Cohomologies of Derived Intersections

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Summary

The intersection of derived schemes carries as structure complex the derived tensor product of structure sheaves of the schemes we are intersecting ([17][10][1]). For intersections of underived schemes, the cohomologies of the intersection structure complex carries important geometric information about the intersection. Computations using these derived tensor products also arise naturally in the context of Fourier-Mukai transforms.

In §2 we provide the background material for this thesis. This includes a construction of the derived tensor product, as well as an overview of the necessary results and definitions for Koszul complexes and local complete intersections.

In §3 we give proofs of results in the literature on the cohomologies of derived intersections. This includes a novel proof of the excess intersection formula in the local complete intersection case.

In §4 we provide new results on the cohomologies of derived intersections in the non-local complete intersection case. In the case that we study, we provide a precise description of the cohomologies as a glued object over the components of the intersection.

Contents

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Chapter 1

Introduction

In any category $\mathcal{C}$ with enough structure, the correct notion of an intersection of subobjects is given by the categorical pullback of their inclusions. Indeed, we would consider an object the intersection of subobjects if it were the largest simultaneous subobject of both objects in consideration. Here, by largest we mean precisely that it satisfies the universal property of a pullback. For subobjects Y and Z of some object X we can define $Y \cap Z$ by

$$\begin{tikzcd} Y \cap Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

In the category of schemes, this tells us that the structure sheaf of such an intersection is given by $\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Z$, distinguishing the intersection subscheme structure on $Y \cap Z$ from the induced reduced closed subscheme structure it inherits naturally from X. If the intersection is transverse, there is no distinction.

Derived Algebraic Geometry is a generalisation of Algebraic Geometry where we replace rings and schemes with their derived counterparts. From the locally ringed space perspective, a derived scheme is a pair $(X, \mathcal{O}_X)$ with X a topological space and $\mathcal{O}_X$ a sheaf of simplicial commutative rings or commutative ring spectra such that $(X, \pi_0\mathcal{O}_X)$ is a scheme and $\pi_k\mathcal{O}_X$ a quasi-coherent $\pi_0\mathcal{O}_X$ module. One can think of this as a topological space X together with a *structure complex* $\mathcal{O}_X$([17][10][1]). We can view classical schemes as derived schemes by considering the structure sheaf as a complex concentrated in degree 0. In this context, the *derived intersection* of $(Y, \mathcal{O}_Y), (Z, \mathcal{O}_Z)$ consists of the topological space $Y \cap Z$ and has $\mathcal{O}_Y \otimes^{\mathbf{L}} \mathcal{O}_Z$, the left derived tensor product of structure sheaves, as structure complex ([17]).

Even for ordinary schemes, some important geometric properties of the intersection is really encoded by the derived intersection. For example, in [13], Serre gave his *intersection multiplicity formula;*

Theorem 1.0.1 (Serre [13])**.** *Let X be a regular variety, and Y, Z subvarieties of complementary dimension whose intersection is 0-dimensional. Then at a point $P \in Y \cap Z$, the intersection multiplicity is given by;*

$$m_P(Y,Z) = \sum_{i=0}^{\infty}(-1)^i \operatorname{length}_{\mathcal{O}_{X,P}}(\operatorname{Tor}_i^{\mathcal{O}_{X,P}}(\mathcal{O}_{Y,P}, \mathcal{O}_{Z,P}))$$

Here $\operatorname{length}_{\mathcal{O}_{X,P}}$ *means length as an $\mathcal{O}_{X,P}$-module.*

The $\operatorname{Tor}_i^{\mathcal{O}_{X,P}}(\mathcal{O}_{Y,P}, \mathcal{O}_{Z,P})$ can be computed by $H^{-i}(\mathcal{O}_{Y,P} \otimes^{\mathbf{L}}_{\mathcal{O}_{X,P}} \mathcal{O}_{Z,P})$, the $-i^{th}$ cohomology of the derived tensor product complex. All of the higher Tors appearing in this formula are necessary to obtain the correct intersection multiplicity. For example, the intersection in $\mathbb{A}^4$ of a union of two planes meeting in a point (e.g. $Y = (xz, xw, yz, yw)$) with a general plane through that point (e.g. $Z = (x-z, y-w)$) should have intersection multiplicity 2, but $\operatorname{length}(\mathcal{O}_{Y,P} \otimes \mathcal{O}_{Z,P}) = 3$, so we need $\operatorname{length}(\operatorname{Tor}^1(\mathcal{O}_{Y,P}, \mathcal{O}_{Z,P})) = 1$ to correct it.

Computing these Tors is hard in general. Indeed, the computation requires finding a flat resolution for $\mathcal{O}_Y$ (up to relabelling), tensoring this resolution with $\mathcal{O}_Z$ and then computing the cohomology of the resulting complex. Alternatively we can find flat resolutions of both $\mathcal{O}_Y$ and $\mathcal{O}_Z$, tensor the resolutions and then compute the cohomologies of the resulting complex. In practice, one would rather work with the stronger notion of a locally free resolution (although over noetherian schemes flat + coherent = locally free [6][III.9.2]) and indeed for computations we would like an explicit locally free resolution. Even better, we would like a locally free resolution which behaves well computationally with respect to the tensor product. An immediate class of examples is supplied by the *Koszul complex* $K^\bullet(\mathcal{E}, s)$, a combinatorially defined complex generated by the data of a map of locally free sheaves $s : \mathcal{E} \to \mathcal{O}_X$. Koszul complexes have the property that $K^\bullet(\mathcal{E}, s) \otimes K^\bullet(\mathcal{F}, t) \cong K^\bullet(\mathcal{E} \oplus \mathcal{F}, s \oplus t)$ (Prop 2.2.5), and so computing the Tors for structure sheaves of subschemes which have Koszul resolutions reduces our problem to computing the cohomologies of Koszul complexes.

There is a class of subschemes whose structure sheaves are resolved by Koszul complexes. In Cohen-Macaulay ambient schemes these are the complete intersections, schemes whose number of defining equations is equal to their codimension (§2.3). Many computations of the Tor sheaves happen locally, and so often we only require that our structure sheaves locally have Koszul resolutions, leading us to consider *local complete intersections*, schemes with the complete intersection property only locally. Computations of the Tor sheaves for *derived self-intersections* of local complete intersection subvarieties go back at least as far as [2], Expose VII. We have the following result;

Theorem 1.0.2 (Theorem 3.1.2). *Let $i : Y \to X$ be a local complete intersection subvariety of a nonsingular variety. Then*

$$\mathrm{Tor}_q^{\mathcal{O}_X}(i_*\mathcal{O}_Y, i_*\mathcal{O}_Y) \cong i_* \bigwedge^q \mathcal{N}^\vee_{Y/X},$$

where $\mathcal{N}^\vee_{Y/X}$ is the conormal bundle of Y in X.

We present a proof of this theorem which follows the classical approach to such problems. We work locally enough to reduce our problem to a computation of Koszul cohomology, then we show that the local isomorphisms that we find glue up to give a global isomorphism. This method is taken to its limit in [12] wherein the author proves the following

Theorem 1.0.3 (Theorem 3.2.4). *Let X be a nonsingular algebraic variety and $Y_1, \dots, Y_n$ be locally complete intersection subvarieties of X such that the intersection $Z := Y_1 \cap \cdots \cap Y_n$ is also a locally complete intersection. Then:*

$$\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_n}) = \bigwedge^q \mathcal{E}_Z$$

where $\mathcal{E}_Z$ is the conormal excess bundle

$$\mathcal{E}_Z := \mathrm{Ker}\left(\bigoplus (\mathcal{N}_{Y_i/X})^\vee|_Z \to \mathcal{N}^\vee_{Z/X}\right).$$

In this thesis, we provide a novel proof of Theorem 1.0.3. In this proof, we demonstrate a global morphism from the derived intersection of the Y_i to the derived self-intersection of the product of the Y_i inside the n-fold product of X. The morphism is given by the adjunction unit for pushforward and pullback by the closed immersion of the intersection. Furthermore the derived self-intersection of the product of the Y_i is a self-intersection of a local complete intersection, for which we have a formula, and the adjunction unit fits the multitors into a long exact sequence *with no assumptions on the intersection at all.* We give a brief description of this long exact sequence. For the closed subscheme $Z \subset X$, there is a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0.$$

For any sheaf $\mathcal{F}$, the pullback-pushforward adjunction unit for the closed immersion $Z \hookrightarrow X$ is given by tensoring by the surjection $\mathcal{O}_X \to \mathcal{O}_Z$. So tensoring the above short exact sequence by $\mathcal{F}$ results in a long exact sequence in cohomology, with every third morphism given by the cohomologies of the adjunction unit morphism.

In our situation where the intersection is itself a local complete intersection, this long exact sequence simplifies enough to prove the result. The original proof method relies on a gluing argument over open affines where we can construct isomorphisms via a comparison with Koszul models. The drawback of this method is that the gluing argument is rather involved, even in the case where the intersection is a local complete intersection in its own right. The benefit of our method is that the gluing argument is packaged up in the self-intersection formula for local complete intersections Theorem 3.1.2, where the gluing argument is much nicer. While it suffices to make a comparison to the self-intersection in the lci intersection case, it is not clear whether or not this method of proof can be extended to the case when the intersection is not lci. A discussion of how such an extension might work is contained in §5.

We are ultimately interested in extending this result to the case where the intersection of the Y_i is no longer a local complete intersection. It can very easily happen that some local complete intersections may intersect in a non local complete intersection. For instance, any subscheme which is not equidimensional is trivially not a local complete intersection. One needs at least as many equations at the codimension of the highest codimension component to define that component, but the whole subscheme has codimension given by the lowest codimension component. When the intersection of the Y_i is not lci, one immediately runs into the local problem of computing Koszul cohomologies, and also faces the difficulty of providing a gluing argument. Without the lci assumption on the Y_i or on Z there is still a natural surjection

$$\bigoplus (\mathcal{N}_{Y_i/X})^{\vee}|_Z \to \mathcal{N}^{\vee}_{Z/X}$$

and so one call always define a *conormal excess sheaf* $\mathcal{E}_Z$ as the kernel of this morphism. In the case that a subscheme Y_i is not lci, the conormal sheaf $\mathcal{N}^{\vee}_{Y/X}$ is not locally free but we can still consider exterior powers of this conormal excess sheaf $\mathcal{E}_Z$. It seems a reasonable question to ask

Question 1.0.4 ([12] Question 3.11)**.** *Is it true, for arbitrary subchemes Y_i of a nonsingular variety X whose intersection we denote by Z, that*

$$\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_n}) \cong \bigwedge^q \mathcal{E}_Z?$$

In §4 we compute some examples that show that this is unfortunately not the case. In those examples, we have an intersection of hyperplanes in affine space whose intersection itself contains a hyperplane. Such examples demonstrate that the multitors of the intersections of divisors may not even form an exterior algebra levelwise, that is to say, the higher multitors are not exterior powers of the lower ones. This leads us to investigate the question of what the multitors of such a non-local complete intersection intersection will be. As a rule of thumb, the complexity of the geometry increases with codimension so we tackle the following problem; Let Y_i $(1 \leq i \leq n)$ be effective Cartier divisors of a nonsingular variety X over an algebraically closed field of characteristic 0. Let D be another effective Cartier divisor in X. Each $\mathcal{O}_{Y_i}$ has a global Koszul resolution $\mathcal{O}(-Y_i) \xrightarrow{\sigma_{Y_i}} \mathcal{O}_X$ and each $\mathcal{O}_{Y_i+D}$ has a global Koszul resolution $\mathcal{O}(-Y_i - D) \xrightarrow{\sigma_{Y_i+D}} \mathcal{O}_X$. We therefore have global models

$$\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_n}) = H^{-q}(\bigotimes K^{\bullet}(\mathcal{O}(-Y_i), \sigma_{Y_i})),$$

$$\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{Y_1+D}, \dots, \mathcal{O}_{Y_n+D}) = H^{-q}(\bigotimes K^{\bullet}(\mathcal{O}(-Y_i - D), \sigma_{Y_i+D})).$$

We prove the following;

Theorem 1.0.5 (Theorem 4.2.2)**.** *With the notation as above, denote the differential of the Koszul complex $K^{\bullet}(\bigoplus \mathcal{O}(-Y_i), \sum \sigma_{Y_i})$ by δ. Assume that $D \cap \mathrm{Ass}_X(\mathrm{Tor}_q(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_n})) =$*

$\emptyset, \forall q$. *Then there is a fibre square in* Mod_X*;*

$$\begin{array}{ccc} \mathrm{Tor}_q(\mathcal{O}_{Y_1+D}, \ldots, \mathcal{O}_{Y_n+D}) & \longrightarrow & (\mathrm{Ker}(\delta^{-q}) \otimes \mathcal{O}(-qD))|_D \\ \downarrow & & \downarrow \\ \mathrm{Tor}_q(\mathcal{O}_{Y_1}, \ldots, \mathcal{O}_{Y_n}) \otimes \mathcal{O}(-qD) & \longrightarrow & (\mathrm{Tor}_q(\mathcal{O}_{Y_1}, \ldots, \mathcal{O}_{Y_n}) \otimes \mathcal{O}(-qD))|_D \end{array}$$

where the bottom horizontal morphism is the projection and the right vertical arrow is induced by the quotient projection $\mathrm{Ker}(\delta^{-q}) \to \mathrm{Ker}(\delta^{-q}) / \mathrm{Im}(\delta^{-q-1})$.

In the case where the intersection of the Y_i is itself a lci, this Theorem reduces to

Corollary 1.0.6. *With the same notation as above, if* $\bigcap Y_i = Z$ *is a local complete intersection and* D *does not contain any of the irreducible components of* Z *then there is a fibre square*

$$\begin{array}{ccc} \mathrm{Tor}_q(\mathcal{O}_{Y_1+D}, \ldots, \mathcal{O}_{Y_n+D}) & \longrightarrow & (\mathrm{Ker}(\delta^{-q}) \otimes \mathcal{O}(-qD))|_D \\ \downarrow & & \downarrow \\ j_* \bigwedge^q \mathcal{E}_Z \otimes \mathcal{O}(-qD) & \longrightarrow & h_*(\bigwedge^q \mathcal{E}_Z|_{Z\cap D}) \otimes \mathcal{O}(-qD), \end{array}$$

where $q : Z \to X, h : Z \cap D \to X$ *are the closed immersions coming from the intersection.*

In effect, this result tells us that the multitor of the $Y_i + D$ is given by a gluing of excess sheaves on each component of the intersection. On the open sets away from each component, our subschemes satisfy the hypotheses of Theorem 1.0.3 and therefore the multitors on these open loci are excess bundles. In our case, the intersection of Z with D has codimension at least 2 in D so these locally free sheaves have unique extensions to all of D, however Example 4.1.4 demonstrates that the naive gluing of these extensions will not yield the correct answer and these more complicated kernel sheaves are required. This example also demonstrates that the multitors do not have an exterior algebra structure.

This is the first known instance of a general multitor formula for an intersection of schemes which is not a local complete intersection. While the setting is somewhat artificial, the nature of the result leads one to hope that a similar formula may be obtained for intersections which decompose into lci components in more generality. We discuss potential generalities in §5.

Chapter 2

Preliminaries

2.1 Derived Categories

The purpose of this section is to give an account of the construction of the derived tensor product of structure sheaves of subvarieties on a scheme X. To this end we define derived categories and functors for general abelian categories, and then more specifically define the derived tensor product for objects in the derived category of quasi-coherent sheaves on a scheme X.

2.1.1 Definition

A derived category is a category that is constructed from the starting data of an abelian category. Hence we include the definition of an abelian category for completeness. Roughly speaking, this is a category over which one can construct a cohomology theory. Abelian categories were first introduced by Grothendieck in [5].

Definition 2.1.1 (Additive Category, [5])**.** Let $\mathcal{A}$ be a category. Then $\mathcal{A}$ is called *additive* if

1. For any objects $A, B \in \mathcal{A}$, $Hom_{\mathcal{A}}(A, B)$ is an abelian group and composition of morphisms is bilinear ($\mathcal{A}$ is enriched over the category of abelian groups),

2. There is a zero object $0 \in \mathcal{A}$, i.e. an object such that $Hom_{\mathcal{A}}(0, 0) = 0$,

3. For any two objects A_1, A_2 there is an object B which is both a direct sum and direct product of A_1 and A_2

Definition 2.1.2 (Abelian Category,[5])**.** Let $\mathcal{A}$ be a category. Then $\mathcal{A}$ is called *abelian* if it is additive and additionally

1. Every morphism admits a kernel and a cokernel,

2. For any morphism $A \xrightarrow{f} B$ there is a canonical decomposition

$$\mathrm{Ker}(f) \to A \to C \cong C' \to B \to \mathrm{Coker}(f)$$

where C is the cokernel of $\mathrm{Ker}(f) \to A$ and C' is the kernel of $B \to \mathrm{Coker}(f)$.

Definition 2.1.3. Let $\mathcal{A}$ be an abelian category. A cochain complex $A^\bullet$ over $\mathcal{A}$ is a collection of objects $A^i \in \mathcal{A}, i \in \mathbb{Z}$ and morphisms $d_A^i : A^i \to A^{i+1}$ such that for each i, $d_A^i \circ d_A^{i-1} = 0$. Pictorially we represent a complex by

$$A^\bullet = \qquad \{\dots \xrightarrow{d_A^{i-2}} A^{i-1} \xrightarrow{d_A^{i-1}} A^i \xrightarrow{d_A^i} A^{i+1} \xrightarrow{d_A^{i+1}} \dots\}$$

We abuse notation and just write d_A or d for each of the differentials when it is clear from the context which complex the differentials belong to.

Definition 2.1.4. A morphism of cochain complexes $f : A^\bullet \to B^\bullet$ is a collection of morphisms $f^i : A^i \to B^i$ such that all of the resulting squares commute, that is, $f^i \circ d_A^{i-1} = d_B^{i-1} \circ f^{i-1}$. Pictorially we represent a morphism of cochain complexes by the commuting diagram

$$\begin{tikzcd}
\dots \arrow[r, "d_A"] & A^{i-1} \arrow[r, "d_A"] \arrow[d, "f^{i-1}"'] & A^i \arrow[r, "d_A"] \arrow[d, "f^i"'] & A^{i+1} \arrow[r, "d_A"] \arrow[d, "f^{i+1}"'] & \dots \\
\dots \arrow[r, "d_B"] & B^{i-1} \arrow[r, "d_B"] & B^i \arrow[r, "d_B"] & B^{i+1} \arrow[r, "d_B"] & \dots
\end{tikzcd}$$

Definition 2.1.5. The complex category $\mathbf{C}(\mathcal{A})$ is defined to be the category with objects given by cochain complexes over $\mathcal{A}$ and with morphisms given by cochain complex morphisms. This is also an abelian category.

Definition 2.1.6. Let $A^\bullet \in \mathbf{C}(\mathcal{A})$ be a cochain complex. The i^{th} cohomology $H^i(A^\bullet)$ of $A^\bullet$ is defined to be the object $\mathrm{Ker}(d_A^i)/\,\mathrm{Im}(d_A^{i-1})$. Note that this is a well defined quotient by the $d_A^i \circ d_A^{i-1} = 0$ condition. We call a complex exact or acyclic if all of its cohomology objects are 0. Any map $f : A^\bullet \to B^\bullet$ of cochain complexes induces morphisms $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ between the cohomologies of the cochain complexes. If the induced maps between the cohomologies are all isomorphisms we call the morphism $f : A^\bullet \to B^\bullet$ a *quasi-isomorphism.*

We wish to study complexes over an abelian category up to quasi-isomorphism, in particular we would like to identify an object of the category $\mathcal{A}$ with any of its resolutions. A resolution $A^\bullet$ of an object A is a complex with cohomology given by $H^0(A^\bullet) = A, H^i(A^\bullet) = 0, \forall i \neq 0$, equipped with an augmentation map $A^\bullet \to A$ or $A \to A^\bullet$ making the resulting extended complex exact.

Definition 2.1.7. We say that two morphisms of complexes $f, g : A^\bullet \to B^\bullet$ are *homotopy equivalent* if there exists a degree -1 morphism $k : A^\bullet \to B^\bullet$ such that $f - g = k \circ d_A + d_B \circ k$.

$$\begin{tikzcd}
\dots \arrow[r, "d_A"] & A^{i-1} \arrow[r, "d_A"] \arrow[d, "f-g"] \arrow[dl, "k"'] & A^i \arrow[r, "d_A"] \arrow[d, "f-g"] \arrow[dl, "k"'] & A^{i+1} \arrow[r, "d_A"] \arrow[d, "f-g"] \arrow[dl, "k"'] & \dots \arrow[dl, "k"'] \\
\dots \arrow[r, "d_B"] & B^{i-1} \arrow[r, "d_B"] & B^i \arrow[r, "d_B"] & B^{i+1} \arrow[r, "d_B"] & \dots
\end{tikzcd}$$

If f and g are homotopy equivalent we write $f \simeq_h g$.

