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Essential spectrum for dissipative Maxwell equations in domains with cylindrical ends

F. Ferraresso [a,*], M. Marletta [b,*]

[a] Dipartimento di Scienze chimiche, fisiche, matematiche e naturali, Via Piandanna 4, Sassari 07100, Italy
[b] School of Mathematics, Cardiff University, Abacws, Senghennydd Road, Cathays, Cardiff CF24 4AG, UK

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ABSTRACT

We consider the Maxwell equations with anisotropic coefficients and non-trivial conductivity in a domain with finitely many cylindrical ends. We assume that the conductivity vanishes at infinity and that the permittivity and permeability tensors converge to non-constant matrices at infinity, which coincide with a positive real multiple of the identity matrix in each of the cylindrical ends. We establish that the essential spectrum of Maxwell system can be decomposed as the union of the essential spectrum of a bounded multiplication operator acting on gradient fields, and the union of the essential spectra of the Maxwell systems obtained by freezing the coefficients to their different limiting values along the several different cylindrical ends of the domain.

1. Introduction

The aim of this article is to analyse the spectrum of the following dissipative Maxwell system with anisotropic permittivity ε and permeability μ, and conductivity $\sigma \neq 0$ in a unbounded domain Ω having multiple cylindrical ends:

$$\begin{cases} -\mathrm{i}\sigma E + \mathrm{i}\,\mathrm{curl}\, H = \omega\varepsilon E + J_e, & \text{in } \Omega, \\ -\mathrm{i}\,\mathrm{curl}\, E = \omega\mu H + J_m & \text{in } \Omega, \\ \nu \times E = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

* Corresponding authors.
E-mail addresses: fferraresso@uniss.it (F. Ferraresso), MarlettaM@cardiff.ac.uk (M. Marletta).

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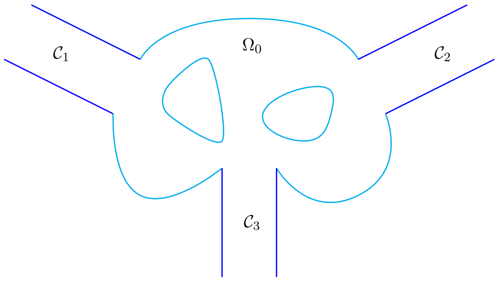

Fig. 1. The domain Ω obtained by gluing three cylindrical ends $\mathcal{C}_i$ to the domain Ω_0.

Here $\omega \in \mathbb{C}$ is the spectral parameter, ν is the outer normal vector to $\partial\Omega$, E, H are, respectively, the electric and the magnetic vector fields, and $J_e \in L^2(\Omega)$, $J_m \in L^2(\Omega)$ are the current sources. More precisely, we wish to study the spectrum of the linear operator pencil $V(\cdot)$ defined by

$$V(\omega) := \begin{pmatrix} -\mathrm{i}\sigma - \omega\varepsilon & \mathrm{i}\,\mathrm{curl} \\ -\mathrm{i}\,\mathrm{curl} & -\omega\mu \end{pmatrix} \tag{1.2}$$

on the domain $\mathrm{dom}(V) := H_0(\mathrm{curl}, \Omega) \otimes H(\mathrm{curl}, \Omega)$. The equation (1.1) is then $V(\omega)(E, H)^t = (J_e, J_m)^t$. Our main interest will be the essential spectrum[1] of V, i.e.

$$\begin{aligned}\sigma_{ess}(V) &= \{\omega \in \mathbb{C} :\ 0 \in \sigma_{ess}(V(\omega))\} \\ &= \{\omega \in \mathbb{C} : \exists u_n \in \mathrm{dom}(V),\ \|u_n\| = 1,\ u_n \rightharpoonup 0,\ \|V(\omega)u_n\| \to 0\}.\end{aligned}$$

The unbounded Lipschitz open set $\Omega \subset \mathbb{R}^3$ is assumed to be the gluing of a bounded Lipschitz domain Ω_0 with the disjoint cylindrical ends $\mathcal{C}_i$, see Fig. 1.

$$\Omega = \Omega_0 \sqcup \left(\bigsqcup_{i=1}^{M} \mathcal{C}_i \right) \tag{1.3}$$

Up to a rigid motion, each cylindrical end has the form

$$\mathcal{C}_i = \{x \in \mathbb{R}^3 : x_1 \in (0, +\infty), (x_2, x_3) \in C_i\}$$

where the cross-sections C_i are bounded Lipschitz domains, possibly not simply connected.

For the selfadjoint case $\sigma \equiv 0$, the Maxwell system on a domain with several cylindrical ends was considered in [26,27]. However the determination of the essential spectrum for the Maxwell system is a great deal more difficult in the non-selfadjoint case. Even for bounded domains it was first considered only as recently as 1997 [22]. Starting with [3], several authors studied the spectrum of Maxwell's equations in unbounded dispersive media, see for instance [4] for the case of the full-space, with general dependence on the spectral parameter; [8] and [9] for the interface between vacuum and a metamaterial.

Compared to [3,6], the novelty of this article lies in allowing both anisotropic conductivities σ, and anisotropic ε, μ, with $\varepsilon, \mu, \sigma \in L^\infty(\Omega, \mathrm{Sym}_3(\mathbb{R}))$, which are non-constant at infinity in the precise sense that there exist real constants $\varepsilon_i > 0$, $\mu_i > 0$, $i = 1, \ldots, M$, such that, for all $\delta > 0$, one has decompositions

$$\varepsilon(x) = \varepsilon_c(x) + \varepsilon_\delta(x) + \sum_{i=1}^{M} \varepsilon_i \chi_{\mathcal{C}_i}(x)\mathbb{I}, \quad \mu(x) = \mu_c(x) + \mu_\delta(x) + \sum_{i=1}^{M} \mu_i \chi_{\mathcal{C}_i}(x)\mathbb{I}, \tag{1.4}$$

[1] The essential spectrum $\sigma_{ess}(T)$, for an operator T, is $\sigma_{e2}(T)$ in the taxonomy of [15].

where $\|\varepsilon_\delta\|_{L^\infty}, \|\mu_\delta\|_{L^\infty} < \delta$, and ε_c, μ_c are compactly supported. This assumption ensures that

$$\lim_{R\to\infty} \sup_{x\in\mathcal{C}_i, |x|>R} \|\varepsilon(x) - \varepsilon_i \mathbb{I}\| = 0, \quad \lim_{R\to\infty} \sup_{x\in\mathcal{C}_i, |x|>R} \|\mu(x) - \mu_i \mathbb{I}\| = 0,$$

for $i = 1, \dots, M$. There then exist isotropic tensors ε_∞, μ_∞ in $C^\infty(\Omega, \mathbb{R}^{3\times3})$ such that

$$\varepsilon_\infty(x) = \varepsilon_i \mathbb{I}, \quad \mu_\infty(x) = \mu_i \mathbb{I} \quad \text{for } x \in \mathcal{C}_i, \tag{1.5}$$

and

$$\lim_{R\to\infty} \sup_{|x|>R} \|\varepsilon(x) - \varepsilon_\infty(x)\| = 0, \quad \lim_{R\to\infty} \sup_{|x|>R} \|\mu(x) - \mu_\infty(x)\| = 0. \tag{1.6}$$

We also assume that

$$\lim_{R\to\infty} \sup_{|x|>R} \|\sigma(x)\| = 0. \tag{1.7}$$

Together, (1.6), (1.7) constitute one of the simplest possible forms of inhomogeneity at infinity.

It was established in [3,6] that the essential spectrum of the Maxwell pencil $V(\cdot)$, in the case where ε_∞, μ_∞ are constant real multiples of the identity matrix, can be expressed as

$$\sigma_{ess}(V) = \sigma_{ess}(P_\nabla \mathcal{W}(\omega) P_\nabla) \cup \sigma_{ess}(V_\infty) \tag{1.8}$$

where $\mathcal{W}(\omega) = \omega(\omega\varepsilon + i\sigma)$, P_∇ is the orthogonal projection in $L^2(\Omega)^3$ onto the closed subspace of gradient fields $\nabla H_0^1(\Omega)$, and V_∞ is defined by

$$V_\infty(\omega) = \begin{pmatrix} -\omega\varepsilon_\infty & \mathrm{i}\,\mathrm{curl} \\ -\mathrm{i}\,\mathrm{curl} & -\omega\mu_\infty \end{pmatrix}; \quad \mathrm{dom}(V_\infty) = H_0(\mathrm{curl}, \Omega) \otimes H(\mathrm{curl}, \Omega). \tag{1.9}$$

A similar result holds for more general dependence of the coefficients upon the frequency ω, see [17]. However it is easy to check that the proofs in all of [3,6,17] break down as soon as ε_∞ and μ_∞ are not constant, indeed even under the simplest form of inhomogeneity considered here, namely (1.4).

The present article deals with this problem. Our main result is the following

Theorem 1.1. *The essential spectrum of the pencil $V(\cdot)$ is given by*

$$\sigma_{ess}(V) = \sigma_{ess}(P_\nabla \mathcal{W}(\omega) P_\nabla) \cup \left(\bigcup_{i=1}^{M} \sigma_{ess}(V_{\infty,i}) \right) \tag{1.10}$$

where $V_{\infty,i}$ is the selfadjoint pencil (1.9) with $\varepsilon_{\infty,i}$ in place of ε_∞, $\mu_{\infty,i}$ in place of μ_∞, and $\mathcal{C}_i$ in place of Ω. In particular,

$$\bigcup_{i=1}^{M} \sigma_{ess}(V_{\infty,i}) \subseteq \mathbb{R},$$

and all non-real parts of the essential spectrum $\sigma_{ess}(V)$ arise from the bordered multiplication pencil $P_\nabla \mathcal{W}(\omega) P_\nabla$.

Remark 1.2. Since all the operators appearing here are J-selfadjoint with respect to a suitable conjugation J, the σ_{ess} in Theorem 1.1 coincides with all the σ_{ek} for $k = 1, 2, 3, 4$ [15].

The proof of Theorem 1.1 requires a sequence of completely non-trivial arguments and is achieved by establishing two 'Decomposition and Simplification' theorems, Theorem 4.8 and Theorem 5.3.

Theorem 4.8 shows that (1.8) continues to hold in the case of non-constant coefficients at infinity. The difficulty here is to prove that the operator $P_\nabla \varepsilon_\infty P_{\ker(\mathrm{div})}$ is compact; note that if instead ε_∞ is constant,

$$P_\nabla \varepsilon_\infty P_{\ker(\mathrm{div})} = \varepsilon_\infty P_\nabla P_{\ker(\mathrm{div})} \equiv 0$$

is identically zero. The required compactness property will be a consequence of an auxiliary Glazman decomposition argument and of the compactness of

$$(-\Delta_\Omega - \lambda)^{-1} : L^2(B) \subset L^2_{\mathrm{comp}}(\Omega) \to H^1(\Omega)$$

where B is any bounded Lipschitz domain, $B \subset \Omega$, see Proposition (3.3). Although probably not new, these results do not seem to be explicitly stated in the literature. They use in a substantial way the celebrated 1995 compactness result of Jerison and Kenig [19], which became pivotal in the regularity theory for Maxwell's equations after the important $H^{1/2}$-regularity theorem of Costabel [13]. Note that Glazman decomposition cannot be applied directly to the original Maxwell system, as the term $\sigma_{ess}(P_\nabla \mathcal{W}(\omega) P_\nabla)$ appearing in (1.10) depends on local behaviour of the coefficients and not on behaviour at infinity.

Theorem 5.3 guarantees that

$$\sigma_{ess}(V_\infty) = \bigcup_{i=1}^{M} \sigma_{ess}(V_{\infty,i}). \tag{1.11}$$

In the literature results of this kind were established for the Laplace operator by using domain decomposition and Dirichlet-to-Neumann maps across an artificial boundary, see *e.g.*, [23]. We mention that the Dirichlet-to-Neumann map (and the corresponding Glazman decomposition [18]) is widely used in the context of scattering theory, see e.g., [1,10–12,24,25], in order to establish spectral asymptotics or upper bounds on the number or resonances. However, the analogues of the Dirichlet-to-Neumann map for Maxwell would be the impedance map (also called Calderón map or electric-to-magnetic map) which in general does not have compact resolvent, see the recent articles [7,16]; the consequent presence of essential spectrum of the impedance map disrupts the main arguments in the proof of the decomposition of the essential spectrum. These additional hurdles are due to our weak assumptions on ε, μ and σ, which are just L^∞-matrix valued functions. For instance, in the case of constant coefficients, and provided that the geometry of Ω is not too complex, other techniques such as TE-TM mode decompositions allow one to explicitly define the electric-to-magnetic map, see e.g., [20]; however, essential spectrum might arise also in this case, see [7].

