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Moment asymptotics for super-Brownian motions

YAOZHONG HU[1,a], XIONG WANG[2,b], PANQIU XIA[3,c] and JIAYU ZHENG[4,d]

[1]*Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1, Canada,* [a]*yaozhong@ualberta.ca*
[2]*Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA,* [b]*xiong_wang@jhu.edu*
[3]*Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849, USA,* [c]*pqxia@auburn.edu*
[4]*Faculty of Computational Mathematics and Cybernetics, Shenzhen MSU-BIT University, Shenzhen, Guangdong, 518172, China,* [d]*jyzheng@smbu.edu.cn*

In this paper, long-time and high-order moment asymptotics for super-Brownian motions (sBm's) are studied. By using a moment formula for sBm's (e.g. (Hu et al, 2023, Theorem 3.1)), precise upper and lower bounds for all positive integer moments at any time $t > 0$ of sBm's for certain initial conditions are achieved. Then, the moment asymptotics as time goes to infinity or as the moment order goes to infinity follow immediately. Additionally, as an application of the two-sided moment bounds, the tail probability estimates of sBm's are obtained.

Keywords: Intermittency; moment asymptotics; moment formula; super-Brownian motion; tail probability; two-sided moment bounds

1. Introduction

Super-Brownian motions (sBm's), also called the Dawson-Watanabe superprocesses, are a class of measure-valued Markov processes (cf. (Dawson, 1993, Etheridge, 2000, Perkins, 2002, etc)) which play a key role in branching processes. Because of their close connection to positive solutions to a class of nonlinear elliptic partial differential equations (cf. Dynkin (2002, 2004)), sBm's have attracted much attention in the past few decades. In the one spatial dimensional situation, it is well-known that, with an initial condition u_0 being a deterministic finite measure on $\mathbb{R}$, an sBm has a density with respect to the Lebesgue measure almost surely as long as $t > 0$. This density is the unique weak (in the probabilistic sense) solution to the stochastic partial differential equation (SPDE),

$$\frac{\partial}{\partial t} u_t(x) = \frac{1}{2}\Delta u_t(x) + \sqrt{u_t(x)}\dot{W}(t,x), \tag{1}$$

where $\dot{W}$ denotes the space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$, i.e.,

$$\mathbb{E}\big(\dot{W}(t,x)\big) = 0 \quad \text{and} \quad \mathbb{E}\big(\dot{W}(t,x)\dot{W}(s,y)\big) = \delta(t-s)\delta(x-y)\,.$$

In the present paper, we explore the moment asymptotics of the one-dimensional sBm as time $t \uparrow \infty$ or moment order $n \uparrow \infty$, under certain initial conditions. The moment asymptotics and related intermittency properties for the solution of SPDEs have been intensively studied under the global Lipschitz assumption on diffusion coefficients (cf. (Bertini and Cancrini, 1995, Carmona and Molchanov, 1994, Chen, 2015, Chen and Dalang, 2015, Chen et al, 2024+, Das and Tsai, 2021, Foondun and Khoshnevisan., 2009, Hu and Wang, 2024, etc)). However, in equation (1), the diffusion coefficient for sBm's is not Lipschitz at 0. Thus these results are not applicable to sBm's. Indeed, from (Hu et al, 2023, Proposition 4.7), one can easily deduce that the n-th moment of an sBm is of at most polynomial growth in t. In comparison with, e.g. the parabolic Anderson model (the square root term $\sqrt{u_t(x)}$ in (1) is replaced by $u_t(x)$) to which the n-th moment of the solution is of the exponential growth (cf. (Chen, 2015, Theorem 1.1)), the growth of the sBm is much slower, which implies that the intermittency property does not hold for sBm's. A few months following the announcement of our paper, a preprint Chen and Xia (2023) delved into the investigation of the asymptomatic behavior within a substantial category of sublinear SPDE's including sBm's. However, as a compromise for its robustness, the approach presented in this preprint yields an upper bound that lacks sharpness when compared to our findings; see the remark "Generality versus sharpness" on (Chen and Xia, 2023, Section 2).

On the other hand, one may regard an sBm satisfying (1) as a special case of the following equation with $\beta = \frac{1}{2}$,

$$\frac{\partial}{\partial t} u_t(x) = \frac{1}{2}\Delta u_t(x) + u_t(x)^{\beta}\dot{W}(t,x), \quad \beta \in [0,1]. \tag{2}$$

It is well-known that if $\beta = 1$, equation (2) is the parabolic Anderson model, whose solution is the exponential of the solution to the Kardar-Parisi-Zhang equation (see Kardar et al (1986)) through the Hopf-Cole's transformation (cf. Bertini and Giacomin (1997), Hairer (2013)). A sequence of results related to the moment asymptotics, such as intermittency, high peaks (Anderson's localization), (macroscopic) multifractality, etc., have been fully studied (cf. (Chen, 2015, Conus et al, 2013, Conus and Khoshnevisan, 2012, etc)). Instead, if $\beta = 0$, equation (2) degenerates to a stochastic heat equation with additive noise. Even if the intermittency property fails in this case, the solution to (2) with $\beta = 0$ is still microscopically multifractal with high peaks (cf. Khoshnevisan et al (2017, 2018)). It is natural to conjecture that sBm's are also microscopically multifractal with high peaks and it is also natural to guess that the corresponding parameters shall be bounded between those for the parabolic Anderson model and those for additive noise case. However, this seems to be a difficult task and we will not address it here in this work.

In the following, we present the hypothesis on initial condition u_0 which is assumed by default in the rest of the paper.

Hypothesis 1. *u_0 is a σ-finite measure on $\mathbb{R}$ fulfilling the following conditions.*

(i) *There exists a constant $p \geq 0$ such that*

$$\int_{\mathbb{R}} \frac{u_0(dx)}{1+|x|^p} < \infty.$$

(ii) *There exist constants $\gamma \geq 0$, $K_2 \geq K_1 > 0$, such that for each $x \in \mathbb{R}$,*

$$K_1 \leq t^{\gamma} \int_{\mathbb{R}} p(t, x-z) u_0(dz) \leq K_2,$$

for all $t \geq C_x$ with a constant $C_x \geq 0$ depending on x.

Before presenting our main results, let us make some remarks on the hypothesis. The Condition (i) assumes that the initial condition is of polynomial growth, which is very mild. In fact, both conditions are not too restrictive. First, in the typical context of superprocess (cf. Perkins (2002)), an sBm can be constructed as the scaling limit of a sequence of branching Brownian motions, where the limit is a random variable taking values in $\mathcal{D}(\mathbb{R}_+; \mathcal{M}_F(\mathbb{R}))$, which is the Skorokhod space of functions on $\mathbb{R}_+$ taking values on $\mathcal{M}_F(\mathbb{R})$—the set of all finite measures on $\mathbb{R}$. This requires that the flat initial condition is also a finite measure. Nevertheless, the initial condition $u_0 \equiv 1$ satisfying both conditions in Hypothesis 1 with $p = 2$, $\gamma = 0$, $K_1 = K_2 = 0$ and $C_x \equiv 0$ for all $x \in \mathbb{R}$, is not a finite measure. In such case, (Konno and Shiga, 1988, Theorem 1.4) ensures the existence, uniqueness, and non-negativity of sBm's starting at an infinite measure fulfilling Hypothesis 1 (i). We also refer readers to (Li and Pu, 2023, Section 2) for a heuristic discussion on this problem.