Lemma 2.1.8. *Let $f, g : A^\bullet \to B^\bullet$ be maps of cochain complexes which are homotopy equivalent. Then*

$$H^i(f) = H^i(g) : H^i(A^\bullet) \to H^i(B^\bullet).$$

Proof. We equivalently want to show that $H^i(f - g) = H^i(k \circ d_A + d_B \circ k) : H^i(A^\bullet) \to H^i(B^\bullet)$ is the zero map. Note that for any $x \in \operatorname{Ker}(d_A^i)$ we have $(k \circ d_A + d_B \circ k)(x) = (d_B \circ k)(x) \in \operatorname{Im}(d_B)$ so the image of x under $H^i(k \circ d_A + d_B \circ k)$ is 0. Since x was arbitrary this shows that the morphism is 0. $\square$

Definition 2.1.9. A morphism $f : A^\bullet \to B^\bullet$ is called a homotopy equivalence of chain complexes if there exists a map $g : B^\bullet \to A^\bullet$ such that $f \circ g \simeq_h 1_{B^\bullet}$ and $g \circ f \simeq_h 1_{A^\bullet}$. We say that two cochain complexes $A^\bullet$ and $B^\bullet$ are homotopy equivalent if there exists a homotopy equivalence $f : A^\bullet \to B^\bullet$.

Part of the motivation for setting up the derived category is to establish a framework to construct derived functors like Tor and Ext. With this in mind, we see that we will want to formally identify objects with their resolutions.

Definition 2.1.10. Let $\mathcal{A}$ be an abelian category. We say an object $P \in \mathcal{A}$ is projective if the functor $Hom_{\mathcal{A}}(P, -)$ is exact (preserves exact complexes). We say an object $I \in \mathcal{A}$ is injective if the functor $Hom_{\mathcal{A}}(-, I)$ is exact.

Definition 2.1.11. A projective resolution of an object $A \in \mathcal{A}$ is a complex $P^\bullet$ with an augmentation morphism $P^\bullet \to A$ such that each P^i is a projective object in $\mathcal{A}$ and the augmented complex $P^\bullet \to A$ is exact. Similarly an injective resolution of an object $A \in \mathcal{A}$ is a complex $I^\bullet$ with an augmentation morphism $A \to I^\bullet$ such that each I^i is an injective object in $\mathcal{A}$ and the augmented complex $A \to I^\bullet$ is exact.

Lemma 2.1.12 ([19])**.** *Suppose $P^\bullet \to A$ and $Q^\bullet \to A$ are two projective resolutions of A. Then $P^\bullet$ and $Q^\bullet$ are homotopy equivalent. Similarly, suppose that $A \to I^\bullet$ and $A \to J^\bullet$ are injective resolutions of A. Then $I^\bullet$ and $J^\bullet$ are homotopy equivalent.*

This is part of the motivation of defining the intermediary *homotopy category.*

Definition 2.1.13. Let $\mathcal{A}$ be an abelian category. Then we define the homotopy category $\mathbf{K}(\mathcal{A})$ to have the same objects as $\mathbf{C}(\mathcal{A})$ and with morphisms given by equivalence classes of morphisms in $\mathbf{C}(\mathcal{A})$ under the equivalence relation of homotopy equivalence.

Now to formally identify objects with their resolutions, we want to introduce formal inverses of the augmentation quasi-isomorphisms. This process of introducing formal inverses is analogous to the process of localisation of a ring at a multiplicative system in commutative algebra.

Definition 2.1.14 ([7] §1.3)**.** Let $\mathcal{C}$ be a category. A collection S of morphisms in $\mathcal{C}$ is called a multiplicative system if it satisfies the following:

1. If $f, g \in S$ and the composition $f \circ g$ is defined, then $f \circ g \in S$. For every $C \in \mathcal{C}, 1_X \in S$.

2. Any diagram

$$\begin{tikzcd} & Z \arrow[d, "s"] \\ X \arrow[r, "u"] & Y \end{tikzcd}$$

with $s \in S$ can be completed to a commutative diagram of the form

$$\begin{tikzcd} W \arrow[r, "v"] \arrow[d, "t"] & Z \arrow[d, "s"] \\ X \arrow[r, "u"] & Y \end{tikzcd}$$

with $t \in S$. This should also hold with all arrows reversed.

3. If $f, g : X \to Y$ are morphisms in $\mathcal{C}$, the following are equivalent;

   - There exists an $s : Y \to Y' \in S$ such that $sf = sg$
   - There exists an $t : X' \to X \in S$ such that $ft = gt$.

Definition 2.1.15 ([7]). If $\mathcal{C}$ is a category and S a multiplicative system in $\mathcal{C}$, then the localisation of $\mathcal{C}$ at S is a category $\mathcal{C}_S$, together with a functor $Q : \mathcal{C} \to \mathcal{C}_S$ such that

1. $Q(s)$ is an isomorphism for every $s \in S$,
2. Any functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for all $s \in S$ factors uniquely through Q.

Lemma 2.1.16. *Let $\mathcal{A}$ be an abelian category. Then the collection of quasi-isomorphisms in the homotopy category $\boldsymbol{K}(\mathcal{A})$ form a multiplicative system.*

Remark 2.1.17. *Note that the collection of quasi-isomorphisms in the complex category $\boldsymbol{C}(\mathcal{A})$ do not form a multiplicative system. Condition 2 of the definition is only satisfied up to homotopy ([9] Lemma 1.4.3.6).*

Definition 2.1.18. Let $\mathcal{A}$ be an abelian category. Then the derived category $\mathbf{D}(\mathcal{A})$ is defined to be the localisation $\mathbf{K}(\mathcal{A})_{Qis}$ of the homotopy category at the collection of quasi-isomorphisms.

We give a more explicit description of this category. The objects of $\mathbf{D}(\mathcal{A})$ are the same as those of $\mathbf{K}(\mathcal{A})$, but a morphism $A \to B$ is an equivalence class of diagrams

$$\begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

where s is a quasi-isomorphism. We write such an equivalence class as $f/s : A \to B$. For details on the equivalence relation and the well-definedness of compositions etc. we refer the reader to ([9], §1.2), ([7],§1.3). Now we have formally identified an object $A \in \mathcal{A}$ with its resolutions, we have a good framework in which to perform cohomological computations. However, the derived category $\mathbf{D}(\mathcal{A})$ (and in fact the homotopy category $\mathbf{K}(\mathcal{A})$ is no longer abelian, so we no longer have short exact sequences of complexes giving rise to a long exact sequence in cohomology. Fortunately, the derived category still has the structure of a *triangulated category*, which enables us to recover all of the cohomological data we require.

Definition 2.1.19 (Triangulated Category, [18])**.** A triangulated category (Δ-category) is an additive category K together with an additive autoequivalence T (called the *translation functor*) and a collection $\mathcal{T}$ of diagrams of the form

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA,$$

satisfying certain axioms. A *triangle* in K is a diagram in $\mathcal{T}$. The axioms are as follows:

1. Any diagram isomorphic to a triangle is a triangle and every diagram of the following form is a triangle

$$A \xrightarrow{id} A \longrightarrow 0 \longrightarrow TA.$$

   Further, every morphism $A \xrightarrow{u} B$ can be completed to a triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA,$$

2. A diagram

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA,$$

   is a triangle if and only if

$$B \xrightarrow{v} C \xrightarrow{w} TA \xrightarrow{-Tu} TB,$$

   is a triangle,

3. For any diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\alpha"] & B \arrow[r, "v"] \arrow[d, "\beta"] & C \arrow[r, "w"] \arrow[d, dashed, "(\exists\gamma)"] & TA \arrow[d, "T\alpha"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA' \end{tikzcd}$$

whose rows are triangles, and with maps α, β given such that $\beta u = u'\alpha$, there exists a morphism $\gamma : C \to C'$ making the entire diagram commute.

4. For any commuting diagram

$$\begin{tikzcd} X_1 \arrow[rr, "u_3"] \arrow[dr, "u_2"'] & & X_2 \arrow[dl, "u_1"] \\ & X_3 & \end{tikzcd}$$

then in any diagram with triangles for rows;

$$\begin{tikzcd} X_1 \arrow[r, "u_3"] \arrow[d, "Id"] & X_2 \arrow[r, "v_3"] \arrow[d, "u_1"] & Z_3 \arrow[r, "w_3"] \arrow[d, dashed, "(\exists m_1)"] & TX_1 \arrow[d, "Id"] \\ X_1 \arrow[r, "u_2"] \arrow[d, "u_3"] & X_3 \arrow[r, "v_2"] \arrow[d, "Id"] & Z_2 \arrow[r, "w_2"] \arrow[d, dashed, "(\exists m_3)"] & TX_1 \arrow[d, "Tu_3"] \\ X_2 \arrow[r, "u_1"] & X_3 \arrow[r, "v_1"] & Z_1 \arrow[r, "w_1"] & TX_2 \end{tikzcd}$$

there exist morphisms m_1, m_3 making the diagram commute and such that

$$Z_3 \xrightarrow{m_1} Z_2 \xrightarrow{m_3} Z_1 \xrightarrow{Tv_3 \circ w_1} TZ_3$$

is a triangle.

As a consequence of these axioms we have that, if in (3) both α and β are isomorphisms, then γ is an isomorphism too. Another consequence is that any composition of morphisms in a triangle is 0. Via repeated application of (2), we have an infinite chain of objects and morphisms, any four consecutive of which form a triangle.

There exists a natural Δ-structure on the homotopy category $\mathbf{K}(\mathcal{A})$.

Definition 2.1.20. Define the translation functor $T : \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ by taking a complex $A^\bullet$ with differential d_A to the complex $T(A^\bullet)$ with terms $T(A)^i = A^{i+1}$ and differential $-d_A$. We denote $T(A^\bullet)$ by $A^\bullet[1]$. It is clear that this is an autoequivalence of $\mathbf{K}(\mathcal{A})$ with inverse denoted $[-1]$.

The triangles in this triangulation are given by all sequences isomorphic to a standard triangle. To define standard triangles, we must introduce the notion of *mapping cone*.

Definition 2.1.21. Given a morphism of complexes $u : A^\bullet \to B^\bullet$, we define the cone of u, $C_u^\bullet$ to be the following complex. It has terms

$$C_u^n = B^n \oplus A^{n+1}$$

and differentials $d^n : C_u^n \to C_u^{n+1}$ given pictorially by

$$\begin{tikzcd} B^n \arrow[d, "d_B^n"] & \oplus & A^{n+1} \arrow[dll, "u^{n+1}"'] \arrow[d, "-d_A^{n+1}"] \\ B^{n+1} & \oplus & A^{n+2}. \end{tikzcd}$$

There are obvious inclusion maps $v : B^\bullet \to C_u^\bullet$ and projection maps $w : C_u^\bullet \to A^\bullet[1]$, and we define the standard triangles of $\mathbf{K}(\mathcal{A})$ to be ones of the form

$$A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C_u^\bullet \xrightarrow{w} A^\bullet[1].$$

The proof that this collection of diagrams satisfies the axioms of a triangulation is omitted, but can be found for instance in ([9]§1.4, [7]§1.2).

There is a relationship between short exact sequences of objects in $\mathbf{C}(\mathcal{A})$ and triangles in $\mathbf{K}(\mathcal{A})$. Consider the short exact sequence of complexes

$$0 \longrightarrow A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \longrightarrow 0,$$

if u_0 is the isomorphism from $A^\bullet$ onto the kernel of v induced by u, then we have a natural short exact sequence of complexes

$$0 \longrightarrow C_{u_0}^\bullet \longrightarrow C_u^\bullet \xrightarrow{\xi} C^\bullet \longrightarrow 0$$

where $\xi^n : C_u^n \to C^n$ is the composition

$$\xi^n = v \circ \pi : C_u^n = B^n \oplus A^{n+1} \to B^n \to C^n.$$

Since $C_{u_0}^\bullet$ is isomorphic to the cone over the identity map on $A^\bullet$, $H^n(C_{u_0}^\bullet) = 0 \quad \forall n$ and we conclude that ξ is a quasi-isomorphism from the resulting long exact sequence on cohomology. It can be shown that any triangle in $\mathbf{K}(\mathcal{A})$ is isomorphic to one so obtained. A consequence of this fact is that applying the cohomology functor to a triangle as above

$$A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C_u^\bullet \xrightarrow{w} A^\bullet[1]$$

induces a long exact sequence on cohomology

$$
\begin{tikzcd}
\cdots \arrow[r] & H^i(A^\bullet) \arrow[r, "H^i(u)"] & H^i(B^\bullet) \arrow[r, "H^i(v)"] & H^i(C^\bullet) \arrow[dll, "H^i(w)"] & \\
& H^{i+1}(A^\bullet) \arrow[r, "H^{i+1}(u)"] & H^{i+1}(B^\bullet) \arrow[r, "H^{i+1}(v)"] & H^{i+1}(C^\bullet) \arrow[r] & \cdots
\end{tikzcd}
$$

Following the category theoretic philosophy, alongside the existence of Δ-categories there are also notions of Δ-functors and Δ-subcategories.

Definition 2.1.22. Let $\mathcal{A}$ be a Δ-category. Then a subcategory $\mathcal{B} \subset \mathcal{A}$ is said to be a $\Delta-$subcategory if it is an additive subcategory, and it is also closed under the Δ-structure. That is, $\mathcal{B}$ is closed under the autoequivalence T and the summit of any triangle with a base in $\mathcal{B}$ lies in $\mathcal{B}$.

Definition 2.1.23 ([9]§1.5)**.** Let $\mathcal{A}_1, \mathcal{A}_2$ be two Δ-categories, with translation functors T_1, T_2 respectively. A (covariant) Δ-functor is defined to be a pair (F, θ) consisting of an additive functor $F : \mathcal{A}_1 \to \mathcal{A}_2$ together with an isomorphism of functors

$$\theta : FT_1 \to T_2F$$

such that for every triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T_1A$$

in $\mathcal{A}_1$, the corresponding diagram

$$FA \xrightarrow{Fu} FB \xrightarrow{Fv} FC \xrightarrow{\theta \circ Fw} T_2FA$$

is a triangle in $\mathcal{A}_2$.

From the definition of localisation there is also a natural inclusion functor $Q : \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ which is the identity on objects and on morphisms $Q(f) = f/1$. We impose a triangulation on $\mathbf{D}(\mathcal{A})$ from the triangulation on $\mathbf{K}(\mathcal{A})$ by insisting that Q be a Δ-functor.

2.1.2 Derived Functors

Part of the whole motivation for setting up the machinery of derived categories was to have a framework in which to understand derived functors. We give here the definition of a derived functor in this context and a result on their existence.

Definition 2.1.24 ([9])**.** Let $\mathcal{A}$ be an abelian category, $\mathbf{J}$ a $\Delta-$subcategory of $\mathbf{K}(\mathcal{A})$, let $\mathbf{D_J}$ be the corresponding derived category, and let

$$Q = Q_{\mathbf{J}} : \mathbf{J} \to \mathbf{D_J}$$

be the canonical $\Delta-$functor. Then we say a $\Delta-$functor $F : \mathbf{J} \to \mathbf{E}$ is left-derivable if there exists a $\Delta-$functor

$$\mathbf{L}F : \mathbf{D_J} \to \mathbf{E}$$

and a morphism of Δ-functors

$$\zeta : \mathbf{L}F \circ Q \to F$$

which is universal with respect to pairs (G, η) where $G : \mathbf{D_J} \to \mathbf{E}$ is a Δ-functor and

$$\eta : G \circ Q \to F,$$

is a morphism of Δ-functors. Similarly, we say a Δ-functor $F : \mathbf{J} \to \mathbf{E}$ is right-derivable if there exists a Δ-functor

$$\mathbf{R}F : \mathbf{D_J} \to \mathbf{E}$$

and a morphism of $\Delta-$functors

$$\zeta : F \to \mathbf{R}F \circ Q$$

which is universal with respect to pairs (G, η) where $G : \mathbf{D_J} \to \mathbf{E}$ is a Δ-functor and

$$\eta : F \to G \circ Q$$

is a morphism of Δ-functors.

Definition 2.1.25. Let $\mathbf{J}$ be a Δ-subcategory of $\mathbf{K}(\mathcal{A})$ and $F : \mathbf{J} \to \mathbf{E}$ a Δ-functor. We say an object $X \in \mathbf{J}$ is *left-F-acyclic* if for any quasi-isomorphism $u : Y \to X$ in $\mathbf{J}$, there exists a quasi-isomorphism $v : Z \to Y$ in $\mathbf{J}$ such that $F(uv)$ is an isomorphism. Similarly, we say that an object $X \in \mathbf{J}$ is *right-F-acyclic* if for any quasi-isomorphism $u : X \to Y$ in $\mathbf{J}$ there exists a quasi-isomorphism $v : Y \to Z$ in $\mathbf{J}$ such that the map $F(vu) : F(X) \to F(Z)$ is an isomorphism.

Definition 2.1.26 ([14])**.** We call an object $I \in \mathbf{K}(\mathcal{A})$ *q-injective* if for every diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"] & I \\ Y & \end{tikzcd}$$

with s a quasi-isomorphism, there is a unique morphism $g : Y \to I$ in $\mathbf{K}(\mathcal{A})$ such that $gs = f$. We say an object $P \in \mathbf{K}(\mathcal{A})$ is q-projective if any diagram of the form

$$\begin{tikzcd} & X \arrow[d, "s"] \\ P \arrow[r, "f"] & Y \end{tikzcd}$$

with s a quasi-isomorphism, there is a unique morphism $g : P \to X$ in $\mathbf{K}(\mathcal{A})$ such that $sg = f$.

Remark 2.1.27. *Note that for the definitions of q-injectivity and q-projectivity, the uniqueness of the lifting morphism is in $\boldsymbol{K}(\mathcal{A})$, and so does not determine a unique morphism of complexes only a unique homotopy class of morphisms of complexes.*

Lemma 2.1.28 ([9] §2.3)**.** *An object $I \in \boldsymbol{K}(\mathcal{A})$ is q-injective if and only if I is right$-F-$acyclic for every $\Delta-$functor $F : \boldsymbol{K}(\mathcal{A}) \to \boldsymbol{E}$. An object $P \in \boldsymbol{K}(\mathcal{A})$ is q-projective if and only if P is left$-F-$acyclic for every Δ-functor $F : \boldsymbol{K}(\mathcal{A}) \to \boldsymbol{E}$.*

Example 2.1.29. Any projective object of $\mathcal{A}$ viewed as a complex in degree 0 is a q-projective object of $\mathbf{K}(\mathcal{A})$. In fact, any complex concentrated in a single degree with a projective object of $\mathcal{A}$ in that degree is q-projective. The q-projective complexes form a Δ-subcategory, so we immediately see that bounded complexes of projective sheaves are q-projective. Similarly with injectives.

Theorem 2.1.30 ([9] Proposition 2.2.6)**.** *Let $\mathcal{A}$ be an abelian category, let $\boldsymbol{J}$ be a $\Delta-$subcategory of $\boldsymbol{K}(\mathcal{A})$, and let F be a Δ-functor from $\boldsymbol{J}$ to a Δ-category $\boldsymbol{E}$. Suppose $\boldsymbol{J}$ contains a family of quasi-isomorphisms $\varphi_X : A_X \to X (X \in \boldsymbol{J})$ such that A_X is left-F-acyclic for all X. Then F has a left-derived functor $(\boldsymbol{L}F, \zeta)$ such that for all $X \in \boldsymbol{J}$,*

$$\boldsymbol{L}F(X) = F(A_X), \ \textit{and} \ \zeta(X) = F(\varphi_X) : F(A_X) = \boldsymbol{L}F(X) \to F(X).$$

Moreover, X is left-F-acyclic $\iff$ $\zeta(X)$ is an isomorphism. Similarly, suppose that $\boldsymbol{J}$ contains a family of quasi-isomorphisms $\psi_X : X \to B_X, (X \in \boldsymbol{J})$ such that B_X is right-F-acyclic for all X. Then F has a right derived functor $(\boldsymbol{R}F, \zeta)$ such that for all $X \in \boldsymbol{J}$,

$$\boldsymbol{R}F(X) = F(B_X), \text{ and } \zeta(X) = F(\psi_X) : F(X) \to F(B_X) = \boldsymbol{L}F(X).$$

Moreover, X is right-F-acyclic $\iff$ $\zeta(X)$ is an isomorphism.

Proposition 2.1.31. *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ and $G : \boldsymbol{D} \to \boldsymbol{E}$ be Δ-functors. Suppose that $\boldsymbol{R}F, \boldsymbol{R}G$, and $\boldsymbol{R}(G \circ F)$ exist. Suppose that for every right-F-acyclic object $X \in \boldsymbol{C}$, $F(X)$ is right-G-acyclic. Then*

$$\boldsymbol{R}G \circ \boldsymbol{R}F \cong \boldsymbol{R}(G \circ F)$$

Similarly for left-derived functors.

