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"Subjectivity and correlation in randomized strategies": Back to the roots[☆]

Françoise Forges[a,*], Indrajit Ray[b]

[a] Université Paris-Dauphine, PSL University, LEDa, France
[b] Cardiff University, Cardiff Business School, United Kingdom

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ABSTRACT

The very first paper published in the *Journal of Mathematical Economics*, "Subjectivity and correlation in randomized strategies", by Aumann, proposes a new approach to strategic form games by taking account of an extraneous space of states of the world, on which every player has a subjective probability distribution and private information. We review some of Aumann's results as well as some properties and extensions of the best known by-product of his seminal paper, the "correlated equilibrium".

1. Introduction

Published fifty years ago, quoted in more than 2500 papers, "Subjectivity and correlation in randomized strategies" is best known for introducing the "correlated equilibrium", a solution concept for games in strategic form, which is by now part of many textbooks on game theory.[1] Aumann (1974)'s seminal article indeed demonstrates the power of correlated strategies on striking examples. However, the term "correlated equilibrium" is first used by Rosenthal (1974), whose motivation is to identify classes of games in which "correlated equilibrium improves [in some precise sense] on Nash equilibrium", a topic suggested by Aumann's examples.

The main goal of Aumann (1974) is rather to investigate the effect of allowing the players of an exogenously given game form (namely, strategies and outcome functions) to randomize over their decisions by means of subjective devices. Aumann writes: "Rather oddly, in spite of the long history of the theory of subjective probability, nobody seems to have examined the consequences of basing mixed strategies on 'subjective' random devices, i.e. devices on the probabilities of whose outcomes people may disagree (such as horse races, elections, etc.)".[2] To formalize these subjective devices, Aumann (1974) embeds the game form in *the* space of *all* states of the world. Every player has a preference on lotteries over the game outcomes, which, according to Savage (1954), can be represented by a utility function over the outcomes and a subjective probability over the states. It is only after describing this "equipment for [subjective] randomizing strategies" that Aumann identifies objective probabilities and correlated strategies as a particular case of the model.

In a second article dealing with correlated equilibrium, Aumann (1987) concentrates on the "objective version" of his model by making the assumption that the players share a *common prior* probability

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* Corresponding author at: Université Paris-Dauphine, PSL University, LEDa, France.
E-mail addresses: francoise.forges@dauphine.psl.eu (F. Forges), rayi1@cardiff.ac.uk (I. Ray).

[1] Just to mention a few: Myerson (1991), Osborne and Rubinstein (1994), Maschler et al. (2020), von Stengel (2022), Dehez (2024).

[2] The quoted sentence may look surprising since the discussion section of Aumann (1974) extensively refers to Harsanyi (1967-1968), where subjective probabilities already appear in a game-theoretical context. Our interpretation is that Aumann's statement reflects the originality of his primary goal, namely, an extension of mixed strategies in strategic form games.

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distribution over the states of the world. Aumann (1987) proposes a notion of Bayes-rational players, who maximize their individual expected utility given their private information on the states of the world. He provides correlated equilibrium with a decision-theoretic foundation (thus independent of Nash equilibrium) by establishing that the correlated equilibrium outcomes coincide with the outcomes that are achievable by players who commonly know that they are Bayes-rational. The key element of the proof is the "canonical representation" – a terminology used in Forges (1985) – of correlated equilibrium. Moulin and Vial (1978) already refer to this result as "a known one (and an easy one to check)". It can be viewed as a particular case of Myerson (1982)'s general "revelation principle" (see Forges et al. (2024)) but is not part of Aumann (1974)'s article.[3]

In Section 2, we describe a version of the basic framework sketched above, which, as Aumann (1987)'s model, is grounded in a strategic form (thus involving utility functions) rather than in a game form. In Section 2.1, we follow in the footsteps of Aumann (1974) by starting with one of his examples, in which a mediator can help the players to improve on Nash equilibrium by privately informing them on correlated events. We go on with Aumann (1987)'s definition of correlated equilibrium and the canonical representation mentioned above. In Section 2.2, we give a brief account of a result that does not deal with "correlated equilibrium" but is representative of the scope of Aumann (1974)'s basic framework, which, as emphasized above, involves a *single* probability space of states of the world. This result is central in some recent papers, e.g., dealing with the "purification" of mixed strategies. Section 2.3 deals with another topic that goes back to Aumann (1974) and has deserved attention recently, the question of improving on Nash equilibrium. As observed by Moulin and Vial (1978), this question leads to a relaxation of the (canonical) correlated equilibrium conditions, which characterizes the "coarse correlated equilibrium".

The Nash equilibria of a game in extensive form do not raise any conceptual problem: one just considers the equilibria of the associated strategic form. The same can obviously be done with correlated equilibria, but the basic model of Aumann (1974, 1987) offers many other opportunities. For instance, the players could get private information on extraneous events gradually, along the course of the game. In Section 3, we briefly review some possible extensions of correlated equilibrium in *multistage games*. Every stage of such a game starts with a move of nature. Then the players receive some information on the past moves (of nature and other players) and make simultaneous choices. This relatively simple model enables us to discuss two offsprings of Aumann's correlated equilibrium: the "autonomous correlated equilibrium" (Forges, 1986) and the "communication equilibrium" (Myerson, 1986b). The first one (defined in Section 3.1) fulfills the requirement of being independent of the underlying exogenously given game; the second one (defined in Section 3.2) is inspired by the relationship between correlated equilibrium and generalized mechanism design (Myerson, 1982).