Therefore, to establish (1.11) we use instead a direct argument based on singular sequences, see Theorem 5.3. The proof of the latter exploits in a substantial way the compactness of the embedding $H_0(\mathrm{curl}, B) \cap H(\mathrm{div}, B)$ into $L^2(B)$ for bounded Lipschitz domains $B \subset \Omega$, the specific geometry of the set Ω, and a novel interpolation inequality of independent interest, see Lemma 6.8.

As a final aside, we mention that although the results on essential spectrum in [6] do not cover the situation of the present paper, the enclosures for the whole spectrum presented there remain valid. Moreover, for a domain Ω with multiple cylindrical ends of the kind we consider, the Dirichlet Laplacian on Ω has a strictly positive infimum $\lambda(\Omega)$ (see (2.3) in Section 2 below); the results in [6] immediately allow us to deduce that $\omega = 0$ is the centre of a slit disc which does not contain any spectrum of $V(\cdot)$.

This article is structured in the following way. In Section 2 we collect some basic facts about system (1.1) and the associated functional spaces. Section 3 contains some auxiliary results about the local compactness of the resolvent of the Dirichlet Laplacian in a unbounded Lipschitz domain; these facts are needed in the proof of Theorem 4.8. Section 4 and Section 5 contain respectively the two main steps in the proof of (1.10): the first step involves applying a Helmholtz-type decomposition to a curl curl-type pencil $\mathcal{L}$ having the same spectrum as the Maxwell pencil V defined in (1.2), see Theorem 4.8. The second step is the Glazman decomposition applied to the constant-coefficient pencil $\mathcal{L}_\infty$ obtained from the previous step, see Theorem 5.3. The proof of the latter requires a subtle estimate on the Jacobian of some vector fields in terms of a suitable curl curl-graph norm. We provide this proof in the Appendix A.

2. Preliminaries and notation

We recall here some basic facts about system (1.1), mainly to fix the notation. The natural domain of the operator $V(\omega)$ defined in (1.2) acting in the Hilbert space of couples of square-integrable vector fields $L^2(\Omega) \times L^2(\Omega) \equiv L^2(\Omega; \mathbb{C}^3) \times L^2(\Omega; \mathbb{C}^3)$ is the union of all the couples $(E, H) \in L^2(\Omega) \times L^2(\Omega)$ such that

$$((-\mathrm{i}\sigma - \omega\varepsilon)E + \mathrm{i}\,\mathrm{curl}\,H) \in L^2(\Omega) \quad \text{and} \quad (-\mathrm{i}\,\mathrm{curl}\,E - \omega\mu H) \in L^2(\Omega);$$

equivalently,

$$\mathrm{curl}\,E \in L^2(\Omega), \quad \mathrm{curl}\,H \in L^2(\Omega), \quad \nu \times E|_{\partial\Omega} = 0.$$

Therefore, E, H belong to Sobolev spaces of $L^2(\Omega)$-functions having distributional curl in $L^2(\Omega)$. More precisely, the magnetic field H and the electric field E lie respectively in

$$H(\mathrm{curl}, \Omega) := \{u \in L^2(\Omega) : \mathrm{curl}\,u \in L^2(\Omega)\},$$
$$H_0(\mathrm{curl}, \Omega) := \{u \in H(\mathrm{curl}, \Omega) : \nu \times u|_{\partial\Omega} = 0\},$$

endowed with the standard norm $\|u\|_{H(\mathrm{curl},\Omega)} := (\|u\|^2 + \|\mathrm{curl}\,u\|^2)^{1/2}$.

Functions $u \in H(\mathrm{curl}, \Omega)$ have a well-defined tangential trace $\nu \times u \in H^{-1/2}(\partial\Omega)$, see [14, Thm. 2, Chp. IX]; the subspace $H_0(\mathrm{curl}, \Omega)$ of $H(\mathrm{curl}, \Omega)$ is then identified by the condition $\nu \times u = 0$ on $\partial\Omega$. Equivalently, $H_0(\mathrm{curl}, \Omega)$ can be defined as the closure of $C_c^\infty(\Omega)$ vector fields with respect to the $H(\mathrm{curl}, \Omega)$-norm. Due to the integration by parts formula

$$\int_\Omega \mathrm{curl}\,u\,\Phi - \int_\Omega u\,\mathrm{curl}\,\Phi = \int_{\partial\Omega} (\nu \times u)\,\Phi \tag{2.1}$$

valid for all smooth vector fields u and Φ, we deduce that $u \in H_0(\mathrm{curl}, \Omega)$ if and only if

$$\int_\Omega (\mathrm{curl}\,u\,\Phi - u\,\mathrm{curl}\,\Phi) = 0, \quad \text{for all } \Phi \in C^\infty(\overline{\Omega}).$$

The symmetric differential expression curl in $L^2(\Omega)$ is associated with two different operators. With an abuse of notation, we denote by curl the operator given by the differential expression curl on the 'maximal domain' $\mathrm{dom}(\mathrm{curl}) := H(\mathrm{curl}, \Omega)$. Its adjoint $\mathrm{curl}_0 = \mathrm{curl}^*$ is given by the differential expression curl on the smaller domain $\mathrm{dom}(\mathrm{curl}_0) := H_0(\mathrm{curl}, \Omega)$.

One defines analogously the space $H(\mathrm{div}, \Omega) = \{u \in L^2(\Omega) : \mathrm{div}\,u \in L^2(\Omega)\}$ of square integrable vector fields $u \in L^2(\Omega)$ having distributional divergence $\mathrm{div}\,u$ in $L^2(\Omega)$, endowed with the canonical norm $\|u\|_{H(\mathrm{div},\Omega)} := (\|u\|^2 + \|\mathrm{div}\,u\|^2)^{1/2}$. In this case, the analogue of formula (2.1) is the more familiar formula

$$\int_{\Omega} \operatorname{div} u \, \varphi - \int_{\Omega} u \nabla\varphi = \int_{\partial\Omega} (\nu \cdot u)\varphi$$

for all vector fields $u \in H(\operatorname{div}, \Omega)$ and scalar functions $\varphi \in H^1(\Omega)$. An important subspace of $L^2(\Omega)$ (and of $H(\operatorname{div}, \Omega)$) is $H(\operatorname{div} 0, \Omega)$, consisting of those $L^2(\Omega)$-vector fields u having null distributional divergence; more explicitly, $u \in H(\operatorname{div} 0, \Omega)$ if and only if

$$\int_{\Omega} u \cdot \nabla\varphi = 0, \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

The space $H(\operatorname{curl}, \Omega) \cap H(\operatorname{div}, \Omega)$ is equipped with the norm $\|u\|_{H(\operatorname{curl},\Omega)\cap H(\operatorname{div},\Omega)} := \|u\|_{H(\operatorname{curl},\Omega)} + \|u\|_{H(\operatorname{div},\Omega)}$. It is important to recall that the space $H_0(\operatorname{curl}, B) \cap H(\operatorname{div}, B)$ is compactly embedded in $L^2(B)$ whenever B is a bounded Lipschitz open set, see [29].

Throughout the paper we will constantly make use of the following orthogonal representation, often called Helmholtz decomposition:

$$L^2(\Omega) = \nabla \dot{H}^1_0(\Omega) \oplus H(\operatorname{div} 0, \Omega), \tag{2.2}$$

where the homogeneous Sobolev space $\dot{H}^1_0(\Omega)$ is defined as the completion of $C_c^\infty(\Omega)$ with respect to the seminorm $\|u\|_{\dot{H}^1(\Omega)} := \|\nabla u\|_{L^2(\Omega)}$. The space $\nabla \dot{H}^1_0(\Omega)$ is therefore the image of $\dot{H}^1_0(\Omega)$ under the gradient. There are two canonical orthogonal projections associated with the decomposition (2.2), given by

$$P_\nabla := P_{\nabla \dot{H}^1_0(\Omega)} : L^2(\Omega) \to \nabla \dot{H}^1_0(\Omega), \qquad P_{\ker \operatorname{div}} := P_{H(\operatorname{div} 0,\Omega)} : L^2(\Omega) \to H(\operatorname{div} 0, \Omega)$$

In the sequel we will often omit the dependence on the domain Ω of the projections P_∇ and $P_{\ker \operatorname{div}}$ in $L^2(\Omega)$; the domain of the projection operators will be clear from the context. Note also that P_∇ admits the classical explicit representation

$$P_\nabla F = \nabla \Delta_\Omega^{-1} \operatorname{div} F$$

where we have denoted with Δ_Ω the Dirichlet Laplacian acting from $\dot{H}^1_0(\Omega)$ to $\dot{H}^{-1}(\Omega) =: (\dot{H}^1_0(\Omega))^*$.

For a general unbounded domain Ω, $H^1_0(\Omega) \subsetneq \dot{H}^1_0(\Omega)$; for instance, in $\mathbb{R}^3$, the vector field $F = (1 + |x|)^\alpha$ is not in $L^2(\mathbb{R}^3)$ for $\alpha \geq -3/2$, but $\nabla F \in L^2(\mathbb{R}^3)$ for $\alpha < -1/2$. Therefore, for $\alpha \in [-3/2, -1/2)$, $F \in \dot{H}^1_0(\mathbb{R}^3) \setminus H^1_0(\mathbb{R}^3)$. However, for the domain with cylindrical ends defined in (1.3), one has $\dot{H}^1_0(\Omega) = H^1_0(\Omega)$. This identity follows from the Poincaré inequality

$$\|u\|_{L^2(\Omega)} \leq C_\Omega \|\nabla u\|_{L^2(\Omega)}, \quad u \in \dot{H}^1_0(\Omega). \tag{2.3}$$

Note that the best Poincaré constant is $C_\Omega = \lambda(\Omega)^{-1/2}$, $\lambda(\Omega)$ being the infimum of the spectrum of the Dirichlet Laplacian in Ω, and $\lambda(\Omega) > 0$.

To see that $\lambda(\Omega) > 0$, add artificial boundaries Γ_i, $i = 1, \dots, M$ as in Fig. 2. Impose Neumann boundary conditions on Γ_i; this operation enlarges the energy space of $-\Delta$ in Ω and decomposes the problem into $M+1$ boundary value problems with mixed boundary conditions of Dirichlet-Neumann type in the domains $\widetilde{\Omega_0}$ and $\widetilde{\mathcal{C}_i}$, $i = 1, \dots, M$, see Fig. 2. In particular, $\lambda(\Omega) \geq \lambda(\Omega; \Gamma)$, where $\lambda(\Omega; \Gamma) := \min\{\lambda(\widetilde{\Omega_0}); \min_{i=1,\dots,M} \lambda(\widetilde{\mathcal{C}_i})\}$. But this last number is strictly positive. Indeed, $\lambda(\widetilde{\Omega_0})$ is the first eigenvalue of a mixed Neumann-Dirichlet problem in the Lipschitz bounded domain $\widetilde{\Omega_0}$, and therefore it is strictly positive; and $\lambda(\widetilde{\mathcal{C}_i}) > 0$ for every $i = 1, \dots, M$, as one can easily check by separation of variables. Altogether, $\lambda(\Omega) > 0$.

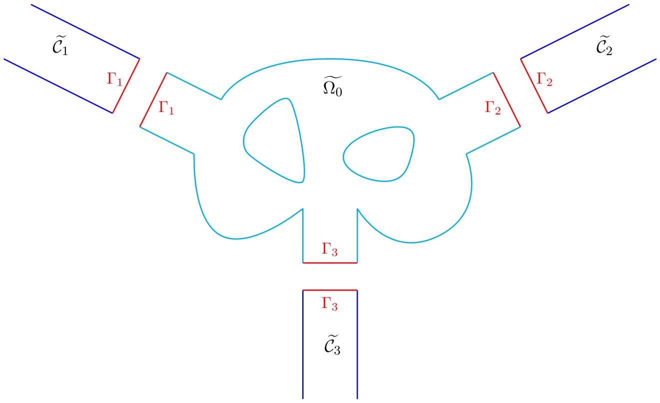

Fig. 2. Imposing Neumann boundary conditions on Γ_i, $i = 1, \dots, 3$, divides the domain Ω in 4 subdomains.