Remark 2.1.32. *This is a modern formulation of the convergence of the Grothendieck spectral sequence [5]. That result says that if $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are two additive and left exact functors between abelian categories such that both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives (every object has an injection to an injective object), and F takes injective objects to G-acyclic objects, then for each object A of $\mathcal{A}$ there is a spectral sequence:*

$$E_2^{p,q} = (\boldsymbol{R}^p G \circ \boldsymbol{R}^q F)(A) \implies \boldsymbol{R}^{p+q}(G \circ F)(A).$$

Example 2.1.33 (Leray spectral sequence)**.** Let $f : X \to Y$ be a continuous map of topological spaces, $\mathcal{A} = \mathrm{Ab}(X), \mathcal{B} = \mathrm{Ab}(Y)$ the category of sheaves of abelian groups on X and Y respectively, and let $\mathcal{C} = \mathrm{Ab}$, the category of abelian groups. Let $F = f_*$ the direct image functor and $G = \Gamma_Y$. Then $G \circ F = \Gamma_Y \circ f_* = \Gamma_X$ and these functors satisfy the hypotheses of the Proposition so we have

$$\mathbf{R}\Gamma_Y(\mathbf{R}f_*\mathcal{F}) = \mathbf{R}\Gamma_X(\mathcal{F})$$

for any sheaf of abelian groups $\mathcal{F}$ on X. The Grothendieck spectral sequence

$$H^p(Y, \mathbf{R}^q f_*\mathcal{F}) \implies H^{p+q}(X, \mathcal{F})$$

is called the *Leray spectral sequence.*

Example 2.1.34 (Local-to-global Ext spectral sequence). Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{A} = \mathcal{B} = \mathrm{Mod}_X$ the category of sheaves of $\mathcal{O}_X$-modules, and $\mathcal{C} = \mathrm{Ab}$ the category of abelian groups. Let $F = \mathcal{H}om_X(\mathcal{F}, -)$ be the sheaf-hom functor for some $\mathcal{O}_X$-module $\mathcal{F}$ and let $G = \Gamma_X$. Then the hypotheses of the proposition are satisfied and

$$\mathbf{R}\Gamma_X(\mathbf{R}\mathcal{H}om_X(\mathcal{F}, -)) = \mathbf{R}Hom_X(\mathcal{F}, -).$$

The Grothendieck spectral sequence for each sheaf of $\mathcal{O}_X$-modules $\mathcal{G}$

$$E_2^{p,q} = H^p(X; \mathcal{E}xt_X^q(\mathcal{F}, \mathcal{G}) \implies \mathrm{Ext}_X^{p+q}(\mathcal{F}, \mathcal{G})$$

is called the local-to-global Ext spectral sequence.

2.1.3 The derived tensor product

From now on, the abelian category we will be considering will be $\mathcal{A} = \mathrm{Mod}_X$, the category of quasi-coherent sheaves on a scheme X. In this case, we denote $\mathbf{C}(\mathcal{A})$ by $\mathbf{C}(X)$, $\mathbf{K}(\mathcal{A})$ by $\mathbf{K}(X)$, and $\mathbf{D}(\mathcal{A})$ by $\mathbf{D}(X)$.

We are trying to investigate the derived tensor product functor. The definition of the derived tensor product is, as suggested by the name, the derived functor associated to the tensor product functor. So we first define the tensor product functor associated to an object in $\mathbf{K}(X)$ for a scheme X.

Definition 2.1.35. For any complexes $A^\bullet, B^\bullet \in \mathbf{C}(X)$, we define the complex $A^\bullet \otimes B^\bullet$ as follows;

$$(A^\bullet \otimes B^\bullet)^n = \bigoplus_{p+q=n} A^p \otimes B^q$$

with differential given on the $A^p \otimes B^q$ summand by

$$d^n|_{A^p \otimes B^q} : A^p \otimes B^q \to (A^{p+1} \otimes B^q) \oplus (A^p \otimes B^{q+1}), \quad a \otimes b \mapsto d_A(a) \otimes b + (-1)^p a \otimes d_B(b).$$

It is readily checked that the functor $F_A : \mathbf{C}(X) \to \mathbf{C}(X)$ sending $B^\bullet \to A^\bullet \otimes B^\bullet$ preserves homotopies and cones over morphisms, and so therefore induces a Δ-functor $\mathbf{K}(X) \to \mathbf{K}(X)$. We would now like to make use of Theorem 2.1.30, so we investigate a class of objects adapted to the tensor product functor.

Definition 2.1.36. [14] A complex $P^\bullet \in \mathbf{C}(X)$ is called q-flat if for every acyclic complex $Q^\bullet \in \mathbf{C}(X)$, $P^\bullet \otimes Q^\bullet$ is acyclic.

Proposition 2.1.37. *For any complex $P^\bullet \in \boldsymbol{C}(X)$, the following conditions are equivalent:*

1. *$P^\bullet$ is q-flat,*
2. *For every quasi-isomorphism $Q_1^\bullet \to Q_2^\bullet$ in $\boldsymbol{C}(X)$, the resulting map $P^\bullet \otimes Q_1^\bullet \to P^\bullet \otimes Q_2^\bullet$ is also a quasi-isomorphism.*

Proof. Let $P^\bullet$ be a cochain complex in $\mathbf{C}(X)$ and suppose that we have a quasi-isomorphism $u : Q_1^\bullet \to Q_2^\bullet$. Now, u being a quasi-isomorphism is equivalent to $C_u^\bullet$ being acyclic. Hence it is clear that $P^\bullet$ being q-flat implies that F_P preserves quasi-isomorphisms. For the other implication, we use the fact that any acyclic complex $C^\bullet$ is the cone over the quasi-isomorphism $0 \to C^\bullet$. $\square$

Example 2.1.38. Any flat $\mathcal{O}_X$ module viewed as a complex in degree 0 is a q-flat object of $\mathbf{D}(X)$. In fact, any complex concentrated in a single degree with a flat $\mathcal{O}_X$-module in that degree is q-flat. The q-flat complexes form a Δ-subcategory, so we immediately see that bounded complexes of flat sheaves are q-flat.

Proposition 2.1.39 ([9]). *Every* Mod_X*-complex $C^\bullet$ is the target of a quasi-isomorphism ψ_C from a q-flat complex $P_{C^\bullet}^\bullet$, which can be constructed to depend functorially on $C^\bullet$, and so that $P_{C^\bullet[1]}^\bullet = P_{C^\bullet}^\bullet[1]$ and $\psi_{C^\bullet[1]} = \psi_{C^\bullet}[1]$.*

Let $A^\bullet$ be any complex in $\mathbf{K}(\mathcal{A})$. We would like to derive the functor

$$Q \circ F_{A^\bullet} : \mathbf{K}(X) \to \mathbf{D}(X),$$

where $F_{A^\bullet}$ is as defined above. To make use of Theorem 2.1.30, we now only need to show that every q-flat complex in $\mathbf{K}(X)$ is left-$F_{A^\bullet}$-acyclic.

Proposition 2.1.40. *Every q-flat complex $P^\bullet \in \boldsymbol{K}(X)$ is left-$F_{A^\bullet}$-acyclic for every $A^\bullet$ in $\boldsymbol{K}(X)$.*

Proof. Suppose we have a quasi-isomorphism $u : Q^\bullet \to P^\bullet$ in $\mathbf{K}(X)$. Consider the quasi-isomorphism $\psi_{Q^\bullet} : P^\bullet_{Q^\bullet} \to Q^\bullet$ coming from Proposition 2.1.39. Then we need to show that $F_{A^\bullet}(u\psi_{Q^\bullet})$ is an isomorphism. We prove more generally that $F_{A^\bullet}$ transforms a quasi-isomorphism between q-flat complexes $u : Q^\bullet \to P^\bullet$ into an isomorphism. To see this, note that there is a quasi-ismorphism $\psi_{A^\bullet} : P^\bullet_{A^\bullet} \to A^\bullet$ as in Proposition 2.1.39, and therefore also quasi-isomorphisms $P^\bullet_{A^\bullet} \otimes Q^\bullet \to F_{A^\bullet}(Q^\bullet)$ and $P^\bullet_{A^\bullet} \otimes P^\bullet \to F_{A^\bullet}(P^\bullet)$ by q-flatness, which become isomorphisms in $\mathbf{D}(X)$. We then have a commuting square in $\mathbf{D}(X)$

$$\begin{tikzcd}
P^\bullet_{A^\bullet} \otimes Q^\bullet \arrow[r, "1\otimes u"] \arrow[d, "\psi_{A^\bullet}\otimes 1"] & P^\bullet_{A^\bullet} \otimes P^\bullet \arrow[d, "\psi_{A^\bullet}\otimes 1"] \\
A^\bullet \otimes Q^\bullet \arrow[r, "1\otimes u"] & A^\bullet \otimes P^\bullet
\end{tikzcd}$$

with both vertical and the top horizontal arrows being isomorphisms. Therefore the bottom arrow is an isomorphism as desired. □

From all of the above results we may deduce the following. Since the functor $F_{A^\bullet}$ defined on objects by $F_{A^\bullet}(B^\bullet) = A^\bullet \otimes B^\bullet$ satisfies all of the conditions of Theorem 2.1.30, it has a left derived functor $\mathbf{L}F_{A^\bullet}$, whose image on $B^\bullet$ we denote by

$$A^\bullet \otimes^{\mathbf{L}} B^\bullet \cong A^\bullet \otimes P^\bullet_{B^\bullet} \cong P^\bullet_{A^\bullet} \otimes P^\bullet_{B^\bullet} \cong P^\bullet_{A^\bullet} \otimes B^\bullet \cong F_{B^\bullet}(A^\bullet),$$

where the isomorphisms in $\mathbf{D}(X)$ are coming from the quasi-isomorphisms in Proposition 2.1.39.

2.2 Koszul complexes

Now that we have defined the derived tensor product, we can see the importance of having explicit q-flat resolutions in understanding the derived tensor product of structure sheaves of subvarieties. In general, even flat sheaves are hard to understand, let alone q-flat complexes. Thankfully by 2.1.38 we know that we may look for resolutions which are made up of flat sheaves. A more readily available and more easily computable class of sheaves are the locally free sheaves. Hence we would like a class of complexes of locally free sheaves which are well behaved with respect to the tensor product functor and which resolve structure sheaves of subvarieties. This class is supplied by Koszul complexes.

Most of the results here can be found in [11] for the algebraic case or [4] for the geometric case.

2.2.1 Koszul complexes in commutative algebra

Let R be a ring (always assumed commutative and unital) and let $s : M \to R$ be a map of R-modules.

Definition 2.2.1. The *Koszul complex* of (M, s) over R is the complex of R-modules concentrated in negative degree whose terms are $K^{-p}(M, s) = \bigwedge_R^p M$ for $p \geq 0$ with differential defined by;

$$m_1 \wedge \cdots \wedge m_p \mapsto \sum_{i=1}^{p} (-1)^i s(m_i)(m_1 \wedge \cdots \wedge \widehat{m_i} \wedge \cdots \wedge m_p)$$

where $\widehat{m_i}$ means to omit the m_i term from the exterior product.

There is an alternative description of the Koszul differential which has the benefit of notational brevity. Given a map $s : M \to R$ there is an induced map $1 \otimes \cdots \otimes 1 \otimes s : M^{\otimes p} \to M^{\otimes p-1}$. There is a natural embedding $\varphi : \bigwedge^p M \to M^{\otimes p}$ given by linearly extending the mapping

$$m_1 \wedge \cdots \wedge m_p \mapsto \sum_{\sigma \in S_p} (-1)^{sgn(\sigma)} m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(p)},$$

where S_p is the group of permutations on p elements. The image of $(1 \otimes \cdots \otimes 1 \otimes s) \circ \varphi$ is contained in the image of the inclusion of $\bigwedge^{p-1} M$. We denote the induced map on the level of exterior powers by $1 \wedge \cdots \wedge s : \bigwedge^p M \to \bigwedge^{p-1} M$. We sometimes abuse notation and write this map as $1 \wedge s$. Then the differential in the Koszul complex $K^\bullet(M, s)$ is given by $1 \wedge s$.

Proposition 2.2.2. *There is a functor $K^\bullet(-)$ from the category whose objects are pairs (M, s) of R-modules with morphism to R and whose morphisms are $f : (M, s) \to (N, t)$ given by a module map $f : M \to N$ such that $t \circ f = s$ to the category of complexes of R-modules. This functor returns $K^\bullet(M, s)$ on the object (M, s).*

Proof. We need to show that given such a morphism $f : (M, s) \to (N, t)$, we get a morphism $K^\bullet(M, s) \to K^\bullet(N, t)$. Note however, that for each n there is a commuting square

$$\begin{tikzcd}
M^{\otimes n} \arrow[r, "1\otimes s"] \arrow[d, "f^{\otimes n}"] & M^{\otimes n-1} \arrow[d, "f^{\otimes n-1}"] \\
N^{\otimes n} \arrow[r, "1\otimes t"] & N^{\otimes n-1}
\end{tikzcd}$$

which together induce the desired chain map on the level of exterior powers. The composition property for morphisms follows directly. $\square$

When $M \cong R^r$ is a free R-module of finite rank, we write $K^\bullet(f_1, \ldots, f_r)$ or $K^\bullet(\underline{f})$ for $K^\bullet(M, s)$, where $f_i = s(e_i)$ is the image of the i^{th} standard generator. This presentation depends on a choice of basis for M, but by the above result, any two choices of isomorphism $M \to R^r$ will result in isomorphic Koszul complexes.

Example 2.2.3. If we take a single element f of the ring R, then $K^\bullet(f)$ is the complex given by

$$R \xrightarrow{f} R$$

where here the R on the right is in degree 0 and the R on the left is in degree -1. All the other terms of this complex are 0 and so omitted.

Example 2.2.4. If we take three elements f_1, f_2, f_3 of a ring R, then we represent the Koszul complex $K^\bullet(f_1, f_2, f_3)$ pictorially by

$$\begin{tikzcd}
 & R \arrow[r, "f_2"] \arrow[dr, "-f_1" near start] & R \arrow[dr, "f_1"] & \\
R \arrow[ur, "f_3"] \arrow[r, "-f_2"] \arrow[dr, "f_1"] & R \arrow[ur, "f_3" near start] \arrow[dr, "-f_1" near start] & R \arrow[r, "f_2"] & R \\
 & R \arrow[ur, "f_3" near start] \arrow[r, "-f_2"'] & R \arrow[ur, "f_3"] &
\end{tikzcd}$$

Here, the columns are the terms of the complex, starting on the right with the degree 0 part. The meaning of having 3 copies of R in a term is $R \oplus R \oplus R$, where we have omitted the direct sum symbol in the above picture. We identify $\bigwedge^2 R^3$ with R^3 by choosing the basis $e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3$, where e_1, e_2, e_3 are standard generators for R^3. There is no need for us to pick distinct elements f_1, f_2, f_3, we may write the above complex for any choices of three elements of R. Later we will also see examples of Koszul complexes of

2 elements and 4 elements. Their pictorial representations are built analogously to the Koszul complex of 3 elements. This could be done for any number of elements but very quickly the diagrams become unwieldy.

Proposition 2.2.5. *Suppose we have two linear maps $s : M \to R, t : N \to R$ of free modules of finite rank into R. Then we have a natural identification*

$$K^\bullet(M, s) \otimes K^\bullet(N, t) \cong K^\bullet(M \oplus N, s \oplus t).$$

In particular if $N \cong R$, we get

$$K^\bullet(M \oplus N, s \oplus t) \cong Cone(t : K^\bullet(M, s) \to K^\bullet(M, s)).$$

Proof. First, choose isomorphisms $M \cong R^r, N \cong R^s$ so that we can rewrite the statements we are trying to prove to the isomorphisms

$$K^\bullet(f_1, \dots, f_r) \otimes K^\bullet(g_1, \dots, g_s) \cong K^\bullet(f_1, \dots, f_r, g_1, \dots g_s)$$

and

$$K^\bullet(f_1, \dots, f_r) \otimes K^\bullet(g) \cong Cone(g : K^\bullet(f_1, \dots, f_r) \to K^\bullet(f_1, \dots, f_r)).$$

For the first isomorphism, we run an argument by induction on $r + s$. Suppose such an isomorphism exists for $r + s - 1$, by the associativity of the tensor product and the inductive hypothesis, we have

$$K^\bullet(f_1, \dots, f_r) \otimes K^\bullet(g_1, \dots, g_s) \cong K^\bullet(f_1, \dots, f_r, g_1, \dots, g_{s-1}) \otimes K^\bullet(g_s).$$

To see that this is isomorphic to $K^\bullet(f_1, \dots, f_r, g_1, \dots, g_s)$ note that the tensor complex has terms

$$\bigwedge^q R^{r+s-1} \oplus \bigwedge^{q-1} R^{r+s-1} \cong \bigwedge^q R^{r+s}.$$

This isomorphism is constructed as follows. We can consider an element $e_{i_1} \wedge \dots \wedge e_{i_q} \in \bigwedge^q R^{r+s-1}$ as an element of $\bigwedge^q R^{r+s}$ directly. An element $e_{i_1} \wedge \dots \wedge e_{i_{q-1}} \in \bigwedge^{q-1} R^{r+s-1}$ gets mapped to the element $e_{i_1} \wedge \dots \wedge e_{i_{q-1}} \wedge e_{r+s} \in \bigwedge^q R^{r+s}$. Under these identifications, it is clear that the differential in the complex $K^\bullet(f_1, \dots, f_r, g_1, \dots, g_{s-1}) \otimes K^\bullet(g_s)$ aligns with the differential in $K^\bullet(f_1, \dots, g_s)$. This argument also proves the second statement. $\square$

From the above results we see that Koszul complexes are explicit free complexes which behave well under tensor products and therefore would be good candidates for q-flat resolutions of structure sheaves. Hence we are interested in finding objects of our derived categories which have (local) Koszul models, i.e. they are resolved by Koszul complexes. The following is a well known criterion for a Koszul complex $K^\bullet(f_1, \dots, f_n)$ to be a resolution of $R/(f_1, \dots, f_n)$.

Definition 2.2.6. We say a sequence of elements $f_1, \dots, f_n \in R$ form a regular sequence in R if f_1 is not a zero-divisor in R and for each $i > 1$, f_i is not a zero-divisor in $R/(f_1, \dots, f_{i-1})R$.

Proposition 2.2.7. *Let $f_1, \dots, f_r$ form a regular sequence in a ring R. Then $H^i(K^\bullet(f_1, \dots, f_r)) = 0$ for all $i \neq 0$ and $H^0(K^\bullet(f_1, \dots, f_r)) = R/(f_1, \dots, f_r)$, i.e. $K^\bullet(f_1, \dots, f_r)$ is a free resolution of $R/(f_1, \dots, f_r)$.*

Proof. By induction. If $r = 1$, then since f_1 is not a zero-divisor in R we can see that the cohomology of the complex

$$K^\bullet(f_1) = \{0 \to R \xrightarrow{f_1} R \to 0\}$$

is as desired. By Proposition 2.2.5 we can write

$$K^\bullet(f_1, \dots, f_r) = Cone(f_r : K^\bullet(f_1, \dots, f_{r-1}) \to K^\bullet(f_1, \dots, f_{r-1})).$$

The upshot of viewing the Koszul complex as a cone is that we have a long exact sequence on cohomology

$$\dots \to H^j(K_{r-1}) \to H^j(K_{r-1}) \to H^j(K_r) \to H^{j+1}(K_{r-1}) \to \dots$$

where K_r is shorthand notation for $K^\bullet(f_1, \dots, f_r)$. By the inductive hypothesis, the only non-zero part of this exact sequence is

$$0 \to H^{-1}(K_r) \to H^0(K_{r-1}) \to H^0(K_{r-1}) \to H^0(K_r) \to 0$$

where $H^0(K_{r-1}) \to H^0(K_{r-1})$ is multiplication by f_r. Since $H^0(K_{r-1}) = R/(f_1, \dots, f_{r-1})R$ and f_r is not a zero-divisor in this quotient ring, multiplication by f_r is injective. Hence we see that $H^{-1}(K_r) = 0$ and also that $H^0(K_r) = (R/(f_1, \dots, f_{r-1}))/(f_r) = R/(f_1, \dots, f_r)$ as desired. $\square$

In general the converse does not hold (see [8]), but for instance in noetherian local rings we do have an equivalence statement between a sequence being regular and its associated Koszul complex being a resolution of the quotient ring [11].

