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Research Paper

Unbounded twisted complexes

Rina Anno [a], Timothy Logvinenko [b,*]

[a] Department of Mathematics, Kansas State University, 138 Cardwell Hall, Manhattan, KS 66506, USA

[b] School of Mathematics, Cardiff University, Senghennydd Road, Cardiff, CF24 4AG, UK

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ABSTRACT

We define unbounded twisted complexes and bicomplexes generalising the notion of a (bounded) twisted complex over a DG category [6]. These need to be considered relative to another DG category $\mathcal{B}$ admitting countable direct sums and shifts. The resulting DG category of unbounded twisted complexes has a fully faithful convolution functor into $\mathbf{Mod}\text{-}\mathcal{B}$ which factors through $\mathcal{B}$ if the latter is closed under twisting. As an application, we rewrite definitions of A_∞-structures in terms of twisted complexes to make them work in an arbitrary monoidal DG category or a DG bicategory.

1. Introduction

The notion of a twisted complex of objects in a DG category was introduced by Bondal and Kapranov [6]. It was used as a tool to study and construct DG enhancements of triangulated categories. A one-sided twisted complex over a DG category $\mathcal{A}$ can be thought of as a lift to $\mathcal{A}$ of a bounded complex of objects in its homotopy category $H^0(\mathcal{A})$. The lift includes the maps in the complex, the homotopies up to which the consecutive

* Corresponding author.
E-mail addresses: ranno@math.ksu.edu (R. Anno), LogvinenkoT@cardiff.ac.uk (T. Logvinenko).

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maps compose to zero, and then the higher homotopies. In addition to the complex of objects in $H^0(\mathcal{A})$, such data specifies a choice of its convolution together with a collection of Postnikov systems computing this convolution [3, §2.4]. Taking the category of one-sided twisted complexes over $\mathcal{A}$ is a DG realisation of taking the triangulated hull of $H^0(\mathcal{A})$. Now twisted complexes are ubiquitous in working with DG categories and their modules [11] [17][13][2][5].

This paper generalises the notion of a twisted complex to include unbounded complexes. The authors came to need it, and expected the generalisation to be straightforward. It turned out to involve numerous subtleties. The purpose of this short note is to write down these subtleties for the benefit of others. We also give the original application we had in mind: rewriting the definitions of A_∞-structures [10][12][14] in terms of twisted complexes. This decouples them from the differential m_1 and allows them to work in an arbitrary monoidal DG category.

We now describe our results in more detail. In §2 we recall the original definition:

Definition 1.1 *([6]).* A *twisted complex* over a DG category $\mathcal{A}$ comprises

- $\forall\, i \in \mathbb{Z}$, an object a_i of $\mathcal{A}$, non-zero for only finite number of i,
- $\forall\, i, j \in \mathbb{Z}$, a degree $i - j + 1$ morphism $\alpha_{ij} \colon a_i \to a_j$ in $\mathcal{A}$,

satisfying

$$(-1)^j d\alpha_{ij} + \sum_k \alpha_{kj} \circ \alpha_{ik} = 0. \tag{1.1}$$

The twisted complex condition should be thought of as follows. We have Yoneda embedding $\mathcal{A} \hookrightarrow \textbf{Mod-}\mathcal{A}$. Consider the object $\bigoplus a_i[-i]$ in $\textbf{Mod-}\mathcal{A}$. The sum $\sum \alpha_{ij}$ is its degree 1 endomorphism. Let d_{nat} be the natural differential on $\bigoplus a_i[-i]$. The condition (1.1) is equivalent to $d_{nat} + \sum \alpha_{ij}$ squaring to zero.

In other words, a twisted complex is the data which modifies d_{nat} to a new differential on $\bigoplus a_i[-i]$. The resulting new object of $\textbf{Mod-}\mathcal{A}$ is called the *convolution* of the twisted complex (a_i, α_{ij}). In the special case of twisted complexes of form $a_0 \to a_1$ the convolution is simply the cone construction.

Degree n morphisms $(a_i, \alpha_{ij}) \to (b_i, \beta_{ij})$ of twisted complexes are collections $\{f_{ij}\}$ of morphisms $f_{ij} \colon a_i \to b_j$ in $\mathcal{A}$ of degree $n + i - j$. Their composition and differentiation are defined so as to ensure that the convolution becomes a fully faithful embedding of the resulting DG category $\operatorname{Tw}\mathcal{A}$ into $\textbf{Mod-}\mathcal{A}$, see Definition 2.2. Indeed, the assignment of the module $\bigoplus a_i[-i]$ to a collection of objects $\{a_i\}$ determines the rest of the definitions of twisted complexes and their morphisms.

These definitions can be replicated for an infinite collection $\{a_i\}$ resulting in a notion of an unbounded twisted complex $\{a_i, \alpha_{ij}\}$ that we are going to call an *absolute unbounded twisted complex* see Definition 3.2. The sum $\sum \alpha_{ij}$ is now infinite and doesn't necessarily define a degree 1 endomorphism of $\bigoplus a_i[-i]$ in $\textbf{Mod-}\mathcal{A}$, so we impose this as an extra condition. It means that only a finite number $\alpha_{ij} \neq 0$ for any $i \in \mathbb{Z}$, and similarly for the components f_{ij} of morphisms of twisted complexes. We again have $\operatorname{Tw}^{\pm}_{abs} \mathcal{A} \hookrightarrow \mathbf{Mod}\text{-}\mathcal{A}$.

However, often the category $\operatorname{Tw}^{\pm}_{abs} \mathcal{A}$ is not what we want. Firstly, when $\mathcal{A}$ is not small the category $\mathbf{Mod}\text{-}\mathcal{A}$ isn't well-defined. The definition of $\operatorname{Tw}^{\pm}_{abs} \mathcal{A}$ is still valid, but to have the convolution functor we need to enlarge the universe to make $\mathcal{A}$ small. More importantly, even small $\mathcal{A}$ might admit countable shifted direct sums $\bigoplus a_i[-i]$ of its objects.

The main subtlety is then that infinite direct sums, unlike finite, *do not commute with the Yoneda embedding* $\mathcal{A} \hookrightarrow \mathbf{Mod}\text{-}\mathcal{A}$ (Example 3.1). The direct sum $\bigoplus a_i[-i]$ assigned to a twisted complex (a_i, α_{ij}) can thus be taken in $\mathbf{Mod}\text{-}\mathcal{A}$ or in $\mathcal{A}$. The former leads to the category $\operatorname{Tw}^{\pm}_{abs} \mathcal{A}$, while the latter to a strictly larger category $\operatorname{Tw}^{\pm}_{\mathcal{A}} \mathcal{A}$ where infinite number of α_{ij} can be non-zero for any $i \in \mathbb{Z}$ as long as $\sum \alpha_{ij}$ is still an endomorphism of $\bigoplus a_i[-i]$ in $\mathcal{A}$. The difference between $\operatorname{Tw}^{\pm}_{abs} \mathcal{A}$ and $\operatorname{Tw}^{\pm}_{\mathcal{A}} \mathcal{A}$ lies only in unbounded twisted complexes. If $\mathcal{A}$ is closed under twisting (Definition 3.5), the convolution functor takes values in $\mathcal{A}$. All these considerations apply when $\mathcal{A} = \mathbf{Mod}\text{-}\mathcal{C}$ for small $\mathcal{C}$ (Example 3.3).

This motivates our §3 where we define unbounded twisted complexes relative to an embedding of $\mathcal{A}$ into another DG category $\mathcal{B}$:

Definition 1.2 *(see Definition 3.4).* Let $\mathcal{A}$ be a DG category with a fully faithful embedding into a DG category $\mathcal{B}$ which has countable direct sums and shifts.

An *unbounded twisted complex* over $\mathcal{A}$ relative to $\mathcal{B}$ consists of

- $\forall\, i \in \mathbb{Z}$, an object a_i of $\mathcal{A}$,
- $\forall\, i, j \in \mathbb{Z}$, a degree $i - j + 1$ morphism $\alpha_{ij} \colon a_i \to a_j$ in $\mathcal{A}$,

satisfying

- $\sum \alpha_{ij}$ is an endomorphism of $\bigoplus_{i \in \mathbb{Z}} a_i[-i]$ in $\mathcal{B}$,
- The twisted complex condition (1.1).

The *DG category* $\operatorname{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A})$ *of unbounded twisted complexes over* $\mathcal{A}$ *relative to* $\mathcal{B}$ is defined in the unique way which yields fully faithful convolution functor

$$\operatorname{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A}) \to \mathbf{Mod}\text{-}\mathcal{B}, \tag{1.2}$$

which sends any (a_i, α_{ij}) to $\bigoplus a_i[-i]$ with its natural differential modified by $\sum \alpha_{ij}$.

Any $\mathcal{B}$ as above is closed under twisting if and only if it admits convolutions of unbounded twisted complexes. In such case for any $\mathcal{A} \subseteq \mathcal{B}$ the convolution functor (1.2) takes values in $\mathcal{B}$ (Lemma 3.7). We can thus use unbounded twisted complexes over non-small categories without running into set-theoretic issues.

In §4 we generalise twisted complexes in another direction and define a *twisted bicomplex* (Definition 4.1) over $\mathcal{A}$. These are bigraded twisted complexes. To work with the unbounded ones, we again fix an embedding of $\mathcal{A}$ into a DG category $\mathcal{B}$. We denote the resulting category of unbounded twisted complexes by $\mathrm{Twbi}^{\pm}_{\mathcal{B}}(\mathcal{A})$. A twisted bicomplex is not a twisted complex of its rows or of its columns. It only becomes one after a sign twist. We write this down explicitly as a pair of functors

$$\mathcal{C}\mathrm{xrow}, \mathcal{C}\mathrm{xcol}\colon\ \ \mathrm{Tw}^{\pm}_{\mathcal{B}}\left(\mathrm{Tw}^{\pm}_{\mathcal{B}}\left(\mathcal{A}\right)\right) \to \mathrm{Twbi}^{\pm}_{\mathcal{B}}(\mathcal{A}).$$

We relate the images of these functors and show that both become isomorphisms if we only work with one-sided twisted complexes and bicomplexes (Proposition 4.3).

Finally, in §5 we give the main application we had in mind: to reformulate and generalise the definitions of A_∞-algebras and modules [12, §2] in terms of twisted complexes. This disposes with the necessity to work explicitly with the operation m_1 (the differential) and makes the definitions work in an arbitrary DG monoidal category $\mathcal{A}$ (or, more generally, a DG bicategory).

In §5.1 we give the resulting definitions. They all ask for the bar constructions of the A_∞-operations to be a twisted complex. Since these constructions involve infinite number of objects, we need the theory of unbounded twisted complexes. These definitions are studied further in [4] whose §3.2 explains at length how they generalise the classical definitions [12, §2] [14]. Note that in bar constructions there is only a finite number of arrows emerging from each element of the twisted complex. Hence, our definitions of an A_∞-algebra or an A_∞-module are independent of the ambient category $\mathcal{B}$ we use to define unbounded twisted complexes. In [4, §6] analogous definitions of A_∞-coalgebras and A_∞-comodules are given in terms of cobar constructions. There, infinite number of arrows can emerge from a single element of twisted complex, and thus the choice of $\mathcal{B}$ matters.

In §5.2 we look at twisted complexes of A_∞-modules. As per §5.1 let $\mathbf{Nod}_\infty$-A be the category of A_∞-modules over an A_∞-algebra A in a monoidal DG category $\mathcal{A}$. We define twisted complexes over $\mathbf{Nod}_\infty$-A neither relative to $\mathbf{Mod}$-($\mathbf{Nod}_\infty$-A) nor to $\mathbf{Nod}_\infty$-A. Instead, we embed $\mathcal{A}$ into a cocomplete closed monoidal DG category $\mathcal{B}$ with convolutions of unbounded twisted complexes and define twisted complexes relative to $\mathbf{Nod}_\infty$-$A^{\mathcal{B}}$, the category of A_∞-A-modules in $\mathcal{B}$. As $\mathbf{Nod}_\infty$-$A^{\mathcal{B}}$ also admits convolutions of unbounded twisted complexes (Corollary 5.14), convolutions of twisted complexes over $\mathbf{Nod}_\infty$-A take values in $\mathbf{Nod}_\infty$-$A^{\mathcal{B}}$. We can always set $\mathcal{B} = \mathbf{Mod}$-$\mathcal{A}$ with the induced monoidal structure [8, §4.5]. However, we may need to choose differently e.g. for A_∞-modules in a category of A_∞-modules.

