $\tau \to +\infty$.

Proof. Since $(A + k)$ is also a maximally dissipative operator, the scalar operator kI can be absorbed in A. So without loss of generality $k = 0$. Introducing the selfadjoint dilation $\mathcal{L}$ on Hilbert space $\mathcal{H} \supset H$ such that

$$(A + \lambda)^{-1} = P_H(\mathcal{L} + \lambda)^{-1}|_H \quad \text{for all} \ \ \lambda \in \ \mathbb{C}_+,$$

where P_H is the orthogonal projection of $\mathcal{H}$ onto H, we have

$$(i\tau)(A+i\tau)^{-1} = P_H(i\tau)(\mathcal{L}+i\tau)^{-1} \quad \text{for all} \;\; \tau > 0.$$

Therefore, for any $h \in H$

$$\begin{aligned} \left\| i\tau(A+i\tau)^{-1}h - h \right\|_H^2 &= \left\| P_H[i\tau(\mathcal{L}+i\tau)^{-1} - I]h \right\|_H^2 \\ &\leq \left\| (\mathcal{L}(\mathcal{L}+i\tau)^{-1}h \right\|_{\mathcal{H}}^2 = \int_{\mathbb{R}} \left| \frac{t}{t+i\tau} \right|^2 d(E_t h, h)_{\mathcal{H}} \end{aligned}$$

with E_t the spectral resolution of $\mathcal{L}$ on $\mathcal{H}$. Using the Lebesgue dominated convergence theorem and the trivial facts that $|\frac{t}{t+i\tau}|^2 \leq 1$ and $\frac{t}{t+i\tau} \to 0$ as $\tau \to \infty$, for all $t \in \mathbb{R}$, we have $\left\| i\tau(A+i\tau)^{-1}h - h \right\|_H \to 0$ as $\tau \to \infty$. $\square$

We now discuss an abstract framework for a maximally dissipative operator and its anti-dissipative adjoint which allows us to introduce Γ-operators associated with the imaginary part of the operator A. For the case of bounded operators this goes back to the work of the Odessa school on operator colligations [3], see also [33]. The following is [6, Lemma 3.1].

Lemma 2.6. *Let A be a maximally dissipative operator on a Hilbert space H. Then there exists a Hilbert space E and an operator $\Gamma : D(A) \to E$ which is bounded in the graph norm of A, has dense range in E and such that for all $u, v \in D(A)$ we have*

$$\langle Au, v \rangle_H - \langle u, Av \rangle_H = i \langle \Gamma u, \Gamma v \rangle_E . \tag{2.2}$$

Similarly, there exists a Hilbert space E_* and an operator $\Gamma_* : D(A^*) \to E_*$ which is bounded in the graph norm, has dense range in E_* and such that for all $u, v \in D(A^*)$ we have

$$\langle A^* u, v \rangle_H - \langle u, A^* v \rangle_H = -i \langle \Gamma_* u, \Gamma_* v \rangle_{E_*} . \tag{2.3}$$

We note that, in general, the dimensions of E and E_* need not coincide. The operator Γ is determined up to unitary transformations, see [6, Lemma 3.3] and [33]. In particular, choosing $\Gamma = Q(A+i)^{-1}$ and $\Gamma_* = Q_*(A^*-i)^{-1}$, where $Q = (I - T^*T)^{1/2}$, $Q_* = (I - TT^*)^{1/2}$, and $T = (A-i)(A+i)^{-1}$ is the Cayley transform of A, gives a mapping between the results presented here and those in [26]. However in many concrete applications - see, e.g., Section 5 below - (2.2) and (2.3) effectively reduce to integrations by parts, with much simpler canonical choices for Γ and Γ_*.

We require two abstract Green identities [6, Lemma 3.5].

Lemma 2.7. *For $\lambda \in \mathbb{C}_+$ and $\mu \in \mathbb{C}_-$ we have*

$$(A+\lambda)^{-1} - (A^*+\mu)^{-1} + (\lambda - \mu)(A^*+\mu)^{-1}(A+\lambda)^{-1} = -i(\Gamma(A+\overline{\mu})^{-1})^*(\Gamma(A+\lambda)^{-1}) \tag{2.4}$$

and

$$(A+\lambda)^{-1} - (A^*+\mu)^{-1} + (\lambda - \mu)(A+\lambda)^{-1}(A^*+\mu)^{-1} = -i(\Gamma_*(A^*+\overline{\lambda})^{-1})^*(\Gamma_*(A^*+\mu)^{-1}) \tag{2.5}$$

A key ingredient in all functional models is a *characteristic function*, see [19]. We next introduce the characteristic function which we first presented in [6, Corollary 4.2 & Lemma 4.3].

Lemma 2.8. *Let $z \in \mathbb{C}_+$. There exists a unique contraction $S(z) : E \to E_*$, analytic in the upper half-plane, such that*

$$S(z)\Gamma u = \Gamma_*(A^* - z)^{-1}(A - z)u \text{ for all } u \in D(A). \tag{2.6}$$

Correspondingly, for $z \in \mathbb{C}_-$ there exists a contraction $S_*(z) : E_* \to E$, analytic in the lower half-plane, such that

$$S_*(z)\Gamma_* u = \Gamma(A - z)^{-1}(A^* - z)u. \tag{2.7}$$

The characteristic function $S(z)$ can be extended on $\mathrm{Ran}\,(\Gamma)$ by (2.6) to all $z \in \rho(A^*)$ and $S_*(z)$ can be extended on $\mathrm{Ran}\,(\Gamma_*)$ by (2.7) to all $z \in \rho(A)$ ([6, Lemma 4.4]). The operator-valued function $S(\cdot)$, defined for $z \in \rho(A^*)$ by (2.6) on $\mathrm{Ran}\,(\Gamma)$ and extended to E by continuity is called the *Štraus characteristic function* of the operator A.

Finally, we gather some useful facts about the characteristic function in a lemma. The proofs can be found in [6, Section 4].

Lemma 2.9. (1) *For $\mu, \widetilde{\mu} \in \rho(A^*)$, we have the following identity:*

$$S(\mu) - S(\widetilde{\mu}) = i(\mu - \widetilde{\mu}) \left(\Gamma_*(A^* - \mu)^{-1} \right) \left(\Gamma(A - \overline{\widetilde{\mu}})^{-1} \right)^* \text{ on } E. \tag{2.8}$$

(2) *$S(z) = S_*^*(\overline{z})$ for $z \in \rho(A^*)$.*
(3) *$S(z)S_*(z) = I_{E_*}$ and $S_*(z)S(z) = I_E$ whenever $z \in \rho(A) \cap \rho(A^*)$.*

(4) *$S(z)$ is unitary for $z \in \mathbb{R} \cap \rho(A)$.*
(5) *If $\sigma(A)$ does not cover the whole upper half plane (or, equivalently, if $\rho(A) \cap \rho(A^*) \neq \emptyset$), then* $\dim E = \dim E_*$.
(6) *For $w, z \in \mathbb{C}_+$, we have*

$$\frac{1}{\bar{w} - z}\left(I_E - S^*(w)S(z)\right) = i\left(\Gamma(A - \bar{w})^{-1}\right)\left(\Gamma(A - \bar{z})^{-1}\right)^* \tag{2.9}$$

and for $w, z \in \mathbb{C}_-$, we have

$$\frac{1}{\bar{w} - z}\left(I_{E_*} - S_*^*(w)S_*(z)\right) = -i\left(\Gamma_*(A^* - \bar{w})^{-1}\right)\left(\Gamma_*(A^* - \bar{z})^{-1}\right)^*. \tag{2.10}$$

(7) *For any $u \in H$, $\mu, z \in \mathbb{C}_-$ we have*

$$\left(\Gamma_*(A^* - \bar{\mu})^{-1}\right)^* S(\bar{z}) = \left[I - (\bar{z} - \mu)(A - \mu)^{-1}\right]\left(\Gamma(A - z)^{-1}\right)^* \tag{2.11}$$

and

$$\left(\Gamma(A - \mu)^{-1}\right)^* S_*(z) = \left[I - (z - \bar{\mu})(A^* - \bar{\mu})^{-1}\right]\left(\Gamma_*(A^* - \bar{z})^{-1}\right)^*. \tag{2.12}$$

3. Absolute continuity of the spectrum

We start with the following important fact to be used frequently over the paper. It is a generalisation of [20, Theorem 1] for the case of general maximally dissipative operators and is in the spirit of operator colligations.

Theorem 3.1. *Let A be a maximally dissipative positive operator in H. Then*

$$\sup_{\varepsilon>0} \int_{\mathbb{R}} \left\|\Gamma(A - k + i\varepsilon)^{-1}u\right\|_E^2 dk \leq 2\pi \|u\|^2$$

and

$$\sup_{\varepsilon>0} \int_{\mathbb{R}} \left\|\Gamma_*(A^* - k - i\varepsilon)^{-1}u\right\|_{E_*}^2 dk \leq 2\pi \|u\|^2.$$

In other words for any vector $u \in H$ the vector valued function $\Gamma(A - z)^{-1}u \in E$ for $\Im z < 0$ belongs to the vector-valued Hardy class $H_2^-(E)$ (see, e.g. [34]) of E-valued analytic functions in the lower half plane. Similarly $\Gamma_*(A^* - z)^{-1}u \in H_2^+(E_*)$, the Hardy class of E_*-valued analytic functions on $\mathbb{C}_+$.

Proof. According to (2.2)

$$\begin{aligned}
&\int_{\mathbb{R}} \left\|(\Gamma(A - k + i\varepsilon)^{-1}u\right\|_E^2 dk = \int_{\mathbb{R}} (\Gamma(A - k + i\varepsilon)^{-1}u, \Gamma(A - k + i\varepsilon)^{-1}u)_E dk \\
&= \frac{1}{i}\int_{\mathbb{R}} [(A(A - k + i\varepsilon)^{-1}u, (A - k + i\varepsilon)^{-1}u)_H - ((A - k + i\varepsilon)^{-1}u, A(A - k + i\varepsilon)^{-1}u)_H] dk \\
&= \frac{1}{i}\int_{\mathbb{R}} \{(u + (k - i\varepsilon)(A - k + i\varepsilon)^{-1}u, (A - k + i\varepsilon)^{-1}u)_H - ((A - k + i\varepsilon)^{-1}u, u + (k - i\varepsilon)(A - k + i\varepsilon)^{-1}))_H\} dk \\
&= \frac{1}{i}\int_{\mathbb{R}} \{(u, (A - k + i\varepsilon)^{-1}u)_H - ((A - k + i\varepsilon)^{-1}u, u)_H - 2i\varepsilon((A - k + i\varepsilon)^{-1}u, (A - k + i\varepsilon)^{-1})_H\} dk \\
&= \int_{\mathbb{R}} \{2\Im(u, (A - k + i\varepsilon)^{-1}u)_H - 2\varepsilon\left\|(A - k + i\varepsilon)^{-1}u\right\|_H^2\} dk \leq 2\int_{\mathbb{R}} \Im(u, (A - k + i\varepsilon)^{-1}u)_H dk.
\end{aligned}$$

Since, by Proposition 2.4, $(u, (A - k + i\varepsilon)^{-1}u)_H = (u, P_H(\mathcal{L} - k + i\varepsilon)^{-1}u)_{\mathcal{H}}$ for the selfadjoint dilation $\mathcal{L}$ of A in the Hilbert space $\mathcal{H} \supseteq H$, we have

$$\begin{aligned}
\int_{\mathbb{R}} \left\|\Gamma(A - k + i\varepsilon)^{-1}u\right\|_E^2 dk &\leq 2\int_{\mathbb{R}} \Im((\mathcal{L} - k - i\varepsilon)^{-1}u, u)_{\mathcal{H}} dk \\
&= 2\int_{\mathbb{R}} \Im \int_{\mathbb{R}} \frac{1}{\lambda - k - i\varepsilon} d(E_\lambda u, u)_{\mathcal{H}} dk = 2\int_{\mathbb{R}} dk \int_{\mathbb{R}} \frac{\varepsilon}{(\lambda - k)^2 + \varepsilon^2} d(E_\lambda u, u)_{\mathcal{H}},
\end{aligned}$$

where E_λ is the spectral resolution of $\mathcal{L}$. Due to positivity of the function and the measure one may use Fubini's Theorem rewriting the last double integral as

$$\int_{\mathbb{R}} d(E_\lambda u, u)_{\mathcal{H}} \left(\int_{\mathbb{R}} dk \frac{2\varepsilon}{(\lambda - k)^2 + \varepsilon^2}\right) = \int_{\mathbb{R}} d(E_\lambda u, u)_{\mathcal{H}}(2\pi) = 2\pi \|u\|_{\mathcal{H}}^2 = 2\pi \|u\|_H^2$$

since in the integral over variables k does not depend on λ (by the shift of variables $k \to k - \lambda$) and is equal to 2π.

The second inequality in the Theorem 3.1 admits exactly the same proof. □

The proof of the theorem includes two identities:

Corollary 3.2. *We have*

$$\text{(i)} \int_{\mathbb{R}} \left\|\Gamma(A-k+i\varepsilon)^{-1}u\right\|_E^2 dk = 2\pi \|u\|_H^2 - 2\varepsilon \int_{\mathbb{R}} \left\|(A-k+i\varepsilon)^{-1}u\right\|_H^2 dk \text{ for all } \varepsilon > 0$$

and

$$\text{(ii)} \int_{\mathbb{R}} \left\|\Gamma_*(A^*-k-i\varepsilon)^{-1}u\right\|_{E_*} dk = 2\pi \|u\|_H^2 - 2\varepsilon \int_{\mathbb{R}} \left\|(A^*-k-i\varepsilon)^{-1}u\right\|_H^2 dk \text{ for all } \varepsilon > 0.$$

In particular, we have useful bounds for an arbitrary maximally dissipative operator A:

$$\sup_{\varepsilon>0} \int_{\mathbb{R}} \varepsilon \left\|(A-k+i\varepsilon)^{-1}u\right\|_H^2 dk \leq \pi \|u\|_H^2$$

and

$$\sup_{\varepsilon>0} \int_{\mathbb{R}} \varepsilon \left\|(A^*-k-i\varepsilon)^{-1}u\right\|_H^2 dk \leq \pi \|u\|_H^2 .$$

The next result is a formulation of the Langer decomposition [15] in a form which will be convenient for our later applications. Equivalent representations of the completely non-selfadjoint subspace may be deduced from expressions from completely-nonunitary parts of contractions, see e.g. [26, Section 6].

Theorem 3.3. *The reducing subspace of the maximally dissipative operator A corresponding to its completely non-selfadjoint part in the Langer decomposition is*

$$H_{cns} = \bigvee \left\{ \bigvee_{\Im\lambda>0} (\Gamma(A+\lambda)^{-1})^* E, \bigvee_{\Im\mu<0} (\Gamma_*(A^*+\mu)^{-1})^* E_*) \right\} \tag{3.1}$$

and its selfadjoint part $H_{sa} := H \ominus H_{cns}$.

Proof. Denote the right hand side of (3.1) by $\mathcal{M}_A$. Our proof consists of two inclusions, identifying H_{sa} with the orthogonal complement of $\mathcal{M}_A$.

We first show: $H_{sa} \subseteq \mathcal{M}_A^{\perp}$. Let $h \in H_{sa}$ then using (2.4), the first abstract Greens function identity, one gets for $\lambda = \overline{\mu} \in \mathbb{C}_+$

$$-i(\Gamma(A+\lambda)^{-1})^*(\Gamma(A+\lambda)^{-1})h = (A+\lambda)^{-1}h - (A^*+\overline{\lambda})^{-1}h + (\lambda-\overline{\lambda})(A^*+\overline{\lambda})^{-1}(A+\lambda)^{-1}h = 0$$

by the Hilbert identity for the selfadjoint operator $A|_{H_{sa}} = A^*|_{H_{sa}}$. Therefore

$$\left\|\Gamma(A+\lambda)^{-1}h\right\|_E^2 = ((\Gamma(A+\lambda)^{-1})^*(\Gamma(A+\lambda))^{-1}h, h)_H = 0,$$

i.e. $\Gamma(A+\lambda)^{-1}h \equiv 0$ for all $\lambda \in \mathbb{C}_+$.

Similarly, using the second identity of (2.5), we have

$$\Gamma_*(A^*+\mu)^{-1}h \equiv 0$$

for $\mu \in \mathbb{C}_-$. The last two conditions on h mean that $h \perp \mathcal{M}_A$, or $h \in \mathcal{M}_A^{\perp}$ proving the inclusion $H_{sa} \subseteq {\mathcal{M}_{\mathcal{A}}}^{\perp}$.

It remains to show: $H_{sa} \supseteq {\mathcal{M}_{\mathcal{A}}}^{\perp}$. Let the vector $h \neq 0$ and $h \in \mathcal{M}_A^{\perp}$, i.e. $\Gamma(A+\lambda)^{-1}h = 0$ and $\Gamma_*(A+\mu)^{-1}h = 0$ for any $\lambda \in \mathbb{C}_+$ and $\mu \in \mathbb{C}_-$. Consider a reducing subspace for A, η_h in H generated by the vector h:

$$\eta_h = \bigvee\{(\bigvee_{\Im\lambda>0} (A+\lambda)^{-1}h), (\bigvee_{\Im\mu<0} (A^*+\mu)^{-1}h)\}.$$

Its reducing property, which follows from the invariance with respect to both resolvents $(A+\tilde{\lambda})^{-1}$ and $(A^*+\tilde{\mu})^{-1}$, $\tilde{\lambda} \in \mathbb{C}_+$ and $\tilde{\mu} \in \mathbb{C}_-$ can be easily proved by using the Hilbert identity for resolvents of A and A^* respectively. Indeed, for $\Im\tilde{\lambda} > 0, \tilde{\lambda} \neq \lambda; \alpha, \beta \in \mathbb{C}$

$$\begin{aligned}
&(A+\tilde{\lambda})^{-1}(\alpha(A+\lambda)^{-1}h + \beta(A^*+\mu)^{-1}h) \\
&= \alpha((A+\lambda)^{-1} - (A+\tilde{\lambda})^{-1})(\tilde{\lambda}-\lambda)^{-1}h + \beta(A+\tilde{\lambda})^{-1}(A^*+\mu)^{-1}h \\
&= \alpha((A+\lambda)^{-1} - (A+\tilde{\lambda})^{-1})(\tilde{\lambda}-\lambda)^{-1}h + \beta(\tilde{\lambda}-\mu)^{-1}[(A^*+\mu)^{-1} - (A+\lambda)^{-1}]h
\end{aligned}$$

is again a linear combination of vectors of the type $(A+\lambda)^{-1}h$ and $(A^*+\mu)^{-1}h$. Here we used formula (2.5), the second abstract Green function identity together with the fact that $\Gamma_*(A^*+\mu)^{-1}h = 0$, $\mu \in \mathbb{C}_-$. Invariance with respect to the resolvent $(A^*+\tilde{\mu})^{-1}$, $\tilde{\mu} \neq \mu$, $\tilde{\mu} \in \mathbb{C}_-$ has a similar proof using the first abstract Green function identity (2.4) and the fact that $\Gamma(A+\lambda)^{-1}h = 0$. The exceptional cases $\tilde{\lambda} = \lambda$ and $\tilde{\mu} = \mu$ are achieved by the limit procedures $\tilde{\lambda} \to \lambda$ and $\tilde{\mu} \to \mu$ respectively.