3. The resolvent of the Laplacian: Glazman decomposition and compactness properties

In this section we prove some properties of the resolvent of the Dirichlet Laplacian in possibly unbounded open sets of $\mathbb{R}^N$, which will be utilised in the proof of Theorem 4.8. These results are unlikely to be new, but we were not able to find them in the literature for the precise cases we require here. Related results can be found in [18,2].

Lemma 3.1. *Let Ω be a (possibly unbounded) connected Lipschitz open set of $\mathbb{R}^N$ and let B, U be Lipschitz open subsets of Ω such that:*

(i) *B is bounded and connected and $B \cap U = \emptyset$;*
(ii) *$\Gamma := \overline{B} \cap \overline{U}$ is a Lipschitz $N-1$ manifold;*
(iii) *$B \cup U \cup \Gamma = \Omega$.*

Assume that $\lambda \in \varrho(-\Delta_\Omega) \cap \varrho(-\Delta_B) \cap \varrho(-\Delta_U)$, $-\Delta_A$ being the Dirichlet Laplacian in $L^2(A)$. Then, with respect to the decomposition $L^2(\Omega) = L^2(B) \oplus L^2(U)$, the following Glazman resolvent decomposition holds

$$(-\Delta_\Omega - \lambda)^{-1} = \begin{pmatrix} \big(\mathbb{I} - P_B(\lambda)\Lambda(\lambda)^{-1}\partial_\nu^B\big)(-\Delta_B - \lambda)^{-1} & -P_B(\lambda)\Lambda(\lambda)^{-1}\partial_\nu^U(-\Delta_U - \lambda)^{-1} \\ -P_U(\lambda)\Lambda(\lambda)^{-1}\partial_\nu^B(-\Delta_B - \lambda)^{-1} & \big(\mathbb{I} - P_U(\lambda)\Lambda(\lambda)^{-1}\partial_\nu^U\big)(-\Delta_U - \lambda)^{-1} \end{pmatrix} \tag{3.1}$$

where ∂_ν^A is the normal derivative taken with respect to the outer normal to ∂A, $\Lambda(\lambda) = M^B(\lambda) + M^U(\lambda)$ in which $M^B(\lambda) = \partial_\nu^B P_B(\lambda)$ and $M^U(\lambda) = \partial_\nu^U P_U(\lambda)$ are the Dirichlet to Neumann maps acting from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$, $P_B(\lambda)$, $P_U(\lambda)$ are the Poisson extension operators mapping $h \in H^{1/2}(\Gamma)$ to $u_B \in H^1(B)$, $u_U \in H^1(U)$, respectively, and u_B, u_U solve respectively

$$\begin{cases} -\Delta u_B - \lambda u_B = 0, & \text{in } B, \\ u_B = 0, & \text{on } \partial B \setminus \Gamma, \\ u_B = h, & \text{on } \Gamma, \end{cases} \qquad \begin{cases} -\Delta u_U - \lambda u_U = 0, & \text{in } U, \\ u_U = 0, & \text{on } \partial U \setminus \Gamma, \\ u_U = h, & \text{on } \Gamma. \end{cases}$$

Proof. This classical result is a form of Glazman decomposition. Let $f \in L^2(\Omega)$. We define $w \in H_0^1(\Omega)$ as the solution of

$$\begin{cases} -\Delta w - \lambda w = f, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases}$$

and we set $w|_\Gamma = h \in H^{1/2}(\Gamma)$. Let $f_B = f|_B$, $f_U = f|_U$. Define

$$\begin{cases} -\Delta w_B - \lambda w_B = f_B, & \text{in } B, \\ w_B = 0, & \text{on } \partial B \setminus \Gamma, \\ w_B = h, & \text{on } \Gamma, \end{cases} \qquad \begin{cases} -\Delta w_U - \lambda w_U = f_U, & \text{in } U, \\ w_U = 0, & \text{on } \partial U \setminus \Gamma, \\ w_U = h, & \text{on } \Gamma. \end{cases}$$

We then write $w_B = P_B(\lambda)h + v_B$, with $v_B \in H^1(B)$ to be determined. Note that

$$(-\Delta - \lambda)w_B = (-\Delta - \lambda)(P_B(\lambda)h + v_B) = (-\Delta - \lambda)v_B = f_B, \quad v_B|_{\partial B} = 0,$$

so v_B solves the Dirichlet problem on B with datum $f|_B$. Similarly, one has $w_U = P_U(\lambda)h + v_U$, with $(-\Delta - \lambda)v_U = f_U$. Now, from the condition $\partial_\nu^B w_B + \partial_\nu^U w_U = 0$ on Γ and the identity $M^B(\lambda) + M^U(\lambda) = \partial_\nu^B P_B(\lambda) + \partial_\nu^U P_U(\lambda)$, we deduce

$$\begin{aligned} 0 &= \partial_\nu^B (P_B(\lambda)h + v_B) + \partial_\nu^U (P_U(\lambda)h + v_U) \\ &= \partial_\nu^B v_B + \partial_\nu^U v_U + (M^B(\lambda) + M^U(\lambda))h \\ &= \partial_\nu^B v_B + \partial_\nu^U v_U + \Lambda(\lambda)h. \end{aligned} \tag{3.2}$$

It is well known that, for $\lambda \in \varrho(-\Delta_B) \cap \varrho(-\Delta_U)$, $\Lambda(\lambda)$ is boundedly invertible as a map from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ if and only if $\lambda \notin \sigma(-\Delta_\Omega)$; since $\lambda \in \varrho(-\Delta_\Omega)$ by assumption, $\Lambda(\lambda) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is invertible, so (3.2) implies

$$h = -\Lambda(\lambda)^{-1}(\partial_\nu^B v_B + \partial_\nu^U v_U);$$

replacing h in the equalities $w_B = P_B(\lambda)h + v_B$ and $w_U = P_U(\lambda)h + v_U$ gives (3.1). □

Remark 3.2. The assumption that B and U are Lipschitz implies that either $\Gamma \cap \partial\Omega = \emptyset$, or Γ intersects $\partial\Omega$ transversally.

Proposition 3.3. *Let Ω be an open, connected Lipschitz domain in $\mathbb{R}^N$, and let $\Omega = B \cup \Gamma \cup U$, with B, Γ, U as in Lemma 3.1. Let $\lambda \in \varrho(-\Delta_\Omega) \cap \varrho(-\Delta_B) \cap \varrho(-\Delta_U)$. Then*

$$(-\Delta_\Omega - \lambda)^{-1} : L^2(B) \subset L^2_{\mathrm{comp}}(\Omega) \to H^1(\Omega)$$

is compact.

Proof. The claim of the theorem is equivalent to the statement that $(-\Delta_\Omega - \lambda)^{-1}\chi_B$ is compact as a map from $L^2(\Omega)$ to $H^1(\Omega)$. Equation (3.1) implies that

$$\begin{aligned} (-\Delta_\Omega - \lambda)^{-1}(\chi_B f) = \chi_B(I - P_B(\lambda)\Lambda(\lambda)^{-1}\partial_\nu^B)(-\Delta_B - \lambda)^{-1}(f\chi_B) \\ - (1 - \chi_B)P_U(\lambda)\Lambda(\lambda)^{-1}\partial_\nu^B(-\Delta_B - \lambda)^{-1}(f\chi_B). \end{aligned} \tag{3.3}$$

First, we consider the term

$$\chi_B(I - P_B(\lambda)\Lambda(\lambda)^{-1}\partial_\nu^B)(-\Delta_B - \lambda)^{-1}(f\chi_B),$$

which lies in $H^1(B)$; the latter space coincides with the restriction to B of elements of $H^1(\Omega)$, since B is Lipschitz. We split this term as

$$\chi_B(I - P_B(\lambda)\Lambda(\lambda)^{-1}\partial_\nu^B)(-\Delta_B - \lambda)^{-1}(f\chi_B) \;=\; \chi_B(-\Delta_B - \lambda)^{-1}\chi_B f$$
$$-\chi_B P_B(\lambda)\Lambda(\lambda)^{-1}\partial_\nu^B(-\Delta_B - \lambda)^{-1}\chi_B f.$$

Since B is a bounded Lipschitz domain, $(-\Delta_B - \lambda)^{-1}\chi_B f \in H^{3/2}(B) \cap H_0^1(B)$, see e.g. [28, Eq.(4)], [19, [Theorem 0.3]. Therefore,

$$\chi_B(-\Delta_B - \lambda)^{-1}\chi_B f \in (H^{3/2}(B) \cap H_0^1(B)) \hookrightarrow H_0^1(B) \tag{3.4}$$

the embedding being compact. Next, we show that the operator

$$\begin{aligned}&\chi_B P_B(\lambda)\Lambda(\lambda)^{-1}\partial_\nu^B(-\Delta_B - \lambda)^{-1}\chi_B \\ &\quad = \chi_B P_B(\lambda)\Lambda(\lambda)^{-1}\partial_\nu^B \iota_{H^{3/2}(B)\to H^1(B)}(-\Delta_B - \lambda)^{-1}\chi_B : L^2(\Omega) \to H^1(B)\end{aligned} \tag{3.5}$$

is compact. As observed above, $(-\Delta_B - \lambda)^{-1}$ maps $L^2(B)$ to $H^{3/2}(B) \cap H_0^1(B)$ and the embedding $\iota_{H^{3/2}(B)\to H^1(B)}$ is compact. Moreover, $\Lambda(\lambda)^{-1}\partial_\nu^B$ is continuous as a map from $H^1(B)$ to $H^{1/2}(\Gamma)$, provided $\lambda \notin (\sigma(-\Delta_\Omega) \cup \sigma(-\Delta_B) \cup \sigma(-\Delta_U))$. The map $P_B(\lambda)$ is continuous from $H^{1/2}(\Gamma)$ to $H^1(B)$. [To see this, let $\lambda \in \varrho(-\Delta_B)$ be fixed and let $h \in H^{1/2}(\Gamma)$. By definition, there exists $\varphi \in H^1(B)$ such that $\mathrm{Tr}_\Gamma\varphi = h$. We find $u \in H_0^1(B)$ by solving

$$-\Delta u - \lambda u = -\Delta\varphi - \lambda\varphi \in (H^1(B))' \subset H^{-1}(B), \quad u|_{\partial B} = 0.$$

Then $P_B(\lambda)h = \varphi - u$, therefore implying that $P_B(\lambda)h \in H^1(B)$.] Thus the compactness of the operator $\chi_B P_B(\lambda)\Lambda(\lambda)^{-1}\partial_\nu^B \iota_{H^{3/2}(B)\to H^1(B)}(-\Delta_B - \lambda)^{-1}\chi_B$ has been established. Combining this with the compactness of the map in (3.4) we see that $\chi_B(I - P_B(\lambda)\Lambda(\lambda)^{-1}\partial_\nu^B)(-\Delta_B - \lambda)^{-1}\chi_B$ is compact from $L^2(\Omega)$ to $H^1(B)$.