Proposition 2.2.8. *Let $s : M \to R$ be a morphism of free R-modules where rank$(M) = n$ and suppose that the image of s is generated by a regular sequence $f_1, \ldots, f_n$. Let $I = (f_1, \ldots, f_n)$. Then I/I^2 is a free R/I-module of rank n isomorphic to $M \otimes R/I$.*

Proof. Truncating the Koszul complex $K^\bullet(M, s)$ there is an exact sequence

$$\bigwedge^2 M \to M \to I \to 0.$$

Tensoring with R/I (which is right-exact) we get an exact sequence

$$\bigwedge^2 M \otimes R/I \to M \otimes R/I \to I/I^2 \to 0.$$

However, the image of $\bigwedge^2 M \otimes R/I \to M \otimes R/I$ factors through the image of s, and therefore vanishes, so we have an isomorphism $I/I^2 \cong (M \otimes R/I)$, i.e. I/I^2 is a free R/I module of rank n. $\square$

Proposition 2.2.9. *Let R be a ring and M an R-module, and suppose we have elements $y_1, \ldots, y_r \in (x_1, \ldots, x_s)$. Then we have*

$$H^{-q}(K^\bullet(x_1, \ldots, x_s, y_1, \ldots, y_r) \otimes M) \cong \bigoplus_{i+j=q} \left(H^{-i}(K^\bullet(x_1, \ldots, x_s) \otimes M) \otimes \bigwedge^j R^r \right)$$

Proof following [3] Corollary 17.20. The first step is to prove that

$$K^\bullet(x_1, \ldots, x_s, y_1, \ldots, y_r) \otimes M \cong K^\bullet(x_1, \ldots, x_s, 0, \ldots, 0) \otimes M.$$

By Proposition 2.2.2 we need only demonstrate an automorphism $R^{s+r} \to R^{s+r}$ commuting with the maps $(x_1, \ldots, x_s, y_1, \ldots, y_r)$ and $(x_1, \ldots, x_s, 0, \ldots, 0)$ to R. Since each $y_i \in (x_1, \ldots, x_s)$, we can express $y_i = \sum \lambda_{ij} x_j$ for each y_i, $\lambda_{ij} \in R$. Let Λ be the matrix with entries λ_{ij}. Then the morphism $R^{s+r} \to R^{s+r}$ represented by matrix

$$\begin{pmatrix} I & 0 \\ \Lambda & I \end{pmatrix}$$

is an isomorphism commuting the given sections. This isomorphism induces the isomorphism of Koszul complexes we are looking for.

The next step is to show that the cohomology of the complex $K^\bullet(x_1, \dots, x_s, 0, \dots, 0; M) \cong K^\bullet(x_1, \dots, x_s; M) \otimes K^\bullet(0, \dots, 0; M)$ is as desired. This follows from the following more general statement. Suppose we are given two complexes $C^\bullet$ and $D^\bullet$ where the differential in $D^\bullet$ is 0. Then there is a canonical isomorphism

$$H^q(C^\bullet \otimes D^\bullet) \cong \bigoplus_{i+j=q} H^i(C^\bullet) \otimes D^j.$$

□

2.2.2 Koszul complexes in geometry

The algebraic notions of §2.2.1 can be interpreted in an algebro-geometric context.

Definition 2.2.10. Let X be a scheme, $\mathcal{E}$ an $\mathcal{O}_X$-module and $s : \mathcal{E} \to \mathcal{O}_X$ a map of $\mathcal{O}_X$-modules. We can define the Koszul complex $K^\bullet(\mathcal{E}, s)$ analogously to the algebraic case. It is a complex concentrated in negative degrees with $K^{-p}(\mathcal{E}, s) = \bigwedge^p \mathcal{E}$ for $p \geq 0$ and differential $1 \wedge \cdots \wedge 1 \wedge s$.

We will always be working in the case that $\mathcal{E}$ is a locally free $\mathcal{O}_X$ module, in which $K^\bullet(\mathcal{E}, s)$ is locally isomorphic to a complex of the form $K^\bullet(f_1, \dots, f_r)$. All of the following results follow directly from working locally and using the results of §2.2.1

Example 2.2.11. Consider an effective Cartier divisor $D \subset X$. Then the ideal sheaf of D is an invertible $\mathcal{O}_X$-module ([6] II Prop 6.18) and comes with a natural inclusion $\mathcal{O}(-D) \to \mathcal{O}_X$. This morphism already forms a Koszul complex (in fact as we will see it is a resolution) analogous to the one from Example 2.2.3.

Proposition 2.2.12. *There is a functor $K^\bullet(-)$ from the category whose objects are pairs $(\mathcal{E}, s)$ of $\mathcal{O}_X$-modules with morphism to $\mathcal{O}_X$ and whose morphisms are $f : (\mathcal{E}, s) \to (\mathcal{F}, t)$ given by a module map $f : \mathcal{E} \to \mathcal{F}$ such that $t \circ f = s$ to the category of complexes of $\mathcal{O}_X$-modules. This functor returns $K^\bullet(\mathcal{E}, s)$ on the object $(\mathcal{E}, s)$.*

Proposition 2.2.13. *Suppose we have two morphisms $s : \mathcal{E} \to \mathcal{O}_X, t : \mathcal{F} \to \mathcal{O}_X$ of locally free $\mathcal{O}_X$-modules of finite rank into $\mathcal{O}_X$. Then we have an identification*

$$K^\bullet(\mathcal{E} \oplus \mathcal{F}, s \oplus t) \cong K^\bullet(\mathcal{E}, s) \otimes K^\bullet(\mathcal{F}, t).$$

Additionally, if $t : \mathcal{L} \to \mathcal{O}_X$ is a map from a line bundle, then we have

$$K^\bullet(\mathcal{E} \oplus \mathcal{L}, s \oplus t) \cong Cone(t : K^\bullet(\mathcal{E}, s) \otimes \mathcal{L} \to K^\bullet(\mathcal{E}, s)).$$

We say that an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ is a regular ideal sheaf if it can locally be generated by a regular sequence of sections of $\mathcal{O}_X$. We call a section of a locally free sheaf $s : \mathcal{E} \to \mathcal{O}_X$ a regular section if its image sheaf is a regular ideal sheaf, locally generated by a number of sections equal to the rank of $\mathcal{E}$. Any regular ideal sheaf is locally the image sheaf of a regular section. Indeed, working affine locally, the ideal $(f_1, \ldots, f_n)$ is the image of the map $R^n \to R$ sending $e_i \mapsto f_i$. We have the following relation between Koszul complexes and regular sections:

Proposition 2.2.14. *For any regular section $s : \mathcal{E} \to \mathcal{O}_X$, $K^\bullet(\mathcal{E}, s)$ is a global locally free resolution of the structure sheaf of $Z(s)$, the zero locus of the section s.*

Proof. $K^\bullet(\mathcal{E}, s)$ is a globally defined complex, its being a resolution is a local property. Locally however $K^\bullet(\mathcal{E}, s)$ looks like $K^\bullet(f_1, \ldots, f_n)$ where $f_1, \ldots, f_n$ is the regular sequence defining $\mathcal{O}_{Z(s)}$. These complexes have no higher cohomology by Prop 2.2.7 and so neither does $K^\bullet(\mathcal{E}, s)$. □

Proposition 2.2.15. *Let $\mathcal{E}$ be a locally free sheaf of rank n on a scheme X. Let $s : \mathcal{E} \to \mathcal{O}_X$ be a regular section and $\mathcal{I} \subset \mathcal{O}_X$ the image sheaf with corresponding closed subscheme Y. Then there is an isomorphism $\varphi_s : \mathcal{N}_{Y/X} \to \mathcal{E}^\vee|_Y$.*

Proof. The section s is regular, meaning $s : \mathcal{E} \to \mathcal{O}_X$ is a morphism of $\mathcal{O}_X$-modules whose image is a regular ideal sheaf $\mathcal{I}$. Then, we can conclude from Proposition 2.2.8 that $(\mathcal{I}/\mathcal{I}^2)|_Y$ is a locally free $\mathcal{O}_Y$-module. In fact the Proposition tells us that the surjective morphism $s : \mathcal{E} \to \mathcal{I}$ becomes an isomorphism when restricted to Y. Dualizing this trivialisation gives an identification $\varphi_s : \mathcal{N}_{Y/X} \to \mathcal{E}^\vee|_Y$. □

Later we will need the following technical lemma on the interaction of exterior products with tensor products of locally free sheaves.

Lemma 2.2.16. *Let $\mathcal{E}, \mathcal{L}$ be locally free sheaves on a scheme X with $rank(\mathcal{L}) = 1$. Then*

$$\bigwedge^k (\mathcal{E} \otimes \mathcal{L}) \cong \left(\bigwedge^k \mathcal{E} \right) \otimes \mathcal{L}^{\otimes k}.$$

Proof. There is a natural morphism $(\mathcal{E} \otimes \mathcal{L})^{\otimes k} \to \bigwedge^k \mathcal{E} \otimes \mathcal{L}^{\otimes k}$. This descends to a morphism $\bigwedge^k(\mathcal{E}\otimes\mathcal{L}) \to \bigwedge^k \mathcal{E}\otimes\mathcal{L}^{\otimes k}$ since any section $(e\otimes l_1)\wedge(e\otimes l_2)$ is locally everywhere 0 (as $\mathcal{L}$ has rank 1) and hence globally 0. This morphism is clearly locally everywhere an isomorphism as $\mathcal{L}$ is an invertible sheaf and therefore is a global isomorphism. □

2.3 Local Complete Intersections

Throughout this section X is a nonsingular variety, although many of the results will hold if X is a Cohen-Macaulay scheme. A closed subscheme $Y \subset X$ is called a local complete intersection (lci) if locally around every closed point $y \in Y$, there is a neighbourhood U of y in X such that $\mathcal{I}_Y(U)$ is generated by $\mathrm{codim}(Y, X)$ elements. Note that this definition makes use of a global notion of codimension and not a local one.

Definition 2.3.1 ([6] II.3)**.** For an irreducible closed subset Z of a scheme X, we defined the codimension of Z in X to be the supremum of integers n such that there exists a chain

$$Z = Z_0 \subset Z_1 \subset \cdots \subset Z_n$$

of distinct closed irreducible subsets of X. For any closed subset $Y \subset X$ of a scheme, we define

$$\mathrm{codim}(Y, X) = \mathrm{inf}_{Z\subset Y}\mathrm{codim}(Z, X)$$

where the infimum is taken over all closed irreducible subsets of Y.

An immediate consequence is that if Y is not equidimensional then it cannot be a lci, because the minimum number of equations required to define the highest codimension components is equal to their codimension, but the codimension of Y is equal to the *smallest* codimension of its irreducible components. In our case where X is a nonsingular variety over k, we make use of the following

Theorem 2.3.2 ([11] Theorem 17.4)**.** *Let $(A, \mathfrak{m})$ be a noetherian Cohen-Macaulay local ring. Then*

1. *For every ideal I in A, we have an equality $ht(I) + dim(A/I) = dim(A)$.*

2. *A sequence $a_1, \ldots, a_r \in \mathfrak{m}$ is regular $\iff$ $ht(a_1, \ldots, a_r) = r$.*

This shows that a local complete intersection is defined by a regular ideal sheaf in a noetherian Cohen-Macaulay scheme, and therefore locally has Koszul resolutions of its structure sheaf. A local complete intersection can be seen as the generalisation of the notion of effecive Cartier divisor to higher codimensions. Indeed, effective Cartier divisors are defined by having regular ideal sheaves locally generated by a single regular element.

We give some simple examples of the sorts of phenomena that are typical of local complete intersections and non-local complete intersections.

Example 2.3.3. Let $X = \mathbb{A}^3_k$ and let $Y = Z(xy, xz, yz)$ the closed subscheme given as the union of the 3 coordinate axes. Then Y is not a local complete intersection in X. Indeed, at the origin, 3 equations are required to carve out Y, but Y only has codimension 2.

Example 2.3.4. ([6] Theorem II.8.17) Any non-singular subvariety of a non-singular variety is a local complete intersection.

Example 2.3.5. The twisted cubic curve in $\mathbb{P}^3$ is a local complete intersection which is not a complete intersection. In homogeneous coordinates $[X : Y : Z : W]$, it is defined by the three equations $C = V(XZ - Y^2, YW - Z^2, XW - YZ)$. There is no pair of homogenous equations which defines this curve, and since the curve clearly has codimension 2 in $\mathbb{P}^3$ it cannot be a complete intersection globally. However, in any of the standard open affine charts in $\mathbb{P}^3$, only 2 of the given equations are necessary. For instance, in U_X (the open affine chart where X is invertible), the ideal $(XZ - Y^2, YW - Z^2, XW - YZ)$ becomes the ideal $(z - y^2, yw - z^2, w - yz) \subset k[y, z, w]$, where $y = Y/X, z = Z/X, w = W/X$. However,

$$yw - z^2 = y(w - yz) - z(z - y^2)$$

so the ideal $(z - y^2, yw - z^2, w - yz) = (z - y^2, w - yz)$. The same kind of analysis can be done for the open affines U_Y, U_Z, and U_W.

Proposition 2.3.6. *Let $i : Y \to X$ be a local complete intersection subvariety of a non-singular variety over k. Then around every point $y \in Y$ there is an open neighbourhood $U \subset X$ such that $i_*\mathcal{O}_Y|_U$ has a free Koszul resolution $K^\bullet(\mathcal{E}, s)$.*

Proof. By Theorem 2.3.2 there exists U an open affine neighbourhood of y such that locally the ideal sheaf of Y is generated by a regular sequence of sections $\mathcal{I}_Y|_U = (f_1, \ldots, f_n)$. Then take the sheafification of the algebraic complex $K^\bullet(f_1, \ldots, f_n)$. $\square$

2.4 Tor-Independence

In the proofs of the main results later, we will have squares

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{u} & Y \end{array}$$

which are pullback squares. These are square where the data of X', v, g are universal with respect to the starting data of the maps $Y' \xrightarrow{u} Y \xleftarrow{f} X$. This means that given any other object Z with morphisms to X and Y' making the resulting square commute, there is a unique morphism $Z \xrightarrow{\varphi} X'$ such that the morphisms from Z to X and Y' factor through φ. Given such a fibre square of schemes we will want to make the identification

$$f^* u_* \cong v_* g^*.$$

Hence we include a brief discussion on the base-change map and recall the notion of Tor-independence. Here we lay out a sketch of some of this technical material but proofs of well-established results are not given. They may be found in [9].

Definition 2.4.1. Let $f : X \to Y$ be a morphism of schemes and let $\mathrm{Mod}_X, \mathrm{Mod}_Y$ denote the category of quasi-coherent $\mathcal{O}_X, \mathcal{O}_Y$ modules respectively. Then the functors

$$f^* : \mathrm{Mod}_Y \to \mathrm{Mod}_X, \quad f_* : \mathrm{Mod}_X \to \mathrm{Mod}_Y$$

are adjoint. The functor f^* is exact on flat objects and f_* is exact on injective objects. Choose functorial families of quasi-isomorphisms

$$A^\bullet \to I^\bullet_{A^\bullet}, \quad P^\bullet_{B^\bullet} \to B^\bullet$$

for all complexes $A^\bullet \in \mathbf{K}(X), B^\bullet \in \mathbf{K}(Y)$ where $I^\bullet_{A^\bullet}$ is q-injective and $P^\bullet_{B^\bullet}$ is q-flat. Then there are derived functors

$$(\mathbf{R}f_*, 1), \quad (\mathbf{L}f^*, 1)$$

with

$$\mathbf{R}f_*(A) = f_*(I^\bullet_{A^\bullet}), \quad \mathbf{L}f^*(B) = f^*(P^\bullet_{B^\bullet}).$$

Furthermore these functors are also adjoint.

Theorem 2.4.2 ([9] §2.6)**.** *Let $\boldsymbol{S}$ be the category of ringed spaces. For each object $X \in \boldsymbol{S}$, set $X^* = X_* = \boldsymbol{D}(X)$, the derived category of the category of $\mathcal{O}_X$-modules. This category is a closed (has an internal hom functor given by $\boldsymbol{R}\mathcal{H}om_{\mathcal{O}_X}$ and is symmetric monoidal) Δ-category with product $\otimes^{\boldsymbol{L}}$ and unit $\mathcal{O}_X$. For $X \xrightarrow{f} Y$ in $\boldsymbol{S}$, write*

$$f^* \quad for \quad \boldsymbol{L}f^* : Y^* \to X^*,$$

$$f_* \quad for \quad \boldsymbol{R}f_* : X_* \to Y^*.$$

Then this defines an adjoint pair $(^*, _*)$ of monoidal Δ-pseudofunctors on $\boldsymbol{S}$.

We have not nearly included as much exposition as is necessary to rigorously parse this theorem, but we state the implications of this general statement that we will use.

Corollary 2.4.3. *For morphisms of schemes $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have functorial isomorphisms*

$$\boldsymbol{R}(g \circ f)_* \cong \boldsymbol{R}g_* \circ \boldsymbol{R}f_*,$$

$$\boldsymbol{L}(g \circ f)^* \cong \boldsymbol{L}f^* \circ \boldsymbol{L}g^*,$$

$$\boldsymbol{L}f^*(A \otimes^{\boldsymbol{L}} B) \cong \boldsymbol{L}f^* A \otimes^{\boldsymbol{L}} \boldsymbol{L}f^* B.$$

We will implicitly be using the first two isomorphisms.

Remark 2.4.4. *This is not really a Corollary of the preceding Theorem, as all of these functorial isomorphisms are necessary ingredients in the proof of the Theorem. We present this result in this fashion since it is succinct.*

For the rest of this section the pushforward and pullback functors are derived and we suppress the $\mathbf{R}, \mathbf{L}$ that usually attends derived functors. It is worth noting that the pushforward by a closed immersion and the pullback by an open immersion are exact functors, in which case their derived functors are isomorphic to the underived functor.

As a consequence of the adjointness of derived pushforward and derived pullback there are always bifunctorial *projection morphisms*;

$$p_1 : f_*F \otimes G \to f_*(F \otimes f^*G)$$

$$p_2 : G \otimes f_*F \to f_*(f^*G \otimes F)$$

given as the respective compositions

$$f_*F \otimes G \xrightarrow{1\otimes\eta} f_*F \otimes f_*f^*G \to f_*(F \otimes f^*G),$$

$$G \otimes f_*F \xrightarrow{\eta\otimes 1} f_*f^*G \otimes f_*F \to f_*(f^*G \otimes F),$$

where $\eta : 1 \to f_*f^*$ is the adjunction unit. We have the following result for when the projection morphisms are isomorphisms.

Definition 2.4.5. A map of schemes $f : X \to Y$ is said to be concentrated if it is quasi-compact and quasi-separated.

Proposition 2.4.6 ([9] Proposition 3.9.4)**.** *Let $f : X \to Y$ be a concentrated scheme-map, let $F \in \boldsymbol{D}(X), G \in \boldsymbol{D}_{qc}(Y)$, where $\boldsymbol{D}_{qc}(Y)$ is the full subcategory of complexes with quasi-coherent cohomology. Then if f is finite-dimensional, or if $F \in \boldsymbol{D}_{qc}(X)$ then both projection maps p_1 and p_2 are isomorphisms.*

If we have a commuting square σ

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"] & Y \end{tikzcd}$$

we define the base change map

$$\beta_\sigma : f^*u_* \to v_*g^*$$

to be the composition

$$f^*u_* \to f^*u_*g_*g^* \to f^*f_*v_*g^* \to v_*g^*$$

where the first and third arrows come from the pullback-pushforward adjunctions and the middle arrow from the commutativity of the square.

Definition 2.4.7. A fibre square of schemes over a scheme S

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{u} & Y \end{array}$$

is said to be *Tor-independent* if for all pairs of points $y' \in Y', x \in X$ such that $u(y') = y = f(x)$ we have

$$Tor_q^{\mathcal{O}_{Y,y}}(u_*\mathcal{O}_{Y',y'}, f_*\mathcal{O}_{X,x}) = 0 \quad \forall q > 0.$$

Proposition 2.4.8. *If the fibre square σ in Definition 2.4.7 has either f or u being a flat morphism, then it is Tor-independent.*

Proof. Follows immediately from the definition of flat morphism, and Tor vanishing on flat modules in either argument. □

The benefit of Tor-independence is that it gives a homological criterion for a fibre square to be a base change square. This is not always true, but we do have the following characterisation of squares with this equivalence.

Theorem 2.4.9 ([9] Theorem 3.10.3)**.** *For any fibre square as in Definition 2.4.7 where the maps are concentrated and the schemes quasi-separated, Tor-independence is equivalent to the base change map for the square being an isomorphism;*

$$f^* u_* \cong v_* g^*$$

All of the maps and objects that we deal with in §§3,4 will satisfy the hypotheses of this Theorem. We state and prove some results on the kinds of fibre squares which are Tor-independent and which we will need to make use of later.

Proposition 2.4.10. *Let X be a scheme. Let $i : Y \to X$ be a closed immersion and $j : U \to X$ an open immersion. Then the fibre square*

$$\begin{array}{ccc} Y \cap U & \xrightarrow{i_U} & U \\ \downarrow{\scriptstyle j_Y} & & \downarrow{\scriptstyle j} \\ Y & \xrightarrow{i} & X \end{array}$$

is Tor-independent.

Proof. Follows from Prop 2.4.8 since open immersions are flat. $\square$

Proposition 2.4.11. *Let X be a Cohen-Macaulay scheme with local complete intersection subschemes Y and Z. If $Y \cap Z$ has the expected codimension then $Y \cap Z$ is also a local complete intersection in X and the following fibre square is Tor-independent*

$$\begin{tikzcd} Y \cap Z \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

Proof. Let the local defining ideals for Y and Z be given by $(f_1, \dots, f_n)$ and $(g_1, \dots, g_m)$ respectively. The ideal $I = (f_1, \dots, f_n, g_1, \dots, g_m)$ cuts out $Y \cap Z$ which can be generated by a regular sequence of length $n+m$ by the properness assumption. From 2.3.2, we know that $ht(I) = n + m$ and also that the sequence $f_1, \dots, f_n, g_1, \dots, g_m$ is regular. Since the $Tor_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$ can be computed locally by tensoring the Koszul resolutions for the structure sheaves, it is computed from the cohomology of $K^\bullet(f_1, \dots, f_n, g_1, \dots, g_m)$. These cohomologies vanish since we have just shown that this sequence is regular. $\square$

Lemma 2.4.12. *Let X be a nonsingular variety and $D \subset X$ an effective Cartier divisor. Let Z be any subvariety of X such that D does not contain any associated points of Z. Then the fibre square of closed immersions*

$$\begin{tikzcd} Z \cap D \arrow[r, "j_D"] \arrow[d, "i_Z"'] \arrow[dr, "h"] & D \arrow[d, "i"] \\ Z \arrow[r, "j"'] & X. \end{tikzcd}$$

is Tor-independent.