The previous solution concepts do not exhaust the possible definitions of correlated equilibrium beyond strategic form games. For instance, the "extensive form correlated equilibrium" of von Stengel and Forges (2008) is tailored to Kuhn (1953)'s model. In the latter, as in the multistage games above, players may have "imperfect" information, in the sense that, when they have to choose an action, they may not know all preceding moves. However, these models assume that players have common knowledge of the rules of the game. The terminology "incomplete" information is often used to qualify situations in which this common knowledge assumption does not hold. Thanks to Harsanyi (1967-1968)'s breakthrough, interactive decision problems with incomplete information are, in practice, analyzed as Bayesian games, namely, games with imperfect information in which a historical move of nature accounts for the players' "type", namely, their private initial information on their ensuing interaction. As we explain in Section 3.3, the combination of the fundamental insights of Harsanyi (1967-1968) and Aumann (1974) gives rise to a variety of legitimate approaches that deserve a survey of their own (see, e.g., Bergemann and Morris (2019) and Forges (2023)).

After reviewing the definition of correlated equilibrium, its immediate properties and some of its extensions, we turn, in Section 4, to the relevance of the solution concept as assessed by some recent laboratory experiments. The first reason to choose this topic among many other possible ones is that it is currently very active. Furthermore, it can easily be handled without adding any technicalities to Aumann (1974)'s basic framework. A main message of Section 4 is that, in experiments in which subjects are invited to follow the recommendations of a mediator, a vast majority of them actually do. However, it appears to be hard to disentangle various possible rationalizations of the subjects' behavior.

Section 5 concludes with a far from exhaustive list of topics that are not covered in our short and focused paper.

2. A basic framework for strategic interaction

In this section, we recall Aumann (1974)'s basic framework, in which an exogenous game *in strategic form* is combined with a space of states of the world.

2.1. Correlation

Before detailing his formal model, Aumann (1974) develops a number of examples to illustrate the impact of subjectivity and correlation. One of them is the following two-person game:

Example 1.

$$\begin{pmatrix} (6,6) & (0,0) & (2,7) \\ (0,0) & (4,4) & (3,0) \\ (7,2) & (0,3) & (0,0) \end{pmatrix}$$

This game has three Nash equilibria: a pure one with payoff (4,4) and two mixed ones with payoff $(\frac{14}{3}, \frac{14}{3})$ and $(\frac{56}{23}, \frac{56}{23})$, respectively. Aumann (1974) modifies the game by adding a "correlation device" that chooses one of three "outputs" A, B, C with probability $\frac{1}{3}$ each. After the output has been chosen, player 1 is just told whether or not A was chosen, and player 2 is just told whether or not C was chosen. In the extended game involving the correlation device, a possible strategy for player 1 is to play his bottom action when he is informed that A was chosen and his top action otherwise. Similarly, a possible strategy for player 2 is to play his right action when he is informed that C was chosen and his left action otherwise. These strategies are mutual best responses and yield the expected payoff of (5,5) – a payoff that is higher than the best Nash equilibrium payoff (and is in particular outside the convex hull of Nash equilibrium payoffs).[4]

The previous example illustrates a typical correlated equilibrium, namely, a Nash equilibrium of the game extended by means of some correlation device. Aumann (1974, 1987)'s formal model involves an exogenously given finite game G in strategic form, namely,

- n players ($i = 1, \dots, n$),
- for every player i, a finite set A_i of actions. Let $A = \prod_i A_i$ be the set of all action profiles,
- for every player i, a utility function $u_i : A \to \mathbb{R}$.

[3] Aumann (1987) refers to a conversation with L. Shapley, which "clarified for him the equivalence between the two definitions of correlated equilibrium".

[4] See Moulin and Vial (1978) for a simpler example in which there is a unique mixed Nash equilibrium and a correlated equilibrium with a higher payoff for both players.

The model is completed with an extraneous information structure $\langle \Omega, (\mathcal{P}_1, \ldots, \mathcal{P}_n), (\Pi_1 \ldots, \Pi_n)\rangle$, namely, a set of states of the world Ω and, for every player i, an information partition $\mathcal{P}_i$ and a subjective probability distribution Π_i over Ω.[5] As a particular case, there may be an objective probability distribution over Ω such that $\Pi_1 = \cdots = \Pi_n = \Pi$, namely, players may have a common prior. The information structure is then denoted as $\langle \Omega, (\mathcal{P}_1, \ldots, \mathcal{P}_n), \Pi \rangle$.

Given such an extraneous information structure, Aumann (1974) *implicitly* defines a correlated equilibrium as a Nash equilibrium in pure strategies of the following extension of the game G:

- a state of the world ω is chosen in Ω according to Π;
- every player i is informed of the cell of $\mathcal{P}_i$ that contains ω;
- every player i chooses an action a_i in A_i;
- every player i receives the utility $u_i(a)$, where $a = (a_i)$.

A pure strategy for player i in this game is a mapping $\sigma_i : \Omega \to A_i$ that is constant over every cell of $\mathcal{P}_i$. A correlated equilibrium is a pure strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$, in which for every i, σ_i is a best response to σ_{-i}.

To describe the canonical representation of correlated equilibrium, let us first go back to the example above. The correlation device and the strategies in the extended game induce the following distribution over the pairs of actions:

$$\begin{pmatrix} 1/3 & 0 & 1/3 \\ 0 & 0 & 0 \\ 1/3 & 0 & 0 \end{pmatrix}$$

Assume that this distribution is used by a mediator to make private recommendations to the players, i.e., that a mediator selects one of the pairs (top, left), (top, right) or (bottom, left) with equal probability and only tells the first (resp., second) component to the row (resp., column) player. Then none of the players can benefit from deviating unilaterally from the recommendation, namely, the obedient strategies form a Nash equilibrium of the game involving the mediator.