Finally, we observe that $(1-\chi_B)P_U(\lambda)\Lambda(\lambda)^{-1}\partial_\nu^B(-\Delta_B - \lambda)^{-1}\chi_B$ is compact. This follows in a completely analogous way to the previous step, since the composition

$$(1 - \chi_B)P_U(\lambda)\Lambda(\lambda)^{-1}\partial_\nu^B \iota_{H^{3/2}(B)\to H^1(B)}(-\Delta_B - \lambda)^{-1}\chi_B$$

is compact, the embedding $\iota_{H^{3/2}(B)\to H^1(B)}$ being compact. □

Remark 3.4. Let the assumptions of Proposition 3.3 hold, and let B be a bounded Lipschitz subdomain of Ω. Let $\mathcal{E}$ be the extension-by-zero operator, mapping a function $f \in L^2(B)$ to $\mathcal{E}f(x) = f(x)$ if $x \in B$, $\mathcal{E}f(x) = 0$ if $x \in \mathbb{R}^N \setminus \overline{B}$. Let χ_B be the characteristic function of B. A consequence of Proposition 3.3 is that

$$P_\nabla|_{H(\mathrm{div},B)} := \nabla\Delta_\Omega^{-1}\chi_B\mathcal{E}\,\mathrm{div} : H(\mathrm{div}, B) \to L^2(\Omega)^3 \tag{3.6}$$

is compact if and only if $0 \in \varrho(\Delta_\Omega)$, as the composition of the continuous operators $\mathcal{E}\,\mathrm{div}\,|_{H(\mathrm{div},B)}$ and $\nabla|_{H_0^1(\Omega)}$ with the compact operator $\Delta_\Omega^{-1}\chi_B : L^2(\Omega) \to H^1(\Omega)$ (which is compact due to Proposition (3.3)). In particular, whenever U is quasi-conical in the sense of [15, Definition 6.1], $0 \in \sigma(\Delta_\Omega)$, hence formula (3.6) is no longer valid; therefore, the proof of Theorem 4.8 fails if the cylindrical ends are replaced by infinite cones of arbitrarily small aperture. Note that the compactness result in Proposition 3.3 is in stark contrast with the lack of compactness of the operator $P_\nabla := \nabla\Delta_\Omega^{-1}\,\mathrm{div} : L^2(B) \to L^2(B)$, where Δ_Ω denotes the Dirichlet Laplacian acting from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$, which is the standard projection on $\nabla H_0^1(B)$, having essential spectrum $\sigma_{ess}(P_\nabla) = \{0, 1\}$.

4. First step: Helmholtz-type decomposition

Let $0 < \varepsilon_{\min} \leq \varepsilon_{\max}$, $0 < \mu_{\min} \leq \mu_{\max}$ and $0 \leq \sigma_{\max}$ be real constants. We assume the following ellipticity-type conditions on ε, μ and σ: for almost all $x \in \Omega$,

$$0<\varepsilon_{\min}\leq\eta\cdot\varepsilon(x)\eta\leq\varepsilon_{\max},\quad 0<\mu_{\min}\leq\eta\cdot\mu(x)\eta\leq\mu_{\max},\quad 0\leq\eta\cdot\sigma(x)\eta\leq\sigma_{\max},\quad \eta\in\mathbb{R}^3,\ |\eta|=1 \tag{4.1}$$

We emphasise that $\varepsilon_{\min} > 0$, $\mu_{\min} > 0$, while the minimal value of $\eta \cdot \sigma\eta$ might be zero.

Definition 4.1. Let $\omega \in \mathbb{C}$. We define

$$\mathcal{L}(\omega) = \operatorname{curl}\mu^{-1}\operatorname{curl}_0 - \omega(\omega\varepsilon + i\sigma) \tag{4.2}$$

with $\operatorname{dom}(\mathcal{L}(\omega)) = \{u \in H_0(\operatorname{curl}, \Omega) : \mathcal{L}(\omega)u \in L^2(\Omega)\}$.

Note that $\operatorname{dom}(\mathcal{L}(\omega))$ is independent of ω, since the conditions (4.1) ensure that it is equal to $\{u \in H_0(\operatorname{curl}, \Omega) : \operatorname{curl}\mu^{-1}\operatorname{curl} u \in L^2(\Omega)\}$.

Before stating our first 'Decomposition and Simplification' theorem, we recall some results from [6] that link the spectrum of the Maxwell pencil V to the spectrum of the pencil $\mathcal{L}$.

Theorem 4.2 *([6], Thm 4.5). The Maxwell pencil $V(\cdot)$ in (1.2) and the quadratic pencil $\mathcal{L}$ in (4.2) satisfy*

$$\varrho(V) \setminus \{0\} = \varrho(\mathcal{L}) \setminus \{0\}, \quad \sigma(V) \setminus \{0\} = \sigma(\mathcal{L}) \setminus \{0\}, \tag{4.3}$$

and the resolvent of $V(\cdot)$ is given by

$$V(\omega)^{-1} = \begin{pmatrix} \omega\mathcal{L}(\omega)^{-1} & \mathrm{i}\overline{\mathcal{L}(\omega)^{-1}\operatorname{curl}}\mu^{-1} \\ -\mathrm{i}\mu^{-1}\operatorname{curl}_0\mathcal{L}(\omega)^{-1} & \omega^{-1}(-\mu^{-1}+\mu^{-1}\overline{\operatorname{curl}_0\mathcal{L}(\omega)^{-1}\operatorname{curl}}\mu^{-1}) \end{pmatrix} \tag{4.4}$$

for $\omega \in \varrho(V)$, in which $\overline{A}$ denotes the closure of an operator A. Moreover the point spectra σ_p, continuous spectra σ_c and residual spectra σ_r satisfy

$$\sigma_p(V) \setminus \{0\} = \sigma_p(\mathcal{L}) \setminus \{0\}, \quad \sigma_c(V) \setminus \{0\} = \sigma_c(\mathcal{L}) \setminus \{0\}, \quad \sigma_r(V) = \sigma_r(\mathcal{L}) = \emptyset,$$

and the essential spectra σ_{ek} [15] satisfy

$$\sigma_{e1}(V) = \sigma_{e2}(V) = \sigma_{e3}(V) = \sigma_{e4}(V) = \sigma_{e1}(\mathcal{L}) = \sigma_{e2}(\mathcal{L}) = \sigma_{e3}(\mathcal{L}) = \sigma_{e4}(\mathcal{L}).$$

The operator $M(\omega) = ((\omega\varepsilon + i\sigma) - \omega\varepsilon_\infty)P_{\ker(\operatorname{div})}$ is compact from $H(\operatorname{curl}, \Omega)$ to $L^2(\Omega)$, as a consequence of the following.

Proposition 4.3 *([6], Prop. 5.1). Let $m : \Omega \to \mathbb{C}^{3\times 3}$ be a locally bounded, tensor-valued function with*

$$\lim_{R\to\infty} \sup_{\|x\|>R} \|m(x)\| = 0. \tag{4.5}$$

Then $mP_{\ker(\operatorname{div})}$ is compact from $(H(\operatorname{curl}, \Omega), \|\cdot\|_{H(\operatorname{curl},\Omega)})$ to $L^2(\Omega)$.

Lemma 4.4. *Let ε_∞ be as in (1.5). Then $P_\nabla\varepsilon_\infty P_{\ker\operatorname{div}}$ is compact from $H(\operatorname{curl}, \Omega)$ to $L^2(\Omega)$.*

Proof. To see this, recall that $P_\nabla = \nabla(-\Delta_{H_0^1\to H^{-1}})^{-1}\operatorname{div}$, where $-\Delta_{H_0^1\to H^{-1}}$ is the Dirichlet Laplacian seen as a bounded operator from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$. Note also that $-\Delta_{H_0^1\to H^{-1}}$ is invertible. Let χ_R be the characteristic function of $B(0,R)$. Choose R so large that $\varepsilon_\infty(x)$ is constant in $x \in \mathcal{C}_i \cap B(0,R)^c$, $i = 1,\dots,M$. Then,

$$\begin{aligned}(1-\chi_R)\operatorname{div}(\varepsilon_\infty P_{\ker\operatorname{div}}F) &= (1-\chi_R)\nabla\varepsilon_\infty \cdot P_{\ker\operatorname{div}}F \\ &= (1-\chi_R)\chi_R\nabla\varepsilon_\infty \cdot P_{\ker\operatorname{div}}F = 0\end{aligned}$$

for any $F \in H(\operatorname{curl},\Omega)$. Thus, $\operatorname{supp}(\operatorname{div}\varepsilon_\infty P_{\ker\operatorname{div}}F) \subset B(0,R)$ is compact in Ω, for every $F \in H(\operatorname{curl},\Omega)$. Recall that since $\varepsilon_\infty(x) = f(x)\mathbb{I}_{3\times 3}$, for a smooth real-valued function f, we have $\operatorname{div}\varepsilon_\infty P_{\ker\operatorname{div}}F = \nabla f \cdot P_{\ker\operatorname{div}}F \in L^2(\Omega)$. Observe that

$$\nabla(-\Delta_{H_0^1\to H^{-1}})^{-1}\chi_R\operatorname{div}\varepsilon_\infty P_{\ker\operatorname{div}}F = \nabla(-\Delta_{\operatorname{dom}(-\Delta)\to L^2(\Omega)})^{-1}\chi_R\operatorname{div}\varepsilon_\infty P_{\ker\operatorname{div}}F. \tag{4.6}$$

It is worth remarking that $(-\Delta_{\operatorname{dom}(-\Delta)\to L^2(\Omega)})^{-1} = (-\Delta_\Omega)^{-1}$ exists as a bounded operator in $L^2(\Omega)$ because the first eigenvalue of $-\Delta_\Omega$ is strictly positive; it can be computed as the minimum of the first eigenvalues of the Laplace operators in the cross-sections of the cylindrical ends, see (2.3). Now (4.6) is an immediate consequence of the fact that a weak solution of the Poisson equation with datum in $L^2(\Omega)$ lies not only in $H_0^1(\Omega)$ but in the domain of the Dirichlet Laplacian $-\Delta_\Omega$. As a consequence of Proposition 3.3, the operator

$$P_\nabla\varepsilon_\infty P_{\ker\operatorname{div}} = \nabla\left[(-\Delta_\Omega)^{-1}\chi_R\right]\operatorname{div}\varepsilon_\infty P_{\ker\operatorname{div}}$$

is compact from $H(\operatorname{curl},\Omega)$ to $L^2(\Omega)$ as the composition of the compact operator $(-\Delta_\Omega)^{-1}\chi_R$ from $L^2(B(0,R)\cap\Omega)$ to $H^1(\Omega)$ and the continuous operators ∇ from $H^1(\Omega)$ to $L^2(\Omega)$, $\operatorname{div}\varepsilon_\infty P_{\ker\operatorname{div}}$ from $L^2(\Omega)$ to $L^2(\Omega)$. □

Definition 4.5. We define quadratic pencils of closed operators acting in the Hilbert space $H(\operatorname{div}0,\Omega)$ equipped with the $L^2(\Omega)^3$-norm

$$\begin{aligned}&L_\mu(\omega) := \operatorname{curl}\mu^{-1}\operatorname{curl}_0 - P_{\ker\operatorname{div}}\varepsilon_\infty\omega^2\,\mathrm{id},\\ &\operatorname{dom}(L_\mu(\omega)) := \{u \in H_0(\operatorname{curl},\Omega)\cap H(\operatorname{div}0,\Omega) : \mu^{-1}\operatorname{curl}u \in H(\operatorname{curl},\Omega)\},\end{aligned} \qquad \omega\in\mathbb{C},$$

and

$$\begin{aligned}&L_\infty(\omega) := \operatorname{curl}\mu_\infty^{-1}\operatorname{curl}_0 - P_{\ker\operatorname{div}}\varepsilon_\infty\omega^2\,\mathrm{id},\\ &\operatorname{dom}(L_\infty(\omega)) := \{u \in H_0(\operatorname{curl},\Omega)\cap H(\operatorname{div}0,\Omega) : \operatorname{curl}u \in H(\operatorname{curl},\Omega)\};\end{aligned} \qquad \omega\in\mathbb{C}.$$

Note that L_∞ can be regarded as a special case of L_μ, namely when $\mu = \mu_\infty\mathrm{id}$.