On the other hand, the delta initial condition is also covered by Hypothesis 1 (cf. Chen and Dalang (2015) for SPDEs with rough initial conditions). In fact, Suppose $u_0 = \delta_z$ with some $z \in \mathbb{R}$, where δ denotes the Dirac delta measure. Then for every $x \in \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{u_0(dx)}{1 + x^p} = \frac{1}{1 + |z|^p} < \infty,$$

with any $p \geq 0$; and letting $\gamma = 1/2$, $K_1 = \frac{1}{2\sqrt{\pi}}$ and $K_2 = \frac{1}{\sqrt{2\pi}}$, whenever $t \geq C_x := \frac{(x-z)^2}{\log 2}$,

$$t^{\frac{1}{2}} \int_{\mathbb{R}} p(t, x - y)\delta_z(dy) = \frac{1}{\sqrt{2\pi}} \exp\Big(- \frac{(x-z)^2}{2t} \Big) \in [K_1, K_2].$$

In other words, Hypothesis 1 is satisfied by $u_0 = \delta_z$.

It is interesting to notice that in this case $C_x \approx Kx^2$. This is true when the initial condition has a (deterministic) density and $u_0(x) \approx |x|^\alpha$ for any $\alpha > -1$. In fact, in this case

$$\int_{\mathbb{R}} p(t, x - z)u_0(z)dz \approx \mathbb{E}\big[|B_t + x|^\alpha\big] = t^{\alpha/2}\mathbb{E}\left[\left|X + \frac{x}{\sqrt{t}}\right|^\alpha\right]$$

for a standard normal X. Because $\mathbb{E}\left[|X + z|^\alpha\right]$ is a continuous function of $x = z$ which is never 0, we see that the condition (ii) is satisfied for $u_0(x) \approx |x|^\alpha$ with $\gamma = \alpha/2$ and with $C_x = Kx^2$.

Now we state our first result of this paper. Specifically, the matching two-sided bound (5) below is applicable to the flat initial condition $u_0 \equiv 1$.

Theorem 1.1. *Let $u = \{u_t(x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}\}$ be the solution to (1). Then, there are positive constants K_* and K^* independent of n, t and x, with $K_* \leq K^*$, such that*

$$\mathbb{E}\big(u_t(x)^n\big) \leq (K^*)^n n! t^{\frac{n-1}{2} - \gamma}, \tag{3}$$

for all $(t, x) \in [C_x \vee 1, \infty) \times \mathbb{R}$ and

$$\mathbb{E}\big(u_t(x)^n\big) \geq K_*^n n! t^{\frac{n-1}{2} - \gamma} \tag{4}$$

for all $(t, x) \in [nC_x, \infty) \times \mathbb{R}$, where $C_x > 0$ is the same as in Hypothesis 1 (ii). Moreover, if $C_x = 0$, then we can write

$$K_*^n \Big(1 + n! t^{\frac{n-1}{2}}\Big) \leq \mathbb{E}\big(u_t(x)^n\big) \leq (K^*)^n \Big(1 + n! t^{\frac{n-1}{2}}\Big), \tag{5}$$

for all $(t, x) \in [1, \infty) \times \mathbb{R}$; and if further assume $\gamma = 0$, then (5) holds for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

The basic tool in the proof of Theorem 1.1 is the moment formula for sBm's derived in Hu et al (2023). Moment formulas for superprocesses have been investigated in diverse setups, exemplified by prior works such as (Dawson and Kurtz, 1982, Hu et al, 2019, Konno and Shiga, 1988, Wang, 1998, etc.)

using a duality argument to Markov processes (see e.g., (Mueller, 2009, Section 3)). Informally, for a measure-valued Markov process $X = \{X_t \in \mathcal{M}_F(\mathbb{R}^d)\}$ satisfying certain properties, given any $m \in \mathbb{N}$ and any test function ϕ in the Schwartz space of functions on $\mathbb{R}^{md}$, there exists a pair of processes (n, f), where $n = \{n_t \in \mathbb{N} \colon t \in \mathbb{R}_+\}$ and $f = \{f_t \in C(\mathbb{R}^{n_t d}) \colon t \in \mathbb{R}_+\}$ with $n_0 = m$ and $f_0 = \phi$, such that

$$\langle X_t^{\otimes m}, \phi \rangle = \langle X_0^{\otimes n_t}, f_t \rangle, \qquad \text{for all } t \geq 0.$$

Regarding the one-dimensional sBm, one may select $\phi = p(\epsilon, x - \cdot)^{\otimes n}$ with $\epsilon > 0$, where

$$p(t,x) \coloneqq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

denotes the heat kernel, and take $\epsilon \downarrow 0$ to get the moment formula for $\mathbb{E}(u_t(x)^n)$. However, a significant challenge arises when attempting to apply this formula. The dual process (n, f) is notably intricate, entailing a sequence of stopping times with jumps, which seems hard to use for deriving sharp estimates, especially for high dimensions. Up to this point, we have not successfully identified a suitable approach to apply this duality formula to get a precise two-sided bound. Therefore, we opt to employ an alternative formula, as outlined in Theorem A.1.

According to the moment formula (27) in the appendix, the n-th moment of an sBm can be represented as the summation of a finite sequence of integrals. Thus it suffices to obtain some sharp bounds for each summand. Fix positive integers $n > n'$ and $(\alpha, \beta, \tau) \in \mathcal{J}_{n,n'}$ as in (27). One may see that the corresponding summand is a space-time integral of heat kernels. Each variable, e.g. z_i, appears at most three times in the integrand, where expressions like $p(t - s_i, x - z_i)^2$ are counted twice. To apply the semi-group property of heat kernels to calculate the integral, we need to show that the product of three heat kernels in terms of z_i is bounded by that of two heat kernels. In fact, for any $(\alpha, \beta, \tau) \in \mathcal{J}_{n,n'}$ with $n \geq 3$ and $n' > 0$, there will be expression $p(t - s_1, x - z_1)^2$ appearing in the integrand (see (Hu et al, 2023, Section 4) for details). Thus, by using the equality

$$p(t,x)^2 = (4\pi t)^{-\frac{1}{2}} p(t/2, x)$$

and the semi-group property of heat kernels, one can exactly calculate the integral in z_1, which involves a factor of the form $p(\frac{1}{2}(t - s_1) + s_1, x - z)$, if $\beta_0 = 0$; or $p(\frac{1}{2}(t - s_1) + (s_1 - s_j), x - z_j)$ with some $j \in \{2, \dots, n'\}$, if $\beta_0 = 1$. In the former case, we find that $p(t - s_2, x - z_2)^2$ appears in the integrand. Thus, one can further proceed with the integration in z_2. On the other hand, assuming $\beta_0 = 1$, it can be proved that either $p(t - s_2, x - z_2)^2$, the same as the case previously discussed, or $p(t - s_2, x - z_2) p(s_1 - s_2, z_1 - z_2)$ appears in the original integrand. Especially, the existence of $p(t - s_2, x - z_2) p(s_1 - s_2, z_1 - z_2)$ implies that $j = 2$ and thus after the integration in z_1, there is the factor

$$p\Big(\frac{1}{2}(t - s_1) + (s_1 - s_2), x - z_2\Big) p(t - s_2, x - z_2),$$

in the remaining integrand. Thanks to the fact that $\frac{1}{2}(t - s_2) \leq \frac{1}{2}(t - s_1) + (s_1 - s_2) \leq t - s_2$, we can formulate the next inequality

$$\begin{aligned} p\Big(\frac{1}{2}(t - s_2), x - z_2\Big) p(t - s_2, x - z_2) &\lesssim p\Big(\frac{1}{2}(t - s_1) + (s_1 - s_2), x - z_2\Big) p(t - s_2, x - z_2) \\ &\lesssim p(t - s_2, x - z_2)^2. \end{aligned}$$

Throughout the paper, $A \lesssim B$ (and $A \gtrsim B$, $A \sim B$) means that there are universal constants $C_1, C_2 \in (0,\infty)$ such that $A \le C_1 B$ (and $A \ge C_2 B$, $C_1 B \le A \le C_2 B$). Notice that

$$p\Big(\frac{1}{2}(t-s_2), x-z_2\Big) p(t-s_2, x-z_2) = (3\pi(t-s_2))^{-\frac{1}{2}} p\Big(\frac{2}{3}(t-s_2), x-z_2\Big).$$

The semi-group property of the heat kernel can be applied again when computing two-sided bounds of the integral in z_2. Hence, one could expect a desired two-sided bound for $\mathbb{E}(u_t(x)^n)$ via typical iteration arguments. The detailed proof is given in Section 2.