We then use the twisted bicomplex techniques we developed in §4 to prove that a twisted complex of A_∞-modules defines an A_∞-module structure on the twisted complex of their underlying objects in a way that gives a fully faithful embedding of the corresponding categories (Proposition 5.12). It follows that the DG category $\mathbf{Nod}_\infty$-A of A_∞-modules over an A_∞-algebra A is pretriangulated (resp. admits convolutions of unbounded twisted complexes) if and only if DG monoidal category $\mathcal{A}$ we work is (resp. does) (Corollary 5.13).

In the Appendix we describe a homotopy transfer of structure for A_∞-modules.

We are aware of an alternative definition of the DG category of unbounded twisted complexes in [7]. It ignores the subtleties we consider by imposing no finiteness conditions on the differentials α_{ij} in twisted complexes and the components f_{ij} of their morphisms. The resulting category admits no convolution functor and is better suited to purposes different from ours.

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2. Preliminaries

2.1. DG categories

For a brief introduction to DG-categories, DG-modules and the technical notions involved we direct the reader to a survey in [1], §2-4. Other nice sources are [9], [16], [17], and [13].

We summarise the key notions relevant to this paper. Throughout the paper we work in a fixed universe $\mathbb{U}$ of sets containing an infinite set. We also fix the base field or commutative ring k we work over.

We define $\mathbf{Mod}\text{-}k$ to be the category of $\mathbb{U}$-small complexes of k-modules. It is a cocomplete closed symmetric monoidal category with monoidal operation $\otimes_k$ and unit k. A *DG category* is a category enriched over $\mathbf{Mod}\text{-}k$. In particular, any DG category is locally small.

If a DG category $\mathcal{A}$ is small, we write $\mathbf{Mod}\text{-}\mathcal{A}$ for the DG category of (right) $\mathcal{A}$-modules. These are functors $\mathcal{A}^{\text{opp}} \to \mathbf{Mod}\text{-}k$, so $\mathbf{Mod}\text{-}\mathcal{A} = \text{DGFun}(\mathcal{A}^{\text{opp}}, \mathbf{Mod}\text{-}k)$. Note that if $\mathcal{A}$ is not small, then $\mathbf{Mod}\text{-}\mathcal{A}$ doesn't make sense. It isn't even a DG category in the above sense - its morphism spaces are no longer small and hence do not lie in $\mathbf{Mod}\text{-}k$.

We can always enlarge our universe $\mathbb{U}$ to a universe $\mathbb{V}$ where $\mathcal{A}$ is small. This enlarges $\mathbf{Mod}\text{-}k$ and hence $\mathbf{Mod}\text{-}\mathcal{A}$ depends on choice of $\mathbb{V}$. However, in this paper we only work with $\mathbf{Mod}\text{-}\mathcal{A}$ as a target for the convolution of twisted complexes over $\mathcal{A}$. For these purposes, the choice of $\mathbb{V}$ doesn't matter - the only part of $\mathbf{Mod}\text{-}\mathcal{A}$ we interact with are countable direct sums of shifts of objects of $\mathcal{A}$ with modified differential.

Thus, when $\mathcal{A}$ is not small, we mean by $\mathbf{Mod}\text{-}\mathcal{A}$ the module category of $\mathcal{A}$ taken in any appropriate enlargement $\mathbb{V}$ of $\mathbb{U}$. Moreover, the constructions in this paper, such as that of the category of twisted complexes over $\mathcal{A}$ taken relative to a DG category $\mathcal{B}$, were devised precisely to enable us to replace $\mathbf{Mod}\text{-}\mathcal{A}$ with something more appropriate when $\mathcal{A}$ is not small.

2.2. Key isomorphism

Let $\mathcal{A}$ be a DG-category, let $E, F \in \mathbf{Mod}\text{-}\mathcal{A}$ and $i, j \in \mathbb{Z}$. The theory of twisted complexes [6] which we summarise in §2.3 depends crucially on the choice of an isomorphism

$$\operatorname{Hom}_{\mathcal{A}}(E, F)[j-i] \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(E[i], F[j]). \tag{2.1}$$

The simplest such isomorphism is:

Definition 2.1. Let $\mathcal{A}$ be a DG-category, let $E, F \in \mathbf{Mod}\text{-}\mathcal{A}$ and $i, j \in \mathbb{Z}$. Define the isomorphism of graded k-modules

$$\psi\colon \operatorname{Hom}_{\mathcal{A}}(E, F)[j-i] \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(E[i], F[j]) \tag{2.2}$$

to be the map which sends any $f \in \operatorname{Hom}^p_{\mathcal{A}}(E, F)$ to itself considered as an element of $\operatorname{Hom}^{p-j+i}(E[i], F[j])$. In other words, forgetting the grading, in every fibre over every $a \in \mathcal{A}$ the map $\psi(f)$ is the same map of k-vector spaces as f.

Note that ψ is *not* compatible with the differentials:

$$d_{\operatorname{Hom}_{\mathcal{A}}(E[i],F[j])} \circ \psi = (-1)^i \psi \circ d_{\operatorname{Hom}_{\mathcal{A}}(E,F)[j-i]}.$$

There are at least two natural ways to fix this. Define

$$\psi_1, \psi_2\colon \operatorname{Hom}_{\mathcal{A}}(E, F)[j-i] \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(E[i], F[j])$$

to be the maps which send $f \in \operatorname{Hom}^p_{\mathcal{A}}(E, F)$ to $(-1)^{ip}\psi(f)$ and $(-1)^{i(p-j+i)}\psi(f)$. The difference between the two lies in whether we multiply i by the degree of f in $\operatorname{Hom}_{\mathcal{A}}(E, F)$ or its degree in $\operatorname{Hom}_{\mathcal{A}}(E, F)[j-i]$.

Both ψ_1 and ψ_2 are isomorphisms of DG k-modules. However, they are incompatible with the composition. By this we mean the following: let $E, F, G \in \mathbf{Mod}\text{-}\mathcal{A}$ and $i, j, k \in \mathbb{Z}$, then e.g. the isomorphism

$$\psi_1(E, i, G, k)\colon \operatorname{Hom}_{\mathcal{A}}(E, G)[k-i] \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(E[i], G[k])$$

is not a composition of $\psi_1(E, i, F, j)$ and $\psi_1(F, j, G, k)$. On the other hand, ψ, while incompatible with differentials, is compatible with composition.

The theory of twisted complexes and its fundamental definitions depend on the choice of an isomorphism (2.1). The definition of the DG category $\operatorname{Tw}(\mathcal{A})$ of twisted complexes over $\mathcal{A}$ is set up so as to ensure that there exists a fully faithful functor $\operatorname{Tw}(\mathcal{A}) \hookrightarrow \mathbf{Mod}\text{-}\mathcal{A}$

called *convolution*, cf. §2.3. This functor is defined using the isomorphism (2.1), thus different choices would lead to different formulas in the definition of $\mathrm{Tw}(\mathcal{A})$.

The incompatibility of ψ with differentials introduces in these formulas a simple sign to every appearance of the differential $d_{\mathcal{A}}$ of $\mathcal{A}$, cf. (2.3) and (2.5). On the other hand, the incompatibility of ψ_1 and ψ_2 with composition introduces into the same formulas a complicated sign to every composition of two morphisms of $\mathcal{A}$.

We choose to use the graded module isomorphism ψ to identify $\mathrm{Hom}_{\mathcal{A}}(E, F)[j - i]$ with $\mathrm{Hom}_{\mathcal{A}}(E[i], F[j])$ when defining twisted complexes. We fix this choice and use it implicitly in the sections below.

2.3. Bounded twisted complexes

Here we summarise some known facts about the usual, bounded twisted complexes. This notion was originally introduced by Bondal and Kapranov in [6]:

Definition 2.2. A *twisted complex* over a DG category $\mathcal{A}$ is a collection of

- $\forall\, i \in \mathbb{Z}$, an object a_i of $\mathcal{A}$, non-zero for only finite number of i,
- $\forall\, i, j \in \mathbb{Z}$, a degree $i - j + 1$ morphism $\alpha_{ij} \colon a_i \to a_j$ in $\mathcal{A}$,

$$
\begin{tikzcd}
a_0 \arrow[loop below, "\alpha_{00}"] \arrow[r, shift left, "\alpha_{01}"] \arrow[rr, bend left, "\alpha_{02}"] \arrow[rrr, bend left, "\alpha_{03}"] \arrow[rrrr, bend left, "\alpha_{04}"] & a_1 \arrow[loop below, "\alpha_{11}"] \arrow[l, shift left, "\alpha_{10}"] \arrow[r, shift left, "\alpha_{12}"] \arrow[rr, bend left, "\alpha_{13}"] \arrow[rrr, bend left, "\alpha_{14}"] & a_2 \arrow[loop below, "\alpha_{22}"] \arrow[l, shift left, "\alpha_{21}"] \arrow[ll, bend left, "\alpha_{20}"] \arrow[r, shift left, "\alpha_{23}"] \arrow[rr, bend left, "\alpha_{24}"] & a_3 \arrow[loop below, "\alpha_{33}"] \arrow[l, shift left, "\alpha_{32}"] \arrow[ll, bend left, "\alpha_{31}"] \arrow[lll, bend left, "\alpha_{30}"] \arrow[r, shift left, "\alpha_{34}"] & a_4 \arrow[loop below, "\alpha_{44}"] \arrow[l, shift left, "\alpha_{43}"] \arrow[ll, bend left, "\alpha_{42}"] \arrow[lll, bend left, "\alpha_{41}"] \arrow[llll, bend left, "\alpha_{40}"]
\end{tikzcd}
$$

satisfying the condition

$$(-1)^j d\alpha_{ij} + \sum_k \alpha_{kj} \circ \alpha_{ik} = 0. \tag{2.3}$$

Define the *DG category* $\mathrm{Tw}(\mathcal{A})$ *of twisted complexes over* $\mathcal{A}$ by setting

$$\mathrm{Hom}^{\bullet}_{\mathrm{Tw}(\mathcal{A})}\big((a_i, \alpha_{ij}), (b_i, \beta_{ij})\big) := \bigoplus_{q,k,l \in \mathbb{Z}} \mathrm{Hom}^q_{\mathcal{A}}(a_k, b_l) \tag{2.4}$$

where each $f \in \operatorname{Hom}^q_{\mathcal{A}}(a_k, b_l)$ has degree $q + l - k$ and

$$df := (-1)^l d_{\mathcal{A}} f + \sum_{m \in \mathbb{Z}} \left(\beta_{lm} \circ f - (-1)^{q+l-k} f \circ \alpha_{mk} \right), \tag{2.5}$$

where $d_{\mathcal{A}}$ is the differential on morphisms in $\mathcal{A}$.

This definition ensures that $\operatorname{Tw}(\mathcal{A})$ is isomorphic to the full subcategory of $\mathbf{Mod}\text{-}\mathcal{A}$ consisting of the DG $\mathcal{A}$-modules whose underlying graded modules are of form $\oplus_{i \in \mathbb{Z}} a_i[-i]$ with only finite number of $a_i \in \mathcal{A}$ non-zero. Indeed:

- The twisted complex condition (2.3) is equivalent to $d_{nat} + \sum_{i,j} \alpha_{ij}$ being another differential on $\oplus_{i \in \mathbb{Z}} a_i[-i]$. Here d_{nat} is its natural differential.
- The Hom-complex (2.5) is defined to have the same underlying graded k-module as

$$\operatorname{Hom}^{\bullet}_{\mathbf{Mod}\text{-}\mathcal{A}} \left(\bigoplus_{k \in \mathbb{Z}} a_k[-k], \bigoplus_{l \in \mathbb{Z}} b_l[-l] \right) = \bigoplus_{k,l \in \mathbb{Z}} \operatorname{Hom}^{\bullet - l + k}_{\mathcal{A}} (a_k, b_l), \tag{2.6}$$

  and the differential (2.5) is defined so as to coincide under this identification with the one obtained on (2.6) by endowing the two direct sums with their new differentials.

We thus have a fully faithful *convolution functor*

$$\operatorname{Conv}\colon \quad \operatorname{Tw}(\mathcal{A}) \hookrightarrow \mathbf{Mod}\text{-}\mathcal{A}$$

which sends each (a_i, α_{ij}) to the $\mathcal{A}$-module $\bigoplus a_i[-i]$ equipped with the new differential $d_{nat} + \sum \alpha_{ij}$. Note, that the existence of this functor can be used as the definition of the category $\operatorname{Tw}(\mathcal{A})$ once one fixes the assignment of the graded module $\bigoplus a_i[-i]$ to any collection $\{a_i\}_{i \in \mathbb{Z}}$.