The previous construction is fully general. Let us define a canonical correlation device for the strategic form game G as a probability distribution p over the set A of action profiles and consider the extended game G_p in which a mediator first selects $a = (a_1, \ldots, a_n) \in A$ according to p and then privately recommends a_i to player i. The probability distribution p defines a *canonical correlated equilibrium* if the obedient strategies (in which every player i chooses the recommended action a_i) form a Nash equilibrium of G_p. Equivalently, p defines a canonical correlated equilibrium if it satisfies the following linear inequalities, for every i, a_i, b_i,

$$\sum_{a_{-i} \in A_{-i}} p(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} p(a_i, a_{-i}) u_i(b_i, a_{-i}). \quad (1)$$

We can now state the *canonical representation of correlated equilibria*, which is referred to as the "revelation principle for strategic form games" in Myerson (1991), Section 6.2 (see also Section 3 of this paper):

The set of all correlated equilibrium outcomes – namely, the set of all probability distributions over A that are induced by some correlated equilibrium (defined by an extraneous information structure $\langle \Omega, (\mathcal{P}_1, \ldots, \mathcal{P}_n), \Pi \rangle$ and a strategy profile σ) – is equal to the set of canonical correlated equilibria.

Nash equilibrium has been used above to define correlated equilibrium. As already mentioned in the introduction, proceeding as to establish the canonical representation, Aumann (1987) gives an equivalent definition, which relies on common knowledge among the players of individual maximization of expected utility by each of them. This result strengthens Aumann (1974)'s original approach, in which no mediator explicitly appears. Bach and Perea (2020) point out that, although equivalent in terms of (ex ante) outcomes, a correlated equilibrium (defined with respect to some information structure) and its canonical counterpart may not entail the same beliefs and optimal actions (given these beliefs) at the interim stage.

A consequence of the canonical representation, together with inequalities (1), is that the set of correlated equilibrium outcomes in a convex polytope, which makes it quite different from the set of Nash equilibrium outcomes. Hart and Schmeidler (1989) and Nau and McCardle (1990) propose elementary proofs of the existence of a correlated equilibrium, based on linear duality. These results are first steps in the understanding of the geometric structure of correlated equilibria (see Nau et al. (2004), Viossat (2010) and the references therein).

2.2. Subjectivity and randomized strategies

As presented in Section 2.1, a correlated equilibrium of the game G consists of *some* Nash equilibrium in *pure strategies* of the extended game obtained by adding *some* extraneous information structure $\langle \Omega, (\mathcal{P}_1, \ldots, \mathcal{P}_n), \Pi \rangle$ to G. It follows that every *mixed* Nash equilibrium of G induces a correlated equilibrium of G, which guarantees the existence of the latter. Alternatively, every Nash equilibrium outcome of G is a canonical correlated equilibrium.

Aumann (1974) does not proceed in this way. As pointed out above, to represent the players' information on "all states of the world", he fixes a *single* information structure $\langle \Omega, (\mathcal{P}_1, \ldots, \mathcal{P}_n), (\Pi_1 \ldots, \Pi_n)\rangle$, with a *subjective* probability Π_i for every player i, $i = 1, \ldots, n$. He then introduces two (non-exclusive) notions. An *i-secret event* is an event that is observable by player i and is independent of the other players' (pooled) information. An *objective event* is an event E such that $\Pi_1(E) = \cdots = \Pi_n(E)$. Using these notions, an "objective mixed strategy" for player i – in the game associated with the single information structure – is defined as a (pure) strategy that bases action choices on observed events that are both i-secret and objective. Using the previous definition, Aumann (1974)'s Proposition 4.3 characterizes the Nash equilibrium payoffs of the original game G as follows:

The set of [mixed] Nash equilibrium payoffs [of G] coincides with the set of "objective mixed equilibrium" payoffs.

A consequence of the above result is that for every game G, the *mixed* Nash equilibrium payoffs of G can be obtained as the *pure* Nash equilibrium payoffs of *the* extension of G that is induced by the *given* underlying information structure. This clearly calls for an assumption expressing that the information structure is sufficiently "rich".[6]

The purification of Nash equilibria is a natural application of Aumann (1974)'s Proposition 4.3, as shown by Greinecker and Podczeck (2015). The scope of the proposition is enlarged by Khan and Zhang (2018) (to Bayesian games with type-independent payoffs) and Greinecker (2023) (to games with a continuum of actions).

Let us keep, as in the previous paragraphs, an information structure $\langle \Omega, (\mathcal{P}_1, \ldots, \mathcal{P}_n), (\Pi_1 \ldots, \Pi_n)\rangle$ with subjective prior probability distributions.[7] Strategies can be defined exactly as in Section 2.1. To define a *subjective* correlated equilibrium, let every player i compute his best response to the other players' strategies by using his subjective prior probability Π_i. The *a posteriori* equilibrium is a straightforward

[5] Our formulation is appropriate if we assume, as in Aumann (1987), that Ω is finite. This restriction is not made in Aumann (1974), for reasons that will be clarified in Section 2.2.

[6] Aumann (1974) formalizes "richness" as the existence, for every player i, of a σ-field $\mathcal{R}_i$ of i-secret events such that every Π_j ($j \neq i$) is non-atomic on $\mathcal{R}_i$. To be consistent with this assumption, player i's information should also be described by a σ-field, rather than by a partition. Aumann (1974) notes that Savage (1954)'s model, which is fundamental to subjective probabilities and utility functions, also makes non-atomicity assumptions.

[7] In what follows, it does not matter whether a single information structure is fixed or not.

refinement of this concept, already proposed by Aumann (1974).[8] Brandenburger and Dekel (1987) show that a posteriori equilibrium is equivalent to correlated rationalizability. Since the latter game theoretic solution concept is based on common knowledge of individual rationality, Brandenburger and Dekel (1987)'s motivation is similar to Aumann (1987)'s one, in which, by contrast, the players' common prior is the main assumption.

2.3. Improving on Nash equilibrium

Example 1 features a correlated equilibrium in which both players obtain a higher payoff than in any Nash equilibrium. Aumann (1974) makes it clear that, in two-person zero-sum games, this phenomenon cannot arise. Moulin and Vial (1978) identify a further class of games in which the Nash equilibrium payoffs cannot be improved upon; they refer to this class as "strategically zero-sum games". Subsequently, Liu (1996) analyzes a Cournot oligopoly with linear demand in which each firm has a constant marginal cost. He proves that the only correlated equilibrium of the induced game is the unique Cournot-Nash equilibrium. Neyman (1997) generalizes Liu's result in a large class of "potential games" (Monderer and Shapley, 1996).[9]

The previous "failure" of correlated equilibrium in benchmark economic models (as a way to improve over Nash equilibrium payoffs) motivates a different use of Aumann's correlation schemes, already initiated by Gérard-Varet and Moulin (1978) and Moulin and Vial (1978), who propose the next example.