Lemma 4.6 *(Lemma 5.3, [6]). The following are true.*

(i) *The operator $L_\mu(\omega)^{-1}\operatorname{curl}$ is closable and bounded from $L^2(\Omega)^3$ to $H(\operatorname{div}0,\Omega)$.*
(ii) *For $\omega = it$ with $t \geq \varepsilon_{\min}^{-1/2}$, the operator $\operatorname{curl}_0 L_\infty(\omega)^{-1}$ is bounded in $H(\operatorname{div}0,\Omega)$, and also as an operator from $H(\operatorname{div}0,\Omega)$ to $H(\operatorname{curl},\Omega)$ with*

$$\|\operatorname{curl}_0 L_\infty(\omega)^{-1}\|_{\mathcal{B}(H(\operatorname{div}0,\Omega),H(\operatorname{curl},\Omega))} \leq \left(\frac{\|\mu_\infty\|_{L^\infty}}{(\inf_{x\in\Omega}\varepsilon_\infty(x))|\omega|^2} + \|\mu_\infty\|_{L^\infty}^2\right)^{1/2}. \tag{4.7}$$

Proposition 4.7 *(Similar to [6], Prop. 5.4). If μ satisfies the limiting assumption (1.6) and L_μ, L_∞ are as in Definition 4.5, then the essential spectra σ_{ek} [15] satisfy $\sigma_{ek}(L_\mu) = \sigma_{ek}(L_\infty) \subset \mathbb{R}$ for $k = 1,2,3,4,5$.*

Proof. This proof is similar to the proof of [6, Prop. 5.4]. First note that L_μ and L_∞ define self-adjoint pencils of operators (that is, $L_\mu(\omega) = L_\mu(\overline{\omega})^*$, for all $\omega \in \mathbb{C}$; similarly for L_∞); hence, $\sigma_{ek}(L_\mu)$ and $\sigma_{ek}(L_\infty)$ are subsets of $\mathbb{R}$. It remains to establish the identity $\sigma_{ek}(L_\mu)=\sigma_{ek}(L_\infty)$. The latter is a consequence of

$$L_\mu(\omega)^{-1} - L_\infty(\omega)^{-1} = (\operatorname{curl}_0(L_\mu(\omega)^*)^{-1})^*(\mu_\infty^{-1} - \mu^{-1})\operatorname{curl}_0(L_\infty(\omega))^{-1} \tag{4.8}$$

valid for all $\omega \in \mathbb{C} \setminus \mathbb{R}$, which can be proved exactly as in the proof of [6, Prop. 5.4]. Note that the right hand side of (4.8) can be rewritten as

$$\begin{aligned}&(\operatorname{curl}_0(L_\mu(\omega)^*)^{-1})^*(\mu_\infty^{-1} - \mu^{-1})\operatorname{curl}_0(L_\infty(\omega))^{-1}\\ &= (\operatorname{curl}_0(L_\mu(\omega)^*)^{-1})^* \underbrace{(\mu_\infty^{-1} - \mu^{-1})P_{\ker\operatorname{div}}}_{\text{compact due to Proposition 4.3}} \operatorname{curl}_0(L_\infty(\omega))^{-1};\end{aligned}$$

therefore, the difference of resolvents $L_\mu(\omega)^{-1} - L_\infty(\omega)^{-1}$ is compact, for all $\omega \in \mathbb{C} \setminus \mathbb{R}$. Theorem IX.2.4 in [15] now implies that $\sigma_{ek}(L_\mu) = \sigma_{ek}(L_\infty)$, $k = 1, 2, 3, 4, 5$. □

We are now ready to state and prove our first 'Decomposition and Simplification' result for the Maxwell essential spectrum in a domain with cylindrical ends.

Theorem 4.8 *(Decomposition and Simplification I). Suppose that σ, ε and μ satisfy the limiting assumptions (1.6), (1.7). Let $P_\nabla := \mathrm{id} - P_{\ker\operatorname{div}}$ be the orthogonal projection from $L^2(\Omega)^3 = \nabla\dot{H}_0^1(\Omega) \oplus H(\operatorname{div} 0, \Omega)$ onto $\nabla\dot{H}_0^1(\Omega)$ and recall that $\mathcal{W}(\omega) := -\omega(\omega\varepsilon + \mathrm{i}\sigma)$, $\omega \in \mathbb{C}$, in $L^2(\Omega)^3$. Then*

$$\sigma_{ek}(\mathcal{L}) = \sigma_{ek}(L_\infty) \cup \sigma_{ek}(P_\nabla \mathcal{W}(\cdot)|_{\nabla\dot{H}_0^1(\Omega)}), \quad k = 1, 2, 3, 4,$$

with $\sigma_{ek}(L_\infty) \subset \mathbb{R}$ and $\sigma_{ek}(P_\nabla \mathcal{W}(\cdot)|_{\nabla\dot{H}_0^1(\Omega)}) \subset \mathrm{i}[-\frac{\sigma_{\max}}{\varepsilon_{\min}}, 0]$.

Proof. Let $\omega \in \mathbb{C}$. By Proposition 4.3 and the limiting assumptions (1.6), (1.7), the operator $M(\omega) := (\omega(\omega\varepsilon + \mathrm{i}\sigma) - \omega^2)P_{\ker(\operatorname{div})}$ in $L^2(\Omega)^3$ is curl_0-compact. This means that if $(E_n)_n \subset \operatorname{dom}(\operatorname{curl}_0) = H_0(\operatorname{curl}, \Omega)$ with curl_0-graph-norm bounded (that is, $\sup_n \|E_n\|_{H(\operatorname{curl},\Omega)} < \infty$), then $(M(\omega)E_n)_n$ is precompact in $L^2(\Omega)^3$. Then it is also T_0-compact with $T_0 = \mu^{-1/2}\operatorname{curl}_0$. Since $\mathcal{L}(\omega) = T_0^*T_0 + \mathcal{W}(\omega)$, bounded sequences whose $\mathcal{L}(\omega)$ graph norms are bounded have bounded T_0-graph norms. Hence $M(\omega)$ is $\mathcal{L}(\omega)$-compact which yields $\sigma_{ek}(\mathcal{L}(\omega)) = \sigma_{ek}(\mathcal{L}(\omega) + M(\omega))$, $k = 1, 2, 3, 4$.

Since $\nabla\dot{H}_0^1(\Omega) \subset \ker(\operatorname{curl}_0) = \ker T_0$ and hence $T_0 P_\nabla = P_\nabla T_0^* = 0$, $\nabla\dot{H}_0^1(\Omega)$ is a reducing subspace for $T_0^*T_0$. Therefore the operator

$$\begin{aligned}\mathcal{T}(\omega) := \mathcal{L}(\omega) + M(\omega) &= T_0^*T_0 - \omega(\omega\varepsilon + \mathrm{i}\sigma)(P_\nabla + P_{\ker\operatorname{div}}) + (\omega(\omega\varepsilon + \mathrm{i}\sigma) - \varepsilon_\infty\omega^2)P_{\ker(\operatorname{div})}\\ &= T_0^*T_0 - \omega(\omega\varepsilon + \mathrm{i}\sigma)P_\nabla - \omega^2\varepsilon_\infty P_{\ker(\operatorname{div})},\end{aligned} \tag{4.9}$$

which is a bounded perturbation of $T_0^*T_0$, admits an operator matrix representation with respect to the decomposition $L^2(\Omega)^3 = \nabla\dot{H}_0^1(\Omega) \oplus H(\operatorname{div} 0, \Omega)$ given by

$$\begin{aligned}\mathcal{T}(\omega) &= \begin{pmatrix} P_\nabla\mathcal{T}(\omega)|_{\nabla\dot{H}_0^1(\Omega)} & P_\nabla\mathcal{T}(\omega)|_{H(\operatorname{div} 0,\Omega)} \\ P_{\ker\operatorname{div}}\mathcal{T}(\omega)|_{\nabla\dot{H}_0^1(\Omega)} & P_{\ker\operatorname{div}}\mathcal{T}(\omega)|_{H(\operatorname{div} 0,\Omega)} \end{pmatrix}\\ &= \begin{pmatrix} P_\nabla(-\omega(\omega\varepsilon + \mathrm{i}\sigma))|_{\nabla\dot{H}_0^1(\Omega)} & -\omega^2 P_\nabla\varepsilon_\infty|_{H(\operatorname{div} 0,\Omega)} \\ P_{\ker\operatorname{div}}(-\omega(\omega\varepsilon + \mathrm{i}\sigma))|_{\nabla\dot{H}_0^1(\Omega)} & P_{\ker\operatorname{div}}(T_0^*T_0 - \omega^2\varepsilon_\infty)|_{H(\operatorname{div} 0,\Omega)} \end{pmatrix}\\ &= \begin{pmatrix} P_\nabla\mathcal{W}(\omega)|_{\nabla\dot{H}_0^1(\Omega)} & -\omega^2 P_\nabla\varepsilon_\infty P_{\ker(\operatorname{div})} \\ P_{\ker\operatorname{div}}\mathcal{W}(\omega)|_{\nabla\dot{H}_0^1(\Omega)} & L_\mu(\omega) \end{pmatrix},\end{aligned} \tag{4.10}$$

with domain $\mathrm{dom}(\mathcal{T}(\omega)) = \nabla \dot{H}^1_0(\Omega) \oplus \mathrm{dom}(L_\mu(\omega))$. Note that

$$P_{\ker \mathrm{div}}(T_0^* T_0 - \omega^2 \varepsilon_\infty)|_{H(\mathrm{div}\, 0,\Omega)} = (T_0^* T_0 - \omega^2 P_{\ker \mathrm{div}} \varepsilon_\infty)|_{H(\mathrm{div}\, 0,\Omega)}$$

since the range of $T_0^* T_0 \subset H(\mathrm{div}\, 0, \Omega)$. Now, due to Lemma 4.4, $P_\nabla \varepsilon_\infty P_{\ker(\mathrm{div})}$ is compact.

Since compact perturbations do not change the essential spectra $\sigma_{ek}(\mathcal{T})$ for $k = 1, 2, 3, 4$, we deduce that $\sigma_{ek}(\mathcal{T}(\omega))$ coincides with the essential spectrum σ_{ek} of

$$\tilde{\mathcal{T}}(\omega) = \begin{pmatrix} P_\nabla \mathcal{W}(\omega)|_{\nabla \dot{H}^1_0(\Omega)} & 0 \\ P_{\ker \mathrm{div}} \mathcal{W}(\omega)|_{\nabla \dot{H}^1_0(\Omega)} & L_\mu(\omega) \end{pmatrix}.$$

Apart from $L_\mu(\omega)$, the other two matrix entries in $\tilde{\mathcal{T}}(\omega)$ are bounded and everywhere defined, and $\sigma_{e2}(L_\mu(\omega)) = \sigma^*_{e2}(L_\mu(\omega))$. Thus [6, Theorem 8.1] implies that

$$\sigma_{e2}(\tilde{\mathcal{T}}(\omega)) = \sigma_{e2}(L_\mu(\omega)) \cup \sigma_{e2}(P_\nabla \mathcal{W}(\omega)|_{\nabla \dot{H}^1_0(\Omega)}) = \sigma_{e2}(L_\infty(\omega)) \cup \sigma_{e2}(P_\nabla \mathcal{W}(\omega)|_{\nabla \dot{H}^1_0(\Omega)})$$

and hence, since $\omega \in \mathbb{C}$ was arbitrary,

$$\sigma_{e2}(\mathcal{L}) = \sigma_{e2}(\mathcal{L} + M) = \sigma_{e2}(\mathcal{T}) = \sigma_{e2}(L_\infty) \cup \sigma_{e2}(P_\nabla \mathcal{W}(\cdot)|_{\nabla \dot{H}^1_0(\Omega)}).$$

By Theorem 4.2 the $\sigma_{ek}(\mathcal{L})$, $k = 1, 2, 3, 4$, all coincide. The same is true of all the $\sigma_{ek}(L_\infty)$ and all the $\sigma_{ek}(P_\nabla \mathcal{W}(\cdot)|_{\nabla \dot{H}^1_0(\Omega)})$, by J-selfadjointness of L_∞ and $P_\nabla \mathcal{W}(\cdot)|_{\nabla \dot{H}^1_0(\Omega)}$. Thus

$$\sigma_{ek}(\mathcal{L}) = \sigma_{ek}(L_\infty) \cup \sigma_{ek}(P_\nabla \mathcal{W}(\cdot)|_{\nabla \dot{H}^1_0(\Omega)}), \quad k = 1, 2, 3, 4. \quad \square$$

5. Second step: Glazman decomposition

Recall that in Definition 4.5 we introduced the pencil L_∞ corresponding to the 'limit at infinity' of the pencil L_μ. In a similar way, now define pencils $L_{\infty,i}$, $i = 1, \dots, M$, obtained from L_∞ by choosing $\varepsilon_\infty = \varepsilon_i$, $\mu_\infty = \mu_i$ where ε_i, μ_i have been defined in (1.5).