A twisted complex is called *one-sided* if $\alpha_{ij} = 0$ for all $i \geq j$.

$$\begin{tikzcd} a_0 \arrow[r, "\alpha_{01}"] \arrow[rr, bend left, "\alpha_{02}"] \arrow[rrr, bend left, "\alpha_{03}"] \arrow[rrrr, bend left, "\alpha_{04}"] & a_1 \arrow[r, "\alpha_{12}"] \arrow[rr, bend left, "\alpha_{13}"] \arrow[rrr, bend left, "\alpha_{14}"] & a_2 \arrow[r, "\alpha_{23}"] \arrow[rr, bend left, "\alpha_{24}"] & a_3 \arrow[r, "\alpha_{34}"] & a_4 \end{tikzcd} \tag{2.7}$$

If (a_i, α_{ij}) is a one-sided twisted complex over $\mathcal{A}$, then (a_i, α_{ij}) is a (usual) complex over $H^0(\mathcal{A})$. Thus one-sided twisted complexes can be considered as homotopy lifts to $\mathcal{A}$ of usual complexes in $H^0(\mathcal{A})$. The full subcategory of $\operatorname{Tw}(\mathcal{A})$ consisting of one-sided twisted complexes is called the *pretriangulated hull* of $\mathcal{A}$ and is denoted $\operatorname{Pre-Tr}(\mathcal{A})$. We say that a DG category is *pretriangulated* (resp. *strongly pretriangulated*) if the natural embedding $\mathcal{A} \hookrightarrow \text{Pre-Tr}(\mathcal{A})$ is a quasi-equivalence (resp. equivalence).

The reason for the term "pretriangulated" is that $H^0(\text{Pre-Tr}(\mathcal{A}))$ is the triangulated hull of $H^0(\mathcal{A})$ in $H^0(\mathbf{Mod}\text{-}\mathcal{A})$, or indeed any $H^0(\mathcal{B})$ for any fully faithful embedding of $\mathcal{A}$ into a pretriangulated DG category $\mathcal{B}$.

3. Unbounded twisted complexes

In this section we generalise the notions in §2.3 to unbounded twisted complexes. The generalisation seems straightforward, but there are subtleties regarding infinite direct sums. Unlike finite direct sums, these are not preserved by all DG-functors. In particular, they are not preserved by the Yoneda embedding

$$\Upsilon \colon \mathcal{A} \hookrightarrow \mathbf{Mod}\text{-}\mathcal{A}, \qquad a \mapsto \operatorname{Hom}_{\mathcal{A}}(-, a) \quad \forall\, a \in \mathcal{A}$$

which we used implicitly in defining the convolution of a twisted complex.

Example 3.1. Let $\{a_i\}_{i\in\mathbb{Z}}$ be objects in $\mathcal{A}$ such that $\bigoplus_{i\in\mathbb{Z}} a_i$ exists in $\mathcal{A}$. Then

$$\bigoplus_{i\in\mathbb{Z}} \Upsilon(a_i) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(-, a_i),$$
$$\Upsilon\left(\bigoplus_{i\in\mathbb{Z}} a_i\right) = \operatorname{Hom}_{\mathcal{A}}(-, \bigoplus_{i\in\mathbb{Z}} a_i),$$

are two different $\mathcal{A}$-modules, with the former being a strict submodule of the latter. Let $b \in \mathcal{A}$, the morphisms from $\Upsilon(b)$ to the former module are the finite sums of $b \to a_i$. On the other hand, the morphisms from $\Upsilon(b)$ to the latter are the morphisms from b to $\bigoplus_{i\in\mathbb{Z}} a_i$, which includes some infinite sums of $b \to a_i$. In particular, if $b = \bigoplus_{i\in\mathbb{Z}} a_i$, then Id_b is the infinite sum of Id_{a_i}.

To define an unbounded twisted complex of objects $\{a_i\}_{i\in\mathbb{Z}}$ of $\mathcal{A}$, we need to choose in which category we take the infinite direct sum $\bigoplus_{i\in\mathbb{Z}} a_i[-i]$. We can always do it in $\mathbf{Mod}\text{-}\mathcal{A}$. Then, proceeding as before, we arrive at the following definition. In it, we allow infinite number of non-zero objects a_i, but then, both for twisted differentials and for the morphisms of twisted complexes, we disallow an infinite number of non-zero maps to emerge from any one object a_i:

Definition 3.2 *(Absolute version).* An *absolute unbounded twisted complex* over a DG category $\mathcal{A}$ consists of

- $\forall\, i \in \mathbb{Z}$, an object a_i of $\mathcal{A}$,
- $\forall\, i, j \in \mathbb{Z}$, a degree $i - j + 1$ morphism $\alpha_{ij} \colon a_i \to a_j$ in $\mathcal{A}$,

satisfying

- For any $i \in \mathbb{Z}$ only finite number of α_{ij} are non-zero,
- The twisted complex condition (2.3).

Define *DG category* $\mathrm{Tw}^{\pm}_{abs}(\mathcal{A})$ *of such twisted complexes over* $\mathcal{A}$ by

$$\mathrm{Hom}^{\bullet}_{\mathrm{Tw}^{\pm}_{abs}(\mathcal{A})}\big((a_i, \alpha_{ij}), (b_i, \beta_{ij})\big) := \bigoplus_{q\in\mathbb{Z}} \prod_{k\in\mathbb{Z}} \bigoplus_{l\in\mathbb{Z}} \mathrm{Hom}^q_{\mathcal{A}}(a_k, b_l) \tag{3.1}$$

where the degree of $\mathrm{Hom}^q_{\mathcal{A}}(a_k, b_l)$ is $q + l - k$ and the differential is defined by (2.5).

As before, this results in the fully faithful convolution functor

$$\mathrm{Conv}\colon\ \mathrm{Tw}^{\pm}_{abs}(\mathcal{A}) \hookrightarrow \mathbf{Mod}\text{-}\mathcal{A}.$$

Apriori, this is the only definition we can make for an arbitrary DG category $\mathcal{A}$. Indeed, unless specifically mentioned otherwise, we write $\mathrm{Tw}^{\pm}(\mathcal{A})$ for $\mathrm{Tw}^{\pm}_{abs}(\mathcal{A})$.

However, in some cases it is useful to define $\mathrm{Tw}^{\pm}(\mathcal{A})$ to be bigger than $\mathrm{Tw}^{\pm}_{abs}(\mathcal{A})$:

Example 3.3. Let $\mathcal{A} = \mathbf{Mod}\text{-}\mathcal{C}$ for some small DG category $\mathcal{C}$. Assign to a collection $\{a_i\}_{i\in\mathbb{Z}}$ the representable $\mathcal{A}$-module $\Upsilon(\bigoplus_{i\in\mathbb{Z}} a_i[-i])$, instead of the non-representable $\mathcal{A}$-module $\bigoplus_{i\in\mathbb{Z}} \Upsilon(a_i[-i])$. This yields the definition of $\mathrm{Tw}^{\pm}(\mathcal{A})$ which is analogous to the one above, except we do allow infinite number of twisted differentials α_{ij} to emerge from a single object a_i as long as $\sum \alpha_{ij}$ defines an endomorphism of $\bigoplus_{i\in\mathbb{Z}} a_i[-i]$ in $\mathcal{A}$, and similarly for morphisms of twisted complexes. As before, this definition ensures that we have the fully faithful convolution functor $\mathrm{Tw}^{\pm}(\mathcal{A}) \hookrightarrow \mathbf{Mod}\text{-}\mathcal{A}$. However, since $\mathcal{A} = \mathbf{Mod}\text{-}\mathcal{C}$ is closed under twisting, this convolution factors through the Yoneda embedding. Thus we have the fully faithful functor

$$\mathrm{Conv}\colon\ \mathrm{Tw}^{\pm}(\mathcal{A}) \to \mathcal{A}.$$

In fact, it is an equivalence, since it has a right inverse - the tautological embedding $\mathcal{A} \hookrightarrow \mathrm{Tw}^{\pm}(\mathcal{A})$ which sends any $a \in \mathcal{A}$ to itself considered as a trivial twisted complex concentrated in degree zero. We thus see that $\mathcal{A} = \mathbf{Mod}\text{-}\mathcal{C}$ is closed under convolutions of all unbounded twisted complexes in $\mathrm{Tw}^{\pm}(\mathcal{A})$.

Finally, even when $\mathcal{A}$ does not admit all small direct sums, there might still be a better category to take these in than $\mathbf{Mod}\text{-}\mathcal{A}$. For example, $\mathcal{A}$ might be a full subcategory of some $\mathbf{Mod}\text{-}\mathcal{C}$ containing some infinite direct sums, but not all of them. Another example, which indeed motivated these considerations, can be found in §5.2. We thus define the following:

Definition 3.4. Let $\mathcal{A}$ be a DG category with a fully faithful embedding into a DG category $\mathcal{B}$ which has countable direct sums and shifts.

An *unbounded twisted complex* over $\mathcal{A}$ relative to $\mathcal{B}$ consists of

- $\forall\, i \in \mathbb{Z}$, an object a_i of $\mathcal{A}$,
- $\forall\, i, j \in \mathbb{Z}$, a degree $i - j + 1$ morphism $\alpha_{ij} \colon a_i \to a_j$ in $\mathcal{A}$,

satisfying

- $\sum \alpha_{ij}$ is an endomorphism of $\bigoplus_{i\in\mathbb{Z}} a_i[-i]$ in $\mathcal{B}$,
- The twisted complex condition (2.3).

Define *DG category* $\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A})$ *of unbounded twisted complexes over* $\mathcal{A}$ *relative to* $\mathcal{B}$ by setting

$$\mathrm{Hom}^{\bullet}_{\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A})}\big((a_i, \alpha_{ij}), (b_i, \beta_{ij})\big) := \mathrm{Hom}^{\bullet}_{\mathcal{B}}\Big(\bigoplus_{k\in\mathbb{Z}} a_k[-k], \bigoplus_{l\in\mathbb{Z}} b_l[-l]\Big) \tag{3.2}$$

with its natural grading and the differential defined by (2.5).

Where the choice of $\mathcal{B}$ is clear or was fixed, we shall write $\mathrm{Tw}^{\pm}(\mathcal{A})$ for $\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A})$.

As before, our definition ensures that we have a fully faithful convolution functor

$$\mathrm{Conv} \colon\ \mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A}) \to \mathbf{Mod}\text{-}\mathcal{B}.$$

We have the commutative square of fully faithful embeddings

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "I"] \arrow[d, "\Upsilon"] & \mathcal{B} \arrow[d, "\Upsilon"] \\ \mathbf{Mod}\text{-}\mathcal{A} \arrow[r, hook, "I^*"] & \mathbf{Mod}\text{-}\mathcal{B}. \end{tikzcd} \tag{3.3}$$

Since all DG-functors preserve finite direct sums, we see that on bounded twisted complexes the convolution functor into $\mathbf{Mod}\text{-}\mathcal{B}$ is simply the composition of the usual convolution functor into $\mathbf{Mod}\text{-}\mathcal{A}$ and I^*.

Observe that setting $\mathcal{B} = \mathbf{Mod}\text{-}\mathcal{A}$ recovers the definition of $\mathrm{Tw}^{\pm}_{abs}(\mathcal{A})$ with the convolution into $\mathbf{Mod}\text{-}\mathcal{A}$. On the other hand, when we have $\mathcal{A} = \mathbf{Mod}\text{-}\mathcal{C}$ for some small DG-category $\mathcal{C}$, setting $\mathcal{B} = \mathcal{A}$ recovers the category constructed in Example 3.3 with its convolution into $\mathbf{Mod}\text{-}\mathcal{A}$ which factors through $\mathcal{A}$.

Definition 3.5. A DG category $\mathcal{B}$ is *closed under twisting* if for all $b \in \mathcal{B}$ and $f \in \mathrm{Hom}^1_{\mathcal{B}}(b, b)$ with $df + f^2 = 0$ the module in $\mathbf{Mod}\text{-}\mathcal{B}$ which has the underlying graded module of $\mathrm{Hom}_{\mathcal{B}}(-, b)$ and the differential $d_{\mathrm{Hom}_{\mathcal{B}}(-,b)} + f$ is representable.

Definition 3.6. A DG category $\mathcal{B}$ *admits convolutions of unbounded twisted complexes* if it admits countable direct sums and shifts and the convolution functor $\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{B}) \hookrightarrow \mathbf{Mod}\text{-}\mathcal{B}$ factors through $\mathcal{B} \hookrightarrow \mathbf{Mod}\text{-}\mathcal{B}$.

We do not need to specify for which unbounded twisted complexes $\mathcal{B}$ admits convolutions, because for the convolution to be representable the infinite direct sum needs to be taken in $\mathcal{B}$ itself. Thus we need to consider unbounded twisted complexes relative to $\mathcal{B}$ itself.

If $\mathcal{B}$ admits convolutions of unbounded twisted complexes, then the convolution

$$\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{B}) \hookrightarrow \mathcal{B}$$

is necessarily an equivalence. It is fully faithful and has a right inverse which sends any $b \in \mathcal{B}$ to itself considered as trivial twisted complex in degree zero.