As a consequence of Theorem 1.1, we can write the following two propositions of the large time and the high moment asymptotics for sBm's. The proofs are trivial, and thus skipped for simplification.

Proposition 1.2. *Let $u = \{u_t(x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ be the solution to (1). Then, for every positive integer n and $x \in \mathbb{R}$,*

$$\lim_{t\uparrow\infty} \frac{\log \mathbb{E}\big(u_t(x)^n\big)}{\log t} = \frac{1}{2}(n-1) - \gamma.$$

Proposition 1.3. *Let $u = \{u_t(x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ be the solution to (1). Then, for every $(t,x) \in [C_x, \infty) \times \mathbb{R}$,*

$$\limsup_{n\uparrow\infty} \frac{\log \mathbb{E}\big(u_t(x)^n\big)}{n \log n} \le 1. \tag{6}$$

Moreover, if $C_x = 0$, then for all $(t,x) \in [1,\infty) \times \mathbb{R}$,

$$\lim_{n\uparrow\infty} \frac{\log \mathbb{E}\big(u_t(x)^n\big)}{n \log n} = 1; \tag{7}$$

and if it is further assumed that $\gamma = 0$, then (7) holds for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

Without the assumption that $C_x = 0$, we only get an upper bound for the high moment asymptotics (see (6)). This is because as in Theorem 1.1, inequality (4) holds only for $t > nC_x$. As $n \uparrow \infty$, $nC_x \uparrow \infty$ as well. Thus it seems not possible to have a lower bound for the high moment asymptotics for fixed t by using Theorem 1.1.

Another application of Theorem 1.1 is to get the following tail estimate of sBm's.

Proposition 1.4. *Let $u = \{u_t(x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ be the solution to (1). Fix $x \in \mathbb{R}$. Then for all $t \ge C_x$,*

$$\limsup_{z\uparrow\infty} \frac{\log \mathbb{P}(u_t(x) > z)}{z} \le -Ct^{-\frac{1}{2}}. \tag{8}$$

Moreover, suppose $C_x = 0$. Then, for all $(t,x) \in [1,\infty) \times \mathbb{R}$,

$$-C_1 t^{-\frac{1}{2}} \le \liminf_{z\uparrow\infty} \frac{\log \mathbb{P}(u_t(x) > z)}{z} \le \limsup_{z\uparrow\infty} \frac{\log \mathbb{P}(u_t(x) > z)}{z} \le -C_2 t^{-\frac{1}{2}}; \tag{9}$$

and if is further assumed that $\gamma = 0$, then (9) holds for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Here, C_1, C_2 and C are positive constants independent of x.

Proposition 1.5. *Let $u = \{u_t(x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ be the solution to (1). Fix $x \in \mathbb{R}$. Then,*

$$-C_1 \le \liminf_{t\uparrow\infty} t^{\frac{1}{2}-\sigma} \log \mathbb{P}\big(u_t(x) > t^{\sigma}\big) \le \limsup_{t\uparrow\infty} t^{\frac{1}{2}-\sigma} \log \mathbb{P}\big(u_t(x) > t^{\sigma}\big) \le -C_2, \tag{10}$$

for any $\sigma \in (\frac{1}{2}, \frac{3}{2})$, where C_1 and C_2 are positive constants independent of x. Moreover, if $C_x = 0$, then (10) holds for any $\sigma > \frac{1}{2}$.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. To this end, several lemmas related to the upper and lower bounds are proved in Sections 2.1 and 2.2 respectively. Then, we complete the proof of Theorem 1.1 in Section 2.3.

2.1. The upper bound

For fixed $(\alpha, \beta, \tau) \in \mathcal{J}_{n,n'}$ (see in Section A), let

$$\begin{aligned} \mathcal{X}_{t,x} := &\prod_{i=1}^{n} \Big(\int_{\mathbb{R}} p(t, x-z) u_0(dz) \Big)^{1-\alpha_i} \int_{\mathbb{R}^{n'}} d\mathbf{z}_{n'} \prod_{i=1}^{n'} \Big(\int_{\mathbb{R}} p(s_i, z_i - z) u_0(dz) \Big)^{1-\beta_i} \\ &\times \prod_{i=1}^{|\alpha|} p(t - s_{\tau(i)}, x - z_{\tau(i)}) \prod_{i=|\alpha|+1}^{2n'} p(s_{\iota_\beta(i-|\alpha|)} - s_{\tau(i)}, z_{\iota_\beta(i-|\alpha|)} - z_{\tau(i)}), \end{aligned} \tag{11}$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. In this subsection, we will prove the next lemma for a sharp upper bound for $\mathcal{X}_{t,x}$.

Lemma 2.1. *Let $\mathcal{X}_{t,x}$ be given as in (11) with some $(\alpha, \beta, \tau) \in \mathcal{J}_{n,n'}$ with positive integer n and non-negative integer $n' < n$. Then,*

$$\mathcal{X}_{t,x} \le (2\pi)^{-\frac{n'}{2}} K_2^{n-n'} t^{-\gamma(n-n')} \prod_{i=1}^{n'} (t - s_i)^{-\frac{1}{2}}, \tag{12}$$

for all $(t,x) \in [C_x, \infty) \times \mathbb{R}$, where $K_2 > 0$ and $C_x \ge 0$ are the same as in Hypothesis 1 (ii).

Proof. Let n be any positive integer, and let $n' = 0$. Then, $(\alpha, \beta, \tau) = (\mathbf{0}_n, \partial, \partial)$, and thus

$$\mathcal{X}_{t,x} = \Big(\int_{\mathbb{R}} p(t, x-z) u_0(dz) \Big)^{n}.$$

Thus, inequality (12) is trivially true under Hypotheses 1. Particularly, it holds for $n = 1$ and all $n' \in \{0, \dots, n-1\} = \{0\}$. Additionally, if $n = 2$ and $n' = 1$, we have

$$\mathcal{X}_{t,x} = \int_{\mathbb{R}} d\mathbf{z}_1 \Big(\int_{\mathbb{R}} p(s_1, z_1 - z) u_0(dz) \Big) p(t - s_1, x - z_1) = \int_{\mathbb{R}} p(t, x-z) u_0(dz).$$

For the same reason, we can easily verify this lemma in such a situation ($n = 2$ and $n' = 1$). This allows us to prove this lemma by mathematical induction in n.

Let $n > 2$ and let $n' \in \{1, \dots, n-1\}$. By definition of τ, for any $\{i_1 < i_2\} \subset \{1, \dots, 2n'\}$ such that $\tau(i_1) = \tau(i_2) = 1$, we know that $i_2 \le |\alpha|$. Therefore, $p(t - s_1, x - z_1)^2$ appears in the integrand of $\mathcal{X}_{t,x}$.