Proof. Tor-independence is a local criterion, so we are reduced to proving the algebraic statement that

$$\mathrm{Tor}_q^R(R/I, R/(x)) = 0 \quad \forall q \geq 1$$

for R a local ring and x a non-zero divisor in R. For $q \geq 2$ this follows from the existence of the free resolution

$$R \xrightarrow{x} R$$

of $R/(x)$. For $q = 1$, we wish to show that the map

$$R/I \xrightarrow{x} R/I$$

is injective. This follows from the fact that D does not contain any associated points of Z. $\square$

Chapter 3

Lci-intersections

In this section we provide proofs for results that exist in the literature on the geometric properties of the derived intersections of local complete intersections whose intersection is also a local complete intersection. We will first give a classical account of the derived self-intersection formula for a local complete intersection. The second part of this section is dedicated to a novel proof of a result by Scala in [12] of an excess intersection formula for lci intersections.

We first prove that within the framework of intersections of local complete intersections the local computations of Koszul cohomologies will glue to agree with a global model for the multitors. To begin with, suppose that $Y_1, \dots, Y_n$ are local complete intersection subvarieties of a scheme X. Then there are global flat resolutions $\mathcal{F}_i^\bullet \to \mathcal{O}_{Y_i}$ for each i since the category of quasi-coherent sheaves on a scheme X has enough flats. We fix a global model of the multitors by defining

$$\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_n}) := H^{-q}(\mathcal{F}_1^\bullet \otimes \cdots \otimes \mathcal{F}_n^\bullet).$$

For $p \in Z = \bigcap Y_i$, let U be a local affine neighbourhood such that each $\mathcal{O}_{Y_i}|_U$ can be resolved by a free Koszul complex $K^\bullet(E_i, s_i)$, which exists because each Y_i is a local complete intersection. Restriction to U is an exact functor which preserves flatness, so the restriction of the augmentation $\mathcal{F}_i^\bullet|_U \to \mathcal{O}_{Y_i}|_U$ is still a quasi-isomorphism. Then since the complexes $K^\bullet(E_i, s_i)$ are free resolutions of $\mathcal{O}_{Y_i}|_U$ (and therefore q-projective), there are morphisms (unique up to homotopy) $\psi_i : K^\bullet(E_i, s_i) \to \mathcal{F}_i^\bullet|_U$ lifting the augmentation to $\mathcal{O}_{Y_i}|_U$. That is, making the following triangle commute;

$$\begin{tikzcd} & \mathcal{F}_i^\bullet|_U \arrow[d] \\ K^\bullet(E_i,s_i) \arrow[ur, dashed, "\psi_i"] \arrow[r] & \mathcal{O}_{Y_i}|_U \end{tikzcd}$$

Necessarily, the ψ_i are quasi-isomorphisms because both of the map from $K^\bullet(E_i,s_i)$ and $\mathcal{F}_i^\bullet|_U$ to $\mathcal{O}_{Y_i}|_U$ are quasi-isomorphisms. These lifts ψ_i together induce a quasi-isomorphism

$$\psi : K^\bullet(E_1,s_1)\otimes\cdots\otimes K^\bullet(E_n,s_n) \cong K^\bullet\left(\bigoplus E_j, \oplus s_j\right) \to \mathcal{F}_1^\bullet|_U\otimes\cdots\otimes\mathcal{F}_n^\bullet|_U,$$

because each $\mathcal{F}_i^\bullet|_U$ is a q-flat complex and tensoring with q-flat complexes preserves quasi-isomorphisms. Since ψ is a quasi-isomorphism, it gives isomorphisms in cohomology

$$H^{-q}(\psi) : H^{-q}\left(K^\bullet\left(\bigoplus E_j, \oplus s_j\right)\right) \to H^{-q}(\mathcal{F}_1^\bullet|_U\otimes\cdots\otimes\mathcal{F}_n^\bullet|_U) \cong \mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{Y_1},\ldots,\mathcal{O}_{Y_n})|_U.$$

As the ψ_i are unique up to homotopy, the $H^{-q}\psi$ are natural isomorphisms on cohomology coming from choices of Koszul resolutions $K^\bullet(E_i,s_i)\to\mathcal{O}_{Y_i}|_U$.

Take another open set $U'\subset X$ such that $U\cap U' = V \neq \emptyset$ and sufficiently local that we construct similar isomorphisms, with local Koszul resolutions $K^\bullet(E_i', s_i') \to \mathcal{O}_{Y_i}|_{U'}$. On V, we have a commuting diagram of lifts unique up to homotopy for each $\mathcal{O}_{Y_i}|_V$

$$\begin{tikzcd} K^\bullet(E_i,s_i)|_V \arrow[r, "\psi_i|_V"] \arrow[dr] & \mathcal{F}_i^\bullet|_V \arrow[d] & K^\bullet(E_i',s_i')|_V \arrow[l, "\psi_i'|_V"'] \arrow[dl] \\ & \mathcal{O}_{Y_i}|_V. & \end{tikzcd}$$

There are therefore diagrams commuting up to homotopy;

$$\begin{tikzcd} K^\bullet(E_i,s_i)|_V \arrow[rr, "\chi_i"] \arrow[dr, "\psi_i|_V"'] & & K^\bullet(E_i',s_i')|_V \arrow[dl, "\psi_i'|_V"] \\ & \mathcal{F}_i^\bullet|_V & \end{tikzcd}$$

of lifts of the augmentations to $\mathcal{O}_{Y_i}|_V$. Hence, setting $E := \bigoplus E_i, E' := \bigoplus E_i'$, the induced quasi-isomorphisms

$$\begin{tikzcd} K^\bullet(E,s)|_V \arrow[rr, "\chi"] \arrow[dr, "\psi|_V"'] & & K^\bullet(E',s')|_V \arrow[dl, "\psi'|_V"] \\ & \bigotimes\mathcal{F}_i^\bullet|_V & \end{tikzcd}$$

also commute up to homotopy and so the induced isomorphisms

$$
\begin{tikzcd}
H^{-q}(K^\bullet(E,s)|_V) \arrow[rr, "H^{-q}\chi"] \arrow[dr, "H^{-q}\psi|_V"'] & & H^{-q}(K^\bullet(E',s')|_V) \arrow[dl, "H^{-q}\psi'|_V"] \\
& \mathrm{Tor}_q(\mathcal{O}_{Y_1},\dots,\mathcal{O}_{Y_n})|_V &
\end{tikzcd}
$$

commute on the nose. The commutativity of these triangles is what we mean when we talk about the fact that the cohomologies of the local Koszul models glue.

For us there are natural choices of representative for the homotopy equivalences $\chi_i : K^\bullet(E_i,s_i)|_V \to K^\bullet(E_i',s_i')|_V$. Namely, since the ideal generated by the images of $s_i|_V$ and $s_i'|_V$ are the same, there is a change of basis morphism $\chi_{i,0} : E_i|_V \to E_i'|_V$ making the triangle

$$
\begin{tikzcd}
E_i|_V \arrow[rr, "\chi_{i,0}"] \arrow[dr, "s_i|_V"'] & & E_i'|_V \arrow[dl, "s_i'|_V"] \\
& \mathcal{O}_V &
\end{tikzcd}
$$

commute. This change of basis morphism extends by the functoriality of the Koszul complex (Proposition 2.2.12) to a morphism of Koszul complexes lifting the augmentations to $\mathcal{O}_{Y_i}|_V$, and hence by the uniqueness property from q-projectivity is a homotopy equivalence.

3.1 Self-Intersection

The goal of this section is to prove the derived self-intersection formula, which we shall make use of to prove a derived intersection formula. We first include a some explicit computations to provide a sense of the kind of result that we would expect.

Example 3.1.1. Consider first the example of the self-intersection of a divisor D. Then locally we are in the situation where we have a ring R and a non-zero-divisor $x \in R$ such that D is the zero-locus of x. In this local setup, $\mathcal{O}_D$ has a Koszul resolution

$$R \xrightarrow{x} R$$

so to compute the cohomologies of the derived self-intersection, we see that we are computing the cohomologies of the Koszul complex

$$\begin{tikzcd} & R \arrow[dr, "x"] & \\ R \arrow[ur, "x"] \arrow[dr, "-x"'] & \oplus & R. \\ & R \arrow[ur, "x"'] & \end{tikzcd}$$

Of course, we have that $H^0(K^\bullet(x,x)) \cong R/(x)$. We compute that $\mathrm{Ker}(\delta^{-1}) = \langle (1,-1) \rangle$, $\mathrm{Im}(\delta^{-2}) = \langle (x,-x) \rangle$ and so

$$H^{-1}(K^\bullet(x,x)) \cong \mathrm{Ker}(\delta^{-1})/\,\mathrm{Im}(\delta^{-2}) \cong \langle (1,-1) \rangle / \langle (x,-x) \rangle \cong R/(x).$$

Here, we see that the cohomology we are picking up comes from the nontrivial relation between x and itself.

Theorem 3.1.2. *Let $i : Y \to X$ be a local complete intersection subscheme of a smooth variety. Then*

$$\mathrm{Tor}_q(i_*\mathcal{O}_Y, i_*\mathcal{O}_Y) \cong i_* \bigwedge^q \mathcal{C}_{Y/X}$$

where $\mathcal{C}_{Y/X} = \mathcal{N}^\vee_{Y/X}$ is the conormal bundle for Y in X.

Proof. Let $\{U_\alpha\}$ be an open cover of X such that on each U_α there is a Koszul resolution $K^\bullet(E_\alpha, s_\alpha)$ of $(i_*)\mathcal{O}_Y|_{U_\alpha}$. For each α we have the following Tor-independent fibre square (Proposition 2.4.10)

$$\begin{tikzcd} Y \cap U_\alpha \arrow[r, "i_\alpha"] \arrow[d, "(j_\alpha)_Y"'] & U_\alpha \arrow[d, "j_\alpha"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

of closed and open immersions. We want to construct local isomorphisms

$$H^{-q}(K^\bullet(E_\alpha, s_\alpha) \otimes (i_*\mathcal{O}_Y)|_{U_\alpha}) \to (i_* \bigwedge^q \mathcal{C}_{Y/X})|_{U_\alpha}$$

for each α, commuting with the map induced by the change of basis morphisms

$$\begin{tikzcd} H^{-q}(K^\bullet(E_\alpha, s_\alpha)|_{U_{\alpha\beta}} \otimes (i_*\mathcal{O}_Y)|_{U_{\alpha\beta}}) \arrow[rr] \arrow[dr] & & H^{-q}(K^\bullet(E_\beta, s_\beta)|_{U_{\alpha\beta}} \otimes (i_*\mathcal{O}_Y)|_{U_{\alpha\beta}}) \arrow[dl] \\ & (i_* \bigwedge^q \mathcal{C}_{Y/X})|_{U_{\alpha\beta}}, & \end{tikzcd}$$

where $U_{\alpha\beta}$ means $U_\alpha \cap U_\beta$. Doing so will prove the statement of the theorem, because the discussion just before §3.1 tells us that there are local isomorphisms of the Tors to Koszul models which commute with the isomorphism on cohomology coming from the change of basis map. Constructing isomorphisms as in the above triangle provides the gluing data for local isomorphisms to glue up to a global one. For each α, the complex $K^\bullet(E_\alpha, s_\alpha) \otimes (i_*\mathcal{O}_Y)|_{U_\alpha}$ has zero differential by Proposition 2.2.15 and has

$$\begin{aligned} H^{-q}(K^\bullet(E_\alpha, s_\alpha) \otimes (i_*\mathcal{O}_Y)|_{U_\alpha}) = K^{-q}(E_\alpha, s_\alpha) \otimes (i_*\mathcal{O}_Y)|_{U_\alpha} &\cong (i_\alpha)_* \bigwedge^q E_\alpha|_{Y\cap U_\alpha} \\ &\cong (i_\alpha)_* \bigwedge^q \mathcal{C}_{Y\cap U_\alpha/U_\alpha}. \end{aligned}$$

This final isomorphism is the restriction of the morphism s_α to $Y \cap U_\alpha$, which becomes an isomorphism onto its image. We now claim that

$$(i_\alpha)_* \bigwedge^q \mathcal{C}_{Y\cap U_\alpha/U_\alpha} \cong (j_\alpha)^* i_* \bigwedge^q \mathcal{C}_{Y/X}.$$

To see this, note that $\mathcal{C}_{Y/X} := i^*\mathcal{I}_Y$, $\mathcal{C}_{Y\cap U_\alpha/U_\alpha} := (i_\alpha)^*\mathcal{I}_{Y\cap U_\alpha} = (j_\alpha \circ i_\alpha)^*\mathcal{I}_Y$. Then there are isomorphisms

$$(i_\alpha)_* \bigwedge^q (j_\alpha \circ i_\alpha)^*\mathcal{I}_Y \cong (i_\alpha)_* \bigwedge^q (i \circ (j_\alpha)_Y)^*\mathcal{I}_Y \cong (i_\alpha)_*(j_\alpha)_Y^* \bigwedge^q \mathcal{C}_{Y/X} \cong (j_\alpha)^* i_* \bigwedge^q \mathcal{C}_{Y/X},$$

where the first isomorphism comes from commutativity of the above square, the second from commutativity of exterior powers with pullbacks, and the third from base change around the square. Then on $U_{\alpha\beta}$ we have the following diagram of isomorphisms;

$$\begin{tikzcd}[column sep=small]
H^{-q}(K^\bullet(E_\alpha, s_\alpha)|_{U_{\alpha\beta}} \otimes (i_*\mathcal{O}_Y)|_{U_{\alpha\beta}}) \arrow[rr] \arrow[d] & & H^{-q}(K^\bullet(E_\beta, s_\beta)|_{U_{\alpha\beta}} \otimes (i_*\mathcal{O}_Y)|_{U_{\alpha\beta}}) \arrow[d] \\
(i_{\alpha\beta})_*(\bigwedge^q \mathcal{C}_{Y\cap U_\alpha/U_\alpha})|_{U_{\alpha\beta}} \arrow[dr] & & (i_{\alpha\beta})_*(\bigwedge^q \mathcal{C}_{Y\cap U_\beta/U_\beta})|_{U_{\alpha\beta}} \arrow[dl] \\
& (i_* \bigwedge^q \mathcal{C}_{Y/X})|_{U_{\alpha\beta}}. &
\end{tikzcd}$$

Since the change of basis morphism $K^\bullet(E_\alpha, s_\alpha)|_{U_{\alpha\beta}} \to K^\bullet(E_\beta, s_\beta)|_{U_{\alpha\beta}}$ commutes the sections

$$\begin{tikzcd}
E_\alpha|_{U_{\alpha\beta}} \arrow[rr] \arrow[dr, "s_\alpha|_{U_{\alpha\beta}}"'] & & E_\beta|_{U_{\alpha\beta}} \arrow[dl, "s_\beta|_{U_{\alpha\beta}}"] \\
& \mathcal{O}_{U_{\alpha\beta}}, &
\end{tikzcd}$$

and the vertical isomorphisms in the pentagonal diagram come from the restrictions of the s_α, we see the diagram commutes. This is exactly what we were trying to show and we conclude that the local isomorphisms glue to give a global isomorphism

$$\operatorname{Tor}_q^{\mathcal{O}_X}(i_*\mathcal{O}_Y, i_*\mathcal{O}_Y) \cong i_* \bigwedge^q \mathcal{C}_{Y/X}.$$

$\square$

3.2 Excess Intersection Formula

We include here a novel proof of the excess intersection formula for lcis. The setup is that a collection of local complete intersections intersect in a local complete intersection. We first include a couple of local Koszul computations in some easy cases.

Example 3.2.1. Consider the transverse intersection of two divisors D_1, D_2, and now intersect again with D_1. Locally, let D_1 be the zero locus of the non-zero-divisor x and D_2 the zero locus of the non-zero-divisor y. Then the cohomologies of the derived intersection $D_1 \cap D_2 \cap D_1$ are given by the cohomologies of the Koszul complex $K^\bullet(x, y, x)$. Let the local open set we are working in be isomorphic to $Spec(R)$. We represent this complex pictorially by

$$\begin{tikzcd}
& R \arrow[r, "y"] \arrow[dr, "-x"] & R \arrow[dr, "x"] & \\
R \arrow[ur, "x"] \arrow[r, "-y"] \arrow[dr, "x"'] & R \arrow[ur, "x"] \arrow[dr, "-x"] & R \arrow[r, "y"] & R \\
& R \arrow[ur, "x"] \arrow[r, "-y"'] & R \arrow[ur, "x"'] &
\end{tikzcd}$$

We of course have that $H^0(K^\bullet(x, y, x)) \cong R/(x, y)$. We compute that $\operatorname{Ker}(\delta^{-1}) = \langle (y, -x, 0), (0, x, -y), (1, 0, -1) \rangle, \operatorname{Im}(\delta^{-2}) = \langle (y, -x, 0), (0, x, -y), (x, 0, -x) \rangle$ and so

$$\begin{aligned}
H^{-1}(K^\bullet(x, y, x)) &= \operatorname{Ker}(\delta^{-1}) / \operatorname{Im}(\delta^{-2}) \\
&= \langle (y, -x, 0), (0, x, -y), (1, 0, -1) \rangle / \langle (y, -x, 0), (0, x, -y), (x, 0, -x) \rangle \\
&\cong R/(x, y),
\end{aligned}$$

where the isomorphism comes from $R \cong \langle(1,0,-1)\rangle$ and both ideals containing $(y,0,-y)$ and $(x,0,-x)$. For the second cohomology, we see that $\mathrm{Ker}(\delta^{-2}) = \langle(x,-y,x)\rangle = \mathrm{Im}(\delta^{-3})$, so $H^{-2}(K^\bullet(x,y,x)) = 0$. We see contained here the rough idea that it is this 'extra' x that contributes to the cohomology, because it is the relation between x and itself that makes the kernel differ from the image of the differentials.

Example 3.2.2. Consider the intersection in affine 2-space of three lines which meet in a common point. Let $X = Spec(k[x,y]), Y_1 = Z(x), Y_2 = Z(y), Y_3 = Z(x-y)$. Then $Y_1 \cap Y_2 \cap Y_3 = Z(x,y)$. The cohomologies of the derived intersection of $Y_1 \cap Y_2 \cap Y_3$ are given by the cohomologies of the Koszul complex $K^\bullet(x,y,x-y)$. Let $R = Spec(k[x,y])$. Then we represent this Koszul complex pictorially by

$$\begin{tikzcd}
 & R \arrow[r, "y"] \arrow[dr, "-x"' near start] & R \arrow[dr, "x"] & \\
R \arrow[ur, "x-y"] \arrow[r, "-y"] \arrow[dr, "x"'] & R \arrow[ur, "x-y" near start] \arrow[dr, "-x" near start] & R \arrow[r, "y"] & R \\
 & R \arrow[ur, "x-y" near start] \arrow[r, "-y"'] & R \arrow[ur, "x-y"'] &
\end{tikzcd}$$

We compute that $\mathrm{Ker}(\delta^{-1}) = \langle(y,-x,0),(x-y,0,-x),(0,x-y,-y),(1,-1,-1)\rangle$, $\mathrm{Im}(\delta^{-2}) = \langle(y,-x,0),(x-y,0,-x),(0,x-y,-y)\rangle$. We therefore have an isomorphism $H^{-1}(K^\bullet(x,y,x-y)) \cong R/(x,y)$. This isomorphism is given by $R \cong \langle(1,-1,-1)\rangle$, with $(x,-x,-x) = (x-y,0,-x)+(y,-x,0)$ and $(y,-y,-y) = (y,-x,0)+(0,x-y,-y)$, therefore both identifying with 0 in the quotient. As before, $H^{-2}(K^\bullet(x,y,x-y)) = 0$. Again, here the extra cohomology we are picking up comes from the nontrivial relation between x, y, and $x-y$.

The excess intersection formula tells us that for an intersection of local complete intersections, whose intersection is also a local complete intersection, the extra cohomological data contained in the derived intersection is given by the exterior powers of the *excess bundle*. This excess bundle is a geometric object encoding the 'extra' conormal directions sitting on the intersection that the induced reduced structure would not see. In effect, it therefore tells us that all of the extra geometric data of the derived intersection is encoded by the first cohomology.