Example 2.

$$\begin{pmatrix} (3,3) & (1,1) & (4,1) \\ (1,4) & (7,2) & (0,0) \\ (1,1) & (0,0) & (2,7) \end{pmatrix}$$

This game (which we denote as G) can be solved by iterative elimination of strictly dominated strategies, leading player 1 to choose his top action and player 2 to choose his left action. This is the unique correlated (and thus Nash) equilibrium of the game. The associated payoff is $(3,3)$. Let us consider the following specific probability distribution over actions:

$$p = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

If a mediator tried to use the distribution p to privately recommend an action to every player (as he would in a canonical correlated equilibrium, see Section 2.1), his advice would not be followed. Indeed, the distribution p does not induce a correlated equilibrium of the game G. Consider however the extended game G'_p in which every player is offered to *commit* to the correlation device p (in which case a pair of actions is selected according to p and implemented in G) or to choose an action by himself in G. Let us check that committing to p is the best response of every player when the other player is committing to p as well. If player 2 commits to p in G, player 1 gets an expected payoff of 4 $(=\frac{1}{3}(3+7+2))$ from committing to p. Should player 1 instead choose his action by himself, he would get a lower expected payoff (namely, $\frac{8}{3}$ $(=\frac{1}{3}(3+1+4))$ from his top action, $\frac{8}{3}$ $(=\frac{1}{3}(1+7+0))$ from his middle action and 1 $(=\frac{1}{3}(1+0+2))$ from his bottom action).

[8] A posteriori equilibrium strengthens the (interim) best response condition by requiring optimality even on null information cells. In the case of objective correlated equilibrium this strengthening makes no difference.

[9] See Viossat (2008) for more on the uniqueness of correlated equilibrium.

The previous example illustrates an equilibrium concept proposed by Moulin and Vial (1978), which is now referred to as "coarse correlated equilibrium".[10] To give a formal definition, let us consider, as in Section 2.1, a strategic form game G and a probability distribution p over the set A of action profiles, which can be interpreted as a correlation device. Let G'_p be the extended game in which every player i can either commit to p or choose an action in A_i. The probability distribution p is a *coarse correlated equilibrium* if committing to p, for every player, is a Nash equilibrium of G'_p.

Equivalently, denoting as p'_i the marginal probability distribution induced by p over A_{-i}, namely, $p'_i(a_{-i}) = \sum_{a_i \in A_i} p(a_i, a_{-i})$, p defines a coarse correlated equilibrium if it satisfies the following linear inequalities, for every i, b_i,

$$\sum_{a\in A} p(a)u_i(a) \geq \sum_{a_{-i}\in A_{-i}} p'_i(a_{-i})u_i(b_i, a_{-i}). \tag{2}$$

In these linear inequalities, the left-hand-side is the expected payoff of player i, when the correlation device is accepted by all players. Note that this is the payoff of the obedient strategy profile if p is a canonical correlated equilibrium, namely, satisfies inequalities (1). More precisely, for any player i and any action b_i of this player, the inequalities (2) can be obtained by summing inequalities (1) over $a_i \in A_i$, which confirms that the set of coarse correlated equilibria is indeed larger than the set of (canonical) correlated equilibria. It is also easy to prove that the set of correlated and coarse correlated equilibria coincide in n−person games where every player has two actions.

Unlike correlated equilibrium, coarse correlated equilibrium requires some commitment from the players, but yet can easily implement outcomes as soon players can be organized to follow the rules (i.e., if and when they choose to accept the correlation device), with the hope of reaching some form of efficiency (see Ray (1996b)). Building on previous work of Ray and Sen Gupta (2013), Moulin et al. (2014) consider an economically relevant class of two-person symmetric games, including duopoly and public good provision games, in which, as suggested above, correlated equilibrium does not offer anything more than Nash equilibrium but coarse correlated equilibria provide substantial improvements.

3. Extending the framework to multistage games

In this section, we describe some extensions of Aumann (1974)'s correlated equilibrium that are appropriate when the players' interaction involves sequential moves. To keep the model simple, we fix a multistage game G^T (as defined in, e.g., Forges (1986), Myerson (1986b) and Sugaya and Wolitzky (2021)) played by n players over T periods.

Let S_i^t and A_i^t $(i = 1,\dots,n,\ t = 1,\dots,T)$ be finite sets. The set S_i^t contains the new signals that player i can receive at stage t; the set A_i^t is the set of actions of player i at stage t. Let $S^t = \prod_i S_i^t$, $S = \prod_t S^t$, $A^t = \prod_i A_i^t$ and $A = \prod_t A^t$. Every period $t = 1,\dots,T$, of G^T starts with a move of nature. Then every player i receives a signal in S_i^t, which captures the new information that player i gets at stage t (i.e., in addition to the information he already had at stage $t-1$); this information is about the past moves of nature as well as of the other players (allowing for some possible delay). Finally, every player i chooses an action $a_i^t \in A_i^t$. The players receive their signals simultaneously and then make their

[10] Moulin and Vial (1978) only refer to a "simple extension of correlation schemes *à la* Aumann". The solution concept is then used under the name "weak correlated equilibrium", see, e.g., Nowak (1988). The terminology "coarse correlated equilibrium" appears in Young (2004)'s book, after the concept has been rediscovered as the "Hannan set" (after Hannan (1957)) in the context of no regret dynamics (see Hart and Mas-Colell (2013)). Of course, Hannan's motivation was not to relax the correlated equilibrium conditions.

moves simultaneously; they do have perfect recall. The description of the game is completed by utility functions

$u_i : S \times A \to \mathbb{R}$.