Definition 5.1. We define quadratic pencils of closed operators acting in the Hilbert space $H(\mathrm{div}\, 0, \mathcal{C}_i)$ equipped with the $L^2(\mathcal{C}_i)$-norm, by

$$L_{\infty,i}(\omega) := \mathrm{curl}\, \mu_i^{-1}\, \mathrm{curl}_0 - \varepsilon_i\, \omega^2\ \mathrm{id},$$
$$\mathrm{dom}(L_{\infty,i}(\omega)) := \{u \in H_0(\mathrm{curl}, \mathcal{C}_i) \cap H(\mathrm{div}\, 0, \mathcal{C}_i) :\ \mathrm{curl}\, u \in H(\mathrm{curl}, \mathcal{C}_i)\},$$

for all $\omega \in \mathbb{C}$, $i = 1, \dots, M$.

We now investigate the relation between $\sigma_{ess}(L_\infty)$ and $\sigma_{ess}(L_{\infty,i})$, $i = 1, \dots, M$. We first point out the following standard result.

Lemma 5.2. *Let B be a Lipschitz bounded set. Let $(u_n)_n$ be a sequence in $H_0(\mathrm{curl}, B) \cap H(\mathrm{div}, B)$, $u_n \rightharpoonup 0$, $\|u_n\|_{L^2(B)} = 1$, $n \in \mathbb{N}$, $\sup_n \|\mathrm{div}\, u_n\|_{L^2(B)} < \infty$. Then $\|\mathrm{curl}\, u_n\|_{L^2(B)} \to \infty$.*

Proof. Assume for a contradiction that $\|\mathrm{curl}\, u_n\|_{L^2(B)} < M$, $n \in \mathbb{N}$. Then the sequence $(u_n)_n \subset H_0(\mathrm{curl}, B) \cap H(\mathrm{div}, B)$ is uniformly bounded with respect to the norm $\|\cdot\|_{L^2(B)} + \|\mathrm{curl} \cdot\|_{L^2(B)} + \|\mathrm{div} \cdot\|_{L^2(B)}$. By compactness of the embedding

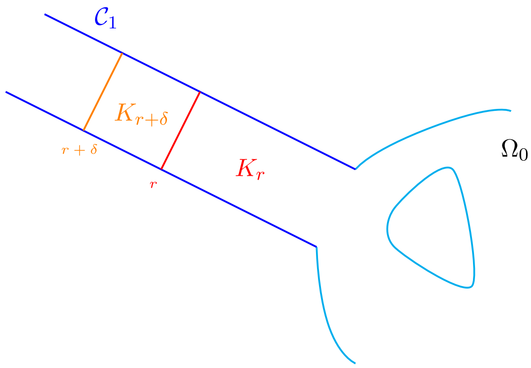

Fig. 3. The domains K_r and $K_{r+\delta}$ in the proof of Theorem 5.3.

$$H_0(\mathrm{curl}, B) \cap H(\mathrm{div}, B) \hookrightarrow L^2(B)$$

(see e.g. [29]) we conclude that u_n has a strongly convergent subsequence in $L^2(B)$. By assumption, $u_n \rightharpoonup 0$, so the strongly convergent subsequence must converge to 0. This is impossible as $\|u_n\|_{L^2(B)} = 1$ for all n. This contradiction proves the result. □

Theorem 5.3 *(Decomposition and Simplification II). The selfadjoint pencils L_∞ and $L_{\infty,i}$ have essential spectra satisfying $\sigma_{ess}(L_\infty) = \bigcup_{i=1}^{M} \sigma_{ess}(L_{\infty,i})$.*

Remark 5.4. The σ_{ess} appearing in Theorem 5.3 may be replaced by σ_{ek}, $k = 1, 2, 3, 4, 5$, which are all equal in the selfadjoint case. In the proof below we use the singular-sequence characterisation of essential spectrum, associated with $\sigma_{e2} = \sigma_{ess}$.

Proof. For simplicity of exposition we assume that $M = 2$; the general case is similar. We first prove that $\sigma_{ess}(L_\infty) \subset \bigcup_{i=1}^{M} \sigma_{ess}(L_{\infty,i})$. If $\omega \in \sigma_{ess}(L_\infty)$ there exist $E_n \in \mathrm{dom}(L_\infty) \subset H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}\, 0, \Omega)$, $\|E_n\| = 1$ $\forall n \in \mathbb{N}$, $E_n \rightharpoonup 0$, such that

$$(\mathrm{curl}\, \mu_\infty^{-1} \,\mathrm{curl} - P_{\ker(\mathrm{div})} \varepsilon_\infty \omega^2) E_n \to 0, \tag{5.1}$$

as $n \to \infty$. Our strategy will be to construct a Weyl singular sequence Ψ_n for one of the operators $L_{\infty,i}$, by suitably truncating the vector fields E_n. However, this is not a trivial task since the truncation of a vector field in $H(\mathrm{div}\, 0, \Omega)$ does not belong to $H(\mathrm{div}\, 0, \mathcal{C}_i)$; for, $\mathrm{div}(E_n\chi) = (\mathrm{div}\, E_n)\chi + E_n \cdot \nabla\chi = E_n \cdot \nabla\chi \neq 0$ for a general non-constant $\chi \in C^\infty(\Omega)$. In order to implement our strategy, we first need to establish the following claim.

Claim. $\|E_n\|_{L^2(K)} \to 0$, $\|\mathrm{curl}\, E_n\|_{L^2(K)} \to 0$ and $\|\mathrm{curl}\, \mu_\infty^{-1} \,\mathrm{curl}\, E_n\|_{L^2(K)} \to 0$ as $n \to \infty$, for every bounded Lipschitz subdomain K of Ω.

To prove this **Claim** we first note that, by boundedness of K, there exists $r > 0$ such that $K \subset K_r = \Omega_0 \sqcup \mathcal{C}_{1,r} \sqcup \mathcal{C}_{2,r}$, where $r \geq 0$, $\mathcal{C}_{i,r}$ is expressed in local coordinates as the set

$$\mathcal{C}_{i,r} := \{y \in \mathcal{C}_i : y_1 \in (0, r), (y_2, y_3) \in C_i\}. \tag{5.2}$$

Let $\delta > 0$ and let $R = r + \delta$. Let $\chi_r \in C^\infty(\overline{\Omega})$ be such that $0 \leq \chi_r \leq 1$, $\chi_r(x) = 1$ in K_r, $\chi_r(x) = 0$ in $\Omega \setminus \overline{K_R}$, see Fig. 3. We may also assume that χ_r depends only on the first local coordinate y_1 in the cylinders $\mathcal{C}_i$.

Note then that for fixed n, $E_n\chi_r \in H_0(\mathrm{curl}, K_R) \cap H(\mathrm{div}, K_R)$; indeed, since χ_r is smooth, $E_n\chi_r \in H(\mathrm{curl}, K_R) \cap H(\mathrm{div}, K_R)$, hence we just need to check the boundary condition $\nu \times (E_n\chi_r) = 0$ on ∂K_R.

But $\partial K_R = (\partial\Omega \cap \partial K_R) \cup (\partial K_R \setminus \overline{\partial\Omega})$; on $(\partial\Omega \cap \partial K_R)$ we have $\nu \times (E_n\chi_r) = \chi_r(\nu \times E_n) = 0$, while on $\partial K_R \setminus \overline{\partial\Omega}$ we have $\chi_r = 0$. Moreover $E_n\chi_r \rightharpoonup 0$ as $n \to \infty$ and

$$\sup_{n\in\mathbb{N}} \|\mathrm{div}(E_n\chi_r)\|_{L^2(K_R)} \le \sup_{n\in\mathbb{N}} \|E_n \cdot \nabla\chi_r\|_{L^2(K_R)} \le \|\nabla\chi_r\|_{L^\infty(K_R)}.$$

Next we show that the sequence $\mathrm{curl}(E_n\chi_r)$ is uniformly bounded in n in the $L^2(K_R)$-norm. First note that

$$\|\mathrm{curl}(E_n\chi_r)\|_{L^2(K_R)} \le \|\mathrm{curl}\, E_n\|_{L^2(K_R)} + \|E_n \times \nabla\chi_r\|_{L^2(K_R\setminus K_r)}. \tag{5.3}$$

Now, (5.1) implies that $(\mu_\infty^{-1}\,\mathrm{curl}\, E_n, \mathrm{curl}\, E_n) - \omega^2(P_{\ker(\mathrm{div})}\varepsilon_\infty E_n, E_n) =: h_n \to 0$; therefore,

$$\inf_{x\in\Omega} |\mu_\infty^{-1}(x)| \|\mathrm{curl}\, E_n\|^2_{L^2(\Omega)} \le \|\mu_\infty^{-1/2}\,\mathrm{curl}\, E_n\|^2_{L^2(\Omega)} \le |\omega|^2\|\varepsilon_\infty\|_\infty + |h_n|,$$

from which the uniform boundedness of $\|\mathrm{curl}\, E_n\|^2_{L^2(\Omega)}$ and of $\|\mathrm{curl}\, E_n\|_{L^2(K_R)}$ follows. Inequality (5.3) finally implies that $\sup_n \|\mathrm{curl}(E_n\chi_r)\|_{L^2(K_R)} < \infty$.

Since $(E_n\chi_r)_n \subset H_0(\mathrm{curl}, K_R) \cap H(\mathrm{div}, K_R)$ has uniformly bounded $H(\mathrm{curl})$- and $H(\mathrm{div})$-norms, Lemma 5.2 implies that $E_n\chi_r \to 0$ in $L^2(K_R)$; in particular, $E_n|_K \to 0$. Therefore (5.1) implies that $\mathrm{curl}\,\mu_\infty^{-1}\,\mathrm{curl}\, E_n|_K \to 0$.

We now prove that $(\mathrm{curl}\, E_n)\chi_r \to 0$ in $L^2(\Omega)$, in particular, $\mathrm{curl}\, E_n|_K \to 0$ in L^2.

Note that

$$(\mathrm{curl}\,\mu_\infty^{-1}\,\mathrm{curl}\, E_n - \omega^2 P_{\ker(\mathrm{div})}\varepsilon_\infty E_n)\chi_r \to 0 \tag{5.4}$$

in $L^2(K_R)$ and $E_n\chi_r \to 0$ due to the previous part of the proof, so in particular $\omega^2 P_{\ker(\mathrm{div})}\varepsilon_\infty(E_n\chi_r) \to 0$. Taking scalar products with E_n in (5.4) and integrating by parts we obtain

$$\int_{K_R} |\mu_\infty^{-1/2}\,\mathrm{curl}\, E_n|^2\chi_r + \int_{K_R} (\mu_\infty^{-1}\,\mathrm{curl}\, E_n \times \nabla\chi_r) \cdot \overline{(E_n\chi_r)} \to 0.$$

Since $E_n\chi_r \to 0$, $\mu_\infty^{-1/2}\,\mathrm{curl}\, E_n \to 0$ in $L^2(K_R)$; hence, $(\mathrm{curl}\, E_n)\chi_r \to 0$. This concludes the proof of the **Claim**.

Because of the **Claim** we have $\|E_n\|_{L^2(K_r)} \to 0$ for any $r > 0$. If $\|E_n\|_{L^2(\mathcal{C}_1)} = 1$ for $n \ge n_0$, then $(E_n)_{n\ge n_0}$ is a Weyl singular sequence such that $L_{\infty,1}(\omega)E_n \to 0$, and the proof is finished. Otherwise, we assume that $\|E_n\|_{L^2(\mathcal{C}_1)} \to c_1$, $\|E_n\|_{L^2(\mathcal{C}_2)} \to c_2$ and $c_1^2 + c_2^2 = 1$, $c_i \in [0, 1]$. Without loss of generality we may also assume that $c_1 > 0$; otherwise just swap c_1 with c_2.

Working in local coordinates, we may assume that

$$\mathcal{C}_1 = \{x \in \mathbb{R}^3 : x_1 \in (0, +\infty), (x_2, x_3) \in C_1\}.$$

Let $R > 0$ be large and let $\Theta_R = (R, +\infty) \times C_1$. Let $\xi_R \in C^\infty(\mathcal{C}_1)$, $0 \le \xi_R(x) \le 1$, $x \in \mathcal{C}_1$, be such that $\xi_R = 1$ in Θ_R, $\chi_R = 0$ in $K_{R/2}$, and $\xi_R(x) = \xi_R(x_1)$ for all $x \in \mathcal{C}_1$.