By Lemma 2.5, we have $h = \lim_{\tau\to+\infty}(A+i\tau)^{-1}(i\tau)h$. This implies that $h \in \eta_h$. Therefore, it is sufficient to prove that the reduced operator $A|_{\eta_h}$ is a selfadjoint operator in η_h, since then $h \in \eta_h \subseteq H_{sa}$. Let us consider the Cayley transform of $A|_{\eta_h}$

$$T_h := (A-i)(A+i)^{-1}|_{\eta_h} = (I-2i(A+i)^{-1})|_{\eta_h}$$

and show that T_h is a unitary operator on η_h. Indeed, noting that $(T_h)^* = (I+2i(A^*-i)^{-1})|_{\eta_h}$, we see

$$\begin{aligned}(T_h)^*T_h &= (I+2i(A^*-i)^{-1})(I-2i(A+i)^{-1}))|_{\eta_h}\\ &= (I+2i(A^*-i)^{-1}-2i(A+i)^{-1}-(2i)^2(A^*-i)^{-1}(A+i)^{-1})|_{\eta_h}.\end{aligned}$$

Again by the first abstract Green function formula (2.4)

$$\begin{aligned}&[(A+\lambda)^{-1}-(A^*+\mu)^{-1}+(\lambda-\mu)(A^*+\mu)^{-1}(A+\lambda)^{-1}]h\\ &= -i(\Gamma(A+\overline{\mu})^{-1})^*)(\Gamma(A+\lambda)^{-1})h = 0\end{aligned}$$

and similarly by the second abstract Green function formula (2.5)

$$\begin{aligned}&[(A+\lambda)^{-1}-(A^*+\mu)^{-1}+(\lambda-\mu)(A+\lambda)^{-1}(A^*+\mu)^{-1}]h\\ &= -i(\Gamma_*(A^*+\overline{\lambda})^{-1})^*(\Gamma_*(A^*+\mu)^{-1})h = 0\end{aligned}$$

and therefore, comparing the last two formulae, we have shown

$$(A+\lambda)^{-1}(A^*+\mu)^{-1}h = (A^*+\mu)^{-1}(A+\lambda)^{-1}h \tag{3.2}$$

for any $\lambda \in \mathbb{C}_+$, $\mu \in \mathbb{C}_-$. So

$$\begin{aligned}(T_h)^*T_h(A+\lambda)^{-1}h &= (I+2i(A^*-i)^{-1}-2i(A+i)^{-1}-(2i)^2(A^*-i)^{-1}(A+i)^{-1})(A+\lambda)^{-1}h\\ &= [(A+\lambda)^{-1}+2i(A+\lambda)^{-1}(A^*-i)^{-1}-\\ &\quad -2i(A+\lambda)^{-1}(A+i)^{-1}-(2i)^2(A^*-i)^{-1}(A+i)^{-1}(A+\lambda)^{-1}]h\\ &= (A+\lambda)^{-1}[h+2i(A^*-i)^{-1}h-2i(A+i)^{-1}h]\\ &\quad -(2i)^2(A^*-i)^{-1}\{(A+i)^{-1}-(A+\lambda)^{-1}\}(\lambda-i)^{-1}h\\ &= (A+\lambda)^{-1}[h+2i(A^*-i)^{-1}h-2i(A+i)^{-1}h]\\ &\quad -(2i)^2\{(A+i)^{-1}-(A+\lambda)^{-1}\}(\lambda-i)^{-1}(A^*-i)^{-1}h\\ &= (A+\lambda)^{-1}[h+2i(A^*-i)^{-1}h-2i(A+i)^{-1}h]\\ &\quad -(2i)^2(A+\lambda)^{-1}(A+i)^{-1}(A^*-i)^{-1}h = (A+\lambda)^{-1}h,\end{aligned}$$

where in the last step the second abstract Green identity (2.5) was used and $h \in M_A^\perp$. Hence

$$(T_h)^*T_h(A+\lambda)^{-1}h = (A+\lambda)^{-1}h, \forall\lambda \in \mathbb{C}_+$$

Similar calculations which are even simpler because we can use (3.2) explicitly, show

$$(T_h)^*T_h(A^*+\mu)^{-1}h = (A^*+\mu)^{-1}h, \forall\mu \in \mathbb{C}_-.$$

Since any element of η_h is a limit of linear combinations of vectors $(A+\lambda)^{-1}h$, and $(A^*+\mu)^{-1}h$ and T_h is a bounded operator we have proved the isometry of T_h: $(T_h)^*T_h = I$ on η_h. The second identity, $T_h(T_h)^* = I$ on η_h, admits a similar proof because all calculations above are symmetric with respect to both A and A^*.

This proves that $A|_{\eta_h}$ is selfadjoint, so $\eta_h \subseteq H_{sa}$. Since $h \in \eta_h$, this shows the required inclusion $\mathcal{M_A}^\perp \subseteq H_{sa}$, completing the proof. □

The next theorem demonstrates one of the deepest results of dilation theory. Its original proof, given in [34], is based on some ideas of a geometric nature. Actually, this theorem was first proven for the case of contractions and their unitary dilations. However the fact can be easily transfered to the dissipative situation using the Cayley transform. Below we suggest a new proof based essentially on Theorem 3.1, i.e. applying operator analytic arguments.

Theorem 3.4. *(B.Sz.-Nagy - C.Foias [34]) The minimal selfadjoint dilation of a completely non-selfadjoint maximally dissipative operator has purely absolutely continuous spectrum covering the whole real line.*

Proof. According to Theorem 3.3 complete non-selfadjointness leads to the fact that

$$H = H_{cns} = \bigvee\{\mathrm{Span}_{\Im\lambda>0}(\Gamma(A+\lambda)^{-1})^*E,\ \ \mathrm{Span}_{\Im\mu<0}(\Gamma_*(A^*+\mu)^{-1})^*E_*\}.$$

Consider two linear sets of test vectors generating H:

$$\mathcal{L}_1 := \mathrm{Span}_{\Im\lambda>0}(\Gamma(A+\lambda)^{-1})^*E;\ \mathcal{L}_2 := \mathrm{Span}_{\Im\mu<0}(\Gamma_*(A^*+\mu)^{-1}E_*).$$

Theorem 3.1 shows that for any

$$u \in H, g \in E, g_* \in E_* :$$
$$(\Gamma(A+\lambda)^{-1}u, g)_E = (u, ((\Gamma(A+\lambda)^{-1})^* g)_H \in H_2^+$$

and

$$(\Gamma_*(A^*+\mu)^{-1}u, g_*)_{E_*} \equiv (u, ((\Gamma_*(A^*+\mu)^{-1})^* g_*)_H \in H_2^- .$$

Introducing the auxiliary parameter $\lambda_0 \in \mathbb{C}_+$ we have by the Hilbert identity

$$(\Gamma(A+\lambda_0)^{-1}(A+\lambda)^{-1}u, g)_E = (\Gamma(A+\lambda_0)^{-1}[(A+\lambda)^{-1}u], g)_E$$
$$= ([(A+\lambda)^{-1}u], (\Gamma(A+\lambda_0)^{-1})^* g)_H \in H_2^+, \forall g \in E$$

as a function of $\lambda \in \mathbb{C}_+$. Similarly,

$$(\Gamma_*(A^*+\mu_0)^{-1}(A^*+\mu)^{-1}u, g_*)_{E_*} = ((A^*+\mu)^{-1}u, (\Gamma_*(A^*+\mu_0)^{-1})^* g_*)_H \in H_2^-, \forall g_* \in E_*,$$

$\Im\mu_0 < 0$, as a function of $\mu \in \mathbb{C}_-$. So introducing the minimal selfadjoint dilation $\mathcal{L}$ of A in $\mathcal{H}$ we proved, assuming without loss of generality $\mathcal{H} \supset H$,

$$(P_H(\mathcal{L}+\lambda)^{-1}u, (\Gamma(A+\lambda_0)^{-1})^* g)_{\mathcal{H}} \in H_2^+, \forall g \in E, \forall u \in H, \forall \lambda_0 \in \mathbb{C}_+$$

as a function of $\lambda \in \mathbb{C}_+$ and

$$(P_H(\mathcal{L}+\mu)^{-1}u, (\Gamma_*(A^*+\mu_0)^{-1})^* g_*)_{\mathcal{H}} \in H_2^-$$

as a function of $\mu \in \mathbb{C}_-$ for arbitrary $u \in H, \mu_0 \in \mathbb{C}_-$ and $g_* \in E_*$. Taking linear combinations of the test vectors one gets

$$(P_H(\mathcal{L}+\lambda)^{-1}u, \phi)_{\mathcal{H}} = ((\mathcal{L}+\lambda)^{-1}u, \phi)_{\mathcal{H}} \in H_2^+, \forall u \in H, \phi \in \mathcal{L}_1$$

and

$$(P_H(\mathcal{L}+\mu)^{-1}u, \psi)_{\mathcal{H}} = ((\mathcal{L}+\mu)^{-1}u, \psi)_{\mathcal{H}} \in H_2^-, \forall u \in H, \psi \in \mathcal{L}_2.$$

Introducing the spectral resolution E_t of $\mathcal{L}$ in $\mathcal{H} \supset H$ we can rewrite our conditions as follows:

$$\int_{\mathbb{R}} \frac{1}{t+\lambda} d(E_t u, \phi)_{\mathcal{H}} \in H_2^+, \forall u \in H,\ \phi \in \mathcal{L}_1$$

and

$$\int_{\mathbb{R}} \frac{1}{t+\mu} d(E_t u, \psi)_{\mathcal{H}} \in H_2^-, \forall u \in H, \psi \in \mathcal{L}_2.$$

By the standard representation theorem for Hardy classes there exist two $L_2(\mathbb{R})$ scalar functions $f_\pm$ (depending on u, ϕ, ψ) such that

$$\int_{\mathbb{R}} \frac{1}{t+\lambda} d(E_t u, \phi)_{\mathcal{H}} = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{f_+(t)}{t+\lambda} dt, \forall \lambda \in \mathbb{C}_+$$

and

$$\int_{\mathbb{R}} \frac{1}{t+\mu} d(E_t u, \psi)_{\mathcal{H}} = -\frac{1}{2\pi} \int_{\mathbb{R}} \frac{f_-(t)}{t+\mu} dt, \forall \mu \in \mathbb{C}_-.$$

Therefore

$$\int_{\mathbb{R}} \frac{1}{t+\lambda} [d(E_t u, \phi)_{\mathcal{H}} - \frac{1}{2\pi i} f_+(t) dt] \equiv 0, \lambda \in \mathbb{C}_+$$

and

$$\int_{\mathbb{R}} \frac{1}{t+\lambda} [d(E_t u, \psi)_{\mathcal{H}} + \frac{1}{2\pi i} f_-(t) dt] \equiv 0, \lambda \in \mathbb{C}_+.$$

The F. and M. Riesz Theorem [14] implies that the complex measures $\{d(E_t u, \phi)_{\mathcal{H}} - \frac{1}{2\pi i} f_+(t) dt\}$ and $\{d(E_t u, \psi)_{\mathcal{H}} + \frac{1}{2\pi i} f_-(t) dt\}$ are both absolutely continuous, i.e. $d(E_t u, \phi)_{\mathcal{H}}$ and $d(E_t u, \psi)_{\mathcal{H}}$ are both absolutely continuous for any $\phi \in \mathcal{L}_1$ and $\psi \in \mathcal{L}_2$ respectively. Summing up, $d(E_t u, v)_{\mathcal{H}}$ is an absolutely continuous measure for any $u \in H$ and any vector $v \in \mathcal{L}_1 + \mathcal{L}_2 =: \mathcal{L}_3$. As a final step let us consider the expression

$$d(E_t(\mathcal{L}+\overline{\gamma})^{-1}(\mathcal{L}+\sigma)^{-1}u, v)_{\mathcal{H}} = \frac{1}{(t+\sigma)(t+\overline{\gamma})} d(E_t u, v)_{\mathcal{H}}$$
$$= d(E_t(\mathcal{L}+\sigma)^{-1}u, (\mathcal{L}+\gamma)^{-1}v)_{\mathcal{H}}$$

which is an absolutely continuous complex measure for any values of non-real parameters σ and γ. Since $\mathcal{L}$ is a minimal selfadjoint dilation of A the Span $\{(\mathcal{L}+\sigma)^{-1}u : u \in H \text{ and } \sigma \in \mathbb{C}\backslash\mathbb{R}\}$ is dense in $\mathcal{H}$. The same is true for Span $\{(\mathcal{L}+\gamma)^{-1}v : v \in \mathcal{L}_3 \text{ and } \gamma \in \mathbb{C}\backslash\mathbb{R}\}$. So the measure $d(E_t x, y)_{\mathcal{H}}$ is pure absolutely continuous for dense sets of vectors x and y in $\mathcal{H}$. Therefore the minimal dilation $\mathcal{L}$ has pure absolutely continuous spectrum.

The spectrum of $\mathcal{L}$ has to cover the whole real line. Indeed, assume that its spectrum has a gap, which includes an interval I. Then the formula in Proposition 2.4 connects the resolvents of the dilation and of the operators A and A^*, showing that all three resolvents admit analytic continuations from the appropriate complex half-plane to the interval I and therefore coincide there. Hence $A = A^*$. □

4. Spectral form of the selfadjoint dilation of a maximally dissipative operator

By von Neumann's general theory of selfadjoint operators in Hilbert spaces (see, e.g. [2]), the abstract Hilbert space can be replaced by the space of L_2-summable functions $\eta(k)$ with $k \in \mathbb{R}$, taking values in auxiliary Hilbert spaces, such that the minimal dilation of the completely non-selfadjoint maximally dissipative operator A is represented in that space by a multiplication operator by the independent variable $k \in \mathbb{R}$.

In this section we will make this procedure explicit, transforming the translational form [20] of $\mathcal{L}$ into its explicit spectral form. Another important feature of the approach is that both transforms to translational form and later to the spectral form will be performed in an explicit way using the operator colligations method, as well as the Strauss characteristic function. The spectral form will be presented in B. Pavlov's version [27], symmetric with respect to both incoming and outgoing subspaces [16, 29]. That form, in our opinion, has some advantages compared to both the standard B. Sz-Nagy-C. Foias form of the selfadjoint (unitary) dilation and also the L. de Branges [9] form.

We first recall (in the notation of [6, Section 5]) the explicit construction procedure of the selfadjoint dilation of a maximally dissipative operator A in a Hilbert space H.

The linear set defined next will be the domain of the selfadjoint dilation of A in the so-called translation form.

Definition 4.1. *Let $\mu \in \mathbb{C}_-$ and $\lambda \in \mathbb{C}_+$ and consider the Hilbert space $\mathcal{H}_{tr} := L_2(\mathbb{R}_-, E_*) \oplus H \oplus L_2(\mathbb{R}_+, E)$. Define the linear subset $\mathcal{D}(\mathcal{L}_{tr})$ by*

$$D(\mathcal{L}_{tr}) = \left\{ U = \begin{pmatrix} v_- \\ u \\ v_+ \end{pmatrix} \in \mathcal{H}_{tr} \ : \ u \in H,\ v_+ \in H^1(\mathbb{R}_+, E),\ v_- \in H^1(\mathbb{R}_-, E_*), \right. \tag{4.1}$$

$$\textbf{(I)} \quad \begin{array}{l} u + (\Gamma_*(A^* + \mu)^{-1})^* v_-(0) \in D(A) \ \textit{and} \\ v_+(0) = S^*(-\mu) v_-(0) + i\Gamma\left(u + (\Gamma_*(A^* + \mu)^{-1})^* v_-(0)\right) \end{array},$$

$$\left. \textbf{(II)} \quad \begin{array}{l} u + (\Gamma(A + \lambda)^{-1})^* v_+(0) \in D(A^*) \ \textit{and} \\ v_-(0) = S(-\overline{\lambda}) v_+(0) - i\Gamma_*\left(u + (\Gamma(A + \lambda)^{-1})^* v_+(0)\right) \end{array} \right\}.$$

Here, $H^1(\mathbb{R}_+, E)$ and $H^1(\mathbb{R}_-, E_*)$ are the Sobolev spaces of vector-valued functions on $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively, taking values on the auxiliary Hilbert spaces E, or correspondingly E_*. The norm of the spaces is given by

$$\int_0^\infty (\left\|v_+'(\xi)\right\|_E^2 + \|v_+(\xi)\|_E^2) d\xi =: \|v_+\|^2_{H^1(\mathbb{R}_+, E)}$$

and

$$\int_{-\infty}^0 (\left\|v_-'(\xi)\right\|_{E_*}^2 + \|v_-(\xi)\|_{E_*}^2) d\xi =: \|v_-\|^2_{H^1(\mathbb{R}_-, E_*)}.$$

It follows that both $v_+(0) := \lim_{\xi \to 0^+} v_+(\xi)$ and $v_-(0) := \lim_{\xi \to 0^-} v_+(\xi)$ are well-defined in the E and E_* topologies, respectively.