A pure strategy for player i in G^T is a sequence of mappings $\sigma_i^t : H_i^t \to A_i^t$, where $H_i^t = \prod_{r=1}^{t} S_i^r$. Denoting as Σ_i^t the set of all such mappings, $\Sigma = \prod_{r=1}^{T} \Sigma_i^r$ is the set of pure strategies of player i in G^T.

3.1. Autonomous correlated equilibrium

If the rules of the multistage game G^T must be strictly followed, as in a board game, no mediator can intervene after the beginning of the game and we can just apply Aumann (1974)'s approach to the strategic form of G^T, in which every player i chooses a strategy in Σ. The corresponding solution concept is the **strategic form correlated equilibrium**. In canonical form, every player i follows a recommendation in Σ_i telling him, before the beginning of G^T, how to play the whole game.

Even if the rules of the game G^T are interpreted in the literal sense, it may be difficult to prevent the players from privately observing extraneous events as the game goes on. To formalize this possibility, let us define an **autonomous correlation device** (Forges, 1986, 1988) for G^T as a set of outputs O_i^t for every player i at (the beginning of) every period t ($i = 1, \dots, n$, $t = 1, \dots, T$), together with transition probability distributions Π^t to choose the outputs at period t, in $O^t = \prod_i O_i^t$, as a function of past outputs (in $\prod_{r=1}^{t-1} O^r$ for $t \geq 2$). Let us define an **autonomous correlated equilibrium** as a Nash equilibrium of the game obtained by adding some autonomous correlated device to G^T.[11]

A canonical representation applies to autonomous correlated equilibria, in a similar way as in the strategic form case. More precisely, in a *canonical* autonomous correlation device, the sets of outputs are $O_i^t = \Sigma_i^t$ ($i = 1, \dots, n$, $t = 1, \dots, T$). We can think of a mediator sending, at (the beginning of) every period t, a recommendation to every player i on how to play *at period* t, as a function of the history available to this player. A canonical autonomous correlated equilibrium is described by *obedience conditions*, expressing that every player prefers to follow the mediator's recommendation *at every stage* as long as the other players do so. The canonical representation of autonomous correlated equilibria states that the set of all autonomous correlated equilibrium outcomes is equal to the set of all *canonical* autonomous correlated equilibrium outcomes.

In the previous canonical representation, the obedience conditions take the convenient form of linear inequalities. Exactly as in the strategic form case, a mediator emerges as a byproduct of the representation. But it is important to keep in mind that, as the strategic form correlated equilibrium, the autonomous correlated equilibrium can be defined without appealing to any mediator, by adding a set Ω of extraneous states of the world to the multistage game G^T and sequences $\mathcal{P}_i^t$ of partitions of Ω describing player i's information on the state at stage t, such that $\mathcal{P}_i^{t+1}$ is finer than $\mathcal{P}_i^t$.

The next example (from Myerson (1986b)) illustrates that the set of autonomous correlated equilibrium outcomes is larger than the set of strategic form correlated equilibrium outcomes.

Example 3. Consider the following two-stage game between two players:

- At stage 1, player 1 chooses between two actions, "out" or "in". If he chooses "out", the game ends, both players get 3.
- At stage 2, if player 1 has chosen "in" at the first stage, both players make simultaneous choices; the utilities are determined by the following matrix (battle of the sexes):

$$\begin{pmatrix} (0,0) & (2,7) \\ (7,2) & (0,0) \end{pmatrix}$$

In the strategic form of this game, player 1's pure strategy "in" at stage 1 and "top" at stage 2 is strictly dominated. Hence, player 2's payoff, at any strategic form correlated equilibrium, cannot exceed 3. An autonomous correlated equilibrium with expected payoff (4.5,4.5) can be achieved with the help of an autonomous correlation device that, between stage 1 and stage 2, tosses a fair coin and reveals the outcome to both players.

Strategic form and autonomous correlated equilibria have been applied to a variety of multistage games with imperfect information, possibly with an infinite horizon. Just to mention a few examples: two-person repeated games with a single informed player (Forges, 1988), repeated games with imperfect monitoring (Lehrer, 1992), stochastic games (Solan, 2001), (Solan and Vieille, 2002), games with public information (Heller et al., 2012).

3.2. Communication equilibrium

Myerson (1986b) (see also Forges (1986) and Myerson (1991)) proposes an extension of correlated equilibrium for multistage games that is inspired by mechanism design: the **communication equilibrium**. The very definition of this solution concept makes use of a mediator who acts as an *intermediary* among players. More precisely, a **communication device** for G^T consists of a set of inputs I_i^t from every player i at every period t, a set of outputs O_i^t to every player i at every period t ($i = 1, \dots, n$, $t = 1, \dots, T$) together with transition probability distributions Π^t to choose the outputs at period t, in $O^t = \prod_i O_i^t$, as a function of past and current inputs, in $\prod_{r=1}^{t} I^r$, and past outputs, in $\prod_{r=1}^{t-1} O^r$ (if $t \geq 2$). A communication equilibrium can then be defined as a Nash equilibrium of the game obtained by adding some communication device to G^T.

In a *canonical* communication device, $I_i^t = S_i^t$, $O_i^t = A_i^t$ ($i = 1, \dots, n$, $t = 1, \dots, T$). Given such a canonical device, at every period t, every player i is invited to report his information in S_i^t to the mediator who then recommends him an action in A_i^t using the probability distribution Π^t. A canonical strategy for player i involves not only to be *obedient* at every period t, as in the previous solution concepts, but also to be *sincere*, a familiar incentive compatibility condition in mechanism design. The canonical representation of communication equilibria states that the set of all communication equilibrium outcomes is equal to the set of all *canonical* communication equilibrium outcomes.[12]

Communication equilibria have been applied to various classes of games, such as repeated games with initial private information (Forges, 1985, 1988) or repeated games with imperfect monitoring (Renault and Tomala, 2004). It is clear that the set of communication equilibrium outcomes of a given multistage game contains its set of autonomous correlated outcomes, which in turn contains its set of strategic form correlated outcomes. A family of results deals with the possible outcome-equivalence of these three solution concepts. Such results can be motivated by the weak implementation of communication equilibrium outcomes by cheap talk. Indeed, a reasonable first step is to make players achieve the coordination effects of a communication equilibrium without revealing any information to a mediator, just by relying on extraneous outputs and exchanging direct messages (see Forges (2009) for a survey).