Lemma 3.7. *Let $\mathcal{B}$ be a DG-category which admits countable direct sums and shifts. The following are equivalent:*

(1) *$\mathcal{B}$ is closed under twisting.*
(2) *$\mathcal{B}$ admits convolutions of unbounded twisted complexes.*
(3) *The embedding $\mathcal{B} \hookrightarrow \mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{B})$ which sends any $b \in \mathcal{B}$ to itself considered as a trivial twisted complex in degree zero is an equivalence.*
(4) *For any DG-category $\mathcal{A}$ with an embedding into $\mathcal{B}$, $\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A}) \to \mathbf{Mod}\text{-}\mathcal{B}$ factors through $\mathcal{B} \hookrightarrow \mathbf{Mod}\text{-}\mathcal{B}$.*

Proof. (1) $\Rightarrow$ (2): This is the same argument as in Example 3.3.

(2) $\Leftrightarrow$ (3): The composition

$$\mathcal{B} \hookrightarrow \mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{B}) \hookrightarrow \mathbf{Mod}\text{-}\mathcal{B}$$

is the Yoneda embedding. Thus $\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{B}) \hookrightarrow \mathbf{Mod}\text{-}\mathcal{B}$ factors through the Yoneda embedding if and only if $\mathcal{B} \hookrightarrow \mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{B})$ admits a right quasi-inverse. A fully faithful functor admits a right quasi-inverse if and only if it is an equivalence.

(2) $\Leftrightarrow$ (4):

The "if" implication is obvious. The "only if' one results from the following commutative triangle of fully faithful functors:

$$\begin{tikzcd} \mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A}) \arrow[r, hook, "\mathrm{Conv}"] \arrow[d, hook] & \mathbf{Mod}\text{-}\mathcal{B}. \\ \mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{B}) \arrow[ur, hook, "\mathrm{Conv}"'] & \end{tikzcd}$$

(2) ⇒ (1):

Let $b \in \mathcal{B}$ and $f \in \mathrm{Hom}^1_{\mathcal{B}}(b,b)$ with $df + f^2 = 0$. Then the complex consisting of b in degree 0 with a single differential f from b to itself is a twisted complex. Its convolution in $\mathbf{Mod}$-$\mathcal{B}$ has the same graded module as b and the differential $d_b + f$. Since $\mathcal{B}$ admits convolutions of twisted complexes, it is representable. $\square$

Finally, for any version of $\mathrm{Tw}^{\pm}(\mathcal{A})$, we define $\mathrm{Tw}^{+}(\mathcal{A})$ and $\mathrm{Tw}^{-}(\mathcal{A})$ to be its full subcategories consisting of all bounded above twisted complexes and all bounded below twisted complexes, respectively. We also define $\text{Pre-Tr}^{\pm}(\mathcal{A})$, $\text{Pre-Tr}^{+}(\mathcal{A})$, and $\text{Pre-Tr}^{-}(\mathcal{A})$ to be the full subcategories of $\mathrm{Tw}^{\pm}(\mathcal{A})$, $\mathrm{Tw}^{+}(\mathcal{A})$, and $\mathrm{Tw}^{-}(\mathcal{A})$ consisting of one-sided twisted complexes.

4. Twisted bicomplexes

The following is a natural generalisation of the notion of a twisted complex:

Definition 4.1. A *twisted bicomplex* (a_{ij}, α_{ijkl}) over a DG category $\mathcal{A}$ comprises

- $\forall\, i,j \in \mathbb{Z}$, an object a_{ij} of $\mathcal{A}$, non-zero for only finite number of pairs (i,j),
- $\forall\, i,j,k,l \in \mathbb{Z}$, a degree $(i+j) - (j+k) + 1$ morphism $\alpha_{ijkl}\colon a_{ij} \to a_{kl}$ in $\mathcal{A}$,

satisfying

$$(-1)^{k+l} d\alpha_{ijkl} + \sum_{m,n} \alpha_{ijmn} \circ \alpha_{mnkl} = 0. \tag{4.1}$$

Define the *DG category* $\mathrm{Twbi}(\mathcal{A})$ *of twisted bicomplexes over* $\mathcal{A}$ by setting

$$\mathrm{Hom}^{\bullet}_{\mathrm{Tw}(\mathcal{A})}\big((a_{ij}, \alpha_{ijkl}), (b_{ij}, \beta_{ijkl})\big) := \bigoplus_{q,k,l,m,n \in \mathbb{Z}} \mathrm{Hom}^q_{\mathcal{A}}(a_{kl}, b_{mn}) \tag{4.2}$$

where each $f \in \mathrm{Hom}^q_{\mathcal{A}}(a_{kl}, b_{mn})$ has degree $q + (m+n) - (k+l)$ and

$$df := (-1)^{m+n} d_{\mathcal{A}} f + \sum_{p,q \in \mathbb{Z}} \Big(\beta_{mnpq} \circ f - (-1)^{q+(m+n)-(k+l)} f \circ \alpha_{pqkl} \Big), \tag{4.3}$$

where $d_{\mathcal{A}}$ is the differential on morphisms in $\mathcal{A}$.

We think of indices i and j of each a_{ij} as the row index and the column index, respectively. We say that a twisted bicomplex is *horizontally one-sided* (resp. *vertically one-sided*) if $\alpha_{ijkl} = 0$ when $l \leq j$ (resp. $k \leq i$). We say that a twisted bicomplex is *one-sided* if it is both vertically and horizontally one-sided.

The categories $\mathrm{Twbi}^{\pm}(\mathcal{A})$, $\mathrm{Twbi}^{+}(\mathcal{A})$ and $\mathrm{Twbi}^{-}(\mathcal{A})$ of absolute unbounded, unbounded below, and unbounded above twisted bicomplexes, as well as their versions relative to another DG category $\mathcal{B}$ are defined similarly to the way they are defined in §3 for twisted complexes. We also use Twbi^{vos}, Twbi^{hos} and Twbi^{os} to denote the full subcategories consisting of one-sided twisted bicomplexes.

Finally, we note that the category of twisted bicomplexes is naturally isomorphic to the category of twisted complexes of twisted complexes, but in two different ways: the complex of sign-twisted rows and the complex of sign-twisted columns. For this result, the twisted complexes over $\mathcal{A}$ need to be considered relative to some $\mathcal{B}$ which admits the convolutions of unbounded twisted complexes, cf. Lemma 3.7. This is always true when $\mathcal{B} = \mathbf{Mod}\text{-}\mathcal{C}$ for some DG-category $\mathcal{C}$.

Definition 4.2. Let $\mathcal{A}$ be DG category and fix its embedding into a DG-category $\mathcal{B}$ which admits convolutions of unbounded twisted complexes. Define

$$\mathcal{C}\mathrm{xrow}\colon\ \mathrm{Tw}^{\pm}_{\mathcal{B}}\left(\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A})\right) \to \mathrm{Twbi}^{\pm}_{\mathcal{B}}(\mathcal{A}), \tag{4.4}$$

$$\mathcal{C}\mathrm{xcol}\colon\ \mathrm{Tw}^{\pm}_{\mathcal{B}}\left(\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A})\right) \to \mathrm{Twbi}^{\pm}_{\mathcal{B}}(\mathcal{A}), \tag{4.5}$$

as follows. Let (E_i, α_{ik}) be an object of $\mathrm{Tw}^{\pm}_{\mathcal{B}}\left(\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A})\right)$. Write

- $E_{i,j}$ for the objects of each E_i and $\alpha_{i,jl} : E_{i,j} \to E_{i,l}$ for its differentials.
- $\alpha_{ik,jl} : E_{i,j} \to E_{k,l}$ for the components of $\alpha_{ik}\colon E_i \to E_k$.

We then define:

- $\mathcal{C}\mathrm{xrow}(E_i, \alpha_{ik})$ to be the twisted bicomplex whose ij-th object is $E_{i,j}$ and whose $ijkl$-th differential is $\alpha_{ik,jl}$ if $i \neq k$ and $\alpha_{ii,jl} + (-1)^i \alpha_{i,jl}$ if $i = k$.
- $\mathcal{C}\mathrm{xcol}(E_i, \alpha_{ik})$ to be the twisted bicomplex whose ij-th object is $E_{j,i}$ and whose $ijkl$-th differential is $\alpha_{jl,ik}$ if $j \neq l$ and $\alpha_{jj,ik} + (-1)^j \alpha_{j,ik}$ if $j = l$.

Similarly, let $f : (E_i, \alpha_{ik}) \to (F_i, \beta_{ik})$ be a morphism in $\mathrm{Tw}^{\pm}_{\mathcal{B}}\left(\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A})\right)$. Write f_{ik} for its $E_i \to F_k$ component, and then $f_{ik,jl}$ for $E_{i,j} \to F_{k,l}$ component of that. We then define:

- $\mathcal{C}\mathrm{xrow}(f)$ to be the bicomplex map whose $ijkl$-th component is $f_{ik,jl}$,
- $\mathcal{C}\mathrm{xcol}(f)$ to be the bicomplex map whose $ijkl$-th component is $f_{jl,ik}$.

In other words, $\mathcal{C}\mathrm{xrow}(E_i, \alpha_{ik})$ is the bicomplex whose i-th row is the twisted complex $(-1)^i E_i$ to whose differentials we further add all the components of α_{ii}. The differentials between i-th and j-th rows for $i \neq j$ are the components of α_{ij}. Similarly, $\mathcal{C}\mathrm{xcol}(E_i, \alpha_{ik})$ is the bicomplex whose columns are $(-1)^i E_i$ modified by α_{ii} and whose intercolumn differentials are α_{ij}. On morphisms, both functors simply map a morphism of complexes of complexes to the bicomplex morphism with the same components.

By our assumption on $\mathcal{B}$, the convolution functor embeds $\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A})$ fully faithfully into $\mathcal{B}$. We thus have a double convolution functor:

$$\mathrm{Conv}\,\mathrm{Conv}\colon\ \mathrm{Tw}^{\pm}_{\mathcal{B}}\left(\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A})\right) \hookrightarrow \mathcal{B}. \tag{4.6}$$

Proposition 4.3. *Let $\mathcal{A}$ be a DG category with a fully faithful functor into a DG category $\mathcal{B}$ which admits convolutions of unbounded twisted complexes. Let $(E_i, \alpha_{ik}) \in \mathrm{Tw}^{\pm}_{\mathcal{B}}\left(\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A})\right)$. Then:*

(1) *Functors $\mathcal{C}\mathrm{xrow}$ and $\mathcal{C}\mathrm{xcol}$ in Definition 4.2 are well-defined. The data they assign to an object of $\mathrm{Tw}^{\pm}_{\mathcal{B}}\left(\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A})\right)$ satisfies the twisted bicomplex condition (4.1) and the finiteness condition of the sum of its differentials being a morphism in $\mathcal{B}$. The data they assign to a morphism satisfies the finiteness condition of the sum of its components being a morphism in $\mathcal{B}$.*

(2) *The following diagram commutes:*

$$\begin{tikzcd} \mathrm{Tw}^{\pm}_{\mathcal{B}}\left(\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A})\right) \arrow[r, "\mathrm{Conv}\,\mathrm{Conv}"] \arrow[d, "\mathcal{C}\mathrm{xrow}\ \ \mathit{or}\ \ \mathcal{C}\mathrm{xcol}"'] & \mathcal{B}. \\ \mathrm{Twbi}^{\pm}_{\mathcal{B}}(\mathcal{A}) \arrow[ur, "\mathrm{Conv}"'] & \end{tikzcd} \tag{4.7}$$

(3) *The following diagram commutes:*

$$\begin{tikzcd} \mathrm{Tw}^{\pm}_{\mathcal{B}}\left(\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A})\right) \arrow[r, "\mathcal{C}\mathrm{xrow}"] \arrow[d, "\mathcal{C}\mathrm{xcol}"'] & \mathrm{Twbi}^{\pm}_{\mathcal{B}}(\mathcal{A}). \\ \mathrm{Twbi}^{\pm}_{\mathcal{B}}(\mathcal{A}) \arrow[ur, "\mathit{Reflect}"'] & \end{tikzcd} \tag{4.8}$$

Here Reflect is a self-inverse automorphism of $\mathrm{Twbi}^{\pm}_{\mathcal{B}}(\mathcal{A})$ which reflects it along the diagonal: a bicomplex (E_{ij}, α_{ijkl}) is sent to $(E_{ji}, (-1)^{\delta_{ik}+\delta_{jl}}\alpha_{jilk})$, while a morphism (f_{ijkl}) is sent to (f_{jilk}).