The original proof in [12] runs along similar lines to the proof of Theorem 3.1.2. That is, work locally enough that every Y_i has a Koszul resolution $K^\bullet(E_i, s_i)$. Compute that locally there are isomorphisms $H^{-q}(K^\bullet(\bigoplus E_i, \oplus s_i)) \to \bigwedge^q \mathcal{E}|_U$ and then show that all of the local triangles on the intersections

$$\begin{tikzcd}
H^{-q}(K^\bullet(\bigoplus E_i, \oplus s_i)|_{U\cap U'}) \arrow[rr] \arrow[dr] & & H^{-q}(K^\bullet(\bigoplus E_i', \oplus s_i')|_{U\cap U'}) \arrow[dl] \\
& \bigwedge^q \mathcal{E}|_{U\cap U'} &
\end{tikzcd}$$

commute. Then conclude that the local isomorphisms

$$\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_n})|_U \to H^{-q}(K^\bullet(\bigoplus E_i, \oplus s_i)) \to \bigwedge^q \mathcal{E}|_U$$

glue to a global isomorphism

$$\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_n}) \to \bigwedge^q \mathcal{E}.$$

Both of these middle steps are non-trivial, and though there is no way to skirt around step one (the local computation), we produce a proof which utilises the work for step 2 (the gluing) done in the proof of Theorem 3.1.2. To do this, we first change our perspective via

Proposition 3.2.3. *Let X be a non-singular scheme over a field k with local complete intersection subschemes $i_j : Y_j \to X$ $(1 \le j \le n)$ with intersection W. Consider the fiber square*

$$\begin{tikzcd}
W \arrow[r, "w"] \arrow[d, "j_1\times\cdots\times j_n"'] & X \arrow[d, "\Delta_X^n"] \\
Y_1 \times \cdots \times Y_n \arrow[r, "(i_1\times\cdots\times i_n)"] & X^{\times n}
\end{tikzcd}$$

where $X^{\times n}$ means the n-fold fibre product of X with itself over k, Δ_X^n is the diagonal morphism, and w is the closed embedding of W into X. Suppressing the notation of right and left derived functors, there is an isomorphism in $\boldsymbol{D}(X)$;

$$(i_1)_*\mathcal{O}_{Y_1} \otimes \cdots \otimes (i_n)_*\mathcal{O}_{Y_n} \cong (\Delta_X^n)^*(i_1 \times \cdots \times i_n)_*\mathcal{O}_{Y_1\times\cdots\times Y_n}.$$

Proof. We prove the statement by induction on n. The case $n = 1$ is clear so suppose $n > 1$. Consider the expanded commutative diagram;

$$\begin{tikzcd}
W \arrow[rr, "j_n"] \arrow[dd, "j_1\times\cdots\times j_n"'] & & Y_n \arrow[r, "i_n"] \arrow[d, "\Delta^n_{Y_n}"] & X \arrow[dd, "\Delta^n_X"] \\
& & Y_n^{\times n} \arrow[d, "(i_n\times\cdots\times i_n)\times 1"] & \\
(Y_1\times\cdots\times Y_{n-1})\times Y_n \arrow[rr, "(i_1\times\cdots\times i_{n-1})\times 1"] & & X^{\times n-1}\times Y_n \arrow[r, "(1\times\cdots\times 1)\times i_n"] & X^{\times n}
\end{tikzcd}$$

where $\Delta^n_{Y_n}$ is the diagonal morphism for Y_n into $Y_n^{\times n}$. Here Δ^n_X is a regular immersion because X is non-singular. By Proposition 2.4.11 the square on the right hand side is Tor-independent. So then

$$(\Delta^n_X)^*(i_1\times\cdots\times i_n)_* = (i_n)_*(\Delta^n_{Y_n})^*(i_n\times\cdots\times i_n\times 1)^*(i_1\times\cdots\times i_{n-1}\times 1)_*.$$

Denote by $\pi_{(1,\ldots,n-1)}$ the projection to the first $n-1$ factors. Then since $\mathcal{O}_{Y_1\times\cdots\times Y_n} = \pi^*_{(1,\ldots,n-1)}\mathcal{O}_{Y_1\times\cdots\times Y_{n-1}}$ we have the identification

$$\begin{aligned}
&(i_n)_*(\Delta^n_{Y_n})^*(i_n\times\cdots\times i_n\times 1)^*(i_1\times\cdots\times i_{n-1}\times 1)_*(\mathcal{O}_{Y_1\times\cdots\times Y_n}) \\
&= (i_n)_*(\Delta^n_{Y_n})^*(i_n\times\cdots\times i_n\times 1)^*(i_1\times\cdots\times i_{n-1}\times 1)_*\pi^*_{(1,\ldots,n-1)}(\mathcal{O}_{Y_1\times\cdots\times Y_{n-1}}).
\end{aligned}$$

There is another commuting diagram;

$$\begin{tikzcd}
(Y_1\times\cdots\times Y_{n-1})\times Y_n \arrow[r, "(i_1\times\cdots\times i_{n-1})\times 1"] \arrow[d, "\pi_{(1,\ldots,n-1)}"] & X^{\times(n-1)}\times Y_n \arrow[d, "\pi_{(1,\ldots,n-1)}"] & Y_n^{\times n} \arrow[l, "(i_n\times\cdots\times i_n)\times 1"'] \arrow[d, "\pi_{(1,\ldots,n-1)}"] \\
Y_1\times\cdots\times Y_{n-1} \arrow[r, "i_1\times\cdots\times i_{n-1}"] & X^{\times(n-1)} & Y_n^{\times(n-1)} \arrow[l, "i_n\times\cdots\times i_n"'] \\
\bigcap_{i=1}^{n-1} Y_i \arrow[u] \arrow[r] & X \arrow[u, "\Delta^{n-1}_X"'] & Y_n. \arrow[l, "i_n"'] \arrow[u, "\Delta^{n-1}_{Y_n}"']
\end{tikzcd}$$

Here now by Proposition 2.4.8 and because projections are flat morphisms the top left hand square is Tor-independent. Hence

$$(i_n\times\cdots\times i_n\times 1)^*(i_1\times\cdots\times i_{n-1}\times 1)_*\pi^*_{(1,\ldots,n-1)} = \pi^*_{(1,\ldots,n-1)}(i_n\times\cdots\times i_n)^*(i_1\times\cdots\times i_{n-1})_*.$$

Furthermore the composed morphism

$$Y_n \xrightarrow{\Delta^n_{Y_n}} Y_n^{\times n} \xrightarrow{\pi_{(1,\ldots,n-1)}} Y_n^{\times(n-1)}$$

and the morphism $\Delta^{n-1}_{Y_n}$ are equal. Hence

$$(\Delta^n_{Y_n})^*\pi^*_{(1,\ldots,n-1)}(i_n\times\cdots\times i_n)^*(i_1\times\cdots\times i_{n-1})_* = (\Delta^{n-1}_{Y_n})^*(i_n\times\cdots\times i_n)^*(i_1\times\cdots\times i_{n-1})_*$$

By commutativity of the bottom right square, we have an equality

$$(\Delta_{Y_n}^{n-1})^*(i_n \times \cdots \times i_n)^*(i_1 \times \cdots \times i_{n-1})_* = (i_n)^*(\Delta_X^{n-1})^*(i_1 \times \cdots \times i_{n-1})_*$$

Combining all of the above identifications, we have shown that there is an isomorphism in $\mathbf{D}(X)$

$$(\Delta_X^n)^*(i_1 \times \cdots \times i_n)_*\mathcal{O}_{Y_1 \times \cdots \times Y_n} \cong (i_n)_*(i_n)^*(\Delta_X^{n-1})^*(i_1 \times \cdots \times i_{n-1})_*(\mathcal{O}_{Y_1 \times \cdots \times Y_{n-1}}).$$

The result then follows from the projection formula and induction. □

Theorem 3.2.4. *Let X be a non-singular variety over an algebraically closed field k and let $Y_1, \ldots, Y_n$ be local complete intersection subvarieties of X. Suppose that $w : W = \bigcap Y_i \to X$ is also a local complete intersection. Then*

$$\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{Y_1}, \ldots, \mathcal{O}_{Y_n}) \cong w_* \bigwedge^q \mathcal{E}_W,$$

where $\mathcal{E}_W$ is defined as the kernel of the natural surjection

$$\bigoplus \mathcal{N}_{Y_i/X}^\vee|_W \to \mathcal{N}_{W/X}^\vee$$

coming from $\mathcal{I}_W = \sum \mathcal{I}_{Y_i}$.

Proof. We prove this statement when $n = 2$ for notational convenience, with the general case proceeding by an equivalent argument. By Proposition 3.2.3 we are looking to compute the cohomologies of the object $\Delta^*(i_1 \times i_2)_*\mathcal{O}_{Y_1 \times Y_2}$. The closed immersion w gives rise to an adjunction unit morphism

$$\eta_w : \Delta^*(i_1 \times i_2)_*\mathcal{O}_{Y_1 \times Y_2} \to w_*w^*\Delta^*(i_1 \times i_2)_*\mathcal{O}_{Y_1 \times Y_2}.$$

We have an isomorphism

$$w_*w^*\Delta^*(i_1 \times i_2)_*\mathcal{O}_{Y_1 \times Y_2} \cong w_*(j_1 \times j_2)^*(i_1 \times i_2)^*(i_1 \times i_2)_*\mathcal{O}_{Y_1 \times Y_2},$$

from commutativity of the square in the statement of Proposition 3.2.3. We can compute the cohomologies of the object on the right, since the morphism $(i_1 \times i_2)$ is a regular immersion. Applying the derived self-intersection formula Theorem 3.1.2, and writing $\mathcal{C}_{Y/X}$ for $\mathcal{N}_{Y/X}^\vee$ etc,we see that there are global isomorphisms

$$H^{-q}((i_1 \times i_2)^*(i_1 \times i_2)_*\mathcal{O}_{Y_1 \times Y_2}) \cong \bigwedge^q \mathcal{C}_{Y_1 \times Y_2/X \times X}.$$

Since these cohomologies are locally free sheaves on $Y_1 \times Y_2$, taking cohomology commutes with the pullback $(j_1 \times j_2)^*$. Additionally w is a closed immersion so w_* is an exact functor and also commutes with cohomology. We therefore compute that there is a global isomorphism

$$\varphi_q : H^{-q}(w_* w^* \Delta^*(i_1 \times i_2)_* \mathcal{O}_{Y_1 \times Y_2}) \to w_*(j_1 \times j_2)^* \bigwedge^q \mathcal{C}_{Y_1 \times Y_2 / X \times X} \cong w_* \bigwedge^q (\mathcal{C}_{Y_1/X} \oplus \mathcal{C}_{Y_2/X})|_W.$$

This isomorphism has the following important property. If U is any open subset on which $\mathcal{O}_{Y_1}$ and $\mathcal{O}_{Y_2}$ have Koszul resolutions $K^\bullet(\mathcal{F}_1, s_1), K^\bullet(\mathcal{F}_2, s_2)$ respectively, then $\varphi_q|_U$ factors as

$$H^{-q}(w_* w^* \Delta^*(i_1 \times i_2)_* \mathcal{O}_{Y_1 \times Y_2})|_U \to w_* H^{-q}(K^\bullet(\mathcal{F}_1 \oplus \mathcal{F}_2, s_1 \oplus s_2)|_{W \cap U}) \to w_* \bigwedge^q (\mathcal{C}_{Y_1/X} \oplus \mathcal{C}_{Y_2/X})|_{W \cap U},$$

where the first morphism is the inverse of the one induced from the $K^\bullet(\mathcal{F}_i, s_i)$ being free resolutions and the second morphism coming from the computation of Koszul cohomology in this case. Since $\mathcal{E}_W$ is a subbundle of $\mathcal{C}_{Y_1/X} \oplus \mathcal{C}_{Y_2/X}$ we have a diagram of global morphisms

$$\begin{tikzcd} H^{-q}(\Delta^*(i_1 \times i_2)_* \mathcal{O}_{Y_1 \times Y_2}) \arrow[r, "H^{-q}\eta_w"] & H^{-q}(w_* w^* \Delta^*(i_1 \times i_2)_* \mathcal{O}_{Y_1 \times Y_2}) \arrow[d, "\varphi_q"] \\ w_* \bigwedge^q \mathcal{E}_W \arrow[r, "\iota"] & w_* \bigwedge^q (\mathcal{C}_{Y_1/X} \oplus \mathcal{C}_{Y_2/X})|_W. \end{tikzcd}$$

with the bottom arrow being the natural inclusion and φ_q the isomorphism above. Now all we need to check is that the image of $w_* \bigwedge^q \mathcal{E}_W$ under $\varphi_q^{-1} \circ \iota$ agrees with the image of $H^{-q}(\Delta^*(i_1 \times i_2)_* \mathcal{O}_{Y_1 \times Y_2})$ under $H^{-q}\eta_w$ and that $H^{-q}\eta_w$ is an injection to conclude the statement of the theorem. Checking that the images of two maps agree and that a map is an injection are both local checks so we may work locally.

We work locally enough that $\mathcal{O}_{Y_1}$ has resolution $K^\bullet(\mathcal{F}_1, s_1)$, $\mathcal{O}_{Y_2}$ has resolution $K^\bullet(\mathcal{F}_2, s_2)$ and $\mathcal{O}_W$ has resolution $K^\bullet(\mathcal{G}, t)$. Then, since the intersection of Y_1 and Y_2 is equal to W, we have

$$K^\bullet(\mathcal{F}_1, s_1) \otimes K^\bullet(\mathcal{F}_2, s_2) \cong K^\bullet(\mathcal{F}_1 \oplus \mathcal{F}_2, s_1 \oplus s_2) \cong K^\bullet(\mathcal{G} \oplus \mathcal{F}, t \oplus 0).$$

We can compute that $H^{-q}(K^\bullet(\mathcal{G} \oplus \mathcal{F}, t \oplus 0)) \cong \bigwedge^q \mathcal{F}|_W$. We assume that we are working locally enough that the short exact sequence of locally free sheaves on W

$$0 \to \mathcal{E}_W \to (\mathcal{C}_{Y_1/X} \oplus \mathcal{C}_{Y_2/X})|_W \to \mathcal{C}_{W/X} \to 0$$

splits. Then we have two split exact sequences

$$0 \to \mathcal{F}|_W \to (\mathcal{F}_1 \oplus \mathcal{F}_2)|_W \to \mathcal{G}|_W \to 0$$

and

$$0 \to \mathcal{E}_W \to (\mathcal{C}_{Y_1/X} \oplus \mathcal{C}_{Y_2/X})|_W \to \mathcal{C}_{W/X} \to 0$$

which have isomorphic middle and rightmost terms. Therefore they are isomorphic split exact sequences. In particular, we have a commuting square

$$\begin{tikzcd}
\mathcal{F}|_W \arrow[r] \arrow[d, "\cong"] & (\mathcal{F}_1 \oplus \mathcal{F}_2)|_W \arrow[d, "\cong"] \\
\mathcal{E}_W \arrow[r] & (\mathcal{C}_{Y_1/X} \oplus \mathcal{C}_{Y_2/X})|_W.
\end{tikzcd}$$

We want now to show that $H^{-q}\eta_w$ is an injection. To do this, we recall that the functor $w_*w^* \cong - \otimes \mathcal{O}_W$. Then we get a long exact sequence on cohomology

$$\begin{tikzcd}
\ldots \arrow[r] & H^{-q}(\Delta^*(i_1 \times i_2)_*\mathcal{O}_{Y_1 \times Y_2} \otimes \mathcal{I}_W) \arrow[r] & H^{-q}(\Delta^*(i_1 \times i_2)_*\mathcal{O}_{Y_1 \times Y_2}) \arrow[dll, "H^{-q}\eta_w"] \\
H^{-q}(w_*w^*\Delta^*(i_1 \times i_2)_*\mathcal{O}_{Y_1 \times Y_2}) \arrow[r] & H^{-q+1}(\Delta^*(i_1 \times i_2)_*\mathcal{O}_{Y_1 \times Y_2} \otimes \mathcal{I}_W) \arrow[r] & \ldots
\end{tikzcd}$$

coming from the short exact sequence

$$0 \to \mathcal{I}_W \to \mathcal{O}_X \to \mathcal{O}_W \to 0.$$

In our local context, we are therefore interested in computing

$$H^{-q}(K^\bullet(\mathcal{G} \oplus \mathcal{F}, t \oplus 0) \otimes \mathcal{I}_W) \cong \bigoplus_{r+s=q} (H^{-r}(K^\bullet(\mathcal{G}, t) \otimes \mathcal{I}_W) \otimes \bigwedge^s \mathcal{F}).$$

Since $K^\bullet(\mathcal{G}, t)$ is a resolution for $\mathcal{O}_W$, we have that $H^{-r}(K^\bullet(\mathcal{G}, t) \otimes \mathcal{I}_W) = \mathrm{Tor}_r(\mathcal{O}_W, \mathcal{I}_W)$. If we compute these by instead resolving the $\mathcal{I}_W$ by a truncated version of $K^\bullet(\mathcal{G}, t)$, we see that $\mathrm{Tor}_r(\mathcal{O}_W, \mathcal{I}_W) \cong \bigwedge^{r+1} \mathcal{G}$. However, from the identification

$$\bigwedge^q (\mathcal{G} \oplus \mathcal{F}) \cong \bigoplus_{r+s=q} \bigwedge^r \mathcal{G} \otimes \bigwedge^s \mathcal{F}$$

we see that

$$H^{-q}(K^\bullet(\mathcal{G} \oplus \mathcal{F}, t \oplus 0) \otimes \mathcal{I}_W) \cong \bigwedge^{q+1} (\mathcal{G} \oplus \mathcal{F})|_W / \bigwedge^{q+1} (\mathcal{F})|_W.$$

This implies that the maps $H^{-q}\eta_w$ are injections and make the squares

$$\begin{array}{ccc} H^{-q}(\Delta^*(i_1 \times i_2)_*\mathcal{O}_{Y_1\times Y_2}) & \xrightarrow{H^{-q}\eta_w} & H^{-q}(w_*w^*\Delta_*(i_1 \times i_2)_*\mathcal{O}_{Y_1\times Y_2}) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ w_* \bigwedge^q \mathcal{F}|_W & \longrightarrow & w_* \bigwedge^q(\mathcal{F} \oplus \mathcal{G})|_W \end{array}$$

commute. Linking our two commuting squares together we have a commuting square

$$\begin{array}{ccc} H^{-q}(\Delta^*(i_1 \times i_2)_*\mathcal{O}_{Y_1\times Y_2}) & \xrightarrow{H^{-q}\eta_w} & H^{-q}(w_*w^*\Delta_*(i_1 \times i_2)_*\mathcal{O}_{Y_1\times Y_2}) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\varphi_q} \\ w_* \bigwedge^q \mathcal{E}|_W & \longrightarrow & w_* \bigwedge^q(\mathcal{C}_{Y_1/X} \oplus \mathcal{C}_{Y_2/X})|_W \end{array}$$

implying that our isomorphism is global and induced by φ_q. □

Chapter 4

Non-lci-intersections

We want to investigate the multitors of intersections which are not lci. For any intersection of schemes $Y_i \subset X$ with intersection Z there is a natural surjection

$$\bigoplus \mathcal{C}_{Y_i/X}|_Z \to \mathcal{C}_{Z/X}$$

. There is thus always a conormal excess sheaf $\mathcal{E}_Z := \mathrm{Ker}(\bigoplus \mathcal{C}_{Y_i/X}|_Z \to \mathcal{C}_{Z/X})$, and it is natural to ask whether or not the multitors for the intersection are given by exterior powers of this sheaf. An obvious example to consider is where the intersection is not equidimensional but in each codimension the corresponding component is lci. As the complexity of the geometry increases with the codimension, the most straightforward case to look at is where we have two components to the intersection, where both are lcis but one has codimension one.

4.1 The affine case

We begin with an algebraic result.

Proposition 4.1.1. *Let R be a ring and $f_1, \ldots, f_n$ a sequence in R. Let x be a non-zero-divisor in R and consider the sequence $xf_1, \ldots, xf_n$. Then*

$$H^q(K^\bullet(xf_1, \ldots, xf_n)) \cong \mathrm{Ker}(\partial^q_{\underline{f}})/x \cdot \mathrm{Im}(\partial^{q-1}_{\underline{f}})$$

where $\partial_{\underline{f}}$ denotes the differential in $K^\bullet(f_1, \ldots, f_n)$.