Case 1. Suppose $\beta_1 = 0$. Recall that $\beta_{n'} = 0$. It follows that $|\beta| \le n' - 2$. On the other hand, we know that $|\alpha| + |\beta| = 2n'$ and $|\alpha| \le n$. Thus, $n' \le n - 2$, and we can write,

$$\mathcal{X}_{t,x} = \mathcal{X}^0_{t,x}\mathcal{X}^1_{t,x},$$

where

$$\mathcal{X}^0_{t,x} := \int_{\mathbb{R}^2} p(t - s_1, x - z_1)^2 p(s_1, z_1 - z) u_0(dz) dz_1,$$

and

$$\begin{aligned}\mathcal{X}^1_{t,x} := &\prod_{i=1}^{n}\Big(\int_{\mathbb{R}} p(t, x - z) u_0(dz)\Big)^{1-\alpha_i} \int_{\mathbb{R}^{n'-1}} dz_2 \cdots dz_{n'} \prod_{i=2}^{n'}\Big(\int_{\mathbb{R}} p(s_i, z_i - z) u_0(dz)\Big)^{1-\beta_i} \\ &\times \prod_{\substack{1\le i\le|\alpha| \\ i\notin\{i_1,i_2\}}} p(t - s_{\tau(i)}, x - z_{\tau(i)}) \prod_{i=|\alpha|+1}^{2n'} p(s_{\iota_\beta(i-|\alpha|)} - s_{\tau(i)}, z_{\iota_\beta(i-|\alpha|)} - z_{\tau(i)}).\end{aligned}$$

Note that

$$p(t - s_1, x - z_1)^2 = (4\pi(t - s_1))^{-\frac{1}{2}} p\Big(\frac{1}{2}(t - s_1), x - z_1\Big).$$

It follows that

$$\begin{aligned}\mathcal{X}^0_{t,x} =& (4\pi(t - s_1))^{-\frac{1}{2}} \int_{\mathbb{R}^2} p\Big(\frac{1}{2}(t - s_1), x - z_1\Big) p(s_1, z_1 - z) u_0(dz) dz_1 \\ =& (4\pi(t - s_1))^{-\frac{1}{2}} \int_{\mathbb{R}} p\Big(\frac{1}{2}(t - s_1) + s_1, x - z\Big) u_0(dz).\end{aligned}$$

Additionally, using the fact that $\frac{1}{2}(t - s_1) + s_1 \le t - s_1 + s_1 = t$, we can show that

$$p\Big(\frac{1}{2}(t - s_1) + s_1, x - z\Big) \le \sqrt{2}\, p(t, x - z).$$

Therefore, for all $(t, x) \in [C_x, \infty) \times \mathbb{R}$,

$$\mathcal{X}^0_{t,x} \le \frac{K_2}{\sqrt{2\pi}} t^{-\gamma} (t - s_1)^{-\frac{1}{2}}. \tag{13}$$

Let $\alpha' := (\alpha_1, \dots, \alpha_{i_1-1}, \alpha_{i_1+1}, \dots \alpha_{i_2-1}, \alpha_{i_2+1}, \dots, \alpha_n)$, and let $\beta' := (\beta_2, \dots, \beta_{n'})$. Then, it can be verified that $[\alpha', \beta'] \in \mathcal{I}_{n-2,n'-1}$ (cf. Section A). Let $\tau' : \{1, \dots, 2(n'-1)\} \to \{1, \dots, n'-1\}$ be given by

$$\tau'(i) := \begin{cases} \tau(i) - 1, & 1 \le i \le i_1 - 1, \\ \tau(i+1) - 1, & i_1 \le i \le i_2 - 2, \\ \tau(i+2) - 1, & i_2 - 1 \le i \le 2(n'-1). \end{cases}$$

Then, we can also show that $\tau' \in \mathcal{K}_{n-2,n'-1}^{\alpha',\beta'}$. Moreover, $\mathcal{X}_{t,x}^1$ can be represented as follows,

$$\mathcal{X}_{t,x}^1 = \prod_{i=1}^{n-2}\Big(\int_{\mathbb{R}} p(t,x-z)u_0(dz)\Big)^{1-\alpha_i'} \int_{\mathbb{R}^{n'-1}} dz_2\cdots dz_{n'} \prod_{i=1}^{n'-1}\Big(\int_{\mathbb{R}} p(s_{i+1},z_{i+1}-z)u_0(dz)\Big)^{1-\beta_i'}$$
$$\times \prod_{i=1}^{|\alpha'|} p(t-s_{\tau'(i)+1},x-z_{\tau'(i)+1}) \prod_{i=|\alpha'|+1}^{2(n'-1)} p(s_{\iota_{\beta'}(i-|\alpha'|)+1}-s_{\tau'(i)+1}, z_{\iota_{\beta'}(i-|\alpha'|)+1}-z_{\tau'(i)+1}).$$

By using the induction hypothesis, we have

$$\mathcal{X}_{t,x}^1 \le (2\pi)^{-\frac{n'-1}{2}} K_2^{n-n'-1} t^{-\gamma(n-n'-1)} \prod_{i=2}^{n'} (t-s_i)^{-\frac{1}{2}}, \tag{14}$$

for all $(t,x) \in [C_x,\infty)\times\mathbb{R}$.

Hence, inequality (12) is a consequence of inequalities (13) and (14), and the proof of this lemma is complete under the assumption that $\beta_1 = 0$.

Case 2. Suppose that $\beta_1 = 1$, then there exists $j \ge 2$ such that $\tau(|\alpha|+1) = j$. Thus we find the following expression in the integrand of $\mathcal{X}_{t,x}$,

$$p(t-s_1,x-z_1)p(t-s_1,x-z_1)p(s_1-s_j,z_1-z_j).$$

Integrating in z_1 and by a similar argument as in Case 1, we have

$$\mathcal{X}_{t,x} \le \frac{1}{\sqrt{2\pi}}(t-s_1)^{-\frac{1}{2}}\mathcal{X}_{t,x}^2,$$

where

$$\begin{aligned}\mathcal{X}_{t,x}^2 :=& \prod_{i=1}^{n}\Big(\int_{\mathbb{R}} p(t,x-z)u_0(dz)\Big)^{1-\alpha_i} \int_{\mathbb{R}^{n'-1}} dz_2\cdots dz_{n'} \prod_{i=2}^{n'}\Big(\int_{\mathbb{R}} p(s_i,z_i-z)u_0(dz)\Big)^{1-\beta_i}\\
&\times p\Big(\frac{1}{2}(t-s_1)+(s_1-s_j),x-z_j\Big) \prod_{\substack{1\le i\le|\alpha|\\ i\notin\{i_1,i_2\}}} p(t-s_{\tau(i)},x-z_{\tau(i)})\\
&\times \prod_{i=|\alpha|+2}^{2n'} p(s_{\iota_\beta(i-|\alpha|)}-s_{\tau(i)}, z_{\iota_\beta(i-|\alpha|)}-z_{\tau(i)})\\
\le& \sqrt{2}\prod_{i=1}^{n}\Big(\int_{\mathbb{R}} p(t,x-z)u_0(dz)\Big)^{1-\alpha_i} \int_{\mathbb{R}^{n'-1}} dz_2\cdots dz_{n'} \prod_{i=2}^{n'}\Big(\int_{\mathbb{R}} p(s_i,z_i-z)u_0(dz)\Big)^{1-\beta_i}\\
&\times p(t-s_j,x-z_j) \prod_{\substack{1\le i\le|\alpha|\\ i\notin\{i_1,i_2\}}} p(t-s_{\tau(i)},x-z_{\tau(i)})\\
&\times \prod_{i=|\alpha|+2}^{2n'} p(s_{\iota_\beta(i-|\alpha|)}-s_{\tau(i)}, z_{\iota_\beta(i-|\alpha|)}-z_{\tau(i)}).\end{aligned} \tag{15}$$