Remark 4.2. *Note that whenever $u \in H$ and $v_-(0) \in E_*$ are such that $[u + (\Gamma_*(A^* + \mu_0)^{-1})^* v_-(0))] \in \mathcal{D}(A)$ for some $\mu_0 \in \mathbb{C}_-$, then $[u + (\Gamma_*(A^* + \mu)^{-1})^* v_-(0)] \in \mathcal{D}(A)$ for all $\mu \in \mathbb{C}_-$ (see* [6, Lemma 5.2]*). Similarly, the second condition $[u + (\Gamma(A + \lambda)^{-1})^* u_+(0)] \in \mathcal{D}(A^*)$ does not depend on the choice of $\lambda \in \mathbb{C}_+$. Although, we denoted vectors in the description of $D(\mathcal{L}_{tr})$ from E_* and E in the form $v_-(0)$ and $v_+(0)$ to be suggestive of their role in applications, both vectors can be chosen arbitrarily in the respective spaces.*

We should mention that in (4.1), conditions **(I)** *and* **(II)** *are equivalent, see e.g.* [6, Lemma 5.4]. *Finally, we see that there are only four free parameters in the domain of $\mathcal{L}_{tr}$. These can be chosen as*

1. *the vector $v_+(0) \in E$*
2. *a vector $h \in \mathcal{D}(A^*)$ such that $u := h - (\Gamma(A + \lambda)^{-1})^* v_+(0)$, one can take $\lambda = i$ here for example;*
3. *two vector-valued functions $w_+ \in H^1(\mathbb{R}_+, E)$ and $w_- \in H^1(\mathbb{R}_-, E_*)$ with $w_+(0) = 0, w_-(0) = 0$.*

Indeed, according to the equivalence of the conditions **(I)** *and* **(II)** *of (4.1), one can choose a vector (v_-, u, v_+) from $\mathcal{D}(\mathcal{L}_{tr})$ such that*

1. $v_+(\xi) := w_+(\xi) + v_+(0)e^{-\xi},\ \xi \geq 0$
2. $v_-(\xi) := w_-(\xi) + (S(-\overline{\lambda})v_+(0) - i\Gamma_* h)e^{\xi},\ \xi \leq 0$

(3) $u := h - (\Gamma(A+\lambda)^{-1})^* v_+(0)$

for any fixed $\lambda \in \mathbb{C}_+$, say $\lambda = i$.

In order to introduce the formula for the dilation $\mathcal{L}_{tr}$ we need the following definition.

Definition 4.3. *Let $\mu \in \mathbb{C}_-$ and $\lambda \in \mathbb{C}_+$ be fixed. For any vector $U = (v_-, u, v_+) \in \mathcal{D}(\mathcal{L}_{tr})$ define two operators T and $T_* : \mathcal{D}(\mathcal{L}_{tr}) \to H$ by*

$$TU := A^*(u + (\Gamma(A+\lambda)^{-1})^* v_+(0)) + \overline{\lambda}(\Gamma(A+\lambda)^{-1})^* v_+(0) \tag{4.2}$$

and

$$T_*U = A(u + (\Gamma_*(A^*+\mu)^{-1})^* v_-(0)) + \overline{\mu}(\Gamma_*(A^*+\mu)^{-1})^* v_-(0) \tag{4.3}$$

We note that $T \equiv T_*$ on $\mathcal{D}(\mathcal{L}_{tr})$ and therefore both are independent of λ and μ, see [6, Lemma 5.7 & Corollary 5.8].

Now the selfadjoint dilation of the maximally dissipative operator A in H can be defined as follows:

Definition 4.4. *For any vector $U = (v_-, u, v_+) \in \mathcal{D}(\mathcal{L}_{tr}) \subset L_2(\mathbb{R}_-, E_*) \oplus H \oplus L_2(\mathbb{R}_+, E)$, set*

$$\mathcal{L}_{tr}U \equiv \mathcal{L}_{tr} \begin{pmatrix} v_- \\ u \\ v_+ \end{pmatrix} = \begin{pmatrix} iv'_- \\ TU \\ iv'_+ \end{pmatrix}.$$

Therefore, the operator $\mathcal{L}_{tr}$ acts, both in the incoming channel

$$D_- = (L_2(\mathbb{R}_-, E_*), 0, 0) \equiv L_2(\mathbb{R}_-, E_*)$$

and in the outgoing one

$$D_+ = (0, 0, L_2(\mathbb{R}_+, E)) \equiv L_2(\mathbb{R}_+, E)$$

in the sense of the Lax-Phillips scattering theory [16], as a first order differentiation operator on v_- and v_+, respectively. This operator, being the generator of the standard shift semigroups on the half lines, gives a justification to the name "translational form" for this realisation of the dilation. The "middle" operator TU explicitly uses the operator A^*, to make a coupling between the two terms $v_\pm(0)$ and of course between both channels.

The main result of [6] is the Theorem 7.6:

Proposition 4.5. *The operator $\mathcal{L}_{tr}$ in the Hilbert space $\mathcal{H}_{tr} = L_2(\mathbb{R}_-, E_*) \oplus H \oplus L_2(\mathbb{R}_+, E)$, defined in Definitions 4.1 and 4.4 is a minimal selfadjoint dilation of the maximally dissipative operator A in H, i.e.*

(1)
$$\mathcal{L}_{tr} = \mathcal{L}^*_{tr},$$

(2) *for $U = (0, u, 0)$*
$$P_H(\mathcal{L}_{tr} - \lambda)^{-1}U = \begin{cases} (A-\lambda)^{-1}u, & \lambda \in \mathbb{C}_-, \\ (A^*-\lambda)^{-1}u, & \lambda \in \mathbb{C}^+, \end{cases}$$

where P_H is the projection onto the second component in $\mathcal{H}_{tr}$: $P_H(v_-, u, v_+) = (0, u, 0)$.

(3) *Define the completely non-selfadjoint subspace H_{cns} of A as in* (3.1) *and its orthogonal complement H_{sa}. Then the subspace $(0, H_{sa}, 0) \subset \mathcal{H}_{tr}$ is a reducing subspace for the dilation $\mathcal{L}_{tr}$, and $\mathcal{L}_{tr}$ restricted to $(0, H_{sa}, 0)$ is $(0, A_{sa}, 0)$, where $A_{sa} := A|_{H_{sa}}$ is the selfadjoint part of A. Further the operator $\mathcal{L}_{tr}$ restricted to the second reducing subspace*
$$L_2(\mathbb{R}_-, E_*) \oplus H_{cns} \oplus L_2(\mathbb{R}_+, E),$$
the orthogonal complement of $(0, H_{sa}, 0)$, is the minimal selfadjoint dilation of $A|_{H_{cns}}$. Moreover
$$\operatorname{clos}\left(\operatorname{Span}_{\lambda \notin \mathbb{R}}(\mathcal{L}_{tr} - \lambda)^{-1} \begin{pmatrix} L_2(\mathbb{R}_-, E_*) \\ 0 \\ L_2(\mathbb{R}_+, E) \end{pmatrix}\right) = \begin{pmatrix} L_2(\mathbb{R}_-, E_*) \\ H_{cns} \\ L_2(\mathbb{R}_+, E) \end{pmatrix}.$$

Now we are ready to transform the translational form of the dilation $\mathcal{L}_{tr}$, given by Proposition 4.5, into its unitarily equivalent spectral form. The part of $\mathcal{L}_{tr}$, corresponding to the completely non-selfadjoint component of A, is presented as the multiplication operator by an independent variable $k \in \mathbb{R}$, in an L_2-space of vector-valued functions on $\mathbb{R}$. The existence of this form is clear from the Foias Theorem 3.4, but our translational form describes it explicitly. This allows us to preserve information about the original form of the operator A under the requisite transformation.

In what follows, we will in some places assume that A is completely non-selfadjoint. This will allow us to ignore the selfadjoint part $A|_{H_{sa}}$ of A, since in the minimal selfadjoint dilation this is reflected by the operator $A|_{H_{sa}}$ on $(0, H_{sa}, 0)$ and can be studied using the classical spectral theorem for selfadjoint operators in Hilbert space. Both parts, $A|_{H_{cns}}$ and $A|_{H_{sa}}$, should be considered independently using these different tools, i.e. the functional model for the first part in $\mathcal{H}_{cns}$ and the standard spectral theorem for the second part in $\mathcal{H}_{sa}$. In view of this we will mainly concentrate on the minimal selfadjoint dilation of a completely non-selfadjoint maximally dissipative operator A, without any loss of generality.

As the first step we consider two maps $\mathcal{F}_\pm$ transforming the translational form of the Hilbert space $\mathcal{H}_{tr} = L_2(\mathbb{R}_-, E_*) \oplus H \oplus L^2(\mathbb{R}_+, E)$ into $L_2(\mathbb{R}, E)$ and $L_2(\mathbb{R}, E_*)$, respectively.

Definition 4.6. *For any vector $(v_-, u, v_+) \in \mathcal{H}_{tr}$ and a.e. $k \in \mathbb{R}$, set*

$$\left[\mathcal{F}_+ \begin{pmatrix} v_- \\ u \\ v_+ \end{pmatrix}\right](k) := -\frac{1}{\sqrt{2\pi}}\Gamma(A - k + i0)^{-1}u + S^*(k)\hat{v}_-(k) + \hat{v}_+(k) \in L_2(\mathbb{R}, E)$$

and

$$\left[\mathcal{F}_- \begin{pmatrix} v_- \\ u \\ v_+ \end{pmatrix}\right](k) := -\frac{1}{\sqrt{2\pi}}\Gamma_*(A^* - k - i0)^{-1}u + S(k)\hat{v}_+(k) + \hat{v}_-(k) \in L_2(\mathbb{R}, E_*),$$

where $S(k) := S(k + i0)$ and $\hat{v}_\pm(k)$ are the Fourier transforms of the vector-valued functions $v_\pm(\xi)$ extended by 0 onto the complementary semiaxis $\mathbb{R}_\mp$:

$$\hat{v}_+(k) = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{i\xi k} v_+(\xi)d\xi,$$
$$\hat{v}_-(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 e^{i\xi k} v_-(\xi)d\xi.$$

Using the canonical identification of an analytic function from the Hardy class in the upper or lower half-plane with its boundary values on the real line $\mathbb{R}$, by the Paley-Wiener theorem [14, 34], we have

$$\hat{v}_\pm(k) \in H_2^\pm := H_2(\mathbb{C}^\pm).$$

Similarly, $\Gamma(A - k + i0)^{-1}u$ and $\Gamma_*(A^* - k - i0)^{-1}u$ are the boundary values of $\Gamma(A - \lambda)^{-1}u$ and $\Gamma_*(A^* - \lambda)^{-1}u$, where $\lambda \to k \mp i0$ in the lower and upper half-plane, respectively. The existence of the non-tangential boundary values of the resolvents and $S(k + i0)$ and $S^*(k + i0)$, in the strong operator topology of the Hilbert spaces E and E_* for a.e. $k \in \mathbb{R}$ is guaranteed by the B. Sz-Nagy Theorem [34]. Since the boundary values $S(k)$ and $S^*(k)$ are contractions, we have that all three terms in the formulae for $\mathcal{F}_\pm$ are $L_2(\mathbb{R}, E)$- or $L_2(\mathbb{R}, E_*)$-vector-valued functions on $\mathbb{R}$. Therefore the maps $\mathcal{F}_\pm$ are well defined on the whole space $\mathcal{H}_{tr}$. Moreover, from Theorem 3.1, we have by the triangle inequality

$$\begin{aligned}
\left\|\mathcal{F}_+ \begin{pmatrix} v_- \\ u \\ v_+ \end{pmatrix}\right\|_{L_2(\mathbb{R},E)} &\leq \|u\|_H + \|S^*\hat{v}_-\|_{L_2(\mathbb{R},E)} + \|\hat{v}_+\|_{L_2(\mathbb{R},E)} \\
&\leq \|u\|_H + \|\hat{v}_-\|_{L_2(\mathbb{R},E_*)} + \|\hat{v}_+\|_{L_2(\mathbb{R},E)} = \|u\|_H + \|v_-\|_{L_2(\mathbb{R}_-,E_*)} + \|v_+\|_{L_2(\mathbb{R}_+,E)} \\
&\leq \sqrt{3}\,\|(v_-, u, v_+)\|_{\mathcal{H}_{tr}}.
\end{aligned}$$

Here we used the Parseval identity [14] and the contraction property of S^*. The case of $\mathcal{F}_-$ can be considered similarly. Thus the maps are bounded operators from $\mathcal{H}_{tr}$ to $L_2(\mathbb{R}, E)$ or to $L_2(\mathbb{R}, E_*)$, respectively.

Following the ideas in the paper [20] we use the maps $\mathcal{F}_\pm$ for the construction of the spectral form of the dilation $\mathcal{L}_{tr}$. We next introduce a new Hilbert space:

$$\mathcal{H}_{sp} := L_2\left(\mathbb{R}, E \oplus E_*; \begin{pmatrix} I_E & S^*(k) \\ S(k) & I_{E_*} \end{pmatrix}\right).$$

Our new version of the functional model Hilbert space $\mathcal{H}_{sp}$, referred to as Pavlov's symmetric form of the functional model, is by definition the closure in a weighted norm of the space of vectors of the form $\begin{pmatrix} \tilde{g} \\ g \end{pmatrix}$, where $\tilde{g} = \tilde{g}(k) \in L_2(\mathbb{R}, E)$ and $g = g(k) \in L_2(\mathbb{R}, E_*)$. The norm of the vector in $\mathcal{H}_{sp}$ is defined as follows.

$$\left\|\begin{pmatrix} \tilde{g} \\ g \end{pmatrix}\right\|^2_{\mathcal{H}_{sp}} := \int_{\mathbb{R}} \left\langle \begin{pmatrix} I_E & S^*(k) \\ S(k) & I_{E_*} \end{pmatrix}\begin{pmatrix} \tilde{g} \\ g \end{pmatrix}, \begin{pmatrix} \tilde{g} \\ g \end{pmatrix}\right\rangle_{E\oplus E_*} dk.$$

Simple algebraic manipulations give the following identities.

$$\left\| \begin{pmatrix} \tilde g \\ g \end{pmatrix} \right\|^2_{\mathcal{H}_{sp}} = \|\tilde g(k) + S^*(k) g(k)\|^2_{L_2(\mathbb{R},E)} + \int_{\mathbb{R}} \langle (I - S(k)S^*(k)) g(k), g(k)\rangle_{E_*}\, dk \tag{4.4}$$

$$= \|S(k)\tilde g(k) + g(k)\|^2_{L_2(\mathbb{R},E_*)} + \int_{\mathbb{R}} \langle (I - S^*(k)S(k)) \tilde g(k), \tilde g(k)\rangle_{E}\, dk. \tag{4.5}$$

Since the boundary values $S^*(k) \equiv (S(k+i0))^*, S(k) \equiv S(k+i0)$ of the contraction $S(z),\ z \in \mathbb{C}_+$ are also contractions, we obviously have

$$\left\| \begin{pmatrix} \tilde g \\ g \end{pmatrix} \right\|_{\mathcal{H}_{sp}} \geq \|\tilde g + S^* g\|_{L_2(\mathbb{R},E)} \quad \text{and} \quad \left\| \begin{pmatrix} \tilde g \\ g \end{pmatrix} \right\|_{\mathcal{H}_{sp}} \geq \|S\tilde g + g\|_{L_2(\mathbb{R},E_*)}, \tag{4.6}$$

where we omitted the arguments of the functions on the right hand side terms for notational convenience. The elements of $\mathcal{H}_{sp}$ are the limits of vectors from $L_2(\mathbb{R},E)) \oplus L_2(\mathbb{R},E_*)$ and we will still denote them as $(\tilde g(k), g(k))$, although this is symbolic, especially since the matrix weight $\begin{pmatrix} I_E & S^*(k) \\ S(k) & I_{E_*} \end{pmatrix}$ may be degenerated at a set of positive Lebesgue measure on $\mathbb{R}$. On the other hand, due to (4.6), the expressions $\tilde h(k) := \tilde g(k) + S^*(k) g(k)$ and $h(k) := S(k)\tilde g(k) + g(k)$ are still $L_2(\mathbb{R},E)$ and $L_2(\mathbb{R},E_*)$ functions respectively, even after taking a closure. Note that the L_2-vector valued functions $(\tilde h(k), h(k))$ form a de Branges [9] version of the functional model [24].

Alternatively, the pairs

$$(\tilde g(k) + S^*(k) g(k), g(k)) = (\tilde h(k), g(k)) \in L_2(\mathbb{R},E) \oplus L_2(\mathbb{R},E_*; (I_{E_*} - S(k)S^*(k))^{1/2})$$

and

$$(S(k)\tilde g(k) + g(k), \tilde g(k)) = (h(k), \tilde g(k)) \in L_2(\mathbb{R},E_*) \oplus L_2(\mathbb{R},E; (I_{E} - S^*(k)S(k))^{1/2})$$

give a transformation to the Sz-Nagy-Foias form of the functional model [34]. We will discuss this in more detail later.

Lemma 4.7. *A vector* $\begin{pmatrix} \tilde g \\ g \end{pmatrix} \in \mathcal{H}_{sp}$ *is uniquely determined by the two vector functions* $\tilde h(k) = \tilde g(k) + S^*(k) g(k)$ *and* $h(k) = S(k)\tilde g(k) + g(k)$.

Proof. Assume that $\tilde h(k) = 0$ and $h(k) = 0$ for almost every $k \in \mathbb{R}$ and recall that the operator functions $S(k)$ and $S^*(k)$ are also only well defined for a.e. $k \in \mathbb{R}$. Then almost everywhere

$$0 = h(k) - S(k)\tilde h(k) = S(k)\tilde g(k) + g(k) - S(k)(\tilde g(k) + S^*(k) g(k)) = (I_{E_*} - S(k)S^*(k)) g(k),$$

and therefore $(I_{E_*} - S(k)S^*(k))^{1/2} g(k) = 0$ almost everywhere. Now comparing this with (4.4), we have that $\left\| \begin{pmatrix} \tilde g \\ g \end{pmatrix} \right\|_{\mathcal{H}_{sp}} = 0$. This is merely a formal proof, but it can be made rigorous via a limiting argument as we now show.