[11] Autonomous correlated equilibria are referred to as "extensive form correlated equilibria" in, e.g., Forges (1986) and Solan (2001), but starting with von Stengel and Forges (2008), the latter terminology is used for another solution concept (see Section 3.3).

[12] When the multistage game takes the form of a – single stage – Bayesian game with a common prior, the canonical representation of communication equilibrium is already established as a general revelation principle in Myerson (1982), see Forges et al. (2024).

3.3. Some other extensions of correlated equilibrium

In contrast with the solution concepts surveyed in Sections 3.1 and 3.2, the *extensive form correlated equilibrium* (von Stengel and Forges, 2008) is specifically adapted to extensive form games *à la* Kuhn (1953). The associated scenario is that a mediator selects a strategy profile (i.e., in Σ) before the beginning of the game (as in a strategic form correlated equilibrium), but every player only learns his recommended move *at the relevant information set*. More precisely, no input from any player is necessary, but at stage t, player i receives a recommendation that depends on his precise information in H_i^t. While, to the best of our knowledge, the extensive form correlated equilibrium still lacks firm foundations, recent research in computer science shows that it has remarkable properties with respect to complexity and learning (see, e.g., Farina et al. (2022) and Anagnostides et al. (2022)).

The extension of correlated equilibrium to Bayesian games and, more generally, to games with incomplete information, is delicate and beyond the scope of this paper. As recalled in the introduction, Harsanyi (1967-1968) proposed to summarize the initial private information of every player into his "type". At a Bayesian equilibrium, every player holds a belief over the other players' types and maximizes his expected utility with respect to his belief, given his own type. If the players' beliefs are generated by a common prior and the utilities are *type-independent*, the model is similar to the one of Section 2.1, thus making correlated equilibrium a particular case of Bayesian equilibrium. Bach and Perea (2017) elaborate on this by proposing an epistemic model in which Bayesian equilibrium and correlated equilibrium are behaviorally equivalent.

When utilities are *type-dependent*, there are basically two approaches to extend Aumann (1974)'s solution concept. The first one consists of constructing a Bayesian game according to Harsanyi (1967-1968)'s methodology and then to apply variants of the correlated equilibrium like the ones of Sections 3.1 and 3.2 (see Forges (1993, 2006)). In this approach (followed, e.g., in Myerson (1991)), a Bayesian game is just handled as a particular multistage game. The other approach consists of foregoing the fragile distinction between *intrinsic* information (generating types) and *extraneous* information (generating correlation of actions). Indeed Aumann's space of all states of the world looks rich enough to capture both. To formalize the second approach, Bergemann and Morris (2016, 2019) distinguish a state-dependent interactive decision problem and an information structure representing the players' private information. This information structure may be the implicit result of players' hierarchies of beliefs, the explicit result of a designer's intervention, etc. They take this flexibility of interpretation into account by considering "robust solutions", namely, the set *all* Nash equilibrium outcomes that can arise from *some* possible private information structure. To characterize this set in a synthetic way, they introduce yet another extension of Aumann (1974)'s solution concept, the "Bayes correlated equilibrium", which we describe below.

Imagine that an *omniscient* mediator, who knows the state in the initial interactive decision problem, privately recommends an action to every player. The corresponding canonical strategy of every player is to obey the mediator's recommendation. If these strategies are best responses to each other, they form, together with the mediator's state dependent correlation device, a canonical "Bayes correlated equilibrium". While this looks like the most permissive extension of Aumann (1974)'s correlated equilibrium, it preserves the following typical feature: the set of outcomes is canonically described by "obedience conditions". The literal interpretation of these conditions is of course that a mediator makes recommendations to the players. However, as emphasized by Bergemann and Morris (2019), in a Bayes correlated equilibrium, the mediator can be viewed as "metaphoric", because his information may stand as a substitute for any information that the players might possibly have.[13]

Makris and Renou (2023) extend the notion of Bayes correlated equilibrium to multistage games, by identifying a "base game", which takes the form of a multistage game (as the game G^T described above), and an "expansion" of the game, in which the players get "additional signals". The resulting solution concepts sharply differ from the ones considered in Sections 3.1 and 3.2, which can be implemented with the help of a mediator who does not *directly* observe the information of the players (in S_i^t) or their chosen actions (in A_i^t.)

4. Experiments on correlation

As we explained in Sections 2.1 and 2.3, the notions of correlated equilibrium and coarse correlated equilibrium can be interpreted by introducing a mediator who every player finds optimal to obey while playing a given game. For correlated equilibrium, players play the recommended actions chosen from a correlation device, while for coarse correlated equilibrium, players accept the correlation device itself.

A fairly well-established literature aims at testing the validity of these two solutions concepts within behavioral context. A number of laboratory experiments are designed to test whether players follow a mediator's advice in a strategic form game. This experimental literature on correlation complements other strands of experiments, such as, coordination on a particular equilibrium in a game, cheap talk prior to a game, observing sunspots before playing a game and so on. For example, in strategic form games, any convex combination (public lottery) over pure Nash equilibrium outcomes can be viewed as a correlated equilibrium. Thus, combining the results from the literature on coordination and correlation, one may easily conjecture that in symmetric 2 x 2 games like Battle of the Sexes, individuals can avoid coordination-failure by following a correlation device that randomly selects one of the two pure Nash equilibria. Indeed, such a conjecture has been confirmed as a result by Cason and Sharma (2007) and Duffy and Feltovich (2010) on correlation, and in other relevant papers on coordination.