Let $\alpha'' := (1, \alpha_1, \dots, \alpha_{i_1-1}, \alpha_{i_1+1}, \dots \alpha_{i_2-1}, \alpha_{i_2+1}, \dots, \alpha_n)$, and let $\beta'' := (\beta_2, \dots, \beta_{n'})$. Then, $[\alpha'', \beta''] \in \mathcal{I}_{n-1,n'-1}$. Let $\tau'' : \{1, \dots, 2(n'-1)\} \to \{1, \dots, n'-1\}$ be given by

$$\tau''(i) := \begin{cases} j-1, & i = 1 \\ \tau(i-1)-1, & 2 \le i \le i_1, \\ \tau(i)-1, & i_1 < i < i_2, \\ \tau(i+1)-1, & i_2 \le i \le |\alpha''|, \\ \tau(i+2)-1, & |\alpha''| < i \le 2(n'-1). \end{cases}$$

Then, $\tau'' \in \mathcal{K}^{\alpha'',\beta''}_{n-1,n'-1}$, and we can rewrite (15) as follows

$$\begin{aligned} \mathcal{X}^2_{t,x} \le & \sqrt{2} \prod_{i=1}^{n-1} \Big(\int_{\mathbb{R}} p(t, x-z) u_0(dz) \Big)^{1-\alpha''_i} \int_{\mathbb{R}^{n'-1}} dz_2 \cdots dz_{n'} \prod_{i=1}^{n'-1} \Big(\int_{\mathbb{R}} p(s_{i+1}, z_{i+1} - z) u_0(dz) \Big)^{1-\beta''_i} \\ & \times \prod_{i=1}^{|\alpha''|} p(t - s_{\tau''(i)+1}, x - z_{\tau'(i)+1}) \\ & \times \prod_{i=|\alpha''|+1}^{2(n'-1)} p(s_{\iota_{\beta''}(i-|\alpha''|)+1} - s_{\tau''(i)+1}, z_{\iota_{\beta''}(i-|\alpha''|)+1} - z_{\tau''(i)+1}). \end{aligned}$$

By induction hypothesis again, we have

$$\mathcal{X}^2_{t,x} \le \sqrt{2} (2\pi)^{-\frac{n'-1}{2}} K_2^{n-n'} t^{-\gamma(n-n'-1)} \prod_{i=2}^{n'} (t - s_i)^{-\frac{1}{2}},$$

for all $(t, x) \in [C_x, \infty) \times \mathbb{R}$. The proof of this lemma is thus complete. □

2.2. The lower bound

Recalling moment formula (27), it follows that for all $n \ge 2$,

$$\begin{aligned} \mathbb{E}\big(u_t(x)^n\big) \ge & \Big(\int_{\mathbb{R}} p(t, x-z) u_0(dz) \Big)^n + \sum_{(\alpha,\beta,\tau) \in \mathcal{J}_{n,n-1}} \prod_{i=1}^{n} \Big(\int_{\mathbb{R}} p(t, x-z) u_0(dz) \Big)^{1-\alpha_i} \\ & \times \int_{\mathbb{T}^t_{n-1}} d\mathbf{s}_{n-1} \int_{\mathbb{R}^{n-1}} d\mathbf{z}_{n-1} \prod_{i=1}^{n-1} \Big(\int_{\mathbb{R}} p(s_i, z_i - z) u_0(dz) \Big)^{1-\beta_i} \\ & \times \prod_{i=1}^{|\alpha|} p(t - s_{\tau(i)}, x - z_{\tau(i)}) \prod_{i=|\alpha|+1}^{2(n-1)} p(s_{\iota_\beta(i-|\alpha|)} - s_{\tau(i)}, z_{\iota_\beta(i-|\alpha|)} - z_{\tau(i)}). \end{aligned}$$

In fact, for every $(\alpha, \beta, \tau) \in \mathcal{J}_{n,n-1}$, we know that $\alpha = \alpha_* := \mathbf{1}_n$ and $\beta = \beta_* := (\mathbf{1}_{n-2}, 0)$, where $\mathbf{1}_n$ denotes the n-dimensional vector with unit coordinates. Thus, we can write

$$\mathbb{E}\big(u_t(x)^n\big) \ge \Big(\int_{\mathbb{R}} p(t, x-z) u_0(dz) \Big)^n$$

$$+\sum_{\tau\in\mathcal{K}^{\alpha_*,\beta_*}_{n,n-1}}\int_{\mathbb{T}^t_{n-1}}d\mathbf{s}_{n-1}\int_{\mathbb{R}^{n-1}}d\mathbf{z}_{n-1}\Big(\int_{\mathbb{R}}p(s_{n-1},z_{n-1}-z)u_0(dz)\Big)$$
$$\times\prod_{i=1}^{n}p(t-s_{\tau(i)},x-z_{\tau(i)})\prod_{i=n+1}^{2(n-1)}p(s_{\iota_\beta(i-n)}-s_{\tau(i)},z_{\iota_\beta(i-n)}-z_{\tau(i)}).\tag{16}$$

Let $(\theta_1,\dots,\theta_n)\in(0,1]^n$. Fix $\tau\in\mathcal{K}^{\alpha_*,\beta_*}_{n,n-1}$, and let

$$\mathcal{Y}_{t,x}:=\int_{\mathbb{R}^{n-1}}d\mathbf{z}_{n-1}\Big(\int_{\mathbb{R}}p(s_{n-1},z_{n-1}-z)u_0(dz)\Big)\prod_{i=1}^{n}p(\theta_i(t-s_{\tau(i)}),x-z_{\tau(i)})$$
$$\times\prod_{i=n+1}^{2(n-1)}p(s_{\iota_\beta(i-n)}-s_{\tau(i)},z_{\iota_\beta(i-n)}-z_{\tau(i)}).\tag{17}$$

The next lemma is the main result of this subsection.

Lemma 2.2. *Let $\mathcal{Y}_{t,x}$ be defined as in (17). Then,*

$$\mathcal{Y}_{t,x}\ge K_1(4\pi)^{-\frac{n}{2}}t^{-\gamma}\prod_{i=1}^{n-1}(t-s_i)^{-\frac{1}{2}}\tag{18}$$

for all $(t,x)\in[(\sum_{i=1}^n\theta_i^{-1})C_x,\infty)\times\mathbb{R}$, where $K_1>0$ and $C_x\ge0$ are the same as in Hypothesis 1 (ii).

Proof. We prove this lemma following similar ideas as in Lemma 2.1. Firstly, suppose $n=2$, we have

$$\begin{aligned}\mathcal{Y}_{t,x}&=\int_{\mathbb{R}}d\mathbf{z}_1\Big(\int_{\mathbb{R}}p(s_1,z_1-z)u_0(dz)\Big)p(\theta_1(t-s_1),x-z_1)\\&=\int_{\mathbb{R}}p(\theta_1(t-s_1)+s_1,x-z)u_0(dz).\end{aligned}$$

Then, it is clear that this lemma holds for $n=2$. In the next step, choose $i_1<i_2$ such that $\tau(i_1)=\tau(i_2)=1$ and choose j such that $\tau(n+1)=j$. Then, $1\le i_1<i_2\le n$ and $j\ge2$, and we can write $\mathcal{Y}_{t,x}$ as follows,

$$\mathcal{Y}_{t,x}=\int_{\mathbb{R}^{n-2}}dz_2\cdots dz_{n-1}\Big(\int_{\mathbb{R}}p(s_{n-1},z_{n-1}-z)u_0(dz)\Big)\prod_{\substack{1\le i\le n\\ i\notin\{i_1,i_2\}}}p(\theta_i(t-s_{\tau(i)}),x-z_{\tau(i)})$$
$$\times\prod_{i=n+2}^{2(n-1)}p(s_{\iota_\beta(i-n)}-s_{\tau(i)},z_{\iota_\beta(i-n)}-z_{\tau(i)})$$
$$\times\int_{\mathbb{R}}dz_1p(\theta_{i_1}(t-s_1),z-z_1)p(\theta_{i_2}(t-s_1),z-z_1)p(s_1-s_j,z_1-z_j).$$