Let

$$\lim_{n\to\infty} \begin{pmatrix} \tilde g_n(k) \\ g_n(k) \end{pmatrix} = \begin{pmatrix} \tilde g(k) \\ g(k) \end{pmatrix}$$

in $\mathcal{H}_{sp}$ with $\tilde g_n(k) \in L_2(\mathbb{R},E)$ and $g_n(k) \in L_2(\mathbb{R},E_*)$. Then setting $\tilde h_n := \tilde g_n + S^* g_n$ and $h_n := S\tilde g_n + g_n$, we have $\tilde h = \lim_{n\to\infty} \tilde h_n = 0$ and $h = \lim_{n\to\infty} h_n = 0$ in the norm of $L_2(\mathbb{R},E)$ and $L_2(\mathbb{R},E_*)$, respectively. So,

$$\begin{aligned} 0 &= h - S\tilde h = \lim_{n\to\infty} (h_n - S\tilde h_n) \\ &= \lim_{n\to\infty} (S\tilde g_n + g_n - S(\tilde g_n + S^* g_n)) \\ &= \lim_{n\to\infty} (g_n - SS^* g_n) = (I_{E_*} - SS^*)^{1/2} (I_{E_*} - SS^*)^{1/2} g. \end{aligned}$$

So $(I_{E_*} - SS^*)^{1/2} g = 0$ as a function from $L_2(\mathbb{R},E_*)$, and the lemma follows from (4.4), as above. $\square$

In what follows we will usually omit this type of argument based on a limiting procedure of approximating elements $\begin{pmatrix} \tilde g \\ g \end{pmatrix} \in \mathcal{H}_{sp}$ by L_2-vector-valued functions and instead proceed formally as we indicated in the proof of the lemma. However, we note that a rigorous proof along the indicated lines can always be performed. Operating with the symbol $\begin{pmatrix} \tilde g \\ g \end{pmatrix}$ (in Pavlov's symmetric form) is often more convenient, especially taking into consideration that the L_2-vector-valued functions $\tilde h(k)$ and $h(k)$ are also only defined for a.e. $k \in \mathbb{R}$.

Now we are ready to formulate the main result.

Theorem 4.8. *Let $\mathcal{L}_{tr}$ be a minimal selfadjoint dilation of a completely non-selfadjoint maximally dissipative operator A in a Hilbert space H. Then there exists a unique transformation Φ of the translational form Hilbert space $\mathcal{H}_{tr} = L_2(\mathbb{R}, E_*) \oplus H \oplus L_2(\mathbb{R}_+, E)$ onto the spectral form of the Hilbert space*

$$\mathcal{H}_{sp} = L_2\left(\mathbb{R}, E \oplus E_*; \begin{pmatrix} I_E & S^*(k) \\ S(k) & I_{E_*} \end{pmatrix}\right)$$

such that the image in $\mathcal{H}_{sp}$ of the translational form of the selfadjoint dilation $\mathcal{L}_{tr}$ is the multiplication operator by the independent variable $k \in \mathbb{R}$ satisfying the explicit formula

$$\left\{ \begin{array}{l} \tilde{g}(k) + S^*(k)g(k) = \mathcal{F}_+(v_-, u, v_+), \\ S(k)\tilde{g}(k) + g(k) = \mathcal{F}_-(v_-, u, v_+), \end{array} \right. \tag{4.7}$$

where $\Phi(v_-, u, v_+) = \begin{pmatrix} \tilde{g} \\ g \end{pmatrix}$.

Remark 4.9. *A slightly more explicit formula for Φ is given in Lemma 4.11 below.*

Proof. The long proof of the theorem is broken down into five steps.

Step 1: Density of a set of test vectors.

We formulate the result as a lemma.

Lemma 4.10. *Let A be a maximally dissipative operator in H. Then the linear set of test functions in the translational version of the Hilbert space $\mathcal{H}_{tr}$*

$$\tau := \text{Span}\left\{ \text{Span}_{\,v_+\in L_2(\mathbb{R}_+,E), v_-\in L_2(\mathbb{R}_-,E_*), e\in E, \lambda\in\mathbb{C}_+} \left(\begin{array}{c} v_- \\ (\Gamma(A+\lambda)^{-1})^* e \\ v_+ \end{array} \right), \right.$$
$$\left. \text{Span}_{\,v_+\in L_2(\mathbb{R}_+,E), v_-\in L_2(\mathbb{R}_-,E_*), e_*\in E_*, \mu\in\mathbb{C}_-} \left(\begin{array}{c} v_- \\ (\Gamma_*(A^*+\mu)^{-1})^* e_* \\ v_+ \end{array} \right) \right\}$$

is a dense set in $\mathcal{H} \ominus (0, H_{sa}, 0)$ where H_{sa} is the selfadjoint part of the Langer decomposition of H. In particular, if A is a completely non-selfadjoint operator then the set τ presented above is dense in the whole space $\mathcal{H}_{tr}$.

Proof. Consider a vector $\begin{pmatrix} f_- \\ u \\ f_+ \end{pmatrix} \in \mathcal{H}_{tr}$ orthogonal to τ. Then, choosing $e = 0$ or $e_* = 0$ we get immediately that $f_- = 0$ and $f_+ = 0$. On the other hand, putting $v_- = 0$ and $v_+ = 0$ we obtain from the orthogonality condition that

(1) $\Gamma(A + \lambda)^{-1}u = 0$ for all $\lambda \in \mathbb{C}_+$,
(2) $\Gamma_*(A^* + \mu)^{-1}u = 0$ for all $\mu \in \mathbb{C}_-$.

By Theorem 3.3, conditions (1) and (2) together imply that $u \in H_{sa}$. □

Step 2: Embedding of the test vectors into $\mathcal{H}_{sp}$.

Let us first define the transformation Φ acting on the test vectors from Lemma 4.10 by the explicit formula in the following lemma:

Lemma 4.11. *Let $\lambda_0 \in \mathbb{C}_+$, $e \in E$, $v_- \in L_2(\mathbb{R}_-, E_*)$ and $v_+ \in L_2(\mathbb{R}_+, E)$. The map*

$$\Phi : \left(\begin{array}{c} v_- \\ (\Gamma(A - \overline{\lambda}_0)^{-1})^* e \\ v_+ \end{array} \right) \to \begin{pmatrix} \tilde{g}(k) \\ g(k) \end{pmatrix} := \left(\begin{array}{c} \hat{v}_+(k) + \dfrac{i}{\sqrt{2\pi}} \dfrac{e}{k - \lambda_0} \\ \hat{v}_-(k) - \dfrac{i}{\sqrt{2\pi}} \dfrac{S(\lambda_0)e}{k - \lambda_0} \end{array} \right)$$

satisfies condition (4.7).

Proof. We have

$$\begin{aligned} S(k)\tilde{g}(k) + g(k) &= S(k)\left(\hat{v}_+(k) + \frac{i}{\sqrt{2\pi}} \frac{e}{k - \lambda_0}\right) + \hat{v}_-(k) - \frac{i}{\sqrt{2\pi}} \frac{S(\lambda_0)e}{k - \lambda_0} \\ &= S(k)\hat{v}_+(k) + \frac{i}{\sqrt{2\pi}} \frac{(S(k) - S(\lambda_0))e}{k - \lambda_0} + \hat{v}_-(k), \end{aligned}$$

and

$$\mathcal{F}_- \left(\begin{array}{c} v_- \\ (\Gamma(A-\overline{\lambda}_0)^{-1})^* e \\ v_+ \end{array} \right) = -\frac{1}{\sqrt{2\pi}} \left(\Gamma_*(A^* - k - i0)^{-1}\right) \left(\Gamma(A-\overline{\lambda}_0)^{-1}\right)^* e + S(k)\hat{v}_+(k) + \hat{v}_-(k).$$

To prove that the two are equal, we need to show

$$\frac{S(k)-S(\lambda_0)}{k-\lambda_0} e = i(\Gamma_*(A^*-k-i0)^{-1})(\Gamma(A-\overline{\lambda}_0)^{-1})^* e \tag{4.8}$$

for all $e \in E$ and a.e.$k \in \mathbb{R}$. In order to do this, we use (2.8) and set $\tilde{\mu} = \lambda_0 \in \mathbb{C}_+, \mu = k + i\varepsilon, \varepsilon > 0$. Letting $\varepsilon \to 0$ we have that for fixed $e \in E$ and a.e. $k \in \mathbb{R}$ the equality (4.8) is valid. We remind the reader that, applied to any $e \in E$ the right hand side term in (2.8) lies in the vector-valued Hardy class as a function of μ by Theorem 3.1. Similarly,

$$\mathcal{F}_+ \left(\begin{array}{c} v_- \\ (\Gamma(A-\overline{\lambda}_0)^{-1})^* e \\ v_+ \end{array} \right) = -\frac{1}{\sqrt{2\pi}} (\Gamma(A-k+i0)^{-1})(\Gamma(A-\overline{\lambda}_0)^{-1})^* e + S^*(k)\hat{v}_-(k) + \hat{v}_+(k).$$

In order to prove that this is equal to $\tilde{g}(k) + S^* g(k)$, we have to show

$$\frac{I_E - S^*(z)S(\lambda_0)}{\overline{z}-\lambda_0} = i(\Gamma(A-\overline{z})^{-1})(\Gamma(A-\overline{\lambda}_0)^{-1})^*$$

for all $\lambda_0, z \in \mathbb{C}_+$, which is exactly (2.9). As $\varepsilon \to 0$ we see that $z = k + i\varepsilon \to k$. Thus both of the terms above converge in the strong topology for a.e. $k \in \mathbb{R}$. This completes the proof. □

Remark 4.12. *From* [6, Lemma 5.4]*, it is easily seen that the test vectors of first type from Lemma 4.10 belong to $\mathcal{D}(\mathcal{L}_{tr})$ provided that $v_+ \in H^1(\mathbb{R}_+, E)$, $v_- \in H^1(\mathbb{R}_-, E_*)$ with $v_+(0) = -e$ and $v_-(0) = -S(\lambda_0)e$, in particular, $v_-(0) = S(\lambda_0)v_+(0)$.*

Concerning the second type of test vectors $(v_-, (\Gamma_*(A^* - \overline{\mu}_0)^{-1})^* e_*, v_+) \in \mathcal{H}_{tr}$ with $e_* \in E_*, \mu_0 \in \mathbb{C}_-$ from Lemma 4.10, we have that they lie in $\mathcal{D}(\mathcal{L}_{tr})$ provided $v_-(0) = -e_*$ and $v_+(0) = S^*(\overline{\mu}_0)v_-(0)$.

Lemma 4.13. *Define the map*

$$\Phi : \left(\begin{array}{c} v_- \\ (\Gamma_*(A^* - \overline{\mu}_0)^{-1})^* e_* \\ v_+ \end{array} \right) \to \left(\begin{array}{c} \tilde{g}(k) \\ g(k) \end{array} \right) = \left(\begin{array}{c} \hat{v}_+(k) + \dfrac{i}{\sqrt{2\pi}} \dfrac{S^*(\overline{\mu}_0)e_*}{k-\mu_0} \\ \hat{v}_-(k) - \dfrac{i}{\sqrt{2\pi}} \dfrac{e_*}{k-\mu_0} \end{array} \right)$$

for any $\mu_0 \in \mathbb{C}_-, e_* \in E_*, v_+ \in L_2(\mathbb{R}_+, E), v_- \in L_2(\mathbb{R}_- E_*)$. This map satisfies the condition (4.7)*.*

Proof. The statement of the lemma and its proof are both completely analogous to those of Lemma 4.11. Therefore we omit the details and simply note that in the proof we use the following identities

$$\frac{S(\overline{\mu}_0) - S(\overline{z})}{\overline{z} - \overline{\mu}_0} = -i(\Gamma_*(A^* - \overline{\mu}_0)^{-1})(\Gamma(A-z)^{-1})^*$$

with $z, \mu_0 \in \mathbb{C}_-$ which is obtained from (2.8), and

$$\frac{S(z)S^*(\overline{\mu}_0) - I_{E_*}}{z-\mu_0} = i(\Gamma_*(A^*-z)^{-1})(\Gamma_*(A^*-\overline{\mu}_0)^{-1})^*$$

with $z \in \mathbb{C}_+$, $\mu_0 \in \mathbb{C}_-$, which is obtained from (2.10) and noting $S(z) = S_*^*(\overline{z})$ from Lemma 2.9. □

In order to avoid unnecessary notation we have used the symbol Φ to denote both the maps from Lemmas 4.11 and 4.13. The construction of the map Φ allows us to extend it by linearity to any finite linear combination of test functions of both of the above types. We will see in the next lemma that this procedure does not lead to any contradiction. Moreover, it justifies the use of the same symbol Φ for both maps. In view of this, the extended embedding maps of the test vectors into the spectral version's Hilbert space $\mathcal{H}_{sp}$ have the form

$$\Phi \left(\begin{array}{c} v_- \\ \sum_j (\Gamma(A-\overline{\lambda}_j)^{-1})^* e_j \\ v_+ \end{array} \right) \to \left(\begin{array}{c} \hat{v}_+(k) + \dfrac{i}{\sqrt{2\pi}} \sum_j \dfrac{e_j}{k-\lambda_j} \\ \hat{v}_-(k) - \dfrac{i}{\sqrt{2\pi}} \sum_j \dfrac{S(\lambda_j)e_j}{k-\lambda_j} \end{array} \right), \tag{4.9}$$

$v_- \in L_2(\mathbb{R}_-, E_*)$, $v_+ \in L_2(\mathbb{R}_+, E)$, $\lambda_j \in \mathbb{C}_+$, $e_j \in E$, $j = 1, 2...N$ and

$$\Phi \begin{pmatrix} v_- \\ \sum_j (\Gamma_*(A^* - \overline{\mu}_j)^{-1})^* e_{*j} \\ v_+ \end{pmatrix} \to \begin{pmatrix} \hat{v}_+(k) + \frac{i}{\sqrt{2\pi}} \sum_j \frac{S^*(\overline{\mu}_j) e_{*j}}{k - \mu_j} \\ \hat{v}_-(k) - \frac{i}{\sqrt{2\pi}} \sum_j \frac{e_{*j}}{k - \mu_j} \end{pmatrix} \tag{4.10}$$

$v_- \in L_2(\mathbb{R}_-, E_*)$, $v_+ \in L_2(\mathbb{R}_+, E)$, $\mu_j \in \mathbb{C}_-$, $e_{*j} \in E_*$, $j = 1, 2, ...N$.

Step 3: Isometry property of the embedding maps.

Lemma 4.14. *The embedding maps given by (4.9) and (4.10) are isometries. By linearity, they generate the isometry map Φ from a dense subset of $\mathcal{H}_{tr} \ominus (0, H_{sa}, 0)$ to $\mathcal{H}_{sp}$.*

Proof. From Lemma 4.10, we know that the linear set generated by test vectors of both types is dense in $\mathcal{H}_{tr} \ominus (0, H_{sa}, 0)$. Therefore it remains to show the isometry property and that (4.9) and (4.10) do not lead to a contradiction.

Consider two test vectors $\begin{pmatrix} v_- \\ \sum_{j=1}^N (\Gamma(A - \overline{\lambda}_j)^{-1})^* e_j \\ v_+ \end{pmatrix}$ and $\begin{pmatrix} w_- \\ \sum_{m=1}^M (\Gamma(A - \overline{\mu}_m)^{-1})^* f_j \\ w_+ \end{pmatrix}$ as on the left hand side of (4.9). In particular, we assume here that $\lambda_j \in \mathbb{C}_+$, $j = 1, ..., N$ and $\mu_m \in \mathbb{C}_+$, $m = 1, 2, ..., M$ and that the vectors $e_j, f_m \in E$ for all values of j and m. Let $\begin{pmatrix} \tilde{g} \\ g \end{pmatrix}$ and $\begin{pmatrix} \tilde{f} \\ f \end{pmatrix}$ denote their images under Φ, respectively.

Then

$$\left\langle \begin{pmatrix} \tilde{g} \\ g \end{pmatrix}, \begin{pmatrix} \tilde{f} \\ f \end{pmatrix} \right\rangle_{\mathcal{H}_{sp}} = \left\langle \begin{pmatrix} I_E & S^* \\ S & I_{E_*} \end{pmatrix} \begin{pmatrix} \tilde{g} \\ g \end{pmatrix}, \begin{pmatrix} \tilde{f} \\ f \end{pmatrix} \right\rangle_{L_2(\mathbb{R}, E \oplus E_*)}$$

$$= \left\langle \hat{v}_+(k) + \frac{i}{\sqrt{2\pi}} \sum_j \frac{e_j}{k - \lambda_j} + S^*(k)\hat{v}_-(k) - \frac{i}{\sqrt{2\pi}} \sum_j \frac{S^*(k)S(\lambda_j)e_j}{k - \lambda_j}, \hat{w}_+(k) + \frac{i}{\sqrt{2\pi}} \sum_m \frac{f_m}{k - \mu_m} \right\rangle_{L_2(\mathbb{R}, E)}$$

$$+ \left\langle S(k)\hat{v}_+(k) + \frac{i}{\sqrt{2\pi}} \sum_j \frac{S(k)e_j}{k - \lambda_j} + \hat{v}_-(k) - \frac{i}{\sqrt{2\pi}} \sum_j \frac{S(\lambda_j)e_j}{k - \lambda_j}, \hat{w}_-(k) - \frac{i}{\sqrt{2\pi}} \sum_m \frac{S(\mu_m)f_m}{k - \mu_m} \right\rangle_{L_2(\mathbb{R}, E_*)}. \tag{4.11}$$

This follows since images of Φ consist of images of finite linear combinations of test vectors of the first type belonging to $L_2(\mathbb{R}, E) \oplus L_2(\mathbb{R}, E_*) \subset \mathcal{H}_{sp}$.