(4) *Functor $\mathcal{C}\mathrm{xrow}$ restricts to the isomorphism*

$$\mathcal{C}\mathrm{xrow}\colon\ \text{Pre-Tr}^{\pm}_{\mathcal{B}}\left(\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A})\right) \xrightarrow{\sim} \mathrm{Twbi}^{\pm,vos}_{\mathcal{B}}(\mathcal{A}),$$

while $\mathcal{C}\mathrm{xcol}$ restricts to the isomorphism

$$\mathcal{C}\mathrm{xcol}\colon\ \text{Pre-Tr}^{\pm}_{\mathcal{B}}\left(\mathrm{Tw}^{\pm}_{\mathcal{B}}(\mathcal{A})\right) \xrightarrow{\sim} \mathrm{Twbi}^{\pm,hos}_{\mathcal{B}}(\mathcal{A}).$$

Both functors restrict to the isomorphisms

$$\mathcal{C}\mathrm{xrow}, \mathcal{C}\mathrm{xcol}\colon\ \text{Pre-Tr}^{\pm}_{\mathcal{B}}\left(\text{Pre-Tr}^{\pm}_{\mathcal{B}}(\mathcal{A})\right) \xrightarrow{\sim} \mathrm{Twbi}^{\pm,os}_{\mathcal{B}}(\mathcal{A}).$$

Proof. Straightforward computation. □

The resulting "monodromy" $\mathcal{C}\mathrm{xrow}^{-1} \circ \mathcal{C}\mathrm{xcol}$ of $\text{Pre-Tr}^{\pm}_{\mathcal{B}}\left(\text{Pre-Tr}^{\pm}_{\mathcal{B}}(\mathcal{A})\right)$ is a non-trivial autoequivalence which takes a complex of complexes and slices up the resulting bigraded data of objects and differentials in the other direction to produce a different complex of complexes out of the same data, while sign-twisting purely horizontal and vertical differentials.

5. Application: A_∞-structures in monoidal DG categories

The main application we had in mind for unbounded twisted complexes is to reformulate and generalise the definitions of A_∞-algebras and modules [12, §2]: we define these structures in an arbitrary DG monoidal category $\mathcal{A}$ (or, more generally, a DG bicategory). This disposes with the necessity to work explicitly with the operation m_1, i.e. the differential.

Traditionally, A_∞-algebra formalism was defined for objects in the DG category **Mod**-k of DG complexes of k-modules with its natural monoidal structure given by the tensor product of complexes [12, §2]. In **Mod**-k the internal differential of each object, that is – its differential as a complex of k-modules, exists as a degree 1 endomorphism of the object. It can therefore be a part of the definition of an A_∞-algebra or A_∞-module in **Mod**-k. This is no longer true if we work with an arbitrary monoidal DG category $\mathcal{A}$. In **Mod**-$\mathcal{A}$ the internal differentials of objects do not appear as their degree 1 endomorphisms. Moreover, if we wanted to try and set up A_∞-formalism to work in $\mathcal{A}$ itself, its objects do not possess an internal differential.

The language of twisted complexes solves both of these problems. It implicitly embeds the objects of $\mathcal{A}$ into **Mod**-$\mathcal{A}$ as Hom-complexes of $\mathcal{A}$. These do have an internal differential: the differential $d_{\mathcal{A}}$ of $\mathcal{A}$. The twisted complex condition (2.3) involves $d_{\mathcal{A}}$ and makes it possible to define an A_∞-algebra or module structure on an object $a \in \mathcal{A}$ while referring explicitly only to operations $\{m_i\}_{i\geq 2}$.

The resulting definitions all ask for the corresponding bar construction of the A_∞-operations to be a twisted complex. These twisted complexes have to be unbounded, thus necessitating the theory developed in this paper and its subtleties. We note that in the bar constructions there is only a finite number of arrows emerging from each element of the twisted complex. Hence, our definitions of an A_∞-algebra or an A_∞-module are independent of the ambient category $\mathcal{B}$ we use to define unbounded twisted complexes. We see another example of these subtleties coming into play when we consider twisted complexes of A_∞-modules in §5.2.

In §5.1 we give the key definitions which are studied further [4]. An interested reader should consult §3.2 of that paper for further explanation of the way in which these definitions generalise the classical ones in [12, §2].

In §5.2 we use the twisted bicomplex techniques we developed in §4 to prove several theorems about twisted complexes of A_∞-modules. We first relate a twisted complex of A_∞-modules to an A_∞-module structure on the twisted complex of their underlying objects. This allows us to show that the DG category $\mathbf{Nod}_\infty\text{-}A$ of A_∞-modules over an A_∞-algebra A is strongly pretriangulated (resp. pretriangulated) if and only if DG monoidal category $\mathcal{A}$ we work in is. Hence if we expand $\mathcal{A}$ to $\mathbf{Mod}\text{-}\mathcal{A}$ with the induced monoidal structure [8, §4.5] all the categories of A_∞-modules over all A_∞-algebras in it become strongly pretriangulated.

5.1. Definitions

Throughout this section we assume that DG monoidal category $\mathcal{A}$ we work with comes with a fixed choice of a monoidal embedding

$$\mathcal{A} \hookrightarrow \mathcal{B}$$

into a closed monoidal DG category $\mathcal{B}$ which admits convolutions of unbounded twisted complexes. Note, that we can always set $\mathcal{B} = \mathbf{Mod}\text{-}\mathcal{A}$ with the induced monoidal structure [8, §4.5], enlarging our universe if necessary when $\mathcal{A}$ is not small. All the unbounded twisted complexes over $\mathcal{A}$ are then defined relative to this ambient category $\mathcal{B}$.

The condition that $\mathcal{B}$ is closed under convolutions of unbounded twisted complexes can be replaced throughout by $\mathcal{B}$ being closed under the convolutions of bounded above twisted complexes and/or bounded below twisted complexes.

Given any object $A \in \mathcal{A}$, we write A^i to denote the i-fold tensor product $A^{\otimes i}$.

Definition 5.1. Let $\mathcal{A}$ be a monoidal DG category, let $A \in \mathcal{A}$ and let $\{m_i\}_{i\geq 2}$ be a collection of degree $2-i$ morphisms $A^i \to A$.

The *(non-augmented) bar-construction* $B^{na}_\infty(A)$ of A is the collection of objects A^{i+1} for all $i \geq 0$ each placed in degree $-i$ and of degree $k-1$ maps $d_{(i+k)i} \colon A^{i+k} \to A^i$ defined by

$$d_{(i+k)i} := (-1)^{(i-1)(k+1)} \sum_{j=0}^{i-1} (-1)^{jk} \operatorname{Id}^{i-j-1} \otimes m_{k+1} \otimes \operatorname{Id}^j . \tag{5.1}$$

$$\begin{tikzcd}[column sep=huge]
\dots \arrow[r, "A^3m_2-A^2m_2A+"', "+Am_2A^2-m_2A^3"' below] \arrow[rr, bend left, "A^2m_3+Am_3A+m_3A^2"] \arrow[rrr, bend left, "Am_4-m_4A"] \arrow[rrrr, bend left, "m_5"] & A^4 \arrow[r, "A^2m_2-Am_2A+m_2A^2"'] \arrow[rr, bend left, "-Am_3-m_3A"] \arrow[rrr, bend left, "m_4"] & A^3 \arrow[r, "Am_2-m_2A"'] \arrow[rr, bend left, "m_3"] & A^2 \arrow[r, "m_2"'] & \underset{\text{deg.}0}{A}
\end{tikzcd} \tag{5.2}$$

We say that the bar construction in Definition 5.1 is non-augmented because in the standard sources on A_∞-algebras [12] two bar constructions are used: the non-augmented one is as in Definition 5.1 and the augmented one would in our context be the direct sum of the non-augmented and $\mathrm{Id}_{\mathcal{A}}$, the unit of the monoidal structure on $\mathcal{A}$. The augmented bar construction has a natural structure of a strictly counital strict coalgebra, a point of view much exploited in [12].

Definition 5.2. Let $\mathcal{A}$ be a monoidal DG category. An A_∞-*algebra* (A, m_i) in $\mathcal{A}$ is an object $A \in \mathcal{A}$ equipped with operations $m_i \colon A^i \to A$ for all $i \geq 2$ which are degree $2 - i$ morphisms in $\mathcal{A}$ such that their non-augmented bar-construction $B^{\mathrm{na}}_\infty(A)$ is a twisted complex over $\mathcal{A}$.

We define morphisms of A_∞-algebras in $\mathcal{A}$ in a similar way:

Definition 5.3. Let (A, m_k) and (B, n_k) be A_∞-algebras in $\mathcal{A}$. Let $(f_i)_{i\geq 1}$ be a collection of degree $1 - i$ morphisms $A^i \to B$.

The *bar-construction* $B_\infty(f_\bullet)$ is the morphism $B^{\mathrm{na}}_\infty(A) \to B^{\mathrm{na}}_\infty(B)$ in $\operatorname{Pre-Tr}^-(\mathcal{A})$ whose $A^{i+k} \to B^i$ component is

$$\sum_{t_1+\dots+t_i=i+k} (-1)^{\sum_{l=2}^{i}(1-t_l)\sum_{n=1}^{l} t_n} f_{t_1} \otimes \dots \otimes f_{t_i}.$$

$$\begin{tikzcd}
\dots \arrow[r] & A^4 \arrow[r] \arrow[d, "f_1 f_1 f_1 f_1"] \arrow[dr, "f_1 f_1 f_2 - f_1 f_2 f_1 + f_2 f_1 f_1"] \arrow[drr, "f_1 f_3 + f_2 f_2 + f_3 f_1"] \arrow[drrr, "f_4"] & A^3 \arrow[r] \arrow[d, "f_1 f_1 f_1"] \arrow[dr, "-f_1 f_2 + f_2 f_1"] \arrow[drr, "f_3"] & A^2 \arrow[r] \arrow[d, "f_1 f_1"] \arrow[dr, "f_2"] & A \arrow[d, "f_1"] \\
\dots \arrow[r] & B^4 \arrow[r] & B^3 \arrow[r] & B^2 \arrow[r] & B.
\end{tikzcd}$$

Definition 5.4. A morphism $f_\bullet \colon (A, m_k) \to (B, n_k)$ of A_∞-algebras is a collection $(f_i)_{i\geq 1}$ of degree $1-i$ morphisms $A^i \to B$ whose bar construction is a closed degree 0 morphism of twisted complexes.

We define left and right A_∞-modules over such $(A, m_\bullet)$ in a similar way:

Definition 5.5. Let (A, m_i) be an A_∞-algebra in a monoidal DG category $\mathcal{A}$. Let $E \in \mathcal{A}$ and let $\{p_i\}_{i\geq 2}$ be a collection of degree $2 - i$ morphisms $E \otimes A^{i-1} \to E$.

The *right module bar-construction* $B_\infty(E)$ of (E, p_i) comprises objects $E \otimes A^i$ for $i \geq 0$ placed in degree $-i$ and degree $1 - k$ maps $E \otimes A^{i+k-1} \to E \otimes A^{i-1}$ defined by

$$d_{(i+k)i} := (-1)^{(i-1)(k+1)} \left(\sum_{j=0}^{i-2} \left((-1)^{jk} \operatorname{Id}^{i-j-1} \otimes m_{k+1} \otimes \operatorname{Id}^{j} \right) + (-1)^{(i-1)k} p_{k+1} \otimes \operatorname{Id}^{i-1} \right). \tag{5.3}$$

$$\begin{tikzcd}
\dots \arrow[r, "{\substack{EA^2m_2 - EAm_2A + \\ + Em_2A^2 - p_2A^3}}"'] \arrow[rr, bend left, "EAm_3 + Em_3A + p_3A^2"] \arrow[rrr, bend left, "Em_4 - p_4A"] \arrow[rrrr, bend left, "p_5"] & EA^3 \arrow[r, "EAm_2 - Em_2A + p_2A^2"'] \arrow[rr, bend left, "-Em_3 - p_3A"] \arrow[rrr, bend left, "p_4"] & EA^2 \arrow[r, "Em_2 - p_2A"'] \arrow[rr, bend left, "p_3"] & EA \arrow[r, "p_2"'] & \underset{\text{deg.}0}{E}
\end{tikzcd}. \tag{5.4}$$

Definition 5.6. For $E \in \mathcal{A}$ and a collection $\{p_i\}_{i\geq 2}$ of degree $2-i$ morphisms $A^{i-1} \otimes E \to E$, its *left module bar-construction* $B_\infty(E)$ comprises objects $A^i \otimes E$ for all $i \geq 0$ placed in degree $-i$ and degree $1-k$ maps $A^{i+k-1} \otimes E \to A^{i-1} \otimes E$ defined by

$$d_{(i+k)i} := (-1)^{(i-1)(k+1)} \left(\sum_{j=1}^{i-1} \left((-1)^{jk} \operatorname{Id}^{i-j-1} \otimes m_{k+1} \otimes \operatorname{Id}^{j} \right) + \operatorname{Id}^{i-1} \otimes p_{k+1} \right). \tag{5.5}$$

$$\begin{tikzcd}
\dots \arrow[r, "{\substack{A^3p_2 - A^2m_2E + \\ + Am_2AE - m_2A^2E}}"'] \arrow[rr, bend left, "A^2p_3 + Am_3E + m_3AE"] \arrow[rrr, bend left, "Ap_4 - m_4E"] \arrow[rrrr, bend left, "p_5"] & A^3E \arrow[r, "A^2p_2 - Am_2E + m_2AE"'] \arrow[rr, bend left, "-Ap_3 - m_3E"] \arrow[rrr, bend left, "p_4"] & A^2E \arrow[r, "Ap_2 - m_2E"'] \arrow[rr, bend left, "p_3"] & AE \arrow[r, "p_2"'] & \underset{\text{deg.}0}{E}
\end{tikzcd}. \tag{5.6}$$

Definition 5.7. Let $\mathcal{A}$ be a monoidal DG category and let (A, m_i) be an A_∞-algebra in $\mathcal{A}$. A *right (resp. left) A_∞-module (E, p_i) over A* is an object $E \in \mathcal{A}$ and a collection $\{p_i\}_{i\geq 2}$ of degree $2-i$ morphisms $E \otimes A^{i-1} \to E$ (resp. $A^{i-1} \otimes E \to E$) such that $B_\infty(E)$ is a twisted complex.