The literature on correlation starts with an analysis, by Moreno and Wooders (1998), of a three-player version of a one-shot matching pennies game, in which two of the players have perfectly aligned interests. They allow their subjects to participate in a round of communication in which each player can freely send any text message to either one other player individually or to both publicly, prior to the start of the game. Their analysis supports the hypothesis that the observed distribution of outcomes is generated by the play of the coalition-proof correlated equilibrium (Moreno and Wooders (1996), see also Section 5 below) of the game, while it clearly rejects the hypothesis of Nash equilibrium play.

The subsequent literature on correlation allows recommendations from a mediator, using a correlation device. This literature mainly uses simple 2 x 2 games that have structures similar to Battle of the Sexes (BoS), Chicken or Hawk and Dove type games to test the robustness of following recommendations using different types of correlation devices. The question is then whether or not players follow recommendations of playing given actions in a game. In laboratory experiments, this question possibly leads to the difficulty of interpreting why players do not follow recommendations. For example, the subjects may not understand the "conditional probabilities" involved in making inference from recommendations or they may not use expected payoffs or they may not believe the player they are matched with is "rational". Thus, observations of failure to follow recommendations give rise to debatable theoretical or behavioral justifications. Fortunately for our theory, Cason and Sharma (2007) and Duffy and Feltovich (2010)

[13] The mediator is easily interpreted in a metaphoric way in the case of strategic form and autonomous correlated equilibria. By contrast, the mediator has an active role in the *literal definition* of communication equilibrium. Nevertheless, as mentioned in Section 3.2 above, under appropriate assumptions, communication equilibrium *outcomes* can be achieved without the help a mediator.

conclude that correlated equilibrium passes the test. The headline message of these papers is that the subjects do follow recommendations from a correlation device, when the device is indeed a correlated equilibrium and that the subjects learn to ignore recommendations from devices that are not correlated equilibria. These recommendations are respectively called the "good" and the "bad" recommendations by Duffy and Feltovich (2010). In their experiment with a Chicken game, they observe that about 75% of good and only 53% of bad recommendations are followed.

Bone et al. (2013), subsequently replaced by Bone et al. (2024a,b), first provide specific criteria to be considered while choosing the payoffs in such games, so that one can compare the individuals' behavior in different treatments (games and correlation devices). They provide full explanation of the required criteria and thereby completely characterize the payoffs in their games based on these theoretical criteria.

Bone et al. (2024a) consider the following parametric version of Chicken, as in Kar et al. (2010), where $a < b < c < d$:

$$\begin{pmatrix} (a,a) & (d,b) \\ (b,d) & (c,c) \end{pmatrix}$$

Henceforth, the first action ("top" for player 1, "left" for player 2) is denoted as X and similarly, the second action ("bottom" for player 1, "right" for player 2) is denoted as Y. The game has two pure Nash equilibria, namely, (X,Y) and (Y,X), and a mixed Nash equilibrium in which each player plays X with probability $\frac{(d-c)}{(d-c)+(b-a)}$.

Bone et al. (2024a) choose two correlation devices, a *public* and a *private* one. The (canonical) *public* device is described below. For every player, given the recommendation he receives, the conditional probability over the other player's recommendation is always 1.

$$\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

Clearly, this correlation device is a canonical correlated equilibrium for every parametric version of Chicken, as it is a convex combination of two pure Nash equilibria, (X,Y) and (Y,X). Bone et al. (2024a)'s second device is shown below :

$$\begin{pmatrix} 0 & 1/3 \\ 1/3 & 1/3 \end{pmatrix}$$

With this canonical correlation device, if player 1 (resp., player 2) is recommended to play Y, the conditional probability that player 2 (resp., player 1) receives a recommendation of X (or Y) is $1/2$.

For a parametric version of Chicken as above, one can characterize the canonical correlated equilibria that maximize the sum of the expected payoffs, often called *utilitarian*. Consider the following direct symmetric correlation device, with $0 < p < \frac{1}{2}$:

$$\begin{pmatrix} 0 & p \\ p & 1-p \end{pmatrix}$$

Under $b + d < 2c$, the utilitarian correlated equilibrium of Chicken is characterized by a device with $p = \frac{(d-c)}{(b-a)+2(d-c)}$. With $b + d = 2c$, however, there are several utilitarian correlated equilibria; any convex combination of two pure NE and any device as above with $\frac{(d-c)}{(b-a)+2(d-c)} \leq p < \frac{1}{2}$ maximizes the sum of expected payoffs.

In their experiment, Bone et al. (2024a) actually use the version of Chicken shown below, in which $a = 2$, $b = 11$, $c = 14$ and $d = 17$, with $b + d = 2c$:

$$\begin{pmatrix} (2,2) & (17,11) \\ (11,17) & (14,14) \end{pmatrix}$$

The pure Nash equilibrium payoffs in this game are (17,11) and (11,17). The mixed Nash equilibrium is $(1/4, 3/4)$, which is different from a "naive" $1/2-1/2$ mix. Moreover, both the public and the private devices as described above are utilitarian for this game; they generate the same expected payoffs, namely, $(14,14)$, which is higher than the mixed strategy Nash equilibrium payoff, $(53/4, 53/4)$.

For the specific choice of these parameters, a couple of other desirable criteria are met. The inequalities (1) expressing that the private device is a correlated equilibrium (see Section 2.1 above) are strict. Also, the conditional expected gains in payoffs from following a recommendation from the private device are the same, for both possible recommendations, X and Y.

Once the previous theoretical criteria are incorporated, one can safely formulate the null hypotheses according to which recommendations should be followed in the same way, whether the public or the private device is used and whether X or Y is recommended. However, Bone et al. (2024a) still observe that players follow recommendation Y significantly more than X (75% as opposed to 65%) in the private device.