To continue the explicit calculation of the last expression (4.11) notice that by the Paley-Wiener Theorem [14],

$$\hat{v}_+(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} e^{ik\xi} v_+(\xi) d\xi \in H_2^+(E),$$

$$\hat{v}_-(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_-} e^{ik\xi} v_-(\xi) d\xi \in H_2^-(E_*),$$

and

$$\frac{S(k)e_j}{k - \lambda_j} = \frac{(S(k) - S(\lambda_j))e_j}{k - \lambda_j} + \frac{S(\lambda_j)e_j}{k - \lambda_j} \tag{4.12}$$

gives the orthogonal decomposition of the vector from $L_2(\mathbb{R}, E_*)$ into the sum of two vector functions from $H_2^+(E_*)$ and $H_2^-(E_*)$, respectively.

Then, omitting the index on the scalar product in the Hilbert spaces $L_2(\mathbb{R}, E)$ and $L_2(\mathbb{R}, E_*)$ and using its linearity properties, we get that

$$\left\langle \begin{pmatrix} \tilde{g} \\ g \end{pmatrix}, \begin{pmatrix} \tilde{f} \\ f \end{pmatrix} \right\rangle_{\mathcal{H}_{sp}} = \langle \hat{v}_+, \hat{w}_+ \rangle + \langle \hat{v}_-, \hat{w}_- \rangle + \frac{1}{2\pi} \left\langle \sum_j \frac{e_j}{k - \lambda_j}, \sum_m \frac{f_m}{k - \mu_m} \right\rangle - \frac{1}{2\pi} \left\langle \sum_j \frac{S^* S(\lambda_j) e_j}{k - \lambda_j}, \sum_m \frac{f_m}{k - \mu_m} \right\rangle$$

$$+ \left\langle S^*(k)\hat{v}_-, \frac{i}{\sqrt{2\pi}} \sum_m \frac{f_m}{k - \mu_m} \right\rangle + \frac{i}{\sqrt{2\pi}} \left\langle \hat{v}_-, \frac{S(\mu_m) f_m}{k - \mu_m} \right\rangle. \tag{4.13}$$

Here we have used that

(1) $\hat{v}_+ \in H_2^+(E) \perp \sum_m \frac{f_m}{k - \mu_m} \in H_2^-(E)$,

(2) $\sum_j \frac{e_j}{k - \lambda_j} \in H_2^-(E) \perp \hat{w}_+ \in H_2^+(E)$,

(3) $S^*(k)\hat{v}_- \in H_2^-(E) \perp \hat{w}_+ \in H_2^+(E)$,

(4) $\sum_j \dfrac{S^*S(\lambda_j)e_j}{k-\lambda_j} \in H_2^-(E) \perp \hat{w}_+ \in H_2^+(E)$,

(5) $S(k)\hat{v}_+ \in H_2^+(E_*) \perp \left(\hat{w}_- - \dfrac{i}{\sqrt{2\pi}}\sum_m \dfrac{S(\mu_m)f_m}{k-\mu_m}\right) \in H_2^-(E_*)$,

(6) $\sum_j \dfrac{S(k)e_j}{k-\lambda_j} - \sum_j \dfrac{S(\lambda_j)e_j}{k-\lambda_j} \in H_2^+(E_*) \perp \left(\hat{w}_- - \dfrac{i}{\sqrt{2\pi}}\sum_m \dfrac{S(\mu_m)f_m}{k-\mu_m}\right) \in H_2^-(E_*)$; this is due to the decomposition (4.12).

The last two terms in (4.13) cancel because

$$
\begin{aligned}
\left\langle S^*(k)\hat{v}_-, \frac{i}{\sqrt{2\pi}}\sum_m \frac{f_m}{k-\mu_m}\right\rangle &= -\frac{i}{\sqrt{2\pi}}\left\langle \hat{v}_-, \sum_m \frac{S(k)f_m}{k-\mu_m}\right\rangle \\
&= -\frac{i}{\sqrt{2\pi}}\left\langle \hat{v}_-, \sum_m \frac{S(\mu_m)f_m}{k-\mu_m}\right\rangle - \frac{i}{\sqrt{2\pi}}\left\langle \hat{v}_-, \sum_m \frac{(S(k)-S(\mu_m))f_m}{k-\mu_m}\right\rangle \\
&= -\frac{i}{\sqrt{2\pi}}\left\langle \hat{v}_-, \sum_m \frac{S(\mu_m)f_m}{k-\mu_m}\right\rangle
\end{aligned}
$$

by the orthogonality of $H_2^-(E_*)$ and $H_2^+(E_*)$.

We have

$$
\begin{aligned}
\left\langle \frac{S^*(k)S(\lambda_j))e_j}{k-\lambda_j}, \frac{f_m}{k-\mu_m}\right\rangle &= \left\langle \frac{S(\lambda_j)e_j}{k-\lambda_j}, \frac{S(k)f_m}{\lambda-\mu_m}\right\rangle \\
&= \left\langle \frac{S(\lambda_j)e_j}{k-\lambda_j}, \frac{(S(k)-S(\mu_m))f_m}{k-\mu_m}\right\rangle + \left\langle \frac{S(\lambda_j)e_j}{k-\lambda_j}, \frac{S(\mu_m)f_m}{k-\mu_m}\right\rangle \\
&= \left\langle \frac{S(\lambda_j)e_j}{k-\lambda_j}, \frac{S(\mu_m)f_m}{k-\mu_m}\right\rangle = \left\langle \frac{S^*(\mu_m)S(\lambda_j)e_j}{k-\lambda_j}, \frac{f_m}{k-\mu_m}\right\rangle
\end{aligned}
$$

by (4.12) and the orthogonality of $H_2^+(E_*)$ and $H_2^-(E_*)$.

Therefore, applying the Parseval identity, (4.13) becomes

$$
\begin{aligned}
\left\langle \begin{pmatrix} \tilde{g} \\ g \end{pmatrix}, \begin{pmatrix} \tilde{f} \\ f \end{pmatrix}\right\rangle_{\mathcal{H}_{sp}} &= \langle v_+, w_+\rangle + \langle v_-, w_-\rangle + \frac{1}{2\pi}\left\langle \sum_j \frac{(I_E - S^*(k)S(\lambda_j))e_j}{k-\lambda_j}, \sum_m \frac{f_m}{k-\mu_m}\right\rangle \\
&= \langle v_+, w_+\rangle + \langle v_-, w_-\rangle + \frac{1}{2\pi}\sum_{j,m}\left\langle \frac{(I_E - S^*(\mu_m)S(\lambda_j))e_j}{k-\lambda_j}, \frac{f_m}{k-\mu_m}\right\rangle.
\end{aligned}
$$

An explicit calculation of the residues yields

$$
\frac{1}{2\pi i}\left\langle \frac{1}{k-\lambda_j}, \frac{1}{k-\mu_m}\right\rangle_{L_2(\mathbb{R})} = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{dk}{(k-\lambda_j)(k-\overline{\mu}_m)} = \frac{1}{\lambda_j - \overline{\mu}_m} \text{ for } \lambda_j, \mu_m \in \mathbb{C}_+.
$$

Hence,

$$
\begin{aligned}
\left\langle \begin{pmatrix} \tilde{g} \\ g \end{pmatrix}, \begin{pmatrix} \tilde{f} \\ f \end{pmatrix}\right\rangle_{\mathcal{H}_{sp}} &= \langle v_+, w_+\rangle + \langle v_-, w_-\rangle + \sum_j \sum_m \frac{i}{\lambda_j - \overline{\mu}_m}\langle (I_E - S^*(\mu_m)S(\lambda_j)e_j, f_m\rangle_E \\
&= \left\langle \begin{pmatrix} v_- \\ \sum_j (\Gamma(A-\overline{\lambda}_j)^{-1})^* e_j \\ v_+ \end{pmatrix}, \begin{pmatrix} w_- \\ \sum_m (\Gamma(A-\overline{\mu}_m)^{-1})^* f_m \\ w_+ \end{pmatrix}\right\rangle_{\mathcal{H}_{tr}}
\end{aligned}
$$

due to (2.9) with $w = \mu_m$ and $z = \lambda_j$.

Similarly one proves that the second map, given in (4.10), is also isometric. The proof does not differ essentially from the previous one and will be omitted. We note, more generally that the proof also follows from interchanging A and A^*.

To complete the proof of Lemma 4.14 it is sufficient to consider two test vectors which are an arbitrary linear combination of vectors of the first and second type:

$$
\overrightarrow{G} := \begin{pmatrix} v_- \\ \sum_j (\Gamma(A-\overline{\lambda}_j)^{-1})^* e_j + \sum_m (\Gamma_*(A^* - \overline{\eta}_m)^{-1})^* e_{*m} \\ v_+ \end{pmatrix}
$$

with $v_- \in L_2(\mathbb{R}_-, E_*)$, $v_+ \in L_2(\mathbb{R}_+, E)$, $\lambda_j \in \mathbb{C}_+$, $e_j \in E$ and $\eta_m \in \mathbb{C}_-$, $e_{*m} \in E_*$ for all j and m.

Decomposing the vector $\overrightarrow{G}$ into the sum of two vectors of the first and second type separately we get

$$\overrightarrow{G} := \overrightarrow{G_1} + \overrightarrow{G_2} := \begin{pmatrix} 0 \\ \sum_j (\Gamma(A-\overline{\lambda}_j)^{-1})^* e_j \\ 0 \end{pmatrix} + \begin{pmatrix} v_- \\ \sum_m (\Gamma_*(A^* - \overline{\eta}_m)^{-1})^* e_{*_m} \\ v_+ \end{pmatrix}.$$

Next we define

$$\Phi\overrightarrow{G} := \Phi\overrightarrow{G_1} + \Phi\overrightarrow{G_2} = \begin{pmatrix} \frac{i}{\sqrt{2\pi}} \sum_j \frac{e_j}{k-\lambda_j} \\ -\frac{i}{\sqrt{2\pi}} \sum_j \frac{S(\lambda_j)e_j}{k-\lambda_j} \end{pmatrix} + \begin{pmatrix} \hat{v}_+(k) + \frac{i}{\sqrt{2\pi}} \sum_m \frac{S^*(\overline{\eta}_m)e_{*m}}{k-\eta_m} \\ \hat{v}_-(k) - \frac{i}{\sqrt{2\pi}} \sum_m \frac{e_{*m}}{k-\eta_m} \end{pmatrix},$$

while a similar notation will be used for the second vector $\overrightarrow{F} := \overrightarrow{F_1} + \overrightarrow{F_2}$. To show that Φ is well-defined, it is sufficient to show that it has the isometry property.

By linearity it is enough to consider just one term of different types in each sum over, i.e.

$$\overrightarrow{G_1} := \begin{pmatrix} 0 \\ (\Gamma(A-\overline{\lambda})^{-1})^* e \\ 0 \end{pmatrix} \quad \text{and} \quad \overrightarrow{G_2} := \begin{pmatrix} v_- \\ (\Gamma_*(A^* - \overline{\eta})^{-1})^* e_* \\ v_+ \end{pmatrix},$$

and similarly

$$\overrightarrow{F_1} := \begin{pmatrix} 0 \\ (\Gamma(A-\overline{\mu})^{-1})^* f \\ 0 \end{pmatrix} \quad \text{and} \quad \overrightarrow{F_2} := \begin{pmatrix} w_- \\ (\Gamma_*(A^* - \overline{\nu})^{-1})^* f_* \\ w_+ \end{pmatrix},$$

with $\mu, \nu \in \mathbb{C}_-$. Now, using that Φ is an isometry on each type of test vectors individually,

$$\begin{aligned} \langle \Phi(\overrightarrow{G_1} + \overrightarrow{G_2}), \Phi(\overrightarrow{F_1} + \overrightarrow{F_2}) \rangle_{\mathcal{H}_{sp}} &= \langle \Phi\overrightarrow{G_1}, \Phi\overrightarrow{F_1} \rangle_{\mathcal{H}_{sp}} + \langle \Phi\overrightarrow{G_2}, \Phi\overrightarrow{F_2} \rangle_{\mathcal{H}_{sp}} + \langle \Phi\overrightarrow{G_1}, \Phi\overrightarrow{F_2} \rangle_{\mathcal{H}_{sp}} + \langle \Phi\overrightarrow{G_2}, \Phi\overrightarrow{F_1} \rangle_{\mathcal{H}_{sp}} \\ &= \langle \overrightarrow{G_1}, \overrightarrow{F_1} \rangle_{H_{tr}} + \langle \overrightarrow{G_2}, \overrightarrow{F_2} \rangle_{H_{tr}} + \langle \Phi\overrightarrow{G_1}, \Phi\overrightarrow{F_2} \rangle_{\mathcal{H}_{sp}} + \overline{\langle \Phi\overrightarrow{F_1}, \Phi\overrightarrow{G_2} \rangle}_{\mathcal{H}_{sp}}. \end{aligned}$$

Therefore, to show that

$$\langle \Phi(\overrightarrow{G_1} + \overrightarrow{G_2}), \Phi(\overrightarrow{F_1} + \overrightarrow{F_2}) \rangle_{\mathcal{H}_{sp}} = \langle \overrightarrow{G_1} + \overrightarrow{G_2}, \overrightarrow{F_1} + \overrightarrow{F_2} \rangle_{\mathcal{H}_{tr}}, \tag{4.14}$$

it is enough to check that $\langle \Phi\overrightarrow{G_1}, \Phi\overrightarrow{F_2} \rangle_{\mathcal{H}_{sp}} = \langle \overrightarrow{G_1}, \overrightarrow{F_2} \rangle_{\mathcal{H}_{tr}}$. We have

$$\begin{aligned} \langle \Phi\overrightarrow{G_1}, \Phi\overrightarrow{F_2} \rangle_{\mathcal{H}_{sp}} &= \left\langle \begin{pmatrix} I_E & S^*(k) \\ S(k) & I_{E_*} \end{pmatrix} \begin{pmatrix} \frac{i}{\sqrt{2\pi}} \frac{e}{k-\lambda} \\ -\frac{i}{\sqrt{2\pi}} \frac{S(\lambda)e}{k-\lambda} \end{pmatrix}, \begin{pmatrix} \hat{w}_+(k) + \frac{i}{\sqrt{2\pi}} \frac{S^*(\overline{\nu})f_*}{k-\nu} \\ \hat{w}_-(k) - \frac{i}{\sqrt{2\pi}} \frac{f_*}{k-\nu} \end{pmatrix} \right\rangle_{L_2(\mathbb{R}, E \oplus E_*)} \\ &= \left\langle \begin{pmatrix} \frac{i}{\sqrt{2\pi}(k-\lambda)} (I_E - S^*(k)S(\lambda))e \\ \frac{i}{\sqrt{2\pi}(k-\lambda)} (S(k) - S(\lambda))e \end{pmatrix}, \begin{pmatrix} \hat{w}_+(k) + \frac{i}{\sqrt{2\pi}} \frac{S^*(\overline{\nu})f_*}{k-\nu} \\ \hat{w}_-(k) - \frac{i}{\sqrt{2\pi}} \frac{f_*}{k-\nu} \end{pmatrix} \right\rangle_{L_2(\mathbb{R}, E \oplus E_*)}. \end{aligned}$$

Since $\frac{1}{k-\lambda}(I_E - S^*(k)S(\lambda))e \in H_2^-(E)$ for $\lambda \in \mathbb{C}_+$, while $\hat{w}_+(k) + \frac{i}{\sqrt{2\pi}} \frac{S^*(\overline{\nu})f_*}{k-\nu} \in H_2^+(E)$ for $\nu \in \mathbb{C}_-$, and $\frac{1}{k-\lambda}(S(k) - S(\lambda))e \in H_2^+(E_*)$ for $\lambda \in \mathbb{C}_+$, while $\hat{w}_-(k) \in H_2^-(E_*)$, many terms in the scalar product vanish and we are left with

$$\langle \Phi\overrightarrow{G_1}, \Phi\overrightarrow{F_2} \rangle_{\mathcal{H}_{sp}} = \left\langle \frac{i}{\sqrt{2\pi}} \frac{1}{k-\lambda} (S(k) - S(\lambda))e, -\frac{i}{\sqrt{2\pi}} \frac{f_*}{k-\nu} \right\rangle_{L_2(\mathbb{R}, E_*)}.$$

Finally, calculating the residue at point $k = \overline{\nu} \in \mathbb{C}_+$, we get

$$\begin{aligned} \langle \Phi\overrightarrow{G_1}, \Phi\overrightarrow{F_2} \rangle_{H_{sp}} &= -\frac{1}{2\pi} \left\langle \frac{(S(k) - S(\lambda))e}{k-\lambda}, \frac{f_*}{k-\nu} \right\rangle_{L_2(\mathbb{R}, E_*)} = -\frac{1}{2\pi} \int_{\mathbb{R}} dk \frac{\langle (S(k) - S(\lambda))e, f_* \rangle_{E_*}}{(k-\lambda)(k-\overline{\nu})} \\ &= \left(-\frac{2\pi i}{2\pi} \right) \frac{\langle (S(\overline{\nu}) - S(\lambda))e, f_* \rangle_{E_*}}{(\overline{\nu} - \lambda)} = -i \left\langle \left(\frac{S(\overline{\nu}) - S(\lambda)}{\overline{\nu} - \lambda} \right) e, f_* \right\rangle_{E_*}. \end{aligned}$$

By (2.8), for $\lambda \in \mathbb{C}_+$ and $\nu \in \mathbb{C}_-$, we have

$$\frac{S(\overline{\nu}) - S(\lambda)}{\overline{\nu} - \lambda} = i(\Gamma_*(A^* - \overline{\nu})^{-1})(\Gamma(A - \overline{\lambda}))^{-1})^*.$$

Hence

$$\begin{aligned}\langle \Phi\overrightarrow{G_1}, \Phi\overrightarrow{F_2}\rangle_{\mathcal{H}_{sp}} &= \langle (\Gamma_*(A^* - \overline{\nu})^{-1})(\Gamma(A - \overline{\lambda}))^{-1})^* e, f_*\rangle_{E_*} \\ &= \langle (\Gamma(A - \overline{\lambda})^{-1})^* e, ((\Gamma_*(A^* - \overline{\nu}))^{-1})^* f_*\rangle_H = \langle \overrightarrow{G_1}, \overrightarrow{F_2}\rangle_{\mathcal{H}_{tr}},\end{aligned}$$

as required. This completes the proof of the lemma. $\square$

Step 4: $\Phi : \mathcal{H}_{tr} \ominus (0, H_{sa}, 0) \to \mathcal{H}_{sp}$ is surjective.