Taking account of the fact that

$$p(s,x)p(t,x)=(2\pi(s+t))^{-\frac{1}{2}}p\Big(\frac{st}{s+t},x\Big),$$

for all $s,t\in\mathbb{R}_+$ and $x\in\mathbb{R}$, we can write

$$\begin{aligned} p(\theta_{i_1}(t-s_1),z-z_1)p(\theta_{i_2}(t-s_1),z-z_1) =& \frac{p\left(\frac{\theta_{i_1}\theta_{i_2}}{\theta_{i_1}+\theta_{i_2}}(t-s_1),z-z_1\right)}{(2\pi(\theta_{i_1}+\theta_{i_2})(t-s_1))^{\frac12}} \\ \geq& (4\pi(t-s_1))^{-\frac12}p\Big(\frac{\theta_{i_1}\theta_{i_2}}{\theta_{i_1}+\theta_{i_2}}(t-s_1),z-z_1\Big). \end{aligned}$$

It follows that

$$\begin{aligned} &\int_{\mathbb{R}} dz_1 p(\theta_{i_1}(t-s_1),z-z_1)p(\theta_{i_2}(t-s_1),z-z_1)p(s_1-s_j,z_1-z_j) \\ \geq& (4\pi(t-s_1))^{-\frac12}p\Big(\frac{\theta_{i_1}\theta_{i_2}}{\theta_{i_1}+\theta_{i_2}}(t-s_1)+(s_1-s_j),z-z_j\Big). \end{aligned}$$

On the other hand, it is clear that

$$\frac{\theta_{i_1}\theta_{i_2}}{\theta_{i_1}+\theta_{i_2}}(t-s_j)\leq\frac{\theta_{i_1}\theta_{i_2}}{\theta_{i_1}+\theta_{i_2}}(t-s_1)+(s_1-s_j)\leq t-s_j.$$

Thus, letting

$$\theta' := \Big[\frac{\theta_{i_1}\theta_{i_2}}{\theta_{i_1}+\theta_{i_2}}(t-s_1)+(s_1-s_j)\Big]/(t-s_j),$$

we have $\theta'\in[\frac{\theta_{i_1}\theta_{i_2}}{\theta_{i_1}+\theta_{i_2}},1]\subset(0,1]$. Furthermore, we can write,

$$\begin{aligned} \mathcal{Y}_{t,x} \geq& (4\pi)^{-\frac12}(t-s_1)^{-\frac12}\int_{\mathbb{R}^{n-2}} dz_2\cdots dz_{n-1}\Big(\int_{\mathbb{R}} p(s_{n-1},z_{n-1}-z)u_0(dz)\Big)p(\theta'(t-s_j),z-z_j) \\ &\times\prod_{\substack{1\leq i\leq n\\ i\notin\{i_1,i_2\}}} p(\theta_i(t-s_{\tau(i)}),x-z_{\tau(i)})\prod_{i=n+2}^{2(n-1)} p(s_{\iota_\beta(i-n)}-s_{\tau(i)},z_{\iota_\beta(i-n)}-z_{\tau(i)}). \end{aligned}$$

Similarly to Case 2 of the proof of Lemma 2.1, we can find $\tau'\in\mathcal{K}^{\alpha_0',\beta_0'}_{n-1,n-2}$, where $\alpha_0'=\mathbf{1}_{n-1}$ and $\beta_0'=(\mathbf{1}_{n-3},0)$ such that

$$\mathcal{Y}_{t,x}\geq(4\pi)^{-\frac12}(t-s_1)^{-\frac12}\mathcal{Y}^0_{t,x},$$

with

$$\begin{aligned} \mathcal{Y}^0_{t,x} =& \int_{\mathbb{R}^{n-2}} dz_2\cdots dz_{n-1}\Big(\int_{\mathbb{R}} p(s_{n-1},z_{n-1}-z)u_0(dz)\Big)\prod_{i=1}^{n-1} p(\theta_i''(t-s_{\tau'(i)+1}),x-z_{\tau'(i)+1}) \\ &\times\prod_{i=n}^{2(n-2)} p(s_{\iota_{\beta'}(i-n-1)+1}-s_{\tau'(i)+1},z_{\iota_{\beta'}(i-n-1)+1}-z_{\tau'(i)+1}), \end{aligned}$$

and

$$\theta_i'' = \begin{cases} \theta', & i = 1 \\ \theta_{i-1}, & 2 \le i \le i_1 - 2, \\ \theta_i, & i_1 - 1 \le i \le i_2 - 1, \\ \theta_{i+1}, & i_2 \le i \le n. \end{cases}$$

Notice that the induction hypothesis implies that

$$\mathcal{Y}^0_{t,x} \ge K_1 (4\pi)^{\frac{n-1}{2}} t^{-\gamma} \prod_{i=2}^{n-1} (t - s_i)^{-\frac{1}{2}}, \tag{19}$$

for all $(t,x) \in [(\sum_{i=1}^{n-1}(\theta_i'')^{-1})C_x, \infty) \times \mathbb{R}$. Recalling the construction of $\{\theta_i'' : i = 1, \ldots, n-1\}$ and the fact that $\theta' \in [\frac{\theta_{i_1}\theta_{i_2}}{\theta_{i_1}+\theta_{i_2}}, 1]$, we have

$$\sum_{i=1}^{n-1} (\theta_i'')^{-1} = \sum_{\substack{1 \le i \le n \\ i \notin \{i_1, i_2\}}} \theta_i^{-1} + (\theta')^{-1} \le \sum_{\substack{1 \le i \le n \\ i \notin \{i_1, i_2\}}} \theta_i^{-1} + (\theta_{i_1}^{-1} + \theta_{i_2}^{-1}) = \sum_{i=1}^{n} \theta_i^{-1}.$$

Thus inequality (19) holds for all $(t,x) \in [(\sum_{i=1}^{n} \theta_i^{-1})C_x, \infty) \times \mathbb{R} \subset [(\sum_{i=1}^{n-1}(\theta_i'')^{-1})C_x, \infty) \times \mathbb{R}$, and inequality (18) is straightforward. The proof of this lemma is complete. ☐

2.3. Completion of the proof of Theorem 1.1

The proof of Theorem 1.1 follows from Lemmas 2.1 and 2.2 and the next lemma of the cardinality of $\mathcal{J}_{n,n'}$.

Lemma 2.3 ((Hu et al, 2023, Lemma 4.3)). *Let $\mathcal{J}_{n,n'}$ be defined as in (26) with some positive integer n and nonnegative integer $n' \le n-1$. Then,*

$$|\mathcal{J}_{n,n'}| = \frac{n!(n-1)!}{2^{n'}(n-n')!(n-n'-1)!},$$

where by convention $0! = 1$.