Bone et al. (2024b) consider two other BoS type games by changing the values of c, with $c < b$, as the payoffs from the outcome (Y,Y). They use recommendations from the public device and observe significant differences in frequencies of the outcome (Y,Y) in various games, a finding that suggests that the players try to achieve the "cooperative" outcome when they do not follow recommendations.

Duffy et al. (2017) consider different treatments to study coordination using perfectly correlated signals in a version of BoS. They also ask how, if at all, players use a coded language that is not directly related to actions in the game to achieve coordination. They use recommendations from a public device as described here and vary treatments with players being either randomly matched or in a fixed match. They find that the number of subjects who coordinate by following recommendations from the public device is significantly larger when the recommendations are canonical (actions) than when they are indirect (codes), especially when the subjects are randomly matched to play the game.

Anbarci et al. (2018) provide a design to test how correlated equilibrium performs in BoS type games with different sets of payoffs (of the two NE outcomes of BoS). They also use the public device described here to send recommendations. Their main result is that when subjects receive recommendations, they are less likely to follow them as payoff asymmetry increases or as the "cost" of not following decreases.

Georgalos et al. (2020) analyze the notion of coarse correlated equilibrium in an experiment using several games very similar to the one in Example 2 in Section 2.3 with a unique correlated equilibrium, achieved by iterative elimination of strictly dominated strategies. They use a correlation device similar to the one shown in Example 2. They observe that the subjects do not commit to this (coarse) correlation device; less than 10% of the pairs (both under randomly matched and fixed matched) accepted the device. However, a large proportion of the subjects (about 70% of them) do accept a lottery, mimicking the device, as an individual choice. Georgalos et al. (2020) also find that, when the subjects do not accept the coarse correlation device, a huge proportion of them (77%) play the game and choose the Nash equilibrium. Furthermore, the proportion of the Nash equilibrium outcome is the highest in the treatment without any correlation device. In this set-up, a possible reason for not accepting the (coarse) correlation device is that the device involves not "winning" in two out of three possible outcomes.

Most of the experiments described so far involve a mediator, which is consistent with the theoretical model developed in Section 2. By contrast, some recent papers (e.g., Arifovic et al. (2019), Friedman et al. (2022)) do not make use of any exogenous coordinating device. Their goal is typically to test whether the joint distribution of players' actions under various treatments is consistent with correlated equilibrium, as predicted by theoretical learning models (see Section 5 below for some references). Cason et al. (2020) propose an experiment in which subjects predict outcomes in 2 x 2 games played by others. They provide evidence showing that subjects form beliefs over other players' strategic behavior that are consistent with some focal correlated actions.

5. Concluding remarks

As could be anticipated from the title and the introduction of this paper, we have concentrated on solution concepts and results that can be easily derived from Aumann (1974)'s basic framework. Many other achievements prompted by Aumann (1974)'s insights would definitely have deserved some space in our paper. Here is a very selective list of topics, with an even more selective list of references:

- Refinements: Dhillon and Mertens (1996) and Myerson (1986a) consider perfect-like correlated equilibrium. Myerson (1986b) and Gerardi and Myerson (2007) go on with sequential-like communication equilibrium (see Section 3.2 above). Sugaya and Wolitzky (2021) push the analysis forward, with a focus on the (failure of the) revelation principle for communication equilibrium in multistage games. In addition to the previous refinements, which aim at capturing the rationality of individual players, some papers have developed notions of coalition-proof correlated equilibria. Milgrom and Roberts (1996), Moreno and Wooders (1996) and Ray (1996a) offer three different ways of restricting correlated equilibria so that no coalition of players has an improving deviation. Einy and Peleg (1995) propose a definition of coalition-proof communication equilibrium.
- Implementation: Kar et al. (2010) illustrate the difficulty of *strongly* implementing correlated equilibria. By contrast, under suitable assumptions, correlated equilibria can be successfully *weakly* implemented by cheap talk. Bárány (1992) and Ben-Porath (1998) indeed show that all correlated equilibrium outcomes of a strategic form game can be achieved as (possibly refined) Nash equilibrium outcomes of an extension of the game in which players exchange direct messages, without relying on extraneous signals. Forges (1990) and Gerardi (2004) pursue the same goal for communication equilibrium (see Section 3.2 above) in Bayesian games. As already mentioned, more details can be found in Forges (2009).
- Learning: Young (2004) provides a detailed account of the variety of learning dynamics that can be applied given a strategic form game. Among them, "regret matching", a simple adaptive procedure introduced by Hart and Mas-Colell (2000), guarantees convergence of the empirical distributions of play to the set of correlated equilibria of the game. Hart and Mas-Colell (2013) gather their other contributions on the topic, including convergence of dynamic procedures to coarse correlated equilibria. Recent studies in computer science (see, e.g., Farina et al. (2022) and Anagnostides et al. (2022)) combine no-regret learning dynamics and extensive form correlated equilibrium (see Section 3.3 above).
- Computational complexity: Starting with Gilboa and Zemel (1989), a number of papers investigate the complexity of algorithms dealing with correlated equilibrium. These papers show that correlated equilibrium behaves much better than Nash equilibrium. Hart and Nisan (2018) strengthen this conclusion by connecting complexity with no-regret dynamics (see above). Huang and von Stengel (2008) confirm that the extensive form correlated equilibrium (see Section 3.3 above), which is widely motivated by computational considerations, indeed behaves well with respect to complexity criteria. As mentioned in the previous paragraph, recent contributions in computer science study correlated equilibrium in conjunction with learning. In another direction, the computer science literature pursues comparisons between correlation (or more generally, communication) devices and randomization devices that are based on quantum signals (see, e.g., La Mura (2005) and Auletta et al. (2021)).

CRediT authorship contribution statement

Françoise Forges: Writing – review & editing, Writing – original draft, Formal analysis, Conceptualization. **Indrajit Ray:** Writing – review & editing, Writing – original draft, Formal analysis, Conceptualization.

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The authors have declared no conflict of interest.

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Data availability

No data was used for the research described in the article.

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