We have shown that the map Φ admits a unique extension as an isometric operator to the closure of all test vectors. According to Lemma 4.10, the closure coincides with $\mathcal{H}_{tr} \ominus (0, H_{sa}, 0)$, where $\mathcal{H}_{sa}$ is the selfadjoint subspace of A in the Langer decomposition. We will use the same letter Φ for the isometric extension of Φ to $\mathcal{H}_{tr} \ominus (0, H_{sa}, 0)$.

Lemma 4.15. *The map Φ, defined on test vectors in* (4.9) *and* (4.10) *and extended by linearity and continuity to $\mathcal{H}_{tr} \ominus (0, H_{sa}, 0)$, has the property that $\overline{\operatorname{Ran}\Phi} = \mathcal{H}_{sp}$.*

Proof. Consider a vector $\begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \in \mathcal{H}_{sp}$ such that $\begin{pmatrix} \tilde{g} \\ g \end{pmatrix}$ is orthogonal to $\Phi \begin{pmatrix} v_- \\ 0 \\ v_+ \end{pmatrix}$ for all $v_- \in L_2(\mathbb{R}_-, E_*), v_+ \in L_2(\mathbb{R}_+, E)$. Clearly the vector $\begin{pmatrix} v_- \\ 0 \\ v_+ \end{pmatrix} \in D(\Phi) = \mathcal{H}_{tr} \ominus (0, H_{sa}, 0)$. Moreover it is simultaneously a test vector of the first and second type, with $e = 0$ or $e_* = 0$, and therefore its image under Φ is easy to calculate. We have, due to $\hat{v}_+$ and $\hat{v}_-$ being L_2-functions,

$$\begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \perp \Phi \begin{pmatrix} v_- \\ 0 \\ v_+ \end{pmatrix} = \begin{pmatrix} \hat{v}_+(k) \\ \hat{v}_-(k) \end{pmatrix} \Leftrightarrow \left\langle \begin{pmatrix} \tilde{g} + S^* g \\ S\tilde{g} + g \end{pmatrix}, \begin{pmatrix} \hat{v}_+ \\ \hat{v}_- \end{pmatrix} \right\rangle_{L_2(\mathbb{R}; E \oplus E_*)} = 0,$$

i.e. $\tilde{g} + S^* g \perp \hat{v}_+$ and $S\tilde{g} + g \perp \hat{v}_-$. Since, by the Paley-Wiener Theorem (see [14]), $\hat{v}_\pm$ run over the whole of the spaces $H_2^+(E)$, $H_2^-(E_*)$, respectively, and $\tilde{g} + S^* g \in L_2(\mathbb{R}, E), \tilde{S}g + g \in L_2(\mathbb{R}, E_*)$ we must have $\tilde{g} + S^* g \in H_2^-(E)$ and $S\tilde{g} + g \in H_2^+(E_*)$.

Additionally, $\begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \perp \Phi \begin{pmatrix} 0 \\ (\Gamma_*(A^* - \overline{\mu})^{-1})^* e_* \\ 0 \end{pmatrix}$ for all $\mu \in \mathbb{C}_-$ and $e_* \in E_*$. According to (4.10), we have

$$\Phi \begin{pmatrix} 0 \\ (\Gamma_*(A^* - \overline{\mu})^{-1})^* e_* \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{iS^*(\overline{\mu}) e_*}{\sqrt{2\pi}(k - \mu)} \\ -\dfrac{ie_*}{\sqrt{2\pi}(k - \mu)} \end{pmatrix}$$

and therefore by our assumption

$$0 = \left\langle \tilde{g} + S^* g, \frac{S^*(\overline{\mu}) e_*}{k - \mu} \right\rangle_{L_2(\mathbb{R}, E)} - \left\langle S\tilde{g} + g, \frac{e_*}{k - \mu} \right\rangle_{L_2(\mathbb{R}, E_*)}.$$

The first term is equal to 0, since $\dfrac{S^*(\overline{\mu}) e_*}{k - \mu} \in H_2^+(E)$ and we have already seen that $g + S^* g \in H_2^-(E)$. Hence

$$\left\langle S\tilde{g} + g, \frac{e_*}{k - \mu} \right\rangle_{L_2(\mathbb{R}, E_*)} = 0 \quad \text{for all } e_* \in E_* \text{ and } \mu \in \mathbb{C}_-,$$

i.e.

$$0 = \frac{1}{2\pi i} \left\langle S\tilde{g} + g, \frac{e_*}{k - \mu} \right\rangle_{L_2(\mathbb{R}, E_*)} = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\langle (S\tilde{g} + g)(k), e_*\rangle_{E_*}}{k - \overline{\mu}}\, dk.$$

The last equality means the Riesz projection P_+ onto H_2^+ of the scalar function $\langle (S\tilde{g} + g)(k), e_*)_{E_*} \in {H_2}^+$ is equal to 0, and hence $\langle (S\tilde{g} + g)(k), e_*)\rangle_{E_*} = 0$ for a.e. $k \in \mathbb{R}$ and for any fixed $e_* \in E$. Choosing a countable orthonormal basis in E_* as vectors e_*, we have $(S\tilde{g} + g)(k) = 0$ for a.e. $k \in \mathbb{R}$, which means $S\tilde{g} + g = 0$.

Similarly, the other condition

$$\begin{pmatrix} \tilde g \\ g \end{pmatrix} \perp \Phi \begin{pmatrix} 0 \\ (\Gamma(A-\overline{\lambda})^{-1})^* e \\ 0 \end{pmatrix} = \frac{i}{\sqrt{2\pi}} \begin{pmatrix} \dfrac{e}{k-\lambda} \\ -\dfrac{S(\lambda)e}{k-\lambda} \end{pmatrix}, \lambda \in \mathbb{C}_+, e \in E,$$

means that

$$\begin{pmatrix} \tilde g + S^* g \\ S\tilde g + g \end{pmatrix} \perp \begin{pmatrix} \dfrac{e}{k-\lambda} \\ -\dfrac{S(\lambda)e}{k-\lambda} \end{pmatrix}$$

in $L_2(\mathbb{R}, E \oplus E_*)$. Since $S\tilde g + g \in H_2^+(E_*)$ and $\dfrac{S(\lambda)e}{k-\lambda} \in H_2^-(E_*)$ for all $\lambda \in \mathbb{C}^+$ this orthogonality condition can be written as

$$\left\langle \tilde g + S^* g, \frac{e}{k-\lambda} \right\rangle_{L_2(\mathbb{R},E)} = 0, \quad \text{for all} \quad e \in E, \lambda \in \mathbb{C}_+,$$

i.e. $P_- \langle \tilde g + S^* g, e\rangle_E = 0$. Since $\langle \tilde g + S^* g, e\rangle)_E \in H_2^-$, we also have $\langle \tilde g + S^* g, e\rangle_E = 0$ for all $e \in E$ or $\tilde g + S^* g = 0$, as in the previous case. In summary this yields $(S\tilde g + g) = 0$ and $(\tilde g + S^* g) = 0$ which, by Lemma 4.7, gives

$$\begin{pmatrix} \tilde g \\ g \end{pmatrix} = 0,$$

which proves the result. □

Now we are ready to prove our previous claim concerning the validity of using a single symbol Φ for both maps of test vectors acting separately on functions of the first and second kind.

Lemma 4.16. *Let two linear sets in $\mathcal{H}_{tr}$ be:*

$$\mathcal{L}_1 := \text{Span} \left\{ \begin{pmatrix} v_- \\ (\Gamma(A-\overline{\lambda})^{-1})^* e \\ v_+ \end{pmatrix} \middle| v_- \in L_2(\mathbb{R}_-, E_*), v_+ \in L_2(\mathbb{R}_+, E), e \in E, \lambda \in \mathbb{C}_+ \right\}$$

and

$$\mathcal{L}_2 := \text{Span} \left\{ \begin{pmatrix} v_- \\ (\Gamma_*(A^*-\overline{\mu})^{-1})^* e_* \\ v_+ \end{pmatrix} \middle| v_- \in L_2(\mathbb{R}_-, E_*), v_+ \in L_2(\mathbb{R}_+, E), e_* \in E_*, \mu \in \mathbb{C}_- \right\},$$

then

$$\overline{\mathcal{L}_1 + \mathcal{L}_2} = \mathcal{H}_{tr} \ominus (0, H_{sa}, 0)$$

and for any vector $\overrightarrow{F} \in \overline{\mathcal{L}}_1 \cap \overline{\mathcal{L}}_2$ we have $\Phi_1 \overrightarrow{F} = \Phi_2 \overrightarrow{F}$ where Φ_1 is a map defined on $\overline{\mathcal{L}}_1$ by (4.9) and Φ_2 is the map defined on $\overline{\mathcal{L}}_2$ by (4.10) after taking the closure of the isometric operators $\Phi_j : \mathcal{L}_j \to \mathcal{H}_{sp}$. $j = 1, 2$.

Proof. That

$$\overline{\mathcal{L}_1 + \mathcal{L}_2} = \mathcal{H}_{tr} \ominus (0, H_{sa}, 0)$$

follows from Lemma 4.10.

Let $\overrightarrow{F} \in \overline{\mathcal{L}}_1 \cap \overline{\mathcal{L}}_2$. Then, using the isometry property of both maps Φ for test vectors of both first and second type, we may extend the identity (4.14) to the closure of both types of vectors, $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$. Let $\overrightarrow{F} = \overrightarrow{F}_1 = \overrightarrow{F}_2$ with $\overrightarrow{F}_1 \in \overline{\mathcal{L}}_1$ and $\overrightarrow{F}_2 \cap \overline{\mathcal{L}}_2$. Then for arbitrary $\overrightarrow{G}_1 \in \overline{\mathcal{L}}_1$ and $\overrightarrow{G}_2 \in \overline{\mathcal{L}}_2$

$$\langle \Phi_1 \overrightarrow{F}_1 - \Phi_2 \overrightarrow{F}_2, \Phi_1 \overrightarrow{G}_1 + \Phi_2 \overrightarrow{G}_2 \rangle_{\mathcal{H}_{sp}} = (\overrightarrow{F}_1 - \overrightarrow{F}_2, \overrightarrow{G}_1 + \overrightarrow{G}_2)_{\mathcal{H}_{tr}} = 0.$$

Since by Lemma 4.16 the set of images of the test vectors from Span $\{\mathcal{L}_1, \mathcal{L}_2\}$ is dense in $\mathcal{H}_{sp}$, we have

$$\Phi_1 \overrightarrow{F} - \Phi_2 \overrightarrow{F} = 0.$$

□

<u>**Step 5:**</u> The intertwining identity

So far we have seen that Theorem 4.8 delivers an isometric linear map Φ of $\mathcal{H}_{tr} \ominus \{O \oplus H_{sa} \oplus O\}$ onto $\mathcal{H}_{sp} = \left(L_2(\mathbb{R}; E \oplus E_*; \begin{pmatrix} I_E & S^*(k) \\ S(k) & I_{E_*} \end{pmatrix} dk \right)$ satisfying conditions (4.7). The formula for Φ is explicit on the set of special test vectors of both the first and second kind. It remains to show that the transform Φ constructed above gives the spectral representation of the minimal selfadjoint dilation $\mathcal{L}$ of the of the completely non-selfadjoint part of the maximally dissipative operator A.

Lemma 4.17. *(The intertwining identity) We have that* $\Phi(\mathcal{L}_{tr}-\lambda)^{-1} = (k-\lambda)^{-1}\Phi$ *on* $\mathcal{H} \ominus \{O \oplus H_{sa} \oplus O\}$ *for all* $\lambda \in \mathbb{C}\backslash\mathbb{R}$.

Proof. Let us assume w.l.o.g. that $H_{sa} = O$, i.e. A is a completely non-selfadjoint operator. Then, Φ being surjective and isometric, we have $\Phi^*\Phi = I_{\mathcal{H}_{tr}}$ and $\Phi\Phi^* = I_{\mathcal{H}_{sp}}$. We will calculate the resolvent $(\mathcal{L}_{tr}-\lambda)^{-1}$ on the whole space $\mathcal{H}_{tr}$.

The equality

$$\Phi(\mathcal{L}_{tr}-\lambda)^{-1} = (k-\lambda)^{-1}\Phi \tag{4.15}$$

is equivalent to

$$\mathcal{F}_{\pm}(\mathcal{L}_{tr}-\lambda)^{-1} = (k-\lambda)^{-1}\mathcal{F}_{\pm}, \tag{4.16}$$

for both signs simultaneously. Indeed, we have checked in Lemma 4.11 and Lemma 4.13 that our map Φ satisfies the condition:

$$\begin{pmatrix} I_E & S^* \\ S & I_{E_*} \end{pmatrix} \Phi \overrightarrow{F} = \begin{pmatrix} \mathcal{F}_+ \overrightarrow{F} \\ \mathcal{F}_- \overrightarrow{F} \end{pmatrix} \tag{4.17}$$

on the test vectors $\overrightarrow{F}$, which generate a dense set in $\mathcal{H}_{tr}$, under the condition $H_{sa} = \{0\}$. Since $\|\Phi\| = 1$ and the map

$$\overrightarrow{F} \mapsto \begin{pmatrix} \mathcal{F}_+ \overrightarrow{F} \\ \mathcal{F}_- \overrightarrow{F} \end{pmatrix}$$

from the set of test vectors to $L_2(\mathbb{R}, E) \oplus L_2(\mathbb{R}, E_*)$ has norm which obviously does not exceed 2, we can extend the equality (4.17) to the whole space $\mathcal{H}_{tr}$.

We remind the reader that the vector $\begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \in \mathcal{H}_{sp}$ is equal to 0 iff $\begin{pmatrix} I_E & S^* \\ S & I_{E_*} \end{pmatrix}\begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = 0$, so the equality (4.16) is equivalent (after multiplication on the left by the matrix function of $k \in \mathbb{R}$ $\begin{pmatrix} I_E & S^*(k) \\ S(k) & I_{E_*} \end{pmatrix}$ and $(k-\lambda)^{-1}$), to the condition (4.15) due to the commutation of the two operations of multiplication by $(k-\lambda)^{-1}$ and by the matrix-function mentioned above.

Consider the vector $\overrightarrow{\mathcal{F}} = \begin{pmatrix} v_- \\ u \\ v_+ \end{pmatrix} \in \mathcal{H}_{tr}$ and denote $(\mathcal{L}_{tr}-\lambda)^{-1}\begin{pmatrix} v_- \\ u \\ v_+ \end{pmatrix} =: \begin{pmatrix} \tilde{v}_- \\ \tilde{u}, \\ \tilde{v}_+ \end{pmatrix} \in \mathcal{H}_{tr}$, $\lambda \notin \mathbb{R}$. Then $(\tilde{v}_-, \tilde{u}, \tilde{v}_+) \in \mathcal{D}(\mathcal{L}_{tr})$ and, by the definition of the dilation,

$$v_- = i\tilde{v}_-' - \lambda\tilde{v}_-, \tag{4.18}$$

$$v_+ = i\tilde{v}_+' - \lambda\tilde{v}_+, \tag{4.19}$$

$$u = T_* \begin{pmatrix} \tilde{v}_- \\ \tilde{u} \\ \tilde{v}_+ \end{pmatrix} - \lambda\tilde{u}. \tag{4.20}$$

Inclusion of $(\tilde{v}_-, \tilde{u}, \tilde{v}_+)$ in $\mathcal{D}(\mathcal{L}_{tr})$ leads additionally to the following facts: $\tilde{v}_+ \in H^1(\mathbb{R}_+, E), \tilde{v}_- \in H^1(\mathbb{R}_-, E_*)$ and

$$\begin{cases} \tilde{u} + (\Gamma_*(A^*+\mu)^{-1})^*\tilde{v}_-(0) \in \mathcal{D}(A), \mu \in \mathbb{C}_-, \\ \tilde{v}_+(0) = S^*(-\mu)\tilde{v}_-(0) + i\Gamma(\tilde{u} + (\Gamma_*(A^*+\mu)^{-1})^*\tilde{v}_-(0)). \end{cases} \tag{4.21}$$

Now we need to prove that for any fixed $\lambda \in \mathbb{C}\backslash\mathbb{R}$

$$\mathcal{F}_{\pm}\begin{pmatrix} \tilde{v}_- \\ \tilde{u} \\ \tilde{v}_+ \end{pmatrix} = (k-\lambda)^{-1}\mathcal{F}_{\pm}\begin{pmatrix} v_- \\ u \\ v_+ \end{pmatrix}. \tag{4.22}$$

Using the Fourier transform for equation (4.18), we get, after extension of $v_-, \tilde{v}_-$ by 0 on the positive half-line, that

$$\hat{v}_-(\xi) = \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{i\xi t}\tilde{v}_-'(t)dt \ - \lambda\hat{\tilde{v}}_-(\xi) = (\xi-\lambda)\hat{\tilde{v}}_-(\xi) + \frac{i}{\sqrt{2\pi}}\tilde{v}_-(0).$$

Similarly, using (4.19), we obtain

$$\hat{v}_+(\xi) = (\xi-\lambda)\hat{\tilde{v}}_+(\xi) - \frac{i}{\sqrt{2\pi}}\tilde{v}_+(0).$$

Using (4.3) for the operator T_* in (4.20), we have, independently of $\mu \in \mathbb{C}_-$,

$$u = A(\tilde{u}+(\Gamma_*(A^*+\mu)^{-1})^*\tilde{v}_-(0)) + \overline{\mu}(\Gamma_*(A^*+\mu)^{-1})^*\tilde{v}_-(0) - \lambda\tilde{u}).$$