Proof. Let $\partial_{\underline{xf}}$ denote the differential in $K^\bullet(xf_1, \ldots, xf_n)$. The first thing to note is that we have $\partial_{\underline{xf}} = x\partial_{\underline{f}}$ as can be seen from the explicit formulation of the differential in a Koszul complex. Then since x is a non-zero-divisor, we have

$$\mathrm{Ker}(\partial^q_{\underline{xf}}) = \mathrm{Ker}(\partial^q_{\underline{f}}), \quad \mathrm{Im}(\partial^q_{\underline{xf}}) = x(\mathrm{Im}(\partial^q_{\underline{f}})).$$

We therefore have the identifications

$$\begin{aligned} H^q(K^\bullet(xf_1, \ldots, xf_n)) &= \mathrm{Ker}(\partial^q_{\underline{xf}})/\,\mathrm{Im}(\partial^{q-1}_{\underline{xf}}) \\ &= \mathrm{Ker}(\partial^q_{\underline{f}})/x(\mathrm{Im}(\partial^{q-1}_{\underline{f}})) \end{aligned}$$

as stated. □

Corollary 4.1.2. *If the sequence $f_1, \ldots, f_n$ is regular, then for any $q < 0$*

$$H^q(K^\bullet(xf_1, \ldots, xf_n)) \cong \mathrm{Ker}(\partial^q_{\underline{f}}) \otimes R/(x).$$

Proof. The sequence $f_1, \ldots, f_n$ being regular implies that $\mathrm{Im}(\partial^{q-1}_{\underline{f}}) = \mathrm{Ker}(\partial^q_{\underline{f}})$ for all $q < 0$ so

$$\begin{aligned} H^q(K^\bullet(xf_1, \ldots, xf_n)) &= \mathrm{Ker}(\partial^q_{\underline{f}})/x(\mathrm{Ker}(\partial^q_{\underline{f}})) \\ &\cong \mathrm{Ker}(\partial^q_{\underline{f}}) \otimes R/(x) \end{aligned}$$

□

Example 4.1.3 (A line through the plane)**.** The first example to consider when talking about complete intersections intersecting in a non-complete intersection is that of the line through the plane in affine 3-space. Indeed this space can be defined by the equations xy, xz, which is not a regular sequence, yet neither xy not xz is a zero-divisor in the polynomial ring in 3 variables.

For this space we have a global Koszul complex given by

$$\begin{tikzcd} & R \arrow[dr, "xy"] & \\ R \arrow[ur, "xz"] \arrow[dr, "-xy"'] & \oplus & R \\ & R \arrow[ur, "xz"'] & \end{tikzcd}$$

where $R = k[x,y,z]$. As always, $H^0(K^\bullet(xy,xz)) \cong R/(xy,xz)$. We compute that $\operatorname{Ker}(\delta^{-1}) = \langle (z,-y)\rangle$ and that $\operatorname{Im}(\delta^{-2}) = \langle (xz,-xy)\rangle$. Hence the quotient

$$H^{-1}(K^\bullet(xy,xz)) = \operatorname{Ker}(\delta^{-1})/\operatorname{Im}(\delta^{-2}) = \langle (z,-y)\rangle/\langle (xz,-xy)\rangle \cong R/(x).$$

With R as before and now $I = (xy,xz)$ an ideal in R we compute the excess sheaf as the kernel of the morphism

$$\phi : (R/I)^2 \to I/I^2$$

given by $e_1 \mapsto xy, e_2 \mapsto xz$. We find that the kernel is generated by the single element $ze_1 - ye_2$, and this ideal is isomorphic to $(R/I)/(x) \cong R/(x)$. We conclude that in this example $i_*\mathcal{E} \cong H^1(K(xy,xz))$ and in general $i_* \bigwedge^k \mathcal{E} \cong H^k(K(xy,xz))$.

Example 4.1.4. We give a higher dimensional example to show that our answer is not in the form of exterior powers. Consider a line through a hyperplane in 4-space, we will use the example of the line $\{y = z = w = 0\}$ and the hyperplane $\{x = 0\}$ in $\mathbb{A}^4_k$, considered as the intersection of the hypersurfaces $\{xy = 0\}, \{xz = 0\}, \{xw = 0\}$. Let $R = k[x,y,z,w]$. We are finding the cohomologies of the Koszul complex $K^\bullet(xy,xz,xw)$. By Proposition 4.1.1 the cohomologies are given by the kernels of the complex tensored with $R/(x)$. However, in an exact complex, the kernel of d_i is isomorphic to the cokernel of d_{i-2} so we see

$$H^{-1} \cong (R/(x))^3/((w,-z,y)), \quad H^{-2} \cong R/(x).$$

We compute the second exterior power of H^{-1}. Pick generators e_1, e_2, e_3 of H^{-1} subject to the relation $ze_2 = we_1 + ye_3$. Then $\bigwedge^2 H^{-1}$ is generated by $e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3$ which are subject to the relations

$$z(e_1 \wedge e_2) = y(e_1 \wedge e_3), z(e_2 \wedge e_3) = w(e_1 \wedge e_3), w(e_1 \wedge e_2) = y(e_2 \wedge e_3).$$

Hence

$$\bigwedge^2 H^{-1} \cong (R/(x))^3/((-z,y,0),(-w,0,y),(0,-w,z)) \cong (y,z,w) \subset R/(x).$$

To see this isomorphism note that the module on the left is $\operatorname{Coker}(d_2) = R^3/\operatorname{Im}(d_2) = R^3/\operatorname{Ker}(d_1) \cong \operatorname{Im}(d_1)$ all tensored with $R/(x)$.

Note that this shows that $\bigwedge^2 H^{-1}$ is a free $R/(x)$-module of rank one *away from the line* $\{y{=}z{=}w{=}0\}$, and therefore coincides with H^{-2} there. So, as expected, the multitors do form an exterior algebra *away from the intersection of the components.*

We'd like to relate $\operatorname{Ker}(\partial^q_{\underline{f}})/x(\operatorname{Im}(\partial^{q-1}_{\underline{f}}))$ with the cohomology of $K^\bullet(f_1, \ldots, f_n)$, to which end we make use of the following general algebraic statement.

Lemma 4.1.5. *Let M be an R-module with submodules P, Q. Then the following commuting square of projections is cartesian:*

$$\begin{array}{ccc} M/(P \cap Q) & \longrightarrow & M/Q \\ \downarrow & & \downarrow \\ M/P & \longrightarrow & M/(P+Q). \end{array}$$

Proof. The given square of projections commutes so there is an induced map to the pullback $M/(P\cap Q) \to X$. An element of X is uniquely determined by a pair $(\pi_P(m), \pi_Q(n))$ such that $\pi_{P+Q}(m) = \pi_{P+Q}(n)$. Then $n = m + p + q$ so $(\pi_P(m), \pi_Q(n)) = (\pi_P(m + p), \pi_Q(m+p))$. Hence $M/(P \cap Q) \to X$ is surjective. Suppose that two elements m, n of M map to the same element of X via $M/(P \cap Q)$. Then $\pi_P(m) = \pi_P(n), \pi_Q(m) = \pi_Q(n)$ so $\pi_{P\cap Q}(m) = \pi_{P\cap Q}(n)$. Hence our map $M/(P \cap Q) \to X$ is injective and so an isomorphism. □

Remark 4.1.6. *Note that Lemma 4.1.5 holds for sheaves of modules on a topological space. The maps are global but checking that $M/(P \cap Q)$ is isomorphic to the pullback is a local computation.*

Proposition 4.1.7. *In the context of Proposition 4.1.1, suppose additionally that x does not belong to any prime ideal in $\operatorname{Ass}_R(\operatorname{Ker}(\partial^q_{\underline{f}})/\operatorname{Im}(\partial^{q-1}_{\underline{f}}))$. Then the object $\operatorname{Ker}(\partial^q_{\underline{f}})/x \cdot \operatorname{Im}(\partial^{q-1}_{\underline{f}})$ fits into a cartesian square of projections*

$$\begin{array}{ccc} \operatorname{Ker}(\partial^q_{\underline{f}})/x \cdot \operatorname{Im}(\partial^{q-1}_{\underline{f}}) & \longrightarrow & \operatorname{Ker}(\partial^q_{\underline{f}}) \otimes R/(x) \\ \downarrow & & \downarrow \\ \operatorname{Ker}(\partial^q_{\underline{f}})/\operatorname{Im}(\partial^{q-1}_{\underline{f}}) & \longrightarrow & \operatorname{Ker}(\partial^q_{\underline{f}})/\operatorname{Im}(\partial^{q-1}_{\underline{f}}) \otimes R/(x). \end{array}$$

Proof. For the sake of notation, denote $\operatorname{Ker}(\partial^q_{\underline{f}})$ and $\operatorname{Im}(\partial^{q-1}_{\underline{f}})$ by Ker and Im respectively. In Lemma 4.1.5, take M to be $\operatorname{Ker}/x\cdot\operatorname{Im}$, $P = \operatorname{Im}/x\cdot\operatorname{Im}$, $Q = x\cdot\operatorname{Ker}/x\cdot\operatorname{Im}$. Since x does not belong to any element of $\operatorname{Ass}_R(\operatorname{Ker}/\operatorname{Im})$, we can equivalently say that $(\operatorname{Im})\cap(x\cdot\operatorname{Ker}) = x \cdot \operatorname{Im}$. To see this note that if $xy \in \operatorname{Im}, y \in \operatorname{Ker}$, then $\overline{xy} = 0$ in $\operatorname{Ker}/\operatorname{Im}$ but the action of x does not kill any elements of $\operatorname{Ker}/\operatorname{Im}$ so $\overline{y} = 0 \in \operatorname{Ker}/\operatorname{Im} \implies y \in \operatorname{Im}$. Therefore

$$P \cap Q = (\operatorname{Im}/x \cdot \operatorname{Im}) \cap (x \cdot \operatorname{Ker})/(x \cdot \operatorname{Im}) = 0$$

so the Lemma applies. The identifications

$$M/P = (\operatorname{Ker}/x{\cdot}\operatorname{Im})/(\operatorname{Im}/x{\cdot}\operatorname{Im}) \cong \operatorname{Ker}/\operatorname{Im}, \quad M/Q = (\operatorname{Ker}/x{\cdot}\operatorname{Im})/(x{\cdot}\operatorname{Ker}/x{\cdot}\operatorname{Im}) \cong \operatorname{Ker}/x{\cdot}\operatorname{Ker}$$

both follow from the third isomorphism theorem. Finally, we need to make an identification

$$M/(P{+}Q) = (\operatorname{Ker}/x{\cdot}\operatorname{Im})/((\operatorname{Im}/x{\cdot}\operatorname{Im}){+}(x{\cdot}\operatorname{Ker}/x{\cdot}\operatorname{Im})) \cong (\operatorname{Ker}/\operatorname{Im})/((x{\cdot}\operatorname{Ker})/(x{\cdot}\operatorname{Im})).$$

The bijection between submodules of $\operatorname{Ker}/x \cdot \operatorname{Im}$ and submodules of Ker containing $x \cdot \operatorname{Im}$ implies that

$$\operatorname{Im}/x \cdot \operatorname{Im} + x \cdot \operatorname{Ker}/x \cdot \operatorname{Im} = (\operatorname{Im} + x \cdot \operatorname{Ker})/x \cdot \operatorname{Im},$$

hence

$$M/(P+Q) \cong \operatorname{Ker}/(\operatorname{Im} + x \cdot \operatorname{Ker}).$$

We obtain a short exact sequence

$$0 \to (\operatorname{Im} + x \cdot \operatorname{Ker})/\operatorname{Im} \to \operatorname{Ker}/\operatorname{Im} \to \operatorname{Ker}/(\operatorname{Im} + x \cdot \operatorname{Ker}) \to 0.$$

From here we make the identifications

$$(\operatorname{Im} + x \cdot \operatorname{Ker})/\operatorname{Im} \cong (x \cdot \operatorname{Ker})/(\operatorname{Im} \cap x \cdot \operatorname{Ker}) = (x \cdot \operatorname{Ker})/(x \cdot \operatorname{Im}).$$

where the isomorphism comes from the second isomorphism theorem. Hence we have the desired identification

$$M/(P+Q) \cong (\operatorname{Ker}/\operatorname{Im})/(x \cdot \operatorname{Ker}/x \cdot \operatorname{Im}) \cong \operatorname{Ker}/\operatorname{Im} \otimes R/(x),$$

since $x \cdot \operatorname{Ker}/x \cdot \operatorname{Im} = x \cdot (\operatorname{Ker}/\operatorname{Im})$ again by $x \cdot \operatorname{Ker} \cap \operatorname{Im} = x \cdot \operatorname{Im}$. $\square$

Remark 4.1.8. *This reduces to Corollary 4.1.2 in the case that $f_1, \dots, f_n$ form a regular sequence, as the bottom row of the above square vanishes under that assumption.*

Remark 4.1.9. *Without the assumption on x not belonging to any prime ideal in $\operatorname{Ass}_R(\operatorname{Ker}(\partial^q_{\underline{f}})/\operatorname{Im}(\partial^{q-1}_{\underline{f}}))$ there is still a fibre square relating $H^{-q}(K^\bullet(xf_1, \dots, xf_n))$ and $H^{-q}(K^\bullet(f_1, \dots, f_n))$. Namely, the result of Lemma 4.1.5 still holds but the objects $M/(P \cap Q)$ and $M/(P+Q)$ become messy in this case. We limit ourselves to the situation where these objects simplify substantially, but it should be noted that there is a more general fibre square.*

Example 4.1.10. We provide an affine example where all 4 corners of the above square are non-zero. For this example, consider the line through the plane in affine 3-space given by the intersection of the zero loci xz, xy and then subsequently intersect with xz again. To compute the first multitor $\mathrm{Tor}_1(K^\bullet(xz, xy, xz))$ we need to make use of the computations of the cohomology and kernels of the differentials of $K^\bullet(z, y, z)$ done in Example 3.2.1. There, we found that

$$H^{-1}(K^\bullet(z, y, z)) \cong k[x] \cong k[x, y, z]^3/\langle(1, 0, 0), (0, 1, 0), (0, 0, y), (0, 0, z)\rangle$$

$$\mathrm{Ker}(d^{-1}) \otimes k[y, z] \cong k[y, z]^3/\langle(1, 1, -y)\rangle \cong k[x, y, z]^3/\langle(x, 0, 0), (0, x, 0), (0, 0, x), (1, 1, -y)\rangle.$$

From Lemma 4.1.5 we see that the bottom right hand of our square is given by

$$H^{-1}(K^\bullet(z, y, z)) \otimes k[y, z] \cong k[x, y, z]^3/\langle(1, 0, 0), (0, 1, 0), (0, 0, x), (0, 0, y), (0, 0, z)\rangle \cong k$$

and the top-left is given by

$$\begin{aligned} H^{-1}(K^\bullet(xz, xy, xz)) &\cong k[x, y, z]^3/\langle(x, 0, 0), (0, x, 0), (0, 0, xy), (0, 0, xz), (1, 1, -y)\rangle \\ &\cong \left(k[y, z]^2 \oplus k[x, y, z]/(xy, xz)\right)/(1, 1, -y). \end{aligned}$$

A heuristic understanding of this computation is that the first multitor consists of 2 copies of the plane and one copy of the whole intersection (line through the plane) where the diagonal direction in the 2 copies of the plane is identified with the y direction in the whole intersection.

4.2 The global case

We want to extend these results to a global setting and make a geometric interpretation. In the proof of Proposition 4.1.1, we made use of the identity $\partial_{\underline{xf}} = x\partial_{\underline{f}}$. We make a similar identification in the global case.

Lemma 4.2.1. *Suppose that we have a non-zero morphism $s : \mathcal{F} \to \mathcal{O}_X$ where $\mathcal{F}$ is a locally free sheaf of constant rank. Let $\mathcal{L}$ be an invertible sheaf with section $\lambda : \mathcal{O}_X \to \mathcal{L}$. Suppose furthermore that there is a map $t : \mathcal{F} \otimes \mathcal{L} \to \mathcal{O}_X$ making the triangle commute*

$$\begin{tikzcd} \mathcal{F}\otimes\mathcal{L} \arrow[rr, "t"] \arrow[dr, "s\otimes 1"'] & & \mathcal{O}_X \arrow[dl, "\lambda"] \\ & \mathcal{L} & \end{tikzcd} \tag{$*$}$$

Denote by ∂ the differential in $K^\bullet(\mathcal{F}, s)$ and by δ the differential in $K^\bullet((\mathcal{F}\otimes\mathcal{L}), t)$. Then for any n, we have the following relationship between $K^\bullet(\mathcal{F}, s)\otimes\mathcal{L}^n$ and $K^\bullet((\mathcal{F}\otimes\mathcal{L}), t)$;

$$\operatorname{Ker}(\partial^{-q}\otimes 1_{\mathcal{L}^n}) = \operatorname{Ker}(\delta^{-q})\otimes\mathcal{L}^{n-q}, \quad \operatorname{Im}(\partial^{-q-1}\otimes 1_{\mathcal{L}^n}) = \operatorname{Im}(\delta^{-q-1})\otimes\lambda(\mathcal{O}_X)\otimes\mathcal{L}^{n-q+1}.$$

Proof. We first claim that for any n

$$\bigwedge^q(\mathcal{F})\otimes\mathcal{L}^n \xrightarrow{\partial^{-q}\otimes 1} \bigwedge^{q-1}(\mathcal{F})\otimes\mathcal{L}^n$$

can be expressed as the image of the morphism

$$\bigwedge^q(\mathcal{F}\otimes\mathcal{L})\otimes\mathcal{O}_X \xrightarrow{\delta^{-q}\otimes\lambda} \bigwedge^{q-1}(\mathcal{F}\otimes\mathcal{L})\otimes\mathcal{L}$$

under the functor $(-\otimes\mathcal{L}^{n-q})$. The commutativity of $(*)$ implies there is a commuting square

$$\begin{tikzcd} (\mathcal{F}\otimes\mathcal{L})^q \arrow[r, "1\otimes t"] \arrow[d, "\sim"] & (\mathcal{F}\otimes\mathcal{L})^{q-1}\otimes\mathcal{O}_X \arrow[r, "1\otimes\lambda"] & (\mathcal{F}\otimes\mathcal{L})^{q-1}\otimes\mathcal{L} \arrow[d, "\sim"] \\ \mathcal{F}^q\otimes\mathcal{L}^q \arrow[rr, "(1\otimes s)\otimes 1"] & & \mathcal{F}^{q-1}\otimes\mathcal{L}^q. \end{tikzcd}$$

By definition of $1\wedge t, 1\wedge s$ and viewing $(\mathcal{F}\otimes\mathcal{L})^q$ as a subsheaf of $\bigwedge^q(\mathcal{F}\otimes\mathcal{L})$ etc (see the beginning of §2.2), there is an induced commuting square of subsheaves

$$\begin{tikzcd} \bigwedge^q(\mathcal{F}\otimes\mathcal{L}) \arrow[r, "1\wedge t"] \arrow[d, "\sim"] & \bigwedge^{q-1}(\mathcal{F}\otimes\mathcal{L})\otimes\mathcal{O}_X \arrow[r, "1\otimes\lambda"] & \bigwedge^{q-1}(\mathcal{F}\otimes\mathcal{L})\otimes\mathcal{L} \arrow[d, "\sim"] \\ (\bigwedge^q\mathcal{F})\otimes\mathcal{L}^q \arrow[rr, "(1\wedge s)\otimes 1"] & & (\bigwedge^{q-1}\mathcal{F})\otimes\mathcal{L}^q. \end{tikzcd}$$

The maps $1\wedge t$ and $1\wedge s$ are the differentials in $K^\bullet(\mathcal{F}\otimes\mathcal{L}, t)$ and $K^\bullet(\mathcal{F}, s)$, respectively, and the induced vertical isomorphisms are from Lemma 2.2.16. Hence applying the functor $(-\otimes\mathcal{L}^{n-q})$ yields the claim. Since $s\neq 0$ and a morphism of line bundles can either be injective or 0, we have that λ is injective by commutativity of $(*)$. As $\bigwedge^{q-1}(\mathcal{F}\otimes\mathcal{L})$ is locally free, $1\otimes\lambda$ is injective. Hence

$$\operatorname{Ker}(\delta^{-q}\otimes\lambda) = \operatorname{Ker}((1\otimes\lambda)\circ(\delta^{-q}\otimes 1_{\mathcal{O}_X})) = \operatorname{Ker}(\delta^{-q}).$$

As $\mathcal{L}$ is also locally free we therefore have

$$\operatorname{Ker}(\partial^{-q} \otimes 1_{\mathcal{L}^n}) = \operatorname{Ker}(\delta^{-q} \otimes \lambda \otimes 1_{\mathcal{L}^{n-q}}) = \operatorname{Ker}(\delta^{-q}) \otimes \mathcal{L}^{n-q}.$$

Similarly, $\operatorname{Im}(\partial^{-q-1} \otimes 1_{\mathcal{L}^n}) = \operatorname{Im}(\delta^{-q-1} \otimes \lambda \otimes 1_{\mathcal{L}^{n-q+1}}) = \operatorname{Im}(\delta^{-q-1}) \otimes \lambda(\mathcal{O}_X) \otimes \mathcal{L}^{n-q+1}$. $\square$

We are now ready to prove the main theorem of this section. Let X be a nonsingular variety over an algebraically closed field of characteristic 0. Let $Y_1, \ldots, Y_n$ be effective Cartier divisors in X. Then there are global Koszul resolutions

$$\mathcal{F}_i^\bullet = \{\mathcal{O}(-Y_i) \xrightarrow{\sigma_{Y_i}} \mathcal{O}_X\} \to \mathcal{O}_{Y_i}$$

which we shall use to model $\operatorname{Tor}_q(\mathcal{O}_{Y_1}, \ldots, \mathcal{O}_{Y_n})$. Let D be another effective Cartier divisor in X. There are global Koszul resolutions

$$\mathcal{G}_i^\bullet = \{\mathcal{O}(-Y_i - D) \xrightarrow{\sigma_{Y_i+D}} \mathcal{O}_X\} \to \mathcal{O}_{Y_i+D}$$

which we shall use to model $\operatorname{Tor}_q(\mathcal{O}_{Y_1+D}, \ldots, \mathcal{O}_{Y_n+D})$.