Definition 5.8. Let $\mathcal{A}$ be a monoidal DG category and let (A, m_i) be an A_∞-algebra in $\mathcal{A}$. Let (E, p_k) and (F, q_k) be right A_∞-modules over A in $\mathcal{A}$.

A *degree j morphism* $f_\bullet \colon (E, p_k) \to (F, q_k)$ of right A_∞-A-modules is a collection $(f_i)_{i\geq 1}$ of degree $j-i+1$ morphisms $E \otimes A^{i-1} \to F$. Its *bar-construction* $B_\infty(f_\bullet)$ is the morphism $B_\infty(E) \to B_\infty(F)$ in $\text{Pre-Tr}^{-}(\mathcal{A})$ whose components are

$$E \otimes A^{i+k-1} \to F \otimes A^{i-1} \colon \; (-1)^{j(i-1)} f_{k+1} \otimes \operatorname{Id}^{i-1}.$$

We illustrate the case when $f_\bullet$ is of odd degree:

$$\begin{tikzcd}
\cdots \arrow[r] & EA^3 \arrow[r] \arrow[d, "-f_1A^3"] \arrow[dr, "f_2A^2"] \arrow[drr, "-f_3A"] \arrow[drrr, "f_4"] & EA^2 \arrow[r] \arrow[d, "f_1A^2"] \arrow[dr, "-f_2A"] \arrow[drr, "f_3"] & EA \arrow[r] \arrow[d, "-f_1A"] \arrow[dr, "f_2"] & E \arrow[d, "f_1"] \\
\cdots \arrow[r] & FA^3 \arrow[r] & FA^2 \arrow[r] & FA \arrow[r] & F.
\end{tikzcd}$$

The corresponding definition for the left A_∞-modules differs only in signs:

Definition 5.9. Let $\mathcal{A}$ be a monoidal DG category and let (A, m_i) be an A_∞-algebra in $\mathcal{A}$. Let (E, p_k) and (F, q_k) be left A_∞-modules over A in $\mathcal{A}$.

A degree j morphism $f_\bullet\colon (E, p_k) \to (F, q_k)$ of left A_∞-A-modules is a collection $(f_i)_{i\geq 1}$ of degree $j-i+1$ morphisms $A^{i-1} \otimes E \to F$. Its *bar-construction* $B_\infty(f_\bullet)$ is the morphism $B_\infty(E) \to B_\infty(F)$ in Pre-Tr$^-(\mathcal{A})$ whose components are

$$A^{i+k-1} \otimes E \to A^{i-1} \otimes F\colon\ (-1)^{(j+k)(i-1)} \operatorname{Id}^{i-1} \otimes f_{k+1}.$$

We define the DG categories of left and right modules over A in the unique way which makes the left and right module bar constructions into faithful DG functors from these categories to Pre-Tr$^-(\mathcal{A})$:

Definition 5.10. Let $\mathcal{A}$ be a monoidal DG category and A be an A_∞-algebra in $\mathcal{A}$. Define the *DG category* $\mathbf{Nod}_\infty$*-A of right A_∞-A-modules in $\mathcal{A}$* by:

- Its objects are right A_∞-A-modules in $\mathcal{A}$,
- For any $E, F \in \operatorname{Ob} \mathbf{Nod}_\infty A$, the complex $\operatorname{Hom}^\bullet_{\mathbf{Nod}_\infty A}(E, F)$ consists of A_∞-morphisms $f_\bullet\colon E \to F$ with their natural grading. The differential and the composition are defined by differentiating and by composing the corresponding twisted complex morphisms.
- The identity morphism of $E \in \mathbf{Nod}_\infty A$ is the morphism $(f_\bullet)$ with $f_1 = \operatorname{Id}_E$ and $f_{\geq 2} = 0$ whose corresponding twisted complex morphism is $\operatorname{Id}_{B_\infty(E)}$.

The *DG category A-*$\mathbf{Nod}_\infty$ *of left A_∞-A-modules in $\mathcal{A}$* is defined analogously.

The letter ‘N’ in $\mathbf{Nod}_\infty$ stands for ‘non-unital’. Both classical A_∞-algebras and our generalisations of them are not required to be equipped with a unitality structure, and A_∞-modules over them are not subject to any unitality constraints.

Similar definitions exist for A_∞-coalgebras and A_∞-comodules, see [4, §6].

5.2. Twisted complexes of A_∞-modules

The notion of an A_∞-module over an A_∞-algebra $(A, m_\bullet)$ in a monoidal DG category $\mathcal{A}$ which we defined in §5.1 differs in several ways from the usual notion which corresponds to the case $\mathcal{A} = \mathbf{Mod}\text{-}k$.

One is that the DG-category of usual A_∞-modules is strongly pretriangulated, while in our generality $\mathbf{Nod}_\infty\text{-}A$ doesn't have to be. In this section we show that $\mathbf{Nod}_\infty\text{-}A$ is strongly pretriangulated if and only if $\mathcal{A}$ is strongly pretriangulated.

First, we need to fix our conventions. As in §5.1 we assume that our monoidal DG category $\mathcal{A}$ comes with a monoidal embedding into a closed monoidal DG category $\mathcal{B}$ which has convolutions of unbounded twisted complexes. Recall that we can always set $\mathcal{B} = \mathbf{Mod}\text{-}\mathcal{A}$ with the induced monoidal structure [8, §4.5].

We define $\operatorname{Tw}^{\pm} \mathcal{A}$ and $\operatorname{Tw}^{\pm} \mathcal{B}$ relative to $\mathcal{B}$. Thus twisted complexes in $\operatorname{Tw}^{\pm} \mathcal{A}$ and $\operatorname{Tw}^{\pm} \mathcal{B}$ can have infinite number of differentials and/or morphism components emerge from a single object, but only if their sum still defines a morphism in $\mathcal{B}$. By Lemma 3.7, since $\mathcal{B}$ admits convolutions of unbounded twisted complexes, the convolution functor $\operatorname{Tw}^{\pm} \mathcal{B} \hookrightarrow \mathcal{B}$ is an equivalence.

Let $(A, m_\bullet)$ be an A_∞-algebra in $\mathcal{A}$. In §5.1 we define the DG category $\mathbf{Nod}_\infty\text{-}A$ of right A_∞-A-modules in $\mathcal{A}$. Using the monoidal embedding we can view $(A, m_\bullet)$ as an A_∞-algebra in $\mathcal{B}$. We write $\mathbf{Nod}_\infty\text{-}A^{\mathcal{B}}$ for the category of right A_∞-A-modules in $\mathcal{B}$. Note that we have tautological embedding $\mathbf{Nod}_\infty\text{-}A \hookrightarrow \mathbf{Nod}_\infty\text{-}A^{\mathcal{B}}$.

We now want to define $\operatorname{Tw}^{\pm} \mathbf{Nod}_\infty\text{-}A$. Working with absolute twisted complexes gives us a convolution into $\mathbf{Mod}\text{-}(\mathbf{Nod}_\infty\text{-}A)$, but it isn't the category we want to work with. Instead, we have an embedding of $\mathbf{Nod}_\infty\text{-}A$ into $\mathbf{Nod}_\infty\text{-}A^{\mathcal{B}}$, and we want $\mathbf{Nod}_\infty\text{-}A^{\mathcal{B}}$ to be the target for the convolution of twisted complexes of $\mathbf{Nod}_\infty\text{-}A$.

The category $\mathbf{Nod}_\infty\text{-}A^{\mathcal{B}}$ is closed under shifts and direct sums because $\mathcal{B}$ is. Indeed, $(E, p_\bullet)[n] = (E[n], (-1)^n p_\bullet)$ and $\oplus_i (E_i, p_{i\bullet}) = (\oplus_i E_i, \sum_i p_{i\bullet})$. To see that $\sum_i p_{i\bullet}$ define an A_∞-module structure on $\oplus_i E_i$, note that since $\mathcal{B}$ is closed monoidal its monoidal structure commutes with infinite direct sums. We thus define both $\operatorname{Tw}^{\pm} \mathbf{Nod}_\infty\text{-}A$ and $\operatorname{Tw}^{\pm} \mathbf{Nod}_\infty\text{-}A^{\mathcal{B}}$ relative to $\mathbf{Nod}_\infty\text{-}A^{\mathcal{B}}$. This yields fully faithful convolution functors from both into $\mathbf{Mod}\text{-}\left(\mathbf{Nod}_\infty\text{-}A^{\mathcal{B}}\right)$. We now want to show that $\mathbf{Nod}_\infty\text{-}A^{\mathcal{B}}$ admits convolutions of unbounded twisted complexes, and thus both convolution functors take values in $\mathbf{Nod}_\infty\text{-}A^{\mathcal{B}}$.

For this, we prove below that a twisted complex of A_∞-modules defines the structure of an A_∞-module on the underlying twisted complex of objects of $\mathcal{A}$. But first, we define what such structure is. The category $\operatorname{Tw}^{\pm} \mathcal{A}$ is not apriori monoidal as a tensor product of two twisted complexes over $\mathcal{A}$ should have as objects direct sums of objects of $\mathcal{A}$. These do not apriori exist in $\mathcal{A}$, but they do exist in $\mathcal{B}$. Indeed, $\operatorname{Tw}^{\pm} \mathcal{B}$ is a monoidal category equivalent to $\mathcal{B}$.

Definition 5.11. Let $A^{\mathrm{Tw}^{\pm}\mathcal{A}}$ denote A considered as a trivial twisted complex in $\mathrm{Tw}^{\pm}\mathcal{A}$. Define $\mathbf{Nod}_{\infty}\text{-}A^{\mathrm{Tw}^{\pm}\mathcal{A}}$ to be the full subcategory of $\mathbf{Nod}_{\infty}\text{-}A^{\mathcal{B}}$ consisting of the A_{∞}-modules whose underlying objects of $\mathcal{B}$ lie in $\mathrm{Tw}^{\pm}\mathcal{A} \subseteq \mathrm{Tw}^{\pm}\mathcal{B} \simeq \mathcal{B}$.

Explicitly, an object of $\mathbf{Nod}_{\infty}\text{-}A^{\mathrm{Tw}^{\pm}\mathcal{A}}$ is a twisted complex (E_i, α_{ij}) over $\mathcal{A}$ together with degree $2-k$ twisted complex morphisms

$$p_k \colon (E_i \otimes A^{k-1}, \alpha_{ij} \otimes \mathrm{Id}) \to (E_i, \alpha_{ij})$$

such that their right-module bar-construction is a twisted complex of twisted complexes $(E_i \otimes A^{k-1}, \alpha_{ij} \otimes \mathrm{Id})$.

We can now state the main result of this subsection:

Proposition 5.12. *There exist fully faithful embeddings of DG-categories:*

$$\Phi\colon\ \mathrm{Tw}^{\pm}\left(\mathbf{Nod}_{\infty}\text{-}A\right) \hookrightarrow \mathbf{Nod}_{\infty}\text{-}A^{\mathrm{Tw}^{\pm}\mathcal{A}}, \tag{5.7}$$

$$\Phi\colon\ \mathrm{Tw}^{\pm}\left(\mathbf{Nod}_{\infty}\text{-}A^{\mathcal{B}}\right) \hookrightarrow \mathbf{Nod}_{\infty}\text{-}A^{\mathcal{B}}. \tag{5.8}$$

These preserve boundedness and one-sidedness of twisted complexes. We can replace $\mathrm{Tw}^{\pm}$ *with any of* Tw^{+}, Tw^{-}, Tw, $\text{Pre-Tr}^{\pm}$, Pre-Tr^{+}, Pre-Tr^{-}, *or* Pre-Tr*.*

Note that the embedding $\mathcal{A} \hookrightarrow \mathcal{B}$ and the convolution functor $\mathrm{Tw}^{\pm}\mathcal{A} \hookrightarrow \mathcal{B}$ induces fully faithful functors from the LHS and the RHS of (5.7) to those of (5.8). Our construction of Φ in the proof below ensures that (5.7) is the restriction of (5.8).