If we fix the value of the parameter $\lambda \in \mathbb{C}_-$ (it is enough to prove (4.24) for λ in a half-plane, as the result on the complementary half-plane follows immediately by taking adjoint operators in (4.24)), the convenient choice of μ is $\mu = -\overline{\lambda} \in \mathbb{C}_-$. Then

$$u = (A-\lambda)[\tilde{u} + (\Gamma_*(A^*-\overline{\lambda})^{-1})^*\tilde{v}_-(0)] \quad \text{or} \quad \tilde{u} = (A-\lambda)^{-1}u - (\Gamma_+(A^*-\overline{\lambda})^{-1})^*\tilde{v}_-(0).$$

Let us first consider the case $\mathcal{F}_+$ in (4.22). Inserting the expression for $\tilde{u}$ from above, we now need to prove that for a.e. $k \in \mathbb{R}$

$$\lim_{\varepsilon\to+0}(-\frac{1}{\sqrt{2\pi}})\Gamma(A-k+i\varepsilon)^{-1}[(A-\lambda)^{-1}u - (\Gamma_*(A^*-\overline{\lambda})^{-1})^*\tilde{v}_-(0)] + S^*(k)\hat{\tilde{v}}_-(k) + \hat{\tilde{v}}_+(k)$$
$$= (k-\lambda)^{-1}[\lim_{\varepsilon\to+0}(-\frac{1}{\sqrt{2\pi}})\Gamma(A-k+i\varepsilon)^{-1}u + S^*(k)\hat{v}_-(k) + \hat{v}_+(k)].$$

Using the Hilbert identity

$$(A-k+i\varepsilon)^{-1}(A-\lambda)^{-1} = \frac{(A-\lambda)^{-1} - (A-k+i\varepsilon)^{-1}}{\lambda-k+i\varepsilon}$$

and substituting the explicit expressions for $\hat{v}_\pm(\xi)$ calculated above, we get that the equality we have to prove can be reduced to the following:

$$\Gamma(A-\lambda)^{-1}u + (k-\lambda)\lim_{\varepsilon\to+0}\Gamma(A-k+i\varepsilon)^{-1}(\Gamma_*(A^*-\overline{\lambda})^{-1})^*\tilde{v}_-(0) + i[\tilde{v}_+(0) - S^*(k)\tilde{v}_-(0)] = 0. \tag{4.23}$$

Taking adjoints in (2.8), with $\mu = \overline{\lambda}$ and $\tilde{\mu} = \overline{z}$, both in $\mathbb{C}_+$ we have that

$$(\lambda - z)\Gamma(A-\overline{z})^{-1}(\Gamma_*(A^*-\overline{\lambda})^{-1})^* = i(S^*(\overline{\lambda}) - S^*(\overline{z})).$$

To complete the proof for $\mathcal{F}_+$ we need to substitute $z := k - i\varepsilon \in \mathbb{C}_-$ and take the limit as $\varepsilon \to +0$ in the strong topology of E_* for a.e. k. Indeed, following this procedure the proof of (4.23) reduces to

$$0 = \Gamma(A-\lambda)^{-1}u - i(S^*(\overline{\lambda}) - S^*(k))\tilde{v}_-(0) + i[\tilde{v}_+(0) - S^*(k)\tilde{v}_-(0)] = \Gamma(A-\lambda)^{-1}u + i\tilde{v}_+(0) - iS^*(\overline{\lambda})\tilde{v}_-(0).$$

Substituting $\mu = -\overline{\lambda}$ into (4.21) and taking into account that $(A-\lambda)^{-1}u = \tilde{u} + (\Gamma_*(A^*-\overline{\lambda})^{-1})^*\tilde{v}_-(0)$, we see that the expression vanishes, as desired. The second equality

$$\mathcal{F}_-\begin{pmatrix}\tilde{v}_-\\ \tilde{u}\\ \tilde{v}_+\end{pmatrix} = (k-\lambda)^{-1}\mathcal{F}_-\begin{pmatrix}v_-\\ u\\ v_+\end{pmatrix}$$

admits a similar proof. Although $T = T_*$ on $\mathcal{D}(\mathcal{L}_{tr})$, it is more convenient to use the operator T from (4.2) for the $\mathcal{F}_-$ case. □

This completes the proof of the main theorem. □

Remark 4.18. *For minimal selfadjoint dilations $\mathcal{L}_{tr}$ of a general maximally dissipative operator A we have immediately from Lemma 4.17 and the Langer decomposition Theorem 3.3 that*

$$(\mathcal{L}_{tr}-\lambda)^{-1} = \Phi^*(k-\lambda)^{-1}\Phi \oplus (A|_{H_{sa}}-\lambda)^{-1}, \tag{4.24}$$

where $\mathcal{L}_{tr}$ is an operator in the translation form space $\mathcal{H}_{tr}$ and the orthogonal sum corresponds to the Langer decomposition

$$\mathcal{H}_{tr} = (\mathcal{H}_{tr} \ominus (0, H_{sa}, 0)) \oplus (0, H_{sa}, 0)$$

and the selfadjoint operator $A|_{H_{sa}}$ acts in $(0, H_{sa}, 0)$ as an operator in the second component.

From the theorem, we get the following corollary.

Corollary 4.19. *We have the dilation property*

$$P_H(\mathcal{L}_{tr}-\lambda)^{-1}|_H = \begin{cases} (A^*-\lambda)^{-1}, & \lambda \in \mathbb{C}_+,\\ (A-\lambda)^{-1}, & \lambda \in \mathbb{C}_-\end{cases}$$

in $\mathcal{H}_{tr}$. If A is completely non-selfadjoint, this can be transformed into the spectral form version

$$P_H\Phi^*(k-\lambda)^{-1}\Phi|_H = \begin{cases} (A^*-\lambda)^{-1}, & \lambda \in \mathbb{C}_+,\\ (A-\lambda)^{-1}, & \lambda \in \mathbb{C}_-,\end{cases}$$

or

$$(\Phi P_H\Phi^*)(k-\lambda)^{-1}|_{\Phi(H)} = \begin{cases} \Phi(A^*-\lambda)^{-1}\Phi^*|_{\Phi(H)}, & \lambda \in \mathbb{C}_+,\\ \Phi(A-\lambda)^{-1}\Phi^*|_{\Phi(H)}, & \lambda \in \mathbb{C}_-.\end{cases}$$

Set $K := \Phi(0, H, 0) \subset \mathcal{H}_{sp}$. If A is a completely non-selfadjoint maximally dissipative operator, then $\Phi P_H \Phi^* = P_K$ is the orthogonal projection on to the subspace K in $\mathcal{H}_{sp}$ and $P_K \dfrac{1}{k-\lambda}$ is unitarily equivalent to the resolvent of A (for $\lambda \in \mathbb{C}_-$) or A^* (if $\lambda \in \mathbb{C}_+$). Explicit calculations (see [27, 28]) give that

$$P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = \begin{pmatrix} \tilde{g} - P_+(\tilde{g} + S^* g) \\ g - P_-(S\tilde{g} + g) \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \in \mathcal{H}_{sp},$$

where $P_\pm$ are Riesz projections onto

$$H_2^+(E) \subset L_2(\mathbb{R}, E) \quad \text{and} \quad H_2^-(E_*) \subset L_2(\mathbb{R}, E_*)$$

respectively. The last formula is well-defined since $\tilde{g} + S^* g \in L_2(\mathbb{R}, E)$ and $S\tilde{g} + g \in L_2(\mathbb{R}, E_*)$ for all $(\tilde{g}, g) \in \mathcal{H}_{sp}$.

Note that the images of Lax-Phillips's incoming and outgoing channels (subspaces) $\mathcal{D}_- = (L_2(\mathbb{R}_-, E_*), 0, 0)$ and $\mathcal{D}_+ = (0, 0, L_2(\mathbb{R}_+, E))$ under Φ are

$$\Phi\mathcal{D}_- = \begin{pmatrix} 0 \\ H_2^-(E_*) \end{pmatrix}, \quad \Phi\mathcal{D}_+ = \begin{pmatrix} H_2^+(E) \\ 0 \end{pmatrix}$$

in a completely symmetric way. This property of Pavlov's version of the functional model is a very convenient tool in model calculations. Note that the minimality of the selfadjoint dilation follows immediately from the minimality of the translation form of the dilation. The last fact holds true in both the completely non-selfadjoint and the general maximally dissipative operator cases.

5. Example: a limit-circle problem

Explicit calculation of the ingredients appearing in the functional model, for concrete examples, can be quite non-trivial. In this section we consider a one-dimensional Schrödinger problem with one regular endpoint and one singular, limit-circle endpoint, and compute expressions for the characteristic function and two other operators appearing in the functional model. We choose a limit-circle endpoint since this allows freedom of choice in the boundary conditions, and hence reveals the different explicit rôles of the boundary conditions and of the imaginary part of the potential. Our calculations allow for a limit-circle-oscillatory endpoint, and hence for spectrum with real part unbounded below; limit-circle non-oscillatory endpoints can be transformed to regular endpoints [22] and are therefore covered by our previous work [6]. As for all limit-circle problems, there is no essential spectrum.

Consider the expression

$$\ell u := -u'' + Q(x)u \quad x \in (0, 1];$$

here we suppose that Q is real-valued, regular at $x = 1$ and limit-circle at $x = 0$. We choose a real-valued basis $\{c, s\}$ of the solution space of the equation $\ell u = 0$ determined by initial conditions $s(1) = 0$, $s'(1) = 1$, together with the Wronskian condition $sc' - s'c \equiv 1$. We associate with the expression ℓ an operator L_B with domain

$$D(L_B) = \{u \in L_2(0, 1) \mid \ell u \in L_2(0, 1), \ \ u(1) = 0, \ \ [u, s](0) + B[u, c](0) = 0\}.$$

Here the square bracket notation denotes the Wronskian, i.e. $[u, s](x) = u(x)s'(x) - u'(x)s(x)$, and values at $x = 0$ are to be interpreted in terms of limits. $B \neq 0$ is a complex number; if B is real then it is well known that L_B is self-adjoint.

Assume that $\lambda = 0$ is not an eigenvalue of L_B. Then we may calculate the resolvent of L_B by the variation of parameters formula: $u = L_B^{-1} f$ if and only if

$$u(x) = c(x) \int_x^1 s(t)f(t)dt + s(x) \int_0^x c(t)f(t)dt + \frac{1}{B} s(x) \int_0^1 s(t)f(t)dt. \tag{5.1}$$

It is then a simple calculation to show that

$$u'(x) = c'(x) \int_x^1 s(t)f(t)dt + s'(x) \int_0^x c(t)f(t)dt + \frac{1}{B} s'(x) \int_0^1 s(t)f(t)dt. \tag{5.2}$$

Now we wish to examine conditions on B to have a dissipative operator L_B. Evidently

$$\langle L_B u, u \rangle = \lim_{x \searrow 0} \left[u'(x)\overline{u(x)} + \int_x^1 \left(|u'(t)|^2 + Q(t)|u(t)|^2 dt \right) \right], \tag{5.3}$$

and so since Q is real-valued it follows that L_B is dissipative if and only if for all $u \in D(L_B)$

$$\lim_{x \searrow 0} \Im(u'(x)\overline{u(x)}) \geq 0.$$

In order to simplify the calculations slightly we observe that if we restrict our attention to functions f which vanish in a neighbourhood of $x = 0$ then, since such f are dense in $L_2(0, 1)$, the resulting $u = L_B^{-1} f$ which we generate will form a core of $D(L_B)$. It is therefore sufficient to check dissipativity on such u. If x is sufficiently small to lie outside the support of f then from (5.1) and (5.2),

$$u(x) = \left(c(x) + \frac{1}{B}s(x)\right)\int_0^1 s(t)f(t)dt, \quad u'(x) = \left(c'(x) + \frac{1}{B}s'(x)\right)\int_0^1 s(t)f(t)dt,$$

and thus, as c and s are real-valued,

$$\Im(u'(x)\overline{u(x)}) = \Im\left(\frac{1}{B}\right)(s'(x)c(x) - s(x)c'(x))\left|\int_0^1 s(t)f(t)dt\right|^2 = -\Im\left(\frac{1}{B}\right)\left|\int_0^1 s(t)f(t)dt\right|^2, \tag{5.4}$$

where in the last step we have used the fact that $sc' - s'c \equiv 1$. Thus L_B is dissipative if and only if $\Im(B) \geq 0$. We will assume $\Im(B) \geq 0$ from now on.

We now cast this example into a boundary-triples framework [5, 10, 11, 12]. Our maximal operator L_{max} is given by the expression

$$L_{max}u = \ell u; \quad D(L_{max}) = \{u \in L_2(0,1) \mid \ell u \in L_2(0,1), \ \ u(1) = 0\},$$

and we wish to compute $\langle L_{max}f, g\rangle - \langle f, L_{max}g\rangle$, for $f, g \in D(L_{max})$. Using the von Neumann decomposition, together with the fact that $[s,c] = 1$, we have, in a neighbourhood of $x = 0$,

$$f(x) = f_0(x) + [f,c](0)s(x) - [f,s](0)c(x), \quad g(x) = g_0(x) + [g,c](0)s(x) - [g,s](0)c(x),$$

in which $f_0, g_0 \in D(L_{max}^*) = \{u \in L_2(0,1) \mid \ell u \in L_2(0,1), \ \ u(1) = 0, \ [u,c](0) = 0, \ [u,s](0) = 0\}$. Also, a straightforward calculation using integration by parts shows that

$$\langle L_{max}f, g\rangle - \langle f, L_{max}g\rangle = -[f,\overline{g}](0).$$

It then follows that

$$\langle L_{max}f, g\rangle - \langle f, L_{max}g\rangle = -[f,\overline{g}](0) = [f,c](0)\overline{[g,s](0)} - [f,s](0)\overline{[g,c](0)}.$$

If we define boundary operators Γ_0, Γ_1 on $D(L_{max})$ by

$$\Gamma_0 f = [f,c](0), \quad \Gamma_1 f = [f,s](0),$$

then the fundamental boundary triple identity can be written in the usual form

$$\langle L_{max}f, g\rangle - \langle f, L_{max}g\rangle = \Gamma_0 f\,\overline{\Gamma_1 g} - \Gamma_1 f\,\overline{\Gamma_0 g}.$$

The boundary condition associated with $D(L_B)$ is $\Gamma_1 u + B\Gamma_0 u = 0$. Using equations (5.1), (5.2) we see that if $u = L_B^{-1}f \in D(L_B)$ then

$$[u,c](x) = \int_0^x c(t)f(t)dt + \frac{1}{B}\int_0^1 s(t)f(t)dt,$$

whence, taking the limit as $x \searrow 0$,

$$\Gamma_0 u = \frac{1}{B}\int_0^1 s(t)f(t)dt.$$

Combining this with (5.3) and (5.4) we find that

$$\Im\langle L_B u, u\rangle = \Im(B)|\Gamma_0 u|^2 = \left|\sqrt{\Im(B)}\Gamma_0 u\right|^2.$$

Let $V \in L_\infty(0,1)$ be an essentially bounded, non-negative function. We define an operator A_B by

$$A_B = L_B + iV; \quad D(A_B) = D(L_B).$$

Then

$$\Im\langle A_B u, u\rangle = \Im\langle L_B u, u\rangle + \langle Vu, u\rangle = \left|\sqrt{\Im(B)}\Gamma_0 u\right|^2 + \langle\sqrt{V}u, \sqrt{V}u\rangle \tag{5.5}$$

If we define a map $\Gamma : D(A_B) \longrightarrow \mathbb{C} \oplus L_2(V^{-1}(\mathbb{R}_+))$ by

$$\Gamma u = \begin{pmatrix} \sqrt{\Im B}\Gamma_0 u \\ \sqrt{V}u \end{pmatrix} = \begin{pmatrix} \sqrt{\Im B}[u,c](0) \\ \sqrt{V}u \end{pmatrix} \tag{5.6}$$

then we have the Lagrange identity

$$\Im\langle A_B u, u\rangle = \langle \Gamma u, \Gamma u\rangle_{\mathbb{C}\oplus L_2(V^{-1}(\mathbb{R}_+))}. \tag{5.7}$$

Note that for this example, we also have

$$\Im\langle A_B^* u, u\rangle = -\langle \Gamma u, \Gamma u\rangle_{\mathbb{C}\oplus L_2(V^{-1}(\mathbb{R}_+))}. \tag{5.8}$$

We are thus in the simple situation $E = E_*$ and $\Gamma_* = \Gamma$.

The characteristic function $S(z)$ is defined by

$$S(z)\Gamma u = \Gamma(A_B^* - z)^{-1}(A_B - z)u, \quad z \in \mathbb{C}_+.$$

We now calculate $S(z)$. The first step is to find an expression for $v := (A_B^* - z)^{-1}(A_B - z)u$. To this end we introduce solutions $\tilde{s}_z$ and $\tilde{\phi}_z$ of the formal adjoint equation

$$-y'' + (Q - iV)y = zy,$$

determined by the conditions

$$\tilde{s}_z(1) = 0,\ \tilde{s}_z'(1) = 1; \quad \Gamma_0\tilde{\phi}_z = -1,\ \Gamma_1\tilde{\phi}_z = \overline{B}. \tag{5.9}$$

The existence of $\tilde{s}_z$, which is an entire function of z, is immediate from standard results on regular initial value problems. The existence of an entire $\tilde{\phi}_z$ is less obvious, but may be proved by using a variation-of-parameters argument. From (5.9), $\tilde{\phi}_z$ satisfies the left-hand boundary condition associated with A_B^*, viz.

$$\Gamma_1\tilde{\phi}_z + \overline{B}\Gamma_0\tilde{\phi}_z = 0. \tag{5.10}$$

Moreover,

$$\tilde{\phi}_z = -\overline{B}c - s + g_z, \tag{5.11}$$

in which g_z is a function such that $\Gamma_0 g_z = 0 = \Gamma_1 g_z$.