We claim that $\Psi_n = E_n\xi_R - P_\nabla(E_n\xi_R) \in H_0(\mathrm{curl}, \mathcal{C}_1) \cap H(\mathrm{div}\, 0, \mathcal{C}_1)$ is a singular sequence for $L_{\infty,1}(\omega)$. We first note that $\mathrm{div}\,\Psi_n = 0$ by construction. Moreover, $\mathrm{curl}\,\Psi_n = \mathrm{curl}(E_n\xi_R) = (\mathrm{curl}\, E_n)\xi_R + E_n \times \nabla\xi_R$, so $\Psi_n \in H_0(\mathrm{curl}, \mathcal{C}_1) \cap H(\mathrm{div}\, 0, \mathcal{C}_1)$ for every n. We now observe that

$$\begin{aligned}\|\Psi_n - E_n\|_{L^2(\mathcal{C}_1)} &\le \|E_n\xi_R - E_n\|_{L^2(K_R)} + \|P_\nabla(E_n\xi_R)\|_{L^2(\mathcal{C}_1)} \\ &\le \|E_n\xi_R - E_n\|_{L^2(K_R)} + \left(\frac{1}{\lambda_1^{-\Delta}(\mathcal{C}_1)}\right)^{1/2} \|E_n \cdot \nabla\xi_R\|_{L^2(K_R)},\end{aligned} \tag{5.5}$$

where in the last inequality of (5.5) we used the identity

$$P_\nabla(E_n\xi_R) = \nabla\Delta_{\mathcal{C}_1}^{-1}\operatorname{div}(E_n\xi_R),$$

$\Delta_{\mathcal{C}_1}^{-1}$ being the inverse Dirichlet Laplace operator acting from $H^{-1}(\mathcal{C}_1)$ to $H_0^1(\mathcal{C}_1)$, the fact that

$$\|\nabla\Delta^{-1}\|^2 = 1/(\lambda_1^{-\Delta}(\mathcal{C}_1)),$$

and the fact that $\operatorname{div}(E_n\xi_R) = E_n\cdot\nabla\xi_R$ is supported in a bounded subdomain of K_R. The right hand-side of (5.5) tends to zero due to the **Claim**, so $\|\Psi_n - E_n\|_{L^2(\mathcal{C}_1)} \to 0$. Thus $\|\Psi_n\|_{L^2(\mathcal{C}_1)} \to c_1 > 0$ as $n\to\infty$; moreover, from $E_n \rightharpoonup 0$ we deduce that $\Psi_n \rightharpoonup 0$ as $n\to\infty$.

It is left to prove that $\mu_1^{-1}\operatorname{curl}\operatorname{curl}\Psi_n - \omega^2\varepsilon_1\Psi_n \to 0$. We have

$$\operatorname{curl}\operatorname{curl}(E_n\xi_R) = (\operatorname{curl}\operatorname{curl}E_n)\xi_R + \nabla\xi_R\times\operatorname{curl}E_n - \nabla E_n(\nabla\xi_R) - E_n\Delta\xi_R + \operatorname{div}E_n\nabla\xi_R + (D^2\xi_R)(E_n)$$

Since ξ_R depends only on x_1 and $\operatorname{div}E_n = 0$ we can further write

$$\operatorname{curl}\operatorname{curl}(E_n\xi_R) = (\operatorname{curl}\operatorname{curl}E_n)\xi_R + \nabla\xi_R\times\operatorname{curl}E_n - \partial_{x_1}\xi_R\partial_{x_1}E_n - \partial_{x_1}^2\xi_R\begin{pmatrix}0\\E_n^2\\E_n^3\end{pmatrix} \tag{5.6}$$

Apart from the first summand, all the terms on the right hand-side of (5.6) are supported in bounded subdomains of $\mathcal{C}_1$, and hence due to the previous **Claim** they tend to zero in $L^2(\mathcal{C}_1)$. This is clear for all the terms except perhaps $\partial_{x_1}\xi_R\partial_{x_1}E$. We show in Lemma 6.8, that there exists a constant $C>0$ such that the following interpolation inequality holds:

$$\begin{aligned}\|\nabla E_n(\nabla\xi_R)\|_{L^2(K_R)} &= \|\partial_{x_1}\xi_R\partial_{x_1}E_n\|_{L^2(K_R)}\\ &\le C\left(\frac{\|E_n\|_{L^2(K_{R+\delta})}}{\delta^2} + \|\operatorname{curl}\operatorname{curl}E_n\|_{L^2(K_{R+\delta})}\right)\|\partial_{x_1}\xi_R\|_{L^\infty(K_R)} \to 0,\end{aligned}$$

as $n\to\infty$, with $R/2 > \delta > 0$. We finally conclude that

$$\mu_1^{-1}\operatorname{curl}\operatorname{curl}\Psi_n - \varepsilon_1\omega^2\Psi_n = (\mu_1^{-1}\operatorname{curl}\operatorname{curl}E_n - \varepsilon_1\omega^2E_n)\xi_R + o(1) \to 0$$

as $n\to\infty$. Hence the sequence $(\Psi_n/\|\Psi_n\|)_{n\in\mathbb{N}}$ is the desired singular sequence for the operator $L_{\infty,1}$ acting in $L^2(\mathcal{C}_1)$.

Conversely, if $\omega\in\sigma_{ess}(L_{\infty,1})$, and $(E_n)\subset H_0(\operatorname{curl},\mathcal{C}_1)\cap H(\operatorname{div}0,\mathcal{C}_1)$ is a Weyl singular sequence such that $L_{\infty,1}(\omega)E_n \to 0$ as $n\to\infty$, define $\Psi_n = \mathcal{E}_0E_n\xi_R - P_\nabla(\mathcal{E}_0E_n\xi_R)$, where $\mathcal{E}_0$ is the extension by zero operator mapping $H_0(\operatorname{curl},\mathcal{C}_1)$ into $H_0(\operatorname{curl},\Omega)$. With arguments similar to the previous part of the proof one can check that the sequence $\Psi_n/\|\Psi_n\|$ is a Weyl singular sequence for the operator $L_\infty(\omega)$ acting in $L^2(\Omega)$. □

Proof of main Theorem 1.1

This follows from the two 'Decomposition and Simplification' theorems, Theorem 4.8 and Theorem 5.3. □

Data availability

All data are included in the text.

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Appendix A. Estimates of the intermediate derivatives

Definition 6.1. Let Ω be an open set of $\mathbb{R}^3$. We define $\mathcal{G}(\Omega) = \{u \in H(\operatorname{div} 0, \Omega) : \operatorname{curl}^2 u \in L^2(\Omega)^3\}$, endowed with the norm

$$\|u\|_{H(\operatorname{curl}^2,\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|\operatorname{curl}^2 u\|^2_{L^2(\Omega)}\right)^{1/2}$$

Remark 6.2. Note that $\mathcal{G}(\Omega) \cap H_0(\operatorname{curl}, \Omega) = \operatorname{dom}(L_\infty)$.

Remark 6.3. Let $W^{k,2}(\Omega)$ be the homogeneous Sobolev space of integrability exponent 2 and regularity index k, endowed with the norm

$$\|u\|_{W^{k,l}(\Omega)} = \left(\|u\|^2 + \sum_{|\alpha|=0}^{k} \|D^\alpha u\|^2\right)^{\frac{1}{2}}$$

In general $\mathcal{G}(\Omega) \nsubseteq W^{2,2}(\Omega)$, $\mathcal{G}(\Omega) \nsubseteq W^{1,2}(\Omega)$. For instance, if Ω is a smooth bounded domain, with $0 \in \partial\Omega$, one can easily construct a counterexample by taking

$$u(x) = \operatorname{curl}\left(\frac{\xi Y_{10}(\xi)}{|x|}\right) = |x|^{-1}\nabla Y_{10}(\xi) \times \xi$$

where $Y_{10}(\xi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cos(\theta)$ is the real spherical harmonic of index $(1, 0)$ and $\xi = x/|x|$. Then $u \in L^2(\Omega)$, by definition one has $\operatorname{div} u = 0$ and

$$\operatorname{curl}^2 u = \Delta(|x|^{-1}\nabla Y_{10}(\xi) \times \xi) = 0,$$

because $|x|^{-1}\nabla Y_{10}(\xi) \times \xi = \frac{\sqrt{2}}{r^2}\mathbf{A_1}(\xi)$, $\mathbf{A_1}(\xi)$ being the first vector spherical harmonic [21, p.350], and

$$\Delta\left(\frac{\sqrt{2}}{r^2}\mathbf{A}_1(\xi)\right) = \frac{\sqrt{2}}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)\right)\mathbf{A}_1 - \frac{\sqrt{2}}{r^2}\frac{2}{r^2}\mathbf{A_1} = 0.$$

However,

$$\frac{\partial u^1}{\partial x_j} \sim \frac{x_j}{|x|^3}(\nabla Y_{10}(\xi) \times \xi) \notin L^2(\Omega), \quad \text{in a neighbourhood of } x = 0$$

and similarly

$$\frac{\partial^2 u^1}{\partial x_j \partial x_i} \notin L^2(\Omega).$$

If Ω is unbounded one can argue in a similar way. It is sufficient to multiply u by a smooth cut-off η such that $\eta(x) = 1$ if $|x| \le 1$ and $\eta(x) = 0$ if $|x| \ge 2$, and then take $u\eta - P_\nabla(u\eta)$.

On the other hand, if $u \in \mathrm{dom}(L_\infty) = \mathcal{G}(\Omega) \cap H_0(\mathrm{curl}, \Omega)$, and Ω is bounded and smooth ($\partial\Omega \in \mathcal{C}^{1,1}$ will do), then by Gaffney inequality

$$\|\nabla u\|_{L^2(\Omega)} \le C(\|\mathrm{curl}\, u\|_{L^2(\Omega)} + \|\mathrm{div}\, u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})$$

and therefore $u \in H^1(\Omega)$.

Notation. Given $\delta > 0$, let $\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \delta\}$. We call Ω_δ the *shrinking* of Ω (of size δ).

The following statement is classical and can be proved by using mollifiers.

Lemma 6.4. *There exists $\eta_\delta \in C^\infty(\mathbb{R}^3)$, $0 \le \eta_\delta(x) \le 1$, $\eta_\delta(x) = 1$ in Ω_δ, $\eta_\delta(x) = 0$ in $\mathbb{R}^3 \setminus \Omega_{\delta/2}$ and $|(D^\alpha \eta_\delta)(x)| \le \frac{C_\alpha}{\delta^{|\alpha|}}$, for all $x \in \mathbb{R}^3$, $\alpha \in (\mathbb{N} \cup \{0\})^3$.*

Lemma 6.5. *For every open set $\Omega \subset \mathbb{R}^3$, $\mathcal{G}(\Omega) \subset W^{1,2}_{\mathrm{loc}}(\Omega)$, and there exists $C > 0$ such that*

$$\|\nabla u\|_{L^2(\Omega_\delta)} \le \frac{1}{\sqrt{2}}\|u\|_{H(\mathrm{curl}^2,\Omega)} + \frac{C}{\delta}\|u\|_{L^2(\Omega\setminus\Omega_\delta)}$$

for all $u \in \mathcal{G}(\Omega)$.

Proof. This proof is classical in the case of the Laplace operator, see e.g. [5, §4.4].