Proof. The bar-construction functor $B_{\infty} : \mathbf{Nod}_{\infty}\text{-}A \to \text{Pre-Tr}^{-}\mathcal{A}$ induces a functor $\mathrm{Tw}^{\pm}(B_{\infty})\colon\ \mathrm{Tw}^{\pm}\mathbf{Nod}_{\infty}\text{-}A \to \mathrm{Tw}^{\pm}\left(\mathrm{Tw}^{\pm}\mathcal{A}\right)$. Composing it with $\mathcal{C}\mathrm{xcol}$ (see Definition 4.2) we get a functor

$$\mathcal{C}\mathrm{xcol} \circ \mathrm{Tw}^{\pm}(B_{\infty})\colon\ \mathrm{Tw}^{\pm}\left(\mathbf{Nod}_{\infty}\text{-}A\right)) \to \mathrm{Twbi}^{\pm}_{\mathcal{B}}(\mathcal{A}).$$

Similarly, composing $B_{\infty}\colon \mathbf{Nod}_{\infty}\text{-}A^{\mathrm{Tw}^{\pm}(\mathcal{A})} \to \text{Pre-Tr}^{-}\left(\mathrm{Tw}^{\pm}(\mathcal{A})\right)$ with $\mathcal{C}\mathrm{xrow}$ gives

$$\mathcal{C}\mathrm{xrow} \circ B_{\infty}\colon \mathbf{Nod}_{\infty}\text{-}A^{\mathrm{Tw}^{\pm}(\mathcal{A})} \to \mathrm{Twbi}^{\pm}_{\mathcal{B}}(\mathcal{A}).$$

The functors B_{∞}, $\mathcal{C}\mathrm{xcol}$, and $\mathcal{C}\mathrm{xrow}$ are injective on objects and faithful. Hence so are $\mathcal{C}\mathrm{xcol} \circ \mathrm{Tw}^{\pm}(B_{\infty})$ and $\mathcal{C}\mathrm{xrow} \circ B_{\infty}$.

This proof is based on the observation that the image of $\mathcal{C}\mathrm{xcol} \circ \mathrm{Tw}^{\pm}(B_{\infty})$ is mapped into the image of $\mathcal{C}\mathrm{xrow} \circ B_{\infty}$ by the automorphism σ of $\mathrm{Twbi}^{\pm}_{\mathcal{B}}(\mathcal{A})$ which multiplies every differential α_{ijkl} and morphism component f_{ijkl} by $(-1)^{ij+kl}$. Thus there exists a unique functor Φ which makes the following square commute:

$$\begin{tikzcd} \mathrm{Tw}^{\pm}\left(\mathbf{Nod}_{\infty}\text{-}A\right) \arrow[r, "\mathcal{C}\mathrm{xcol} \circ \mathrm{Tw}^{\pm}(B_{\infty})"] \arrow[d, "\Phi"] & \mathrm{Twbi}^{\pm}_{\mathcal{B}}(\mathcal{A}) \arrow[d, "\sigma"] \\ \mathbf{Nod}_{\infty}\text{-}A^{\mathrm{Tw}^{\pm}(\mathcal{A})} \arrow[r, "\mathcal{C}\mathrm{xrow} \circ B_{\infty}"] & \mathrm{Twbi}^{\pm}_{\mathcal{B}}(\mathcal{A}). \end{tikzcd} \tag{5.9}$$

Explicitly, Φ has the following description. Let $((E_i, p_{i\bullet}), \alpha_{ij\bullet}) \in \mathrm{Tw}^{\pm}\left(\mathbf{Nod}_{\infty}\text{-}A\right)$. Set $P_{k+1}\colon (E_i, \alpha_{ij1}) \otimes A^k \to (E_i, \alpha_{ij1})$ to be the morphism of twisted complexes whose components are $(-1)^{ik}\alpha_{ijk} + \delta_{ij}(-1)^{i(k+1)}p_{i,k+1}$. Then

$$\Phi((E_i, \alpha_{ij\bullet})) = ((E_i, \alpha_{ij1}), P_\bullet) \tag{5.10}$$

$$\Phi((f_{ij\bullet})) = \left((-1)^{i\bullet} f_{ij\bullet}\right). \tag{5.11}$$

Taking the above as the definition of Φ in (5.7), one can now verify by direct computation that Φ is well-defined and that it makes (5.9) commute. Its fully faithfullness follows immediately from (5.11).

An identical bicomplex argument applies to $\mathrm{Tw}^{\pm}\left(\mathbf{Nod}_{\infty}\text{-}A^{\mathcal{B}}\right)$ leading to the identical definition of Φ in (5.8) via the same formulas (5.10) and (5.11) which is well-defined and fully faithful for the same reasons. $\square$

It follows that $\mathbf{Nod}_{\infty}\text{-}A$ is pretriangulated if and only if $\mathcal{A}$ is:

Corollary 5.13. *The natural embedding*

$$\mathbf{Nod}_{\infty}\text{-}A \hookrightarrow \mathrm{Tw}^{\pm}\left(\mathbf{Nod}_{\infty}\text{-}A\right) \tag{5.12}$$

is an equivalence (resp. quasi-equivalence) if and only if $\mathcal{A} \hookrightarrow \mathrm{Tw}^{\pm}(\mathcal{A})$ *is. The same holds if* $\mathrm{Tw}^{\pm}$ *is replaced by any of its subcategories* $\mathrm{Tw}^{\bullet}$ *or* $\text{Pre-Tr}^{\bullet}$.

Note that $\mathcal{A} \hookrightarrow \mathrm{Tw}^{\pm}\mathcal{A}$ is never an equivalence. Let $\{a_i\}$ be a twisted complex with zero differentials. We define $\mathrm{Tw}^{\pm}\mathcal{A}$ relative to $\mathbf{Mod}\text{-}\mathcal{A}$, so any morphism from some $b \in \mathcal{A}$ to $\{a_i\}$ only has a finite number of non-zero components $b \to a_i$. So it can not have a right inverse. Thus, by above Corollary, $\mathbf{Nod}_{\infty}\text{-}A$ never admits the convolutions of unbounded twisted complexes taken relative to $\mathbf{Nod}_{\infty}\text{-}A^{\mathcal{B}}$.

Proof. All arguments in this proof work identically if we replace $\mathrm{Tw}^{\pm}$ with any of its full subcategories $\mathrm{Tw}^{\bullet}$ or $\text{Pre-Tr}^{\bullet}$.

"If": Since (5.12) is fully faithful, it is an equivalence (resp. quasi-equivalence) if so is its composition $\mathbf{Nod}_{\infty}\text{-}A \hookrightarrow \mathbf{Nod}_{\infty}\text{-}A^{\mathrm{Tw}^{\pm}\mathcal{A}}$ with (5.7).

If $\mathcal{A} \hookrightarrow \mathrm{Tw}^{\pm}(\mathcal{A})$ is an equivalence, it is clear that so is $\mathbf{Nod}_{\infty}\text{-}A \hookrightarrow \mathbf{Nod}_{\infty}\text{-}A^{\mathrm{Tw}^{\pm}\mathcal{A}}$. If $\mathcal{A} \hookrightarrow \mathrm{Tw}^{\pm}(\mathcal{A})$ is only a quasi-equivalence, $\mathbf{Nod}_{\infty}\text{-}A \hookrightarrow \mathbf{Nod}_{\infty}\text{-}A^{\mathrm{Tw}^{\pm}\mathcal{A}}$ is also a quasi-equivalence, but this requires more work. Let $((E_i, \alpha_{ij}), p_\bullet) \in \mathbf{Nod}_{\infty}\text{-}A^{\mathrm{Tw}^{\pm}\mathcal{A}}$. By assumption, (E_i, α_{ij}) is homotopy equivalent in $\mathrm{Tw}^{\pm}(\mathcal{A})$ to some $F \in \mathcal{A}$. By the homotopy transfer of structure (Theorem A.2) we can transfer the A_∞-structure $p_\bullet$ from (E_i, α_{ij}) to F. We thus obtain an A_∞-A-module $(F, q_\bullet)$ homotopy equivalent to $((E_i, \alpha_{ij}), p_\bullet)$, as desired.

"Only if": The forgetful functor $\mathbf{Nod}_\infty\text{-}A \to \mathcal{A}$ has a right inverse: the functor $\mathcal{A} \to \mathbf{Nod}_\infty\text{-}A$ which sends any object $a \in \mathcal{A}$ to $(a, p_\bullet)$ with $p_i = 0$ for all i and sends any morphism $f\colon a \to b$ to $(f_\bullet)$ with $f_1 = f$ and $f_i = 0$ for $i > 1$. $\square$

Since $\mathcal{B} \hookrightarrow \operatorname{Tw}^{\pm} \mathcal{B}$ is an equivalence, the same argument as in Corollary 5.13 gives:

Corollary 5.14. $\mathbf{Nod}_\infty$-$A^{\mathcal{B}}$ *has convolutions of unbounded twisted complexes.*

It follows by Lemma 3.7, that the convolutions of unbounded twisted complexes in $\mathbf{Nod}_\infty$-A and $\mathbf{Nod}_\infty$-$A^{\mathcal{B}}$ take value in $\mathbf{Nod}_\infty$-$A^{\mathcal{B}}$, as desired.

Data availability

No data was used for the research described in the article.

Appendix A. Homotopy transfers of structure for A_∞-modules

In [15] Markl described the homotopy transfer of structure for (the usual) A_∞-algebras over a commutative ring. In this section we give its analogue for A_∞-modules over an A_∞-category. One of the reasons to write this down in detail, is to convince ourselves that it works just the same for our new notion of A_∞-modules over an A_∞-algebra in a DG monoidal category $\mathcal{A}$ introduced in §5.

Another is that the bar construction for morphisms of modules, unlike that for morphisms of algebras, is additive. As result, we can give simple explicit formulas for the transfer in terms of the bar construction.

First we describe the homotopy transfer of structure for classical A_∞-modules.

Let $\mathcal{A}$ be a small A_∞-category over a commutative ring k in the sense of [12] As usual, denote by $k_{\mathcal{A}}$ the minimal k-linear category with the same objects as $\mathcal{A}$: $\operatorname{Hom}_{k_{\mathcal{A}}}(a, b)$ is 0 when $a \neq b$ and k when $a = b$. The category of graded $k_{\mathcal{A}}$-$k_{\mathcal{A}}$-bimodules has a natural monoidal structure given by $\otimes_k$. The A_∞-category $\mathcal{A}$ can be naturally viewed as an A_∞-algebra in this category, and A_∞-$\mathcal{A}$-modules — as A_∞-modules over this algebra in the category of graded $k_{\mathcal{A}}$-modules.

For our purposes, it is more natural to consider $\mathcal{A}$ to be an A_∞-algebra in the category $k_{\mathcal{A}}$-$\mathbf{Mod}$-$k_{\mathcal{A}}$ of differentially graded $k_{\mathcal{A}}$-$k_{\mathcal{A}}$-bimodules. In other words, what is usually known as the operation m_1 becomes the intrinsic data of the differential of the DG-$k_{\mathcal{A}}$-$k_{\mathcal{A}}$-bimodule $\mathcal{A}$. The structure $m_\bullet$ of an A_∞-algebra on this DG-bimodule consists then of $k_{\mathcal{A}}$-$\mathbf{Mod}$-$k_{\mathcal{A}}$ maps

$$m_i\colon \mathcal{A}^{\otimes_k i} \to \mathcal{A} \qquad , \deg(m_i) = 2 - i,\ i \geq 2.$$

Write $B_\infty \mathcal{A}$ for the *bar construction* of $\mathcal{A}$ [12, §1.2.2]. It is a DG-coalgebra in $k_\mathcal{A}$-**Mod**-$k_\mathcal{A}$ equal as a graded coalgebra to the free tensor coalgebra $\bigoplus_{i=0}^{\infty} \mathcal{A}^{\otimes_k i}$.