We also define the function $\tilde{M}(z)$ by

$$\Gamma_1\tilde{s}_z = \tilde{M}(z)\Gamma_0\tilde{s}_z; \tag{5.12}$$

this means

$$\tilde{M}(z) = \lim_{x\searrow 0}\frac{\tilde{s}_z(x)s'(x) - \tilde{s}_z'(x)s(x)}{\tilde{s}_z(x)c'(x) - \tilde{s}_z'(x)c(x)}.$$

Note that the denominator does not vanish, as $\Gamma_0\tilde{s}_z = 0$ would imply that $z \in \mathbb{C}_+$ is an eigenvalue of the anti-dissipative operator A_∞^* with eigenfunction $\tilde{s}_z$, which is impossible. The equation $v = (A_B^* - z)^{-1}(A_B - z)u$ is equivalent to $(A_B^* - z)v = (A_B - z)u$, which means that

$$-(v-u)'' + (Q - iV)(v - u) - z(v - u) = 2iVu.$$

We have $v(1) = 0 = u(1)$ and so variation of parameters yields, for some constant $a \in \mathbb{C}$,

$$v(x) = u(x) + \frac{\tilde{\phi}_z(x)\int_x^1 \tilde{s}_z(t)2iV(t)u(t)dt + \tilde{s}_z(x)\int_0^x \tilde{\phi}_z(t)2iV(t)u(t)dt}{[\tilde{\phi}_z, \tilde{s}_z]} + a\tilde{s}_z(x). \tag{5.13}$$

The value of a is determined by imposing the condition $v \in D(A_B^*)$, which means

$$\Gamma_1 v + \overline{B}\Gamma_0 v = 0.$$

Before doing this, however, we manipulate the denominator $[\tilde{\phi}_z, \tilde{s}_z]$ appearing in (5.13). In view of (5.11) we have

$$[\tilde{\phi}_z, \tilde{s}_z] = [-\overline{B}c - s, \tilde{s}_z] = \overline{B}\Gamma_0\tilde{s}_z + \Gamma_1\tilde{s}_z = (\overline{B} + \tilde{M}(z))\Gamma_0\tilde{s}_z.$$

Thus eqn. (5.13) becomes

$$v(x) = u(x) + \frac{\tilde{\phi}_z(x)\int_x^1 \tilde{s}_z(t)2iV(t)u(t)dt + \tilde{s}_z(x)\int_0^x \tilde{\phi}_z(t)2iV(t)u(t)dt}{(\overline{B} + \tilde{M}(z))\Gamma_0\tilde{s}_z} + a\tilde{s}_z(x). \tag{5.14}$$

From (5.14), bearing in mind that $(\Gamma_1 + \overline{B}\Gamma_0)\tilde{\phi}_z = 0$ and $(\Gamma_1 + B\Gamma_0)u = 0$, it follows that

$$\Gamma_1 v + \overline{B}\Gamma_0 v = -2i\Im(B)\Gamma_0 u + a(\tilde{M}(z) + \overline{B})\Gamma_0\tilde{s}_z,$$

whence, since $\Gamma_1 v + \overline{B}\Gamma_0 v = 0$, we have

$$a = \frac{2i\Im(B)\Gamma_0 u}{(\overline{B} + \tilde{M}(z))\Gamma_0\tilde{s}_z},$$

and

$$v(x) = u(x) + \frac{\tilde{\phi}_z(x)\int_x^1 \tilde{s}_z(t)2iV(t)u(t)dt + \tilde{s}_z(x)\int_0^x \tilde{\phi}_z(t)2iV(t)u(t)dt}{(\overline{B} + \tilde{M}(z))\Gamma_0\tilde{s}_z} + \frac{2i\Im(B)\Gamma_0 u}{(\overline{B} + \tilde{M}(z))\Gamma_0\tilde{s}_z}\tilde{s}_z(x). \tag{5.15}$$

In particular, recalling that $\Gamma_0\tilde{\phi}_z = -1$, see eqn. (5.9), it now follows that

$$\Gamma_0 v = \left\{\frac{B + \tilde{M}(z)}{\overline{B} + \tilde{M}(z)}\right\}\Gamma_0 u - \frac{1}{(\overline{B} + \tilde{M}(z))\Gamma_0\tilde{s}_z}\int_0^1 \tilde{s}_z 2iVu.$$

Observing that $Vu = \sqrt{V}\sqrt{V}u$, the characteristic function can be written as a 2×2 block operator matrix,

$$S(z) = \begin{pmatrix} S_{11}(z) & S_{12}(z) \\ S_{21}(z) & S_{22}(z) \end{pmatrix},$$

in which

(5.16)

$$\left.\begin{aligned} S_{11}(z) = \left\{ \frac{B+\tilde{M}(z)}{\overline{B}+\tilde{M}(z)} \right\} = 1 + \frac{2i\Im(B)}{\overline{B}+\tilde{M}(z)}, \quad S_{12}(z)\bullet = \frac{-2i\sqrt{\Im B}}{(\overline{B}+\tilde{M}(z))\Gamma_0\tilde{s}_z}\int_0^1 \tilde{s}_z\sqrt{V}\bullet, \\ S_{21}(z) = \left\{ \frac{2i\sqrt{\Im B}\sqrt{V}\tilde{s}_z}{(\overline{B}+\tilde{M}(z))\Gamma_0\tilde{s}_z} \right\}, \\ S_{22}(z)\bullet = I\bullet + \frac{2i\sqrt{V}}{(\overline{B}+\tilde{M}(z))\Gamma_0\tilde{s}_z}\left\{ \tilde{\phi}_z\int_x^1 \tilde{s}_z\sqrt{V}\bullet + \tilde{s}_z\int_0^x \tilde{\phi}_z\sqrt{V}\bullet \right\} = I\bullet + 2i\sqrt{V(x)}\int_0^1 G(x,t;z)\sqrt{V(t)}\bullet(t)dt, \end{aligned}\right\}$$

in which

$$G(x,t;z) = \frac{\tilde{\phi}_z(\min(x,t))\,\tilde{s}_z(\max(x,t))}{(\overline{B}+\tilde{M}(z))\Gamma_0\tilde{s}_z}. \tag{5.17}$$

Remark 5.1. *Since $\tilde{M}(z)\Gamma_0\tilde{s}_z = \Gamma_1\tilde{s}_z$, see (5.12), the condition $(\overline{B}+\tilde{M}(z))\Gamma_0\tilde{s}_z = 0$ is equivalent to $\overline{B}\Gamma_0\tilde{s}_z + \Gamma_1\tilde{s}_z = 0$, which happens precisely when $\tilde{s}_z$ is an eigenfunction of A_B^*. Since the singular point associated with the differential expression for L_B is of limit circle type, L_B has empty essential spectrum. The same is true of A_B and A_B^* since these are relatively compact perturbations of L_B. The singularities of $S(z)$ are therefore precisely the eigenvalues of A_B^*. If $\Im B > 0$ then these lie strictly in the lower half-plane.*

The other two main ingredients which appear in the functional model, and for which explicit expressions can be found in terms of solutions of initial value problems and M-functions, are the operators $\Gamma(A_B - z)^{-1}$ and $\Gamma_*(A_B^* - z)^{-1}$. Calculating these quantities is not more difficult than calculating the characteristic function S itself. We illustrate this by obtaining an expression for $\Gamma(A_B - z)^{-1}$. The ingredients required are the solutions s_z and ϕ_z of the equation

$$-y'' + (Q + iV)y = zy, \quad z \in \mathbb{C}_-,$$

determined by the conditions

$$s_z(1) = 0,\ s_z'(1) = 1; \quad \Gamma_0\phi_z = -1,\ \Gamma_1\phi_z = B. \tag{5.18}$$

The conditions on ϕ_z ensure that

$$\phi_z = -Bc - s + g_z, \tag{5.19}$$

in which $\Gamma_0 g_z = 0 = \Gamma_1 g_z$.

We also define the function $M(z)$ by

$$\Gamma_1 s_z = M(z)\Gamma_0 s_z;$$

this means

$$M(z) = \lim_{x\searrow 0}\frac{s_z(x)s'(x) - s_z'(x)s(x)}{s_z(x)c'(x) - s_z'(x)c(x)}.$$

A calculation similar to (but simpler than) the one which leads to eqn. (5.14) shows that the resolvent $(A_B - z)^{-1}$ is given by

$$((A_B - z)^{-1}f)(x) = \frac{\phi_z(x)\int_x^1 s_z(t)f(t)dt + s_z(x)\int_0^x \phi_z(t)f(t)dt}{(B+M(z))\Gamma_0 s_z}, \tag{5.20}$$

and so, remembering that $\Gamma_0\phi_z = -1$,

$$\Gamma_0(A_B - z)^{-1}f = \frac{-1}{(B+M(z))\Gamma_0 s_z}\int_0^1 s_z(t)f(t)dt.$$

$$(\Gamma(A_B - z)^{-1}f)(x) = \begin{pmatrix} \dfrac{-\sqrt{\Im B}}{(B+M(z))\Gamma_0 s_z}\displaystyle\int_0^1 s_z(t)f(t)dt \\[2ex] \sqrt{V(x)}\,\dfrac{\phi_z(x)\int_x^1 s_z(t)f(t)dt + s_z(x)\int_0^x \phi_z(t)f(t)dt}{(B+M(z))\Gamma_0 s_z} \end{pmatrix}. \tag{5.21}$$

For reference, we mention the corresponding expression for $\Gamma(A_B^*-z)^{-1}$, viz.

$$(\Gamma(A_B^*-z)^{-1}f)(x)=\left(\begin{array}{c}\dfrac{-\sqrt{\Im B}}{(\overline{B}+\tilde{M}(z))\Gamma_0\tilde{s}_z}\displaystyle\int_0^1\tilde{s}_z(t)f(t)dt\\ \\ \sqrt{V(x)}\,\dfrac{\tilde{\phi}_z(x)\int_x^1\tilde{s}_z(t)f(t)dt+\tilde{s}_z(x)\int_0^x\tilde{\phi}_z(t)f(t)dt}{(\overline{B}+\tilde{M}(z))\Gamma_0\tilde{s}_z}\end{array}\right).$$

The expression for the map Φ given in Lemma 4.11 shows how it acts upon vectors whose middle component is of the form $(\Gamma(A_B-\overline{\lambda_0})^{-1})^*e$ for some $e\in E$, while the corresponding formula in Lemma 4.13 gives the action of Φ upon vectors with middle component of the form $(\Gamma_*(A_B^*-\overline{\mu_0})^{-1})^*e_*$ for some $e_*\in E_*$. It is therefore useful to have expressions in our current example for the inverses $((\Gamma(A_B-\overline{\lambda_0})^{-1})^*)^{-1}$ and $((\Gamma_*(A_B^*-\overline{\mu_0})^{-1})^*)^{-1}$, which we now obtain.

Firstly, we may write (5.21) in the form

$$(\Gamma(A_B-z)^{-1}f)(x)=\left(\begin{array}{c}\sqrt{\Im B}\langle f,g\rangle_{L_2(0,1)}\\ \\ \sqrt{V(x)}((A_B-z)^{-1}f)(x)\end{array}\right)\in\mathbb{C}\oplus L_2(V^{-1}(\mathbb{R}_+)),$$

in which

$$g(\cdot)=\overline{\left\{\frac{-s_z(\cdot)}{(B+M(z))\Gamma_0 s_z}\right\}}.$$

A simple calculation shows that for any test vector $e=\begin{pmatrix}c\\u\end{pmatrix}\in\mathbb{C}\oplus L_2(V^{-1}(\mathbb{R}_+))$,

$$\left\langle\Gamma(A_B-z)^{-1}f,\begin{pmatrix}c\\u\end{pmatrix}\right\rangle_{\mathbb{C}\oplus L_2(V^{-1}(\mathbb{R}_+))}=\left\langle f,\ g(\cdot)\sqrt{\Im B}\,c+(A_B-z)^{-*}\sqrt{V}u\right\rangle_{L_2(0,1)},$$

so that

$$(\Gamma(A_B-z)^{-1})^*e=(\Gamma(A_B-z)^{-1})^*\begin{pmatrix}c\\u\end{pmatrix}=\overline{\left\{\frac{-s_z(\cdot)}{(B+M(z))\Gamma_0 s_z}\right\}}\sqrt{\Im B}\,c+(A_B-z)^{-*}\sqrt{V}u. \tag{5.22}$$

The inverse $((\Gamma(A_B-z)^{-1})^*)^{-1}$ can now be found. Denoting $\varphi:=(\Gamma(A_B-z)^{-1})^*\begin{pmatrix}c\\u\end{pmatrix}$, we observe that since the term $(A_B-z)^{-*}\sqrt{V}u$ lies in $\ker(\Gamma_1+\overline{B}\Gamma_0)$, while $(\Gamma_1+\overline{B}\Gamma_0)\overline{s_z}=\overline{(\Gamma_1+B\Gamma_0)s_z}=\overline{(B+M(z))\Gamma_0 s_z}$, we obtain

$$c=-\frac{1}{\sqrt{\Im B}}(\Gamma_1+\overline{B}\Gamma_0)\varphi.$$

Furthermore, we know that $\left(-\dfrac{d^2}{dx^2}+Q-iV-\overline{z}\right)\overline{s_z}=0$, while

$$\left(-\frac{d^2}{dx^2}+Q-iV-\overline{z}\right)(A_B-z)^{-*}\sqrt{V}u=\sqrt{V}u,$$

and so

$$u=\frac{1}{\sqrt{V}}P_{V^{-1}(\mathbb{R}_+)}\left(-\frac{d^2}{dx^2}+Q-iV-\overline{z}\right)\varphi,$$

in which $P_{V^{-1}(\mathbb{R}_+)}$ is the orthogonal projection from $L_2(0,1)$ to $L_2(V^{-1}(\mathbb{R}_+))$. Finally we arrive at the expression

$$((\Gamma(A_B-\overline{\lambda_0})^{-1})^*)^{-1}\varphi=\left(\begin{array}{c}-\frac{1}{\sqrt{\Im B}}(\Gamma_1+\overline{B}\Gamma_0)\varphi\\ \frac{1}{\sqrt{V}}P_{V^{-1}(\mathbb{R}_+)}\left(-\frac{d^2}{dx^2}+Q-iV-\lambda_0\right)\varphi\end{array}\right)=:\begin{pmatrix}e_1\\e_2\end{pmatrix}=:e. \tag{5.23}$$

The formula

$$((\Gamma_*(A_B^*-\overline{\mu_0})^{-1})^*)^{-1}\varphi_*=\left(\begin{array}{c}-\frac{1}{\sqrt{\Im B}}(\Gamma_1+B\Gamma_0)\varphi_*\\ \frac{1}{\sqrt{V}}P_{V^{-1}(\mathbb{R}_+)}\left(-\frac{d^2}{dx^2}+Q+iV-\mu_0\right)\varphi_*\end{array}\right)=:\begin{pmatrix}e_{1,*}\\e_{2,*}\end{pmatrix}=:e_*. \tag{5.24}$$

is proved similarly. Eqn. (5.23) can be used to compute $\Phi\begin{pmatrix}v_-\\\varphi\\v_+\end{pmatrix}$ for any $\varphi=((\Gamma(A_B-\overline{\lambda_0})^{-1})^*)e$, $e\in E$, $v_-\in L_2(\mathbb{R}_-,E_*)$ and $v_+\in L_2(\mathbb{R}_+,E)$ using the expression in Lemma 4.11. Similarly, (5.24) allows the computation of $\Phi\begin{pmatrix} v_- \\ \varphi \\ v_+ \end{pmatrix}$ for any $\varphi = ((\Gamma(A_B - \overline{\mu_0})^{-1})^*)e_*$, $e_* \in E_*$, $v_- \in L_2(\mathbb{R}_-, E_*)$ and $v_+ \in L_2(\mathbb{R}_+, E)$ using the expression in Lemma 4.13.

We obtain

$$\Phi\left(\begin{array}{c} v_- \\ \varphi = ((\Gamma(A_B - \overline{\lambda_0})^{-1})^*)e \\ v_+ \end{array}\right) = \begin{pmatrix} \hat{v}_+(k) \\ \hat{v}_-(k) \end{pmatrix} + \frac{i}{\sqrt{2\pi}(k-\lambda_0)}\begin{pmatrix} e \\ -S(\lambda_0)e \end{pmatrix}.$$

The quantity $S(\lambda_0)e$ is computed using the 2×2 block operator matrix expression for S in (5.16). We write explicitly only the most complicated quantity, namely

$$\begin{aligned} S_{22}(\lambda_0)e_2 = e_2 + T_{22}(\lambda_0)e_2 &= \frac{1}{\sqrt{V(x)}}P_{V^{-1}(\mathbb{R}_+)}\left(-\frac{d^2}{dx^2} + Q(x) - iV(x) - \lambda_0\right)\varphi(x) \\ &+ 2i\sqrt{V(x)}\int_0^1 G(x,t;\lambda_0)P_{V^{-1}(\mathbb{R}_+)}\left(-\frac{d^2}{dt^2} + Q(t) - iV(t) - \lambda_0\right)\varphi(t)dt, \end{aligned}$$

in which the Green's function G is

$$G(x,t;z) = \frac{\tilde{\phi}_z(\min(x,t))\,\tilde{s}_z(\max(x,t))}{(\overline{B} + \tilde{M}(z))\Gamma_0\tilde{s}_z}.$$

Similarly,

$$\Phi\left(\begin{array}{c} v_- \\ \varphi_* = ((\Gamma_*(A_B^* - \overline{\mu_0})^{-1})^*)e_* \\ v_+ \end{array}\right) = \begin{pmatrix} \hat{v}_+(k) \\ \hat{v}_-(k) \end{pmatrix} + \frac{i}{\sqrt{2\pi}(k-\mu_0)}\begin{pmatrix} S^*(\overline{\mu_0})e_* \\ -e_* \end{pmatrix},$$

and one may show that

$$\begin{aligned} S_{22}^*(\overline{\mu_0})e_{2,*} = e_{2,*} + T_{22}^*(\overline{\mu_0})e_{2,*} &= \frac{1}{\sqrt{V(x)}}P_{V^{-1}(\mathbb{R}_+)}\left(-\frac{d^2}{dx^2} + Q(x) + iV(x) - \mu_0\right)\varphi_*(x) \\ &- 2i\sqrt{V(x)}\int_0^1 \overline{G(x,t;\overline{\mu_0})}P_{V^{-1}(\mathbb{R}_+)}\left(-\frac{d^2}{dt^2} + Q(t) + iV(t) - \mu_0\right)\varphi_*(t)dt. \end{aligned}$$

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