Proof of Theorem 1.1. The case $n = 1$ is trivial. Thus we assume that $n \ge 2$. By Theorem A.1, Lemmas 2.1 and 2.2 with $\theta_1 = \cdots = \theta_n = 1$, we can write

$$\mathbb{E}\big(u_t(x)^n\big) \le \sum_{n'=0}^{n-1} (2\pi)^{-\frac{n'}{2}} K_2^{n-n'} t^{-\gamma(n-n')} |\mathcal{J}_{n,n'}| f_{n'}(t), \tag{20}$$

for all $(t,x) \in [C_x, \infty) \times \mathbb{R}$; and

$$\mathbb{E}\big(u_t(x)^n\big) \ge K_1 (4\pi)^{-\frac{n}{2}} t^{-\gamma} |\mathcal{J}_{n,n-1}| f_{n-1}(t), \tag{21}$$

for all $(t,x) \in [nC_x, \infty)$, where

$$f_{n'}(t) := \int_{\mathbb{T}^t_{n'}} d\mathbf{s}_{n'} \prod_{i=1}^{n'} (t-s_i)^{-\frac{1}{2}} = \frac{2^{n'}}{\Gamma(n')} t^{\frac{1}{2}n'}.$$

Then, inequalities (3) and (4) follow from (20), (21), Lemma 2.3 and Stirling's formula. We remark here that the requirement $t \geq 1$ in (3) is used to avoid the possible blowup around 0 because of $t^{-\gamma(n-n')}$ in (20). Additionally, in case $C_x = 0$, taking account of inequality (16), we can combine (20) and (21) as follows

$$K_1^n + K_1(4\pi)^{-\frac{n}{2}} |\mathcal{J}_{n,n-1}| f_{n-1}(t) \leq \mathbb{E}\big(u_t(x)^n\big) \leq \sum_{n'=0}^{n-1} (2\pi)^{-\frac{n'}{2}} K_2^{n-n'} |\mathcal{J}_{n,n'}| f_{n'}(t),$$

which is true for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. This yields inequality (5) for all $(t,x) \in [1,\infty) \times \mathbb{R}$. If furthermore, $\gamma = 0$, then $t^{\gamma(n-n')}$ in (20) disappears. As a consequence, (5) is valid for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. The proof of this theorem is complete. □

3. Proofs of Propositions 1.4 and 1.5

In this section, we will provide the proofs of Propositions 1.4 and 1.5.

Proof of Proposition 1.4. In the first step, we prove inequality (8). Applying inequality (3), with $\alpha := (2K^*\sqrt{t})^{-1}$, we can write for any $x \in \mathbb{R}$,

$$\mathbb{E}\big(\exp(\alpha u_t(x))\big) = \sum_{n=0}^{\infty} \frac{1}{n!} \alpha^n \mathbb{E}\big(u_t(x)^n\big) \leq 2t^{-\frac{1}{2}-\gamma} < \infty.$$

Thus by using Markov's inequality, we get

$$\mathbb{P}(u_t(x) > z) \leq 2e^{-\alpha z} t^{-\frac{1}{2}-\gamma}, \tag{22}$$

and thus

$$\frac{\log \mathbb{P}(u_t(x) > z)}{z} \leq -\alpha + \frac{\log 2 - (\frac{1}{2}+\gamma)\log(t)}{z} \to -\alpha = -(2K^*\sqrt{t})^{-1},$$

as $z \uparrow \infty$. This proves inequality (8).

In the next step, we will show the next inequality under the assumption that $C_x = 0$,

$$-C_1 t^{-\frac{1}{2}} \leq \liminf_{z\uparrow\infty} \frac{\log \mathbb{P}(|u_t(x)| > z)}{z}. \tag{23}$$

By using inequality (5) and the Paley-Zygmund inequality (cf. (Khoshnevisan, 2014, Lemma 7.3)), we can write

$$\mathbb{P}\Big(u_t(x) > \frac{1}{2} K_*(n!)^{\frac{1}{n}} t^{\frac{n-1}{2n}}\Big) \geq \mathbb{P}\Big(u_t(x) \geq \frac{1}{2}\mathbb{E}\big(u_t(x)^n\big)^{\frac{1}{n}}\Big) \geq \Big(1 - \frac{1}{2^n}\Big)^2 \frac{\big[\mathbb{E}\big(u_t(x)^n\big)\big]^2}{\mathbb{E}\big(u_t(x)^{2n}\big)}$$

$$\geq \left(1-\frac{1}{2^n}\right)^2 \frac{K_*^{2n}(1+n!t^{\frac{1}{2}(n-1)})^2}{(K^*)^{2n}(1+(2n)!t^{\frac{1}{2}(2n-1)})}.$$

If $t \geq 1$, then $(n!)^2 t^{n-1} \leq (2n)!t^{\frac{1}{2}(2n-1)}$ for all $n \geq 1$; otherwise if $t \in (0,1)$, $(n!)^2 t^{n-1} \leq (2n)!t^{\frac{1}{2}(2n-1)}$ still holds for n large enough. As a result, we can write

$$\mathbb{P}\Big(u_t(x) > \frac{1}{2}K_*(n!)^{\frac{1}{n}}t^{\frac{n-1}{2n}}\Big) \geq \frac{9K_*^{2n}\left(1+2n!t^{\frac{1}{2}(n-1)}+(n!)^2t^{n-1}\right)}{16(K^*)^{2n}(1+(2n)!t^{\frac{1}{2}(2n-1)})} \geq \frac{9K_*^{2n}(n!t^{\frac{1}{2}(n-1)})^2}{16(K^*)^{2n}(2n)!t^{\frac{1}{2}(2n-1)}},$$

for n large enough. Taking account of Stirling's formula, we find that

$$\frac{(n!)^2}{(2n)!} \geq \frac{2\pi n(n/e)^{2n}e^{\frac{2}{12n+1}}}{\sqrt{4\pi n}(2n/e)^{2n}e^{\frac{1}{24n}}} = \sqrt{\pi n}2^{-2n}e^{\frac{36n-1}{24n(12n+1)}} \geq 2^{-2n},$$

for all $n \geq 1$. It follows that

$$\mathbb{P}\Big(u_t(x) \geq \frac{1}{2}K_*(n!)^{\frac{1}{n}}t^{\frac{n-1}{2n}}\Big) \geq \left(\frac{K_*}{2K^*}\right)^{2n}t^{-\frac{1}{2}}. \tag{24}$$

Let $z = z(n) := \frac{1}{2}K_*(n!)^{\frac{1}{n}}t^{\frac{n-1}{2n}}$. Then,

$$\frac{\log \mathbb{P}(u_t(x) > z)}{z} \geq \frac{\log \mathbb{P}\Big(u_t(x) \geq \frac{1}{2}K_*(n!)^{\frac{1}{n}}t^{\frac{n-1}{2n}}\Big)}{z} \geq \frac{4n\log\left(\frac{K_*}{2K^*}\right) - \log t}{K_*(n!)^{\frac{1}{n}}t^{\frac{n-1}{2n}}}.$$

Notice that $0 < K_* \leq K^* < 2K^*$, and by Stirling's formula, $(n!)^{\frac{1}{n}} \geq (2\pi n)^{\frac{1}{2n}}e^{\frac{1}{n(12n+1)}} \times \frac{n}{e} \geq \frac{n}{e}$. Thus we can further deduce that

$$\frac{\log \mathbb{P}(u_t(x) > z)}{z} \geq -\left(\frac{4e\log\left(\frac{2K^*}{K_*}\right)}{K_*}t^{\frac{1}{2}-\frac{1}{2n}} + \frac{e\log t}{K_* n t^{\frac{n-1}{2n}}}\right),$$

for n large enough. Let $n \uparrow \infty$ (and thus $z \uparrow \infty$), we get inequality (23) with $C_1 = 4e\log(2K^*/K_*)/K_* > 0$. This proves inequality (9) for all $(t,x) \in [1,\infty) \times \mathbb{R}$. Note that the above arguments do not require $t \geq 1$ once inequality (5) is satisfied. Therefore, assuming $\gamma = 0$, inequality (9) holds for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. The proof of this theorem is complete. □

In the proof of inequality (23), lower bounds for moments of sBm's of all orders are used. This prevents us from deploying the same method to get a lower bound as in (9) without assuming $C_x = 0$. However, if replacing z in (9) by t^σ with some $\sigma > \frac{1}{2}$, we can deduce the next theorem (cf. Iscoe and Lee (1993), Lee and Remillard (1995) for related results).