Let P be a right A_∞-module over $\mathcal{A}$. We consider it, again, as a DG-$k_\mathcal{A}$-module with an A_∞-structure $\mu_\bullet$ given by **Mod**-$k_\mathcal{A}$ maps

$$\mu_i : P \otimes_k \mathcal{A}^{\otimes_k (i-1)} \to P, \qquad \deg(\mu_i) = 2 - i,\ i \geq 2.$$

Write $B_\infty P$ for the *bar construction* of $\mathcal{A}$ [12, §2.3.3]. It is a DG-$B_\infty\mathcal{A}$-comodule equal as a graded comodule to the free comodule $P \otimes_k B_\infty \mathcal{A}$.

Suppose that there is another $k_\mathcal{A}$-module Q and two morphisms

$$f \colon P \to Q,$$
$$g \colon Q \to P,$$

in **Mod**-$k_\mathcal{A}$ such that $gf = \operatorname{Id} + dh$ for some degree -1 map $h : P \to P$.

Theorem A.1. *A homotopy transfer of A_∞-structure from P to Q exists:*

(1) *A structure $\nu_\bullet$ of a right A_∞-module over A on Q;*
(2) *A closed degree zero map of A_∞-modules $\phi_\bullet : P \to Q$ extending f;*
(3) *A closed degree zero map of A_∞-modules $\psi_\bullet : P \to Q$ extending g;*
(4) *A degree -1 map of A_∞-modules $H_\bullet \colon P \to P$ extending h, such that*

$$\psi_\bullet \circ \phi_\bullet = \operatorname{Id} + d(H_\bullet).$$

To prove this, we use the bar-construction as a fully faithful embedding of the DG-category of A_∞-$\mathcal{A}$-modules into the DG-category of DG-$B_\infty\mathcal{A}$-comodules. See [12, §2.3.3]. We give a brief summary. Let E and F be two A_∞-$\mathcal{A}$-modules. Let $x_\bullet \colon E \to F$ be a degree n morphism of A_∞-modules. By definition, it is an arbitrary collection of degree $n + 1 - i$ maps

$$x_i : E \otimes_k \mathcal{A}^{\otimes_k (i-1)} \to F \qquad i \geq 1,$$

in **Mod**-$k_\mathcal{A}$. Such collection is equivalent to the data of a degree n **Mod**-$k_\mathcal{A}$ map

$$x_\bullet \colon B_\infty E \to F,$$

because as a graded module $B_\infty E$ is just $E \otimes_k B_\infty \mathcal{A}$. By universal property of free graded comodules, there is a bijective correspondence of $k_\mathcal{A}$-module morphisms

$$B_\infty E \to F,$$

with the morphisms of DG-$B_\infty\mathcal{A}$-comodules

$$B_\infty E \to B_\infty F.$$

It sends $x_\bullet$ to the map

$$\bar{x}\colon E \otimes_k B_\infty \mathcal{A} \xrightarrow{\mathrm{Id} \otimes \Delta} E \otimes_k B_\infty \mathcal{A} \otimes_k B_\infty \mathcal{A} \xrightarrow{x_\bullet \otimes \mathrm{Id}} F \otimes_k B_\infty \mathcal{A}, \tag{A.1}$$

and, conversely, sends any morphism $\bar{x}$ of DG-$B_\infty\mathcal{A}$-comodules to

$$x_\bullet\colon E \otimes_k B_\infty \mathcal{A} \xrightarrow{\bar{x}} F \otimes_k B_\infty \mathcal{A} \xrightarrow{\mathrm{Id} \otimes \epsilon} F.$$

Here Δ and ϵ are the comultiplication and the counit of $B_\infty\mathcal{A}$.

Proof. By our convention, the A_∞-structure $\mu_\bullet$ on P is a collection of μ_i for $i \geq 2$. View it as the data of a degree one A_∞-morphism $\mu_\bullet\colon P \to P$ with $\mu_1 = 0$. The differential on the bar construction $B_\infty P$ is

$$\bar{\mu} + d_{\mathrm{nat}},$$

where $\bar{\mu}$ is the bar construction of the A_∞-morphism $\mu_\bullet$ as per (A.1) and d_{nat} is the natural differential on the tensor product $P \otimes_k B_\infty\mathcal{A}$. We then have

$$(\bar{\mu} + d_{\mathrm{nat}})^2 = \bar{\mu}^2 + \bar{\mu} \circ d_{\mathrm{nat}} + d_{\mathrm{nat}} \circ \bar{\mu} = \bar{\mu^2} + d(\bar{\mu}),$$

and therefore

$$d_{\mathrm{nat}}(\bar{\mu}) = -\bar{\mu}^2.$$

Here $d_{\mathrm{nat}}(-)$ denotes the differentiation as an endomorphism of $P \otimes_k B_\infty A$, as opposed as an endomorphism of $B_\infty P$.

Now define

$$\bar{\rho} = \bar{\mu} + \bar{\mu}\bar{h}\bar{\mu} + \bar{\mu}\bar{h}\bar{\mu}\bar{h}\bar{\mu} + \ldots$$

where $\bar{h}$ is the bar construction of h viewed as a strict A_∞-morphism. Note that

$$\overline{\rho}\overline{h}\overline{\mu} = \overline{\mu}\overline{h}\overline{\rho} = \overline{\rho} - \overline{\mu},$$
$$d_{\mathrm{nat}}(\bar{\rho}) = \bar{\rho}^2 - \bar{\rho}\overline{dh}\bar{\rho}.$$

Since

$$d_{\mathrm{nat}}(\bar{h}) = \overline{dh} = \overline{g}\overline{f} - \mathrm{Id},$$

we conclude that

$$d_{\text{nat}}(\bar{\rho}) = -\bar{\rho}\bar{g}\bar{f}\bar{\rho}.$$

Define the DG-$B_\infty\mathcal{A}$-module morphism $\bar{\nu}\colon Q \otimes_k B_\infty\mathcal{A} \to Q \otimes_k B_\infty\mathcal{A}$ by

$$\bar{\nu} = \bar{f}\bar{\rho}\bar{g}.$$

Since f and g are closed of degree 0, we have

$$d_{\text{nat}}(\bar{\nu}) = \bar{f} d_{\text{nat}}(\bar{\rho})\bar{g} = -\bar{f}\bar{\rho}\bar{g}\bar{f}\overline{\rho g} = -\bar{\nu}^2.$$

We therefore have

$$(\bar{\nu} + d_{\text{nat}})^2 = 0,$$

so $\bar{\nu} + d_{\text{nat}}$ defines a new differential on $Q \otimes_n B_\infty\mathcal{A}$. Let $\nu_\bullet$ be the corresponding structure of A_∞-$\mathcal{A}$-module on Q.

Define next

$$\begin{aligned} \bar{\phi} &= \bar{f}(\operatorname{Id} + \bar{\rho}\bar{h}), \\ \bar{\psi} &= (\operatorname{Id} + \bar{h}\bar{\rho})\bar{g}, \\ \bar{H} &= \bar{h}(\operatorname{Id} + \bar{\rho}\bar{h}) = (\operatorname{Id} + \bar{h}\bar{\rho})\bar{h}. \end{aligned}$$

We have:

$$d\overline{f} = (\overline{\nu} + d_{\text{nat}})f - f(\overline{\mu} + d_{\text{nat}}) = \overline{\nu}\overline{f} - \overline{f}\overline{\mu} + d_{\text{nat}}(f) = \overline{f}\overline{\rho g}\overline{f} - \overline{f}\overline{\mu} = \overline{f}(\overline{\rho g}\overline{f} - \overline{\mu}),$$

and similarly:

$$\begin{aligned} d\overline{g} &= \cdots = (\overline{\mu} - \overline{g}\overline{f}\overline{\rho})\overline{g}, \\ d\overline{h} &= \cdots = \overline{\mu}\overline{h} + \overline{h}\overline{\mu} + \overline{g}\overline{f} - \operatorname{Id}, \\ d(\operatorname{Id} + \overline{\rho}\overline{h}) &= \cdots = \left(\overline{\mu} - \overline{\rho g}\overline{f}\right)\left(\operatorname{Id} + \overline{\rho}\overline{h}\right), \\ d(\operatorname{Id} + \overline{h}\overline{\rho}) &= \cdots = \left(\overline{\mu} - \overline{g}\overline{f}\overline{\rho}\right)\left(\operatorname{Id} + \overline{h}\overline{\rho}\right). \end{aligned}$$

We thus finally compute

$$\begin{aligned} d(\bar{\phi}) &= d\left(\overline{f}\right)\left(\overline{\rho}\overline{h} + \operatorname{Id}\right) + \overline{f} d\left(\overline{\rho}\overline{h} + \operatorname{Id}\right) = \\ &= \overline{f}\left(\overline{\rho g}\overline{f} - \overline{\mu}\right)\left(\overline{\rho}\overline{h} + \operatorname{Id}\right) + \overline{f}\left(\overline{\mu} - \overline{\rho g}\overline{f}\right)\left(\operatorname{Id} + \overline{\rho}\overline{h}\right) = 0, \\ d(\bar{\psi}) &= \cdots = 0, \\ d(\overline{H}) &= \cdots = \overline{\psi\phi} - \operatorname{Id}, \end{aligned}$$

as desired. $\square$

It follows that the transfer of structure across a homotopy equivalence produces a homotopy equivalent A_∞-module:

Corollary A.1. *If f and g are mutually inverse homotopy equivalences, then the A_∞-$\mathcal{A}$-module $(Q, \nu_\bullet)$ obtained from $(P, \mu_\bullet)$ by the homotopy transfer of structure along (f, g) is homotopy equivalent to $(P, \mu_\bullet)$.*

Proof. As part of the homotopy transfer of structure constructed in Theorem A.1, we have obtained a closed degree zero A_∞-morphism

$$\phi_\bullet\colon (P, \mu_\bullet) \to (Q, \nu_\bullet)$$

extending f, i.e. $\phi_1 = f$. Thus ϕ_1 is a homotopy equivalence of DG-$k_{\mathcal{A}}$-modules, and therefore $\phi_\bullet$ is a homotopy equivalence of A_∞-$\mathcal{A}$-modules [12, 2.4.1.1]. □

The method we used to prove Theorem A.1 can be easily applied to the notion of A_∞-algebras and A_∞-modules in a DG monoidal category introduced in §5:

Theorem A.2. *Let $(A, m_\bullet)$ be an A_∞-algebra in a monoidal DG category $\mathcal{A}$. Let $(a, p_\bullet) \in$ $\mathbf{Nod}_\infty$-A. Let b be an object of $\mathcal{A}$. Suppose there exist morphisms $f\colon a \to b$ and $g\colon b \to a$ in $\mathcal{A}$ such that $gf = \mathrm{Id} + dh$ for some degree -1 morphism $h\colon a \to a$.*

Then a homotopy transfer of A_∞-structure from $(a, p_\bullet)$ to b exists:

(1) *A structure $q_\bullet$ of an A_∞-A-module on b;*
(2) *A closed degree 0 map of A_∞-A-modules $\phi_\bullet : (a, p_\bullet) \to (b, q_\bullet)$ extending f;*
(3) *A closed degree 0 map of A_∞-A-modules $\psi_\bullet : (b, q_\bullet) \to (a, p_\bullet)$ extending g;*
(4) *A degree -1 map of A_∞-A-modules $H_\bullet\colon (a, p_\bullet) \to (a, p_\bullet)$ extending h with*

$$\psi_\bullet \circ \phi_\bullet = \mathrm{Id} + d(H_\bullet).$$

Proof. For A_∞-A-modules, the bar construction was defined in §5.1 as a DG-functor

$$B_\infty\colon \mathbf{Nod}_\infty(T) \to \text{Pre-Tr}^-(\mathcal{A}).$$

Take the data $p_\bullet$ of the A_∞-T-module structure on a and consider it as the data of a morphism of A_∞-A-modules with $p_1 = 0$. Let $\bar{p}$ be its bar construction:

$$\bar{p}\colon B_\infty(a, p_\bullet) \to B_\infty(a, p_\bullet).$$

Consider now two twisted complexes: $B_\infty(a, p_\bullet)$ and $B_\infty(a, 0)$. They have the same objects. Denote by $d_{\mathrm{nat}}(-)$ the operation of differentiating an endomorphism of $B_\infty(a, p_\bullet)$ as if it was an endomorphism of $B_\infty(a, 0)$. Since the differential on $B_\infty(a, p_\bullet)$ is the sum of the differential on $B_\infty(a, 0)$ and $\bar{p}$, we have

$$d_{\text{nat}}(\bar{p}) = -\bar{p}^2.$$

We can now proceed in the same way and with the same computations as in the proof of Theorem A.1, only with all bar constructions being twisted complexes instead of DG-comodules. □

Corollary A.2. *If f and g are mutually inverse homotopy equivalences in $\mathcal{A}$, then the A_∞-A-module $(b, q_\bullet)$ obtained from $(a, p_\bullet)$ by the homotopy transfer of structure along (f, g) is homotopy equivalent to $(a, p_\bullet)$.*

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