Theorem 4.2.2. *With the notation as above, denote the differential of the Koszul complex* $K^\bullet(\bigoplus \mathcal{O}(-Y_i), \sum \sigma_{Y_i})$ *by* δ. *Assume that* $D \cap \operatorname{Ass}_X(\operatorname{Tor}_q(\mathcal{O}_{Y_1}, \ldots, \mathcal{O}_{Y_n})) = \emptyset, \forall q$. *Then there is a fibre square in* Mod_X*;*

$$\begin{tikzcd}
\operatorname{Tor}_q(\mathcal{O}_{Y_1+D}, \ldots, \mathcal{O}_{Y_n+D}) \arrow[r] \arrow[d] & (\operatorname{Ker}(\delta^{-q}) \otimes \mathcal{O}(-qD))|_D \arrow[d] \\
\operatorname{Tor}_q(\mathcal{O}_{Y_1}, \ldots, \mathcal{O}_{Y_n}) \otimes \mathcal{O}(-qD) \arrow[r] & (\operatorname{Tor}_q(\mathcal{O}_{Y_1}, \ldots, \mathcal{O}_{Y_n}) \otimes \mathcal{O}(-qD))|_D
\end{tikzcd} \quad (\dagger)$$

where the bottom horizontal morphism is the projection and the right vertical arrow is induced by the quotient projection $\operatorname{Ker}(\delta^{-q}) \to \operatorname{Ker}(\delta^{-q})/\operatorname{Im}(\delta^{-q-1})$.

Proof. Note that there is a canonical isomorphism $\mathcal{O}(-Y_i - D) \to \mathcal{O}(-Y_i) \otimes \mathcal{O}(-D)$ such that the triangle

$$\begin{tikzcd}
\mathcal{O}(-Y_i - D) \arrow[rr, "\sim"] \arrow[dr, "\sigma_{Y_i+D}"'] & & \mathcal{O}(-Y_i) \otimes \mathcal{O}(-D) \arrow[dl, "\sigma_{Y_i} \otimes \sigma_D"] \\
& \mathcal{O}_X &
\end{tikzcd}$$

commutes ([15] Tag 0C4S). Then, by an abuse of notation, there is a commuting triangle

$$\begin{tikzcd}
\mathcal{O}(-Y_i) \otimes \mathcal{O}(-D) \arrow[rr, "\sigma_{Y_i} \otimes 1"] \arrow[dr, "\sigma_{Y_i+D}"'] & & \mathcal{O}_X \otimes \mathcal{O}(-D) \arrow[dl, "1 \otimes \sigma_D"] \\
& \mathcal{O}_X &
\end{tikzcd}$$

and tensoring with $\mathcal{O}(D)$ we get a commuting triangle

$$\begin{tikzcd} \mathcal{O}(-Y_i) \arrow[rr, "\sigma_{Y_i}"] \arrow[dr, "\sigma_{Y_i+D}\otimes 1"'] & & \mathcal{O}_X \arrow[dl, "\lambda_D"] \\ & \mathcal{O}(D) & \end{tikzcd}$$

where $\lambda_D : \mathcal{O}_X \to \mathcal{O}(D)$ is the defining section. This data fits into the hypotheses of our technical Lemma 4.2.1, with $\mathcal{F} = \bigoplus \mathcal{O}(-Y_i - D), \mathcal{L} = \mathcal{O}(D), s = \sum \sigma_{Y_i+D}, t = \sum \sigma_{Y_i}$, and $\lambda = \lambda_D$. Therefore applying the Lemma with $n = 0$ with the notation ∂ the differential in $K^\bullet(\bigoplus \mathcal{O}(-Y_i - D), \sum \sigma_{Y_i+D}$ and δ the differential in $K^\bullet(\bigoplus \mathcal{O}(-Y_i), \sum \sigma_{Y_i})$, we have

$$\operatorname{Ker}(\partial^{-q}) = \operatorname{Ker}(\delta^{-q}) \otimes \mathcal{O}(-qD), \quad \operatorname{Im}(\partial^{-q-1}) = \operatorname{Im}(\delta^{-q-1}) \otimes \lambda_D(\mathcal{O}_X) \otimes \mathcal{O}((1-q)D).$$

However, we note that

$$\operatorname{Tor}_q(\mathcal{O}_{Y_1+D}, \dots, \mathcal{O}_{Y_n+D}) = H^{-q}(K^\bullet(\bigoplus \mathcal{O}(-Y_i - D), \sum \sigma_{Y_i+D}))$$

and so we have

$$\begin{aligned} \operatorname{Tor}_q(\mathcal{O}_{Y_1+D}, \dots, \mathcal{O}_{Y_n+D}) = \frac{\operatorname{Ker}(\partial^{-q})}{\operatorname{Im}(\partial^{-q-1})} &= \frac{\operatorname{Ker}(\delta^{-q}) \otimes \mathcal{O}(-qD)}{\operatorname{Im}(\delta^{-q-1}) \otimes \lambda(\mathcal{O}_X) \otimes \mathcal{O}((1-q)D)} \\ &\cong \frac{\operatorname{Ker}(\delta^{-q}) \otimes \mathcal{O}(D)}{\operatorname{Im}(\delta^{-q-1}) \otimes \lambda_D(\mathcal{O}_X)} \otimes \mathcal{O}((1-q)D) \\ &\cong \frac{\operatorname{Ker}(\delta^{-q})}{\operatorname{Im}(\delta^{-q-1}) \otimes \mathcal{O}(-D)} \otimes \mathcal{O}(-qD). \end{aligned}$$

We define $\mathcal{H}^q$ to be $\frac{\operatorname{Ker}(\delta^{-q})}{\operatorname{Im}(\delta^{-q-1})\otimes\mathcal{O}(-D)}$ and prove that it is the pullback in the square $(\dagger)\otimes\mathcal{O}(qD)$. Since tensoring with a line bundle is an equivalence of categories this will prove the claim. To see $\mathcal{H}^q$ as the pullback in the diagram $(\dagger)\otimes\mathcal{O}(qD)$, we first construct a morphism from $\mathcal{H}^q$ to the pullback and then check that it is locally an isomorphism. We note that quotienting by the inclusions of $\operatorname{Ker}(\delta^{-q}) \otimes \mathcal{O}(-D)$ and $\operatorname{Im}(\delta^{-q-1})$ into $\operatorname{Ker}(\delta^{-q})$ induces projections

$$\mathcal{H}^q \to \operatorname{Ker}(\delta^{-q}) \otimes \mathcal{O}_D \cong i_* i^* \operatorname{Ker}(\delta^{-q}) = \operatorname{Ker}(\delta^{-q})|_D$$

$$\mathcal{H}^q \to \operatorname{Ker}(\delta^{-q}) / \operatorname{Im}(\delta^{-q-1}) = \operatorname{Tor}_q(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_n})$$

These projections are the morphisms we use in the square $(\dagger)$. With these morphisms, the square $(\dagger)$ locally becomes the commutative square in Proposition 4.1.7, so $(\dagger)$ commutes globally. We therefore have a global induced map from $\mathcal{H}^q$ to the pullback. Since locally the square (†) becomes the square in 4.1.7, we need to check the condition that x does not belong to any prime ideal in $\mathrm{Ass}_R(\mathrm{Ker}(\delta^{-q})/\operatorname{Im}(\delta^{-q-1}))$, where x is the local defining section for D. However, we assumed that D did not contain any of the associated points of $\mathrm{Tor}_q(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_n})$ and so this condition is satisfied. Therefore, by Proposition 4.1.7, the square (†) is locally a cartesian square. Thus our global morphism from $\mathcal{H}^q$ to the pullback is locally an isomorphism, and so a global one. □

Remark 4.2.3. *As in the affine case, there is a statement that can be made without the assumption on D and associated points of the multitors for $Y_1, \dots, Y_n$. However, in that case the multitor of the $Y_i + D$ only have a quotient surjection onto the pullback object in a diagram involving the multitors of the Y_i. Furthermore, the object in the bottom right of the fibre square becomes harder to describe.*

Corollary 4.2.4. *With the same notation as in Theorem 4.2.2, if $\bigcap Y_i = Z$ is a local complete intersection and D does not contain any of the irreducible components of Z then there is a fibre square*

$$\begin{array}{ccc} \mathrm{Tor}_q(\mathcal{O}_{Y_1+D}, \dots, \mathcal{O}_{Y_n+D}) & \longrightarrow & (\mathrm{Ker}(\delta^{-q}) \otimes \mathcal{O}(-qD))|_D \\ \downarrow & & \downarrow \\ j_* \bigwedge^q \mathcal{E}_Z \otimes \mathcal{O}(-qD) & \longrightarrow & h_*(\bigwedge^q \mathcal{E}_Z|_{Z\cap D}) \otimes \mathcal{O}(-qD), \end{array}$$

where $q : Z \to X, h : Z \cap D \to X$ are the closed immersions coming from the intersection.

Proof. By Theorem 3.2.4, we have that $\mathrm{Tor}_q(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_n}) \cong j_* \bigwedge^q \mathcal{E}_Z$, where $\mathcal{E}_Z$ is the excess bundle. We also claim that $j_* \bigwedge^q \mathcal{E}_Z|D \cong h_*(\bigwedge^q \mathcal{E}_Z|_{Z\cap D})$. Consider the fibre square

$$\begin{tikzcd} D \cap Z \arrow[r, "i_Z"] \arrow[d, "j_D"] \arrow[dr, "h"] & Z \arrow[d, "j"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

then we are claiming that the base change morphism applied to $\bigwedge^q \mathcal{E}_Z$ is an isomorphism. Since the sheaf $\bigwedge^q \mathcal{E}_Z$ is locally free, this is indeed true. Finally, we have only assumed that D doesn't contain any of the irreducible components of Z. However, by Lemma 4.2.5, this means that D doesn't contain any of the associated points of $\mathcal{O}_Z$. Additionally, since the multitors are given by locally free sheaves on Z, their associated points are also just given by the associated points on $\mathcal{O}_Z$, so we can apply Theorem 4.2.2 to conclude the statement. $\square$

Lemma 4.2.5. *Let Z be a local complete intersection subscheme of a noetherian Cohen-Macaulay (CM) scheme X. Then the only associated points for Z are the generic points of its irreducible components.*

Proof. By [11][Thm 17.3], if A is a local, noetherian, CM ring and $f_1, \ldots, f_n$ are a regular sequence in A, then $A/(f_1, \ldots, f_n)$ is a CM-module over A. Additionally, CM modules have no embedded associated primes. Now, by [11][Thm 6.2] if a module M over A has the property that M_P has no embedded primes over A_P for every prime ideal P, then M has no embedded prime ideals over A. Hence the structure sheaf of Z has no embedded associated points, and therefore its associated points are only the generic points of its irreducible components. $\square$

Corollary 4.2.6. *Suppose that in the context of Theorem* 4.2.2*, $Y_1 = Y_2 = \cdots = Y_n = Y$. Then the resulting fibre square is isomorphic to the fibre square*

$$\begin{array}{ccc} \operatorname{Tor}_q(\mathcal{O}_{Y+D}, \ldots, \mathcal{O}_{Y+D}) & \longrightarrow & \operatorname{Tor}_q(\mathcal{O}_D, \ldots, \mathcal{O}_D) \otimes \mathcal{O}(-qY) \\ \downarrow & & \downarrow \\ \operatorname{Tor}_q(\mathcal{O}_Y, \ldots, \mathcal{O}_Y) \otimes \mathcal{O}(-qD) & \longrightarrow & \operatorname{Tor}_q(\mathcal{O}_Y, \ldots, \mathcal{O}_Y) \otimes \mathcal{O}(-qD)|_D. \end{array}$$

Moreover, in this case

$$\begin{aligned} \operatorname{Tor}_q(\mathcal{O}_Y, \ldots, \mathcal{O}_Y) \otimes \mathcal{O}(-qD)|_D &\cong \operatorname{Tor}_q(\mathcal{O}_Y, \ldots, \mathcal{O}_Y) \otimes \mathcal{O}(-qD)|_{D\cap Y} \\ &\cong \operatorname{Tor}_q(\mathcal{O}_D, \ldots, \mathcal{O}_D) \otimes \mathcal{O}(-qY)|_{D\cap Y}. \end{aligned}$$

Proof. The claim here is that

$$(\operatorname{Ker}(\delta^{-q}) \otimes \mathcal{O}(-qD))|_D \cong \operatorname{Tor}_q(\mathcal{O}_D, \ldots, \mathcal{O}_D) \otimes \mathcal{O}(-qY),$$

where δ^{-q} is the differential in $K^\bullet(\bigoplus_{i=1}^n \mathcal{O}(-Y), \sum \sigma_Y)$. Note that by the multi-self-intersection formula (which is a special case of Theorem 3.2.4)

$$\operatorname{Tor}_q(\mathcal{O}_D, \ldots, \mathcal{O}_D) \cong \bigwedge^q \mathcal{O}_D^{n-1}(-D) \cong (\bigwedge^q \mathcal{O}_D^{n-1}) \otimes \mathcal{O}_D(-qD).$$

To compute the kernels of the complex $K^\bullet(\bigoplus_{i=1}^n \mathcal{O}(-Y), \sum \sigma_Y)$ note that there is a change of basis isomorphism

$$K^\bullet(\bigoplus_{i=1}^{n} \mathcal{O}(-Y), \sum \sigma_Y) \cong K^\bullet(\mathcal{O}(-Y), \sigma_Y) \otimes K^\bullet(\bigoplus_{i=1}^{n-1} \mathcal{O}(-Y), 0).$$

There is therefore a canonical isomorphism

$$\mathrm{Ker}(\delta^q) \cong \mathcal{O}_X \otimes \bigwedge^q \mathcal{O}(-Y)^{n-1} \cong \bigwedge^q \mathcal{O}_X^{n-1} \otimes \mathcal{O}(-qY)$$

where the $\bigwedge^q \mathcal{O}(-Y)^{n-1}$ here appear as the kernel sheaves of the complex $K^\bullet(\bigoplus_{i=1}^{n-1} \mathcal{O}(-Y), 0)$. With $j : D \to X$ being the closed immersion inclusion of the divisor D we therefore have the identifications;

$$\begin{aligned}(\mathrm{Ker}(\delta^{-q}) \otimes \mathcal{O}(-qD))|_D = j_*j^*(\mathrm{Ker}(\delta^{-q}) \otimes \mathcal{O}(-qD)) &\cong j_*j^*(\bigwedge^q \mathcal{O}_X^{n-1} \otimes \mathcal{O}(-qY) \otimes \mathcal{O}(-qD)) \\ &\cong j_*(\bigwedge^q \mathcal{O}_D^{n-1} \otimes \mathcal{O}_D(-qY) \otimes \mathcal{O}_D(-qD)) \\ &\cong j_*(\bigwedge^q \mathcal{O}_D(-D)^{n-1} \otimes \mathcal{O}_D(-qY)) \\ &\cong \mathrm{Tor}_q(\mathcal{O}_D, \ldots, \mathcal{O}_D) \otimes \mathcal{O}(-qY)\end{aligned}$$

where the last isomorphism follows from the projection formula. The assertion of the equivalences at the end of the statment of the Corollary are trivially verified from the computations in the proof. □

Remark 4.2.7. *If we take the divisor Y to be the empty divisor (though this is not an effective Cartier divisor the proofs do hold with this degenerate case), this Corollary yields an alternative proof to the self-intersection formula for a divisor D. There is no local gluing argument required, but there wouldn't be in the classical proof anyway since the Koszul resolutions for a divisor are global.*

Chapter 5

Further Work

We wish to make some remarks about generalisations of the previous results. Let X be a non-singular variety and $Y_1, \ldots, Y_n$ local complete intersection subvarieties whose intersection $\bigcap Y_i$ can be decomposed into a union $\bigcup Z_j$ of local complete intersection subvarieties (not necessarily irreducible). Pick any Z_j. Then on the set $U_j = X \setminus \bigcup_{k \neq j} Z_k$ we are in the situation of Theorem 3.2.4. Thus

$$\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{Y_1}, \ldots, \mathcal{O}_{Y_l})|_{U_j} \cong i_{Z_j *} \bigwedge^q \mathcal{E}_{Z_j}|_{U_j}$$

where i_{Z_j} is the inclusion of Z_j in X. Hence, whatever the Tor_q are globally, its restriction to each component *away from any other component* is given as the exterior powers of the excess bundle for that component, and the intersections of the components are the only places where there is interesting behaviour. However, there is an imprecision with this statement, which is that the component-wise excess sheaves $\mathcal{E}_{Z_j}$ are only defined on the complement of the other components. While there are cases where one may extend these excess sheaves to locally free sheaves on the whole of Z_j, even in our case this will not yield the correct answer. Indeed even in the situation of Theorem 4.2.2, if there is no excess on Z, $\mathrm{Tor}_q^X(\mathcal{O}_{Y_1}, \ldots, \mathcal{O}_{Y_n}) \cong (\mathrm{Ker}(\delta^{-q}) \otimes \mathcal{O}(-qD))|_D$ which is not necessarily a locally free $\mathcal{O}_D$-module. Given the nature of our answer, it seems natural to believe that there will be projections from the global multitor to *coherent* extensions of the excess bundles of each Z_i. We conjecture that the multitors will form limits over systems consisting of these projections and further projections to coherent sheaves supported on the intersections of the Z_j.

Ideally, we would like to be able to make use of the technique of the proof of Theorem 3.2.4 in cases where the intersection of the Y_i is no longer an lci. However, following the logic of the proof, one would still need to be able to compare the local Koszul cohomologies to the restrictions of some global object in the case one is studying. Suppose that one found a global sheaf $\mathcal{G}$ on X and morphism $f : \mathcal{G} \to w_* \bigwedge^q \bigoplus \mathcal{C}_{Y_i/X}|_W$ such that there is an open affine cover of X for which there are commuting diagrams on each element U of the cover;

$$\begin{tikzcd} H^{-q}(\Delta^*(i_1 \times i_2)_* \mathcal{O}_{Y_1 \times Y_2})|_U \arrow[r, "H^{-q}\eta_w|_U"] \arrow[d, "\cong"] & H^{-q}(w_* w^* \Delta_*(i_1 \times i_2)_* \mathcal{O}_{Y_1 \times Y_2})|_U \arrow[d, "\varphi_q|_U"] \\ \mathcal{G}|_U \arrow[r, "f|_U"] & w_* \bigwedge^q (\mathcal{C}_{Y_1/X} \oplus \mathcal{C}_{Y_2/X})|_W . \end{tikzcd}$$

We want to show that the local vertical isomorphisms on the left glue up to give a global isomorphism $H^{-q}(\Delta^*(i_1 \times i_2)_* \mathcal{O}_{Y_1 \times Y_2}) \cong \mathcal{G}$. The classical approach is to show that the restriction of the isomorphisms on open sets U and V agree on $U \cap V$. However, let P be the global categorical pullback in the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\cong"] & H^{-q}(w_* w^* \Delta_*(i_1 \times i_2)_* \mathcal{O}_{Y_1 \times Y_2}) \arrow[d, "\varphi_q"] \\ \mathcal{G} \arrow[r, "f"] & w_* \bigwedge^q (\mathcal{C}_{Y_1/X} \oplus \mathcal{C}_{Y_2/X})|_W . \end{tikzcd}$$

The left vertical arrow is an isomorphism by the properties of pullbacks in abelian categories. Since taking pullbacks commutes with restricting to open sets, by the universal property of pullbacks on $U \cap V$ we have two isomorphisms

$$H^{-q}(\Delta^*(i_1 \times i_2)_* \mathcal{O}_{Y_1 \times Y_2})|_{U \cap V} \to P|_{U \cap V} \leftarrow H^{-q}(\Delta^*(i_1 \times i_2)_* \mathcal{O}_{Y_1 \times Y_2})|_{U \cap V}$$

coming from the restrictions of the induced isomorphisms on U and V. These isomorphisms have the property that when post-composed with the morphism

$$P_{U \cap V} \to H^{-q}(w_* w^* \Delta^*(i_1 \times i_2)_* \mathcal{O}_{Y_1 \times Y_2})|_{U \cap V}$$

they are equal. To conclude that our isomorphisms agree then, it would be sufficient to know that our post-composition morphism is injective. This is what occurs in Theorem 3.2.4. In cases where our post-composition morphism is not injective, the analysis of whether or not the local isomorphisms agree will have to be investigated more closely.

The next case to consider would be one in which the intersection consists of two components again, but this time where the higher dimensional component has codimension 2. The difficulty of this case can be seen even in the affine setting. One could just take the union of any codimension 2 subvariety with a local complete intersection, or one may take a transverse intersection of a divisor + lci intersection with another divisor. This second case would be easier to study, since we have a cone description of Koszul complexes. However, it is not clear to us how to proceed along a general argument at present.

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