Proof of Proposition 1.5. Let $\sigma > 0$. Replacing z by t^σ in (22), and using the fact that $2e^{-\alpha z}t^{-\frac{1}{2}-\gamma}$ is decreasing in t, we can deduce the upper bound in (10). Next, we will show the lower bound in (10). Assume $\sigma > \frac{1}{2}$. For any t, let

$$n = n(t) := \left\lfloor \frac{8et^{\sigma-\frac{1}{2}}}{K_*} \right\rfloor. \tag{25}$$

Then, $\lim_{t\uparrow\infty} n(t) = \infty$ and

$$\lim_{t\uparrow\infty} t^{-\frac{1}{2n}} = \lim_{t\uparrow\infty} t^{-K_*/(16et^{\sigma-\frac{1}{2}})} = 1.$$

Thus, for all t large enough, $n \geq 2$ and $t^{-\frac{1}{2n}} \geq \frac{1}{2}$. Using the fact that $(n!)^{\frac{1}{n}} \geq \frac{n}{e}$, we can write

$$\frac{1}{2}K_*(n!)^{\frac{1}{n}} t^{\frac{n-1}{2n}} \geq \frac{K_* n}{4e} t^{\frac{1}{2}} \geq \frac{K_*}{4e}\left(\frac{8et^{\sigma-\frac{1}{2}}}{K_*} - 1\right) t^{\frac{1}{2}} \geq t^{\sigma}.$$

In case $C_x = 0$, it follows from inequality (24),

$$\mathbb{P}(u_t(x) > t^{\sigma}) \geq \mathbb{P}\left(u_t(x) > \frac{1}{2}K_*(n!)^{\frac{1}{n}} t^{\frac{n-1}{2n}}\right) \geq \left(\frac{K_*}{2K^*}\right)^{2n} t^{-\frac{1}{2}},$$

and thus

$$t^{\frac{1}{2}-\sigma} \log \mathbb{P}(u_t(x) > t^{\sigma}) \geq -t^{\frac{1}{2}-\sigma}\left(2\left\lfloor \frac{8et^{\sigma-\frac{1}{2}}}{K_*} \right\rfloor \log\left(\frac{2K^*}{K_*}\right) + \frac{1}{2}\log(t)\right).$$

This proves the lower bound in (10) with $C_1 = 16eK_*^{-1}\log(2K^*/K_*) > 0$ for all $\sigma > \frac{1}{2}$. On the other hand, without assuming $C_x = 0$, to apply inequality (24), one needs $t \geq nC_x$ (see Theorem 1.1). This requirement is fulfilled if $\frac{1}{2} < \sigma < \frac{3}{2}$, as (25) implies that t and n, to which we aim to apply (24), satisfy

$$t \geq \frac{K_*}{8e} n^{\frac{1}{\sigma-1/2}} \geq n,$$

for t and thus n large enough. The proof of Proposition 1.5 is complete. $\square$

Appendix A: A moment formula for sBm's

In this section, we provide a moment formula, cited from (Hu et al, 2023, Theorem 4.1), for sBm's. Let n be any positive integer, and let $n' < n$ be any nonnegative integer. We denote by $\mathcal{I}_{n,n'}$ the collection of multi-indexes $[\alpha,\beta] = [(\alpha_1,\ldots,\alpha_n),(\beta_1,\ldots,\beta_{n'})] \in \{0,1\}^{n+n'}$ satisfying

(i) $\beta_{n'} = 0$.
(ii) Let $|\alpha| = \sum_{i=1}^n \alpha_i$ and let $|\beta| = \sum_{i=1}^{n'} \beta_i$. Then $|\alpha| + |\beta| = 2n'$.

In particular, if $n' = 0$, then $\alpha = \mathbf{0}_n$ is the 0-vector in $\mathbb{R}^n$ and β should be a "0-dimensional" vector. In this case we write $[\alpha,\beta] = [\mathbf{0}_n,\partial]$. Fix $[\alpha,\beta] \in \mathcal{I}_{n,n'}$. We introduce a map $\iota_\alpha : \{1,\ldots,|\alpha|\} \to \{1,\ldots,n\}$ by

$$\iota_\alpha(i) = j_i,$$

where the index j_i is such that α_{j_i} is the i-th nonzero coordinate of α for all $i = 1,\ldots,|\alpha|$. The map $\iota_\beta : \{1,\ldots,|\beta|\} \to \{1,\ldots,n'\}$ is defined in a similar way.

Given $[\alpha,\beta] \in \mathcal{I}_{n,n'}$, let $\mathcal{K}_{n,n'}^{\alpha,\beta}$ be the collection of maps $\tau : \{1,\ldots,|\alpha|+|\beta| = 2n'\} \to \{1,\ldots,n'\}$ satisfying the following properties,

(i) For any $k \in \{1,\ldots,n'\}$, there exist $1 \leq i_1 < i_2 \leq 2n'$ such that $\tau(i_1) = \tau(i_2) = k$.
(ii) For all $i \in \{|\alpha|+1,\ldots,2n'\}$, $\tau(i) > \iota_\beta(i - |\alpha|)$.

If $n' = 0$, we denote $\tau = \partial$. Finally, we write

$$\mathcal{J}_{n,n'} = \{(\alpha, \beta, \tau) : [\alpha, \beta] \in \mathcal{I}_{n,n'}, \tau \in \mathcal{K}^{\alpha,\beta}_{n,n'}\}. \tag{26}$$

Especially, $\mathcal{J}_{n,0} = \{(\mathbf{0}_n, \partial, \partial)\}$.

Theorem A.1. *Suppose that $u_0 \in \mathcal{M}_F(\mathbb{R})$. Let n be a positive integer. Then, for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, the following identity holds,*

$$\begin{aligned}\mathbb{E}\big(u_t(x)^n\big) = \sum_{n'=0}^{n-1} \sum_{(\alpha,\beta,\tau)\in\mathcal{J}_{n,n'}} \prod_{i=1}^{n} \Big(\int_{\mathbb{R}} p(t, x-z) u_0(dz)\Big)^{1-\alpha_i} \\ \times \int_{\mathbb{T}^t_{n'}} d\mathbf{s}_{n'} \int_{\mathbb{R}^{n'}} d\mathbf{z}_{n'} \prod_{i=1}^{n'} \Big(\int_{\mathbb{R}} p(s_i, z_i - z) u_0(dz)\Big)^{1-\beta_i} \prod_{i=1}^{|\alpha|} p(t - s_{\tau(i)}, x - z_{\tau(i)}) \\ \times \prod_{i=|\alpha|+1}^{2n'} p(s_{\iota_\beta(i-|\alpha|)} - s_{\tau(i)}, z_{\iota_\beta(i-|\alpha|)} - z_{\tau(i)}),\end{aligned} \tag{27}$$

where the set $\mathcal{J}_{n,n'}$ of triples (α, β, τ) is defined as (26),

$$\mathbb{T}^t_{n'} = \big\{\mathbf{s}_{n'} = (s_1, \ldots, s_{n'}) \in [0,t]^{n'} : 0 < s_{n'} < s_{n'-1} < \cdots < s_1 < t\big\}.$$

Remark A.2. Theorem A.1 also holds for cases when $u_0 \in C_b(\mathbb{R})$. In fact, if one examines the proof of (Hu et al, 2023, Theorem 4.1), the initial condition does not matter, as long as the integrals appearing in the formula are all finite.

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