Let first Ω be bounded and $u \in C^2(\Omega) \cap \mathcal{G}(\Omega)$. For such a function we immediately see that

$$\frac{\partial u^i}{\partial x_j}, \frac{\partial^2 u^i}{\partial x_j^2} \in L^2(\Omega_\delta), \quad \frac{\partial u^i}{\partial x_j}\eta_\delta, \frac{\partial^2 u^i}{\partial x_j^2}\eta_\delta \in L^2(\Omega),$$

for all $i, j \in \{1, 2, 3\}$, where η_δ is the smooth cut-off function introduced in Lemma 6.4. We omit the superscript i in the following calculations. We have

$$\left\|\frac{\partial u}{\partial x_j}\right\|^2_{L^2(\Omega_\delta)} \le \int_\Omega \left|\frac{\partial u}{\partial x_j}\right|^2 \eta_\delta = -\int_\Omega u\frac{\partial^2 \bar{u}}{\partial x_j^2}\eta_\delta - \int_\Omega u\frac{\partial \bar{u}}{\partial x_j}\frac{\partial \eta_\delta}{\partial x_j}$$

and similarly

$$\left\|\frac{\partial u}{\partial x_j}\right\|^2_{L^2(\Omega_\delta)} \le \int_\Omega \left|\frac{\partial u}{\partial x_j}\right|^2 \eta_\delta = -\int_\Omega \bar{u}\frac{\partial^2 u}{\partial x_j^2}\eta_\delta - \int_\Omega \bar{u}\frac{\partial u}{\partial x_j}\frac{\partial \eta_\delta}{\partial x_j}.$$

We then deduce that

$$\begin{aligned}\left\|\frac{\partial u}{\partial x_j}\right\|^2_{L^2(\Omega_\delta)} &\le -\frac{1}{2}\int_\Omega \left(\bar{u}\frac{\partial^2 u}{\partial x_j^2} + u\frac{\partial^2 \bar{u}}{\partial x_j^2}\right)\eta_\delta + \frac{1}{2}\int_\Omega |u|^2\frac{\partial^2 \eta_\delta}{\partial x_j^2} \\ &\le \frac{1}{2}\int_\Omega (|u|^2 + |\mathrm{curl}^2\, u|^2) + \frac{C^2}{\delta^2}\int_{\Omega\setminus\Omega_\delta} |u|^2.\end{aligned}$$

Let now $u \in \mathcal{G}(\Omega)$. Then there exists a sequence $(u_k)_k \subset C_c^\infty(\Omega)$ such that $u_k \to u$ in $\mathcal{G}(\Omega)_{\mathrm{loc}}$. Let K be a compact subset of Ω, and let O be an open set, $K \subset O_\delta \subset \overline{O} \subset \Omega$, where O_δ is the shrinking of O of size $\delta > 0$. A standard Cauchy sequence argument and the previous part of the proof implies the validity of

$$\|\nabla u\|_{L^2(K)} \le \frac{1}{\sqrt{2}}(\|u\|_{L^2(O)} + \|\mathrm{curl}^2\, u\|_{L^2(O)}) + \frac{C}{\delta}\|u\|_{L^2(O\setminus O_\delta)}.$$

Choose now $O = \Omega_\varepsilon \cap B(0, 1/\varepsilon)$ and let $\varepsilon \to 0$. We deduce

$$\|\nabla u\|_{L^2(K)} \leq \frac{1}{\sqrt{2}}(\|u\|_{L^2(\Omega)} + \|\operatorname{curl}^2 u\|_{L^2(\Omega)}) + \frac{C}{\delta}\|u\|_{L^2(\Omega\setminus\Omega_\delta)}$$

and the result follows by allowing K to expand to fill Ω_δ. □

Definition 6.6. Let $\mathcal{V}(\Omega) = \{u \in L^2(\Omega)^3 : \operatorname{curl}^2 u \in L^2(\Omega)^3\} \supset \mathcal{G}(\Omega)$, equipped with the norm $\|\cdot\|_{H(\operatorname{curl}^2,\Omega)}$.

Lemma 6.7. *Given a bounded domain B, $C^\infty(\overline{B})$ is dense in $\mathcal{V}(B)$ with respect to $\|\cdot\|_{H(\operatorname{curl}^2,B)}$, and consequently $C^\infty(\overline{B}) \cap H(\operatorname{div} 0, B)$ is dense in $\mathcal{G}(B)$ with respect to $\|\cdot\|_{H(\operatorname{curl}^2,B)}$.*

Proof. Let $u \in \mathcal{V}(B) \cap C^\infty(\overline{B})^\perp$. Then

$$(u, \phi) + (\operatorname{curl}^2 u, \operatorname{curl}^2 \phi) = 0, \quad \text{for all } \phi \in C^\infty(\overline{B}).$$

Let $u_1 = \operatorname{curl}^2 u$. Since

$$(u, \phi) + (u_1, \operatorname{curl}^2 \phi) = 0, \quad \text{for all } \phi \in C^\infty(\overline{B}),$$

$u_1 \in \mathcal{V}(B)$ and $\operatorname{curl}^2 u_1 = -u$. We then have

$$-(\operatorname{curl}^2 u_1, \phi) + (u_1, \operatorname{curl}^2 \phi) = 0, \quad \text{for all } \phi \in C^\infty(\overline{B}).$$

By integration by parts we see that

$$-(\operatorname{curl}^2 u_1, \phi) + (u_1, \operatorname{curl}^2 \phi) = -\int_{\partial B} [(\operatorname{curl} u_1 \times \nu) \cdot \phi - (u_1 \times \nu) \cdot \operatorname{curl} \phi] = 0$$

and since ϕ is arbitrary in $C^\infty(B)$, hence by density in $H^2(\operatorname{curl}, B)$, we deduce that $(\operatorname{curl} u_1 \times \nu) = (u_1 \times \nu) = 0$ on ∂B, so $u_1 \in H^2_0(\operatorname{curl}, B)$ and therefore there exist $\Phi_k \in C^\infty_c(B)$, $\Phi_k \to u_1$ in $H^2(\operatorname{curl}, B)$. We then deduce that

$$(u, \phi) + (\operatorname{curl}^2 u, \operatorname{curl}^2 \phi) = \lim_{k\to\infty} -(\operatorname{curl}^2 \Phi_k, \phi) + (\Phi_k, \operatorname{curl}^2 \phi) = 0, \quad \text{for all } \phi \in \mathcal{V}(B),$$

hence $u \in \mathcal{V}(B) \cap \mathcal{V}(B)^\perp = \{0\}$.

The density of $C^\infty(B) \cap H(\operatorname{div} 0, B)$ in $\mathcal{G}(B)$ now follows by noting that if $u_0 \in \mathcal{G}(\Omega) \cap (C^\infty(B) \cap H(\operatorname{div} 0, B))^\perp$ then

$$(u_0, \varphi) + (\operatorname{curl}^2 u_0, \operatorname{curl}^2 \varphi) = 0, \quad \text{for all } \varphi \in C^\infty(B) \cap H(\operatorname{div} 0, B).$$

Adding any element of $\nabla H^1_0(B)$ to φ does not change this orthogonality relation, hence the claim follows as in the proof of Lemma 6.3 in [6]. □

Lemma 6.8. *Let $\Omega = \Omega_0 \sqcup \mathcal{C}_1 \sqcup \mathcal{C}_2$. Let $E \in \mathcal{G}(\Omega)$. Let $R > 0$ be fixed and let $K_R = \Omega_0 \sqcup \mathcal{C}_{1,r} \sqcup \mathcal{C}_{2,r}$ with $\mathcal{C}_{i,r}$ as in (5.2). Let $0 \leq \xi_R \leq 1$ be a smooth function such that $\xi_R = 0$ in $K_{R/2}$ and $\xi_R = 1$ in $\Theta_R = \Omega \cap K^c_R$. Then*

$$\|(\nabla\xi_R)^T \nabla E\|_{L^2(K_R)} \leq C\left(\frac{\|E\|_{L^2(K_{R+\delta})}}{\delta^2} + \|\operatorname{curl}^2 E\|_{L^2(K_{R+\delta})}\right)\|\nabla\xi_R\|_{L^\infty(K_R)}.$$

Proof. Upon a change of coordinates we may assume that the cylinder $\mathcal{C}_1$ coincides with $(0,+\infty)\times C_1$. Since $\xi_R(x)=\xi_R(x_1)$, we have $(\nabla\xi_R)^T\nabla E=\partial_{x_1}E\partial_{x_1}\xi_R$. Therefore it is sufficient to estimate $\|\partial_{x_1}E\|_{L^2(K_R\setminus K_{R/2})}$. Let us assume for the moment that $E\in C^\infty(K_R)\cap H(\operatorname{div}0,K_R)$. Let η_δ a smooth cutoff function, as in Lemma 6.4, depending only on x_1, such that $\eta_\delta=1$ in $K_R\setminus K_{R/2}$, $\eta_\delta=0$ in $(K_{R+\delta}\setminus K_{R/2-\delta})^c$. Then

$$\begin{aligned}\left\|\frac{\partial E}{\partial x_1}\right\|^2_{L^2(K_R\cap\operatorname{supp}\nabla\xi_R)}&\le\int\limits_{K_{R+\delta}\setminus K_{R/2-\delta}}\left|\frac{\partial E}{\partial x_1}\right|^2\eta_\delta\\&=-\int\limits_{K_{R+\delta}}E\cdot\frac{\partial^2\bar E}{\partial x_1^2}\eta_\delta-\int\limits_{K_{R+\delta}}E\cdot\frac{\partial\bar E}{\partial x_1}\frac{\partial\eta_\delta}{\partial x_1}+\int\limits_{\partial\Omega\cap\partial\operatorname{supp}\eta_\delta}E\frac{\partial E}{\partial x_1}\nu_1\eta_\delta\,d\sigma\end{aligned}$$

Now note that

$$\int\limits_{\partial\Omega\cap\partial\operatorname{supp}\eta_\delta}E\frac{\partial E}{\partial x_1}\nu_1\eta_\delta\,d\sigma=0$$

since either $\nu_1=0$ or $\eta_\delta=0$ on the boundary. We can then proceed as in Lemma 6.5 and deduce that

$$\begin{aligned}\left\|\frac{\partial E}{\partial x_1}\right\|^2_{L^2(K_R\cap\operatorname{supp}\nabla\xi_R)}&\le\frac12\int\limits_{K_{R+\delta}}(|E|^2+|\operatorname{curl}^2E|^2)+\frac{C^2}{\delta^2}\int\limits_{K_{R+\delta}\setminus K_R}|u|^2\\&\le\frac12\int\limits_{K_{R+\delta}}(|E|^2+|\operatorname{curl}^2E|^2)+\frac{C^2}{\delta^2}\int\limits_{K_{R+\delta}\setminus K_R}|u|^2.\end{aligned}$$

If now E is just in $\mathcal{G}(\Omega)$, due to Lemma 6.7 for fixed n there exists a sequence $(\Psi_k)_{k\in\mathbb{N}}\subset C^\infty(K_{R+\delta})\cap H(\operatorname{div}0,K_{R+\delta})$ such that $\Psi_k\to E$ and $\operatorname{curl}^2\Psi_k\to\operatorname{curl}^2E$ in $L^2(K_{R+\delta})$. We then deduce from the previous part of the proof that for every $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that

$$\left\|\frac{\partial\Psi_k}{\partial x_1}-\frac{\partial\Psi_l}{\partial x_1}\right\|^2_{L^2(K_R\cap\operatorname{supp}\nabla\xi_R)}\le\frac12\int\limits_{K_{R+\delta}}(|\Psi_k-\Psi_l|^2+|\operatorname{curl}^2(\Psi_k-\Psi_l)|^2)+\frac{C^2}{\delta^2}\int\limits_{K_{R+\delta}}|\Psi_k-\Psi_l|^2<\varepsilon$$

for every $k,l\ge N$. By completeness of $L^2(K_R\cap\operatorname{supp}\nabla\xi_R)$ we deduce that there exists $w\in L^2(K_R\cap\operatorname{supp}\nabla\xi_R)$ such that $\partial_{x_1}\Psi_k\to w$. By closedness of the weak derivative ∂_{x_1} it follows that $w=\partial_{x_1}E$. The first part of the proof implies that

$$\left\|\frac{\partial\Psi_k}{\partial x_1}\right\|^2_{L^2(K_R\cap\operatorname{supp}\nabla\xi_R)}\le\frac12\int\limits_{K_{R+\delta}}(|\Psi_k|^2+|\operatorname{curl}^2\Psi_k|^2)+\frac{C^2}{\delta^2}\int\limits_{K_{R+\delta}}|\Psi_k|^2$$

and passing to the limit as $k\to\infty$ we deduce that

$$\left\|\frac{\partial E}{\partial x_1}\right\|^2_{L^2(K_R\cap\operatorname{supp}\nabla\xi_R)}\le\frac12\int\limits_{K_{R+\delta}}(|E|^2+|\operatorname{curl}^2E|^2)+\frac{C^2}{\delta^2}\int\limits_{K_{R+\delta}}|E|^2.\quad\square$$

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